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TABLES OF SOME INDEFINITE INTEGRALS

OF BESSEL FUNCTIONS OF INTEGER ORDER

Integrals of the type

$$\int x J_0^2(x)\,dx \qquad \text{or} \qquad \int x J_0(ax) J_0(bx)\,dx$$

are well-known.

Most of the following integrals are not found in the widely used tables of GRADSTEIN/RYSHIK, BATEMAN/ERDÉLYI, ABRAMOWITZ/ STEGUN, PRUDNIKOV/BRYCHKOV/MARICHEV or JAHNKE/EMDE/LÖSCH. The goal of this table was to get tables for practicians. So the integrals should be expressed by Bessel and Struve functions. Indeed, there occured some exceptions. Generally, integrals of the type $\int x^\mu J_\nu(x)\,dx$ may be written with Lommel functions, see [8], 10 -74, or [3], III .
In many cases reccurence relations define more integrals in a simple way.

Partially the integrals may be found by MAPLE as well. In some cases MAPLE gives results with hypergeometric functions, see also [2], 9.6., or [4].
Some well-known integrals are included for completeness.

Here $Z_\nu(x)$ denotes some Bessel function or modified Bessel function of the first or second kind. Partially the functions $Y_\nu(x)$ [sometimes called Neumann's functions or Weber's functions and denoted by $N_\nu(x)$] and the Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are also considered. The same holds for the modified Bessel function of the second kind $K_\nu(x)$.

When a formula is continued in the next line, then the last sign '+' or '-' is repeated in the beginning of the new line.

On page 509 the used special functions and defined functions are described.

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1. Integrals with one Bessel Function:

See also [10], 2. .

1.1. $x^n Z_\nu(x)$ with integer values of n

1.1.1. Integrals of the type $\int x^{2n} Z_0(x)\, dx$

Let

$$\Phi(x) \;=\; \frac{\pi x}{2}\left[J_1(x)\cdot \mathbf{H}_0(x) - J_0(x)\cdot \mathbf{H}_1(x)\right]\;,$$

where $\mathbf{H}_\nu(x)$ denotes the Struve function, see [1], chapter 11.1.7, 11.1.8 and 12.
And let

$$\Psi(x) \;=\; \frac{\pi x}{2}\left[I_0(x)\cdot \mathbf{L}_1(x) - I_1(x)\cdot \mathbf{L}_0(x)\right]$$

be defined with the modified Struve function $\mathbf{L}_\nu(x)$.
Furthermore, let

$$\Phi_Y(x) \;=\; \frac{\pi x}{2}\left[Y_1(x)\cdot \mathbf{H}_0(x) - Y_0(x)\cdot \mathbf{H}_1(x)\right]\;,$$

$$\Phi_H^{(1)}(x) \;=\; \frac{\pi x}{2}\left[H_1^{(1)}(x)\cdot \mathbf{H}_0(x) - H_0^{(1)}(x)\cdot \mathbf{H}_1(x)\right]\;,$$

$$\Phi_H^{(2)}(x) \;=\; \frac{\pi x}{2}\left[H_1^{(2)}(x)\cdot \mathbf{H}_0(x) - H_0^{(2)}(x)\cdot \mathbf{H}_1(x)\right]$$

and

$$\Psi_K(x) \;=\; \frac{\pi x}{2}\left[K_0(x)\cdot \mathbf{L}_1(x) + K_1(x)\cdot \mathbf{L}_0(x)\right]$$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ and simultaneously $\Phi(x)$ by $\Phi_Y(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$ and $\Phi_H^{(p)}(x)$.

Well-known integrals:

$$\int J_0(x)\, dx \;=\; xJ_0(x) + \Phi(x) \;=\; \Lambda_0(x)$$

$$\int I_0(x)\, dx \;=\; xI_0(x) + \Psi(x) \;=\; \Lambda_0^*(x)$$

$$\int K_0(x)\, dx \;=\; xK_0(x) + \Psi_K(x)$$

The new-defined function $\Lambda_0(x)$ is discussed in 1.2.11 a) on page 122 and so is $\Lambda_0^*(x)$ on page 124.
See also [1], 11.1 .

$$\int Y_0(x)\, dx \;=\; xY_0(x) + \Phi_Y(x)$$

$$\int H_0^{(p)}(x)\, dx \;=\; xH_0^{(p)}(x) + \Phi_H^{(p)}(x)\;, \quad p = 1,\, 2$$

$$\int x^2\, J_0(x)\, dx \;=\; x^2 J_1(x) - \Phi(x)$$

$$\int x^2\, I_0(x)\, dx \;=\; x^2 I_1(x) + \Psi(x) \qquad\qquad * E *$$

$$\int x^2\, K_0(x)\, dx \;=\; -x^2 K_1(x) + \Psi_K(x)$$

$$\int x^4\, J_0(x)\, dx \;=\; (x^4 - 9x^2)J_1(x) + 3x^3 J_0(x) + 9\Phi(x)$$

$$\int x^4\, I_0(x)\, dx \;=\; (x^4 + 9x^2)I_1(x) - 3x^3 I_0(x) + 9\Psi(x)$$

$$\int x^4\, K_0(x)\, dx \;=\; -(x^4 + 9x^2)K_1(x) - 3x^3 K_0(x) + 9\Psi_K(x)$$

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$$\int x^6 \, J_0(x) \, dx \;=\; (x^6 - 25x^4 + 225x^2)J_1(x) + (5x^5 - 75x^3)J_0(x) - 225\Phi(x)$$

$$\int x^6 \, I_0(x) \, dx \;=\; (x^6 + 25x^4 + 225x^2)I_1(x) - (5x^5 + 75x^3)I_0(x) + 225\Psi(x)$$

$$\int x^6 \, K_0(x) \, dx \;=\; -(x^6 + 25x^4 + 225x^2)K_1(x) - (5x^5 + 75x^3)K_0(x) + 225\Psi_K(x) \quad \text{and so on.}$$

$$\int x^8 \, J_0(x) \, dx \;=\; (x^8 - 49x^6 + 1\,225x^4 - 11\,025x^2)J_1(x) + (7x^7 - 245x^5 + 3\,675x^3)J_0(x) + 11\,025\Phi(x)$$

$$\int x^8 \, I_0(x) \, dx \;=\; (x^8 + 49x^6 + 1\,225x^4 + 11\,025x^2)I_1(x) - (7x^7 + 245x^5 + 3\,675x^3)I_0(x) + 11\,025\Psi(x)$$

$$\int x^{10} \, J_0(x) \, dx \;=\; (x^{10} - 81x^8 + 3\,969x^6 - 99\,225x^4 + 893\,025)J_1(x)+$$

$$+(9x^9 - 567x^7 + 19\,845x^5 - 297\,675x^3)J_0(x) - 893\,025\Phi(x)$$

$$\int x^{10} \, I_0(x) \, dx \;=\; (x^{10} + 81x^8 + 3\,969x^6 + 99\,225x^4 + 893\,025)I_1(x)-$$

$$-(9x^9 + 567x^7 + 19\,845x^5 + 297\,675x^3)I_0(x) + 893\,025\Psi(x)$$

$$\int x^{12} \, J_0(x) \, dx \;=\; (11x^{11} - 1\,089x^9 + 68\,607x^7 - 2\,401\,245x^5 + 36\,018\,675x^3)J_0(x)+$$

$$+(x^{12} - 121x^{10} + 9\,801x^8 - 480\,249x^6 + 12\,006\,225x^4 - 108\,056\,025x^2)J_1(x) + 108\,056\,025\Phi(x)$$

$$\int x^{12} \, I_0(x) \, dx \;=\; (x^{12} + 121x^{10} + 9\,801x^8 + 480\,249x^6 + 12\,006\,225x^4 + 108\,056\,025x^2)I_1(x)-$$

$$-(11x^{11} + 1\,089x^9 + 68\,607x^7 + 2\,401\,245x^5 + 36\,018\,675x^3)I_0(x) + 108\,056\,025\Psi(x)$$

Let
$$n!! \quad = \quad \begin{cases} 2 \cdot 4 \cdot \ldots \cdot (n-2) \cdot n & , \quad n = 2m \\ 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-2) \cdot n & , \quad n = 2m+1 \end{cases}$$

and $n!! = 1$ in the case $n \le 0$.

<u>General formulas:</u>

$$\int x^{2n} \, J_0(x) \, dx = \left(\sum_{k=0}^{n-2} (-1)^k \frac{[(2n-1)!!]^2 \, x^{2n-2k-1}}{[(2n-1-2k)!!] \cdot [(2n-3-2k)!!]} \right) J_0(x)+$$

$$+ \left(\sum_{k=0}^{n-1} (-1)^k \left[\frac{(2n-1)!!}{(2n-1-2k)!!} \right]^2 x^{2n-2k} \right) J_1(x) + (-1)^n \, [(2n-1)!!]^2 \, \Phi(x) =$$

$$= \left(\sum_{k=0}^{n-2} (-1)^k \frac{[(2n)!]^2 \cdot (n-k)! \cdot (n-k-1)! \cdot x^{2n-2k-1}}{2^{2k+1} \cdot (n!)^2 \cdot (2n-2k)! \cdot (2n-2-2k)!} \right) J_0(x)+$$

$$+ \left(\sum_{k=0}^{n-1} (-1)^k \left[\frac{(2n)! \cdot (n-k)!}{2^k \cdot (n!) \cdot (2n-2k)!} \right]^2 x^{2n-2k} \right) J_1(x) + (-1)^n \left[\frac{(2n)!}{2^n \cdot n!} \right]^2 \Phi(x)$$

and

$$\int x^{2n} \, I_0(x) \, dx = \left(\sum_{k=0}^{n-1} \left[\frac{(2n-1)!!}{(2n-1-2k)!!} \right]^2 x^{2n-2k} \right) I_1(x)-$$

$$- \left(\sum_{k=0}^{n-2} \frac{[(2n-1)!!]^2 \, x^{2n-2k-1}}{[(2n-1-2k)!!] \cdot [(2n-3-2k)!!]} \right) I_0(x) + [(2n-1)!!]^2 \, \Psi(x) =$$

$$= \left(\sum_{k=0}^{n-1} \left[\frac{(2n)! \cdot (n-k)!}{2^k \cdot (n!) \cdot (2n-2k)!} \right]^2 x^{2n-2k} \right) I_1(x)-$$

$$- \left(\sum_{k=0}^{n-2} \frac{[(2n)!]^2 \cdot (n-k)! \cdot (n-k-1)! \cdot x^{2n-2k-1}}{2^{2k+1} \cdot (n!)^2 \cdot (2n-2k)! \cdot (2n-2-2k)!} \right) I_0(x) + \left[\frac{(2n)!}{2^n \cdot n!} \right]^2 \Psi(x)$$

Recurrence formulas:

$$\int x^{2n+2} J_0(x)\,dx \;=\; (2n+1)x^{2n+1} J_0(x) + x^{2n+2} J_1(x) - (2n+1)^2 \int x^{2n} J_0(x)\,dx$$

$$\int x^{2n+2} I_0(x)\,dx \;=\; -(2n+1)x^{2n+1} I_0(x) + x^{2n+2} I_1(x) + (2n+1)^2 \int x^{2n} I_0(x)\,dx$$

$$\int x^{2n+2} K_0(x)\,dx \;=\; -(2n+1)x^{2n+1} K_0(x) - x^{2n+2} K_1(x) + (2n+1)^2 \int x^{2n} K_0(x)\,dx$$

In the case $n < 0$ the previous formulas give

$$\int \frac{J_0(x)}{x^2}\,dx \;=\; J_1(x) - \frac{x^2+1}{x} J_0(x) - \Phi(x)$$

$$\int \frac{I_0(x)}{x^2}\,dx \;=\; \frac{x^2-1}{x} I_0(x) - I_1(x) + \Psi(x)$$

$$\int \frac{K_0(x)}{x^2}\,dx \;=\; \frac{x^2-1}{x} K_0(x) + K_1(x) + \Psi_K(x)$$

$$\int \frac{J_0(x)}{x^4}\,dx \;=\; \frac{1}{9}\left[\frac{x^4+x^2-3}{x^3} J_0(x) - \frac{x^2-1}{x^2} J_1(x) + \Phi(x) \right] \qquad * E *$$

$$\int \frac{I_0(x)}{x^4}\,dx \;=\; \frac{1}{9}\left[\frac{x^4-x^2-3}{x^3} I_0(x) - \frac{x^2+1}{x^2} I_1(x) + \Psi(x) \right] \qquad * E *$$

$$\int \frac{K_0(x)}{x^4}\,dx \;=\; \frac{1}{9}\left[\frac{x^4-x^2-3}{x^3} K_0(x) + \frac{x^2+1}{x^2} K_1(x) + \Psi_K(x) \right]$$

$$\int \frac{J_0(x)}{x^6}\,dx \;=\; \frac{1}{225}\left[\frac{x^4-x^2+9}{x^4} J_1(x) - \frac{x^6+x^4-3x^2+45}{x^5} J_0(x) - \Phi(x) \right]$$

$$\int \frac{I_0(x)}{x^6}\,dx \;=\; \frac{1}{225}\left[\frac{x^6-x^4-3x^2-45}{x^5} I_0(x) - \frac{x^4+x^2+9}{x^4} I_1(x) + \Psi(x) \right]$$

$$\int \frac{K_0(x)}{x^6}\,dx \;=\; \frac{1}{225}\left[\frac{x^6-x^4-3x^2-45}{x^5} K_0(x) + \frac{x^4+x^2+9}{x^4} K_1(x) + \Psi_K(x) \right] \quad \text{and so on.}$$

$$\int \frac{J_0(x)}{x^8}\,dx \;=\; \frac{1}{11\,025}\left[\frac{x^8+x^6-3x^4+45x^2-1\,575}{x^7} J_0(x) - \frac{x^6-x^4+9x^2-225}{x^6} J_1(x) + \Phi(x) \right]$$

$$\int \frac{I_0(x)}{x^8}\,dx \;=\; \frac{1}{11\,025}\left[\frac{x^8-x^6-3x^4-45x^2-1\,575}{x^7} I_0(x) - \frac{x^6+x^4+9x^2+225}{x^6} I_1(x) + \Psi(x) \right]$$

$$\int \frac{J_0(x)}{x^{10}}\,dx \;=\; \frac{1}{893\,025}\left[\frac{x^8-x^6+9x^4-225x^2+11\,025}{x^8} J_1(x) - \right.$$

$$\left. - \frac{x^{10}+x^8-3x^6+45x^4-1\,575x^2+99\,225}{x^9} J_0(x) - \Phi(x) \right]$$

$$\int \frac{I_0(x)}{x^{10}}\,dx \;=\; \frac{1}{893\,025}\left[\frac{x^{10}-x^8-3x^6-45x^4-1\,575x^2-99\,225}{x^9} I_0(x) - \right.$$

$$-\frac{x^8 + x^6 + 9x^4 + 225x^2 + 11\,025}{x^8} I_1(x) + \Psi(x)\Bigg]$$

$$\int \frac{J_0(x)}{x^{12}}\,dx = \frac{1}{108\,056\,025}\Bigg[\frac{x^{12} + x^{10} - 3x^8 + 45x^6 - 1\,575x^4 + 99\,225x^2 - 9\,823\,275}{x^{11}} J_0(x) -$$

$$-\frac{x^{10} - x^8 + 9x^6 - 225x^4 + 11\,025x^2 - 893\,025}{x^{10}} J_1(x) + \Phi(x)\Bigg]$$

$$\int \frac{I_0(x)}{x^{12}}\,dx = \frac{1}{108\,056\,025}\Bigg[\frac{x^{12} - x^{10} - 3x^8 - 45x^6 - 1\,575x^4 - 99\,225x^2 - 9\,823\,275}{x^{11}} I_0(x) -$$

$$-\frac{x^{10} + x^8 + 9x^6 + 225x^4 + 11\,025x^2 + 893\,025}{x^{10}} I_1(x) + \Psi(x)\Bigg]$$

General formula: With $n!!$ as defined on page 8 holds

$$\int \frac{J_0(x)\,dx}{x^{2n}} = \frac{(-1)^n}{[(2n-1)!!]^2}\Bigg[\left(x + \sum_{k=0}^{n-1}(-1)^k \cdot (2k+1)!! \cdot (2k-1)!! \cdot x^{-2k-1}\right)J_0(x)-$$

$$-\left(1 - \sum_{k=0}^{n-2}(-1)^k \cdot [(2k+1)!!]^2\, x^{-2k-2}\right)J_1(x) + \Phi(x)\Bigg] =$$

$$= \frac{(-1)^n \cdot 2^{2n} \cdot (n!)^2}{(2n)!}\Bigg\{\left(x + \sum_{k=0}^{n-1}(-1)^k \frac{(2k+2)! \cdot (2k)!}{2^{2k+1} \cdot (k+1)! \cdot k! \cdot x^{2k+1}}\right)J_0(x)-$$

$$-\left(1 - \sum_{k=0}^{n-2}\frac{(-1)^k}{x^{2k+2}}\left[\frac{(2k+2)!}{2^{k+1} \cdot (k+1)!}\right]^2\right)J_1(x) + \Phi(x)\Bigg\}$$

With obviously modifications one gets the the formulas for the integrals $\int x^{-2n} I_0(x)\,dx$ and $\int x^{-2n} K_0(x)\,dx$.

1.1.2. Integrals of the type $\int x^{2n+1} Z_0(x)\,dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$.

$$\int x\, J_0(x)\,dx \;=\; x J_1(x)$$

$$\int x\, I_0(x)\,dx \;=\; x I_1(x)$$

$$\int x\, K_0(x)\,dx \;=\; -x K_1(x)$$

$$\int x^3\, J_0(x)\,dx \;=\; x\left[2x J_0(x) + (x^2 - 4) J_1(x)\right]$$

$$\int x^3\, I_0(x)\,dx \;=\; x\left[(x^2 + 4) I_1(x) - 2x I_0(x)\right]$$

$$\int x^3\, K_0(x)\,dx \;=\; -x\left[(x^2 + 4) K_1(x) + 2x K_0(x)\right]$$

$$\int x^5\, J_0(x)\,dx \;=\; x\left[(4x^3 - 32x) J_0(x) + (x^4 - 16x^2 + 64) J_1(x)\right]$$

$$\int x^5\, I_0(x)\,dx \;=\; x\left[(x^4 + 16x^2 + 64) I_1(x) - (4x^3 + 32x) I_0(x)\right]$$

$$\int x^5\, K_0(x)\,dx \;=\; -x\left[(x^4 + 16x^2 + 64) K_1(x) + (4x^3 + 32x) K_0(x)\right]$$

$$\int x^7\, J_0(x)\,dx \;=\; x\left[(6x^5 - 144x^3 + 1\,152x) J_0(x) + (x^6 - 36x^4 + 576x^2 - 2\,304) J_1(x)\right]$$

$$\int x^7\, I_0(x)\,dx \;=\; x\left[(x^6 + 36x^4 + 576x^2 + 2\,304) I_1(x) - (6x^5 + 144x^3 + 1\,152x) I_0(x)\right]$$

$$\int x^7\, K_0(x)\,dx \;=\; -x\left[(x^6 + 36x^4 + 576x^2 + 2\,304) K_1(x) + (6x^5 + 144x^3 + 1\,152x) K_0(x)\right]$$

$$\int x^9\, J_0(x)\,dx \;=$$
$$= x\left[(8x^7 - 384x^5 + 9\,216x^3 - 73\,728x) J_0(x) + (x^8 - 64x^6 + 2\,304x^4 - 36\,864x^2 + 147\,456) J_1(x)\right]$$

$$\int x^9\, I_0(x)\,dx \;=$$
$$= x\left[(x^8 + 64x^6 + 2\,304x^4 + 36\,864x^2 + 147\,456) I_1(x) - (8x^7 + 384x^5 + 9\,216x^3 + 73\,728x) I_0(x)\right]$$

$$\int x^9\, K_0(x)\,dx \;=$$
$$= -x\left[(x^8 + 64x^6 + 2\,304x^4 + 36\,864x^2 + 147\,456) K_1(x) + (8x^7 + 384x^5 + 9\,216x^3 + 73\,728x) K_0(x)\right]$$

Let

$$\int x^m J_0(x)\,dx = x[P_m(x)J_0(x) + Q_m(x)J_1(x)] \quad \text{and} \quad \int x^m I_0(x)\,dx = x[Q_m^*(x)I_1(x) - P_m^*(x)I_0(x)]\,,$$

$$\int x^m K_0(x)\,dx = -x[Q_m^*(x)K_1(x) + P_m^*(x)K_0(x)]\,,$$

then holds

$P_{11}(x) = 10\,x^9 - 800\,x^7 + 38400\,x^5 - 921600\,x^3 + 7372800\,x$
$Q_{11}(x) = x^{10} - 100\,x^8 + 6400\,x^6 - 230400\,x^4 + 3686400\,x^2 - 14745600$
$P_{11}^*(x) = 10\,x^9 + 800\,x^7 + 38400\,x^5 + 921600\,x^3 + 7372800\,x$ *E*
$Q_{11}^*(x) = x^{10} + 100\,x^8 + 6400\,x^6 + 230400\,x^4 + 3686400\,x^2 + 14745600$

$P_{13}(x) = 12\,x^{11} - 1440\,x^9 + 115200\,x^7 - 5529600\,x^5 + 132710400\,x^3 - 1061683200\,x$

$Q_{13}(x) = x^{12} - 144\,x^{10} + 14400\,x^8 - 921600\,x^6 + 33177600\,x^4 - 530841600\,x^2 + 2123366400$

$P_{13}^{*}(x) = 12\,x^{11} + 1440\,x^9 + 115200\,x^7 + 5529600\,x^5 + 132710400\,x^3 + 1061683200\,x$ *E*

$Q_{13}^{*}(x) = x^{12} + 144\,x^{10} + 14400\,x^8 + 921600\,x^6 + 33177600\,x^4 + 530841600\,x^2 + 2123366400$

$P_{15}(x) = 14\,x^{13} - 2352\,x^{11} + 282240\,x^9 - 22579200\,x^7 + 1083801600\,x^5 - 26011238400\,x^3 + 208089907200\,x$

$Q_{15}(x) =$
$x^{14} - 196\,x^{12} + 28224\,x^{10} - 2822400\,x^8 + 180633600\,x^6 - 6502809600\,x^4 + 104044953600\,x^2 - 416179814400$

$P_{15}^{*}(x) =$
$= 14\,x^{13} - 2352\,x^{11} + 282240\,x^9 + 22579200\,x^7 + 1083801600\,x^5 + 26011238400\,x^3 + 208089907200\,x$ *E*

$Q_{15}^{*}(x) =$
$x^{14} + 196\,x^{12} + 28224\,x^{10} + 2822400\,x^8 + 180633600\,x^6 + 6502809600\,x^4 + 104044953600\,x^2 + 416179814400$

Recurrence formulas:

$$\int x^{2n+1}\,J_0(x)\,dx \;=\; 2nx^{2n}\,J_0(x) + x^{2n+1}\,J_1(x) - 4n^2\int x^{2n-1}\,J_0(x)\,dx$$

$$\int x^{2n+1}\,I_0(x)\,dx \;=\; -2nx^{2n}\,I_0(x) + x^{2n+1}\,I_1(x) + 4n^2\int x^{2n-1}\,I_0(x)\,dx$$

$$\int x^{2n+1}\,K_0(x)\,dx \;=\; -2nx^{2n}\,K_0(x) - x^{2n+1}\,K_1(x) + 4n^2\int x^{2n-1}\,K_0(x)\,dx$$

General formula: With $n!!$ as defined on page 8 holds

$$\int x^{2n+1}\,J_0(x)\,dx = \left(\sum_{k=0}^{n-1}(-1)^k\,\frac{[(2n)!!]^2\,x^{2n-2k}}{[(2n-2k)!!]\cdot[(2n-2k-2)!!]}\right)J_0(x)+$$

$$+\left(\sum_{k=0}^{n}(-1)^k\left[\frac{(2n)!!}{(2n-2k)!!}\right]^2 x^{2n+1-2k}\right)J_1(x) =$$

$$=\left(\sum_{k=0}^{n-1}(-1)^k\,\frac{2^{2k+1}\cdot(n!)^2\,x^{2n-2k}}{(n-k)!\cdot(n-k-1)!}\right)J_0(x) + \left(\sum_{k=0}^{n}(-1)^k\left[\frac{2^k\cdot n!}{(n-k)!}\right]^2 x^{2n+1-2k}\right)J_1(x)\;.$$

With obviously modifications one gets the the formulas for the integrals $\int x^{2n+1}\,I_0(x)\,dx$ and $\int x^{2n+1}\,K_0(x)\,dx$.

1.1.3. Integrals of the type $\int x^{-2n-1} \cdot Z_0(x)\,dx$

The basic integral

$$\int \frac{J_0(x)\,dx}{x} \quad \text{can be expressed by} \quad \int_0^x \frac{1-J_0(t)}{t}\,dt \quad \text{or} \quad -\int_x^\infty \frac{J_0(t)\,dt}{t} = Ji_0(x)\ ,$$

see [1], equation 11.1.19 and the following formulas. There are given asymptotic expansions and polynomial approximations as well. Tables of these functions may be found by [1], [11.13] or [11.22]. The function $Ji_0(x)$ is introduced and discussed in [9].

For fast computations of this integrals one should use approximations with Chebyshev polynomials, see [2], tables 9.3 .

I got the information from F. Nouguier, that there is an error in a formula in [9], p. 278. The true formula is

$$Ji_0(x) - \ln x = \frac{\sin \pi x}{\pi x}(\gamma - \ln 2) + \frac{2x \sin \pi x}{\pi} \sum_{s=1}^\infty \frac{(-1)^{s-1}}{s^2 - x^2}\left[Ji_0(s) - \ln s\right]\ .$$

The power series in

$$\int \frac{I_0(x)\,dx}{x} \quad = \quad \ln x \ + \ \sum_{k=1}^\infty \frac{1}{2k \cdot (k!)^2}\left(\frac{x}{2}\right)^{2k} \qquad\qquad *E*$$

can be used without numerical problems.

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$.

$$\int \frac{J_0(x)\,dx}{x^3} = -\frac{J_0(x)}{2x^2} + \frac{J_1(x)}{4x} - \frac{1}{4}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_0(x)\,dx}{x^3} = -\frac{I_0(x)}{2x^2} - \frac{I_1(x)}{4x} + \frac{1}{4}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,dx}{x^5} = \left(\frac{1}{32x^2} - \frac{1}{4x^4}\right)J_0(x) + \left(-\frac{1}{64x} + \frac{1}{16x^3}\right)J_1(x) + \frac{1}{64}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_0(x)\,dx}{x^5} = -\left(\frac{1}{32x^2} + \frac{1}{4x^4}\right)I_0(x) - \left(\frac{1}{64x} + \frac{1}{16x^3}\right)I_1(x) + \frac{1}{64}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,dx}{x^7} = \frac{-x^4 + 8x^2 - 192}{1152\,x^6}J_0(x) + \frac{x^4 - 4x^2 + 64}{2304\,x^5}J_1(x) - \frac{1}{2304}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_0(x)\,dx}{x^7} = -\frac{x^4 + 8x^2 + 192}{1152\,x^6}I_0(x) - \frac{x^4 + 4x^2 + 64}{2304\,x^5}I_1(x) + \frac{1}{2304}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,dx}{x^9} =$$

$$= \frac{x^6 - 8x^4 + 192x^2 - 9216}{73728\,x^8}J_0(x) + \frac{-x^6 + 4x^4 - 64x^2 + 2304}{147456\,x^7}J_1(x) + \frac{1}{147456}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_0(x)\,dx}{x^9} =$$

$$= -\frac{x^6 + 8x^4 + 192x^2 + 9216}{73728\,x^8}I_0(x) - \frac{x^6 + 4x^4 + 64x^2 + 2304}{147456\,x^7}I_1(x) + \frac{1}{147456}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,dx}{x^{11}} = \frac{-x^8 + 8x^6 - 192x^4 + 9216x^2 - 737280}{7372800\,x^{10}}J_0(x)+$$

$$+\frac{x^8 - 4x^6 + 64x^4 - 2304x^2 + 147456}{14745600\,x^9}J_1(x) - \frac{1}{14745600}\int \frac{J_0(x)\,dx}{x}$$

$$*E* \qquad \int \frac{I_0(x)\,dx}{x^{11}} = -\frac{x^8 + 8x^6 + 192x^4 + 9216x^2 + 737280}{7372800\,x^{10}}I_0(x)-$$

$$-\frac{x^8 + 4x^6 + 64x^4 + 2304x^2 + 147456}{14745600\,x^9}I_1(x) + \frac{1}{14745600}\int \frac{I_0(x)\,dx}{x}$$

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Descending recurrence formulas:

$$\int x^{-2n-1} J_0(x)\,dx = \frac{1}{4n^2}\left[x^{-2n+1}J_1(x) - 2nx^{-2n}J_0(x) - \int x^{-2n+1}J_0(x)\,dx\right]$$

$$\int x^{-2n-1} I_0(x)\,dx = \frac{1}{4n^2}\left[-x^{-2n+1}I_1(x) - 2nx^{-2n}I_0(x) + \int x^{-2n+1}I_0(x)\,dx\right]$$

General formula: With $n!!$ as defined on page 8 holds

$$\int \frac{J_0(x)\,dx}{x^{2n+1}} =$$

$$= \frac{(-1)^n}{[(2n)!!]^2}\left\{\left(\sum_{k=0}^{n-1}(-1)^k\frac{(2k+2)!!\cdot(2k)!!}{x^{2k+2}}\right)J_0(x) - \left(\sum_{k=0}^{n-1}(-1)^k\frac{[(2k)!!]^2}{x^{2k+1}}\right)J_1(x) + \int \frac{J_0(x)\,dx}{x}\right\} = \quad *E*$$

$$= \frac{(-1)^n}{2^{2n}\cdot(n!)^2}\left\{\left(\sum_{k=0}^{n-1}(-1)^k\frac{2^{2k+1}\cdot(k+1)!\cdot k!}{x^{2k+2}}\right)J_0(x) - \left(\sum_{k=0}^{n-1}(-1)^k\frac{2^{2k}\cdot(k!)^2}{x^{2k+1}}\right)J_1(x) + \int \frac{J_0(x)\,dx}{x}\right\} \quad *E*$$

With obviously modifications one gets the the formula for the integral $\int x^{-2n-1} I_0(x)\,dx$.

1.1.4. Integrals of the type $\int x^{2n} Z_1(x)\,dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}$, $p = 1, 2$.

$$\int J_1(x)\,dx \;=\; -\,J_0(x)$$

$$\int I_1(x)\,dx \;=\; I_0(x)$$

$$\int K_1(x)\,dx \;=\; -K_0(x)$$

$$\int x^2\, J_1(x)\,dx \;=\; x\,[2J_1(x) - x\,J_0(x)]$$

$$\int x^2\, I_1(x)\,dx \;=\; x\,[x\,I_0(x) - 2I_1(x)]$$

$$\int x^2\, K_1(x)\,dx \;=\; -x\,[x\,K_0(x) + 2K_1(x)]$$

$$\int x^4\, J_1(x)\,dx \;=\; x\,\big[(4x^2 - 16)\,J_1(x) - (x^3 - 8x)\,J_0(x)\big]$$

$$\int x^4\, I_1(x)\,dx \;=\; x\,\big[(x^3 + 8x)\,I_0(x) - (4x^2 + 16)\,I_1(x)\big]$$

$$\int x^4\, K_1(x)\,dx \;=\; -x\,\big[(x^3 + 8x)\,K_0(x) + (4x^2 + 16)\,K_1(x)\big]$$

$$\int x^6\, J_1(x)\,dx \;=\; x\,\big[(6x^4 - 96x^2 + 384)\,J_1(x) - (x^5 - 24x^3 + 192x)\,J_0(x)\big]$$

$$\int x^6\, I_1(x)\,dx \;=\; x\,\big[(x^5 + 24x^3 + 192x)\,I_0(x) - (6x^4 + 96x^2 + 384)\,I_1(x)\big]$$

$$\int x^6\, K_1(x)\,dx \;=\; -x\,\big[(x^5 + 24x^3 + 192x)\,K_0(x) + (6x^4 + 96x^2 + 384)\,K_1(x)\big]$$

$$\int x^8\, J_1(x)\,dx \;=$$
$$= x\,\big[(8x^6 - 288x^4 + 4\,608x^2 - 18\,432)\,J_1(x) - (x^7 - 48x^5 + 1\,152x^3 - 9\,216x)\,J_0(x)\big]$$
$$\int x^8\, I_1(x)\,dx \;=$$
$$= x\,\big[(x^7 + 48x^5 + 1\,152x^3 + 9\,216x)\,I_0(x) - (8x^6 + 288x^4 + 4\,608x^2 + 18\,432)\,I_1(x)\big]$$
$$\int x^8\, K_1(x)\,dx \;=$$
$$= -x\,\big[(x^7 + 48x^5 + 1\,152x^3 + 9\,216x)\,K_0(x) + (8x^6 + 288x^4 + 4\,608x^2 + 18\,432)\,K_1(x)\big]$$

$$\int x^{10}\, J_1(x)\,dx \;=\; x\,\big[(10x^8 - 640x^6 + 23\,040x^4 - 368\,640x^2 + 1\,474\,560)\,J_1(x) -$$
$$-\,(x^9 - 80x^7 + 3\,840x^5 - 92\,160x^3 + 737\,280x)\,J_0(x)\big]$$

$$\int x^{10}\, I_1(x)\,dx \;=\; x\,\big[(x^9 + 80x^7 + 3\,840x^5 + 92\,160x^3 + 737\,280x)\,I_0(x) -$$
$$-(10x^8 + 640x^6 + 23\,040x^4 + 368\,640x^2 + 1\,474\,560)\,I_1(x)\big]$$

$$\int x^{10}\, K_1(x)\,dx \;=\; -x\,\big[(x^9 + 80x^7 + 3\,840x^5 + 92\,160x^3 + 737\,280x)\,K_0(x) +$$

$$+(10x^8 + 640x^6 + 23\,040x^4 + 368\,640x^2 + 1\,474\,560)\,K_1(x)\big]$$

Let

$$\int x^m J_1(x)\,dx = x[Q_m(x)J_1(x) - P_m(x)J_0(x)] \quad \text{and} \quad \int x^m I_1(x)\,dx = x[P_m^*(x)I_0(x) - Q_m^*(x)I_1(x)]\,,$$

$$\int x^m K_1(x)\,dx = -x[P_m^*(x)I_0(x) + Q_m^*(x)I_1(x)]\,,$$

then holds

$P_{12}(x) = x^{11} - 120\,x^9 + 9600\,x^7 - 460800\,x^5 + 11059200\,x^3 - 88473600\,x$
$Q_{12}(x) = 12\,x^{10} - 1200\,x^8 + 76800\,x^6 - 2764800\,x^4 + 44236800\,x^2 - 176947200$
$P_{12}^*(x) = x^{11} + 120\,x^9 + 9600\,x^7 + 460800\,x^5 + 11059200\,x^3 + 88473600\,x$
$Q_{12}^*(x) = 12\,x^{10} + 1200\,x^8 + 76800\,x^6 + 2764800\,x^4 + 44236800\,x^2 + 176947200$ *E*

$P_{14}(x) = x^{13} - 168\,x^{11} + 20160\,x^9 - 1612800\,x^7 + 77414400\,x^5 - 1857945600\,x^3 + 14863564800\,x$
$Q_{14}(x) = 14\,x^{12} - 2016\,x^{10} + 201600\,x^8 - 12902400\,x^6 + 464486400\,x^4 - 7431782400\,x^2 + 29727129600$
$P_{14}^*(x) = x^{13} + 168\,x^{11} + 20160\,x^9 + 1612800\,x^7 + 77414400\,x^5 + 1857945600\,x^3 + 14863564800\,x$
$Q_{14}^*(x) = 14\,x^{12} + 2016\,x^{10} + 201600\,x^8 + 12902400\,x^6 + 464486400\,x^4 + 7431782400\,x^2 + 29727129600$

Recurrence formulas:

$$\int x^{2n+2} J_1(x)\,dx = -x^{2n+2} J_0(x) + (2n+2)x^{2n+1} J_1(x) - 4n(n+1)\int x^{2n} J_1(x)\,dx$$

$$\int x^{2n+2} I_1(x)\,dx = x^{2n+2} I_0(x) - (2n+2)x^{2n+1} I_1(x) + 4n(n+1)\int x^{2n} I_1(x)\,dx$$

$$\int x^{2n+2} K_1(x)\,dx = -x^{2n+2} K_0(x) - (2n+2)x^{2n+1} K_1(x) + 4n(n+1)\int x^{2n} K_1(x)\,dx$$

General formula: With $n!!$ as defined on page 8 holds

$$\int x^{2n} J_1(x)\,dx = \left(\sum_{k=0}^{n-1} (-1)^k \frac{[(2n)!!]\cdot[(2n-2)!!]\cdot x^{2n-1-2k}}{[(2n-2-2k)!!]^2}\right) J_1(x)-$$

$$-\left(\sum_{k=0}^{n-1} (-1)^k \frac{(2n)!!\cdot(2n-2)!!\cdot x^{2n-2k}}{[(2n-2k)!!]\cdot[(2n-2-2k)!!]}\right) J_0(x) =$$

$$= \left(\sum_{k=0}^{n-1} (-1)^k \frac{2^{2k+1}\cdot(n!)\cdot(n-1)!\cdot x^{2n-1-2k}}{[(n-1-k)!]^2}\right) J_1(x)-$$

$$-\left(\sum_{k=0}^{n-1} (-1)^k \frac{2^{2k}\cdot n!\cdot(n-1)!!\cdot x^{2n-2k}}{(n-k)!\cdot(n-1-k)!}\right) J_0(x)$$

With obviously modifications one gets the the formulas for the integrals $\int x^{2n} I_1(x)\,dx$ and $\int x^{2n} K_1(x)\,dx$.

1.1.5. Integrals of the type $\int x^{-2n} \cdot Z_1(x)\,dx$

About the integrals

$$\int \frac{J_0(x)\,dx}{x} \qquad \text{and} \qquad \int \frac{I_0(x)\,dx}{x}$$

see 1.1.3, page 13.

In the following formulas $J_0(x)$ may be substituted by $Y_0(x)$ and simultaneously $J_1(x)$ by $Y_1(x)$.

$$\int \frac{J_1(x)\,dx}{x^2} = -\frac{1}{2x}\,J_1(x) + \frac{1}{2}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^2} = -\frac{1}{2x}\,I_1(x) + \frac{1}{2}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^4} = -\frac{1}{8\,x^2}\,J_0(x) + \frac{x^2-4}{16\,x^3}\,J_1(x) - \frac{1}{16}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^4} = -\frac{1}{8\,x^2}\,I_0(x) - \frac{x^2+4}{16\,x^3}\,I_1(x) + \frac{1}{16}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^6} =$$

$$= \frac{x^2-8}{192\,x^4}\,J_0(x) + \frac{-x^4+4\,x^2-64}{384\,x^5}\,J_1(x) + \frac{1}{384}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^6} = -\frac{x^2+8}{192\,x^4}\,I_0(x) - \frac{x^4+4\,x^2+64}{384\,x^5}\,I_1(x) + \frac{1}{384}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^8} =$$

$$= \frac{-x^4+8\,x^2-192}{9216\,x^6}\,J_0(x) + \frac{x^6-4\,x^4+64\,x^2-2304}{18432\,x^7}\,J_1(x) - \frac{1}{18432}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^8} = -\frac{x^4+8\,x^2+192}{9216\,x^6}\,I_0(x) - \frac{x^6+4\,x^4+64\,x^2+2304}{18432\,x^7}\,I_1(x) + \frac{1}{18432}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^{10}} =$$

$$= \frac{x^6-8\,x^4+192\,x^2-9216}{737280\,x^8}\,J_0(x) + \frac{-x^8+4\,x^6-64\,x^4+2304\,x^2-147456}{1474560\,x^9}\,J_1(x)+$$

$$+\frac{1}{1474560}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^{10}} =$$

$$= -\frac{x^6+8\,x^4+192\,x^2+9216}{737280\,x^8}\,I_0(x) - \frac{x^8+4\,x^6+64\,x^4+2304\,x^2+147456}{1474560\,x^9}\,I_1(x)+$$

$$+\frac{1}{1474560}\int \frac{I_0(x)\,dx}{x}$$

Recurrence formulas:

$$*E* \qquad \int \frac{J_1(x)\,dx}{x^{2n+2}} = -\frac{J_0(x)}{4n(n+1)x^{2n}} - \frac{J_1(x)}{(2n+2)x^{2n+1}} - \frac{1}{4n(n+1)}\int \frac{J_1(x)\,dx}{x^{2n}}$$

$$*E* \qquad \int \frac{I_1(x)\,dx}{x^{2n+2}} = -\frac{I_0(x)}{4n(n+1)x^{2n}} - \frac{I_1(x)}{(2n+2)x^{2n+1}} + \frac{1}{4n(n+1)}\int \frac{I_1(x)\,dx}{x^{2n}}$$

General formula: With $n!!$ as defined on page 8 holds

$$\int \frac{J_1(x)\,dx}{x^{2n}} = \frac{(-1)^{n+1}}{(2n)!! \cdot (2n-2)!!} \cdot$$

$$\cdot \left\{ \left(\sum_{k=0}^{n-2} (-1)^k \frac{(2k+2)!! \cdot (2k)!!}{x^{2k+2}} \right) J_0(x) - \left(\sum_{k=0}^{n-1} (-1)^k \frac{[(2k)!!]^2}{x^{2k+1}} \right) J_1(x) + \int \frac{J_0(x)\,dx}{x} \right\} =$$

$$= \frac{(-1)^{n+1}}{2^{2n-1} \cdot n! \cdot (n-1)!} \cdot$$

$$\cdot \left[\left(\sum_{k=0}^{n-2} (-1)^k \frac{2^{2k+1} \cdot (k+1)! \cdot k!}{x^{2k+2}} \right) J_0(x) - \left(\sum_{k=0}^{n-1} (-1)^k \frac{2^{2k} \cdot (k!)^2}{x^{2k+1}} \right) J_1(x) + \int \frac{J_0(x)\,dx}{x} \right]$$

With obviously modifications one gets the the formula for the integral $\int x^{-2n} I_1(x)\,dx$.

1.1.6. Integrals of the type $\int x^{2n+1} Z_1(x)\,dx$

$\Phi(x)$, $\Phi_Y(x)$, $\Psi(x)$ and $\Psi_K(x)$ are the same as in 1.1.1, page 7 .

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ and simultaneously $\Phi(x)$ by $\Phi_Y(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$ and $\Phi_H^{(p)}(x)$.

$$\int x\, J_1(x)\,dx \;=\; \Phi(x)$$

$$\int x\, I_1(x)\,dx \;=\; -\Psi(x)$$

$$\int x\, K_1(x)\,dx \;=\; \Psi_K(x)$$

$$\int x^3\, J_1(x)\,dx \;=\; 3x^2\, J_1(x) - x^3\, J_0(x) - 3\Phi(x)$$

$$\int x^3\, I_1(x)\,dx \;=\; -3x^2\, I_1(x) + x^3\, I_0(x) - 3\Psi(x)$$

$$\int x^3\, K_1(x)\,dx \;=\; -3x^2\, K_1(x) - x^3\, K_0(x) + 3\Psi_K(x)$$

$$\int x^5\, J_1(x)\,dx \;=\; (5x^4 - 45x^2)\, J_1(x) - (x^5 - 15x^3)\, J_0(x) + 45\Phi(x)$$

$$\int x^5\, I_1(x)\,dx \;=\; -(5x^4 + 45x^2)\, I_1(x) + (x^5 + 15x^3)\, I_0(x) - 45\Psi(x)$$

$$\int x^5\, K_1(x)\,dx \;=\; -(5x^4 + 45x^2)\, K_1(x) - (x^5 + 15x^3)\, K_0(x) + 45\Psi_K(x)$$

$$\int x^7\, J_1(x)\,dx \;=\; (7x^6 - 175x^4 + 1\,575x^2)\, J_1(x) - (x^7 - 35x^5 + 525x^3)\, J_0(x) - 1\,575\Phi(x) \qquad *E*$$

$$\int x^7\, I_1(x)\,dx \;=\; -(7x^6 + 175x^4 + 1\,575x^2)\, I_1(x) + (x^7 + 35x^5 + 525x^3)\, I_0(x) - 1\,575\Psi(x)$$

$$\int x^7\, K_1(x)\,dx \;=\; -(7x^6 + 175x^4 + 1\,575x^2)\, K_1(x) - (x^7 + 35x^5 + 525x^3)\, K_0(x) + 1\,575\Psi_K(x)$$

$$\int x^9\, J_1(x)\,dx =$$

$$= (9x^8 - 441x^6 + 11\,025x^4 - 99\,225x^2)\, J_1(x) - (x^9 - 63x^7 + 2\,205x^5 - 33\,075x^3)\, J_0(x) + 99\,225\, \Phi(x)$$

$$\int x^9\, I_1(x)\,dx =$$

$$= -(9x^8 + 441x^6 + 11\,025x^4 + 99\,225x^2)\, I_1(x) + (x^9 + 63x^7 + 2\,205x^5 + 33\,075x^3)\, I_0(x) - 99\,225\, \Psi(x)$$

$$\int x^9\, K_1(x)\,dx =$$

$$= -(9x^8 + 441x^6 + 11\,025x^4 + 99\,225x^2)\, K_1(x) - (x^9 + 63x^7 + 2\,205x^5 + 33\,075x^3)\, K_0(x) + 99\,225\, \Psi(x)$$

General formula: With $n!!$ as defined on page 8 holds

$$\int x^{2n+1}\, J_1(x)\,dx = \left(\sum_{k=0}^{n-1} (-1)^k\, \frac{(2n+1)!! \cdot (2n-1)!! \cdot x^{2n-2k}}{[(2n-1-2k)!!]^2} \right) J_1(x) -$$

$$- \left(\sum_{k=0}^{n-1} (-1)^k\, \frac{(2n+1)!! \cdot (2n-1)!! \cdot x^{2n+1-2k}}{(2n+1-2k)!! \cdot (2n-1-2k)!!} \right) J_0(x) + (-1)^n \cdot (2n+1)!! \cdot (2n-1)!!\, \Phi(x) =$$

$$= \left(\sum_{k=0}^{n-1} (-1)^k \frac{(2n+2)! \cdot (2n)! \cdot [(n-k)!]^2 \cdot x^{2n-2k}}{2^{2k+1} \cdot (n+1)! \cdot n! \cdot [(2n-2k)!]^2} \right) J_1(x)$$

$$- \left(\sum_{k=0}^{n-1} (-1)^k \frac{(2n+2)! \cdot (2n)! \cdot (n+1-k)! \cdot (n-k)! \cdot x^{2n+1-2k}}{2^{2k} \cdot (n+1)! \cdot n! \cdot (2n+2-2k)! \cdot (2n-2k)!} \right) J_0(x) +$$

$$+ (-1)^n \frac{(2n+2)! \cdot (2n)!}{2^{2n+1} \cdot (n+1)! \cdot n!} \Phi(x)$$

With obviously modifications one gets the the formulas for the integrals $\int x^{2n+1} I_1(x)\, dx$ and $\int x^{2n+1} K_1(x)\, dx$.

Recurrence formulas:

$$\int x^{2n+1} J_1(x)\, dx = -x^{2n+1} J_0(x) + (2n+1)x^{2n} J_1(x) - (2n-1)(2n+1) \int x^{2n-1} J_1(x)\, dx$$

$$\int x^{2n+1} I_1(x)\, dx = x^{2n+1} I_0(x) - (2n+1)x^{2n} I_1(x) + (2n-1)(2n+1) \int x^{2n-1} I_1(x)\, dx$$

$$\int x^{2n+1} K_1(x)\, dx = -x^{2n+1} K_0(x) - (2n+1)x^{2n} K_1(x) + (2n-1)(2n+1) \int x^{2n-1} K_1(x)\, dx$$

Descending:

$$\int \frac{J_1(x)\, dx}{x^{2n+1}} = -\frac{J_0(x)}{(4n^2-1)x^{2n-1}} - \frac{J_1(x)}{(2n+1)x^{2n}} - \frac{1}{4n^2-1} \int \frac{J_1(x)\, dx}{x^{2n-1}}$$

$$\int \frac{I_1(x)\, dx}{x^{2n+1}} = -\frac{I_0(x)}{(4n^2-1)x^{2n-1}} - \frac{I_1(x)}{(2n+1)x^{2n}} + \frac{1}{4n^2-1} \int \frac{I_1(x)\, dx}{x^{2n-1}}$$

$$\int \frac{K_1(x)\, dx}{x^{2n+1}} = \frac{K_0(x)}{(4n^2-1)x^{2n-1}} - \frac{K_1(x)}{(2n+1)x^{2n}} + \frac{1}{4n^2-1} \int \frac{K_1(x)\, dx}{x^{2n-1}}$$

$$\int \frac{J_1(x)}{x}\, dx = x \cdot J_0(x) - J_1(x) + \Phi(x)$$

$$\int \frac{I_1(x)}{x}\, dx = x \cdot I_0(x) - I_1(x) + \Psi(x)$$

$$\int \frac{K_1(x)}{x}\, dx = -x \cdot K_0(x) - K_1(x) - \Psi_K(x)$$

$$\int \frac{J_1(x)}{x^3}\, dx = \frac{1}{3}\left[\frac{x^2-1}{x^2} J_1(x) - \frac{x^2+1}{x} J_0(x) - \Phi(x) \right]$$

$$\int \frac{I_1(x)}{x^3}\, dx = \frac{1}{3}\left[-\frac{x^2+1}{x^2} I_1(x) + \frac{x^2-1}{x} I_0(x) + \Psi(x) \right]$$

$$\int \frac{K_1(x)}{x^3}\, dx = \frac{1}{3}\left[-\frac{x^2+1}{x^2} K_1(x) - \frac{x^2-1}{x} K_0(x) - \Psi_K(x) \right]$$

$$\int \frac{J_1(x)}{x^5}\, dx = \frac{1}{45}\left[\frac{x^4+x^2-3}{x^3} J_0(x) - \frac{x^4-x^2+9}{x^4} J_1(x) + \Phi(x) \right]$$

$$\int \frac{I_1(x)}{x^5}\, dx = \frac{1}{45}\left[\frac{x^4-x^2-3}{x^3} I_0(x) - \frac{x^4+x^2+9}{x^4} I_1(x) + \Psi(x) \right]$$

$$\int \frac{K_1(x)}{x^5}\, dx = \frac{1}{45}\left[-\frac{x^4-x^2-3}{x^3} K_0(x) - \frac{x^4+x^2+9}{x^4} K_1(x) - \Psi_K(x) \right]$$

$$\int \frac{J_1(x)}{x^7}\, dx = \frac{1}{1\,575}\left[\frac{x^6-x^4+9x^2-225}{x^6} J_1(x) - \frac{x^6+x^4-3x^2+45}{x^5} J_0(x) - \Phi(x) \right]$$

$$\int \frac{I_1(x)}{x^7}\, dx \;=\; \frac{1}{1\,575}\left[-\frac{x^6 + x^4 + 9x^2 + 225}{x^6}I_1(x) + \frac{x^6 - x^4 - 3x^2 - 45}{x^5}I_0(x) + \Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^7}\, dx \;=\; \frac{1}{1\,575}\left[-\frac{x^6 + x^4 + 9x^2 + 225}{x^6}K_1(x) - \frac{x^6 - x^4 - 3x^2 - 45}{x^5}K_0(x) - \Psi_k(x)\right]$$

$$\int \frac{J_1(x)}{x^9}\, dx \;=$$

$$= \frac{1}{99\,225}\left[\frac{x^8 + x^6 - 3x^4 + 45x^2 - 1\,575}{x^7}J_0(x) - \frac{x^8 - x^6 + 9x^4 - 225x^2 + 11\,025}{x^8}J_1(x) + \Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^9}\, dx \;=$$

$$= \frac{1}{99\,225}\left[\frac{x^8 - x^6 - 3x^4 - 45x^2 - 1\,575}{x^7}I_0(x) - \frac{x^8 + x^6 + 9x^4 + 225x^2 + 11\,025}{x^8}I_1(x) + \Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^9}\, dx \;=$$

$$= \frac{1}{99\,225}\left[-\frac{x^8 - x^6 - 3x^4 - 45x^2 - 1\,575}{x^7}K_0(x) - \frac{x^8 + x^6 + 9x^4 + 225x^2 + 11\,025}{x^8}I_1(x) - \Psi_K(x)\right]$$

$$\int \frac{J_1(x)}{x^{11}}\, dx \;=\; \frac{1}{9\,823\,275}\left[\frac{x^{10} - x^8 + 9x^6 - 225x^4 + 11\,025x^2 - 893\,025}{x^{10}}J_1(x)-\right.$$

$$\left.-\frac{x^{10} + x^8 - 3x^6 + 45x^4 - 1\,575x^2 + 99\,225}{x^9}J_0(x) - \Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^{11}}\, dx \;=\; \frac{1}{9\,823\,275}\left[-\frac{x^{10} + x^8 + 9x^6 + 225x^4 + 11\,025x^2 + 893\,025}{x^{10}}I_1(x)+\right.$$

$$\left.+\frac{x^{10} - x^8 - 3x^6 - 45x^4 - 1\,575x^2 - 99\,225}{x^9}I_0(x) + \Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^{11}}\, dx \;=\; \frac{1}{9\,823\,275}\left[-\frac{x^{10} + x^8 + 9x^6 + 225x^4 + 11\,025x^2 + 893\,025}{x^{10}}K_1(x)-\right.$$

$$\left.-\frac{x^{10} - x^8 - 3x^6 - 45x^4 - 1\,575x^2 - 99\,225}{x^9}K_0(x) + \Psi_K(x)\right]$$

<u>General formula:</u> With $n!!$ as defined on page 8 holds

$$\int \frac{J_1(x)\, dx}{x^{2n+1}} = \frac{(-1)^n}{(2n+1)!! \cdot (2n-1)!!}\left\{\left(x + \sum_{k=0}^{n-1}\frac{(-1)^k \cdot (2k+1)!! \cdot (2k-1)!!}{x^{2k+1}}\right)J_0(x)-\right.$$

$$\left.-\left(1 - \sum_{k=0}^{n-1}(-1)^k\frac{[(2k+1)!!]^2}{x^{2k+2}}\right)J_1(x) + \Phi(x)\right\} =$$

$$= \frac{2^{2n+1} \cdot (n+1)! \cdot n!}{(2n+2)! \cdot (2n)!}\left\{\left(x - \sum_{k=0}^{n-1}(-1)^k\frac{(2k+2)! \cdot (2k)!}{2^{2k+1} \cdot (k+1)! \cdot k! \cdot x^{2k+1}}\right)J_0(x)-\right.$$

$$\left.-\left(1 - \sum_{k=0}^{n-1}(-1)^k\frac{[(2k+2)!]^2}{2^{2k+2} \cdot [(k+1)!]^2 \cdot x^{2k+2}}\right)J_1(x) + \Phi(x)\right\}$$

With obviously modifications one gets the the formulas for the integrals $\int x^{-2n-1}I_1(x)\, dx$ and $\int x^{-2n-1}K_1(x)\, dx$.

1.1.7. Integrals of the type $\int x^n Z_\nu(x)\,dx$, $\nu > 1$:

From the well-known recurrence relations one gets immadiately

$$\int J_{\nu+1}(x)\,dx \;=\; -2J_\nu(x) + \int J_{\nu-1}(x)\,dx \quad \text{and} \quad \int I_{\nu+1}(x)\,dx \;=\; 2I_\nu(x) - \int I_{\nu-1}(x)\,dx \ .$$

With this formulas follows

$$\int_0^x J_{2\nu}(t)\,dt = \Lambda_0(x) - 2\sum_{\kappa=1}^n J_{2\kappa-1}(x) \ , \qquad \int_0^x J_{2\nu+1}(t)\,dt = 1 - J_0(x) - 2\sum_{\kappa=1}^n J_{2\kappa}(x)$$

$$\int_0^x I_{2\nu}(t)\,dt = (-1)^n \Lambda_0^*(x) + 2\sum_{\kappa=1}^n (-1)^{n+\kappa} I_{2\kappa-1}(x) \ , \quad \int_0^x I_{2\nu+1}(t)\,dt = (-1)^n [I_0(x)-1] + 2\sum_{\kappa=1}^n (-1)^{n+\kappa} I_{2\kappa}(x)$$

The integrals $\Lambda_0(x)$ and $\Lambda_0^*(x)$ are defined on page 7 and discussed on page 122 and 124.
Holds

$$\int Y_{2\nu}(x)\,dx = xY_0(x) + \Phi_Y(x) - 2\sum_{\kappa=1}^n Y_{2\kappa-1}(x) \ , \quad \int Y_{2\nu+1}(x)\,dx = -Y_0(x) - 2\sum_{\kappa=1}^n Y_{2\kappa-1}(x)$$

$$\int H_{2\nu}^{(1)}(x)\,dx = xH_0^{(1)}(x) + \Phi_H^{(1)}(x) - 2\sum_{\kappa=1}^n H_{2\kappa-1}^{(1)}(x) \ , \quad \int H_{2\nu+1}^{(1)}(x)\,dx = -H_0^{(1)}(x) - 2\sum_{\kappa=1}^n H_{2\kappa-1}^{(1)}(x)$$

$$\int H_{2\nu}^{(2)}(x)\,dx = xH_0^{(2)}(x) + \Phi_H^{(2)}(x) - 2\sum_{\kappa=1}^n H_{2\kappa-1}^{(2)}(x) \ , \quad \int H_{2\nu+1}^{(2)}(x)\,dx = -H_0^{(2)}(x) - 2\sum_{\kappa=1}^n H_{2\kappa-1}^{(2)}(x)$$

$$\int K_{2\nu}(x)\,dx = (-1)^n \left\{ xK_0(x) + \frac{\pi x}{2}\left[K_0(x)\mathbf{L}_1(x) + K_1(x)\mathbf{L}_0(x)\right] \right\} + 2\sum_{\kappa=1}^n (-1)^{n+\kappa} K_{2\kappa-1}(x) \ ,$$

$$\int K_{2\nu+1}(x)\,dx = (-1)^{n+1} K_0(x) + 2\sum_{\kappa=1}^n (-1)^{n+\kappa+1} K_{2\kappa}(x)$$

About the functions $\Phi_Y(x)$, $\Phi_H^{(1)}(x)$, $\Phi_H^{(2)}(x)$ see page 7.
Further on, holds

$$\int_0^x t\,J_{2\nu+1}(t)\,dt = (2\nu+1)\Lambda_0(x) - x\left[J_0(x) + 2\sum_{\kappa=1}^\nu J_{2\kappa}(x)\right] - 4\sum_{\kappa=0}^{\nu-1}(\nu-\kappa)J_{2\kappa+1}(x)$$

$$\int_0^x t\,J_{2\nu}(t)\,dt = -x\left[J_1(x) + 2\sum_{\kappa=1}^{\nu-1}J_{2\kappa+1}(x)\right] + 2\nu[1 - J_0(x)] - 4\sum_{\kappa=1}^{\nu-1}(\nu-\kappa)J_{2\kappa}(x)$$

$$\int_0^x t\,I_{2\nu+1}(t)\,dt = (-1)^{\nu+1}\left[(2\nu+1)\Lambda_0^*(x) - xI_0(x) - 2x\sum_{\kappa=1}^\nu(-1)^\kappa I_{2\kappa}(x) - 4\sum_{\kappa=0}^{\nu-1}(-1)^\kappa(\nu-\kappa)I_{2\kappa+1}(x)\right]$$

$$\int_0^x t\,I_{2\nu}(t)\,dt = (-1)^{\nu+1}\left[xI_1(x) + 2x\sum_{\kappa=1}^{\nu-1}(-1)^\kappa I_{2\kappa+1}(x) + 2\nu[1 - I_0(x)] - 4\sum_{\kappa=1}^{\nu-1}(-1)^\kappa(\nu-\kappa)I_{2\kappa}(x)\right]$$

Some of the previous sums may cause numerical problems, if x is located near 0. For instance, the sum

$$\int_0^x t\,I_6(t)\,dt = xJ_1(x) - 2xJ_3(x) + 2xJ_5(x) + 6 - 6J_0(x) + 8J_2(x) - 4J_4(x)$$

gives with $x = 0.3$

$$0.045\,508\,152\,001 - 0.000\,339\,402\,714 + 0.000\,000\,381\,114 + 6 - 6.135\,761\,276\,110 + 0.090\,676\,901\,288-$$

$$-0.000\,084\,755\,400 = 6.136\,185\,434\,403 - 6.136\,185\,434\,224 = 0.000\,000\,000\,179 \ ,$$

which means the loss of 10 decimal digits.

For that reason the value of such integrals should be computed by the power series or other formulas. See also the following remark.

In the following the integrals are expressed by $Z_0(x)$ and $Z_1(x)$.

Integrals with $-2 \leq n \leq 4$ are written explicitely: at first $n = 0$, 1, 2, 3, 4, after them $n = -1$, -2. In the other cases the functions $\mathcal{P}_\nu^{(n)}(x)$, $\mathcal{Q}_\nu^{(n)}(x)$ and the coefficients $\mathcal{R}_\nu^{(n)}$, $\mathcal{S}_\nu^{(n)}$ describe the integral

$$\int x^n \cdot J_\nu(x)\, dx = \mathcal{P}_\nu^{(n)}(x)\, J_0(x) + \mathcal{Q}_\nu^{(n)}(x) J_1(x) + \mathcal{R}_\nu^{(n)}\, \Lambda_0(x) + \mathcal{S}_\nu^{(n)} \int \frac{J_0(x)\, dx}{x} \ .$$

Furthermore, let

$$\int x^n \cdot I_\nu^*(x)\, dx = \mathcal{P}_\nu^{(n),*}(x)\, I_0(x) + \mathcal{Q}_\nu^{(n),*}(x) I_1(x) + \mathcal{R}_\nu^{(n),*}\, \Lambda_0^*(x) + \mathcal{S}_\nu^{(n),*} \int \frac{I_0(x)\, dx}{x} \ .$$

Concerning $\int x^{-1} \cdot Z_0(x)\, dx$ see 1.1.3., page 13.

Simple recurrence formula:

$$\int x^n \cdot J_{\nu+1}(x)\, dx \quad = \quad 2\nu \int x^{n-1} \cdot J_\nu(x)\, dx - \int x^n \cdot J_{\nu-1}(x)\, dx$$

$$\int x^n \cdot I_{\nu+1}(x)\, dx \quad = \quad -2\nu \int x^{n-1} \cdot J_\nu(x)\, dx + \int x^n \cdot J_{\nu-1}(x)\, dx$$

The integrals of $x^n Z_0(x)$ and $x^n Z_1(x)$ to start this recurrences are already described.

Remark:

Let $F_\nu^{(m)}(x)$ denote the antiderivative of $x^m Z_\nu(x)$ as given in the following tables. They do not exist in the point $x = 0$ in the case $\nu + m < 0$. However, even if $\nu + m \geq 0$ the value of $F_\nu^{(m)}(0)$ sometimes turns out to be a limit of the type $\infty - \infty$. For instance, holds

$$\int \frac{J_3(x)\, dx}{x^2} = \frac{J_0(x)}{x^2} - \frac{2J_1(x)}{x^3} = F_3^{(-2)}(x) \quad \text{with} \quad \lim_{x \to 0} F_3^{(-2)}(x) = -\frac{1}{8} \ .$$

With $L_{\nu,m} = \lim_{x \to 0} F_\nu^{(m)}(x)$ for the Bessel functions $J_\nu(x)$ and $L_{\nu,m}^*$ for the modified Bessel functions $I_\nu(x)$ one has the following limits in the tables of integrals (The values $L_{\nu,m} = 0$ are omitted.):

$L_{2,-1} = -1/2$, $L_{2,-1}^* = 1/2$

$L_{3,0} = -1$, $L_{3,-2} = -1/8$; $L_{3,0}^* = -1$, $L_{3,-2}^* = 1/8$

$L_{4,1} = -4$, $L_{4,-1} = -1/4$, $L_{4,-3} = -1/48$; $L_{4,1}^* = 4$, $L_{4,-1}^* = -1/4$, $L_{4,-3}^* = 1/48$

$L_{5,2} = -24$, $L_{5,0} = -1$, $L_{5,-2} = -1/24$, $L_{5,-4} = -1/384$;
$L_{5,2}^* = -24$, $L_{5,0}^* = 1$, $L_{5,-2}^* = -1/24$, $L_{5,-4}^* = 1/384$

$L_{6,3} = -192$, $L_{6,1} = -6$, $L_{6,-1} = -1/6$, $L_{6,-3} = -1/192$, $L_{6,-5} = -1/3840$;
$L_{6,3}^* = 192$, $L_{6,1}^* = -6$, $L_{6,-1}^* = 1/6$, $L_{6,-3}^* = -1/192$, $L_{6,-5}^* = 1/3840$

$L_{7,4} = -1920$, $L_{7,2} = -48$, $L_{7,0} = -1$, $L_{7,-2} = -1/48$, $L_{7,-4} = -1/1920$, $L_{7,-6} = -1/46080$;
$L_{7,4}^* = -1920$, $L_{7,2}^* = 48$, $L_{7,0}^* = -1$, $L_{7,-2}^* = 1/48$, $L_{7,-4}^* = -1/1920$, $L_{7,-6}^* = 1/46080$

$L_{8,5} = -23040$, $L_{8,3} = -480$, $L_{8,1} = -8$, $L_{8,-1} = -1/8$, $L_{8,-3} = -1/480$, $L_{8,-5} = -1/23040$;
$L_{8,5}^* = 23040$, $L_{8,3}^* = -480$, $L_{8,1}^* = 8$, $L_{8,-1}^* = -1/8$, $L_{8,-3}^* = 1/480$, $L_{8,-5}^* = -1/23040$

$L_{9,6} = -322560$, $L_{9,4} = -5760$, $L_{9,2} = -80$, $L_{9,0} = -1$, $L_{9,-2} = -1/80$, $L_{9,-4} = -1/5760$,
$L_{9,-6} = -1/322560$;
$L_{9,6}^* = -322560$, $L_{9,4}^* = 5760$, $L_{9,2}^* = -80$, $L_{9,0}^* = 1$, $L_{9,-2}^* = -1/80$, $L_{9,-4}^* = 1/5760$,
$L_{9,-6}^* = -1/322560$

$L_{10,7} = -5160960$, $L_{10,5} = -80640$, $L_{10,3} = -960$, $L_{10,1} = -10$, $L_{10,-1} = -1/10$, $L_{10,-3} = -1/960$,
$L_{10,-5} = -1/80640$;
$L_{10,7}^* = 5160960$, $L_{10,5}^* = -80640$, $L_{10,3}^* = 960$, $L_{10,1}^* = -10$, $L_{10,-1}^* = 1/10$, $L_{10,-3}^* = -1/960$,
$L_{10,-5}^* = 1/80640$

In the described cases of limits of the type $\infty - \infty$ the numerical computation of $F_\nu^{(m)}(x)$ causes difficulties, if $0 < x << 1$. Then it is preferable to use the power series, which has a fast convergengence for such values of x. With $m + \nu \geq 0$ holds

$$\int_0^x t^m J_\nu(t)\, dt = \frac{x^{m+\nu+1}}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k! \cdot (\nu + k)! \cdot 4^k \cdot (m + \nu + 1 + 2k)}$$

and

$$\int_0^x t^m I_\nu(t)\, dt = \frac{x^{m+\nu+1}}{2^\nu} \sum_{k=0}^{\infty} \frac{x^{2k}}{k! \cdot (\nu + k)! \cdot 4^k \cdot (m + \nu + 1 + 2k)} \ .$$

From this one has

$$F_\nu^{(m)}(x) = L_{\nu,m} + \int_0^x t^m J_\nu(t)\, dt \quad \text{and} \quad F_\nu^{*,(m)}(x) = L_{\nu,m}^* + \int_0^x t^m I_\nu(t)\, dt \ .$$

For instance,

$$\int_{0.002}^3 \frac{J_3(x)}{x^2}\, dx = \frac{J_0(x)}{x^2} - \frac{2J_1(x)}{x^3} \ \Big|_{0.002}^3 =$$

$$= (-0.0288946616557703820 - 0.0251154784093286266) - (249999.750000062500 - 249999.875000020834) =$$

$$= -0.0540101400650990086 - (-0.124999958334^*) = 0.070989818269$$

It was a loss of seven decimal digits at $x = 0.002$. This value may be found without problems by the power series:

$$F_3^{(-2)}(0.002) =$$

$$= -\frac{1}{8} + 5 \cdot 10^{-7} \,[0.0833333333333333335 - 1.04166666666666667 \cdot 10^{-8} + 6.94444444444444445 \cdot 10^{-16} - \ldots] =$$

$$= -\frac{1}{8} + 4.16666614583336806 \cdot 10^{-8} = -0.124999958333338542$$

In the previous value, signed by *, the last digit should be 3 instead of 4 and the result had to finish with 8.

The integrals with $I_\nu(x)$ may be computed in the same way.

This method can be used even if $\nu + m < 0$. For instance,

$$\int_{0.002}^3 \frac{J_4(x)}{x^7}\, dx = \int_{0.002}^1 \frac{J_4(x)}{x^7}\, dx + \int_1^3 \frac{J_4(x)}{x^7}\, dx$$

and the second integral is given in the following tables. For the first one holds with the power series of the function $J_4(x)$

$$\int_{0.002}^1 \frac{J_4(x)}{x^7}\, dx =$$

$$= \int_{0.002}^1 \frac{1}{x^7} \left(\frac{1}{384} x^4 - \frac{1}{7680} x^6 + \frac{1}{368640} x^8 - \frac{1}{30965760} x^{10} + \frac{1}{3963617280} x^{12} - \frac{1}{713451110400} x^{14} + \ldots \right) dx =$$

$$= \int_{0.002}^1 \left(\frac{1}{384\, x^3} - \frac{1}{7680\, x} + \frac{1}{368640} x - \frac{1}{30965760} x^3 + \frac{1}{3963617280} x^5 - \frac{1}{713451110400} x^7 + \ldots \right) dx =$$

$$= -\frac{1}{768\, x^2} - \frac{\ln x}{7680} + \frac{1}{737280} x^2 - \frac{1}{123863040} x^4 + \frac{1}{23781703680} x^6 - \frac{1}{5707608883200} x^8 + \ldots \Big|_{0.002}^1 =$$

$$= (-0.001302083 - 0.0 + 1.3563 \cdot 10^{-6} - 8.07343 \cdot 10^{-9} + 4.20491 \cdot 10^{-11} - 0.17520 \cdot 10^{-13} + \ldots)$$

$$-(-325.52083333333333 + 0.00080919376281538 9549 + 5.4253472 \cdot 10^{-12} - 1.2917 \cdot 10^{-19} + 0.26911 \cdot 10^{-27} - \ldots) =$$

$$= -0.00130073502808721678 - (-325.520024139565093) = 325.518723404537006 \ .$$

Here are no differences of nearly the same values.

$\mathbf{Z_2(x)}$:

$$\int J_2(x)\,dx = -2J_1(x) + \Lambda_0(x)$$

$$\int I_2(x)\,dx = 2I_1(x) - \Lambda_0^*(x)$$

$$\int x\,J_2(x)\,dx = -2J_0(x) - xJ_1(x)$$

$$\int x\,I_2(x)\,dx = -2I_0(x) + xI_1(x)$$

$$\int x^2\,J_2(x)\,dx = -3xJ_0(x) - x^2 J_1(x) + 3\Lambda_0(x)$$

$$\int x^2\,I_2(x)\,dx = -3xI_0(x) + x^2 I_1(x) + 3\Lambda_0^*(x)$$

$$\int x^3\,J_2(x)\,dx = -4x^2 J_0(x) - \left(x^2 - 8\right)xJ_1(x)$$

$$\int x^3\,I_2(x)\,dx = -4x^2 I_0(x) + \left(x^2 + 8\right)xI_1(x)$$

$$\int x^4\,J_2(x)\,dx = -5x\left(x^2 - 3\right)J_0(x) - \left(x^2 - 15\right)x^2 J_1(x) - 15\Lambda_0(x)$$

$$\int x^4\,I_2(x)\,dx = -5x\left(x^2 + 3\right)I_0(x) + \left(x^2 + 15\right)x^2 I_1(x) + 15\Lambda_0^*(x)$$

$$\int \frac{J_2(x)\,dx}{x} = -\frac{J_1(x)}{x}$$

$$\int \frac{I_2(x)\,dx}{x} = \frac{I_1(x)}{x}$$

$$\int \frac{J_2(x)\,dx}{x^2} = \frac{1}{3x}\,J_0(x) - \frac{x^2 + 2}{3x^2}\,J_1(x) + \frac{1}{3}\,\Lambda_0(x)$$

$$\int \frac{I_2(x)\,dx}{x^2} = -\frac{1}{3x}\,I_0(x) - \frac{x^2 - 2}{3x^2}\,I_1(x) + \frac{1}{3}\,\Lambda_0^*(x)$$

$$\mathcal{P}_2^{(5)}(x) = -6\left(x^2 - 8\right)x^2\,, \quad \mathcal{Q}_2^{(5)}(x) = -\left(x^4 - 24\,x^2 + 96\right)x\,, \quad \mathcal{R}_2^{(5)} = 0\,, \quad \mathcal{S}_2^{(5)} = 0$$

$$\mathcal{P}_2^{(5),*}(x) = -6\left(x^2 + 8\right)x^2\,, \quad \mathcal{Q}_2^{(5),*}(x) = x^5 + 24\,x^3 + 96x\,, \quad \mathcal{R}_2^{(5),*} = 0\,, \quad \mathcal{S}_2^{(5),*} = 0$$

$$\mathcal{P}_2^{(6)}(x) = -7\left(x^4 - 15\,x^2 + 45\right)x\,, \quad \mathcal{Q}_2^{(6)}(x) = -\left(x^4 - 35\,x^2 + 315\right)x^2\,, \quad \mathcal{R}_2^{(6)} = 315\,, \quad \mathcal{S}_2^{(6)} = 0$$

$$\mathcal{P}_2^{(6),*}(x) = -7(x^5 + 15\,x^3 + 45\,x)\,, \quad \mathcal{Q}_2^{(6),*}(x) = x^6 + 35\,x^4 + 315\,x^2\,, \quad \mathcal{R}_2^{(6),*} = 315\,, \quad \mathcal{S}_2^{(6),*} = 0$$

$$\mathcal{P}_2^{(7)}(x) = -8\left(x^4 - 24\,x^2 + 192\right)x^2\,, \quad \mathcal{Q}_2^{(7)}(x) = -\left(x^6 - 48\,x^4 + 768\,x^2 - 3072\right)x\,, \quad \mathcal{R}_2^{(7)} = 0\,, \quad \mathcal{S}_2^{(7)} = 0$$

$$\mathcal{P}_2^{(7),*}(x) = -(8\,x^4 + 192\,x^2 + 1536)\,x^2\,, \quad \mathcal{Q}_2^{(7),*}(x) = x^7 + 48\,x^5 + 768\,x^3 + 3072\,x\,, \quad \mathcal{R}_2^{(7),*} = 0\,, \quad \mathcal{S}_2^{(7),*} = 0$$

$$\mathcal{P}_2^{(8)}(x) = -9\left(x^6 - 35\,x^4 + 525\,x^2 - 1575\right)x\,, \quad \mathcal{Q}_2^{(8)}(x) = -\left(x^6 - 63\,x^4 + 1575\,x^2 - 14175\right)x^2\,,$$
$$\mathcal{R}_2^{(8)} = -14175\,, \quad \mathcal{S}_2^{(8)} = 0$$

$$\mathcal{P}_2^{(8),*}(x) = -(9\,x^7 + 315\,x^5 + 4725\,x^3 + 14175\,x)\,, \quad \mathcal{Q}_2^{(8),*}(x) = x^8 + 63\,x^6 + 1575\,x^4 + 14175\,x^2\,,$$
$$\mathcal{R}_2^{(8),*} = 14175\,, \quad \mathcal{S}_2^{(8),*} = 0$$

$$\mathcal{P}_2^{(9)}(x) = -10\left(x^6 - 48\,x^4 + 1152\,x^2 - 9216\right)x^2\,, \quad \mathcal{Q}_2^{(9)}(x) = -\left(x^8 - 80\,x^6 + 2880\,x^4 - 46080\,x^2 + 184320\right)\,,$$
$$\mathcal{R}_2^{(9)} = 0\,, \quad \mathcal{S}_2^{(9)} = 0$$

$$\mathcal{P}_2^{(9),*}(x) = -(10\,x^8+480\,x^6+11520\,x^4+92160\,x^2)\,, \quad \mathcal{Q}_2^{(9),*}(x) = x^9+80\,x^7+2880\,x^5+46080\,x^3+184320\,x\,,$$

$$\mathcal{R}_2^{(9),*} = 0\,, \quad \mathcal{S}_2^{(9),*} = 0$$

$$\mathcal{P}_2^{(10)}(x) = -11\,\left(x^8-63\,x^6+2205\,x^4-33075\,x^2+99225\right)x\,,$$

$$\mathcal{Q}_2^{(10)}(x) = -\left(x^8-99\,x^6+4851\,x^4-121275\,x^2+1091475\right)x^2\,, \quad \mathcal{R}_2^{(10)} = 1091475\,, \quad \mathcal{S}_2^{(10)} = 0$$

$$\mathcal{P}_2^{(10),*}(x) = -(11\,x^9+693\,x^7+24255\,x^5+363825\,x^3+1091475\,x)\,,$$

$$\mathcal{Q}_2^{(10),*}(x) = x^{10}+99\,x^8+4851\,x^6+121275\,x^4+1091475\,x^2\,, \quad \mathcal{R}_2^{(10),*} = 1091475\,, \quad \mathcal{S}_2^{(10),*} = 0$$

$$\mathcal{P}_2^{(-3)}(x) = \frac{1}{4\,x^2}\,, \quad \mathcal{Q}_2^{(-3)}(x) = -\frac{x^2+4}{8\,x^3}\,, \quad \mathcal{R}_2^{(-3)} = 0\,, \quad \mathcal{S}_2^{(-3)} = \frac{1}{8}$$

$$\mathcal{P}_2^{(-3),*}(x) = -\frac{1}{4x^2}\,, \quad \mathcal{Q}_2^{(-3),*}(x) = -\frac{x^2-4}{8\,x^3}\,, \quad \mathcal{R}_2^{(-3),*} = 0\,, \quad \mathcal{S}_2^{(-3),*} = \frac{1}{8}$$

$$\mathcal{P}_2^{(-4)}(x) = -\frac{x^2-3}{15\,x^3}\,, \quad \mathcal{Q}_2^{(-4)}(x) = \frac{x^4-x^2-6}{15\,x^4}\,, \quad \mathcal{R}_2^{(-4)} = -\frac{1}{15}\,, \quad \mathcal{S}_2^{(-4)} = 0$$

$$\mathcal{P}_2^{(-4),*}(x) = -\frac{x^2+3}{15\,x^3}\,, \quad \mathcal{Q}_2^{(-4),*}(x) = -\frac{x^4+x^2-6}{15\,x^4}\,, \quad \mathcal{R}_2^{(-4),*} = \frac{1}{15}\,, \quad \mathcal{S}_2^{(-4),*} = 0$$

$$\mathcal{P}_2^{(-5)}(x) = -\frac{x^2-8}{48\,x^4}\,, \quad \mathcal{Q}_2^{(-5)}(x) = \frac{x^4-4\,x^2-32}{96\,x^5}\,, \quad \mathcal{R}_2^{(-5)} = 0\,, \quad \mathcal{S}_2^{(-5)} = -\frac{1}{96}$$

$$\mathcal{P}_2^{(-5),*}(x) = -\frac{x^2+8}{48\,x^4}\,, \quad \mathcal{Q}_2^{(-5),*}(x) = -\frac{x^4+4\,x^2-32}{96\,x^5}\,, \quad \mathcal{R}_2^{(-5),*} = 0\,, \quad \mathcal{S}_2^{(-5),*} = \frac{1}{96}$$

$$\mathcal{P}_2^{(-6)}(x) = \frac{x^4-3\,x^2+45}{315\,x^5}\,, \quad \mathcal{Q}_2^{(-6)}(x) = -\frac{x^6-x^4+9\,x^2+90}{315\,x^6}\,, \quad \mathcal{R}_2^{(-6)} = \frac{1}{315}\,, \quad \mathcal{S}_2^{(-6)} = 0$$

$$\mathcal{P}_2^{(-6),*}(x) = -\frac{x^4+3\,x^2+45}{315x^5}\,, \quad \mathcal{Q}_2^{(-6),*}(x) = -\frac{x^6+x^4+9\,x^2-90}{315x^6}\,, \quad \mathcal{R}_2^{(-6),*} = \frac{1}{315}\,, \quad \mathcal{S}_2^{(-6),*} = 0$$

$\underline{\mathbf{Z}_3(\mathbf{x})}$:

$$\int J_3(x)\,dx = J_0(x) - \frac{4}{x}\,J_1(x)$$

$$\int I_3(x)\,dx = I_0(x) - \frac{4}{x}\,I_1(x)$$

$$\int x\,J_3(x)\,dx = xJ_0(x) - 8J_1(x) + 3\Lambda_0(x)$$

$$\int x\,I_3(x)\,dx = xI_0(x) - 8I_1(x) + 3\Lambda_0^*(x)$$

$$\int x^2\,J_3(x)\,dx = (x^2-8)J_0(x) - 6xJ_1(x)$$

$$\int x^2\,I_3(x)\,dx = (x^2+8)I_0(x) - 6xI_1(x)$$

$$\int x^3\,J_3(x)\,dx = \left(x^2-15\right)x\,J_0(x) - 7x^2J_1(x) + 15\Lambda_0(x)$$

$$\int x^3\,I_3(x)\,dx = (x^2+15)x\,I_0(x) - 7x^2\,I_1(x) - 15\Lambda_0^*(x)$$

$$\int x^4\,J_3(x)\,dx = \left(x^2-24\right)x^2J_0(x) - 8\left(x^2-6\right)xJ_1(x)$$

$$\int x^4\,I_3(x)\,dx = \left(x^2+24\right)x^2I_0(x) - 8\left(x^2+6\right)xI_1(x)$$

$$\int \frac{J_3(x)\,dx}{x} = \frac{4}{3x}\,J_0(x) - \frac{x^2+8}{3x^2}\,J_1(x) + \frac{1}{3}\,\Lambda_0(x)$$

$$\int \frac{I_3(x)\,dx}{x} = \frac{4}{3x}\,I_0(x) + \frac{x^2-8}{3x^2}\,I_1(x) - \frac{1}{3}\Lambda_0^*(x)$$

$$\int \frac{J_3(x)\,dx}{x^2} = \frac{J_0(x)}{x^2} - \frac{2J_1(x)}{x^3}$$

$$\int \frac{I_3(x)\,dx}{x^2} = \frac{I_0(x)}{x^2} - \frac{2I_1(x)}{x^3}$$

$$\mathcal{P}_3^{(5)}(x) = x^5 - 35\,x^3 + 105\,x\,, \quad \mathcal{Q}_3^{(5)}(x) = -(9\,x^4 - 105\,x^2)\,, \quad \mathcal{R}_3^{(5)} = -105\,, \quad \mathcal{S}_3^{(5)} = 0$$

$$\mathcal{P}_3^{(5),*}(x) = x^5 + 35\,x^3 + 105\,x\,, \quad \mathcal{Q}_3^{(5),*}(x) = -(9\,x^4 + 105\,x^2)\,, \quad \mathcal{R}_3^{(5),*} = -105\,, \quad \mathcal{S}_3^{(5),*} = 0$$

$$\mathcal{P}_3^{(6)}(x) = x^6 - 48\,x^4 + 384\,x^2\,, \quad \mathcal{Q}_3^{(6)}(x) = -(10\,x^5 - 192\,x^3 + 768\,x)\,, \quad \mathcal{R}_3^{(6)} = 0\,, \quad \mathcal{S}_3^{(6)} = 0$$

$$\mathcal{P}_3^{(6),*}(x) = x^6 + 48\,x^4 + 384\,x^2\,, \quad \mathcal{Q}_3^{(6),*}(x) = -(10\,x^5 + 192\,x^3 + 768\,x)\,, \quad \mathcal{R}_3^{(6),*} = 0\,, \quad \mathcal{S}_3^{(6),*} = 0$$

$$\mathcal{P}_3^{(7)}(x) = x^7 - 63\,x^5 + 945\,x^3 - 2835\,x\,, \quad \mathcal{Q}_3^{(7)}(x) = -(11\,x^6 - 315\,x^4 + 2835\,x^2)\,, \quad \mathcal{R}_3^{(7)} = 2835\,, \quad \mathcal{S}_3^{(7)} = 0$$

$$\mathcal{P}_3^{(7),*}(x) = x^7 + 63\,x^5 + 945\,x^3 + 2835\,x\,, \quad \mathcal{Q}_3^{(7),*}(x) = -(11\,x^6 + 315\,x^4 + 2835\,x^2)\,, \quad \mathcal{R}_3^{(7),*} = -2835\,, \quad \mathcal{S}_3^{(7),*} = 0$$

$$\mathcal{P}_3^{(8)}(x) = x^8 - 80\,x^6 + 1920\,x^4 - 15360\,x^2\,, \quad \mathcal{Q}_3^{(8)}(x) = -(12\,x^7 - 480\,x^5 + 7680\,x^3 - 30720\,x)\,,$$
$$\mathcal{R}_3^{(8)} = 0\,, \quad \mathcal{S}_3^{(8)} = 0$$

$$\mathcal{P}_3^{(8),*}(x) = x^8 + 80\,x^6 + 1920\,x^4 + 15360\,x^2\,, \quad \mathcal{Q}_3^{(8),*}(x) = -(12\,x^7 + 480\,x^5 + 7680\,x^3 + 30720\,x)\,,$$
$$\mathcal{R}_3^{(8),*} = 0\,, \quad \mathcal{S}_3^{(8),*} = 0$$

$$\mathcal{P}_3^{(9)}(x) = x^9 - 99\,x^7 + 3465\,x^5 - 51975\,x^3 + 155925\,x\,,$$
$$\mathcal{Q}_3^{(9)}(x) = -(13\,x^8 - 693\,x^6 + 17325\,x^4 - 155925\,x^2)\,, \quad \mathcal{R}_3^{(9)} = -155925\,, \quad \mathcal{S}_3^{(9)} = 0$$

$$\mathcal{P}_3^{(9),*}(x) = x^9 + 99\,x^7 + 3465\,x^5 + 51975\,x^3 + 155925\,x\,,$$
$$\mathcal{Q}_3^{(9),*}(x) = -(13\,x^8 + 693\,x^6 + 17325\,x^4 + 155925\,x^2)\,, \quad \mathcal{R}_3^{(9),*} = -155925\,, \quad \mathcal{S}_3^{(9),*} = 0$$

$$\mathcal{P}_3^{(10)}(x) = x^{10} - 120\,x^8 + 5760\,x^6 - 138240\,x^4 + 1105920\,x^2\,,$$
$$\mathcal{Q}_3^{(10)}(x) = -(14\,x^9 - 960\,x^7 + 34560\,x^5 - 552960\,x^3 + 2211840\,x)\,, \quad \mathcal{R}_3^{(10)} = 0\,, \quad \mathcal{S}_3^{(10)} = 0$$

$$\mathcal{P}_3^{(10),*}(x) = x^{10} + 120\,x^8 + 5760\,x^6 + 138240\,x^4 + 1105920\,x^2\,,$$
$$\mathcal{Q}_3^{(10),*}(x) = -(14\,x^9 + 960\,x^7 + 34560\,x^5 + 552960\,x^3 + 2211840\,x)\,, \quad \mathcal{R}_3^{(10),*} = 0\,, \quad \mathcal{S}_3^{(10),*} = 0$$

$$\mathcal{P}_3^{(-3)}(x) = \frac{x^2 + 12}{15\,x^3}\,, \quad \mathcal{Q}_3^{(-3)}(x) = -\frac{x^4 - x^2 + 24}{15\,x^4}\,, \quad \mathcal{R}_3^{(-3)} = \frac{1}{15}\,, \quad \mathcal{S}_3^{(-3)} = 0$$

$$\mathcal{P}_3^{(-3),*}(x) = -\frac{x^2 - 12}{15\,x^3}\,, \quad \mathcal{Q}_3^{(-3),*}(x) = -\frac{x^4 + x^2 + 24}{15\,x^4}\,, \quad \mathcal{R}_3^{(-3),*} = \frac{1}{15}\,, \quad \mathcal{S}_3^{(-3),*} = 0$$

$$\mathcal{P}_3^{(-4)}(x) = \frac{x^2 + 16}{24\,x^4}\,, \quad \mathcal{Q}_3^{(-4)}(x) = -\frac{x^4 - 4\,x^2 + 64}{48\,x^5}\,, \quad \mathcal{R}_3^{(-4)} = 0\,, \quad \mathcal{S}_3^{(-4)} = \frac{1}{48}$$

$$\mathcal{P}_3^{(-4),*}(x) = -\frac{x^2 - 16}{24\,x^4}\,, \quad \mathcal{Q}_3^{(-4),*}(x) = -\frac{x^4 + 4\,x^2 + 64}{48\,x^5}\,, \quad \mathcal{R}_3^{(-4),*} = 0\,, \quad \mathcal{S}_3^{(-4),*} = \frac{1}{48}$$

$$\mathcal{P}_3^{(-5)}(x) = -\frac{x^4 - 3\,x^2 - 60}{105\,x^5}\,, \quad \mathcal{Q}_3^{(-5)}(x) = \frac{x^6 - x^4 + 9\,x^2 - 120}{105\,x^6}\,, \quad \mathcal{R}_3^{(-5)} = -\frac{1}{105}\,, \quad \mathcal{S}_3^{(-5)} = 0$$

$$\mathcal{P}_3^{(-5),*}(x) = -\frac{x^4 + 3\,x^2 - 60}{105\,x^5}\,, \quad \mathcal{Q}_3^{(-5),*}(x) = \frac{x^6 + x^4 + 9\,x^2 + 120}{105\,x^6}\,, \quad \mathcal{R}_3^{(-5),*} = -\frac{1}{105}\,, \quad \mathcal{S}_3^{(-5),*} = 0$$

$$\mathcal{P}_3^{(-6)}(x) = -\frac{x^4 - 8\,x^2 - 192}{384\,x^6}\,, \quad \mathcal{Q}_3^{(-6)}(x) = \frac{x^6 - 4\,x^4 + 64\,x^2 - 768}{768\,x^7}\,, \quad \mathcal{R}_3^{(-6)} = 0\,, \quad \mathcal{S}_3^{(-6)} = -\frac{1}{768}$$

$$\mathcal{P}_3^{(-6),*}(x) = -\frac{x^4 + 8\,x^2 - 192}{384\,x^6}\,, \quad \mathcal{Q}_3^{(-6),*}(x) = -\frac{x^6 + 4\,x^4 + 64\,x^2 + 768}{768\,x^7}\,, \quad \mathcal{R}_3^{(-6),*} = 0\,, \quad \mathcal{S}_3^{(-6),*} = \frac{1}{768}$$

$\underline{\mathbf{Z_4(x)}}$:

$$\int J_4(x)\, dx = \frac{8J_0(x)}{x} - \frac{16J_1(x)}{x^2} + \Lambda_0(x)$$

$$\int I_4(x)\, dx = -\frac{8I_0(x)}{x} + \frac{16I_1(x)}{x^2} + \Lambda_0^*(x)$$

$$\int x\, J_4(x)\, dx = 8J_0(x) + \frac{x^2 - 24}{x} J_1(x)$$

$$\int x\, I_4(x)\, dx = -8I_0(x) + \frac{x^2 + 24}{x} I_1(x)$$

$$\int x^2\, J_4(x)\, dx = 9xJ_0(x) + (x^2 - 48)J_1(x) + 15\Lambda_0(x)$$

$$\int x^2\, I_4(x)\, dx = -9xI_0(x) + (x^2 + 48)I_1(x) - 15\Lambda_0^*(x)$$

$$\int x^3\, J_4(x)\, dx = (10\, x^2 - 48)J_0(x) + \left(x^2 - 44\right) xJ_1(x)$$

$$\int x^3\, I_4(x)\, dx = -(10\, x^2 + 48)I_0(x) + \left(x^2 + 44\right) xI_1(x)$$

$$\int x^4\, J_4(x)\, dx = \left(11\, x^2 - 105\right) xJ_0(x) + \left(x^2 - 57\right) x^2 J_1(x) + 105\Lambda_0(x)$$

$$\int x^4\, I_4(x)\, dx = -\left(11\, x^2 + 105\right) xI_0(x) + \left(x^2 + 57\right) x^2 I_1(x) + 105\Lambda_0^*(x)$$

$$\int \frac{J_4(x)\, dx}{x} = \frac{6J_0(x)}{x^2} + \frac{x^2 - 12}{x^3} J_1(x)$$

$$\int \frac{I_4(x)\, dx}{x} = -\frac{6J_0(x)}{x^2} + \frac{x^2 + 12}{x^3} J_1(x)$$

$$\int \frac{J_4(x)\, dx}{x^2} = \frac{x^2 + 72}{15x^3} J_0(x) - \frac{x^4 - 16\, x^2 + 144}{15x^4} J_1(x) + \frac{1}{15}\Lambda_0(x)$$

$$\int \frac{I_4(x)\, dx}{x^2} = \frac{x^2 - 72}{15x^3} I_0(x) + \frac{x^4 + 16\, x^2 + 144}{15x^4} I_1(x) - \frac{1}{15}\Lambda_0^*(x)$$

$$\mathcal{P}_4^{(5)}(x) = 12\, x^4 - 192\, x^2\,, \quad \mathcal{Q}_4^{(5)}(x) = x^5 - 72\, x^3 + 384\, x\,, \quad \mathcal{R}_4^{(5)} = 0\,, \quad \mathcal{S}_4^{(5)} = 0$$

$$\mathcal{P}_4^{(5),*}(x) = -(12\, x^4 + 192\, x^2)\,, \quad \mathcal{Q}_4^{(5),*}(x) = x^5 + 72\, x^3 + 384\, x\,, \quad \mathcal{R}_4^{(5),*} = 0\,, \quad \mathcal{S}_4^{(5),*} = 0$$

$$\mathcal{P}_4^{(6)}(x) = 13\, x^5 - 315\, x^3 + 945\, x\,, \quad \mathcal{Q}_4^{(6)}(x) = x^6 - 89\, x^4 + 945\, x^2\,,$$

$$\mathcal{R}_4^{(6)} = -945\,, \quad \mathcal{S}_4^{(6)} = 0$$

$$\mathcal{P}_4^{(6),*}(x) = -(13\, x^5 + 315\, x^3 + 945\, x)\,, \quad \mathcal{Q}_4^{(6),*}(x) = x^6 + 89\, x^4 + 945\, x^2\,,$$

$$\mathcal{R}_4^{(6),*} = 945\,, \quad \mathcal{S}_4^{(6),*} = 0$$

$$\mathcal{P}_4^{(7)}(x) = 14\, x^6 - 480\, x^4 + 3840\, x^2\,, \quad \mathcal{Q}_4^{(7)}(x) = x^7 - 108\, x^5 + 1920\, x^3 - 7680\, x\,,$$

$$\mathcal{R}_4^{(7)} = 0\,, \quad \mathcal{S}_4^{(7)} = 0$$

$$\mathcal{P}_4^{(7),*}(x) = -(14\, x^6 + 480\, x^4 + 3840\, x^2)\,, \quad \mathcal{Q}_4^{(7),*}(x) = x^7 + 108\, x^5 + 1920\, x^3 + 7680\, x\,,$$

$$\mathcal{R}_4^{(7),*} = 0\,, \quad \mathcal{S}_4^{(7),*} = 0$$

$$\mathcal{P}_4^{(8)}(x) = 15\, x^7 - 693\, x^5 + 10395\, x^3 - 31185\, x\,, \quad \mathcal{Q}_4^{(8)}(x) = x^8 - 129\, x^6 + 3465\, x^4 - 31185\, x^2\,,$$

$$\mathcal{R}_4^{(8)} = 31185\,, \quad \mathcal{S}_4^{(8)} = 0$$

$$\mathcal{P}_4^{(8),*}(x) = -(15\,x^7 + 693\,x^5 + 10395\,x^3 + 31185\,x)\,, \quad \mathcal{Q}_4^{(8),*}(x) = x^8 + 129\,x^6 + 3465\,x^4 + 31185\,x^2\,,$$

$$\mathcal{R}_4^{(8),*} = 31185\,, \quad \mathcal{S}_4^{(8),*} = 0$$

$$\mathcal{P}_4^{(9)}(x) = 16\,x^8 - 960\,x^6 + 23040\,x^4 - 184320\,x^2\,,$$

$$\mathcal{Q}_4^{(9)}(x) = x^9 - 152\,x^7 + 5760\,x^5 - 92160\,x^3 + 368640\,x\,, \quad \mathcal{R}_4^{(9)} = 0\,, \quad \mathcal{S}_4^{(9)} = 0$$

$$\mathcal{P}_4^{(9),*}(x) = -(16\,x^8 + 960\,x^6 + 23040\,x^4 + 184320\,x^2)\,,$$

$$\mathcal{Q}_4^{(9),*}(x) = x^9 + 152\,x^7 + 5760\,x^5 + 92160\,x^3 + 368640\,x\,, \quad \mathcal{R}_4^{(9),*} = 0\,, \quad \mathcal{S}_4^{(9),*} = 0$$

$$\mathcal{P}_4^{(10)}(x) = 17\,x^9 - 1287\,x^7 + 45045\,x^5 - 675675\,x^3 + 2027025\,x\,,$$

$$\mathcal{Q}_4^{(10)}(x) = x^{10} - 177\,x^8 + 9009\,x^6 - 225225\,x^4 + 2027025\,x^2\,,$$

$$\mathcal{R}_4^{(10)} = -2027025\,, \quad \mathcal{S}_4^{(10)} = 0$$

$$\mathcal{P}_4^{(10),*}(x) = -(17\,x^9 + 1287\,x^7 + 45045\,x^5 + 675675\,x^3 + 2027025\,x)\,,$$

$$\mathcal{Q}_4^{(10),*}(x) = x^{10} + 177\,x^8 + 9009\,x^6 + 225225\,x^4 + 2027025\,x^2\,,$$

$$\mathcal{R}_4^{(10),*} = 2027025\,, \quad \mathcal{S}_4^{(10),*} = 0$$

$$\mathcal{P}_4^{(-3)}(x) = \frac{4}{x^4}\,, \quad \mathcal{Q}_4^{(-3)}(x) = \frac{x^2 - 8}{x^5}\,, \quad \mathcal{R}_4^{(-3)} = 0\,, \quad \mathcal{S}_4^{(-3)} = 0$$

$$\mathcal{P}_4^{(-3),*}(x) = -\frac{4}{x^4}\,, \quad \mathcal{Q}_4^{(-3),*}(x) = \frac{x^2 + 8}{x^5}\,, \quad \mathcal{R}_4^{(-3),*} = 0\,, \quad \mathcal{S}_4^{(-3),*} = 0$$

$$\mathcal{P}_4^{(-4)}(x) = \frac{x^4 - 3\,x^2 + 360}{105\,x^5}\,, \quad \mathcal{Q}_4^{(-4)}(x) = -\frac{x^6 - x^4 - 96\,x^2 + 720}{105 x^6}\,,$$

$$\mathcal{R}_4^{(-4)} = \frac{1}{105}\,, \quad \mathcal{S}_4^{(-4)} = 0$$

$$\mathcal{P}_4^{(-4),*}(x) = -\frac{x^4 + 3\,x^2 + 360}{105 x^5}\,, \quad \mathcal{Q}_4^{(-4),*}(x) = -\frac{x^6 + x^4 - 96\,x^2 - 720}{105 x^6}\,,$$

$$\mathcal{R}_4^{(-4),*} = \frac{1}{105}\,, \quad \mathcal{S}_4^{(-4),*} = 0$$

$$\mathcal{P}_4^{(-5)}(x) = \frac{x^4 - 8\,x^2 + 576}{192\,x^6}\,, \quad \mathcal{Q}_4^{(-5)}(x) = -\frac{x^6 - 4\,x^4 - 320\,x^2 + 2304}{384\,x^7}\,,$$

$$\mathcal{R}_4^{(-5)} = 0\,, \quad \mathcal{S}_4^{(-5)} = \frac{1}{384}$$

$$\mathcal{P}_4^{(-5),*}(x) = -\frac{x^4 + 8\,x^2 + 576}{192\,x^6}\,, \quad \mathcal{Q}_4^{(-5),*}(x) = -\frac{x^6 + 4\,x^4 - 320\,x^2 - 2304}{384\,x^7}\,,$$

$$\mathcal{R}_4^{(-5),*} = 0\,, \quad \mathcal{S}_4^{(-5),*} = \frac{1}{384}$$

$$\mathcal{P}_4^{(-6)}(x) = -\frac{x^6 - 3\,x^4 + 45\,x^2 - 2520}{945\,x^7}\,, \quad \mathcal{Q}_4^{(-6)}(x) = \frac{x^8 - x^6 + 9\,x^4 + 720\,x^2 - 5040}{945\,x^8}\,,$$

$$\mathcal{R}_4^{(-6)} = -\frac{1}{945}\,, \quad \mathcal{S}_4^{(-6)} = 0$$

$$\mathcal{P}_4^{(-6),*}(x) = -\frac{x^6 + 3\,x^4 + 45\,x^2 + 2520}{945\,x^7}\,, \quad \mathcal{Q}_4^{(-6),*}(x) = \frac{x^8 + x^6 + 9\,x^4 - 720\,x^2 - 5040}{945\,x^8}\,,$$

$$\mathcal{R}_4^{(-6),*} = \frac{1}{945}\,, \quad \mathcal{S}_4^{(-6),*} = 0$$

$\underline{\mathbf{Z}_5(\mathbf{x})}$:

$$\int J_5(x)\,dx = -\frac{x^2 - 48}{x^2}\,J_0(x) + \frac{12\,x^2 - 96}{x^3}\,J_1(x)$$

$$\int I_5(x)\,dx = \frac{x^2+48}{x^2}I_0(x) - \frac{12\,x^2+96}{x^3}I_1(x)$$

$$\int x\,J_5(x)\,dx = -\frac{x^2-64}{x}J_0(x) + \frac{8\,x^2-128}{x^2}J_1(x) + 5\Lambda_0(x)$$

$$\int x\,I_5(x)\,dx = \frac{x^2+64}{x}I_0(x) - \frac{8\,x^2+128}{x^2}I_1(x) - 5\Lambda_0^*(x)$$

$$\int x^2\,J_5(x)\,dx = -(x^2-72)J_0(x) + \frac{14\,x^2-192}{x}J_1(x)$$

$$\int x^2\,I_5(x)\,dx = (x^2+72)I_0(x) - \frac{14\,x^2+192}{x}I_1(x)$$

$$\int x^3\,J_5(x)\,dx = -(x^3-87\,x)J_0(x) + (15\,x^2-384)J_1(x) + 105\Lambda_0(x)$$

$$\int x^3\,I_5(x)\,dx = (x^3+87\,x)I_0(x) - (15\,x^2+384)I_1(x) + 105\Lambda_0^*(x)$$

$$\int x^4\,J_5(x)\,dx = -(x^4-104\,x^2+384)J_0(x) + (16\,x^3-400\,x)J_1(x)$$

$$\int x^4\,I_5(x)\,dx = (x^4+104\,x^2+384)I_0(x) - (16\,x^3+400\,x)I_1(x)$$

$$\int \frac{J_5(x)\,dx}{x} = -\frac{4\,x^2-192}{5\,x^3}J_0(x) - \frac{x^4-56\,x^2+384}{5\,x^4}J_1(x) + \frac{1}{5}\,\Lambda_0(x)$$

$$\int \frac{I_5(x)\,dx}{x} = \frac{4\,x^2+192}{5\,x^3}I_0(x) - \frac{x^4+56\,x^2+384}{5\,x^4}I_1(x) + \frac{1}{5}\Lambda_0^*(x)$$

$$\int \frac{J_5(x)\,dx}{x^2} = -\frac{x^2-32}{x^4}J_0(x) + \frac{10\,x^2-64}{x^5}J_1(x)$$

$$\int \frac{I_5(x)\,dx}{x^2} = \frac{x^2+32}{x^4}I_0(x) - \frac{10\,x^2+64}{x^5}I_1(x)$$

$$\mathcal{P}_5^{(5)}(x) = -(x^5 - 123\,x^3 + 945\,x)\,, \quad \mathcal{Q}_5^{(5)}(x) = 17\,x^4 - 561\,x^2\,, \quad \mathcal{R}_5^{(5)} = 945\,, \quad \mathcal{S}_5^{(5)} = 0$$

$$\mathcal{P}_5^{(5),*}(x) = x^5 + 123\,x^3 + 945\,x\,, \quad \mathcal{Q}_5^{(5),*}(x) = -(17\,x^4 + 561\,x^2)\,, \quad \mathcal{R}_5^{(5),*} = -945\,, \quad \mathcal{S}_5^{(5),*} = 0$$

$$\mathcal{P}_5^{(6)}(x) = -(x^6 - 144\,x^4 + 1920\,x^2)\,, \quad \mathcal{Q}_5^{(6)}(x) = 18\,x^5 - 768\,x^3 + 3840\,x\,,$$
$$\mathcal{R}_5^{(6)} = 0\,, \quad \mathcal{S}_5^{(6)} = 0$$

$$\mathcal{P}_5^{(6),*}(x) = x^6 + 144\,x^4 + 1920\,x^2\,, \quad \mathcal{Q}_5^{(6),*}(x) = -(18\,x^5 + 768\,x^3 + 3840\,x)\,,$$
$$\mathcal{R}_5^{(6),*} = 0\,, \quad \mathcal{S}_5^{(6),*} = 0$$

$$\mathcal{P}_5^{(7)}(x) = -(x^7 - 167\,x^5 + 3465\,x^3 - 10395\,x)\,, \quad \mathcal{Q}_5^{(7)}(x) = 19\,x^6 - 1027\,x^4 + 10395\,x^2\,,$$
$$\mathcal{R}_5^{(7)} = -10395\,, \quad \mathcal{S}_5^{(7)} = 0$$

$$\mathcal{P}_5^{(7),*}(x) = x^7 + 167\,x^5 + 3465\,x^3 + 10395\,x\,, \quad \mathcal{Q}_5^{(7),*}(x) = -(19\,x^6 + 1027\,x^4 + 10395\,x^2)\,,$$
$$\mathcal{R}_5^{(7),*} = -10395\,, \quad \mathcal{S}_5^{(7),*} = 0$$

$$\mathcal{P}_5^{(8)}(x) = -(x^8 - 192\,x^6 + 5760\,x^4 - 46080\,x^2)\,, \quad \mathcal{Q}_5^{(8)}(x) = 20\,x^7 - 1344\,x^5 + 23040\,x^3 - 92160\,x\,,$$
$$\mathcal{R}_5^{(8)} = 0\,, \quad \mathcal{S}_5^{(8)} = 0$$

$$\mathcal{P}_5^{(8),*}(x) = x^8 + 192\,x^6 + 5760\,x^4 + 46080\,x^2\,, \quad \mathcal{Q}_5^{(8),*}(x) = -(20\,x^7 + 1344\,x^5 + 23040\,x^3 + 92160\,x)\,,$$
$$\mathcal{R}_5^{(8),*} = 0\,, \quad \mathcal{S}_5^{(8),*} = 0$$

$$\mathcal{P}_5^{(9)}(x) = -(x^9 - 219\,x^7 + 9009\,x^5 - 135135\,x^3 + 405405\,x)\,,$$
$$\mathcal{Q}_5^{(9)}(x) = 21\,x^8 - 1725\,x^6 + 45045\,x^4 - 405405\,x^2\,, \quad \mathcal{R}_5^{(9)} = 405405\,, \quad \mathcal{S}_5^{(9)} = 0$$

$$\mathcal{P}_5^{(9),*}(x) = x^9 + 219\,x^7 + 9009\,x^5 + 135135\,x^3 + 405405\,x\,,$$

$$\mathcal{Q}_5^{(9),*}(x) = -(21\,x^8 + 1725\,x^6 + 45045\,x^4 + 405405\,x^2)\,, \quad \mathcal{R}_5^{(9),*} = -405405\,, \quad \mathcal{S}_5^{(9),*} = 0$$

$$\mathcal{P}_5^{(10)}(x) = -(x^{10} - 248\,x^8 + 13440\,x^6 - 322560\,x^4 + 2580480\,x^2)\,,$$

$$\mathcal{Q}_5^{(10)}(x) = 22\,x^9 - 2176\,x^7 + 80640\,x^5 - 1290240\,x^3 + 5160960\,x\,, \quad \mathcal{R}_5^{(10)} = 0\,, \quad \mathcal{S}_5^{(10)} = 0$$

$$\mathcal{P}_5^{(10),*}(x) = x^{10} + 248\,x^8 + 13440\,x^6 + 322560\,x^4 + 2580480\,x^2\,,$$

$$\mathcal{Q}_5^{(10),*}(x) = -(22\,x^9 + 2176\,x^7 + 80640\,x^5 + 1290240\,x^3 + 5160960\,x)\,, \quad \mathcal{R}_5^{(10),*} = 0\,, \quad \mathcal{S}_5^{(10),*} = 0$$

$$\mathcal{P}_5^{(-3)}(x) = \frac{x^4 - 108\,x^2 + 2880}{105x^5}\,, \quad \mathcal{Q}_5^{(-3)}(x) = -\frac{x^6 - x^4 - 936\,x^2 + 5760}{105x^6}\,, \quad \mathcal{R}_5^{(-3)} = \frac{1}{105}\,, \quad \mathcal{S}_5^{(-3)} = 0$$

$$\mathcal{P}_5^{(-3),*}(x) = \frac{x^4 + 108\,x^2 + 2880}{105x^5}\,, \quad \mathcal{Q}_5^{(-3),*}(x) = \frac{x^6 + x^4 - 936\,x^2 - 5760}{105x^6}\,, \quad \mathcal{R}_5^{(-3),*} = -\frac{1}{105}\,, \quad \mathcal{S}_5^{(-3),*} = 0$$

$$\mathcal{P}_5^{(-4)}(x) = -\frac{x^2 - 24}{x^6}\,, \quad \mathcal{Q}_5^{(-4)}(x) = \frac{8\,x^2 - 48}{x^7}\,, \quad \mathcal{R}_5^{(-4)} = 0\,, \quad \mathcal{S}_5^{(-4)} = 0$$

$$\mathcal{P}_5^{(-4),*}(x) = \frac{x^2 + 24}{x^6}\,, \quad \mathcal{Q}_5^{(-4),*}(x) = -\frac{8\,x^2 + 48}{x^7}\,, \quad \mathcal{R}_5^{(-4),*} = 0\,, \quad \mathcal{S}_5^{(-4),*} = 0$$

$$\mathcal{P}_5^{(-5)}(x) = \frac{x^6 - 3\,x^4 - 900\,x^2 + 20160}{945\,x^7}\,, \quad \mathcal{Q}_5^{(-5)}(x) = -\frac{x^8 - x^6 + 9\,x^4 - 6840\,x^2 + 40320}{945\,x^8}\,,$$

$$\mathcal{R}_5^{(-5)} = \frac{1}{945}\,, \quad \mathcal{S}_5^{(-5)} = 0$$

$$\mathcal{P}_5^{(-5),*}(x) = -\frac{x^6 + 3\,x^4 - 900\,x^2 - 20160}{945\,x^7}\,, \quad \mathcal{Q}_5^{(-5),*}(x) = -\frac{x^8 + x^6 + 9\,x^4 + 6840\,x^2 + 40320}{945\,x^8}\,,$$

$$\mathcal{R}_5^{(-5),*} = \frac{1}{945}\,, \quad \mathcal{S}_5^{(-5),*} = 0$$

$$\mathcal{P}_5^{(-6)}(x) = \frac{x^6 - 8\,x^4 - 1728\,x^2 + 36864}{1920\,x^8}\,, \quad \mathcal{Q}_5^{(-6)}(x) = -\frac{x^8 - 4\,x^6 + 64\,x^4 - 25344\,x^2 + 147456}{3840\,x^9}\,,$$

$$\mathcal{R}_5^{(-6)} = 0\,, \quad \mathcal{S}_5^{(-6)} = \frac{1}{3840}$$

$$\mathcal{P}_5^{(-6),*}(x) = -\frac{x^6 + 8\,x^4 - 1728\,x^2 - 36864}{1920\,x^8}\,, \quad \mathcal{Q}_5^{(-6),*}(x) = -\frac{x^8 + 4\,x^6 + 64\,x^4 + 25344\,x^2 + 147456}{3840\,x^9}\,,$$

$$\mathcal{R}_5^{(-6),*} = 0\,, \quad \mathcal{S}_5^{(-6),*} = \frac{1}{3840}$$

<u>$\mathbf{Z_6(x)}$:</u>

$$\int J_6(x)\,dx = -\frac{16\,x^2 - 384}{x^3}J_0(x) - \frac{2\,x^4 - 128\,x^2 + 768}{x^4}J_1(x) + \Lambda_0(x)$$

$$\int I_6(x)\,dx = -\frac{16\,x^2 + 384}{x^3}I_0(x) + \frac{2\,x^4 + 128\,x^2 + 768}{x^4}I_1(x) - \Lambda_0^*(x)$$

$$\int x\,J_6(x)\,dx = -\frac{18\,x^2 - 480}{x^2}J_0(x) - \frac{x^4 - 144\,x^2 + 960}{x^3}J_1(x)$$

$$\int x\,I_6(x)\,dx = -\frac{18\,x^2 + 480}{x^2}I_0(x) + \frac{x^4 + 144\,x^2 + 960}{x^3}I_1(x)$$

$$\int x^2\,J_6(x)\,dx = -\frac{19\,x^2 - 640}{x}J_0(x) - \frac{x^4 - 128\,x^2 + 1280}{x^2}J_1(x) + 35\Lambda_0(x)$$

$$\int x^2\,I_6(x)\,dx = -\frac{19\,x^2 + 640}{x}I_0(x) + \frac{x^4 + 128\,x^2 + 1280}{x^2}I_1(x) + 35\Lambda_0^*(x)$$

$$\int x^3\,J_6(x)\,dx = -(20\,x^2 - 768)J_0(x) - \frac{x^4 - 184\,x^2 + 1920}{x}J_1(x)$$

$$\int x^3\, I_6(x)\, dx = -(20\,x^2 + 768)I_0(x) + \frac{x^4 + 184\,x^2 + 1920}{x}I_1(x)$$

$$\int x^4\, J_6(x)\, dx = -(21\,x^3 - 975\,x)J_0(x) - (x^4 - 207\,x^2 + 3840)J_1(x) + 945\Lambda_0(x)$$

$$\int x^4\, I_6(x)\, dx = -(21\,x^3 + 975\,x)I_0(x) + (x^4 + 207\,x^2 + 3840)I_1(x) - 945\Lambda_0^*(x)$$

$$\int \frac{J_6(x)\, dx}{x} = -\frac{16\,x^2 - 320}{x^4}J_0(x) - \frac{x^4 - 112\,x^2 + 640}{x^5}J_1(x)$$

$$\int \frac{I_6(x)\, dx}{x} = -\frac{16\,x^2 + 320}{x^4}I_0(x) + \frac{x^4 + 112\,x^2 + 640}{x^5}I_1(x)$$

$$\int \frac{J_6(x)\, dx}{x^2} = \frac{x^4 - 528\,x^2 + 9600}{35\,x^5}J_0(x) - \frac{x^6 + 34\,x^4 - 3456\,x^2 + 19200}{35\,x^6}J_1(x) + \frac{1}{35}\Lambda_0(x)$$

$$\int \frac{I_6(x)\, dx}{x^2} = -\frac{x^4 + 528\,x^2 + 9600}{35\,x^5}I_0(x) - \frac{x^6 - 34\,x^4 - 3456\,x^2 - 19200}{35\,x^6}I_1(x) + \frac{1}{35}\Lambda_0^*(x)$$

$$\mathcal{P}_6^{(5)}(x) = -(22\,x^4 - 1232\,x^2 + 3840)\,, \quad \mathcal{Q}_6^{(5)}(x) = -(x^5 - 232\,x^3 + 4384\,x)\,,$$
$$\mathcal{R}_6^{(5)} = 0\,, \quad \mathcal{S}_6^{(5)} = 0$$

$$\mathcal{P}_6^{(5),*}(x) = -(22\,x^4 + 1232\,x^2 + 3840)\,, \quad \mathcal{Q}_6^{(5),*}(x) = 22\,x^4 + 1232\,x^2 + 3840\,,$$
$$\mathcal{R}_6^{(5),*} = 0\,, \quad \mathcal{S}_6^{(5),*} = 0$$

$$\mathcal{P}_6^{(6)}(x) = -(23\,x^5 - 1545\,x^3 + 10395\,x)\,, \quad \mathcal{Q}_6^{(6)}(x) = -(x^6 - 259\,x^4 + 6555\,x^2)\,,$$
$$\mathcal{R}_6^{(6)} = 10395\,, \quad \mathcal{S}_6^{(6)} = 0$$

$$\mathcal{P}_6^{(6),*}(x) = -(23\,x^5 + 1545\,x^3 + 10395\,x)\,, \quad \mathcal{Q}_6^{(6),*}(x) = x^6 + 259\,x^4 + 6555\,x^2\,,$$
$$\mathcal{R}_6^{(6),*} = 10395\,, \quad \mathcal{S}_6^{(6),*} = 0$$

$$\mathcal{P}_6^{(7)}(x) = -(24\,x^6 - 1920\,x^4 + 23040\,x^2)\,, \quad \mathcal{Q}_6^{(7)}(x) = -(x^7 - 288\,x^5 + 9600\,x^3 - 46080\,x)\,,$$
$$\mathcal{R}_6^{(7)} = 0\,, \quad \mathcal{S}_6^{(7)} = 0$$

$$\mathcal{P}_6^{(7),*}(x) = -(24\,x^6 + 1920\,x^4 + 23040\,x^2)\,, \quad \mathcal{Q}_6^{(7),*}(x) = x^7 + 288\,x^5 + 9600\,x^3 + 46080\,x\,,$$
$$\mathcal{R}_6^{(7),*} = 0\,, \quad \mathcal{S}_6^{(7),*} = 0$$

$$\mathcal{P}_6^{(8)}(x) = -(25\,x^7 - 2363\,x^5 + 45045\,x^3 - 135135\,x)\,, \quad \mathcal{Q}_6^{(8)}(x) = -(x^8 - 319\,x^6 + 13735\,x^4 - 135135\,x^2)\,,$$
$$\mathcal{R}_6^{(8)} = -135135\,, \quad \mathcal{S}_6^{(8)} = 0$$

$$\mathcal{P}_6^{(8),*}(x) = -(25\,x^7 + 2363\,x^5 + 45045\,x^3 + 135135\,x)\,, \quad \mathcal{Q}_6^{(8),*}(x) = x^8 + 319\,x^6 + 13735\,x^4 + 135135\,x^2\,,$$
$$\mathcal{R}_6^{(8),*} = 135135\,, \quad \mathcal{S}_6^{(8),*} = 0$$

$$\mathcal{P}_6^{(9)}(x) = -(26\,x^8 - 2880\,x^6 + 80640\,x^4 - 645120\,x^2)\,,$$
$$\mathcal{Q}_6^{(9)}(x) = -(x^9 - 352\,x^7 + 19200\,x^5 - 322560\,x^3 + 1290240\,x)\,, \quad \mathcal{R}_6^{(9)} = 0\,, \quad \mathcal{S}_6^{(9)} = 0$$

$$\mathcal{P}_6^{(9),*}(x) = -(26\,x^8 + 2880\,x^6 + 80640\,x^4 + 645120\,x^2)\,,$$
$$\mathcal{Q}_6^{(9),*}(x) = x^9 + 352\,x^7 + 19200\,x^5 + 322560\,x^3 + 1290240\,x\,, \quad \mathcal{R}_6^{(9),*} = 0\,, \quad \mathcal{S}_6^{(9),*} = 0$$

$$\mathcal{P}_6^{(10)}(x) = -(27\,x^9 - 3477\,x^7 + 135135\,x^5 - 2027025\,x^3 + 6081075\,x)\,,$$
$$\mathcal{Q}_6^{(10)}(x) = -(x^{10} - 387\,x^8 + 26259\,x^6 - 675675\,x^4 + 6081075\,x^2)\,, \quad \mathcal{R}_6^{(10)} = 6081075\,, \quad \mathcal{S}_6^{(10)} = 0$$

$$\mathcal{P}_6^{(10),*}(x) = -(27\,x^9 + 3477\,x^7 + 135135\,x^5 + 2027025\,x^3 + 6081075\,x)\,,$$
$$\mathcal{Q}_6^{(10),*}(x) = x^{10} + 387\,x^8 + 26259\,x^6 + 675675\,x^4 + 6081075\,x^2\,, \quad \mathcal{R}_6^{(10),*} = 6081075\,, \quad \mathcal{S}_6^{(10),*} = 0$$

$$\mathcal{P}_6^{(-3)}(x) = -\frac{14\,x^2 - 240}{x^6}\,, \quad \mathcal{Q}_6^{(-3)}(x) = -\frac{x^4 - 88\,x^2 + 480}{x^7}\,, \quad \mathcal{R}_6^{(-3)} = 0\,, \quad \mathcal{S}_6^{(-3)} = 0$$

$$\mathcal{P}_6^{(-3),*}(x) = -\frac{14\,x^2 + 240}{x^6}\,, \quad \mathcal{Q}_6^{(-3),*}(x) = \frac{x^4 + 88\,x^2 + 480}{x^7}\,, \quad \mathcal{R}_6^{(-3),*} = 0\,, \quad \mathcal{S}_6^{(-3),*} = 0$$

$$\mathcal{P}_6^{(-4)}(x) = \frac{x^6 - 3\,x^4 - 12240\,x^2 + 201600}{945\,x^7}\,, \quad \mathcal{Q}_6^{(-4)}(x) = -\frac{x^8 - x^6 + 954\,x^4 - 74880\,x^2 + 403200}{945\,x^8}\,,$$

$$\mathcal{R}_6^{(-4)} = \frac{1}{945}\,, \quad \mathcal{S}_6^{(-4)} = 0$$

$$\mathcal{P}_6^{(-4),*}(x) = \frac{x^6 + 3\,x^4 - 12240\,x^2 - 201600}{945\,x^7}\,, \quad \mathcal{Q}_6^{(-4),*}(x) = \frac{x^8 + x^6 + 954\,x^4 + 74880\,x^2 + 403200}{945\,x^8}\,,$$

$$\mathcal{R}_6^{(-4),*} = -\frac{1}{945}\,, \quad \mathcal{S}_6^{(-4),*} = 0$$

$$\mathcal{P}_6^{(-5)}(x) = -\frac{12\,x^2 - 192}{x^8}\,, \quad \mathcal{Q}_6^{(-5)}(x) = -\frac{x^4 - 72\,x^2 + 384}{x^9}\,, \quad \mathcal{R}_6^{(-5)} = 0\,, \quad \mathcal{S}_6^{(-5)} = 0$$

$$\mathcal{P}_6^{(-5),*}(x) = -\frac{12\,x^2 + 192}{x^8}\,, \quad \mathcal{Q}_6^{(-5),*}(x) = \frac{x^4 + 72\,x^2 + 384}{x^9}\,, \quad \mathcal{R}_6^{(-5),*} = 0\,, \quad \mathcal{S}_6^{(-5),*} = 0$$

$$\mathcal{P}_6^{(-6)}(x) = \frac{x^8 - 3\,x^6 + 45\,x^4 - 115920\,x^2 + 1814400}{10395\,x^9}\,,$$

$$\mathcal{Q}_6^{(-6)}(x) = -\frac{x^{10} - x^8 + 9\,x^6 + 10170\,x^4 - 685440\,x^2 + 3628800}{10395\,x^{10}}\,,$$

$$\mathcal{R}_6^{(-6)} = \frac{1}{10395}\,, \quad \mathcal{S}_6^{(-6)} = 0$$

$$\mathcal{P}_6^{(-6),*}(x) = -\frac{x^8 + 3\,x^6 + 45\,x^4 + 115920\,x^2 + 1814400}{10395\,x^9}\,,$$

$$\mathcal{Q}_6^{(-6),*}(x) = -\frac{x^{10} + x^8 + 9\,x^6 - 10170\,x^4 - 685440\,x^2 - 3628800}{10395\,x^{10}}\,,$$

$$\mathcal{R}_6^{(-6),*} = \frac{1}{10395}\,, \quad \mathcal{S}_6^{(-6),*} = 0$$

$\underline{\mathbf{Z_7(x)}}$:

$$\int J_7(x)\,dx = \frac{x^4 - 240\,x^2 + 3840}{x^4}J_0(x) - \frac{24\,x^4 - 1440\,x^2 + 7680}{x^5}J_1(x)$$

$$\int I_7(x)\,dx = \frac{x^4 + 240\,x^2 + 3840}{x^4}I_0(x) - \frac{24\,x^4 + 1440\,x^2 + 7680}{x^5}I_1(x)$$

$$\int x\,J_7(x)\,dx = \frac{x^4 - 256\,x^2 + 4608}{x^3}J_0(x) - \frac{32\,x^4 - 1664\,x^2 + 9216}{x^4}J_1(x) + 7\Lambda_0(x)$$

$$\int x\,I_7(x)\,dx = \frac{x^4 + 256\,x^2 + 4608}{x^3}I_0(x) - \frac{32\,x^4 + 1664\,x^2 + 9216}{x^4}I_1(x) + 7\Lambda_0^*(x)$$

$$\int x^2\,J_7(x)\,dx = \frac{x^4 - 288\,x^2 + 5760}{x^2}J_0(x) - \frac{26\,x^4 - 1920\,x^2 + 11520}{x^3}J_1(x)$$

$$\int x^2\,I_7(x)\,dx = \frac{x^4 + 288\,x^2 + 5760}{x^2}I_0(x) - \frac{26\,x^4 + 1920\,x^2 + 11520}{x^3}I_1(x)$$

$$\int x^3\,J_7(x)\,dx = \frac{x^4 - 315\,x^2 + 7680}{x}J_0(x) - \frac{27\,x^4 - 1920\,x^2 + 15360}{x^2}J_1(x) + 315\Lambda_0(x)$$

$$\int x^3\,I_7(x)\,dx = \frac{x^4 + 315\,x^2 + 7680}{x}I_0(x) - \frac{27\,x^4 + 1920\,x^2 + 15360}{x^2}I_1(x) - 315\Lambda_0^*(x)$$

$$\int x^4\,J_7(x)\,dx = (x^4 - 344\,x^2 + 9600)J_0(x) - \frac{28\,x^4 - 2608\,x^2 + 23040}{x}J_1(x)$$

$$\int x^4 \, I_7(x) \, dx = (x^4 + 344\, x^2 + 9600) I_0(x) - \frac{28\, x^4 + 2608\, x^2 + 23040}{x} I_1(x)$$

$$\int \frac{J_7(x)\, dx}{x} = \frac{8\, x^4 - 1536\, x^2 + 23040}{7\, x^5} J_0(x) - \frac{x^6 + 160\, x^4 - 8832\, x^2 + 46080}{7\, x^6} J_1(x) + \frac{1}{7}\Lambda_0(x)$$

$$\int \frac{I_7(x)\, dx}{x} = \frac{8\, x^4 + 1536\, x^2 + 23040}{7\, x^5} I_0(x) + \frac{x^6 - 160\, x^4 - 8832\, x^2 - 46080}{7\, x^6} I_1(x) - \frac{1}{7}\Lambda_0^*(x)$$

$$\int \frac{J_7(x)\, dx}{x^2} = \frac{x^4 - 200\, x^2 + 2880}{x^6} J_0(x) - \frac{22\, x^4 - 1120\, x^2 + 5760}{x^7} J_1(x)$$

$$\int \frac{I_7(x)\, dx}{x^2} = \frac{x^4 + 200\, x^2 + 2880}{x^6} I_0(x) - \frac{22\, x^4 + 1120\, x^2 + 5760}{x^7} I_1(x)$$

$$\mathcal{P}_7^{(5)}(x) = x^5 - 375\, x^3 + 12645\, x \,, \quad \mathcal{Q}_7^{(5)}(x) = -(29\, x^4 - 3045\, x^2 + 46080)\,,$$
$$\mathcal{R}_7^{(5)} = 10395\,, \quad \mathcal{S}_7^{(5)} = 0$$

$$\mathcal{P}_7^{(5),*}(x) = x^5 + 375\, x^3 + 12645\, x \,, \quad \mathcal{Q}_7^{(5),*}(x) = -(29\, x^4 + 3045\, x^2 + 46080)\,,$$
$$\mathcal{R}_7^{(5),*} = 10395\,, \quad \mathcal{S}_7^{(5),*} = 0$$

$$\mathcal{P}_7^{(6)}(x) = x^6 - 408\, x^4 + 16704\, x^2 - 46080 \,, \quad \mathcal{Q}_7^{(6)}(x) = -(30\, x^5 - 3552\, x^3 + 56448\, x)\,,$$
$$\mathcal{R}_7^{(6)} = 0\,, \quad \mathcal{S}_7^{(6)} = 0$$

$$\mathcal{P}_7^{(6),*}(x) = x^6 + 408\, x^4 + 16704\, x^2 + 46080 \,, \quad \mathcal{Q}_7^{(6),*}(x) = -(30\, x^5 + 3552\, x^3 + 56448\, x)\,,$$
$$\mathcal{R}_7^{(6),*} = 0\,, \quad \mathcal{S}_7^{(6),*} = 0$$

$$\mathcal{P}_7^{(7)}(x) = x^7 - 443\, x^5 + 22005\, x^3 - 135135\, x \,, \quad \mathcal{Q}_7^{(7)}(x) = -(31\, x^6 - 4135\, x^4 + 89055\, x^2)\,,$$
$$\mathcal{R}_7^{(7)} = 135135\,, \quad \mathcal{S}_7^{(7)} = 0$$

$$\mathcal{P}_7^{(7),*}(x) = x^7 + 443\, x^5 + 22005\, x^3 + 135135\, x \,, \quad \mathcal{Q}_7^{(7),*}(x) = -(31\, x^6 + 4135\, x^4 + 89055\, x^2)\,,$$
$$\mathcal{R}_7^{(7),*} = -135135\,, \quad \mathcal{S}_7^{(7),*} = 0$$

$$\mathcal{P}_7^{(8)}(x) = x^8 - 480\, x^6 + 28800\, x^4 - 322560\, x^2 \,, \quad \mathcal{Q}_7^{(8)}(x) = -(32\, x^7 - 4800\, x^5 + 138240\, x^3 - 645120\, x)\,,$$
$$\mathcal{R}_7^{(8)} = 0\,, \quad \mathcal{S}_7^{(8)} = 0$$

$$\mathcal{P}_7^{(8),*}(x) = x^8 + 480\, x^6 + 28800\, x^4 + 322560\, x^2 \,, \quad \mathcal{Q}_7^{(8),*}(x) = -(32\, x^7 + 4800\, x^5 + 138240\, x^3 + 645120\, x)\,,$$
$$\mathcal{R}_7^{(8),*} = 0\,, \quad \mathcal{S}_7^{(8),*} = 0$$

$$\mathcal{P}_7^{(9)}(x) = x^9 - 519\, x^7 + 37365\, x^5 - 675675\, x^3 + 2027025\, x \,,$$
$$\mathcal{Q}_7^{(9)}(x) = -(33\, x^8 - 5553\, x^6 + 209865\, x^4 - 2027025\, x^2)\,, \quad \mathcal{R}_7^{(9)} = -2027025\,, \quad \mathcal{S}_7^{(9)} = 0$$

$$\mathcal{P}_7^{(9),*}(x) = x^9 + 519\, x^7 + 37365\, x^5 + 675675\, x^3 + 2027025\, x \,,$$
$$\mathcal{Q}_7^{(9),*}(x) = -(33\, x^8 + 5553\, x^6 + 209865\, x^4 + 2027025\, x^2)\,, \quad \mathcal{R}_7^{(9),*} = -2027025\,, \quad \mathcal{S}_7^{(9),*} = 0$$

$$\mathcal{P}_7^{(10)}(x) = x^{10} - 560\, x^8 + 48000\, x^6 - 1290240\, x^4 + 10321920\, x^2 \,,$$
$$\mathcal{Q}_7^{(10)}(x) = -(34\, x^9 - 6400\, x^7 + 311040\, x^5 - 5160960\, x^3 + 20643840\, x)\,, \quad \mathcal{R}_7^{(10)} = 0\,, \quad \mathcal{S}_7^{(10)} = 0$$

$$\mathcal{P}_7^{(10),*}(x) = x^{10} + 560\, x^8 + 48000\, x^6 + 1290240\, x^4 + 10321920\, x^2 \,,$$
$$\mathcal{Q}_7^{(10),*}(x) = -(34\, x^9 + 6400\, x^7 + 311040\, x^5 + 5160960\, x^3 + 20643840\, x)\,, \quad \mathcal{R}_7^{(10),*} = 0\,, \quad \mathcal{S}_7^{(10),*} = 0$$

$$\mathcal{P}_7^{(-3)}(x) = \frac{x^6 + 312\, x^4 - 57600\, x^2 + 806400}{315\, x^7} \,,$$
$$\mathcal{Q}_7^{(-3)}(x) = -\frac{x^8 - x^6 + 6624\, x^4 - 316800\, x^2 + 1612800}{315\, x^8} \,, \quad \mathcal{R}_7^{(-3)} = \frac{1}{315}\,, \quad \mathcal{S}_7^{(-3)} = 0$$

$$\mathcal{P}_7^{(-3),*}(x) = \frac{x^6 - 312\,x^4 - 57600\,x^2 - 806400}{315\,x^7}\,,$$

$$\mathcal{Q}_7^{(-3),*}(x) = -\frac{x^8 + x^6 + 6624\,x^4 + 316800\,x^2 + 1612800}{315\,x^8}\,,\quad \mathcal{R}_7^{(-3),*} = \frac{1}{315}\,,\quad \mathcal{S}_7^{(-3),*} = 0$$

$$\mathcal{P}_7^{(-4)}(x) = \frac{x^4 - 168\,x^2 + 2304}{x^8}\,,\quad \mathcal{Q}_7^{(-4)}(x) = -\frac{20\,x^4 - 912\,x^2 + 4608}{x^9}\,,\quad \mathcal{R}_7^{(-4)} = 0\,,\quad \mathcal{S}_7^{(-4)} = 0$$

$$\mathcal{P}_7^{(-4),*}(x) = \frac{x^4 + 168\,x^2 + 2304}{x^8}\,,\quad \mathcal{Q}_7^{(-4),*}(x) = -\frac{20\,x^4 + 912\,x^2 + 4608}{x^9}\,,\quad \mathcal{R}_7^{(-4),*} = 0\,,\quad \mathcal{S}_7^{(-4),*} = 0$$

$$\mathcal{P}_7^{(-5)}(x) = \frac{x^8 - 3\,x^6 + 10440\,x^4 - 1612800\,x^2 + 21772800}{10395\,x^9}\,,$$

$$\mathcal{Q}_7^{(-5)}(x) = -\frac{x^{10} - x^8 + 9\,x^6 + 197280\,x^4 - 8668800\,x^2 + 43545600}{10395\,x^{10}}\,,\quad \mathcal{R}_7^{(-5)} = \frac{1}{10395}\,,\quad \mathcal{S}_7^{(-5)} = 0$$

$$\mathcal{P}_7^{(-5),*}(x) = \frac{x^8 + 3\,x^6 + 10440\,x^4 + 1612800\,x^2 + 21772800}{10395\,x^9}\,,$$

$$\mathcal{Q}_7^{(-5),*}(x) = \frac{x^{10} + x^8 + 9\,x^6 - 197280\,x^4 - 8668800\,x^2 - 43545600}{10395\,x^{10}}\,,\quad \mathcal{R}_7^{(-5),*} = -\frac{1}{10395}\,,\quad \mathcal{S}_7^{(-5),*} = 0$$

$$\mathcal{P}_7^{(-6)}(x) = \frac{x^4 - 144\,x^2 + 1920}{x^{10}}\,,\quad \mathcal{Q}_7^{(-6)}(x) = -\frac{18\,x^4 - 768\,x^2 + 3840}{x^{11}}\,,\quad \mathcal{R}_7^{(-6)} = 0\,,\quad \mathcal{S}_7^{(-6)} = 0$$

$$\mathcal{P}_7^{(-6),*}(x) = \frac{x^4 + 144\,x^2 + 1920}{x^{10}}\,,\quad \mathcal{Q}_7^{(-6),*}(x) = -\frac{18\,x^4 + 768\,x^2 + 3840}{x^{11}}\,,\quad \mathcal{R}_7^{(-6),*} = 0\,,\quad \mathcal{S}_7^{(-6),*} = 0$$

$\underline{\mathbf{Z_8(x)}}$:

$$\int J_8(x)\,dx = \frac{32\,x^4 - 3456\,x^2 + 46080}{x^5}J_0(x) - \frac{448\,x^4 - 18432\,x^2 + 92160}{x^6}J_1(x) + \Lambda_0(x)$$

$$\int I_8(x)\,dx = -\frac{32\,x^4 + 3456\,x^2 + 46080}{x^5}I_0(x) + \frac{448\,x^4 + 18432\,x^2 + 92160}{x^6}I_1(x) + \Lambda_0^*(x)$$

$$\int x\,J_8(x)\,dx = \frac{32\,x^4 - 3840\,x^2 + 53760}{x^4}J_0(x) + \frac{x^6 - 480\,x^4 + 21120\,x^2 - 107520}{x^5}J_1(x)$$

$$\int x\,I_8(x)\,dx = -\frac{32\,x^4 + 3840\,x^2 + 53760}{x^4}I_0(x) + \frac{x^6 + 480\,x^4 + 21120\,x^2 + 107520}{x^5}I_1(x)$$

$$\int x^2\,J_8(x)\,dx = \frac{33\,x^4 - 4224\,x^2 + 64512}{x^3}J_0(x) + \frac{x^6 - 576\,x^4 + 24576\,x^2 - 129024}{x^4}J_1(x) + 63\Lambda_0(x)$$

$$\int x^2\,I_8(x)\,dx = -\frac{33\,x^4 + 4224\,x^2 + 64512}{x^3}I_0(x) + \frac{x^6 + 576\,x^4 + 24576\,x^2 + 129024}{x^4}I_1(x) - 63\Lambda_0^*(x)$$

$$\int x^3\,J_8(x)\,dx = \frac{34\,x^4 - 4800\,x^2 + 80640}{x^2}J_0(x) + \frac{x^6 - 548\,x^4 + 28800\,x^2 - 161280}{x^3}J_1(x)$$

$$\int x^3\,I_8(x)\,dx = -\frac{34\,x^4 + 4800\,x^2 + 80640}{x^2}I_0(x) + \frac{x^6 + 548\,x^4 + 28800\,x^2 + 161280}{x^3}I_1(x)$$

$$\int x^4\,J_8(x)\,dx = \frac{35\,x^4 - 5385\,x^2 + 107520}{x}J_0(x) + \frac{x^6 - 585\,x^4 + 30720\,x^2 - 215040}{x^2}J_1(x) + 3465\Lambda_0(x)$$

$$\int x^4\,I_8(x)\,dx = -\frac{35\,x^4 + 5385\,x^2 + 107520}{x}I_0(x) + \frac{x^6 + 585\,x^4 + 30720\,x^2 + 215040}{x^2}I_1(x) + 3465\Lambda_0^*(x)$$

$$\int \frac{J_8(x)\,dx}{x} = \frac{30\,x^4 - 3120\,x^2 + 40320}{x^6}J_0(x) + \frac{x^6 - 420\,x^4 + 16320\,x^2 - 80640}{x^7}J_1(x)$$

$$\int \frac{I_8(x)\,dx}{x} = -\frac{30\,x^4 + 3120\,x^2 + 40320}{x^6}I_0(x) + \frac{x^6 + 420\,x^4 + 16320\,x^2 + 80640}{x^7}I_1(x)$$

$$\int \frac{J_8(x)\,dx}{x^2} = \frac{x^6 + 1824\,x^4 - 178560\,x^2 + 2257920}{63x^7}J_0(x)-$$

$$-\frac{x^8 - 64\,x^6 + 24768\,x^4 - 921600\,x^2 + 4515840}{63x^8}J_1(x) + \frac{1}{63}\Lambda_0(x)$$

$$\int \frac{I_8(x)\,dx}{x^2} = \frac{x^6 - 1824\,x^4 - 178560\,x^2 - 2257920}{63x^7}I_0(x)+$$

$$+\frac{x^8 + 64\,x^6 + 24768\,x^4 + 921600\,x^2 + 4515840}{63x^8}I_1(x) - \frac{1}{63}\Lambda_0^*(x)$$

$$\mathcal{P}_8^{(5)}(x) = 36\,x^4 - 6048\,x^2 + 138240\,, \quad \mathcal{Q}_8^{(5)}(x) = \frac{x^6 - 624\,x^4 + 40896\,x^2 - 322560}{x}\,,$$

$$\mathcal{R}_8^{(5)} = 0\,, \quad \mathcal{S}_8^{(5)} = 0$$

$$\mathcal{P}_8^{(5),*}(x) = -(36\,x^4 + 6048\,x^2 + 138240)\,, \quad \mathcal{Q}_8^{(5),*}(x) = \frac{x^6 + 624\,x^4 + 40896\,x^2 + 322560}{x}\,,$$

$$\mathcal{R}_8^{(5),*} = 0\,, \quad \mathcal{S}_8^{(5),*} = 0$$

$$\mathcal{P}_8^{(6)}(x) = 37\,x^5 - 6795\,x^3 + 187425\,x\,, \quad \mathcal{Q}_8^{(6)}(x) = x^6 - 665\,x^4 + 49185\,x^2 - 645120\,,$$

$$\mathcal{R}_8^{(6)} = 135135\,, \quad \mathcal{S}_8^{(6)} = 0$$

$$\mathcal{P}_8^{(6),*}(x) = -(37\,x^5 + 6795\,x^3 + 187425\,x)\,, \quad \mathcal{Q}_8^{(6),*}(x) = x^6 + 665\,x^4 + 49185\,x^2 + 645120\,,$$

$$\mathcal{R}_8^{(6),*} = -135135\,, \quad \mathcal{S}_8^{(6),*} = 0$$

$$\mathcal{P}_8^{(7)}(x) = 38\,x^6 - 7632\,x^4 + 256896\,x^2 - 645120\,, \quad \mathcal{Q}_8^{(7)}(x) = x^7 - 708\,x^5 + 59328\,x^3 - 836352\,x\,,$$

$$\mathcal{R}_8^{(7)} = 0\,, \quad \mathcal{S}_8^{(7)} = 0$$

$$\mathcal{P}_8^{(7),*}(x) = -(38\,x^6 + 7632\,x^4 + 256896\,x^2 + 645120)\,, \quad \mathcal{Q}_8^{(7),*}(x) = x^7 + 708\,x^5 + 59328\,x^3 + 836352\,x\,,$$

$$\mathcal{R}_8^{(7),*} = 0\,, \quad \mathcal{S}_8^{(7),*} = 0$$

$$\mathcal{P}_8^{(8)}(x) = 39\,x^7 - 8565\,x^5 + 353115\,x^3 - 2027025\,x\,, \quad \mathcal{Q}_8^{(8)}(x) = x^8 - 753\,x^6 + 71625\,x^4 - 1381905\,x^2\,,$$

$$\mathcal{R}_8^{(8)} = 2027025\,, \quad \mathcal{S}_8^{(8)} = 0$$

$$\mathcal{P}_8^{(8),*}(x) = -(39\,x^7 + 8565\,x^5 + 353115\,x^3 + 2027025\,x)\,, \quad \mathcal{Q}_8^{(8),*}(x) = x^8 + 753\,x^6 + 71625\,x^4 + 1381905\,x^2\,,$$

$$\mathcal{R}_8^{(8),*} = 2027025\,, \quad \mathcal{S}_8^{(8),*} = 0$$

$$\mathcal{P}_8^{(9)}(x) = 40\,x^8 - 9600\,x^6 + 483840\,x^4 - 5160960\,x^2\,,$$

$$\mathcal{Q}_8^{(9)}(x) = x^9 - 800\,x^7 + 86400\,x^5 - 2257920\,x^3 + 10321920\,x\,, \quad \mathcal{R}_8^{(9)} = 0\,, \quad \mathcal{S}_8^{(9)} = 0$$

$$\mathcal{P}_8^{(9),*}(x) = -(40\,x^8 + 9600\,x^6 + 483840\,x^4 + 5160960\,x^2)\,,$$

$$\mathcal{Q}_8^{(9),*}(x) = x^9 + 800\,x^7 + 86400\,x^5 + 2257920\,x^3 + 10321920\,x\,, \quad \mathcal{R}_8^{(9),*} = 0\,, \quad \mathcal{S}_8^{(9),*} = 0$$

$$\mathcal{P}_8^{(10)}(x) = 41\,x^9 - 10743\,x^7 + 658245\,x^5 - 11486475\,x^3 + 34459425\,x\,,$$

$$\mathcal{Q}_8^{(10)}(x) = x^{10} - 849\,x^8 + 104001\,x^6 - 3613785\,x^4 + 34459425\,x^2\,, \quad \mathcal{R}_8^{(10)} = -34459425\,, \quad \mathcal{S}_8^{(10)} = 0$$

$$\mathcal{P}_8^{(10),*}(x) = -(41\,x^9 + 10743\,x^7 + 658245\,x^5 + 11486475\,x^3 + 34459425\,x)\,,$$

$$\mathcal{Q}_8^{(10),*}(x) = x^{10} + 849\,x^8 + 104001\,x^6 + 3613785\,x^4 + 34459425\,x^2\,, \quad \mathcal{R}_8^{(10),*} = 34459425\,, \quad \mathcal{S}_8^{(10),*} = 0$$

$$\mathcal{P}_8^{(-3)}(x) = \frac{28\,x^4 - 2592\,x^2 + 32256}{x^8}\,, \quad \mathcal{Q}_8^{(-3)}(x) = \frac{x^6 - 368\,x^4 + 13248\,x^2 - 64512}{x^9}\,,$$

$$\mathcal{R}_8^{(-3)} = 0\,, \quad \mathcal{S}_8^{(-3)} = 0$$

$$\mathcal{P}_8^{(-3),*}(x) = -\frac{28\,x^4 + 2592\,x^2 + 32256}{x^8}\,, \quad \mathcal{Q}_8^{(-3),*}(x) = \frac{x^6 + 368\,x^4 + 13248\,x^2 + 64512}{x^9}\,,$$

$$\mathcal{R}_8^{(-3),*} = 0\,, \quad \mathcal{S}_8^{(-3),*} = 0$$

$$\mathcal{P}_8^{(-4)}(x) = \frac{x^8 - 3\,x^6 + 93600\,x^4 - 8265600\,x^2 + 101606400}{3465\,x^9}\,,$$

$$\mathcal{Q}_8^{(-4)}(x) = -\frac{x^{10} - x^8 - 3456\,x^6 + 1195200\,x^4 - 41932800\,x^2 + 203212800}{3465\,x^{10}}\,,$$

$$\mathcal{R}_8^{(-4)} = \frac{1}{3465}\,,\quad \mathcal{S}_8^{(-4)} = 0$$

$$\mathcal{P}_8^{(-4),*}(x) = -\frac{x^8 + 3\,x^6 + 93600\,x^4 + 8265600\,x^2 + 101606400}{3465\,x^9}\,,$$

$$\mathcal{Q}_8^{(-4),*}(x) = -\frac{x^{10} + x^8 - 3456\,x^6 - 1195200\,x^4 - 41932800\,x^2 - 203212800}{3465\,x^{10}}\,,$$

$$\mathcal{R}_8^{(-4),*} = \frac{1}{3465}\,,\quad \mathcal{S}_8^{(-4),*} = 0$$

$$\mathcal{P}_8^{(-5)}(x) = \frac{26\,x^4 - 2208\,x^2 + 26880}{x^{10}}\,,\quad \mathcal{Q}_8^{(-5)}(x) = \frac{x^6 - 324\,x^4 + 11136\,x^2 - 53760}{x^{11}}\,,$$

$$\mathcal{R}_8^{(-5)} = 0\,,\quad \mathcal{S}_8^{(-5)} = 0$$

$$\mathcal{P}_8^{(-5),*}(x) = -\frac{26\,x^4 + 2208\,x^2 + 26880}{x^{10}}\,,\quad \mathcal{Q}_8^{(-5),*}(x) = \frac{x^6 + 324\,x^4 + 11136\,x^2 + 53760}{x^{11}}\,,$$

$$\mathcal{R}_8^{(-5),*} = 0\,,\quad \mathcal{S}_8^{(-5),*} = 0$$

$$\mathcal{P}_8^{(-6)}(x) = \frac{x^{10} - 3\,x^8 + 45\,x^6 + 3376800\,x^4 - 277603200\,x^2 + 3353011200}{135135\,x^{11}}\,,$$

$$\mathcal{Q}_8^{(-6)}(x) = -\frac{x^{12} - x^{10} + 9\,x^8 - 135360\,x^6 + 41227200\,x^4 - 1393459200\,x^2 + 6706022400}{135135\,x^{12}}\,,$$

$$\mathcal{R}_8^{(-6)} = \frac{1}{135135}\,,\quad \mathcal{S}_8^{(-6)} = 0$$

$$\mathcal{P}_8^{(-6),*}(x) = \frac{x^{10} + 3\,x^8 + 45\,x^6 - 3376800\,x^4 - 277603200\,x^2 - 3353011200}{135135\,x^{11}}\,,$$

$$\mathcal{Q}_8^{(-6),*}(x) = \frac{x^{12} + x^{10} + 9\,x^8 + 135360\,x^6 + 41227200\,x^4 + 1393459200\,x^2 + 6706022400}{135135\,x^{12}}\,,$$

$$\mathcal{R}_8^{(-6),*} = -\frac{1}{135135}\,,\quad \mathcal{S}_8^{(-6),*} = 0$$

$\underline{\mathbf{Z}_9(\mathbf{x})}$:

$$\int J_9(x)\,dx = -\frac{x^6 - 720\,x^4 + 53760\,x^2 - 645120}{x^6}J_0(x) + \frac{40\,x^6 - 8160\,x^4 + 268800\,x^2 - 1290240}{x^7}J_1(x)$$

$$\int I_9(x)\,dx = \frac{x^6 + 720\,x^4 + 53760\,x^2 + 645120}{x^6}I_0(x) - \frac{40\,x^6 + 8160\,x^4 + 268800\,x^2 + 1290240}{x^7}I_1(x)$$

$$\int x\,J_9(x)\,dx =$$

$$= -\frac{x^6 - 768\,x^4 + 59904\,x^2 - 737280}{x^5}J_0(x) + \frac{32\,x^6 - 8832\,x^4 + 304128\,x^2 - 1474560}{x^6}J_1(x) + 9\Lambda_0(x)$$

$$\int x\,I_9(x)\,dx =$$

$$= \frac{x^6 + 768\,x^4 + 59904\,x^2 + 737280}{x^5}I_0(x) - \frac{32\,x^6 + 8832\,x^4 + 304128\,x^2 + 1474560}{x^6}I_1(x) - 9\Lambda_0^*(x)$$

$$\int x^2\,J_9(x)\,dx = -\frac{x^6 - 800\,x^4 + 67200\,x^2 - 860160}{x^4}J_0(x) + \frac{42\,x^6 - 9600\,x^4 + 349440\,x^2 - 1720320}{x^5}J_1(x)$$

$$\int x^2\,I_9(x)\,dx = \frac{x^6 + 800\,x^4 + 67200\,x^2 + 860160}{x^4}I_0(x) - \frac{42\,x^6 + 9600\,x^4 + 349440\,x^2 + 1720320}{x^5}I_1(x)$$

$$\int x^3\,J_9(x)\,dx = -\frac{x^6 - 843\,x^4 + 75264\,x^2 - 1032192}{x^3}J_0(x) +$$

$$+\frac{43\,x^6 - 11136\,x^4 + 408576\,x^2 - 2064384}{x^4}J_1(x) + 693\Lambda_0(x)$$

$$\int x^3\,I_9(x)\,dx = \frac{x^6 + 843\,x^4 + 75264\,x^2 + 1032192}{x^3}I_0(x)-$$

$$-\frac{43\,x^6 + 11136\,x^4 + 408576\,x^2 + 2064384}{x^4}I_1(x) + 693\Lambda_0^*(x)$$

$$\int x^4\,J_9(x)\,dx = -\frac{x^6 - 888\,x^4 + 86400\,x^2 - 1290240}{x^2}J_0(x) + \frac{44\,x^6 - 11376\,x^4 + 483840\,x^2 - 2580480}{x^3}J_1(x)$$

$$\int x^4\,I_9(x)\,dx = \frac{x^6 + 888\,x^4 + 86400\,x^2 + 1290240}{x^2}I_0(x) - \frac{44\,x^6 + 11376\,x^4 + 483840\,x^2 + 2580480}{x^3}I_1(x)$$

$$\int \frac{J_9(x)\,dx}{x} = -\frac{8\,x^6 - 6144\,x^4 + 437760\,x^2 - 5160960}{9x^7}J_0(x)-$$

$$-\frac{x^8 - 352\,x^6 + 67968\,x^4 - 2165760\,x^2 + 10321920}{9x^8}J_1(x) + \frac{1}{9}\Lambda_0(x)$$

$$\int \frac{I_9(x)\,dx}{x} = \frac{8\,x^6 + 6144\,x^4 + 437760\,x^2 + 5160960}{9x^7}I_0(x)-$$

$$-\frac{x^8 + 352\,x^6 + 67968\,x^4 + 2165760\,x^2 + 10321920}{9x^8}I_1(x) + \frac{1}{9}\Lambda_0^*(x)$$

$$\int \frac{J_9(x)\,dx}{x^2} = -\frac{x^6 - 648\,x^4 + 44352\,x^2 - 516096}{x^8}J_0(x) + \frac{38\,x^6 - 7008\,x^4 + 217728\,x^2 - 1032192}{x^9}J_1(x)$$

$$\int \frac{I_9(x)\,dx}{x^2} = \frac{x^6 + 648\,x^4 + 44352\,x^2 + 516096}{x^8}I_0(x) - \frac{38\,x^6 + 7008\,x^4 + 217728\,x^2 + 1032192}{x^9}I_1(x)$$

$$\mathcal{P}_9^{(5)}(x) = -\frac{x^6 - 935\,x^4 + 98805\,x^2 - 1720320}{x} \ , \quad \mathcal{Q}_9^{(5)}(x) = \frac{45\,x^6 - 12405\,x^4 + 537600\,x^2 - 3440640}{x^2} \ ,$$

$$\mathcal{R}_9^{(5)} = 45045 \ , \quad \mathcal{S}_9^{(5)} = 0$$

$$\mathcal{P}_9^{(5),*}(x) = \frac{x^6 + 935\,x^4 + 98805\,x^2 + 1720320}{x} \ , \quad \mathcal{Q}_9^{(5),*}(x) = -\frac{45\,x^6 + 12405\,x^4 + 537600\,x^2 + 3440640}{x^2} \ ,$$

$$\mathcal{R}_9^{(5),*} = -45045 \ , \quad \mathcal{S}_9^{(5),*} = 0$$

$$\mathcal{P}_9^{(6)}(x) = -(x^6 - 984\,x^4 + 113472\,x^2 - 2257920) \ , \quad \mathcal{Q}_9^{(6)}(x) = \frac{46\,x^6 - 13536\,x^4 + 710784\,x^2 - 5160960}{x} \ ,$$

$$\mathcal{R}_9^{(6)} = 0 \ , \quad \mathcal{S}_9^{(6)} = 0$$

$$\mathcal{P}_9^{(6),*}(x) = x^6 + 984\,x^4 + 113472\,x^2 + 2257920 \ , \quad \mathcal{Q}_9^{(6),*}(x) = -\frac{46\,x^6 + 13536\,x^4 + 710784\,x^2 + 5160960}{x} \ ,$$

$$\mathcal{R}_9^{(6),*} = 0 \ , \quad \mathcal{S}_9^{(6),*} = 0$$

$$\mathcal{P}_9^{(7)}(x) = -(x^7 - 1035\,x^5 + 130725\,x^3 - 3133935\,x) \ , \quad \mathcal{Q}_9^{(7)}(x) = 47\,x^6 - 14775\,x^4 + 876015\,x^2 - 10321920 \ ,$$

$$\mathcal{R}_9^{(7)} = 2027025 \ , \quad \mathcal{S}_9^{(7)} = 0$$

$$\mathcal{P}_9^{(7),*}(x) = x^7 + 1035\,x^5 + 130725\,x^3 + 3133935\,x \ , \quad \mathcal{Q}_9^{(7),*}(x) = -(47\,x^6 + 14775\,x^4 + 876015\,x^2 + 10321920) \ ,$$

$$\mathcal{R}_9^{(7),*} = 2027025 \ , \quad \mathcal{S}_9^{(7),*} = 0$$

$$\mathcal{P}_9^{(8)}(x) = -(x^8 - 1088\,x^6 + 150912\,x^4 - 4432896\,x^2 + 10321920) \ ,$$

$$\mathcal{Q}_9^{(8)}(x) = 48\,x^7 - 16128\,x^5 + 1087488\,x^3 - 14026752\,x \ , \quad \mathcal{R}_9^{(8)} = 0 \ , \quad \mathcal{S}_9^{(8)} = 0$$

$$\mathcal{P}_9^{(8),*}(x) = x^8 + 1088\,x^6 + 150912\,x^4 + 4432896\,x^2 + 10321920 \ ,$$

$$\mathcal{Q}_9^{(8),*}(x) = -(48\,x^7 + 16128\,x^5 + 1087488\,x^3 + 14026752\,x) \ , \quad \mathcal{R}_9^{(8),*} = 0 \ , \quad \mathcal{S}_9^{(8),*} = 0$$

$$\mathcal{P}_9^{(9)}(x) = -(x^9 - 1143\,x^7 + 174405\,x^5 - 6325515\,x^3 + 34459425\,x) \ ,$$

$$\mathcal{Q}_9^{(9)}(x) = 49\,x^8 - 17601\,x^6 + 1355865\,x^4 - 24137505\,x^2 \;, \quad \mathcal{R}_9^{(9)} = 34459425 \;, \quad \mathcal{S}_9^{(9)} = 0$$

$$\mathcal{P}_9^{(9),*}(x) = x^9 + 1143\,x^7 + 174405\,x^5 + 6325515\,x^3 + 34459425\,x \;,$$

$$\mathcal{Q}_9^{(9),*}(x) = -(49\,x^8 + 17601\,x^6 + 1355865\,x^4 + 24137505\,x^2) \;,$$

$$\mathcal{R}_9^{(9),*} = -34459425 \;, \quad \mathcal{S}_9^{(9),*} = 0$$

$$\mathcal{P}_9^{(10)}(x) = -(x^{10} - 1200\,x^8 + 201600\,x^6 - 9031680\,x^4 + 92897280\,x^2) \;,$$

$$\mathcal{Q}_9^{(10)}(x) = 50\,x^9 - 19200\,x^7 + 1693440\,x^5 - 41287680\,x^3 + 185794560\,x \;,$$

$$\mathcal{R}_9^{(10)} = 0 \;, \quad \mathcal{S}_9^{(10)} = 0$$

$$\mathcal{P}_9^{(10),*}(x) = x^{10} + 1200\,x^8 + 201600\,x^6 + 9031680\,x^4 + 92897280\,x^2 \;,$$

$$\mathcal{Q}_9^{(10),*}(x) = -(50\,x^9 + 19200\,x^7 + 1693440\,x^5 + 41287680\,x^3 + 185794560\,x) \;,$$

$$\mathcal{R}_9^{(10),*} = 0 \;, \quad \mathcal{S}_9^{(10),*} = 0$$

$$\mathcal{P}_9^{(-3)}(x) = \frac{x^8 - 696\,x^6 + 426240\,x^4 - 28224000\,x^2 + 325140480}{693\,x^9} \;,$$

$$\mathcal{Q}_9^{(-3)}(x) = -\frac{x^{10} - x^8 - 25632\,x^6 + 4521600\,x^4 - 137733120\,x^2 + 650280960}{693\,x^{10}} \;,$$

$$\mathcal{R}_9^{(-3)} = \frac{1}{693} \;, \quad \mathcal{S}_9^{(-3)} = 0$$

$$\mathcal{P}_9^{(-3),*}(x) = \frac{x^8 + 696\,x^6 + 426240\,x^4 + 28224000\,x^2 + 325140480}{693\,x^9} \;,$$

$$\mathcal{Q}_9^{(-3),*}(x) = \frac{x^{10} + x^8 - 25632\,x^6 - 4521600\,x^4 - 137733120\,x^2 - 650280960}{693\,x^{10}} \;,$$

$$\mathcal{R}_9^{(-3),*} = -\frac{1}{693} \;, \quad \mathcal{S}_9^{(-3),*} = 0$$

$$\mathcal{P}_9^{(-4)}(x) = -\frac{x^6 - 584\,x^4 + 37632\,x^2 - 430080}{x^{10}} \;, \quad \mathcal{Q}_9^{(-4)}(x) = \frac{36\,x^6 - 6096\,x^4 + 182784\,x^2 - 860160}{x^{11}} \;,$$

$$\mathcal{R}_9^{(-4)} = 0 \;, \quad \mathcal{S}_9^{(-4)} = 0$$

$$\mathcal{P}_9^{(-4),*}(x) = \frac{x^6 + 584\,x^4 + 37632\,x^2 + 430080}{x^{10}} \;, \quad \mathcal{Q}_9^{(-4),*}(x) = -\frac{36\,x^6 + 6096\,x^4 + 182784\,x^2 + 860160}{x^{11}} \;,$$

$$\mathcal{R}_9^{(-4),*} = 0 \;, \quad \mathcal{S}_9^{(-4),*} = 0$$

$$\mathcal{P}_9^{(-5)}(x) = \frac{x^{10} - 3\,x^8 - 45000\,x^6 + 24998400\,x^4 - 1574899200\,x^2 + 17882726400}{45045\,x^{11}} \;,$$

$$\mathcal{Q}_9^{(-5)}(x) = -\frac{x^{12} - x^{10} + 9\,x^8 - 1576800\,x^6 + 257443200\,x^4 - 7620480000\,x^2 + 35765452800}{45045\,x^{12}} \;,$$

$$\mathcal{R}_9^{(-5)} = \frac{1}{45045} \;, \quad \mathcal{S}_9^{(-5)} = 0$$

$$\mathcal{P}_9^{(-5),*}(x) = -\frac{x^{10} + 3\,x^8 - 45000\,x^6 - 24998400\,x^4 - 1574899200\,x^2 - 17882726400}{45045\,x^{11}} \;,$$

$$\mathcal{Q}_9^{(-5),*}(x) = -\frac{x^{12} + x^{10} + 9\,x^8 + 1576800\,x^6 + 257443200\,x^4 + 7620480000\,x^2 + 35765452800}{45045\,x^{12}} \;,$$

$$\mathcal{R}_9^{(-5),*} = \frac{1}{45045} \;, \quad \mathcal{S}_9^{(-5),*} = 0$$

$$\mathcal{P}_9^{(-6)}(x) = -\frac{x^6 - 528\,x^4 + 32640\,x^2 - 368640}{x^{12}} \;,$$

$$\mathcal{Q}_9^{(-6)}(x) = \frac{34\,x^6 - 5376\,x^4 + 157440\,x^2 - 737280}{x^{13}} \;, \quad \mathcal{R}_9^{(-6)} = 0 \;, \quad \mathcal{S}_9^{(-6)} = 0$$

$$\mathcal{P}_9^{(-6),*}(x) = \frac{x^6 + 528\,x^4 + 32640\,x^2 + 368640}{x^{12}} \ ,$$

$$\mathcal{Q}_9^{(-6),*}(x) = -\frac{34\,x^6 + 5376\,x^4 + 157440\,x^2 + 737280}{x^{13}} \ , \quad \mathcal{R}_9^{(-6),*} = 0 \ , \quad \mathcal{S}_9^{(-6),*} = 0$$

$\underline{\mathbf{Z}_{10}(\mathbf{x})}$:

$$\int J_{10}(x)\,dx = -\frac{48\,x^6 - 15744\,x^4 + 921600\,x^2 - 10321920}{x^7} J_0(x) -$$

$$-\frac{2\,x^8 - 1152\,x^6 + 154368\,x^4 - 4423680\,x^2 + 20643840}{x^8} J_1(x) + \Lambda_0(x)$$

$$\int I_{10}(x)\,dx = -\frac{48\,x^6 + 15744\,x^4 + 921600\,x^2 + 10321920}{x^7} I_0(x) +$$

$$+\frac{2\,x^8 + 1152\,x^6 + 154368\,x^4 + 4423680\,x^2 + 20643840}{x^8} I_1(x) - \Lambda_0^*(x)$$

$$\int x\,J_{10}(x)\,dx = -\frac{50\,x^6 - 16800\,x^4 + 1021440\,x^2 - 11612160}{x^6} J_0(x) -$$

$$-\frac{x^8 - 1200\,x^6 + 168000\,x^4 - 4945920\,x^2 + 23224320}{x^7} J_1(x)$$

$$\int x\,I_{10}(x)\,dx = -\frac{50\,x^6 + 16800\,x^4 + 1021440\,x^2 + 11612160}{x^6} I_0(x) +$$

$$+\frac{x^8 + 1200\,x^6 + 168000\,x^4 + 4945920\,x^2 + 23224320}{x^7} I_1(x)$$

$$\int x^2\,J_{10}(x)\,dx = -\frac{51\,x^6 - 18048\,x^4 + 1142784\,x^2 - 13271040}{x^5} J_0(x)$$

$$-\frac{x^8 - 1152\,x^6 + 183552\,x^4 - 5603328\,x^2 + 26542080}{x^6} J_1(x) + 99\Lambda_0(x)$$

$$\int x^2\,I_{10}(x)\,dx = -\frac{51\,x^6 + 18048\,x^4 + 1142784\,x^2 + 13271040}{x^5} I_0(x) +$$

$$+\frac{x^8 + 1152\,x^6 + 183552\,x^4 + 5603328\,x^2 + 26542080}{x^6} I_1(x) + 99\Lambda_0^*(x)$$

$$\int x^3\,J_{10}(x)\,dx = -\frac{52\,x^6 - 19200\,x^4 + 1290240\,x^2 - 15482880}{x^4} J_0(x) -$$

$$-\frac{x^8 - 1304\,x^6 + 201600\,x^4 - 6451200\,x^2 + 30965760}{x^5} J_1(x)$$

$$\int x^3\,I_{10}(x)\,dx = -\frac{52\,x^6 + 19200\,x^4 + 1290240\,x^2 + 15482880}{x^4} I_0(x) +$$

$$+\frac{x^8 + 1304\,x^6 + 201600\,x^4 + 6451200\,x^2 + 30965760}{x^5} I_1(x)$$

$$\int x^4\,J_{10}(x)\,dx = -\frac{53\,x^6 - 20559\,x^4 + 1462272\,x^2 - 18579456}{x^3} J_0(x) -$$

$$-\frac{x^8 - 1359\,x^6 + 231168\,x^4 - 7569408\,x^2 + 37158912}{x^4} J_1(x) + 9009\Lambda_0(x)$$

$$\int x^4\,I_{10}(x)\,dx = -\frac{53\,x^6 + 20559\,x^4 + 1462272\,x^2 + 18579456}{x^3} I_0(x) +$$

$$+\frac{x^8 + 1359\,x^6 + 231168\,x^4 + 7569408\,x^2 + 37158912}{x^4} I_1(x) - 9009\Lambda_0^*(x)$$

$$\int \frac{J_{10}(x)\,dx}{x} = -\frac{48\,x^6 - 14784\,x^4 + 838656\,x^2 - 9289728}{x^8} J_0(x) -$$

$$-\frac{x^8 - 1104\,x^6 + 142464\,x^4 - 3999744\,x^2 + 18579456}{x^9}J_1(x)$$

$$\int \frac{I_{10}(x)\,dx}{x} = -\frac{48\,x^6 + 14784\,x^4 + 838656\,x^2 + 9289728}{x^8}I_0(x)+$$

$$+\frac{x^8 + 1104\,x^6 + 142464\,x^4 + 3999744\,x^2 + 18579456}{x^9}I_1(x)$$

$$\int \frac{J_{10}(x)\,dx}{x^2} = \frac{x^8 - 4656\,x^6 + 1376640\,x^4 - 76124160\,x^2 + 836075520}{99x^9}J_0(x)-$$

$$-\frac{x^{10} + 98\,x^8 - 104832\,x^6 + 13075200\,x^4 - 361267200\,x^2 + 1672151040}{99x^{10}}J_1(x) + \frac{1}{99}\Lambda_0(x)$$

$$\int \frac{I_{10}(x)\,dx}{x^2} = -\frac{x^8 + 4656\,x^6 + 1376640\,x^4 + 76124160\,x^2 + 836075520}{99x^9}I_0(x)-$$

$$-\frac{x^{10} - 98\,x^8 - 104832\,x^6 - 13075200\,x^4 - 361267200\,x^2 - 1672151040}{99x^{10}}I_1(x) + \frac{1}{99}\Lambda_0^*(x)$$

$$\mathcal{P}_{10}^{(5)}(x) = -\frac{54\,x^6 - 22032\,x^4 + 1693440\,x^2 - 23224320}{x^2}\ ,$$

$$\mathcal{Q}_{10}^{(5)}(x) = -\frac{x^8 - 1416\,x^6 + 245664\,x^4 - 9031680\,x^2 + 46448640}{x^3}\ ,\quad \mathcal{R}_{10}^{(5)} = 0\ ,\quad \mathcal{S}_{10}^{(5)} = 0$$

$$\mathcal{P}_{10}^{(5),*}(x) = -\frac{54\,x^6 + 22032\,x^4 + 1693440\,x^2 + 23224320}{x^2}\ ,$$

$$\mathcal{Q}_{10}^{(5),*}(x) = \frac{x^8 + 1416\,x^6 + 245664\,x^4 + 9031680\,x^2 + 46448640}{x^3}\ ,\quad \mathcal{R}_{10}^{(5),*} = 0\ ,\quad \mathcal{S}_{10}^{(5),*} = 0$$

$$\mathcal{P}_{10}^{(6)}(x) = -\frac{55\,x^6 - 23625\,x^4 + 1965915\,x^2 - 30965760}{x}\ ,$$

$$\mathcal{Q}_{10}^{(6)}(x) = -\frac{x^8 - 1475\,x^6 + 272475\,x^4 - 10321920\,x^2 + 61931520}{x^2}\ ,$$

$$\mathcal{R}_{10}^{(6)} = 675675\ ,\quad \mathcal{S}_{10}^{(6)} = 0$$

$$\mathcal{P}_{10}^{(6),*}(x) = -\frac{55\,x^6 + 23625\,x^4 + 1965915\,x^2 + 30965760}{x}\ ,$$

$$\mathcal{Q}_{10}^{(6),*}(x) = \frac{x^8 + 1475\,x^6 + 272475\,x^4 + 10321920\,x^2 + 61931520}{x^2}\ ,$$

$$\mathcal{R}_{10}^{(6),*} = 675675\ ,\quad \mathcal{S}_{10}^{(6),*} = 0$$

$$\mathcal{P}_{10}^{(7)}(x) = -(56\,x^6 - 25344\,x^4 + 2299392\,x^2 - 41287680)\ ,$$

$$\mathcal{Q}_{10}^{(7)}(x) = -\frac{x^8 - 1536\,x^6 + 302976\,x^4 - 13630464\,x^2 + 92897280}{x}\ ,\quad \mathcal{R}_{10}^{(7)} = 0\ ,\quad \mathcal{S}_{10}^{(7)} = 0$$

$$\mathcal{P}_{10}^{(7),*}(x) = -(56\,x^6 + 25344\,x^4 + 2299392\,x^2 + 41287680)\ ,$$

$$\mathcal{Q}_{10}^{(7),*}(x) = \frac{x^8 + 1536\,x^6 + 302976\,x^4 + 13630464\,x^2 + 92897280}{x}\ ,\quad \mathcal{R}_{10}^{(7),*} = 0\ ,\quad \mathcal{S}_{10}^{(7),*} = 0$$

$$\mathcal{P}_{10}^{(8)}(x) = -(57\,x^7 - 27195\,x^5 + 2706165\,x^3 - 58437855\,x)\ ,$$

$$\mathcal{Q}_{10}^{(8)}(x) = -(x^8 - 1599\,x^6 + 337575\,x^4 - 17150175\,x^2 + 185794560)\ ,$$

$$\mathcal{R}_{10}^{(8)} = 34459425\ ,\quad \mathcal{S}_{10}^{(8)} = 0$$

$$\mathcal{P}_{10}^{(8),*}(x) = -(57\,x^7 + 27195\,x^5 + 2706165\,x^3 + 58437855\,x)\ ,$$

$$\mathcal{Q}_{10}^{(8),*}(x) = x^8 + 1599\,x^6 + 337575\,x^4 + 17150175\,x^2 + 185794560\ ,$$

$$\mathcal{R}_{10}^{(8),*} = -34459425\ ,\quad \mathcal{S}_{10}^{(8),*} = 0$$

$$\mathcal{P}_{10}^{(9)}(x) = -(58\,x^8 - 29184\,x^6 + 3200256\,x^4 - 84953088\,x^2 + 185794560)\ ,$$

$$\mathcal{Q}_{10}^{(9)}(x) = -(x^9 - 1664\,x^7 + 376704\,x^5 - 21832704\,x^3 + 262803456\,x)\ , \quad \mathcal{R}_{10}^{(9)} = 0\ , \quad \mathcal{S}_{10}^{(9)} = 0$$

$$\mathcal{P}_{10}^{(9),*}(x) = -(58\,x^8 + 29184\,x^6 + 3200256\,x^4 + 84953088\,x^2 + 185794560)\ ,$$

$$\mathcal{Q}_{10}^{(9),*}(x) = x^9 + 1664\,x^7 + 376704\,x^5 + 21832704\,x^3 + 262803456\,x\ , \quad \mathcal{R}_{10}^{(9),*} = 0\ , \quad \mathcal{S}_{10}^{(9),*} = 0$$

$$\mathcal{P}_{10}^{(10)}(x) = -(59\,x^9 - 31317\,x^7 + 3797535\,x^5 - 125345745\,x^3 + 654729075\,x)\ ,$$

$$\mathcal{Q}_{10}^{(10)}(x) = -(x^{10} - 1731\,x^8 + 420819\,x^6 - 28019355\,x^4 + 468934515\,x^2)\ ,$$

$$\mathcal{R}_{10}^{(10)} = 654729075\ , \quad \mathcal{S}_{10}^{(10)} = 0$$

$$\mathcal{P}_{10}^{(10),*}(x) = -(59\,x^9 + 31317\,x^7 + 3797535\,x^5 + 125345745\,x^3 + 654729075\,x)\ ,$$

$$\mathcal{Q}_{10}^{(10),*}(x) = x^{10} + 1731\,x^8 + 420819\,x^6 + 28019355\,x^4 + 468934515\,x^2\ ,$$

$$\mathcal{R}_{10}^{(10),*} = 654729075\ , \quad \mathcal{S}_{10}^{(10),*} = 0$$

$$\mathcal{P}_{10}^{(-3)}(x) = -\frac{46\,x^6 - 13104\,x^4 + 709632\,x^2 - 7741440}{x^{10}}\ ,$$

$$\mathcal{Q}_{10}^{(-3)}(x) = -\frac{x^8 - 1016\,x^6 + 122976\,x^4 - 3354624\,x^2 + 15482880}{x^{11}}\ , \quad \mathcal{R}_{10}^{(-3)} = 0\ , \quad \mathcal{S}_{10}^{(-3)} = 0$$

$$\mathcal{P}_{10}^{(-3),*}(x) = -\frac{46\,x^6 + 13104\,x^4 + 709632\,x^2 + 7741440}{x^{10}}\ ,$$

$$\mathcal{Q}_{10}^{(-3),*}(x) = \frac{x^8 + 1016\,x^6 + 122976\,x^4 + 3354624\,x^2 + 15482880}{x^{11}}\ , \quad \mathcal{R}_{10}^{(-3),*} = 0\ , \quad \mathcal{S}_{10}^{(-3),*} = 0$$

$$\mathcal{P}_{10}^{(-4)}(x) = \frac{x^{10} - 3\,x^8 - 405360\,x^6 + 111484800\,x^4 - 5933813760\,x^2 + 64377815040}{9009\,x^{11}}\ ,$$

$$\mathcal{Q}_{10}^{(-4)}(x) = -\frac{x^{12} - x^{10} + 9018\,x^8 - 8784000\,x^6 + 1035820800\,x^4 - 27962081280\,x^2 + 128755630080}{9009\,x^{12}}\ ,$$

$$\mathcal{R}_{10}^{(-4)} = \frac{1}{9009}\ , \quad \mathcal{S}_{10}^{(-4)} = 0$$

$$\mathcal{P}_{10}^{(-4),*}(x) = \frac{x^{10} + 3\,x^8 - 405360\,x^6 - 111484800\,x^4 - 5933813760\,x^2 - 64377815040}{9009\,x^{11}}\ ,$$

$$\mathcal{Q}_{10}^{(-4),*}(x) = \frac{x^{12} + x^{10} + 9018\,x^8 + 8784000\,x^6 + 1035820800\,x^4 + 27962081280\,x^2 + 128755630080}{9009\,x^{12}}\ ,$$

$$\mathcal{R}_{10}^{(-4),*} = -\frac{1}{9009}\ , \quad \mathcal{S}_{10}^{(-4),*} = 0$$

$$\mathcal{P}_{10}^{(-5)}(x) = -\frac{44\,x^6 - 11712\,x^4 + 614400\,x^2 - 6635520}{x^{12}}\ ,$$

$$\mathcal{Q}_{10}^{(-5)}(x) = -\frac{x^8 - 936\,x^6 + 107904\,x^4 - 2887680\,x^2 + 13271040}{x^{13}}\ ,$$

$$\mathcal{R}_{10}^{(-5)} = 0\ , \quad \mathcal{S}_{10}^{(-5)} = 0$$

$$\mathcal{P}_{10}^{(-5),*}(x) = -\frac{44\,x^6 + 11712\,x^4 + 614400\,x^2 + 6635520}{x^{12}}\ ,$$

$$\mathcal{Q}_{10}^{(-5),*}(x) = \frac{x^8 + 936\,x^6 + 107904\,x^4 + 2887680\,x^2 + 13271040}{x^{13}}\ ,$$

$$\mathcal{R}_{10}^{(-5),*} = 0\ , \quad \mathcal{S}_{10}^{(-5),*} = 0$$

$$\mathcal{P}_{10}^{(-6)}(x) = \frac{x^{12} - 3\,x^{10} + 45\,x^8 - 29055600\,x^6 + 7506172800\,x^4 - 388949299200\,x^2 + 4184557977600}{675675\,x^{13}}\ ,$$

$$\mathcal{Q}_{10}^{(-6)}(x) =$$

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$$-\frac{x^{14} - x^{12} + 9\,x^{10} + 675450\,x^8 - 607420800\,x^6 + 68660524800\,x^4 - 1824038092800\,x^2 + 8369115955200}{675675\,x^{14}}\ ,$$

$$\mathcal{R}_{10}^{(-6)} = \frac{1}{675675}\ ,\quad \mathcal{S}_{10}^{(-6)} = 0$$

$$\mathcal{P}_{10}^{(-6),*}(x) = -\frac{x^{12} + 3\,x^{10} + 45\,x^8 + 29055600\,x^6 + 7506172800\,x^4 + 388949299200\,x^2 + 4184557977600}{675675\,x^{13}}\ ,$$

$$\mathcal{Q}_{10}^{(-6),*}(x) =$$

$$-\frac{x^{14} + x^{12} + 9\,x^{10} - 675450\,x^8 - 607420800\,x^6 - 68660524800\,x^4 - 1824038092800\,x^2 - 8369115955200}{675675\,x^{14}}\ ,$$

$$\mathcal{R}_{10}^{(-6),*} = \frac{1}{675675}\ ,\quad \mathcal{S}_{10}^{(-6),*} = 0$$

1.1.8. Second Antiderivatives of $x^n Z_\nu(x)$:

$\Phi(x)$, $\Psi(x)$ and $\Psi_K(x)$ are the same as in I., page 7.

a) $x^{2n+1} Z_0(x)$:

With the functions $\Phi(x)$, $\Psi(x)$ and $\Psi_K(x)$ as defined on page ??? holds:

$$x\,J_0(x) = \frac{d^2\,\Phi(x)}{dx^2} \;, \quad x\,I_0(x) = -\frac{d^2\,\Psi(x)}{dx^2} \;, \quad x\,K_0(x) = -\frac{d^2\,\Psi_K(x)}{dx^2}$$

$$x^3\,J_0(x) = \frac{d^2}{dx^2}\left[-x^3\,J_0(x) + 5x^2\,J_1(x) - 9\,\Phi(x)\right]$$

$$x^3\,I_0(x) = \frac{d^2}{dx^2}\left[x^3\,I_0(x) - 5x^2\,I_1(x) + 9\,\Psi(x)\right]$$

$$x^3\,K_0(x) = \frac{d^2}{dx^2}\left[x^3\,K_0(x) + 5x^2\,K_1(x) - 9\,\Psi_K(x)\right]$$

$$x^5\,J_0(x) = \frac{d^2}{dx^2}\left[(-x^5 + 43x^3)\,J_0(x) + (9x^4 - 161x^2)\,J_1(x) + 225\Phi(x)\right]$$

$$x^5\,I_0(x) = \frac{d^2}{dx^2}\left[(x^5 + 43\,x^3)\,I_0(x) - (9\,x^4 + 161\,x^2)\,I_1(x) + 225\,\Psi(x)\right]$$

$$x^5\,K_0(x) = \frac{d^2}{dx^2}\left[(x^5 + 43\,x^3)\,K_0(x) + (9\,x^4 + 161\,x^2)\,K_1(x) - 225\,\Psi_K(x)\right]$$

$$x^7\,J_0(x) = \frac{d^2}{dx^2}\left[(-x^7 + 101\,x^5 - 2523\,x^3)\,J_0(x) + (13\,x^6 - 649\,x^4 + 8721\,x^2)\,J_1(x) - 11025\,\Phi(x)\right]$$

$$x^7\,I_0(x) = \frac{d^2}{dx^2}\left[(x^7 + 101\,x^5 + 2523\,x^3)\,I_0(x) - (13\,x^6 + 649\,x^4 + 8721\,x^2)\,I_1(x) + 11025\,\Psi(x)\right]$$

$$x^7\,K_0(x) = \frac{d^2}{dx^2}\left[(x^7 + 101\,x^5 + 2523\,x^3)\,K_0(x) + (13\,x^6 + 649\,x^4 + 8721\,x^2)\,K_1(x) - 11025\,\Psi_K(x)\right]$$

The formulas for $I_1(x)$ and $K_1(x)$ vary in two signs.

$$x^9\,J_0(x) = \frac{d^2}{dx^2}\left[(-x^9 + 183\,x^7 - 10629\,x^5 + 223947\,x^3)\,J_0(x)+\right.$$

$$\left. +(17\,x^8 - 1665\,x^6 + 62361\,x^4 - 745569\,x^2)\,J_1(x) + 893025\,\Phi(x)\right]$$

$$x^9\,I_0(x) = \frac{d^2}{dx^2}\left[(x^9 + 183\,x^7 + 10629\,x^5 + 223947\,x^3)\,I_0(x)-\right.$$

$$\left. -(17\,x^8 + 1665\,x^6 + 62361\,x^4 + 745569\,x^2)\,I_1(x) + 893025\,\Psi(x)\right]$$

$$x^{11}\,J_0(x) = \frac{d^2}{dx^2}\left[(-x^{11} + 289\,x^9 - 30207\,x^7 + 1479645\,x^5 - 28645875\,x^3)\,J_0(x)+\right.$$

$$\left. +(21\,x^{10} - 3401\,x^8 + 249849\,x^6 - 8319825\,x^4 + 93310425\,x^2)\,J_1(x) - 108056025\,\Phi(x)\right]$$

$$x^{11}\,I_0(x) = \frac{d^2}{dx^2}\left[(x^{11} + 289\,x^9 + 30207\,x^7 + 1479645\,x^5 + 28645875\,x^3)\,I_0(x)-\right.$$

$$\left. -(21\,x^{10} + 3401\,x^8 + 249849\,x^6 + 8319825\,x^4 + 93310425\,x^2)\,I_1(x) + 108056025\,\Psi(x)\right]$$

$$x^{13}\,J_0(x) = \frac{d^2}{dx^2}\left[(-x^{13} + 419\,x^{11} - 68841\,x^9 + 6064983\,x^7 - 273100005\,x^5 + 5025472875\,x^3)\,J_0(x)+\right.$$

$$\left. +(25\,x^{12} - 6049\,x^{10} + 734769\,x^8 - 47984481\,x^6 + 1498210425\,x^4 - 16138101825\,x^2)\,J_1(x) + 18261468225\,\Phi(x)\right]$$

$$x^{13}\,I_0(x) = \frac{d^2}{dx^2}\left[(x^{13} + 419\,x^{11} + 68841\,x^9 + 6064983\,x^7 + 273100005\,x^5 + 5025472875\,x^3)\,I_0(x)-\right.$$

$$\left. -(25\,x^{12} + 6049\,x^{10} + 734769\,x^8 + 47984481\,x^6 + 1498210425\,x^4 + 16138101825\,x^2)\,I_1(x) + 18261468225\,\Psi(x)\right]$$

$$x^{15}\,J_0(x) = \frac{d^2}{dx^2}\left[(-x^{15} + 573\,x^{13} - 136035\,x^{11} + 18830025\,x^9 - 1524979575\,x^7 + 65296102725\,x^5-\right.$$

$$-1161520209675\,x^3)\,J_0(x) + (29\,x^{14} - 9801\,x^{12} + 1778625\,x^{10} - 192049425\,x^8 + 11758658625\,x^6 -$$
$$-352491752025\,x^4 + 3692650536225\,x^2)\,J_1(x) - 4108830350625\,\Phi(x)\big]$$

$$x^{15}\,I_0(x) = \frac{d^2}{dx^2}\,\big[(x^{15} + 573\,x^{13} + 136035\,x^{11} + 18830025\,x^9 + 1524979575\,x^7 + 65296102725\,x^5 +$$
$$+1161520209675\,x^3\,I_0(x) - (29\,x^{14} + 9801\,x^{12} + 1778625\,x^{10} + 192049425\,x^8 + 11758658625\,x^6 +$$
$$+352491752025\,x^4 + 3692650536225\,x^2\,I_1(x) + 4108830350625\,\Psi(x)\big]$$

Recurrence relations:

$$x^{2n+3}\,J_0(x) = \frac{d^2}{dx^2}\left\{x^{2n}\left[\frac{2n(2n+1)(4n+5)}{4n+1}\,x - x^3\right]J_0(x) + (4n+5)x^{2n+2}\,J_1(x)\right\} -$$
$$-\frac{x^{2n-1}[(4n+3)(8n^2+12n+3)x^2 + 4n^2(2n+1)^2(4n+5)]\,J_1(x)}{4n+1}$$

$$x^{2n+3}\,I_0(x) = \frac{d^2}{dx^2}\left\{x^{2n}\left[\frac{2n(2n+1)(4n+5)}{4n+1}\,x + x^3\right]I_0(x) - (4n+5)x^{2n+2}\,I_1(x)\right\} +$$
$$+\frac{x^{2n-1}[(4n+3)(8n^2+12n+3)x^2 - 4n^2(2n+1)^2(4n+5)]\,I_1(x)}{4n+1}$$

$$x^{2n+3}\,K_0(x) = \frac{d^2}{dx^2}\left\{x^{2n}\left[\frac{2n(2n+1)(4n+5)}{4n+1}\,x + x^3\right]K_0(x) + (4n+5)x^{2n+2}\,K_1(x)\right\} +$$
$$+\frac{x^{2n-1}[(4n+3)(8n^2+12n+3)x^2 - 4n^2(2n+1)^2(4n+5)]\,K_1(x)}{4n+1}$$

b) $x^{2n}\,Z_1(x)$:

With the functions $\Phi(x)$, $\Psi(x)$ and $\Psi_K(x)$ as defined on page 7 holds:

$$J_1(x) = \frac{d^2}{dx^2}\left[-x\,J_0(x) - \Phi(x)\right]\,,\quad I_1(x) = \frac{d^2}{dx^2}\left[x\,I_0(x) + \Psi(x)\right]\,,\quad K_1(x) = \frac{d^2}{dx^2}\left[-x\,K_0(x) - \Psi_K(x)\right]$$

$$x^2\,J_1(x) = \frac{d^2}{dx^2}\left[-x^2\,J_1(x) + 3\,\Phi(x)\right]$$

$$x^2\,I_1(x) = \frac{d^2}{dx^2}\left[x^2\,I_1(x) + 3\,\Psi(x)\right]$$

$$x^2\,K_1(x) = \frac{d^2}{dx^2}\left[x^2\,K_1(x) - 3\,\Psi_K(x)\right]$$

$$x^4\,J_1(x) = \frac{d^2}{dx^2}\left[-7x^3\,J_0(x) + (-x^4 + 29\,x^2)\,J_1(x) - 45\,\Phi(x)\right]$$

$$x^4\,I_1(x) = \frac{d^2}{dx^2}\left[-7\,I_0(x) + (x^4 + 29x^2)\,I_1(x) + 45\,\Psi(x)\right]$$

$$x^4\,K_1(x) = \frac{d^2}{dx^2}\left[7x^3\,K_0(x) + (x^4 + 29x^2)\,K_1(x) - 45\,\Psi_K(x)\right]$$

$$x^6\,J_1(x) = \frac{d^2}{dx^2}\left[(-11\,x^5 + 333\,x^3)\,J_0(x) + (-x^6 + 79\,x^4 - 1191\,x^2)\,J_1(x) + 1575\,\Phi(x)\right]$$

$$x^6\,I_1(x) = \frac{d^2}{dx^2}\left[-(11\,x^5 + 333\,x^3)\,I_0(x) + (x^6 + 79\,x^4 + 1191\,x^2)\,I_1(x) + 1575\,\Psi(x)\right]$$

$$x^6\,K_1(x) = \frac{d^2}{dx^2}\left[(11\,x^5 + 333\,x^3)\,K_0(x) + (x^6 + 79\,x^4 + 1191\,x^2)\,K_1(x) - 1575\,\Psi_K(x)\right]$$

The formulas for $I_1(x)$ and $K_1(x)$ vary in two signs.

$$x^8\,J_1(x) =$$

$$= \frac{d^2}{dx^2} \left[(-15\,x^7 + 1053\,x^5 - 23859\,x^3)\,J_0(x) + (-x^8 + 153\,x^6 - 6417\,x^4 + 80793\,x^2)\,J_1(x) - 99225\,\Phi(x) \right]$$

$$x^8 I_1(x) =$$

$$= \frac{d^2}{dx^2} \left[-(15\,x^7 + 1053\,x^5 + 23859\,x^3)\,I_0(x) + (x^8 + 153\,x^6 + 6417\,x^4 + 80793\,x^2)\,I_1(x) - 99225\,\Psi(x) \right]$$

$$x^{10} J_1(x) = \frac{d^2}{dx^2} \left[(-19\,x^9 + 2397\,x^7 - 126135\,x^5 + 2537145\,x^3)\,J_0(x) + \right.$$

$$\left. + (-x^{10} + 251\,x^8 - 20619\,x^6 + 722835\,x^4 - 8348715\,x^2)\,J_1(x) + 9823275\,\Phi(x) \right]$$

$$x^{10} I_1(x) = \frac{d^2}{dx^2} \left[-(19\,x^9 + 2397\,x^7 + 126135\,x^5 + 2537145\,x^3)\,I_0(x) + \right.$$

$$\left. + (x^{10} + 251\,x^8 + 20619\,x^6 + 722835\,x^4 + 8348715\,x^2)\,I_1(x) + 9823275\,\Psi(x) \right]$$

$$x^{12} J_1(x) = \frac{d^2}{dx^2} \left[(-23\,x^{11} + 4557\,x^9 - 431091\,x^7 + 20156985\,x^5 - 379769175\,x^3)\,J_0(x) + \right.$$

$$+ (-x^{12} + 373\,x^{10} - 50613\,x^8 + 3478437\,x^6 - 111844125\,x^4 + 1227781125\,x^2)\,J_1(x) -$$

$$\left. - 1404728325\,\Phi(x) \right]$$

$$x^{12} I_1(x) = \frac{d^2}{dx^2} \left[-(23\,x^{11} + 4557\,x^9 + 431091\,x^7 + 20156985\,x^5 + 379769175\,x^3)\,I_0(x) + \right.$$

$$+ (x^{12} + 373\,x^{10} + 50613\,x^8 + 3478437\,x^6 + 111844125\,x^4 + 1227781125\,x^2)\,I_1(x) +$$

$$\left. + 1404728325\,\Psi(x) \right]$$

$$x^{14} J_1(x) =$$

$$= \frac{d^2}{dx^2} \left[(-27\,x^{13} + 7725\,x^{11} - 1147815\,x^9 + 96504345\,x^7 - 4229210475\,x^5 + 76443776325\,x^3)\,J_0(x) + \right.$$

$$+ (-x^{14} + 519\,x^{12} - 105135\,x^{10} + 11943135\,x^8 - 752944815\,x^6 + 23003997975\,x^4 - 244194893775\,x^2)\,J_1(x) +$$

$$\left. + 273922023375\,\Phi(x) \right]$$

$$x^{14} I_1(x) =$$

$$= \frac{d^2}{dx^2} \left[-(27\,x^{13} + 7725\,x^{11} + 1147815\,x^9 + 96504345\,x^7 + 4229210475\,x^5 + 76443776325\,x^3)\,I_0(x) + \right.$$

$$+ (x^{14} + 519\,x^{12} + 105135\,x^{10} + 11943135\,x^8 + 752944815\,x^6 + 23003997975\,x^4 + 244194893775\,x^2)\,I_1(x) +$$

$$\left. + 273922023375\,\Psi(x) \right]$$

$$x^{16} J_1(x) = \frac{d^2}{dx^2} \left[(-31\,x^{15} + 12093\,x^{13} - 2594835\,x^{11} + 342689625\,x^9 - 27008454375\,x^7 + 1136044984725\,x^5 - \right.$$

$$- 19953933471675\,x^3)\,J_0(x) + (-x^{16} + 689\,x^{14} - 194841\,x^{12} + 33059025\,x^{10} - 3445473825\,x^8 + 206400006225\,x^6 -$$

$$\left. - 6096404738025\,x^4 + 63191238930225\,x^2)\,J_1(x) - 69850115960625\,\Phi(x) \right]$$

$$x^{16} I_1(x) = \frac{d^2}{dx^2} \left[-(31\,x^{15} + 12093\,x^{13} + 2594835\,x^{11} + 342689625\,x^9 + 27008454375\,x^7 + 1136044984725\,x^5 + \right.$$

$$+ 19953933471675\,x^3)\,I_0(x) + (x^{16} + 689\,x^{14} + 194841\,x^{12} + 33059025\,x^{10} + 3445473825\,x^8 + 206400006225\,x^6 +$$

$$\left. + 6096404738025\,x^4 + 63191238930225\,x^2)\,I_1(x) + 69850115960625\,\Psi(x) \right]$$

Recurrence relations:

$$x^{2n+2} J_1(x) = \frac{d^2}{dx^2} \left\{ \left[\frac{2n(4n+3)(2n+1)}{4n-1} - x^2 \right] x^{2n}\,J_1(x) - (4n+3)\,x^{2n+1}\,J_0(x) \right\} -$$

$$- \frac{(4n+1)(8n^2+4n-3)x^2 + 4n(n-1)(4n+3)(2n+1)(2n-1)}{4n-1}\,x^{2n-2}\,J_1(x)$$

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$$x^{2n+2} I_1(x) = \frac{d^2}{dx^2} \left\{ \left[\frac{2n(4n+3)(2n+1)}{4n-1} + x^2 \right] x^{2n} I_1(x) - (4n+3) x^{2n+1} I_0(x) \right\} +$$

$$+ \frac{(4n+1)(8n^2+4n-3)x^2 - 4n(n-1)(4n+3)(2n+1)(2n-1)}{4n-1} x^{2n-2} I_1(x)$$

$$x^{2n+2} K_1(x) = \frac{d^2}{dx^2} \left\{ \left[\frac{2n(4n+3)(2n+1)}{4n-1} + x^2 \right] x^{2n} K_1(x) + (4n+3) x^{2n+1} K_0(x) \right\} +$$

$$+ \frac{(4n+1)(8n^2+4n-3)x^2 - 4n(n-1)(4n+3)(2n+1)(2n-1)}{4n-1} x^{2n-2} K_1(x)$$

1.1.9. Higher antiderivatives:

$\Phi(x)$, $\Psi(x)$ and $\Psi_K(x)$ are the same as in I., page 7. See also [1], 11.2. .

$$
\begin{aligned}
J_0(x) \;=\;& \frac{d^2}{dx^2}\left\{x^2 J_0(x) - x J_1(x) + x\,\Phi(x)\right\} \\[4pt]
=\;& \frac{d^3}{dx^3}\left\{\frac{x^3}{2} J_0(x) - \frac{x^2}{2} J_1(x) + \frac{x^2-1}{2}\,\Phi(x)\right\} \\[4pt]
=\;& \frac{d^4}{dx^4}\left\{\frac{x^4-2x^2}{6} J_0(x) - \frac{x^3-4x}{6} J_1(x) + \frac{x^3-3x}{6}\,\Phi(x)\right\} \\[4pt]
=\;& \frac{d^5}{dx^5}\left\{\frac{x^5-5x^3}{24} J_0(x) - \frac{x^4-7x^2}{24} J_1(x) + \frac{x^4-6x^2+9}{24}\,\Phi(x)\right\} \\[4pt]
=\;& \frac{d^6}{dx^6}\left\{\frac{x^6-9x^4+32x^2}{120} J_0(x) - \frac{x^5-11x^3+64x}{120} J_1(x) + \frac{x^5-10x^3+45x}{120}\,\Phi(x)\right\} \\[4pt]
=\;& \frac{d^7}{dx^7}\left\{\frac{x^7-14x^5+117x^3}{720} J_0(x) - \frac{x^6-16x^4+159x^2}{720} J_1(x) + \frac{x^6-15x^4+135x^2-225}{720}\,\Phi(x)\right\}
\end{aligned}
$$

$$
\begin{aligned}
I_0(x) \;=\;& \frac{d^2}{dx^2}\left\{x^2 I_0(x) - x I_1(x) + x\,\Psi(x)\right\} \\[4pt]
=\;& \frac{d^3}{dx^3}\left\{\frac{x^3}{2} I_0(x) - \frac{x^2}{2} I_1(x) + \frac{x^2+1}{2}\,\Psi(x)\right\} \\[4pt]
=\;& \frac{d^4}{dx^4}\left\{\frac{x^4+2x^2}{6} I_0(x) - \frac{x^3+4x}{6} I_1(x) + \frac{x^3+3x}{6}\,\Psi(x)\right\} \\[4pt]
=\;& \frac{d^5}{dx^5}\left\{\frac{x^5+5x^3}{24} I_0(x) - \frac{x^4+7x^2}{24} I_1(x) + \frac{x^4+6x^2+9}{24}\,\Psi(x)\right\} \\[4pt]
=\;& \frac{d^6}{dx^6}\left\{\frac{x^6+9x^4+32x^2}{120} I_0(x) - \frac{x^5+11x^3+64x}{120} I_1(x) + \frac{x^5+10x^3+45x}{120}\,\Psi(x)\right\} \\[4pt]
=\;& \frac{d^7}{dx^7}\left\{\frac{x^7+14x^5+117x^3}{720} I_0(x) - \frac{x^6+16x^4+159x^2}{720} I_1(x) + \frac{x^6+15x^4+135x^2+225}{720}\,\Psi(x)\right\}
\end{aligned}
$$

$$
\begin{aligned}
K_0(x) \;=\;& \frac{d^2}{dx^2}\left\{x^2 K_0(x) - x K_1(x) + x\,\Psi_K(x)\right\} \\[4pt]
=\;& \frac{d^3}{dx^3}\left\{\frac{x^3}{2} K_0(x) - \frac{x^2}{2} K_1(x) + \frac{x^2+1}{2}\,\Psi_K(x)\right\} \\[4pt]
=\;& \frac{d^4}{dx^4}\left\{\frac{x^4+2x^2}{6} K_0(x) - \frac{x^3+4x}{6} K_1(x) + \frac{x^3+3x}{6}\,\Psi_K(x)\right\} \quad \ldots
\end{aligned}
$$

The formulas for $K_0(x)$ are similar to such for $I_0(x)$.

Let

$$
J_0(x) \;=\; \frac{d^n}{dx^n}\left\{\frac{A_n(x)\,J_0(x) - B_n(x)\,J_1(x) + C_n(x)\,\Phi(x)}{(n-1)!}\right\},
$$

then holds

$$
\begin{aligned}
A_8 &= x^8 - 20\,x^6 + 291\,x^4 - 1152\,x^2 \\
B_8 &= x^7 - 22\,x^5 + 345\,x^3 - 2304\,x \\
C_8 &= x^7 - 21\,x^5 + 315\,x^3 - 1575\,x
\end{aligned}
$$

$$
\begin{aligned}
A_9 &= x^9 - 27\,x^7 + 599\,x^5 - 5541\,x^3 \\
B_9 &= x^8 - 29\,x^6 + 667\,x^4 - 7407\,x^2 \\
C_9 &= x^8 - 28\,x^6 + 630\,x^4 - 6300\,x^2 + 11025
\end{aligned}
$$

$$
\begin{aligned}
A_{10} &= x^{10} - 35\,x^8 + 1095\,x^6 - 17613\,x^4 + 73728\,x^2 \\
B_{10} &= x^9 - 37\,x^7 + 1179\,x^5 - 20583\,x^3 + 147456\,x \\
C_{10} &= x^9 - 36\,x^7 + 1134\,x^5 - 18900\,x^3 + 99225\,x
\end{aligned}
$$

$$
\begin{aligned}
A_{11} &= x^{11} - 44\,x^9 + 1842\,x^7 - 45180\,x^5 + 439605\,x^3 \\
B_{11} &= x^{10} - 46\,x^8 + 1944\,x^6 - 49770\,x^4 + 581535\,x^2 \\
C_{11} &= x^{10} - 45\,x^8 + 1890\,x^6 - 47250\,x^4 + 496125\,x^2 - 893025
\end{aligned}
$$

$$
\begin{aligned}
A_{12} &= x^{12} - 54\,x^{10} + 2912\,x^8 - 100770\,x^6 + 1702215\,x^4 - 7372800\,x^2 \\
B_{12} &= x^{11} - 56\,x^9 + 3034\,x^7 - 107640\,x^5 + 1973205\,x^3 - 14745600\,x \\
C_{12} &= x^{11} - 55\,x^9 + 2970\,x^7 - 103950\,x^5 + 1819125\,x^3 - 9823275\,x
\end{aligned}
$$

$A_{13} = x^{13} - 65\,x^{11} + 4386\,x^9 - 203202\,x^7 + 5231565\,x^5 - 52454925\,x^3$

$B_{13} = x^{12} - 67\,x^{10} + 4530\,x^8 - 213174\,x^6 + 5731245\,x^4 - 68891175\,x^2$

$C_{13} = x^{12} - 66\,x^{10} + 4455\,x^8 - 207900\,x^6 + 5457375\,x^4 - 58939650\,x^2 + 108056025$

$A_{14} = x^{14} - 77\,x^{12} + 6354\,x^{10} - 379386\,x^8 + 13778685\,x^6 - 239546025\,x^4 + 1061683200\,x^2$

$B_{14} = x^{13} - 79\,x^{11} + 6522\,x^9 - 393462\,x^7 + 14661765\,x^5 - 276270075\,x^3 + 2123366400\,x$

$C_{14} = x^{13} - 78\,x^{11} + 6435\,x^9 - 386100\,x^7 + 14189175\,x^5 - 255405150\,x^3 + 1404728325\,x$

$A_{15} = x^{15} - 90\,x^{13} + 8915\,x^{11} - 666348\,x^9 + 32399703\,x^7 - 858134970\,x^5 + 8776408725\,x^3$

$B_{15} = x^{14} - 92\,x^{12} + 9109\,x^{10} - 685728\,x^8 + 33896331\,x^6 - 937030500\,x^4 + 11465661375\,x^2$

$C_{15} = x^{14} - 91\,x^{12} + 9009\,x^{10} - 675675\,x^8 + 33108075\,x^6 - 893918025\,x^4 + 9833098275\,x^2 -$
$\qquad - 18261468225$

$A_{16} = x^{16} - 104\,x^{14} + 12177\,x^{12} - 1113480\,x^{10} + 69776595\,x^8 - 2606612400\,x^6 + 46180633275\,x^4 -$
$\qquad - 208089907200\,x^2$

$B_{16} = x^{15} - 106\,x^{13} + 12399\,x^{11} - 1139580\,x^9 + 72217215\,x^7 - 2767649850\,x^5 + 53076402225\,x^3 -$
$\qquad - 416179814400\,x$

$C_{16} = x^{15} - 105\,x^{13} + 12285\,x^{11} - 1126125\,x^9 + 70945875\,x^7 - 2681754075\,x^5 + 49165491375\,x^3 -$
$\qquad - 273922023375\,x$

$A_{17} = x^{17} - 119\,x^{15} + 16257\,x^{13} - 1785015\,x^{11} + 140033835\,x^9 - 7003021725\,x^7 +$
$\qquad + 188956336275\,x^5 - 1959828398325\,x^3$

$B_{17} = x^{16} - 121\,x^{14} + 16509\,x^{12} - 1819485\,x^{10} + 143878995\,x^8 - 7315353675\,x^6 +$
$\qquad + 205891548975\,x^4 - 2550046679775\,x^2$

$C_{17} = x^{16} - 120\,x^{14} + 16380\,x^{12} - 1801800\,x^{10} + 141891750\,x^8 - 7151344200\,x^6 + 196661965500\,x^4 -$
$\qquad - 2191376187000\,x^2 + 4108830350625$

$A_{18} = x^{18} - 135\,x^{16} + 21281\,x^{14} - 2762727\,x^{12} + 265161195\,x^{10} - 17089901325\,x^8 +$
$\qquad + 650296717875\,x^6 - 11675732422725\,x^4 + 53271016243200\,x^2$

$B_{18} = x^{17} - 137\,x^{15} + 21565\,x^{13} - 2807469\,x^{11} + 271036755\,x^9 - 17667841275\,x^7 +$
$\qquad + 689522590575\,x^5 - 13385846919375\,x^3 + 106542032486400\,x$

$C_{18} = x^{17} - 136\,x^{15} + 21420\,x^{13} - 2784600\,x^{11} + 268017750\,x^9 - 17367550200\,x^7 +$
$\qquad + 668650682700\,x^5 - 12417798393000\,x^3 + 69850115960625\,x$

$A_{19} = x^{19} - 152\,x^{17} + 27384\,x^{15} - 4148856\,x^{13} + 478163970\,x^{11} - 38580445800\,x^9 +$
$\qquad + 1965171423600\,x^7 - 53722981355400\,x^5 + 563060968600725\,x^3$

$B_{19} = x^{18} - 154\,x^{16} + 27702\,x^{14} - 4206042\,x^{12} + 486902160\,x^{10} - 39605788350\,x^8 +$
$\qquad + 2050859043450\,x^6 - 58451723167950\,x^4 + 730304613424575\,x^2$

$C_{19} = x^{18} - 153\,x^{16} + 27540\,x^{14} - 4176900\,x^{12} + 482431950\,x^{10} - 39076987950\,x^8 +$
$\qquad + 2005952048100\,x^6 - 55880092768500\,x^4 + 628651043645625\,x^2 - 1187451971330625$

$A_{20} = x^{20} - 170\,x^{18} + 34710\,x^{16} - 6069258\,x^{14} + 827072928\,x^{12} - 81642319470\,x^{10} +$
$\qquad + 5359374805050\,x^8 - 206598479046750\,x^6 + 3746290783676175\,x^4 - 17259809262796800\,x^2$

$B_{20} = x^{19} - 172\,x^{17} + 35064\,x^{15} - 6141348\,x^{13} + 839759706\,x^{11} - 83394724500\,x^9 +$
$\qquad + 5536741392000\,x^7 - 218854228527900\,x^5 + 4287004731290925\,x^3 - 34519618525593600\,x$

$C_{20} = x^{19} - 171\,x^{17} + 34884\,x^{15} - 6104700\,x^{13} + 833291550\,x^{11} - 82495863450\,x^9 +$
$\qquad + 5444726987700\,x^7 - 212344352520300\,x^5 + 3981456609755625\,x^3 - 22561587455281875\,x$

To get the functions for $I_0(x)$ or $K_0(x)$ one has to change all '-' in the fractions to '+' .
The higher antiderivatives of $J_1(x)$, $I_1(x)$ and $K_1(x)$ follow from the previous tables and the formulas
$J_1(x) = -J_0'(x)$ and $I_1(x) = I_0'(x)$, $K_1(x) = -K_0'(x)$.

1.1.10. Some Integrals of the Type $x^{2n+1} Z_1(x^2 + \alpha)/(x^2 + \alpha)$

$$\int \frac{x^5 J_1(x^2 + i)\, dx}{x^2 + i} = \frac{1}{2}\left[-(x^2 - i)\, J_0(x^2 + i) + J_1(x^2 + i)\right]$$

$$\int \frac{x^5 J_1(x^2 - i)\, dx}{x^2 - i} = -\frac{1}{2}\left[(x^2 + i)\, J_0(x^2 - i) - J_1(x^2 - i)\right]$$

$$\int \frac{x^5 I_1(x^2 + 1)\, dx}{x^2 + 1} = \frac{1}{2}\left[(x^2 - 1)\, I_0(x^2 + 1) - I_1(x^2 + 1)\right]$$

$$\int \frac{x^5 I_1(x^2 - 1)\, dx}{x^2 - 1} = \frac{1}{2}\left[(x^2 + 1)\, I_0(x^2 - 1) - I_1(x^2 - 1)\right]$$

$$\int \frac{x^5 K_1(x^2 + 1)\, dx}{x^2 + 1} = -\frac{1}{2}\left[(x^2 - 1)\, K_0(x^2 + 1) + K_1(x^2 + 1)\right]$$

$$\int \frac{x^5 K_1(x^2 - 1)\, dx}{x^2 - 1} = -\frac{1}{2}\left[(x^2 + 1)\, K_0(x^2 - 1) + K_1(x^2 - 1)\right]$$

$$\int \frac{x^7 J_1(x^2 + \sqrt{3}\, i)\, dx}{x^2 + \sqrt{3}\, i} = \frac{1}{2}\left[(-x^4 + \sqrt{3}\, i\, x + 3)\, J_0(x^2 + \sqrt{3}\, i) + (2x^2 - \sqrt{3}\, i) J_1(x^2 + \sqrt{3}\, i)\right]$$

$$\int \frac{x^7 J_1(x^2 - \sqrt{3}\, i)\, dx}{x^2 - \sqrt{3}\, i} = \frac{1}{2}\left[(-x^4 - \sqrt{3}\, i\, x + 3)\, J_0(x^2 - \sqrt{3}\, i) + (2x^2 + \sqrt{3}\, i) J_1(x^2 - \sqrt{3}\, i)\right]$$

$$\int \frac{x^7 I_1(x^2 + \sqrt{3})\, dx}{x^2 + \sqrt{3}} = \frac{1}{2}\left[(x^4 - \sqrt{3}x^2 + 3)\, I_0(x^2 + \sqrt{3}) + (-2\, x^2 + \sqrt{3}) I_1(x^2 + \sqrt{3})\right]$$

$$\int \frac{x^7 I_1(x^2 - \sqrt{3})\, dx}{x^2 - \sqrt{3}} = \frac{1}{2}\left[(x^4 + \sqrt{3}x^2 + 3)\, I_0(x^2 - \sqrt{3}) - (2\, x^2 + \sqrt{3}) I_1(x^2 - \sqrt{3})\right]$$

$$\int \frac{x^7 K_1(x^2 + \sqrt{3})\, dx}{x^2 + \sqrt{3}} = \frac{1}{2}\left[-(x^4 - \sqrt{3}x^2 + 3)\, K_0(x^2 + \sqrt{3}) + (-2\, x^2 + \sqrt{3}) K_1(x^2 + \sqrt{3})\right]$$

$$\int \frac{x^7 K_1(x^2 - \sqrt{3})\, dx}{x^2 - \sqrt{3}} = -\frac{1}{2}\left[(x^4 + \sqrt{3}x^2 + 3)\, K_0(x^2 - \sqrt{3}) + (2\, x^2 + \sqrt{3}) K_1(x^2 - \sqrt{3})\right]$$

$\sqrt{2\sqrt{3} - 3} = 0.68125\ 00386$

$$\int \frac{x^9 J_1(x^2 + \sqrt{2\sqrt{3} - 3}\,)\, dx}{x^2 + \sqrt{2\sqrt{3} - 3}} =$$

$$= \frac{1}{2}\left[\left(-x^6 + \sqrt{2\sqrt{3} - 3}\, x^4 + 6\, x^2 - 2\sqrt{3}x^2 + 2\sqrt{2\sqrt{3} - 3}\sqrt{3}\right) J_0\left(x^2 + \sqrt{2\sqrt{3} - 3}\,\right) + \right.$$

$$\left. + \left(3\, x^4 - 2\sqrt{2\sqrt{3} - 3}\, x^2 - 6 + 2\sqrt{3}\right) J_1\left(x^2 + \sqrt{2\sqrt{3} - 3}\,\right)\right]$$

$$\int \frac{x^9 J_1(x^2 - \sqrt{2\sqrt{3} - 3}\,)\, dx}{x^2 - \sqrt{2\sqrt{3} - 3}} =$$

$$= \frac{1}{2}\left[-\left(x^6 + \sqrt{2\sqrt{3} - 3}\, x^4 - 6\, x^2 + 2\sqrt{3}x^2 + 2\sqrt{2\sqrt{3} - 3}\sqrt{3}\right) J_0\left(x^2 - \sqrt{2\sqrt{3} - 3}\,\right) + \right.$$

$$\left. + \left(3\, x^4 + 2\sqrt{2\sqrt{3} - 3}\, x^2 - 6 + 2\sqrt{3}\right) J_1\left(x^2 - \sqrt{2\sqrt{3} - 3}\,\right)\right]$$

$$\int \frac{x^9 I_1(x^2 + \sqrt{2\sqrt{3} - 3}\, i)\, dx}{x^2 + \sqrt{2\sqrt{3} - 3}\, i} =$$

$$= \frac{1}{2}\left[\left(x^6 - \sqrt{2\sqrt{3} - 3}\, i\, x^4 + 6\, x^2 - 2\sqrt{3}x^2 + 2i\sqrt{2\sqrt{3} - 3}\sqrt{3}\right) I_0\left(x^2 + \sqrt{2\sqrt{3} - 3}\, i\,\right) + \right.$$

$$+ \left(-3\,x^4 + 2i\,\sqrt{2\sqrt{3}-3}\,x^2 - 6 + 2\sqrt{3} \right) I_1\left(x^2 + \sqrt{2\sqrt{3}-3}\,i \right) \Bigg]$$

$$\int \frac{x^9\,I_1(x^2 - \sqrt{2\sqrt{3}-3}\,i)\,dx}{x^2 - \sqrt{2\sqrt{3}-3}\,i} =$$

$$= \frac{1}{2}\left[\left(x^6 + \sqrt{2\sqrt{3}-3}\,i\,x^4 + 6\,x^2 - 2\sqrt{3}x^2 - 2i\,\sqrt{2\sqrt{3}-3}\sqrt{3} \right) I_0\left(x^2 - \sqrt{2\sqrt{3}-3}\,i \right) + \right.$$

$$\left. + \left(-3\,x^4 - 2i\,\sqrt{2\sqrt{3}-3}\,x^2 - 6 + 2\sqrt{3} \right) I_1\left(x^2 - \sqrt{2\sqrt{3}-3}\,i \right) \right]$$

$$\int \frac{x^9\,K_1(x^2 + \sqrt{2\sqrt{3}-3}\,i)\,dx}{x^2 + \sqrt{2\sqrt{3}-3}\,i} =$$

$$= \frac{1}{2}\left[\left(-x^6 + \sqrt{2\sqrt{3}-3}\,i\,x^4 - 6\,x^2 + 2\sqrt{3}x^2 - 2\sqrt{2\sqrt{3}-3}\sqrt{3}\,i \right) K_0\left(x^2 + \sqrt{2\sqrt{3}-3}\,i \right) + \right.$$

$$\left. + \left(-3\,x^4 + 2\sqrt{2\sqrt{3}-3}\,i\,x^2 - 6 + 2\sqrt{3} \right) K_1\left(x^2 + \sqrt{2\sqrt{3}-3}\,i \right) \right]$$

$$\int \frac{x^9\,K_1(x^2 - \sqrt{2\sqrt{3}-3}\,i)\,dx}{x^2 - \sqrt{2\sqrt{3}-3}\,i} =$$

$$= \frac{1}{2}\left[\left(-x^6 - \sqrt{2\sqrt{3}-3}\,i\,x^4 - 6\,x^2 + 2\sqrt{3}x^2 + 2\sqrt{2\sqrt{3}-3}\sqrt{3}\,i \right) K_0\left(x^2 - \sqrt{2\sqrt{3}-3}\,i \right) + \right.$$

$$\left. + \left(-3\,x^4 - 2\sqrt{2\sqrt{3}-3}\,i\,x^2 - 6 + 2\sqrt{3} \right) K_1\left(x^2 - \sqrt{2\sqrt{3}-3}\,i \right) \right]$$

$$\sqrt{2\sqrt{3}+3} = 2.54245\,97568$$

$$\int \frac{x^9\,J_1(x^2 + \sqrt{2\sqrt{3}+3}\,i)\,dx}{x^2 + \sqrt{2\sqrt{3}+3}\,i} =$$

$$= \frac{1}{2}\left[\left(-x^6 + \sqrt{2\sqrt{3}+3}\,i\,x^4 + 6\,x^2 + 2\sqrt{3}x^2 - 2\sqrt{2\sqrt{3}+3}\,\sqrt{3}\,i \right) J_0\left(x^2 + \sqrt{2\sqrt{3}+3}\,i \right) + \right.$$

$$\left. + \left(3\,x^4 - 2\sqrt{2\sqrt{3}+3}\,i\,x^2 - 6 - 2\sqrt{3} \right) J_1\left(x^2 + \sqrt{2\sqrt{3}+3}\,i \right) \right]$$

$$\int \frac{x^9\,J_1(x^2 - \sqrt{2\sqrt{3}+3}\,i)\,dx}{x^2 - \sqrt{2\sqrt{3}+3}\,i} =$$

$$= \frac{1}{2}\left[\left(-x^6 - \sqrt{2\sqrt{3}+3}\,i\,x^4 + 6\,x^2 + 2\sqrt{3}x^2 + 2\sqrt{2\sqrt{3}+3}\,\sqrt{3}\,i \right) J_0\left(x^2 - \sqrt{2\sqrt{3}+3}\,i \right) + \right.$$

$$\left. + \left(3\,x^4 + 2\sqrt{2\sqrt{3}+3}\,i\,x^2 - 6 - 2\sqrt{3} \right) J_1\left(x^2 - \sqrt{2\sqrt{3}+3}\,i \right) \right]$$

$$\int \frac{x^9\,I_1(x^2 + \sqrt{2\sqrt{3}+3})\,dx}{x^2 + \sqrt{2\sqrt{3}+3}} =$$

$$= \frac{1}{2}\left[\left(x^6 - \sqrt{2\sqrt{3}+3}\,x^4 + 6\,x^2 + 2\sqrt{3}x^2 - 2\sqrt{2\sqrt{3}+3}\,\sqrt{3} \right) I_0\left(x^2 + \sqrt{2\sqrt{3}+3} \right) - \right.$$

$$\left. - \left(3\,x^4 - 2\sqrt{2\sqrt{3}+3}\,x^2 + 6 + 2\sqrt{3} \right) I_1\left(x^2 + \sqrt{2\sqrt{3}+3} \right) \right]$$

$$\int \frac{x^9\,I_1(x^2 - \sqrt{2\sqrt{3}+3})\,dx}{x^2 - \sqrt{2\sqrt{3}+3}} =$$

$$= \frac{1}{2}\left[\left(x^6 + \sqrt{2\sqrt{3}+3}\,x^4 + 6\,x^2 + 2\sqrt{3}x^2 + 2\sqrt{2\sqrt{3}+3}\,\sqrt{3} \right) I_0\left(x^2 - \sqrt{2\sqrt{3}+3} \right) - \right.$$

$$- \left(3\,x^4 + 2\sqrt{2\sqrt{3}+3}\,x^2 + 6 + 2\sqrt{3} \right) I_1 \left(x^2 - \sqrt{2\sqrt{3}+3}\, \right) \Big]$$

$$\int \frac{x^9\,K_1(x^2 + \sqrt{2\sqrt{3}+3}\,)\,dx}{x^2 + \sqrt{2\sqrt{3}+3}} =$$

$$= -\frac{1}{2} \left[\left(x^6 - \sqrt{2\sqrt{3}+3}\,x^4 + 6\,x^2 + 2\sqrt{3}x^2 - 2\sqrt{2\sqrt{3}+3}\,\sqrt{3} \right) K_0 \left(x^2 + \sqrt{2\sqrt{3}+3}\, \right) + \right.$$

$$\left. + \left(3\,x^4 - 2\sqrt{2\sqrt{3}+3}\,x^2 + 6 + 2\sqrt{3} \right) K_1 \left(x^2 + \sqrt{2\sqrt{3}+3}\, \right) \right]$$

$$\int \frac{x^9\,K_1(x^2 - \sqrt{2\sqrt{3}+3}\,)\,dx}{x^2 - \sqrt{2\sqrt{3}+3}} =$$

$$= -\frac{1}{2} \left[\left(x^6 + \sqrt{2\sqrt{3}+3}\,x^4 + 6\,x^2 + 2\sqrt{3}x^2 + 2\sqrt{2\sqrt{3}+3}\,\sqrt{3} \right) K_0 \left(x^2 - \sqrt{2\sqrt{3}+3}\, \right) + \right.$$

$$\left. + \left(3\,x^4 + 2\sqrt{2\sqrt{3}+3}\,x^2 + 6 + 2\sqrt{3} \right) K_1 \left(x^2 - \sqrt{2\sqrt{3}+3}\, \right) \right]$$

$$\sqrt{2\sqrt{10}-5} = 1.15089\ 32706$$

$$\int \frac{x^{11}\,J_1(x^2 + \sqrt{2\sqrt{10}-5}\,)\,dx}{x^2 + \sqrt{2\sqrt{10}-5}} = \frac{1}{2} \left[\left(-x^8 + \sqrt{2\sqrt{10}-5}\,x^6 + 13\,x^4 - 2\sqrt{10}x^4 - 4\sqrt{2\sqrt{10}-5}\,x^2 + \right. \right.$$

$$\left. + 2\sqrt{2\sqrt{10}-5}\,\sqrt{10}x^2 + 6\sqrt{10} - 30 \right) J_0 \left(x^2 + \sqrt{2\sqrt{10}-5}\, \right) +$$

$$+ \left(4\,x^6 - 3\sqrt{2\sqrt{10}-5}\,x^4 - 26\,x^2 + 4\sqrt{10}x^2 + 4\sqrt{2\sqrt{10}-5} - 2\sqrt{2\sqrt{10}-5}\,\sqrt{10} \right) J_1 \left(x^2 + \sqrt{2\sqrt{10}-5}\, \right) \Big]$$

$$\int \frac{x^{11}\,J_1(x^2 - \sqrt{2\sqrt{10}-5}\,)\,dx}{x^2 - \sqrt{2\sqrt{10}-5}} = \frac{1}{2} \left[\left(-x^8 - \sqrt{2\sqrt{10}-5}\,x^6 + 13\,x^4 - 2\sqrt{10}x^4 + 4\sqrt{2\sqrt{10}-5}\,x^2 - \right. \right.$$

$$\left. - 2\sqrt{2\sqrt{10}-5}\,\sqrt{10}x^2 + 6\sqrt{10} - 30 \right) J_0 \left(x^2 - \sqrt{2\sqrt{10}-5}\, \right) +$$

$$+ \left(4\,x^6 + 3\sqrt{2\sqrt{10}-5}\,x^4 - 26\,x^2 + 4\sqrt{10}x^2 - (4 - 2\sqrt{10})\sqrt{2\sqrt{10}-5} \right) J_1 \left(x^2 - \sqrt{2\sqrt{10}-5}\, \right) \Big]$$

$$\int \frac{x^{11}\,I_1(x^2 + \sqrt{2\sqrt{10}-5}\,i)\,dx}{x^2 + \sqrt{2\sqrt{10}-5}\,i} = \frac{1}{2} \left[\left(x^8 - \sqrt{2\sqrt{10}-5}\,i\,x^6 + 13\,x^4 - 2\sqrt{10}x^4 - 4\sqrt{2\sqrt{10}-5}\,i\,x^2 + \right. \right.$$

$$\left. + 2\sqrt{2\sqrt{10}-5}\,\sqrt{10}\,i\,x^2 - 6\sqrt{10} + 30 \right) I_0 \left(x^2 + \sqrt{2\sqrt{10}-5}\,i\, \right) +$$

$$+ \left(-4\,x^6 + 3\sqrt{2\sqrt{10}-5}\,i\,x^4 - 26\,x^2 + 4\sqrt{10}x^2 + (4 - 2\sqrt{10})\sqrt{2\sqrt{10}-5}\,i \right) I_1 \left(x^2 + \sqrt{2\sqrt{10}-5}\,i\, \right) \Big]$$

$$\int \frac{x^{11}\,I_1(x^2 - \sqrt{2\sqrt{10}-5}\,i)\,dx}{x^2 - \sqrt{2\sqrt{10}-5}\,i} = \frac{1}{2} \left[\left(x^8 + \sqrt{2\sqrt{10}-5}\,i\,x^6 + 13\,x^4 - 2\sqrt{10}x^4 + 4\sqrt{2\sqrt{10}-5}\,i\,x^2 + \right. \right.$$

$$\left. - 2\sqrt{2\sqrt{10}-5}\,\sqrt{10}\,i\,x^2 - 6\sqrt{10} + 30 \right) I_0 \left(x^2 - \sqrt{2\sqrt{10}-5}\,i\, \right) +$$

$$+ \left(-4\,x^6 - 3\sqrt{2\sqrt{10}-5}\,i\,x^4 - 26\,x^2 + 4\sqrt{10}x^2 - (4 - 2\sqrt{10})\sqrt{2\sqrt{10}-5}\,i \right) I_1 \left(x^2 - \sqrt{2\sqrt{10}-5}\,i\, \right) \Big]$$

$$\int \frac{x^{11}\,K_1(x^2 + \sqrt{2\sqrt{10}-5}\,i)\,dx}{x^2 + \sqrt{2\sqrt{10}-5}\,i} = \frac{1}{2} \left[\left(-x^8 + \sqrt{2\sqrt{10}-5}\,i\,x^6 - 13\,x^4 + 2\sqrt{10}x^4 + 4\sqrt{2\sqrt{10}-5}\,i\,x^2 - \right. \right.$$

$$-2\sqrt{2\sqrt{10}-5}\sqrt{10}\,i\,x^2+6\sqrt{10}-30\Big)\,K_0\left(x^2+\sqrt{2\sqrt{10}-5}\,i\,\right)+$$

$$+\left(-4x^6+3\sqrt{2\sqrt{10}-5}\,i\,x^4-26\,x^2+4\sqrt{10}x^2+(4-2\sqrt{10})\sqrt{2\sqrt{10}-5}\,i\,\right)K_1\left(x^2+\sqrt{2\sqrt{10}-5}\,i\,\right)\Big]$$

$$\int\frac{x^{11}\,K_1(x^2-\sqrt{2\sqrt{10}-5}\,i)\,dx}{x^2-\sqrt{2\sqrt{10}-5}\,i}=\frac{1}{2}\left[\left(-x^8-\sqrt{2\sqrt{10}-5}\,i\,x^6-13\,x^4+2\sqrt{10}x^4-4\sqrt{2\sqrt{10}-5}\,i\,x^2+\right.\right.$$

$$\left.+2\sqrt{2\sqrt{10}-5}\sqrt{10}\,i\,x^2+6\sqrt{10}-30\right)K_0\left(x^2-\sqrt{2\sqrt{10}-5}\,i\,\right)+$$

$$+\left(-4x^6-3\sqrt{2\sqrt{10}-5}\,i\,x^4-26\,x^2+4\sqrt{10}x^2+(2\sqrt{10}-4)\sqrt{2\sqrt{10}-5}\,i\,\right)K_1\left(x^2-\sqrt{2\sqrt{10}-5}\,i\,\right)\Big]$$

$$\boxed{\lambda=\sqrt{2\sqrt[3]{25}+4\sqrt[3]{5}+5}=4.20570\,31830}$$

$$\int\frac{x^{13}\,J_1(x^2+\lambda i\,)\,dx}{x^2+\lambda i}=\frac{1}{2}\left\{-x^{10}+\lambda i\,x^8+2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6-\lambda i\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4-\right.$$

$$-[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2+12\lambda i\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\,J_0(x^2+\lambda i\,)+\frac{1}{2}\left\{5x^8-4\lambda i\,x^6-6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4+\right.$$

$$+4\lambda i\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2+36\sqrt[3]{5}+24\sqrt[3]{25}+120\Big\}\,J_1(x^2+\lambda i\,)$$

$$\int\frac{x^{13}\,J_1(x^2-\lambda i\,)\,dx}{x^2-\lambda i}=\frac{1}{2}\left\{-x^{10}-\lambda i\,x^8+2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6+\lambda i\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4-\right.$$

$$-[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2-12\lambda i\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\,J_0(x^2-\lambda i\,)+\frac{1}{2}\left\{5x^8+4\lambda i\,x^6-6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4-\right.$$

$$-4\lambda i\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2+36\sqrt[3]{5}+24\sqrt[3]{25}+120\Big\}\,J_1(x^2-\lambda i\,)$$

$$\int\frac{x^{13}\,I_1(x^2+\lambda\,)\,dx}{x^2+\lambda}=\frac{1}{2}\left\{x^{10}-\lambda x^8+2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6-\lambda\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4+\right.$$

$$+[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2-12\lambda\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\,J_0(x^2+\lambda\,)+\frac{1}{2}\left\{-5x^8+4\lambda x^6-6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4+\right.$$

$$+4\lambda\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2-36\sqrt[3]{5}-24\sqrt[3]{25}-120\Big\}\,I_1(x^2+\lambda\,)$$

$$\int\frac{x^{13}\,I_1(x^2-\lambda\,)\,dx}{x^2-\lambda}=\frac{1}{2}\left\{x^{10}+\lambda x^8+2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6+\lambda\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4+\right.$$

$$+[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2+12\lambda\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\,J_0(x^2-\lambda\,)-\frac{1}{2}\left\{5x^8+4\lambda\,x^6 6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4+\right.$$

$$+4\lambda\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2+36\sqrt[3]{5}+24\sqrt[3]{25}+120\Big\}\,I_1(x^2-\lambda\,)$$

$$\int\frac{x^{13}\,K_1(x^2+\lambda\,)\,dx}{x^2+\lambda}=\frac{1}{2}\left\{-x^{10}+\lambda x^8-2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6+\lambda\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4-\right.$$

$$-[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2+12\lambda\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\,K_0(x^2+\lambda\,)+\frac{1}{2}\left\{-5x^8+4\lambda x^6-6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4+\right.$$

$$+4\lambda\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2-36\sqrt[3]{5}-24\sqrt[3]{25}-120\Big\}\,K_1(x^2+\lambda\,)$$

$$\int\frac{x^{13}\,K_1(x^2-\lambda)\,dx}{x^2-\lambda}=-\frac{1}{2}\left\{x^{10}+\lambda x^8+2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6+\lambda\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4+\right.$$

$$+[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2+12\lambda\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\,K_0(x^2-\lambda)-\frac{1}{2}\left\{5x^8+4\lambda\,x^6 6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4+\right.$$

$$+4\lambda\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2+36\sqrt[3]{5}+24\sqrt[3]{25}+120\Big\}\,K_1(x^2-\lambda)$$

$$\mu = \sqrt{5 - 2\sqrt[3]{5} - \sqrt[3]{25} + \sqrt{3}\left(2\sqrt[3]{5} - \sqrt[3]{25}\right)i} \; = 0.35429\,92342 + 1.21222\,83526\,i$$

$$\int \frac{x^{13}\,J_1(x^2 + \mu i)\,dx}{x^2 + \mu i} = \frac{1}{2}\left\{-x^{10} + \mu i x^8 + \left[(2\sqrt{3}\,i - 2)\sqrt[3]{5} - (1 + \sqrt{3}\,i)\sqrt[3]{25} + 20\right]x^6 + \right.$$

$$+ \left[2(\sqrt{3} + i)\sqrt[3]{5} + (i - \sqrt{3})\sqrt[3]{25} - 8i\right]\mu\,x^4 + \left[18(1 - \sqrt{3}\,i)\sqrt[3]{5} + 12(1 + \sqrt{3}\,i)\sqrt[3]{25} - 120\right]x^2 -$$

$$\left. -6\left[(\sqrt{3} + i)\sqrt[3]{5} - (\sqrt{3} - i)\sqrt[3]{25}\right]\mu\right\} J_0(x^2 + \mu i) +$$

$$+ \frac{1}{2}\left\{5x^8 - 4\mu i x^6 + 3\left[2(1 - \sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25} - 20\right]x^4 - \right.$$

$$-2\left[2(\sqrt{3} + i)\sqrt[3]{5} - (\sqrt{3} - i)\sqrt[3]{25} - 8i\right]\mu\,x^2 + \left[18(\sqrt{3}\,i - 1)\sqrt[3]{5} - 12(1 + \sqrt{3}\,i)\sqrt[3]{25} + 120\right]\right\} J_1(x^2 + \mu i)$$

$$\int \frac{x^{13}\,J_1(x^2 - \mu i)\,dx}{x^2 - \mu i} = \frac{1}{2}\left\{-x^{10} - \mu i x^8 + \left[(2\sqrt{3}\,i - 2)\sqrt[3]{5} - (1 + \sqrt{3}\,i)\sqrt[3]{25} + 20\right]x^6 - \right.$$

$$-\left[2(\sqrt{3} + i)\sqrt[3]{5} + (i - \sqrt{3})\sqrt[3]{5} - 8\right]\mu\,x^4 + \left[18(1 - \sqrt{3}\,i)\sqrt[3]{5} + 12(1 + \sqrt{3}\,i)\sqrt[3]{25} - 120\right]x^2 +$$

$$\left. +6\left[(1 + \sqrt{3}\,i)\sqrt[3]{5} - (\sqrt{3} - i)\sqrt[3]{25}\right]\mu\right\} J_0(x^2 - \mu i) +$$

$$+ \frac{1}{2}\left\{5x^8 + 4\mu i x^6 + \left[6(1 - \sqrt{3}\,i)\sqrt[3]{5} + 3(1 + \sqrt{3}\,i)\sqrt[3]{25} - 60\right]x^4 + \right.$$

$$+2\left[2(\sqrt{3} + i)\sqrt[3]{5} - (\sqrt{3} - i)\sqrt[3]{25} - 8i\right]\mu\,x^2 + \left[18(\sqrt{3}\,i - 1)\sqrt[3]{5} - 12(1 + \sqrt{3}\,i)\sqrt[3]{25} + 120\right]\right\} J_1(x^2 - \mu i)$$

$$\int \frac{x^{13}\,I_1(x^2 + \mu)\,dx}{x^2 + \mu} = \frac{1}{2}\left\{x^{10} - \mu\,x^8 + \left[(2\sqrt{3}\,i - 2)\sqrt[3]{5} - (1 + \sqrt{3}\,i)\sqrt[3]{25} + 20\right]x^6 + \right.$$

$$+ \left[(2 - 2\sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25} - 8\right]\mu\,x^4 + \left[18(\sqrt{3}\,i - 1)\sqrt[3]{5} - 12(1 + \sqrt{3}\,i)\sqrt[3]{25} + 120\right]x^2 + $$

$$\left. +6\left[(1 - \sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25}\right]\mu\right\} I_0(x^2 + \mu) +$$

$$+ \frac{1}{2}\left\{-5x^8 + 4\mu\,x^6 + \left[6(1 - \sqrt{3}\,i)\sqrt[3]{5} + 3(1 + \sqrt{3}\,i)\sqrt[3]{25} - 60\right]x^4 + \right.$$

$$+2\left[2(\sqrt{3}\,i - 1)\sqrt[3]{5} - (1 + \sqrt{3}\,i)\sqrt[3]{25} + 8\right]\mu\,x^2 + \left[18(1 - \sqrt{3}\,i)\sqrt[3]{5} + 12(1 + \sqrt{3}\,i)\sqrt[3]{25} - 120\right]\right\} I_1(x^2 + \mu)$$

$$\int \frac{x^{13}\,I_1(x^2 - \mu)\,dx}{x^2 - \mu} = \frac{1}{2}\left\{x^{10} + \mu\,x^8 + \left[(2\sqrt{3}\,i - 2)\sqrt[3]{5} - (1 + \sqrt{3}\,i)\sqrt[3]{25} + 20\right]x^6 - \right.$$

$$-\left[(2 - 2\sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25} - 8\right]\mu\,x^4 + \left[18(\sqrt{3}\,i - 1)\sqrt[3]{5} - 12(1 + \sqrt{3}\,i)\sqrt[3]{25} + 120\right]x^2 - $$

$$\left. -6\left[(1 - \sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25}\right]\mu\right\} I_0(x^2 - \mu) +$$

$$+ \frac{1}{2}\left\{-5x^8 - 4\mu\,x^6 + \left[6(1 - \sqrt{3}\,i)\sqrt[3]{5} + 3(1 + \sqrt{3}\,i)\sqrt[3]{25} - 60\right]x^4 - \right.$$

$$-2\left[2(1 - \sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25} - 8\right]\mu\,x^2 + \left[18(1 - \sqrt{3}\,i)\sqrt[3]{5} + 12(1 + \sqrt{3}\,i)\sqrt[3]{25} - 120\right]\right\} I_1(x^2 - \mu)$$

$$\int \frac{x^{13}\,K_1(x^2 + \mu)\,dx}{x^2 + \mu} = \frac{1}{2}\left\{-x^{10} + \mu\,x^8 - \left[(2\sqrt{3}\,i - 2)\sqrt[3]{5} - (1 + \sqrt{3}\,i)\sqrt[3]{25} + 20\right]x^6 - \right.$$

$$-\left[(2 - 2\sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25} - 8\right]\mu\,x^4 - \left[18(\sqrt{3}\,i - 1)\sqrt[3]{5} - 12(1 + \sqrt{3}\,i)\sqrt[3]{25} + 120\right]x^2 - $$

$$\left. -6\left[(1 - \sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25}\right]\mu\right\} K_0(x^2 + \mu) +$$

$$+ \frac{1}{2}\left\{-5x^8 + 4\mu\,x^6 + \left[6(1 - \sqrt{3}\,i)\sqrt[3]{5} + 3(1 + \sqrt{3}\,i)\sqrt[3]{25} - 60\right]x^4 + \right.$$

$$+2\left[2(\sqrt{3}\,i - 1)\sqrt[3]{5} - (1 + \sqrt{3}\,i)\sqrt[3]{25} + 8\right]\mu\,x^2 + \left[18(1 - \sqrt{3}\,i)\sqrt[3]{5} + 12(1 + \sqrt{3}\,i)\sqrt[3]{25} - 120\right]\right\} K_1(x^2 + \mu)$$

$$\int \frac{x^{13} K_1(x^2 - \mu)\, dx}{x^2 - \mu} = \frac{1}{2}\left\{ -x^{10} - \mu\, x^8 + \left[(2 - 2\sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25} - 20\right] x^6 - \right.$$

$$- \left[(2 - 2\sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25} - 8\right] \mu\, x^4 + \left[18(1 - \sqrt{3}\,i)\sqrt[3]{5} + 12(1 + \sqrt{3}\,i)\sqrt[3]{25} - 120\right] x^2 +$$

$$\left. +6\left[(1 - \sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25}\right] \mu \right\} K_0(x^2 - \mu) +$$

$$+ \frac{1}{2}\left\{ -5x^8 - 4\mu\, x^6 + \left[6(1 - \sqrt{3}\,i)\sqrt[3]{5} + 3(1 + \sqrt{3}\,i)\sqrt[3]{25} - 60\right] x^4 - \right.$$

$$-2\left[2(1 - \sqrt{3}\,i)\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25} - 8\right] \mu\, x^2 + \left[18(1 - \sqrt{3}\,i)\sqrt[3]{5} + 12(1 + \sqrt{3}\,i)\sqrt[3]{25} - 120\right] \right\} K_1(x^2 - \mu)$$

$$\boxed{\nu = \sqrt{5 - 2\sqrt[3]{5} - \sqrt[3]{25} - \sqrt{3}\,(2\sqrt[3]{5} - \sqrt[3]{25})\, i} \ = 0.35429\,92342 - 1.21222\,83526\, i}$$

$$\int \frac{x^{13} J_1(x^2 + \nu i)\, dx}{x^2 + \nu i} = \frac{1}{2}\left\{ -x^{10} + \nu i x^8 - \left[(2 + 2\sqrt{3}\,i)\sqrt[3]{5} + (1 - \sqrt{3}\,i)\sqrt[3]{25} - 20\right] x^6 + \right.$$

$$+ \left[(2(i - \sqrt{3})\sqrt[3]{5} + (\sqrt{3} + i)\sqrt[3]{25} - 8i\right] \nu\, x^4 + \left[18(1 + \sqrt{3}\,i)\sqrt[3]{5} + 12(1 - \sqrt{3}\,i)\sqrt[3]{25} - 120\right] x^2 -$$

$$\left. -6\left[(i - \sqrt{3})\sqrt[3]{5} + (1 + \sqrt{3}\,i)\sqrt[3]{25}\right] \nu \right\} J_0(x^2 + \nu i) +$$

$$+ \frac{1}{2}\left\{ 5x^8 - 4\nu i x^6 + \left[(6 + 6\sqrt{3}\,i)\sqrt[3]{5} + 3(1 - \sqrt{3}\,i)\sqrt[3]{25} - 60\right] x^4 - \right.$$

$$-2\left[2(i - \sqrt{3}\,i)\sqrt[3]{5} + (\sqrt{3} + i)\sqrt[3]{25} - 8i\right] \nu\, x^2 - \left[18(\sqrt{3}\,i + 1)\sqrt[3]{5} + 12(1 - \sqrt{3}\,i)\sqrt[3]{25} - 120\right] \right\} J_1(x^2 + \nu i)$$

$$\int \frac{x^{13} J_1(x^2 - \nu i)\, dx}{x^2 - \nu i} = \frac{1}{2}\left\{ -x^{10} - \nu i x^8 - \left[(2 + 2\sqrt{3}\,i)\sqrt[3]{5} + (1 - \sqrt{3}\,i)\sqrt[3]{25} - 20\right] x^6 - \right.$$

$$- \left[2(i - \sqrt{3})\sqrt[3]{5} + (\sqrt{3} + i)\sqrt[3]{25} - 8i\right] \nu\, x^4 + \left[18(1 + \sqrt{3}\,i)\sqrt[3]{5} + 12(1 - \sqrt{3}\,i)\sqrt[3]{25} - 120\right] x^2 +$$

$$\left. +6\left[(i - \sqrt{3})\sqrt[3]{5} + (\sqrt{3} + i)\sqrt[3]{25}\right] \nu \right\} J_0(x^2 - \nu i) +$$

$$+ \frac{1}{2}\left\{ 5x^8 + 4\nu i x^6 + \left[6(1 + \sqrt{3}\,i)\sqrt[3]{5} - 3(1 - \sqrt{3}\,i)\sqrt[3]{25} - 60\right] x^4 + \right.$$

$$+2\left[2(i - \sqrt{3})\sqrt[3]{5} + (\sqrt{3} + i)\sqrt[3]{25} - 8i\right] \nu\, x^2 - \left[18(\sqrt{3}\,i + 1)\sqrt[3]{5} - 12(1 - \sqrt{3}\,i)\sqrt[3]{25} - 120\right] \right\} J_1(x^2 - \nu i)$$

$$\int \frac{x^{13} I_1(x^2 + \nu)\, dx}{x^2 + \nu} = \frac{1}{2}\left\{ x^{10} - \nu x^8 - \left[(2 + 2\sqrt{3}\,i)\sqrt[3]{5} + (1 - \sqrt{3}\,i)\sqrt[3]{25} - 20\right] x^6 + \right.$$

$$+ \left[(2\sqrt{3}\,i + 2)\sqrt[3]{5} + (1 - \sqrt{3}\,i)\sqrt[3]{25} - 8\right] \nu\, x^4 - \left[18(1 + \sqrt{3}\,i)\sqrt[3]{5} + 12(1 - \sqrt{3}\,i)\sqrt[3]{25} - 120\right] x^2 +$$

$$\left. +6\left[(\sqrt{3}\,i + 1)\sqrt[3]{5} + (1 - \sqrt{3}\,i)\sqrt[3]{25}\right] \nu \right\} I_0(x^2 + \nu) +$$

$$+ \frac{1}{2}\left\{ -5x^8 + 4\nu\, x^6 + \left[(6 + 6\sqrt{3}\,i)\sqrt[3]{5} + 3(1 - \sqrt{3}\,i)\sqrt[3]{25} - 60\right] x^4 - \right.$$

$$-2\left[2(1 + \sqrt{3}\,i)\sqrt[3]{5} - (\sqrt{3}\,i - 1)\sqrt[3]{25} - 8\right] \nu\, x^2 + \left[18(\sqrt{3}\,i + 1)\sqrt[3]{5} + 12(1 - \sqrt{3}\,i)\sqrt[3]{25} - 120\right] \right\} I_1(x^2 + \nu)$$

$$\int \frac{x^{13} I_1(x^2 - \nu)\, dx}{x^2 - \nu} = \frac{1}{2}\left\{ x^{10} + \nu x^8 - \left[(2 + 2\sqrt{3}\,i)\sqrt[3]{5} + (1 - \sqrt{3}\,i)\sqrt[3]{25} - 20\right] x^6 - \right.$$

$$- \left[(2\sqrt{3}\,i + 2)\sqrt[3]{5} + (1 - \sqrt{3}\,i)\sqrt[3]{25} - 8\right] \nu\, x^4 - \left[18(1 + \sqrt{3}\,i)\sqrt[3]{5} + 12(1 - \sqrt{3}\,i)\sqrt[3]{25} - 120\right] x^2 -$$

$$\left. -6\left[(\sqrt{3}\,i + 1)\sqrt[3]{5} + (1 - \sqrt{3}\,i)\sqrt[3]{25}\right] \nu \right\} I_0(x^2 - \nu) +$$

$$+ \frac{1}{2}\left\{ -5x^8 - 4\nu\, x^6 + \left[6(1 + \sqrt{3}\,i)\sqrt[3]{5} + 3(1 - \sqrt{3}\,i)\sqrt[3]{25} - 60\right] x^4 + \right.$$

$$+2\left[2(1+\sqrt{3}\,i)\sqrt[3]{5}+(\sqrt{3}\,i-1)\sqrt[3]{25}-8\right]\nu\,x^2+\left[18(\sqrt{3}\,i+1)\sqrt[3]{5}+12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\right]\Big\}\,I_1(x^2-\nu)$$

$$\int\frac{x^{13}\,K_1(x^2+\nu)\,dx}{x^2+\nu}=\frac{1}{2}\left\{-x^{10}+\nu x^8+\left[(2+2\sqrt{3}\,i)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}-20\right]x^6-\right.$$

$$-\left[(2\sqrt{3}\,i+2)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}-8\right]\nu\,x^4+\left[18(1+\sqrt{3}\,i)\sqrt[3]{5}+12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\right]x^2-$$

$$-6\left[(\sqrt{3}\,i+1)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}\right]\nu\Big\}\,K_0(x^2+\nu)+$$

$$+\frac{1}{2}\left\{-5x^8+4\nu\,x^6+\left[6(1+\sqrt{3}\,i)\sqrt[3]{5}+3(1-\sqrt{3}\,i)\sqrt[3]{25}-60\right]x^4-\right.$$

$$-2\left[2(1+\sqrt{3}\,i)\sqrt[3]{5}-(\sqrt{3}\,i-1)\sqrt[3]{25}-8\right]\nu\,x^2+\left[18(\sqrt{3}\,i+1)\sqrt[3]{5}+12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\right]\Big\}\,K_1(x^2+\nu)$$

$$\int\frac{x^{13}\,K_1(x^2-\nu)\,dx}{x^2-\nu}=\frac{1}{2}\left\{-x^{10}-\nu x^8+\left[(2+2\sqrt{3}\,i)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}-20\right]x^6+\right.$$

$$+\left[(2\sqrt{3}\,i+2)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}-8\right]\nu\,x^4+\left[18(1+\sqrt{3}\,i)\sqrt[3]{5}+12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\right]x^2+$$

$$+6\left[(\sqrt{3}\,i+1)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}\right]\nu\Big\}\,K_0(x^2-\nu)+$$

$$+\frac{1}{2}\left\{-5x^8-4\nu\,x^6+\left[6(1+\sqrt{3}\,i)\sqrt[3]{5}+3(1-\sqrt{3}\,i)\sqrt[3]{25}-60\right]x^4+\right.$$

$$+2\left[2(1+\sqrt{3}\,i)\sqrt[3]{5}-(\sqrt{3}\,i-1)\sqrt[3]{25}-8\right]\nu\,x^2+\left[18(\sqrt{3}\,i+1)\sqrt[3]{5}+12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\right]\Big\}\,K_1(x^2-\nu)$$

1.2. Elementary Function and Bessel Function

1.2.1. Integrals of the type $\int x^{n+1/2} \cdot Z_\nu(x)\, dx$

With the Lommel functions $s_{\mu,\nu}$ (see [7], 8.57, or [8], 10-7) holds:

$$\int \sqrt{x}\, J_0(x)\, dx = \sqrt{x}\, J_1(x) - \frac{x}{4}\left[2\,s_{-1/2,1}(x)\, J_0(x) + s_{-3/2,0}(x)\, J_1(x)\right] \ ,$$

$$\int \sqrt{x}\, J_1(x)\, dx = \frac{x}{2}\left[s_{-1/2,0}(x)\, J_1(x) - 2\,s_{1/2,0}(x)\, J_0(x)\right] .$$

$$x = t^2 \quad \Longrightarrow \quad \int x^{(2n-1)/2}\, J_\nu(x)\, dx = 2\int t^{2n}\, J_\nu(t^2)\, dt$$

Differential equations:

$$\int \sqrt{x}\, J_0(x)\, dx = y(x) \quad \Longrightarrow \quad x^2\, y''' + \left(x^2 + \frac{1}{4}\right) y' \;=\; 0$$

$$\int \sqrt{x}\, J_1(x)\, dx = z(x) \quad \Longrightarrow \quad x^2\, z''' + \left(x^2 - \frac{3}{4}\right) z' \;=\; 0$$
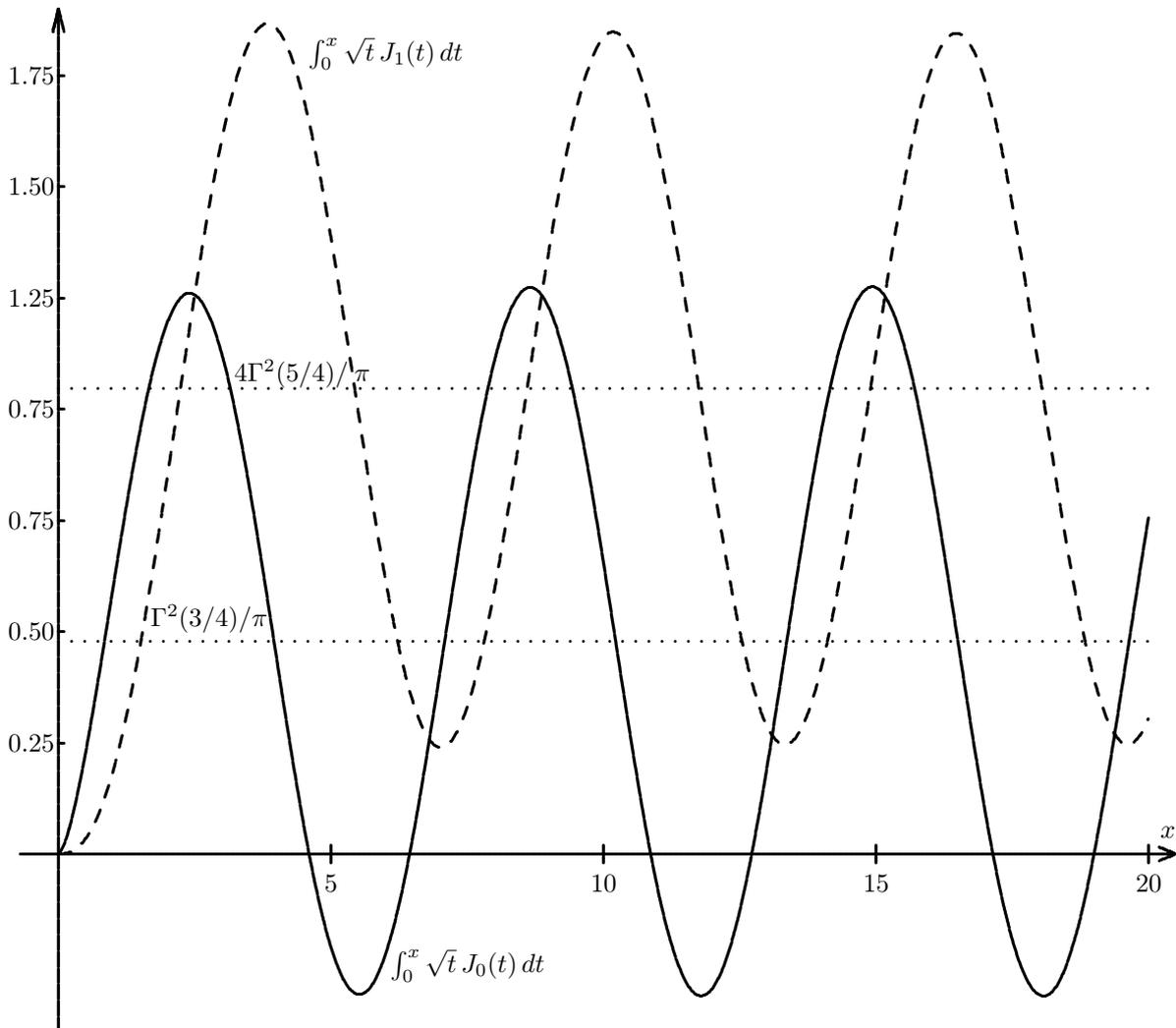
Function $J_0(x)$:

Approximation by Chebyshev polynomials, based on [2], 9.7.:

For $0 \leq x \leq 8$ holds

$$\int_0^x \sqrt{t}\, J_0(t)\, dt \quad \approx \quad \sqrt{x} \cdot \sum_{k=0}^{17} p_k\, T_{2k+1}\left(\frac{x}{8}\right)$$

with the following coefficients:

k	p_k	k	p_k
0	0.29396 17718 67412 06150	9	-0.00000 03762 56494 36038
1	-0.09593 33355 26137 75008	10	0.00000 00146 30781 61468
2	0.39583 39734 26816 07917	11	-0.00000 00004 68853 72762
3	-0.26902 16631 32696 96017	12	0.00000 00000 12605 05370
4	0.07963 03030 17678 07362	13	-0.00000 00000 00288 52942
5	-0.01366 63037 73087 91164	14	0.00000 00000 00005 69318
6	0.00155 43936 32776 04348	15	-0.00000 00000 00000 09787
7	-0.00012 67196 60682 08202	16	0.00000 00000 00000 00148
8	0.00000 77998 70507 77089	17	-0.00000 00000 00000 00002

This approximation differs from the true function as shown:
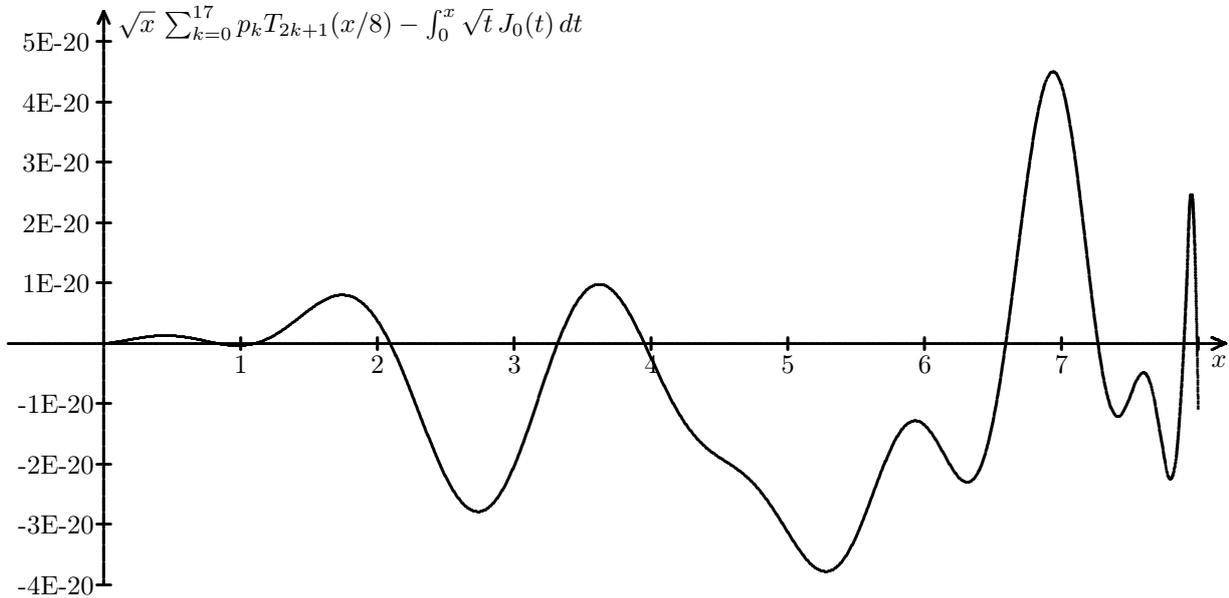
Asymptotic expansions for $x \to +\infty$:

$$\int_0^x \sqrt{t}\, J_0(t)\, dt \sim \frac{\Gamma^2(3/4)}{\pi} + \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{a_k}{x^k} \sin\left(x - \frac{2k+1}{4}\pi\right)$$

$$\frac{\Gamma^2(3/4)}{\pi} = 0.477\ 988\ 797\ 486\ 125$$

$$a_0 = 1\ , \quad a_1 = \frac{1}{8}\ , \quad a_2 = \frac{25}{128}\ , \quad a_3 = \frac{475}{1024}\ , \quad a_4 = \frac{49275}{32768}\ , \quad a_5 = \frac{1636335}{262144}\ , \quad a_6 = \frac{133308045}{4194304}\ ,$$

$$a_7 = \frac{6456759075}{33554432}\ , \quad a_8 = \frac{2905671971475}{2147483648}\ , \quad a_9 = \frac{186381860485275}{17179869184}\ , \ \ldots$$

k	a_k	a_k/a_{k-1}	k	a_k	a_k/a_{k-1}
0	1.000 000 000	-	5	6.242 122 650	4.1510
1	0.125 000 000	0.1250	6	31.783 114 67	5.0917
2	0.195 312 500	1.5625	7	192.426 415 5	6.0544
3	0.463 867 188	2.3750	8	1 353.058 951	7.0316
4	1.503 753 662	3.2418	9	10 848.852 14	8.0180

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Let

$$D_{0,n}(x) = \frac{\Gamma^2(3/4)}{\pi} + \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n} \frac{a_k}{x^k} \, \sin\left(x - \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t} \, J_0(t) \, dt \,,$$

then its first maximum and minimum values of interest are $D_{0,n}(x_{i,n}^*)$.
In the case $x > x_{i,n}^*$ holds $|D_{0,n}(x)| < |D_{0,n}(x_{i,n}^*)|$.

$n=0,\ i=$	1	2	3	4	5	6	7	8	9	10
$x_{i,0}^*$	1.143	4.058	7.146	10.264	13.394	16.527	19.664	22.801	25.940	29.079
$10^3 D_{0,0}(x_i^*)$	50.87	-21.784	13.293	-9.4707	7.3310	-5.9717	5.0342	-4.3496	3.8281	-3.4179

$n=1,\ i=$	1	2	3	4	5	6	7	8	9	10
$x_{i,1}^*$	2.473	5.561	8.681	11.812	14.947	18.085	21.223	24.363	27.503	30.643
$10^4 D_{0,1}(x_i^*)$	158.930	-43.0361	19.1585	-10.6855	6.7779	-4.6705	3.4092	-2.5962	2.0420	-1.6478

$n=2,\ i=$	3	4	5	6	7	8	9	10	11	12
$x_{i,2}^*$	7.100	10.233	13.369	16.508	19.647	22.787	25.927	29.068	32.209	35.350
$10^5 D_{0,2}(x_i^*)$	-85.8809	31.2037	-14.5367	7.8808	-4.7310	3.0556	-2.0851	1.4850	-1.0945	0.8296

$n=3,\ i=$	4	5	6	7	8	9	10	11	12	13
$x_{i,3}^*$	11.793	14.932	18.072	21.213	24.353	27.494	30.635	33.777	36.918	40.059
$10^7 D_{0,3}(x_i^*)$	548.133	-222.407	106.176	-56.7759	33.0053	-20.4558	13.3362	-9.0584	6.3648	-4.6012

$n=4,\ i=$	5	6	7	8	9	10	11	12	13	14
$x_{i,4}^*$	16.499	19.649	22.780	25.922	29.063	32.204	35.345	38.487	41.628	44.770
$10^8 D_{0,4}(x_i^*)$	-369.198	158.653	-76.8752	40.7761	-23.2093	13.9787	-8.8181	5.7815	-3.9164	2.7284

For $8 \le x \le 30$ the special approximation holds:

$$\int_0^x \sqrt{t} \, J_0(t) \, dt \approx 0.477\,988\,797\,935 + \sum_{k=0}^{9} \frac{c_k^{(0)}}{x^k} \, \sin\left(x - \frac{2k+1}{4}\pi\right)$$

with

$$-5.0 \cdot 10^{-9} < 0.477\,988\,797\,935 + \sum_{k=0}^{9} \frac{c_k^{(0)}}{x^k} \, \sin\left(x - \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t} \, J_0(t) \, dt < 6 \cdot 10^{-9} \,.$$

k	$c_k^{(0)}$	k	$c_k^{(0)}$
1	0.797 884 516 538	6	4.733 533 047 132
2	0.099 735 119 074	7	18.859 909 855 69
3	0.155 775 947 720	8	99.846 038 227 04
4	0.369 550 899 387	9	256.775 583 671 0
5	1.170 795 416 963	10	1 527.508 571 668

Function $J_1(x)$:

Approximation by Chebyshev polynomials, based on [2], 9.7.:

$$\int_0^x \sqrt{t}\, J_1(t)\, dt \quad \approx \quad \sqrt{x} \cdot \sum_{k=0}^{17} q_k\, T_{2k}\left(\frac{x}{8}\right) , \qquad 0 \le x \le 8$$

with the coefficients:

k	q_k	k	q_k
0	0.37975 25427 04720 47384	9	0.00000 17101 75937 74175
1	-0.24153 71053 32677 35417	10	-0.00000 00740 94524 46089
2	-0.12554 99442 21699 83184	11	0.00000 00026 16214 44822
3	0.31360 55017 12763 75964	12	-0.00000 00000 76805 72461
4	-0.14432 03488 73845 84716	13	0.00000 00000 01905 60154
5	0.03274 98779 87894 78550	14	-0.00000 00000 00040 50286
6	-0.00458 83639 64558 05653	15	0.00000 00000 00000 74601
7	0.00044 24559 29648 31876	16	-0.00000 00000 00000 01203
8	-0.00003 13683 81557 99050	17	0.00000 00000 00000 00017

Difference between approximation and true frunction:

$$\sqrt{x}\sum_{k=0}^{17} q_k T_{2k}(x/8) - \int_0^x \sqrt{t}\, J_1(t)\, dt$$
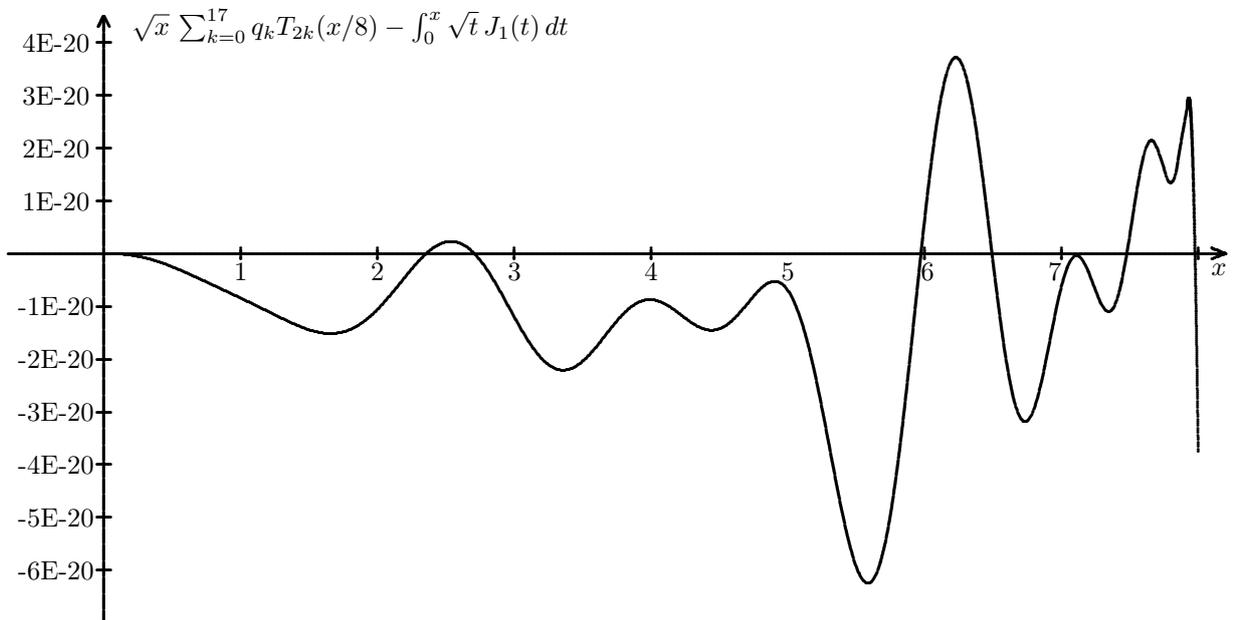
Asymptotic expansion for $x \to \infty$:

$$\int_0^x \sqrt{t}\, J_1(t)\, dt \ \sim \ \frac{4\Gamma^2(5/4)}{\pi} - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{b_k}{x^k}\, \sin\left(x + \frac{2k+1}{4}\pi\right)$$

$$\frac{4\Gamma^2(5/4)}{\pi} \ = \ 1.046\,049\,620\,053\,102$$

$$b_0 = 1\,, \quad b_1 = \frac{3}{8}\,, \quad b_2 = -\frac{63}{128}\,, \quad b_3 = \frac{1113}{1024}\,, \quad b_4 = -\frac{111573}{32768}\,, \quad b_5 = \frac{3643101}{262144}\,, \quad b_6 = -\frac{294285915}{4194304}\,,$$

$$b_7 = \frac{14192615745}{33554432}\,, \quad b_8 = -\frac{6373074947085}{2147483648}\,, \quad b_9 = \frac{408344927902065}{17179869184}\,, \ \ldots$$

k	b_k	$|b_k/b_{k-1}|$	k	b_k	$|b_k/b_{k-1}|$
0	1.000 000 000	-	5	13.897 327 42	4.0815
1	0.375 000 000	0.3750	6	-70.163 229 70	5.0487
2	-0.492 187 500	1.3125	7	422.972 910 0	6.0284
3	1.086 914 063	2.2083	8	-2 967.694 284	7.0163
4	-3.404 937 744	3.1327	9	23 768.803 10	8.0092

Let
$$D_{1,n}(x) = \frac{4\Gamma^2(5/4)}{\pi} - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n} \frac{b_k}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t}\, J_1(t)\, dt\,,$$

then its first maximum and minimum values of interest are $D_{1,n}(x_{i,n}^*)$.
In the case $x > x_{i,n}^*$ holds $|D_{1,n}(x)| < |D_{1,n}(x_{i,n}^*)|$.

$n = 0,\ i =$	1	2	3	4	5	6	7	8	9	10
$x_{i,0}^*$	2.470	6.470	8.675	11.807	14.943	18.081	21.220	24.360	27.500	30.641
$10^3 D_{1,0}(x_i^*)$	-98.4511	50.5537	-33.4821	24.9168	-19.8073	16.4242	-14.0227	12.2312	-10.8443	9.7390

$n = 1,\ i =$	2	3	4	5	6	7	8	9	10	11
$x_{i,1}^*$	3.974	7.097	10.230	13.367	15.506	16.645	22.786	25.926	29.067	32.208
$10^4 D_{1,1}(x_i^*)$	-194.456	70.4454	-35.5457	21.2582	-14.0945	10.0128	-7.4731	5.7878	-4.6133	3.7625

$n = 2,\ i =$	3	4	5	6	7	8	9	10	11	12
$x_{i,2}^*$	8.654	12.654	14.931	18.071	21.212	24.353	27.494	30.635	33.776	36.917
$10^5 D_{1,2}(x_i^*)$	117.286	-48.9485	24.7672	-14.1826	8.8517	-5.8853	4.1071	-2.9778	2.2269	-1.7083

$n = 3,\ i =$	4	5	6	7	8	9	10	11	12	13
$x_{i,3}^*$	13.358	16.498	19.639	22.780	29.921	29.063	32.204	35.345	38.487	41.628
$10^7 D_{0,3}(x_i^*)$	-773.343	342.802	-173.926	97.2278	-58.4641	37.2099	-24.7842	17.1337	-12.2177	8.9436

$n = 4,\ i =$	4	5	6	7	8	9	10	11	12	13
$x_{i,4}^*$	11.785	14.926	18.067	21.208	24.349	27.491	30.632	33.774	36.915	40.057
$10^7 D_{1,4}(x_i^*)$	409.325	-133.167	53.0102	-24.2927	12.3500	-6.7990	3.9863	-2.4597	1.5831	-1.0557

For $8 \le x \le 30$ holds

$$\int_0^x \sqrt{t}\, J_1(t)\, dt \approx 1.046\,049046\,618046\,299 + \sum_{k=0}^{9} \frac{c_k^{(1)}}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right)$$

with

$$-7.0 \cdot 10^{-9} < 1.046\,049\,046\,618 + \sum_{k=0}^{9} \frac{c_k^{(1)}}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t}\, J_0(t)\, dt < 5 \cdot 10^{-9}\,.$$

k	$c_k^{(1)}$	k	$c_k^{(1)}$
1	-0.797 884 661 354	6	-10.914 764 692 58
2	-0.299 206 773 536	7	40.086 327 439 47
3	0.392 563 542 201	8	-264.350 611 778 9
4	-0.867 123 732 390	9	464.583 909 606 6
5	2.645 254 609 577	10	-5 043.243 969 567

Modified Bessel Functions:

$$\int_0^x \sqrt{t}\, I_0(t)\, dt = 2\sqrt{x} \sum_{k=0}^{\infty} \frac{x^{2n+1}}{4^k \cdot (k!)^2 \cdot (4k+3)} =$$

$$= 2\sqrt{x}\left(\frac{x}{3} + \frac{x^3}{28} + \frac{x^5}{704} + \frac{x^7}{34\,560} + \frac{x^9}{2\,801\,664} + \frac{x^{11}}{339\,148\,800} + \frac{x^{13}}{57\,330\,892\,800} + \frac{x^{15}}{12\,901\,574\,246\,400} + \ldots\right)$$

$$\int_0^x \sqrt{t}\, I_1(t)\, dt = \sqrt{x} \sum_{k=0}^{\infty} \frac{x^{2n+2}}{4^k \cdot (k!)^2 \cdot (4k+5) \cdot (k+1)} =$$

$$= \sqrt{x}\left(\frac{x^2}{5} + \frac{x^4}{72} + \frac{x^6}{2\,496} + \frac{x^8}{156\,672} + \frac{x^{10}}{15\,482\,880} + \frac{x^{12}}{2\,211\,840\,000} + \frac{x^{14}}{431\,043\,379\,200} + \ldots\right)$$

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<u>1.2.1. b) Integrals:</u>

$$\int x^{3/2} J_0(x)\, dx = x^{3/2} J_1(x) - \frac{1}{2} \int \sqrt{x}\, J_1(x)\, dx$$

$$\int x^{3/2} J_1(x)\, dx = -x^{3/2} J_0(x) + \frac{3}{2} \int \sqrt{x}\, J_0(x)\, dx$$

$$\int x^{3/2} I_0(x)\, dx = x^{3/2} I_1(x) - \frac{1}{2} \int \sqrt{x}\, I_1(x)\, dx$$

$$\int x^{3/2} I_1(x)\, dx = x^{3/2} I_0(x) - \frac{3}{2} \int \sqrt{x}\, I_0(x)\, dx$$

$$\int x^{5/2} J_0(x)\, dx = \sqrt{x} \left[\frac{3x}{2} J_0(x) + x^2 J_1(x) \right] - \frac{9}{4} \int \sqrt{x}\, J_0(x)\, dx$$

$$\int x^{5/2} J_1(x)\, dx = \sqrt{x} \left[-x^2 J_0(x) + \frac{5}{2} x J_1(x) \right] - \frac{5}{4} \int \sqrt{x}\, J_1(x)\, dx$$

$$\int x^{5/2} I_0(x)\, dx = \sqrt{x} \left[-\frac{3x}{2} I_0(x) + x^2 I_1(x) \right] + \frac{9}{4} \int \sqrt{x}\, I_0(x)\, dx$$

$$\int x^{5/2} I_1(x)\, dx = \sqrt{x} \left[x^2 J_0(x) - \frac{5}{2} x I_1(x) \right] + \frac{5}{4} \int \sqrt{x}\, I_1(x)\, dx$$

$$\int x^{7/2} J_0(x)\, dx = \sqrt{x} \left[\frac{5x^2}{2} J_0(x) + \left(x^3 - \frac{25}{4} x \right) J_1(x) \right] + \frac{25}{8} \int \sqrt{x}\, J_1(x)\, dx$$

$$\int x^{7/2} J_1(x)\, dx = \sqrt{x} \left[\left(-x^3 + \frac{21}{4} x \right) J_0(x) + \frac{7}{2} x^2 J_1(x) \right] - \frac{63}{8} \int \sqrt{x}\, J_0(x)\, dx$$

$$\int x^{7/2} I_0(x)\, dx = \sqrt{x} \left[-\frac{5x^2}{2} I_0(x) + \left(x^3 + \frac{25}{4} x \right) I_1(x) \right] - \frac{25}{8} \int \sqrt{x}\, I_1(x)\, dx$$

$$\int x^{7/2} I_1(x)\, dx = \sqrt{x} \left[\left(x^3 + \frac{21}{4} x \right) I_0(x) - \frac{7}{2} x^2 I_1(x) \right] - \frac{63}{8} \int \sqrt{x}\, I_0(x)\, dx$$

$$\int x^{9/2} J_0(x)\, dx = \sqrt{x} \left[\left(\frac{7}{2} x^3 - \frac{147}{8} x \right) J_0(x) + \left(x^4 - \frac{49}{4} x^2 \right) J_1(x) \right] + \frac{441}{16} \int \sqrt{x}\, J_0(x)\, dx$$

$$\int x^{9/2} J_1(x)\, dx = \sqrt{x} \left[\left(-x^4 + \frac{45}{4} x^2 \right) J_0(x) + \left(\frac{9}{2} x^3 - \frac{225}{8} x \right) J_1(x) \right] + \frac{225}{16} \int \sqrt{x}\, J_1(x)\, dx$$

$$\int x^{9/2} I_0(x)\, dx = \sqrt{x} \left[-\left(\frac{7}{2} x^3 + \frac{147}{8} x \right) I_0(x) + \left(x^4 + \frac{49}{4} x^2 \right) I_1(x) \right] + \frac{441}{16} \int \sqrt{x}\, I_0(x)\, dx$$

$$\int x^{9/2} I_1(x)\, dx = \sqrt{x} \left[\left(x^4 + \frac{45}{4} x^2 \right) I_0(x) - \left(\frac{9}{2} x^3 + \frac{225}{8} x \right) I_1(x) \right] + \frac{225}{16} \int \sqrt{x}\, I_1(x)\, dx$$

To find $\int x^{(2n+1)/2} I_\nu(x)\, dx$ with $n > 4$ use the recurrence formulas (see page 67).

$$\int x^{11/2} J_0(x)\, dx = \sqrt{x} \left[\left(\frac{9}{2} x^4 - \frac{405}{8} x^2 \right) J_0(x) + \left(x^5 - \frac{81}{4} x^3 + \frac{2025}{16} x \right) J_1(x) \right] - \frac{2025}{32} \int \sqrt{x}\, J_1(x)\, dx$$

$$\int x^{11/2} J_1(x)\, dx = \sqrt{x} \left[\left(-x^5 + \frac{77}{4} x^3 - \frac{1617}{16} x \right) J_0(x) + \left(\frac{11}{2} x^4 - \frac{539}{8} x^2 \right) J_1(x) \right] + \frac{4851}{32} \int \sqrt{x}\, J_0(x)\, dx$$

$$\int x^{13/2} J_0(x)\, dx = \sqrt{x} \left[\left(\frac{11}{2} x^5 - \frac{847}{8} x^3 + \frac{17787}{32} x \right) J_0(x) + \left(x^6 - \frac{121}{4} x^4 + \frac{5929}{16} x^2 \right) J_1(x) \right] -$$

$$- \frac{53361}{64} \int \sqrt{x}\, J_0(x)\, dx$$

$$\int x^{13/2} J_1(x)\,dx = \sqrt{x}\left[\left(-x^6 + \frac{117}{4}x^4 - \frac{5265}{16}x^2\right)J_0(x) + \left(\frac{13}{2}x^5 - \frac{1053}{8}x^3 + \frac{26325}{32}x\right)J_1(x)\right] -$$
$$-\frac{26325}{64}\int \sqrt{x}\,J_1(x)\,dx$$

$*E*$
$$\int x^{15/2} J_0(x)\,dx = \sqrt{x}\left[\left(\frac{13}{2}x^6 - \frac{1521}{8}x^4 + \frac{68445}{32}x^2\right)J_0(x)+\right.$$
$$\left.+\left(x^7 - \frac{169}{4}x^5 + \frac{13689}{16}x^3 - \frac{342225}{64}x\right)J_1(x)\right] + \frac{342225}{128}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{15/2} J_1(x)\,dx = \sqrt{x}\left[\left(-x^7 + \frac{165}{4}x^5 - \frac{12705}{16}x^3 + \frac{266805}{64}x\right)J_0(x)+\right.$$
$$\left.+\left(\frac{15}{2}x^6 - \frac{1815}{8}x^4 + \frac{88935}{32}x^2\right)J_1(x)\right] - \frac{800415}{128}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{17/2} J_0(x)\,dx = \sqrt{x}\left[\left(\frac{15}{2}x^7 - \frac{2475}{8}x^5 + \frac{190575}{32}x^3 - \frac{4002075}{128}x\right)J_0(x)+\right.$$
$$\left.+\left(x^8 - \frac{225}{4}x^6 + \frac{27225}{16}x^4 - \frac{1334025}{64}x^2\right)J_1(x)\right] + \frac{12006225}{256}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{17/2} J_1(x)\,dx = \sqrt{x}\left[\left(-x^8 + \frac{221}{4}x^6 - \frac{25857}{16}x^4 + \frac{1163565}{64}x^2\right)J_0(x)+\right.$$
$$\left.+\left(\frac{17}{2}x^7 - \frac{2873}{8}x^5 + \frac{232713}{32}x^3 - \frac{5817825}{128}x\right)J_1(x)\right] + \frac{5817825}{256}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{19/2} J_0(x)\,dx = \sqrt{x}\left[\left(\frac{17}{2}x^8 - \frac{3757}{8}x^6 + \frac{439569}{32}x^4 - \frac{19780605}{128}x^2\right)J_0(x)+\right.$$
$$\left.+\left(x^9 - \frac{289}{4}x^7 + \frac{48841}{16}x^5 - \frac{3956121}{64}x^3 + \frac{98903025}{256}x\right)J_1(x)\right] - \frac{98903025}{512}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{19/2} J_1(x)\,dx = \sqrt{x}\left[\left(-x^9 + \frac{285}{4}x^7 - \frac{47025}{16}x^5 + \frac{3620925}{64}x^3 - \frac{76039425}{256}x\right)J_0(x)+\right.$$
$$\left.+\left(\frac{19}{2}x^8 - \frac{4275}{8}x^6 + \frac{517275}{32}x^4 - \frac{25346475}{128}x^2\right)J_1(x)\right] + \frac{228118275}{512}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{21/2} J_0(x)\,dx =$$
$$\sqrt{x}\left[\left(\frac{19}{2}x^9 - \frac{5415}{8}x^7 + \frac{893475}{32}x^5 - \frac{68797575}{128}x^3 + \frac{1444749075}{512}x\right)J_0(x)+\right.$$
$$\left.+\left(x^{10} - \frac{361}{4}x^8 + \frac{81225}{16}x^6 - \frac{9828225}{64}x^4 + \frac{481583025}{256}x^2\right)J_1(x)\right] -$$
$$-\frac{4334247225}{1024}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{21/2} J_1(x)\,dx = \sqrt{x}\left[\left(-x^{10} + \frac{357}{4}x^8 - \frac{78897}{16}x^6 + \frac{9230949}{64}x^4 - \frac{415392705}{256}x^2\right)J_0(x)+\right.$$
$$+\left(\frac{21}{2}x^9 - \frac{6069}{8}x^7 + \frac{1025661}{32}x^5 - \frac{83078541}{128}x^3 + \frac{2076963525}{512}x\right)J_1(x)\bigg] - \frac{2076963525}{1024}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{23/2} J_0(x)\,dx = \sqrt{x}\left[\left(\frac{21}{2}x^{10} - \frac{7497}{8}x^8 + \frac{1656837}{32}x^6 - \frac{193849929}{128}x^4 + \frac{8723246805}{512}x^2\right)J_0(x)+\right.$$
$$\left.+\left(x^{11} - \frac{441}{4}x^9 + \frac{127449}{16}x^7 - \frac{21538881}{64}x^5 + \frac{1744649361}{256}x^3 - \frac{43616234025}{1024}x\right)J_1(x)\right] +$$

$$+\frac{43616234025}{2048}\int \sqrt{x}\, J_1(x)\, dx$$

$$\int x^{23/2}\, J_1(x)\, dx =$$

$$= \sqrt{x}\left[\left(-x^{11} + \frac{437}{4}x^9 - \frac{124545}{16}x^7 + \frac{20549925}{64}x^5 - \frac{1582344225}{256}x^3 + \frac{33229228725}{1024}x\right)J_0(x)+\right.$$

$$\left.+\left(\frac{23}{2}x^{10} - \frac{8303}{8}x^8 + \frac{1868175}{32}x^6 - \frac{226049175}{128}x^4 + \frac{11076409575}{512}x^2\right)J_1(x)\right]-$$

$$-\frac{99687686175}{2048}\int \sqrt{x}\, J_0(x)\, dx$$

$$\int x^{25/2}\, J_0(x)\, dx =$$

$$= \sqrt{x}\left[\left(\frac{23}{2}x^{11} - \frac{10051}{8}x^9 + \frac{2864535}{32}x^7 - \frac{472648275}{128}x^5 + \frac{36393917175}{512}x^3 - \frac{764272260675}{2048}x\right)J_0(x)+\right.$$

$$\left.+\left(x^{12} - \frac{529}{4}x^{10} + \frac{190969}{16}x^8 - \frac{42968025}{64}x^6 + \frac{5199131025}{256}x^4 - \frac{254757420225}{1024}x^2\right)J_1(x)\right]+$$

$$+\frac{2292816782025}{4096}\int \sqrt{x}\, J_0(x)\, dx$$

$$\int x^{25/2}\, J_1(x)\, dx =$$

$$= \sqrt{x}\left[\left(-x^{12} + \frac{525}{4}x^{10} - \frac{187425}{16}x^8 + \frac{41420925}{64}x^6 - \frac{4846248225}{256}x^4 + \frac{218081170125}{1024}x^2\right)J_0(x)+\right.$$

$$+\left(\frac{25}{2}x^{11} - \frac{11025}{8}x^9 + \frac{3186225}{32}x^7 - \frac{538472025}{128}x^5 + \frac{43616234025}{512}x^3 - \frac{1090405850625}{2048}x\right)J_1(x)\right]+$$

$$+\frac{1090405850625}{4096}\int \sqrt{x}\, J_1(x)\, dx$$

$$\int x^{27/2}\, J_0(x)\, dx =$$

$$= \sqrt{x}\left[\left(\frac{25}{2}x^{12} - \frac{13125}{8}x^{10} + \frac{4685625}{32}x^8 - \frac{1035523125}{128}x^6 + \frac{121156205625}{512}x^4 - \frac{5452029253125}{2048}x^2\right)J_0(x)+\right.$$

$$+\left(x^{13} - \frac{625}{4}x^{11} + \frac{275625}{16}x^9 - \frac{79655625}{64}x^7 + \frac{13461800625}{256}x^5 - \frac{1090405850625}{1024}x^3+\right.$$

$$\left.\left.+\frac{27260146265625}{4096}x\right)J_1(x)\right] - \frac{27260146265625}{8192}\int \sqrt{x}\, J_1(x)\, dx$$

$$\int x^{27/2}\, J_1(x)\, dx = \sqrt{x}\left[\left(-x^{13} + \frac{621}{4}x^{11} - \frac{271377}{16}x^9 + \frac{77342445}{64}x^7 - \frac{12761503425}{256}x^5+\right.\right.$$

$$\frac{982635763725}{1024}x^3 - \frac{20635351038225}{4096}x\right)J_0(x) + \left(\frac{27}{2}x^{12} - \frac{14283}{8}x^{10} + \frac{5156163}{32}x^8 - \frac{1160136675}{128}x^6+\right.$$

$$\left.\left.+\frac{140376537675}{512}x^4 - \frac{6878450346075}{2048}x^2\right)J_1(x)\right] + \frac{61906053114675}{8192}\int \sqrt{x}\, J_0(x)\, dx$$

$$\int x^{29/2}\, J_0(x)\, dx = \sqrt{x}\left[\left(\frac{27}{2}x^{13} - \frac{16767}{8}x^{11} + \frac{7327179}{32}x^9 - \frac{2088246015}{128}x^7 + \frac{344560592475}{512}x^5-\right.\right.$$

$$-\frac{26531165620575}{2048}x^3 + \frac{557154478032075}{8192}x\right)J_0(x) + \left(x^{14} - \frac{729}{4}x^{12} + \frac{385641}{16}x^{10} - \frac{139216401}{64}x^8+$$

$$+\frac{31323690225}{256}\,x^6 - \frac{3790166517225}{1024}\,x^4 + \frac{185718159344025}{4096}\,x^2\Big)\,J_1(x)\Big] - \frac{1671463434096225}{16384}\int\sqrt{x}\,J_0(x)\,dx$$

$$\int x^{29/2}\,J_1(x)\,dx = \sqrt{x}\left[\Big(-x^{14} + \frac{725}{4}\,x^{12} - \frac{380625}{16}\,x^{10} + \frac{135883125}{64}\,x^8 - \frac{30030170625}{256}\,x^6 +\right.$$

$$+\frac{3513529963125}{1024}\,x^4 - \frac{158108848340625}{4096}\,x^2\Big)\,J_0(x) + \Big(\frac{29}{2}\,x^{13} - \frac{18125}{8}\,x^{11} + \frac{7993125}{32}\,x^9 - \frac{2310013125}{128}\,x^7 +$$

$$+\frac{390392218125}{512}\,x^5 - \frac{31621769668125}{2048}\,x^3 + \frac{790544241703125}{8192}\,x\Big)\,J_1(x)\Big] - \frac{790544241703125}{16384}\int\sqrt{x}\,J_1(x)\,dx$$

$$\int \frac{J_0(x)}{\sqrt{x}}\,dx = 2\sqrt{x}J_0(x) + 2\int \sqrt{x}J_1(x)\,dx$$

$$\int \frac{J_1(x)}{\sqrt{x}}\,dx = -2\sqrt{x}J_1(x) + 2\int \sqrt{x}J_0(x)\,dx$$

$$\int \frac{I_0(x)}{\sqrt{x}}\,dx = 2\sqrt{x}I_0(x) - 2\int \sqrt{x}J_1(x)\,dx$$

$$\int \frac{I_1(x)}{\sqrt{x}}\,dx = -2\sqrt{x}I_1(x) + 2\int \sqrt{x}I_0(x)\,dx$$

$$\int x^{-3/2}\cdot J_0(x)\,dx = \frac{\sqrt{x}}{x}[-2J_0(x) + 4xJ_1(x)] - 4\int \sqrt{x}J_0(x)\,dx$$

$$\int x^{-3/2}\cdot J_1(x)\,dx = \frac{\sqrt{x}}{3x}[4xJ_0(x) - 2J_1(x)] + \frac{4}{3}\int \sqrt{x}J_1(x)\,dx$$

$$\int x^{-3/2}\cdot I_0(x)\,dx = -\frac{\sqrt{x}}{x}[2I_0(x) + 4xI_1(x)] + 4\int \sqrt{x}I_0(x)\,dx$$

$$\int x^{-3/2}\cdot I_1(x)\,dx = \frac{\sqrt{x}}{3x}[4xI_0(x) - 2I_1(x)] - \frac{4}{3}\int \sqrt{x}I_1(x)\,dx$$

$$\int x^{-5/2}\cdot J_0(x)\,dx = \frac{\sqrt{x}}{9x^2}\left[\left(-8x^2 - 6\right)J_0(x) + 4xJ_1(x)\right] - \frac{8}{9}\int \sqrt{x}J_1(x)\,dx$$

$$\int x^{-5/2}\cdot J_1(x)\,dx = \frac{\sqrt{x}}{5x^2}[-4xJ_0(x) + (8x^2 - 2)J_1(x)] - \frac{8}{5}\int \sqrt{x}J_0(x)\,dx$$

$$\int x^{-5/2}\cdot I_0(x)\,dx = \frac{\sqrt{x}}{9x^2}\left[\left(8x^2 - 6\right)I_0(x) - 4xI_1(x)\right] - \frac{8}{9}\int \sqrt{x}I_1(x)\,dx$$

$$\int x^{-5/2}\cdot I_1(x)\,dx = \frac{\sqrt{x}}{5x^2}[-4xI_0(x) - (8x^2 + 2)I_1(x)] + \frac{8}{5}\int \sqrt{x}I_0(x)\,dx$$

$$\int x^{-7/2}\cdot J_0(x)\,dx = \frac{\sqrt{x}}{25\,x^3}[(8\,x^2 - 10)J_0(x) + (-16\,x^3 + 4\,x)J_1(x)] + \frac{16}{25}\int \sqrt{x}J_0(x)\,dx$$

$$\int x^{-7/2}\cdot J_1(x)\,dx = \frac{\sqrt{x}}{63\,x^3}[(-16\,x^3 - 12\,x)J_0(x) + (8\,x^2 - 18)J_1(x)] - \frac{16}{63}\int \sqrt{x}J_1(x)\,dx$$

$$\int x^{-7/2}\cdot I_0(x)\,dx = -\frac{\sqrt{x}}{25\,x^3}[(8\,x^2 + 10)I_0(x) + (16\,x^3 + 4\,x)I_1(x)] + \frac{16}{25}\int \sqrt{x}I_0(x)\,dx$$

$$\int x^{-7/2}\cdot I_1(x)\,dx = \frac{\sqrt{x}}{63\,x^3}[(16\,x^3 - 12\,x)I_0(x) - (8\,x^2 + 18)I_1(x)] - \frac{16}{63}\int \sqrt{x}I_1(x)\,dx$$

To find $\int x^{-(2n+1)/2}\,I_\nu(x)\,dx$ with $n > 4$ use the recurrence formulas (see page 67).

$$\int x^{-9/2}\cdot J_0(x)\,dx = \frac{\sqrt{x}}{441\,x^4}\left[(32\,x^4 + 24\,x^2 - 126)\,J_0(x) + (-16\,x^3 + 36\,x)J_1(x)\right] + \frac{32}{441}\int \sqrt{x}J_1(x)\,dx$$

$$\int x^{-9/2} \cdot J_1(x)\, dx = \frac{\sqrt{x}}{225\,x^4}[(16\,x^3 - 20\,x)J_0(x) + (-32\,x^4 + 8\,x^2 - 50)J_1(x)] + \frac{32}{225}\int \sqrt{x}\,J_0(x)\, dx$$

$$\int x^{-11/2} \cdot J_0(x)\, dx = \frac{\sqrt{x}}{2025\,x^5}[(-32\,x^4 + 40\,x^2 - 450)J_0(x) + (64\,x^5 - 16\,x^3 + 100\,x)J_1(x)] - \frac{64}{2025}\int \sqrt{x}\,J_0(x)\, dx$$

$$\int x^{-11/2} \cdot J_1(x)\, dx = \frac{\sqrt{x}}{4851\,x^5}[(64\,x^5 + 48\,x^3 - 252\,x)J_0(x) + (-32\,x^4 + 72\,x^2 - 882)J_1(x)] + \frac{64}{4851}\int \sqrt{x}\,J_1(x)\, dx$$

$$\int x^{-13/2} \cdot J_0(x)\, dx =$$

$$= \frac{\sqrt{x}}{53361\,x^6}\left[(-128\,x^6 - 96\,x^4 + 504\,x^2 - 9702)J_0(x) + (64\,x^5 - 144\,x^3 + 1764\,x)J_1(x)\right] - \frac{128}{53361}\int \sqrt{x}\,J_1(x)\, dx$$

$$\int x^{-13/2} \cdot J_1(x)\, dx = \frac{\sqrt{x}}{26325\,x^6}[(-64\,x^5 + 80\,x^3 - 900\,x)J_0(x) +$$

$$+ (128\,x^6 - 32\,x^4 + 200\,x^2 - 4050)J_1(x)] - \frac{128}{26325}\int \sqrt{x}\,J_0(x)\, dx$$

$$\int x^{-15/2} \cdot J_0(x)\, dx = \frac{\sqrt{x}}{342225\,x^7}[(128\,x^6 - 160\,x^4 + 1800\,x^2 - 52650)J_0(x) +$$

$$+ (-256\,x^7 + 64\,x^5 - 400\,x^3 + 8100\,x)J_1(x)] + \frac{256}{342225}\int \sqrt{x}\,J_0(x)\, dx$$

$$\int x^{-15/2} \cdot J_1(x)\, dx = \frac{\sqrt{x}}{800415\,x^7}[(-256\,x^7 - 192\,x^5 + 1008\,x^3 - 19404\,x)J_0(x) +$$

$$+ (128\,x^6 - 288\,x^4 + 3528\,x^2 - 106722)J_1(x)] - \frac{256}{800415}\int \sqrt{x}\,J_1(x)\, dx$$

$$\int x^{-17/2} \cdot J_0(x)\, dx = \frac{\sqrt{x}}{12006225\,x^8}[(512\,x^8 + 384\,x^6 - 2016\,x^4 + 38808\,x^2 - 1600830)J_0(x) +$$

$$+ (-256\,x^7 + 576\,x^5 - 7056\,x^3 + 213444\,x)J_1(x)] + \frac{512}{12006225}\int \sqrt{x}\,J_1(x)\, dx$$

$$\int x^{-17/2} \cdot J_1(x)\, dx = \frac{\sqrt{x}}{5817825\,x^8}[(256\,x^7 - 320\,x^5 + 3600\,x^3 - 105300\,x)J_0(x) +$$

$$+ (-512\,x^8 + 128\,x^6 - 800\,x^4 + 16200\,x^2 - 684450)J_1(x)] + \frac{512}{5817825}\int \sqrt{x}\,J_0(x)\, dx$$

$$\int x^{-19/2} \cdot J_0(x)\, dx = \frac{\sqrt{x}}{98903025\,x^9}[(-512\,x^8 + 640\,x^6 - 7200\,x^4 + 210600\,x^2 - 11635650)J_0(x) +$$

$$+ (1024\,x^9 - 256\,x^7 + 1600\,x^5 - 32400\,x^3 + 1368900\,x)J_1(x)] - \frac{1024}{98903025}\int \sqrt{x}\,J_0(x)\, dx$$

$$\int x^{-19/2} \cdot J_1(x)\, dx = \frac{\sqrt{x}}{228118275\,x^9}[(1024\,x^9 + 768\,x^7 - 4032\,x^5 + 77616\,x^3 - 3201660\,x)J_0(x) +$$

$$+ (-512\,x^8 + 1152\,x^6 - 14112\,x^4 + 426888\,x^2 - 24012450)J_1(x)] + \frac{1024}{228118275}\int \sqrt{x}\,J_1(x)\, dx \quad * E *$$

$$\int x^{-21/2} \cdot J_0(x)\, dx =$$

$$= \frac{\sqrt{x}}{4334247225\,x^{10}}[(-2048\,x^{10} - 1536\,x^8 + 8064\,x^6 - 155232\,x^4 + 6403320\,x^2 - 456236550)J_0(x) +$$

$$+(1024\,x^9 - 2304\,x^7 + 28224\,x^5 - 853776\,x^3 + 48024900\,x)J_1(x)] - \frac{2048}{4334247225} + \frac{1024}{228118275}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{-21/2}\cdot J_1(x)\,dx = \frac{\sqrt{x}}{2076963525\,x^{10}}[(-1024\,x^9 + 1280\,x^7 - 14400\,x^5 + 421200\,x^3 - 23271300\,x)J_0(x)+$$

$$+(2048\,x^{10} - 512\,x^8 + 3200\,x^6 - 64800\,x^4 + 2737800\,x^2 - 197806050)J_1(x)] - \frac{2048}{2076963525}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{-23/2}\cdot J_0(x)\,dx =$$

$$= \frac{\sqrt{x}}{43616234025\,x^{11}}[(2048\,x^{10} - 2560\,x^8 + 28800\,x^6 - 842400\,x^4 + 46542600\,x^2 - 4153927050)J_0(x)+$$

$$+(-4096\,x^{11} + 1024\,x^9 - 6400\,x^7 + 129600\,x^5 - 5475600\,x^3 + 395612100\,x)J_1(x)] + \frac{4096}{43616234025}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{-23/2}\cdot J_1(x)\,dx =$$

$$= \frac{\sqrt{x}}{99687686175\,x^{11}}[(-4096\,x^{11} - 3072\,x^9 + 16128\,x^7 - 310464\,x^5 + 12806640\,x^3 - 912473100\,x)J_0(x)+$$

$$+(2048\,x^{10} - 4608\,x^8 + 56448\,x^6 - 1707552\,x^4 + 96049800\,x^2 - 8668494450)J_1(x)] - \frac{4096}{99687686175}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{-25/2}\cdot J_0(x)\,dx = \frac{\sqrt{x}}{2292816782025\,x^{12}}[(8192\,x^{12} + 6144\,x^{10} - 32256\,x^8 + 620928\,x^6 - 25613280\,x^4+$$

$$+1824946200\,x^2 - 199375372350)J_0(x) + (-4096\,x^{11} + 9216\,x^9 - 112896\,x^7 + 3415104\,x^5 - 192099600\,x^3+$$

$$+17336988900\,x)J_1(x)] + \frac{8192}{2292816782025}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{-25/2}\cdot J_1(x)\,dx =$$

$$= \frac{\sqrt{x}}{1090405850625\,x^{12}}[(4096\,x^{11} - 5120\,x^9 + 57600\,x^7 - 1684800\,x^5 + 93085200\,x^3 - 8307854100\,x)J_0(x)+$$

$$+(-8192\,x^{12} + 2048\,x^{10} - 12800\,x^8 + 259200\,x^6 - 10951200\,x^4 + 791224200\,x^2 - 87232468050)J_1(x)]+$$

$$+\frac{8192}{1090405850625}\int \sqrt{x}\,J_0(x)\,dx$$

1.2.1. c) Recurrence Formulas:

$$\int x^{n+5/2}\,J_0(x)\,dx = x^{n+3/2}\left[\left(n+\frac{3}{2}\right)J_0(x) + x\,J_1(x)\right] - \left(n+\frac{3}{2}\right)^2\int x^{n+1/2}\,J_0(x)\,dx$$

$$\int x^{n+5/2}\,J_1(x)\,dx = x^{n+3/2}\left[\left(n+\frac{5}{2}\right)J_1(x) - x\,J_0(x)\right] - \frac{(2n+1)(2n+5)}{4}\int x^{n+1/2}\,J_1(x)\,dx$$

$$\int x^{n+5/2}\,I_0(x)\,dx = x^{n+3/2}\left[x\,I_1(x) - \left(n+\frac{3}{2}\right)I_0(x)\right] + \left(n+\frac{3}{2}\right)^2\int x^{n+1/2}\,I_0(x)\,dx$$

$$\int x^{n+5/2}\,I_1(x)\,dx = x^{n+3/2}\left[x\,I_0(x) - \left(n+\frac{5}{2}\right)I_1(x)\right] + \frac{(2n+1)(2n+5)}{4}\int x^{n+1/2}\,I_1(x)\,dx$$

1.2.2. Integrals of the type $\int x^n\, e^{\pm x} \cdot \left\{ \begin{array}{c} I_\nu(x) \\ K_\nu x \end{array} \right\} dx$

See also [1], 11.3.

a) Integrals with e^x :

$$\int e^x\, I_0(x)\, dx \;=\; x e^x\, [I_0(x) - I_1(x)] \;, \qquad \int \frac{e^x \cdot I_1(x)\, dx}{x} = e^x\, [I_0(x) - I_1(x)]$$

$$\int e^x\, K_0(x)\, dx = x e^x\, [K_0(x) + K_1(x)] \;, \qquad \int \frac{e^x\, K_1(x)\, dx}{x} = -e^x\, [K_0(x) + K_1(x)]$$

$$\int e^x\, I_1(x)\, dx \;=\; e^x\, [(1 - x)I_0(x) + x I_1(x)]$$

$$\int e^x\, K_1(x)\, dx = -e^x\, [(1 - x)K_0(x) - x K_1(x)]$$

$$\int x e^x\, I_0(x)\, dx \;=\; \frac{x e^x}{3}\, [x I_0(x) + (1 - x)I_1(x)]$$

$$\int x\, e^x\, K_0(x)\, dx = \frac{x e^x}{3}\, [x\, K_0(x) + (x - 1)\, K_1(x)]$$

$$\int x\, e^x\, I_1(x)\, dx \;=\; \frac{x e^x}{3}\, [-x I_0(x) + (2 + x)I_1(x)]$$

$$\int x\, e^x\, K_1(x)\, dx = \frac{x e^x}{3}\, [x\, K_0(x) + (x + 2)\, K_1(x)]$$

$$\int x^2 e^x\, I_0(x)\, dx \;=\; \frac{x e^x}{15}\, [(2\,x + 3\,x^2)I_0(x) + (-4 + 4\,x - 3\,x^2)I_1(x)]$$

$$\int x^2\, e^x\, K_0(x)\, dx = \frac{x e^x}{15}\, [(3x^2 + 2x)\, K_0(x) + (3\,x^2 - 4\,x + 4)\, K_1(x)]$$

$$\int x^2 e^x\, I_1(x)\, dx \;=\; \frac{x e^x}{5}\, [(x - x^2)I_0(x) + (-2 + 2x + x^2)I_1(x)]$$

$$\int x^2\, e^x\, K_1(x)\, dx = \frac{x e^x}{5}\, [(x^2 - x)\, K_0(x) + (x^2 + 2x - 2)\, K_1(x)]$$

$$\int x^3 e^x\, I_0(x)\, dx \;=\; \frac{x e^x}{35}\, [(-6\,x + 6\,x^2 + 5\,x^3)I_0(x) + (12 - 12\,x + 9\,x^2 - 5\,x^3)I_1(x)]$$

$$\int x^3\, e^x\, K_0(x)\, dx = \frac{x e^x}{35}\, [(5\,x^3 + 6\,x^2 - 6\,x)\, K_0(x) + (5\,x^3 - 9\,x^2 + 12\,x - 12)\, K_1(x)]$$

$$\int x^3 e^x\, I_1(x)\, dx \;=\; \frac{x e^x}{35}\, [(-8\,x + 8\,x^2 - 5\,x^3)I_0(x) + (16 - 16\,x + 12\,x^2 + 5\,x^3)I_1(x)]$$

$$\int x^3\, e^x\, K_1(x)\, dx = \frac{x e^x}{35}\, [(5\,x^3 - 8\,x^2 + 8\,x)\, K_0(x) + (5\,x^3 + 12\,x^2 - 16\,x + 16)\, K_1(x)]$$

$$\int x^4 e^x\, I_0(x)\, dx \;=\; \frac{x e^x}{315}\, [(96\,x - 96\,x^2 + 60\,x^3 + 35\,x^4)I_0(x) + (-192 + 192\,x - 144\,x^2 + 80\,x^3 - 35\,x^4)I_1(x)]$$

$$\int x^4\, e^x\, K_0(x)\, dx = \frac{x e^x}{315}\, [(35\,x^4 + 60\,x^3 - 96\,x^2 + 96\,x)\, K_0(x) + (35\,x^4 - 80\,x^3 + 144\,x^2 - 192\,x + 192)\, K_1(x)]$$

$$\int x^4 e^x\, I_1(x)\, dx \;=\; \frac{x e^x}{63}\, [(24\,x - 24\,x^2 + 15\,x^3 - 7\,x^4)I_0(x) + (-48 + 48\,x - 36\,x^2 + 20\,x^3 + 7\,x^4)I_1(x)]$$

$$\int x^4\, e^x\, K_1(x)\, dx = \frac{x e^x}{63}\, [(7\,x^4 - 15\,x^3 + 24\,x^2 - 24\,x)\, K_0(x) + (7\,x^4 + 20\,x^3 - 36\,x^2 + 48\,x - 48)\, K_1(x)]$$

$$\int x^5 e^x\, I_0(x)\, dx \;=\; \frac{x e^x}{693}\, [(-480\,x + 480\,x^2 - 300\,x^3 + 140\,x^4 + 63\,x^5)I_0(x) +$$
$$+ (960 - 960\,x + 720\,x^2 - 400\,x^3 + 175\,x^4 - 63\,x^5)I_1(x)]$$

$$\int x^5\, e^x\, K_0(x)\, dx = \frac{xe^x}{693} \left[(63\, x^5 + 140\, x^4 - 300\, x^3 + 480\, x^2 - 480\, x)\, K_0(x) + \right.$$
$$\left. + (63\, x^5 - 175\, x^4 + 400\, x^3 - 720\, x^2 + 960\, x - 960)\, K_1(x) \right]$$

$$\int x^5 e^x\, I_1(x)\, dx = \frac{xe^x}{231} \left[(-192\, x + 192\, x^2 - 120\, x^3 + 56\, x^4 - 21\, x^5) I_0(x) + \right.$$
$$\left. + (384 - 384\, x + 288\, x^2 - 160\, x^3 + 70\, x^4 + 21\, x^5) I_1(x) \right]$$

$$\int x^5\, e^x\, K_1(x)\, dx = \frac{xe^x}{231} \left[(21\, x^5 - 56\, x^4 + 120\, x^3 - 192\, x^2 + 192\, x)\, K_0(x) + \right.$$
$$\left. + (21\, x^5 + 70\, x^4 - 160\, x^3 + 288\, x^2 - 384\, x + 384)\, K_1(x) \right]$$

$$\int x^6 e^x\, I_0(x)\, dx = \frac{xe^x}{1001} \left[(1920\, x - 1920\, x^2 + 1200\, x^3 - 560\, x^4 + 210 x^5 + 77 x^6) I_0(x) + \right.$$
$$\left. + (-3840 + 3840\, x - 2880\, x^2 + 1600\, x^3 - 700\, x^4 + 252 x^5 - 77 x^6) I_1(x) \right]$$

$$\int x^6\, e^x\, K_0(x)\, dx = \frac{xe^x}{1001} \left[(77\, x^6 + 210\, x^5 - 560\, x^4 + 1200\, x^3 - 1920\, x^2 + 1920\, x)\, K_0(x) + \right.$$
$$\left. + (77\, x^6 - 252\, x^5 + 700\, x^4 - 1600\, x^3 + 2880\, x^2 - 3840\, x + 3840)\, K_1(x) \right]$$

$$\int x^6 e^x\, I_1(x)\, dx = \frac{xe^x}{429} \left[(960\, x - 960\, x^2 + 600\, x^3 - 280\, x^4 + 105\, x^5 - 33\, x^6) I_0(x) + \right.$$
$$\left. + (-1920 + 1920\, x - 1440\, x^2 + 800\, x^3 - 350\, x^4 + 126\, x^5 + 33\, x^6) I_1(x) \right]$$

$$\int x^6\, e^x\, K_1(x)\, dx = \frac{xe^x}{429} \left[(33\, x^6 - 105\, x^5 + 280\, x^4 - 600\, x^3 + 960\, x^2 - 960\, x)\, K_0(x) + \right.$$
$$\left. + (33\, x^6 + 126\, x^5 - 350\, x^4 + 800\, x^3 - 1440\, x^2 + 1920\, x - 1920)\, K_1(x) \right]$$

$$\int x^7 e^x\, I_0(x)\, dx = \frac{xe^x}{2145} \left[(-13440\, x + 13440\, x^2 - 8400\, x^3 + 3920\, x^4 - 1470\, x^5 + 462\, x^6 + 143\, x^7) I_0(x) + \right.$$
$$\left. + (26880 - 26880\, x + 20160\, x^2 - 11200\, x^3 + 4900\, x^4 - 1764\, x^5 + 539\, x^6 - 143\, x^7) I_1(x) \right]$$

$$\int x^7\, e^x\, K_0(x)\, dx = \frac{xe^x}{2145} \left[(143\, x^7 + 462\, x^6 - 1470\, x^5 + 3920\, x^4 - 8400\, x^3 + 13440\, x^2 - 13440\, x)\, K_0(x) + \right.$$
$$\left. + (143\, x^7 - 539\, x^6 + 1764\, x^5 - 4900\, x^4 + 11200\, x^3 - 20160\, x^2 + 26880\, x - 26880)\, K_1(x) \right]$$

$$\int x^7 e^x\, I_1(x)\, dx = \frac{xe^x}{2145} \left[(-15360\, x + 15360\, x^2 - 9600\, x^3 + 4480\, x^4 - 1680\, x^5 + 528\, x^6 - 143\, x^7) I_0(x) + \right.$$
$$\left. + (30720 - 30720\, x + 23040\, x^2 - 12800\, x^3 + 5600\, x^4 - 2016\, x^5 + 616\, x^6 + 143\, x^7) I_1(x) \right]$$

$$\int x^7\, e^x\, K_1(x)\, dx = \frac{xe^x}{2145} \left[(143\, x^7 - 528\, x^6 + 1680\, x^5 - 4480\, x^4 + 9600\, x^3 - 15360\, x^2 + 15360\, x)\, K_0(x) + \right.$$
$$\left. + (143\, x^7 + 616\, x^6 - 2016\, x^5 + 5600\, x^4 - 12800\, x^3 + 23040\, x^2 - 30720\, x + 30720)\, K_1(x) \right]$$

Recurrence formulas:

$$\int x^n e^x I_0(x)\, dx = \frac{x^n e^x}{2n+1}[(n+x)I_0(x) - xI_1(x)] - \frac{n^2}{2n+1} \int x^{n-1} e^x I_0(x)\, dx \quad (*)$$

$$\int x^n e^x I_1(x)\, dx = \frac{x^n e^x}{2n+1}[(n+1-x)I_0(x) + xI_1(x)] - \frac{n(n+1)}{2n+1} \int x^{n-1} e^x I_0(x)\, dx \quad (*)$$

The last formula refers to $I_0(x)$ instead of $I_1(x)$.

$$\int x^n e^x K_0(x)\, dx = \frac{x^n e^x}{2n+1}[(n+x)K_0(x) + xK_1(x)] - \frac{n^2}{2n+1} \int x^{n-1} e^x K_0(x)\, dx$$

$$\int x^n e^x K_1(x)\, dx = \frac{x^n e^x}{2n+1}[(x-n-1)K_0(x) + xK_1(x)] + \frac{n(n+1)}{2n+1} \int x^{n-1} e^x K_0(x)\, dx$$

The last formula refers to $K_0(x)$ instead of $K_1(x)$.

b) Integrals with e^{-x} :

$$\int e^{-x} I_0(x)\, dx \;=\; xe^{-x}\left[I_0(x) + I_1(x)\right] \;, \qquad \int \frac{e^{-x} \cdot I_1(x)\, dx}{x} = e^{-x}\left[I_0(x) + I_1(x)\right]$$

$$\int e^{-x} K_0(x)\, dx = xe^{-x}\left[K_0(x) - K_1(x)\right] \;, \qquad \int \frac{e^{-x} K_1(x)\, dx}{x} = e^{-x}\left[K_0(x) - K_1(x)\right]$$

$$\int e^{-x} I_1(x)\, dx \;=\; e^{-x}\left[(1+x)I_0(x) + xI_1(x)\right]$$

$$\int e^{-x} K_1(x)\, dx = -e^{-x}\left[(1+x)\,K_0(x) - x\,K_1(x)\right]$$

$$\int xe^{-x} I_0(x)\, dx \;=\; \frac{xe^{-x}}{3}\left[xI_0(x) + (1+x)I_1(x)\right]$$

$$\int x\,e^{-x} K_0(x)\, dx = \frac{xe^{-x}}{3}\left[x\,K_0(x) - (x+1)\,K_1(x)\right]$$

$$\int xe^{-x} I_1(x)\, dx \;=\; \frac{xe^{-x}}{3}\left[xI_0(x) + (-2+x)I_1(x)\right]$$

$$\int x\,e^{-x} K_1(x)\, dx = \frac{xe^{-x}}{3}\left[-x\,K_0(x) + (x-2)\,K_1(x)\right]$$

$$\int x^2 e^{-x} I_0(x)\, dx \;=\; \frac{xe^{-x}}{15}\left[(-2\,x + 3\,x^2)I_0(x) + (4 + 4\,x + 3\,x^2)I_1(x)\right]$$

$$\int x^2\, e^{-x} K_0(x)\, dx = \frac{xe^{-x}}{15}\left[(3\,x^2 - 2\,x)\,K_0(x) - (3\,x^2 + 4\,x + 4)\,K_1(x)\right]$$

$$\int x^2 e^{-x} I_1(x)\, dx \;=\; \frac{xe^{-x}}{5}\left[(x + x^2)I_0(x) + (-2 - 2x + x^2)I_1(x)\right]$$

$$\int x^2\, e^{-x} K_1(x)\, dx = \frac{xe^{-x}}{5}\left[-(x^2 + x)\,K_0(x) + (x^2 - 2\,x - 2)\,K_1(x)\right]$$

$$\int x^3 e^{-x} I_0(x)\, dx \;=\; \frac{xe^{-x}}{35}\left[(-6\,x - 6\,x^2 + 5\,x^3)I_0(x) + (12 + 12\,x + 9\,x^2 + 5\,x^3)I_1(x)\right]$$

$$\int x^3\, e^{-x} K_0(x)\, dx = \frac{xe^{-x}}{35}\left[(5\,x^3 - 6\,x^2 - 6\,x)\,K_0(x) - (5\,x^3 + 9\,x^2 + 12\,x + 12)\,K_1(x)\right]$$

$$\int x^3 e^{-x} I_1(x)\, dx \;=\; \frac{xe^{-x}}{35}\left[(8\,x + 8\,x^2 + 5\,x^3)I_0(x) + (-16 - 16\,x - 12\,x^2 + 5\,x^3)I_1(x)\right]$$

$$\int x^3\, e^{-x} K_1(x)\, dx = \frac{xe^{-x}}{35}\left[-(5\,x^3 + 8\,x^2 + 8\,x)\,K_0(x) + (5\,x^3 - 12\,x^2 - 16\,x - 16)\,K_1(x)\right]$$

$$\int x^4 e^{-x} I_0(x)\, dx \;=\; \frac{xe^{-x}}{315}\left[(-96\,x - 96\,x^2 - 60\,x^3 + 35\,x^4)I_0(x) + (192 + 192\,x + 144\,x^2 + 80\,x^3 + 35\,x^4)I_1(x)\right]$$

$$\int x^4\, e^{-x} K_0(x)\, dx =$$

$$= \frac{xe^{-x}}{315}\left[(35\,x^4 - 60\,x^3 - 96\,x^2 - 96\,x)\,K_0(x) - (35\,x^4 + 80\,x^3 + 144\,x^2 + 192\,x + 192)\,K_1(x)\right]$$

$$\int x^4 e^{-x} I_1(x)\, dx \;=\; \frac{xe^{-x}}{63}\left[(24\,x + 24\,x^2 + 15\,x^3 + 7\,x^4)I_0(x) + (-48 - 48\,x - 36\,x^2 - 20\,x^3 + 7\,x^4)I_1(x)\right]$$

$$\int x^4\, e^{-x} K_1(x)\, dx =$$

$$= \frac{xe^{-x}}{63}\left[-(7\,x^4 + 15\,x^3 + 24\,x^2 + 24\,x)\,K_0(x) + (7\,x^4 - 20\,x^3 - 36\,x^2 - 48\,x - 48)\,K_1(x)\right]$$

$$\int x^5 e^{-x} I_0(x)\, dx \;=\; \frac{xe^{-x}}{693}\Big[(-480\,x - 480\,x^2 - 300\,x^3 - 140\,x^4 + 63\,x^5)I_0(x)+$$
$$+(960 + 960\,x + 720\,x^2 + 400\,x^3 + 175\,x^4 + 63\,x^5)I_1(x)\Big]$$

$$\int x^5 e^{-x} K_0(x)\, dx = \frac{xe^{-x}}{693}\Big[(63\,x^5 - 140\,x^4 - 300\,x^3 - 480\,x^2 - 480\,x)\,K_0(x)-$$
$$-(63\,x^5 + 175\,x^4 + 400\,x^3 + 720\,x^2 + 960\,x + 960)\,K_1(x)\Big]$$

$$\int x^5 e^{-x} I_1(x)\, dx \;=\; \frac{xe^{-x}}{231}\Big[(192\,x + 192\,x^2 + 120\,x^3 + 56\,x^4 + 21\,x^5)I_0(x)+$$
$$+(-384 - 384\,x - 288\,x^2 - 160\,x^3 - 70\,x^4 + 21\,x^5)I_1(x)\Big]$$

$$\int x^5 e^{-x} K_1(x)\, dx = \frac{xe^{-x}}{231}\Big[-(21\,x^5 + 56\,x^4 + 120\,x^3 + 192\,x^2 + 192\,x)\,K_0(x)+$$
$$+(21\,x^5 - 70\,x^4 - 160\,x^3 - 288\,x^2 - 384\,x - 384)\,K_1(x)\Big]$$

$$\int x^6 e^{-x} I_0(x)\, dx \;=\; \frac{xe^{-x}}{1001}\Big[(-1920\,x - 1920\,x^2 - 1200\,x^3 - 560\,x^4 - 210x^5 + 77x^6)I_0(x)+$$
$$+ (3840 + 3840\,x + 2880\,x^2 + 1600\,x^3 + 700\,x^4 + 252x^5 + 77x^6)I_1(x)\Big]$$

$$\int x^6 e^{-x} K_0(x)\, dx = \frac{xe^{-x}}{1001}\Big[(77\,x^6 - 210\,x^5 - 560\,x^4 - 1200\,x^3 - 1920\,x^2 - 1920\,x)\,K_0(x)-$$
$$-(77\,x^6 + 252\,x^5 + 700\,x^4 + 1600\,x^3 + 2880\,x^2 + 3840\,x + 3840)\,K_1(x)\Big]$$

$$\int x^6 e^{-x} I_1(x)\, dx \;=\; \frac{xe^{-x}}{429}\Big[(960\,x + 960\,x^2 + 600\,x^3 + 280\,x^4 + 105\,x^5 + 33\,x^6)I_0(x)+$$
$$+(-1920 - 1920\,x - 1440\,x^2 - 800\,x^3 - 350\,x^4 - 126\,x^5 + 33\,x^6)I_1(x)\Big]$$

$$\int x^6 e^{-x} K_1(x)\, dx = \frac{xe^{-x}}{429}\Big[-(33\,x^6 + 105\,x^5 + 280\,x^4 + 600\,x^3 + 960\,x^2 + 960\,x)\,K_0(x)+$$
$$+(33\,x^6 - 126\,x^5 - 350\,x^4 - 800\,x^3 - 1440\,x^2 - 1920\,x - 1920)\,K_1(x)\Big]$$

$$\int x^7 e^{-x} I_0(x)\, dx \;=\; \frac{xe^{-x}}{2145}\Big[(-13440\,x - 13440\,x^2 - 8400\,x^3 - 3920\,x^4 - 1470\,x^5 - 462\,x^6 + 143\,x^7)I_0(x)+$$
$$+(26880 + 26880\,x + 20160\,x^2 + 11200\,x^3 + 4900\,x^4 + 1764\,x^5 + 539\,x^6 + 143\,x^7)I_1(x)\Big]$$

$$\int x^7 e^{-x} K_0(x)\, dx =$$
$$= \frac{xe^{-x}}{2145}\Big[(143\,x^7 - 462\,x^6 - 1470\,x^5 - 3920\,x^4 - 8400\,x^3 - 13440\,x^2 - 13440\,x)\,K_0(x)-$$
$$-(143\,x^7 + 539\,x^6 + 1764\,x^5 + 4900\,x^4 + 11200\,x^3 + 20160\,x^2 + 26880\,x + 26880)\,K_1(x)\Big]$$

$$\int x^7 e^{-x} I_1(x)\, dx \;=\; \frac{xe^{-x}}{2145}\Big[(15360\,x + 15360\,x^2 + 9600\,x^3 + 4480\,x^4 + 1680\,x^5 + 528\,x^6 + 143\,x^7)I_0(x)+$$
$$+(-30720 - 30720\,x - 23040\,x^2 - 12800\,x^3 - 5600\,x^4 - 2016\,x^5 - 616\,x^6 + 143\,x^7)I_1(x)\Big]$$

$$\int x^7 e^{-x} K_1(x)\, dx =$$
$$= \frac{xe^{-x}}{2145}\Big[-(143\,x^7 + 528\,x^6 + 1680\,x^5 + 4480\,x^4 + 9600\,x^3 + 15360\,x^2 + 15360\,x)\,K_0(x)+$$
$$+(143\,x^7 - 616\,x^6 - 2016\,x^5 - 5600\,x^4 - 12800\,x^3 - 23040\,x^2 - 30720\,x - 30720)\,K_1(x)\Big]$$

Recurrence formulas:

$$\int x^n e^{-x} I_0(x)\, dx = \frac{x^n e^{-x}}{2n+1}[(x-n)I_0(x) + xI_1(x)] + \frac{n^2}{2n+1}\int x^{n-1} e^{-x} I_0(x)\, dx \quad (*)$$

$$\int x^n e^{-x} I_1(x)\, dx = \frac{x^n e^{-x}}{2n+1}[(n+1+x)I_0(x) + xI_1(x)] - \frac{n(n+1)}{2n+1}\int x^{n-1} e^{-x} I_0(x)\, dx \quad (*)$$

The last formula refers to $I_0(x)$ instead of $I_1(x)$.

$$\int x^n e^{-x} K_0(x)\, dx = \frac{x^n e^{-x}}{2n+1}[(x-n)K_0(x) - xK_1(x)] + \frac{n^2}{2n+1}\int x^{n-1} e^{-x} K_0(x)\, dx$$

$$\int x^n e^{-x} K_1(x)\, dx = \frac{x^n e^{-x}}{2n+1}[-(x+n+1)K_0(x) + xK_1(x)] + \frac{n(n+1)}{2n+1}\int x^{n-1} e^{-x} K_0(x)\, dx$$

The last formula refers to $K_0(x)$ instead of $K_1(x)$.

1.2.3. Integrals of the type $\int x^n \cdot \left\{ \begin{array}{c} \sinh \\ \cosh \end{array} \right\} x \cdot I_\nu(x)\, dx$

$$\int \frac{\sinh x\, I_1(x)\, dx}{x} = x\, \cosh x\, I_0(x) - \sinh x\, I_1(x)$$

$$\int \frac{\cosh x\, I_1(x)\, dx}{x} = x\, \sinh x\, I_0(x) - \cosh x\, I_1(x)$$

$$\int \sinh x\, I_0(x)\, dx = x\, \sinh x\, I_0(x) - x\, \cosh x\, I_1(x)$$

$$\int \cosh x\, I_0(x)\, dx = x\, \cosh x\, I_0(x) - x\, \sinh x\, I_1(x)$$

$$\int \sinh x\, I_1(x)\, dx = \sinh x\, I_0(x) - x\, \cosh x\, I_0(x) + x\, \sinh x\, I_1(x)$$

$$\int \cosh x\, I_1(x)\, dx = -x\, \sinh x\, I_0(x) + \cosh x\, I_0(x) + x\, \cosh x\, I_1(x)$$

$$\int x\, \sinh x\, I_0(x)\, dx = \frac{x^2}{3}\, \sinh x\, I_0(x) + \frac{x}{3}\, \sinh x\, I_1(x) - \frac{x^2}{3}\, \cosh x\, I_1(x)$$

$$\int x\, \cosh x\, I_0(x)\, dx = \frac{x^2}{3}\, \cosh x\, I_0(x) - \frac{x^2}{3}\, \sinh x\, I_1(x) + \frac{x}{3}\, \cosh x\, I_1(x)$$

$$\int x\, \sinh x\, I_1(x)\, dx = -\frac{x^2}{3}\, \cosh x\, I_0(x) + \frac{x^2}{3}\, \sinh x\, I_1(x) + \frac{2x}{3}\, \cosh x\, I_1(x)$$

$$\int x\, \cosh x\, I_1(x)\, dx = -\frac{x^2}{3}\, \sinh x\, I_0(x) + \frac{2x}{3}\, \sinh x\, I_1(x) + \frac{x^2}{3}\, \cosh x\, I_1(x)$$

$$\int x^2\, \sinh x\, I_0(x)\, dx = \frac{x^3}{5}\, \sinh x\, I_0(x) + \frac{2x^2}{15}\, \cosh x\, I_0(x) + \frac{4x^2}{15}\, \sinh x\, I_1(x) - \frac{3x^3 + 4x}{15}\, \cosh x\, I_1(x)$$

$$\int x^2\, \cosh x\, I_0(x)\, dx = \frac{2x^2}{15}\, \sinh x\, I_0(x) + \frac{x^3}{5}\, \cosh x\, I_0(x) - \frac{3\,x^3 + 4\,x}{15}\, \sinh x\, I_1(x) + \frac{4x^2}{15}\, \cosh x\, I_1(x)$$

$$\int x^2\, \sinh x\, I_1(x)\, dx = \frac{x^2}{5}\, \sinh x\, I_0(x) - \frac{x^3}{5}\, \cosh x\, I_0(x) + \frac{x^3 - 2\,x}{5}\, \sinh x\, I_1(x) + \frac{2x^2}{5}\, \cosh x\, I_1(x)$$

$$\int x^2\, \cosh x\, I_1(x)\, dx = -\frac{x^3}{5}\, \sinh x\, I_0(x) + \frac{x^2}{5}\, \cosh x\, I_0(x) + \frac{2x^2}{5}\, \sinh x\, I_1(x) + \frac{x^3 - 2\,x}{5}\, \cosh x\, I_1(x)$$

$$\int x^3\, \sinh x\, I_0(x)\, dx =$$

$$= \frac{5\,x^4 - 6\,x^2}{35}\, \sinh x\, I_0(x) + \frac{6x^3}{35}\, \cosh x\, I_0(x) + \frac{9\,x^3 + 12\,x}{35}\, \sinh x\, I_1(x) - \frac{5\,x^4 + 12\,x^2}{35}\, \cosh x\, I_1(x)$$

$$\int x^3\, \cosh x\, I_0(x)\, dx =$$

$$= \frac{6x^3}{35}\, \sinh x\, I_0(x) + \frac{5\,x^4 - 6\,x^2}{35}\, \cosh x\, I_0(x) - \frac{5\,x^4 + 12\,x^2}{35}\, \sinh x\, I_1(x) + \frac{9\,x^3 + 12\,x}{35}\, \cosh x\, I_1(x)$$

$$\int x^3\, \sinh x\, I_1(x)\, dx =$$

$$= \frac{8x^3}{35}\, \sinh x\, I_0(x) - \frac{5\,x^4 + 8\,x^2}{35}\, \cosh x\, I_0(x) + \frac{5\,x^4 - 16\,x^2}{35}\, \sinh x\, I_1(x) + \frac{12\,x^3 + 16\,x}{35}\, \cosh x\, I_1(x)$$

$$\int x^3\, \cosh x\, I_1(x)\, dx =$$

$$= -\frac{5\,x^4 + 8\,x^2}{35}\, \sinh x\, I_0(x) + \frac{8x^3}{35}\, \cosh x\, I_0(x) + \frac{12\,x^3 + 16\,x}{35}\, \sinh x\, I_1(x) + \frac{5\,x^4 - 16\,x^2}{35}\, \cosh x\, I_1(x)$$

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$$\int x^4 \sinh x\, I_0(x)\, dx = \frac{35\, x^5 - 96\, x^3}{315}\ \sinh x\, I_0(x) + \frac{20\, x^4 + 32\, x^2}{105}\ \cosh x\, I_0(x) +$$

$$+ \frac{80\, x^4 + 192\, x^2}{315}\ \sinh x\, I_1(x) - \frac{35 x^5 + 144 x^3 + 192 x}{315}\ \cosh x\, I_1(x)$$

$$\int x^4 \cosh x\, I_0(x)\, dx = \frac{20\, x^4 + 32\, x^2}{105}\ \sinh x\, I_0(x) + \frac{35\, x^5 - 96\, x^3}{315}\ \cosh x\, I_0(x) -$$

$$- \frac{35\, x^5 + 144\, x^3 + 192\, x}{315}\ \sinh x\, I_1(x) + \frac{80\, x^4 + 192\, x^2}{315}\ \cosh x\, I_1(x)$$

$$\int x^4 \sinh x\, I_1(x)\, dx = \frac{5\, x^4 + 8\, x^2}{21}\ \sinh x\, I_0(x) - \frac{7\, x^5 + 24\, x^3}{63}\ \cosh x\, I_0(x) +$$

$$+ \frac{7\, x^5 - 36\, x^3 - 48\, x}{63}\ \sinh x\, I_1(x) + \frac{20\, x^4 + 48\, x^2}{63}\ \cosh x\, I_1(x)$$

$$\int x^4 \cosh x\, I_1(x)\, dx = -\frac{7\, x^5 + 24\, x^3}{63}\ \sinh x\, I_0(x) + \frac{5\, x^4 + 8\, x^2}{21}\ \cosh x\, I_0(x) +$$

$$+ \frac{20\, x^4 + 48\, x^2}{63}\ \sinh x\, I_1(x) + \frac{7\, x^5 - 36\, x^3 - 48\, x}{63}\ \cosh x\, I_1(x)$$

$$\int x^5 \sinh x\, I_0(x)\, dx = \frac{21\, x^6 - 100\, x^4 - 160\, x^2}{231}\ \sinh x\, I_0(x) + \frac{140\, x^5 + 480\, x^3}{693}\ \cosh x\, I_0(x) +$$

$$+ \frac{175\, x^5 + 720\, x^3 + 960\, x}{693}\ \sinh x\, I_1(x) - \frac{63\, x^6 + 400\, x^4 + 960\, x^2}{693}\ \cosh x\, I_1(x)$$

$$\int x^5 \cosh x\, I_0(x)\, dx = \frac{140\, x^5 + 480\, x^3}{693}\ \sinh x\, I_0(x) + \frac{21\, x^6 - 100\, x^4 - 160\, x^2}{231}\ \cosh x\, I_0(x) -$$

$$- \frac{63\, x^6 + 400\, x^4 + 960\, x^2}{693}\ \sinh x\, I_1(x) + \frac{175\, x^5 + 720\, x^3 + 960\, x}{693}\ \cosh x\, I_1(x)$$

$$\int x^5 \sinh x\, I_1(x)\, dx = \frac{56\, x^5 + 192\, x^3}{231}\ \sinh x\, I_0(x) - \frac{7\, x^6 + 40\, x^4 + 64\, x^2}{77}\ \cosh x\, I_0(x) +$$

$$+ \frac{21\, x^6 - 160\, x^4 - 384\, x^2}{231}\ \sinh x\, I_1(x) + \frac{70\, x^5 + 288\, x^3 + 384\, x}{231}\ \cosh x\, I_1(x)$$

$$\int x^5 \cosh x\, I_1(x)\, dx = -\frac{7\, x^6 + 40\, x^4 + 64\, x^2}{77}\ \sinh x\, I_0(x) + \frac{56\, x^5 + 192\, x^3}{231}\ \cosh x\, I_0(x) +$$

$$+ \frac{70\, x^5 + 288\, x^3 + 384\, x}{231}\ \sinh x\, I_1(x) + \frac{21\, x^6 - 160\, x^4 - 384\, x^2}{231}\ \cosh x\, I_1(x)$$

$$\int x^6 \sinh x\, I_0(x)\, dx = \frac{77\, x^7 - 560\, x^5 - 1920\, x^3}{1001}\ \sinh x\, I_0(x) + \frac{210\, x^6 + 1200\, x^4 + 1920\, x^2}{1001}\ \cosh x\, I_0(x) +$$

$$+ \frac{252\, x^6 + 1600\, x^4 + 3840\, x^2}{1001}\ \sinh x\, I_1(x) - \frac{77\, x^7 + 700\, x^5 + 2880\, x^3 + 3840\, x}{1001}\ \cosh x\, I_1(x)$$

$$\int x^6 \cosh x\, I_0(x)\, dx = \frac{210\, x^6 + 1200\, x^4 + 1920\, x^2}{1001}\ \sinh x\, I_0(x) + \frac{77\, x^7 - 560\, x^5 - 1920\, x^3}{1001}\ \cosh x\, I_0(x) -$$

$$- \frac{77\, x^7 + 700\, x^5 + 2880\, x^3 + 3840\, x}{1001}\ \sinh x\, I_1(x) + \frac{252\, x^6 + 1600\, x^4 + 3840\, x^2}{1001}\ \cosh x\, I_1(x)$$

$$\int x^6 \sinh x\, I_1(x)\, dx = \frac{35\, x^6 + 200\, x^4 + 320\, x^2}{143}\ \sinh x\, I_0(x) - \frac{33\, x^7 + 280\, x^5 + 960\, x^3}{429}\ \cosh x\, I_0(x) +$$

$$+ \frac{33\, x^7 - 350\, x^5 - 1440\, x^3 - 1920\, x}{429}\ \sinh x\, I_1(x) + \frac{126\, x^6 + 800\, x^4 + 1920\, x^2}{429}\ \cosh x\, I_1(x)$$

$$\int x^6 \cosh x\, I_1(x)\, dx = -\frac{33\, x^7 + 280\, x^5 + 960\, x^3}{429}\ \sinh x\, I_0(x) + \frac{35\, x^6 + 200\, x^4 + 320\, x^2}{143}\ \cosh x\, I_0(x) +$$

$$+ \frac{126\, x^6 + 800\, x^4 + 1920\, x^2}{429}\ \sinh x\, I_1(x) + \frac{33\, x^7 - 350\, x^5 - 1440\, x^3 - 1920\, x}{429}\ \cosh x\, I_1(x)$$

Recurrence formulas:

$$\int x^{n+1} \sinh x \cdot I_0(x)\, dx =$$

$$= \frac{x^{n+1}}{2n+3} \left[x \sinh x \cdot I_0(x) + (n+1) \cosh x \cdot I_0(x) - x \cosh x \cdot I_1(x) \right] - \frac{(n+1)^2}{2n+3} \int x^n \cosh x \cdot I_0(x)\, dx$$

$$\int x^{n+1} \cosh x \cdot I_0(x)\, dx =$$

$$= \frac{x^{n+1}}{2n+3} \left[(n+1) \sinh x \cdot I_0(x) + x \cosh x \cdot I_0(x) - x \sinh x \cdot I_1(x) \right] - \frac{(n+1)^2}{2n+3} \int x^n \sinh x \cdot I_0(x)\, dx$$

$$\int x^{n+1} \cosh x \cdot I_0(x)\, dx =$$

$$= \frac{x^{n+1}}{2n+3} \left[(n+2) \sinh x \cdot I_0(x) - x \cosh x \cdot I_0(x) + x \sinh x \cdot I_1(x) \right] - \frac{(n+1)(n+2)}{2n+3} \int x^n \sinh x \cdot I_0(x)\, dx$$

$$\int x^{n+1} \cosh x \cdot I_0(x)\, dx =$$

$$= \frac{x^{n+1}}{2n+3} \left[-x \sinh x \cdot I_0(x) + (n+2) \cosh x \cdot I_0(x) + x \cosh x \cdot I_1(x) \right] - \frac{(n+1)}{2n+3} \int x^n \cosh x \cdot I_0(x)\, dx$$

1.2.4. Integrals of the type $\int x^n \cdot \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} x \cdot J_\nu(x)\, dx$

See also [1], 11.3.

$$\int \frac{\sin x \cdot J_0(x)\, dx}{x} \;=\; -\sin x\, J_1(x) - \cos x\, J_0(x)$$

$$\int \frac{\cos x \cdot J_0(x)\, dx}{x} \;=\; \sin x\, J_0(x) - \cos x\, J_1(x)$$

$$\int \sin x \cdot J_0(x)\, dx \;=\; x[\sin x \cdot J_0(x) - \cos x \cdot J_1(x)]$$

$$\int \cos x \cdot J_0(x)\, dx \;=\; x[\cos x \cdot J_0(x) + \sin x \cdot J_1(x)]$$

$$\int \sin x \cdot J_1(x)\, dx \;=\; (x \cos x - \sin x)J_0(x) + x \sin x \cdot J_1(x)$$

$$\int \cos x \cdot J_1(x)\, dx \;=\; -(x \sin x + \cos x)J_0(x) + x \cos x \cdot J_1(x)$$

$$\int x \sin x \cdot J_0(x)\, dx \;=\; \frac{x^2}{3} \sin x \cdot J_0(x) + \frac{x \sin x - x^2 \cos x}{3} \cdot J_1(x)$$

$$\int x \cos x \cdot J_0(x)\, dx \;=\; \frac{x^2}{3} \cos x \cdot J_0(x) + \frac{x^2 \sin x - x \cos x}{3} \cdot J_1(x)$$

$$\int x \sin x \cdot J_1(x)\, dx \;=\; \frac{x^2}{3} \cdot \cos x \cdot J_0(x) + \frac{x^2 \sin x - 2x \cos x}{3} \cdot J_1(x)$$

$$\int x \cos x \cdot J_1(x)\, dx \;=\; -\frac{x^2}{3} \sin x \cdot J_0(x) + \frac{2x \sin x + x^2 \cos x}{3} \cdot J_1(x)$$

$$\int x^2 \sin x \cdot J_0(x)\, dx \;=\; \frac{1}{15} \left\{ \left[3 x^3 \sin x - 2 x^2 \cos x \right] \cdot J_0(x) + \left[4 x^2 \sin x + (4 x - 3 x^3) \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^2 \cos x \cdot J_0(x)\, dx \;=\; \frac{1}{15} \left\{ \left[2 x^2 \sin x + 3 x^3 \cos x \right] \cdot J_0(x) + \left[(-4 x + 3 x^3) \sin x + 4 x^2 \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^2 \sin x \cdot J_1(x)\, dx \;=\; \frac{1}{5} \left\{ \left[-x^2 \sin x + x^3 \cos x \right] \cdot J_0(x) + \left[(2 x + x^3) \sin x - 2 x^2 \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^2 \cos x \cdot J_1(x)\, dx \;=\; \frac{1}{5} \left\{ \left[-x^3 \sin x - x^2 \cos x \right] \cdot J_0(x) + \left[2 x^2 \sin x + (2 x + x^3) \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^3 \sin x \cdot J_0(x)\, dx \;=\; \frac{1}{35} \left\{ \left[(6 x^2 + 5 x^4) \sin x - 6 x^3 \cos x \right] \cdot J_0(x) + \right.$$
$$+ \left. \left[(-12 x + 9 x^3) \sin x + (12 x^2 - 5 x^4) \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^3 \cos x \cdot J_0(x)\, dx \;=\; \frac{1}{35} \left\{ \left[6 x^3 \sin x + (6 x^2 + 5 x^4) \cos x \right] \cdot J_0(x) + \right.$$
$$+ \left. \left[(-12 x^2 + 5 x^4) \sin x + (-12 x + 9 x^3) \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^3 \sin x \cdot J_1(x)\, dx \;=\; \frac{1}{35} \left\{ \left[-8x^3 \sin x + (-8 x^2 + 5 x^4) \cos x \right] \cdot J_0(x) + \right.$$
$$+ \left. \left[(16 x^2 + 5 x^4) \sin x + (16 x - 12 x^3) \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^3 \cos x \cdot J_1(x)\, dx \;=\; \frac{1}{35} \left\{ \left[(8 x^2 - 5 x^4) \sin x - 8 x^3 \cos x \right] \cdot J_0(x) + \right.$$
$$+ \left. \left[(-16 x + 12 x^3) \sin x + (16 x^2 + 5 x^4) \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^4 \sin x \cdot J_0(x)\, dx \;=\; \frac{1}{315} \left\{ \left[(96 x^3 + 35 x^5) \sin x + (96 x^2 - 60 x^4) \cos x \right] \cdot J_0(x) + \right.$$
$$+ \left. \left[(-192 x^2 + 80 x^4) \sin x + (-192 x + 144 x^3 - 35 x^5) \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^4 \cos x \cdot J_0(x)\, dx \;=\; \frac{1}{315}\left\{\left[(-96\,x^2 + 60\,x^4)\sin x + (96\,x^3 + 35\,x^5)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(192\,x - 144\,x^3 + 35\,x^5)\sin x + (-192\,x^2 + 80\,x^4)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^4 \sin x \cdot J_1(x)\, dx \;=\; \frac{1}{315}\left\{\left[(120x^2 - 75x^4)\sin x + (-120\,x^3 + 35\,x^5)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(-240\,x + 180\,x^3 + 35\,x^5)\sin x + (240\,x^2 - 100\,x^4)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^4 \cos x \cdot J_1(x)\, dx \;=\; \frac{1}{315}\left\{\left[(120\,x^3 - 35\,x^5)\sin x + (120\,x^2 - 75\,x^4)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(-240\,x^2 + 100\,x^4)\sin x + (-240\,x + 180\,x^3 + 35\,x^5)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^5 \sin x \cdot J_0(x)\, dx \;=\; \frac{1}{693}\left\{\left[(-480\,x^2 + 300\,x^4 + 63\,x^6)\sin x + (480\,x^3 - 140\,x^5)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(960\,x - 720\,x^3 + 175\,x^5)\sin x + (-960\,x^2 + 400\,x^4 - 63\,x^6)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^5 \cos x \cdot J_0(x)\, dx \;=\; \frac{1}{693}\left\{\left[(-480\,x^3 + 140\,x^5)\sin x + (-480\,x^2 + 300\,x^4 + 63\,x^6)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(960\,x^2 - 400\,x^4 + 63\,x^6)\sin x + (960\,x - 720\,x^3 + 175\,x^5)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^5 \sin x \cdot J_1(x)\, dx \;=\; \frac{1}{231}\left\{\left[(192x^3 - 56x^5)\sin x + (192\,x^2 - 120\,x^4 + 21\,x^6)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(-384\,x^2 + 160\,x^4 + 21\,x^6)\sin x + (-384\,x + 288\,x^3 - 70\,x^5)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^5 \cos x \cdot J_1(x)\, dx \;=\; \frac{1}{231}\left\{\left[(-192\,x^2 + 120\,x^4 - 21\,x^6)\sin x + (192\,x^3 - 56\,x^5)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(384\,x - 288\,x^3 + 70\,x^5)\sin x + (-384\,x^2 + 160\,x^4 + 21\,x^6)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^6 \sin x \cdot J_0(x)\, dx \;=$$

$$=\; \frac{1}{1001}\left\{\left[(-1920\,x^3 + 560\,x^5 + 77\,x^7)\sin x + (-1920\,x^2 + 1200\,x^4 - 210\,x^6)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(3840\,x^2 - 1600\,x^4 + 252\,x^6)\sin x + (3840\,x - 2880\,x^3 + 700\,x^5 - 77\,x^7)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^6 \cos x \cdot J_0(x)\, dx \;=$$

$$=\; \frac{1}{1001}\left\{\left[(1920\,x^2 - 1200\,x^4 + 210\,x^6)\sin x + (-1920\,x^3 + 560\,x^5 + 77\,x^7)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(-3840\,x + 2880\,x^3 - 700\,x^5 + 77\,x^7)\sin x + (3840\,x^2 - 1600\,x^4 + 252\,x^6)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^6 \sin x \cdot J_1(x)\, dx \;=$$

$$=\; \frac{1}{429}\left\{\left[(-960x^2 + 600x^4 - 105x^6)\sin x + (960\,x^3 - 280\,x^5 + 33\,x^7)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(1920\,x - 1440\,x^3 + 350\,x^5 + 33\,x^7)\sin x + (-1920\,x^2 + 800\,x^4 - 126\,x^6)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^6 \cos x \cdot J_1(x)\, dx \;=$$

$$=\; \frac{1}{429}\left\{\left[(-960\,x^3 + 280\,x^5 - 33\,x^7)\sin x + (-960\,x^2 + 600\,x^4 - 105\,x^6)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(1920\,x^2 - 800\,x^4 + 126\,x^6)\sin x + (1920\,x - 1440\,x^3 + 350\,x^5 + 33\,x^7)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^7 \sin x \cdot J_0(x)\, dx \;=$$

$$=\; \frac{1}{2145}\left\{\left[(13440\,x^2 - 8400\,x^4 + 1470\,x^6 + 143\,x^8)\sin x + (-13440\,x^3 + 3920\,x^5 - 462\,x^7)\cos x\right]\cdot J_0(x)+\right.$$

$$\left.+\left[(-26880\,x + 20160\,x^3 - 4900\,x^5 + 539\,x^7)\sin x + (26880\,x^2 - 11200\,x^4 + 1764\,x^6 - 143\,x^8)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^7 \cos x \cdot J_0(x)\, dx \;=\;$$

$$=\; \frac{1}{2145} \left\{ \left[(13440\, x^3 - 3920\, x^5 + 462\, x^7) \sin x + (13440\, x^2 - 8400\, x^4 + 1470\, x^6 + 143\, x^8) \cos x \right] \cdot J_0(x) + \right.$$

$$\left. +\; \left[(-26880\, x^2 + 11200\, x^4 - 1764\, x^6 + 143\, x^8) \sin x + (-26880\, x + 20160\, x^3 - 4900\, x^5 + 539\, x^7) \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^7 \sin x \cdot J_1(x)\, dx \;=\;$$

$$=\; \frac{1}{2145} \left\{ \left[(-15360 x^3 + 4480 x^5 - 528 x^7) \sin x + (-15360\, x^2 + 9600\, x^4 - 1680\, x^6 + 143\, x^8) \cos x \right] \cdot J_0(x) + \right.$$

$$\left. +\; \left[(30720\, x^2 - 12800\, x^4 + 2016\, x^6 + 143\, x^8) \sin x + (30720\, x - 23040\, x^3 + 5600\, x^5 - 616\, x^7) \cos x \right] \cdot J_1(x) \right\}$$

$$\int x^7 \cos x \cdot J_1(x)\, dx \;=\;$$

$$=\; \frac{1}{2145} \left\{ \left[(15360\, x^2 - 9600\, x^4 + 1680\, x^6 - 143\, x^8) \sin x + (-15360\, x^3 + 4480\, x^5 - 528\, x^7) \cos x \right] \cdot J_0(x) + \right.$$

$$\left. +\; \left[(-30720\, x + 23040\, x^3 - 5600\, x^5 + 616\, x^7) \sin x + (30720\, x^2 - 12800\, x^4 + 2016\, x^6 + 143\, x^8) \cos x \right] \cdot J_1(x) \right\}$$

Recurrence formulas:
Let
$$S_n^{(\nu)} = \int x^n\, \sin x \cdot J_\nu(x)\, dx \quad , \qquad C_n^{(\nu)} = \int x^n\, \cos x \cdot J_\nu(x)\, dx$$

and
$$\sigma_n^{(\nu)} = x^n\, \sin x \cdot J_\nu(x) \quad , \qquad \gamma_n^{(\nu)} = x^n\, \cos x \cdot J_\nu(x) \, ,$$

then holds

$$S_n^{(0)} \;=\; \frac{n^2 C_{n-1}^{(0)} - n\gamma_n^{(0)} + \sigma_{n+1}^{(0)} - \gamma_{n+1}^{(1)}}{2n+1} \, , \quad S_n^{(1)} \;=\; \frac{n(n+1) S_{n-1}^{(0)} - (n+1)\sigma_n^{(0)} + \gamma_{n+1}^{(0)} + \sigma_{n+1}^{(1)}}{2n+1} \, ,$$

$$C_n^{(0)} \;=\; \frac{n\sigma_n^{(0)} - n^2 S_{n-1}^{(0)} + \gamma_{n+1}^{(0)} + \sigma_{n+1}^{(1)}}{2n+1} \, , \quad C_n^{(1)} \;=\; \frac{n(n+1) C_{n-1}^{(0)} - (n+1)\gamma_n^{(0)} - \sigma_{n+1}^{(0)} + \gamma_{n+1}^{(1)}}{2n+1} \, .$$

1.2.5. Integrals of the type $\int x^n \cdot e^{ax} \cdot Z_\nu(x)\,dx$

a) General facts:

Holds

$$\int e^{ax}\, J_0(x)\,dx \;=\; \int e^{-a\cdot(-x)}\, J_0(-x)\,dx\;,$$

therefore the integral on the left hand side is discussed, assuming $x \geq 0$ and treating the cases $a > 0$ and $a < 0$ separately.

Let $\mathfrak{H}_\nu(x,a)$ denote the following functions:

$$\mathfrak{H}_1(x,a) \;=\; \sum_{k=1}^{\infty} b_k(a)\, x^k \quad , \qquad \mathfrak{H}_0(x,a) \;=\; \sum_{k=1}^{\infty} c_k(a)\, x^k$$

with

$$b_1(a) = 1\;, \quad b_2(a) = 0\;, \quad b_{k+2}(a) = -\frac{a(1+2k)b_{k+1}(a) + (1+a^2)b_k(a)}{k(k+2)}\;, \quad k \geq 1$$

and

$$c_k(a) = -(k+1)b_{k+1}(a) - a\,b_k(a)\;.$$

Then holds with $a \in \mathbb{R}$

$$\int_0^x e^{at}\, J_0(t)\,dt \;=\; e^{ax}\,[\; \mathfrak{H}_1(x,a)\, J_0(x) + \; \mathfrak{H}_0(x,a)\, J_1(x)]\;.$$

In the case $a = 0$ one has with the Struve functions

$$\mathfrak{H}_1(x,0) = x - \frac{\pi x}{2}\,\mathbf{H}_1(x)\;, \qquad \mathfrak{H}_0(x,0) = \frac{\pi x}{2}\,\mathbf{H}_0(x)\;.$$

First terms of the power series:

$$\mathfrak{H}_1(x,a) = x - (a^2+1)\left[\frac{x^3}{3} - \frac{5a}{24}\,x^4 + \frac{27\,a^2 - 8}{360}\,x^5 - \frac{7a\left(8\,a^2 - 7\right)}{2\,880}\,x^6 + \frac{400\,a^4 - 691\,a^2 + 64}{100\,800}\,x^7 - \right.$$

$$-\frac{a\left(1\,080\,a^4 - 3076\,a^2 + 849\right)}{1\,612\,800}\,x^8 + \frac{9\,800\,a^6 - 41\,484\,a^4 + 22\,767\,a^2 - 1024}{101\,606\,400}\,x^9 -$$

$$-\frac{11a\left(1\,792\,a^6 - 10\,536\,a^4 + 9\,588\,a^2 - 1\,289\right)}{1\,625\,702\,400}\,x^{10} +$$

$$+\frac{217\,728\,a^8 - 1\,695\,080\,a^6 + 2\,303\,364\,a^4 - 617\,289\,a^2 + 16\,384}{160\,944\,537\,600}\,x^{11} -$$

$$-\frac{13\left(67\,200\,a^8 - 668\,576\,a^6 + 1\,266\,744\,a^4 - 564\,120\,a^2 + 44\,815\right)a}{6\,437\,781\,504\,000}\,x^{12} + \dots \left.\right]$$

and

$$\mathfrak{H}_0(x,a) = -ax + (a^2+1)\left[x^2 - \frac{a}{2}\,x^3 + \frac{3\,a^2 - 2}{18}\,x^4 - \frac{a\left(12\,a^2 - 23\right)}{288}\,x^5 + \frac{60\,a^4 - 223\,a^2 + 32}{7\,200}\,x^6 -\right.$$

$$-\frac{a\left(40\,a^4 - 242\,a^2 + 103\right)}{28\,800}\,x^7 + \frac{280\,a^6 - 2\,494\,a^4 + 2\,103\,a^2 - 128}{1\,411\,200}\,x^8 -$$

$$-\frac{a\left(2240\,a^6 - 27512\,a^4 + 38356\,a^2 - 6967\right)}{90\,316\,800}\,x^9 + \frac{20160\,a^8 - 326008\,a^6 + 677076\,a^4 - 244839\,a^2 + 8192}{7\,315\,660\,800}\,x^{10} -$$

$$-\frac{\left(40\,320\,a^8 - 829\,424\,a^6 + 2\,397\,216\,a^4 - 1438\,890\,a^2 + 143\,995\right)}{146\,313\,216\,000}\,x^{11} +$$

$$+\frac{443\,520\,a^{10} - 11\,300\,944\,a^8 + 43\,320\,176\,a^6 - 38\,861\,430\,a^4 + 7\,756\,835\,a^2 - 163\,840}{17\,703\,899\,136\,000}\,x^{12} - \dots \left.\right]\;.$$
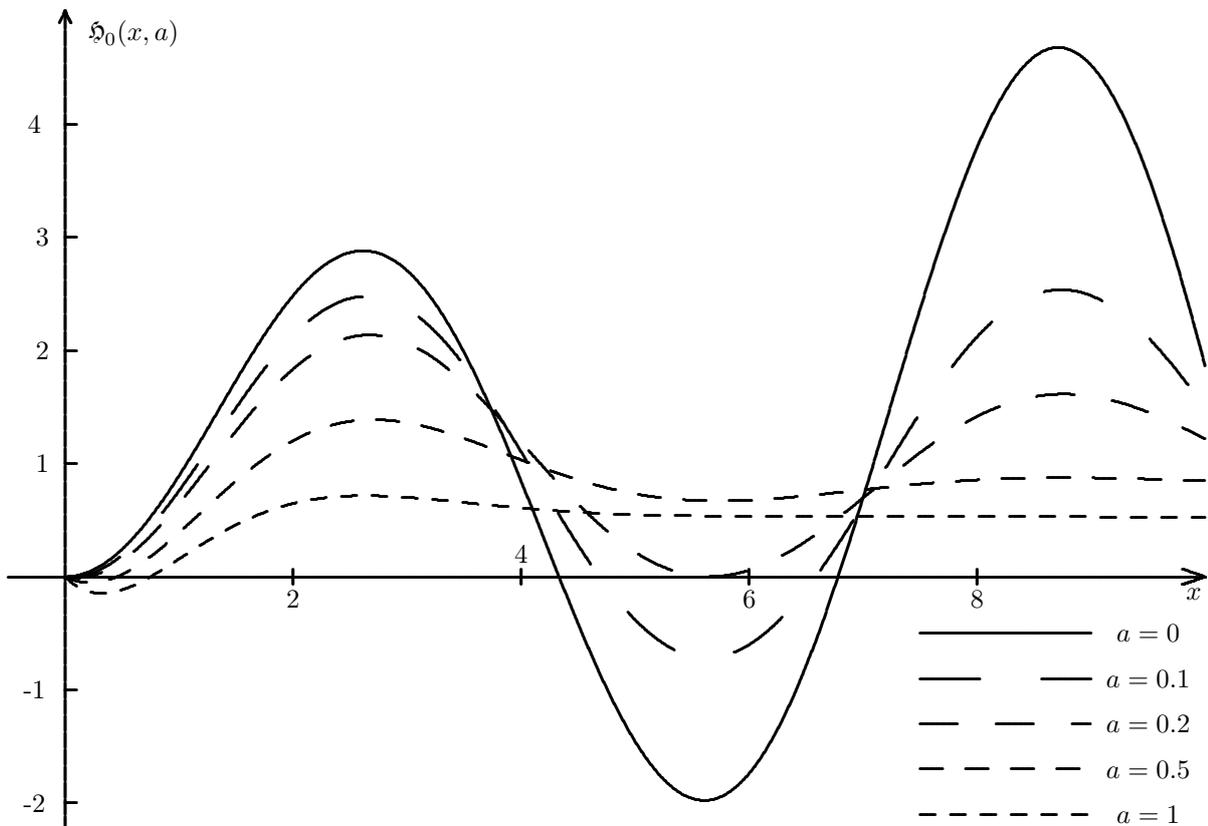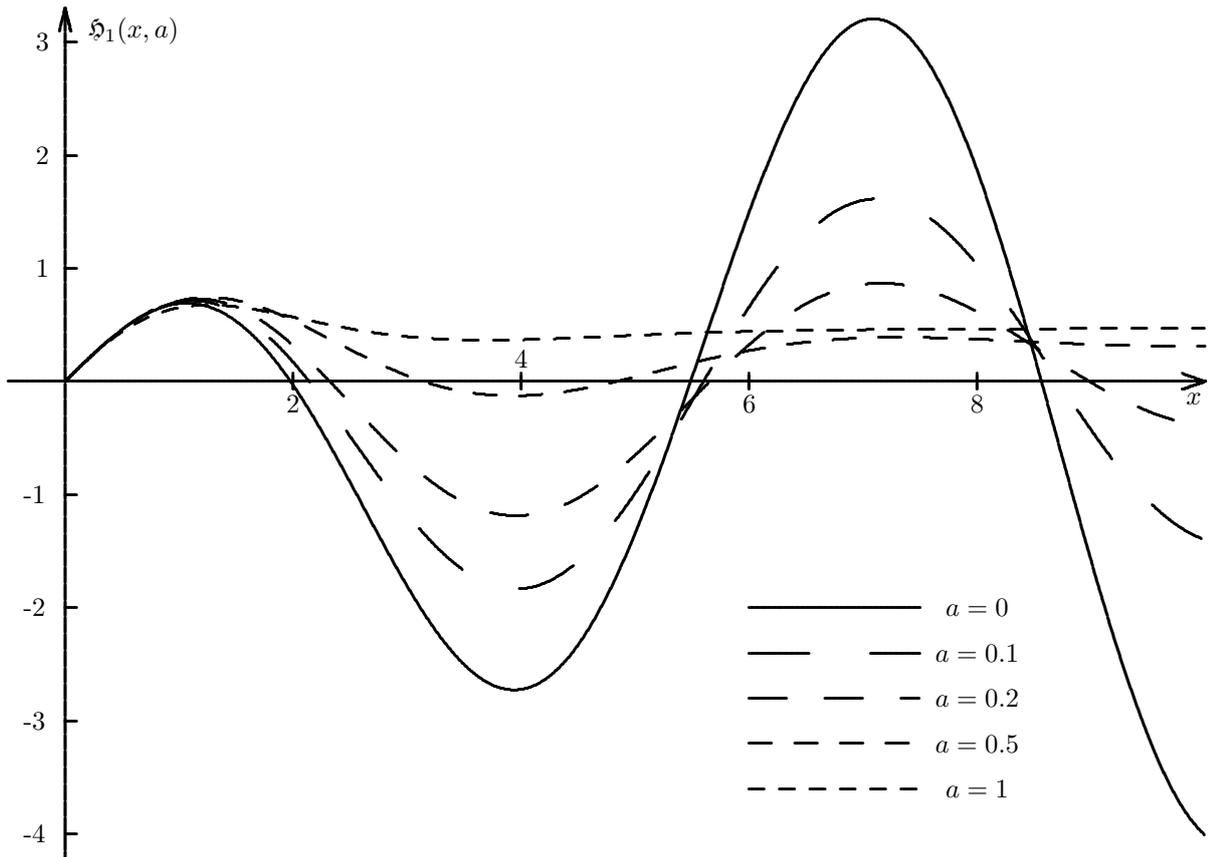
The special case $a = 1$:

$$\int_0^x e^t J_0(t)\, dt = e^x \left[\left(x - \frac{2x^3}{3} + \frac{5}{12}\, x^4 - \frac{19}{180}\, x^4 + \frac{7\, x^6}{1440}\, x^4 + \frac{227\, x^7}{50400} - \frac{1147\, x^8}{806400} + \frac{9941\, x^9}{50803200} + \ldots \right) J_0(x) + \right.$$

$$\left. + \left(-x + 2x^2 - x^3 + \frac{x^4}{9} + \frac{11}{144}\, x^5 - \frac{131\, x^6}{3600} + \frac{11\, x^7}{1600} - \frac{239\, x^8}{705600} - \frac{2039\, x^9}{15052800} + \ldots \right) J_1(x) \right]$$

k	$b_k(1)$	$b_k(1)$	$c_k(1)$	$c_k(1)$
1	1	1.000000000000000	-1	-1.000000000000000
2	0	0.000000000000000	2	2.000000000000000
3	$-2/3$	-0.666666666666667	-1	-1.000000000000000
4	$\frac{5}{12}$	0.416666666666667	$1/9$	0.111111111111111
5	$-\frac{19}{180}$	-0.105555555555556	$\frac{11}{144}$	0.076388888888889
6	$\frac{7}{1440}$	0.004861111111111	$-\frac{131}{3600}$	-0.036388888888889
7	$\frac{227}{50400}$	0.004503968253968	$\frac{11}{1600}$	0.006875000000000
8	$-\frac{1147}{806400}$	-0.001422371031746	$-\frac{239}{705600}$	-0.000338718820862
9	$\frac{9941}{50803200}$	0.000195676650290	$-\frac{2039}{15052800}$	-0.000135456526361
10	$-\frac{979}{162570240}$	-0.000006022012393	$\frac{134581}{3657830400}$	0.000036792575183
11	$-\frac{225107}{80472268800}$	-0.000002797323890	$-\frac{313217}{73156608000}$	-0.000004281458758
12	$\frac{1898819}{3218890752000}$	0.000000589898554	$\frac{1194317}{8851949568000}$	0.000000134921352
13	$-\frac{8554759}{153433792512000}$	-0.000000055755377	$\frac{16109741}{424893579264000}$	0.000000037914767
14	$\frac{14077813}{11047233060864000}$	0.000000001274329	$-\frac{517397957}{71807014895616000}$	-0.000000007205396
15	$\frac{18928541}{47871343263744000}$	0.000000000395404	$\frac{1217807483}{2010596417077248000}$	0.000000000605695
16	$-\frac{402561241}{6433908534647193600}$	-0.000000000062569	$-\frac{422808761}{30158946256158720000}$	-0.000000000014019
17	$\frac{36957033251}{8203233381675171840000}$	0.000000000004505	$-\frac{23427152899}{77206902415766323320000}$	-0.000000000003034
18	$-\frac{21450103637}{262503468213605498880000}$	-0.000000000000082	$\frac{76121087023}{17163688306274205696000}$	0.000000000000444
19	$-\frac{1614496500769}{84788620232994576138240000}$	-0.000000000000019	$-\frac{781266674809}{26775353757787760885760000}$	-0.000000000000029
20	$\frac{4906209165197}{20349268855918698273176000}$	0.000000000000002	$\frac{5157087816757}{966590270656138167975936000}$	0.000000000000001

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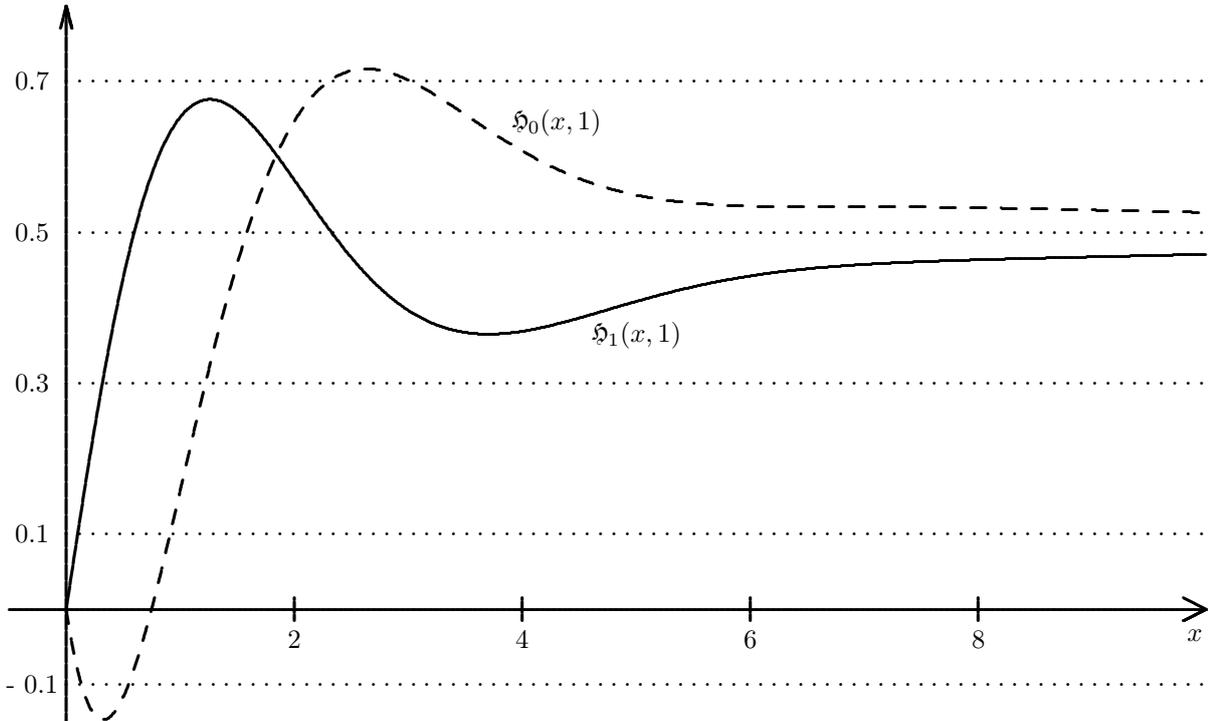
Asymptotic formulas for $x \to +\infty$ in the case $a > 0$:

$$\mathfrak{H}_1(x,a) \sim \frac{a}{1+a^2} - \frac{1}{(1+a^2)^2\, x} - \frac{3a}{(1+a^2)^3\, x^2} - \frac{3\left(4\,a^2-1\right)}{(1+a^2)^4\, x^3} - \frac{15a\left(4\,a^2-3\right)}{(1+a^2)^5\, x^4} -$$

$$-\frac{360\,a^4 - 540\,a^2 + 45}{(1+a^2)^6\, x^5} - \frac{315a\left(8\,a^4 - 20\,a^2 + 5\right)}{(1+a^2)^7\, x^6} - \frac{20160\,a^6 - 75600\,a^4 + 37800\,a^2 - 1575}{(1+a^2)^8\, x^7} - \cdots$$

$$\mathfrak{H}_0(x,a) \sim \frac{1}{1+a^2} + \frac{a}{(1+a^2)^2\, x} + \frac{2\,a^2-1}{(1+a^2)^3\, x^2} + \frac{3\,a\left(2\,a^2-3\right)}{(1+a^2)^4\, x^3} + \frac{24\,a^4 - 72\,a^2 + 9}{(1+a^2)^5\, x^4} +$$

$$+\frac{15\,a\left(8\,a^4 - 40\,a^2 + 15\right)}{(1+a^2)^6\, x^5} + \frac{720\,a^6 - 5400\,a^4 + 4050\,a^2 - 225}{(1+a^2)^7\, x^6} + \frac{315\,a\left(16\,a^6 - 168\,a^4 + 210\,a^2 - 35\right)}{(1+a^2)^8\, x^7} + \cdots$$

The greater a the better these formulas. They cannot be used with $a = 0$.
The following tables show some relative errors. x_k denotes consecutive maxima or minima of this difference.

$$D_0(x) = \frac{a J_0(x) + J_1(x)}{1+a^2} - \int_0^x e^{a(t-x)}\, J_0(t)\, dt \quad :$$

$a = 0.1$		$a = 0.3$		$a = 1$		$a = 3$	
x_k	$D_0(x_k)$	x_k	$D_0(x_k)$	x_k	$D_0(x_k)$	x_k	$D_0(x_k)$
60.226	-4.072E-3	13.386	-3.089E-3	2.399	-1.085E-1	1.757	-1.101E-2
63.622	-1.710E-4	15.961	-1.866E-2	6.200	1.879E-2	5.495	2.073E-3
66.565	-2.720E-3	19.421	5.504E-3	9.308	-1.068E-2	8.759	-1.015E-3
69.863	4.271E-4	22.413	-7.812E-3	12.481	6.748E-3	11.956	6.306E-4
72.883	-1.942E-3	25.632	5.035E-3	15.639	-4.767E-3	15.129	-4.404E-4
76.120	6.932E-4	28.741	-4.770E-3	18.791	3.595E-3	18.290	3.299E-4
79.188	-1.478E-3	31.900	3.866E-3	21.941	-2.835E-3	21.446	-2.590E-4
82.387	7.898E-4	35.036	-3.439E-3	25.089	2.310E-3	24.598	2.104E-4
85.485	-1.188E-3	38.182	2.988E-3	28.235	-1.929E-3	27.748	-1.753E-4
88.661	8.025E-4	41.323	-2.666E-3	31.381	1.642E-3	30.896	1.490E-4
91.776	-9.978E-4	44.466	2.382E-3	34.525	-1.420E-3	34.043	-1.286E-4
91.776	-9.978E-4	47.608	-2.151E-3	37.670	1.244E-3	37.189	1.126E-4

$$D_1(x) = \left(\frac{a}{1+a^2} - \frac{1}{(1+a^2)^2\, x}\right) J_0(x) + \left(\frac{1}{1+a^2} + \frac{a}{(1+a^2)^2\, x}\right) J_1(x) - \int_0^x e^{a(t-x)}\, J_0(t)\, dt \quad :$$

$a = 0.1$		$a = 0.3$		$a = 1$		$a = 3$	
x_k	$D_1(x_k)$	x_k	$D_1(x_k)$	x_k	$D_1(x_k)$	x_k	$D_1(x_k)$
111.782	-1.972E-5	29.538	-2.864E-4	2.061	-5.622E-2	1.816	-3.195E-3
115.333	-4.255E-6	33.053	7.320E-5	6.836	2.161E-3	5.526	2.229E-4
118.158	-1.252E-5	36.018	-1.159E-4	9.729	-1.174E-3	8.831	-6.917E-5
121.548	-3.783E-7	39.248	7.100E-5	12.930	5.554E-4	12.050	3.163E-5
124.497	-8.467E-6	42.348	-6.756E-5	16.086	-3.201E-4	15.237	-1.750E-5
127.789	1.493E-6	45.509	5.293E-5	19.240	2.033E-4	18.408	1.087E-5
130.814	-6.122E-6	48.642	-4.618E-5	22.391	-1.384E-4	21.571	-7.286E-6
134.046	2.317E-6	51.788	3.894E-5	25.540	9.919E-5	24.728	5.164E-6
137.118	-4.708E-6	54.928	-3.381E-5	28.687	-7.393E-5	27.882	-3.817E-6
140.313	2.604E-6	58.070	2.933E-5	31.833	5.683E-5	31.033	2.915E-6

If $x \to +\infty$, then the follwing direct asymptotic formula holds in the case $a > 0$:

$$\int_0^x e^{at} \cdot J_0(t)\,dt \ \sim \ \frac{e^{ax}}{\sqrt{\pi x}} \left[\sum_{k=0}^{\infty} \frac{\lambda_k}{x^k} \sin x + \sum_{k=0}^{\infty} \frac{\mu_k}{x^k} \cos x \right]$$

with

$$\lambda_0 = \frac{a+1}{a^2+1}\,, \quad \mu_0 = \frac{a-1}{a^2+1}$$

$$\lambda_1 = \frac{a^3 + 3\,a^2 + 9\,a - 5}{8\,(a^2+1)^2}\,, \quad \mu_1 = \frac{-a^3 + 3\,a^2 - 9\,a - 5}{8\,(a^2+1)^2}$$

$$\lambda_2 = \frac{-9\,a^5 + 15\,a^4 + 30\,a^3 + 270\,a^2 - 345\,a - 129}{128\,(a^2+1)^3}$$

$$\mu_2 = \frac{-9\,a^5 - 15\,a^4 + 30\,a^3 - 270\,a^2 - 345\,a + 129}{128\,(a^2+1)^3}$$

$$\lambda_3 = \frac{-75\,a^7 - 105\,a^6 - 105\,a^5 + 525\,a^4 + 5775\,a^3 - 12075\,a^2 - 9555\,a + 2655}{1024\,(a^2+1)^4}$$

$$\mu_3 = \frac{75\,a^7 - 105\,a^6 + 105\,a^5 + 525\,a^4 - 5775\,a^3 - 12075\,a^2 + 9555\,a + 2655}{1024\,(a^2+1)^4}$$

$$\lambda_4 = \left[32768\,(a^2+1)^5\right]^{-1} \cdot [3675\,a^9 - 4725\,a^8 + 11340\,a^7 - 8820\,a^6 + 92610\,a^5 + 727650\,a^4 -$$
$$-1984500\,a^3 - 2407860\,a^2 + 1371195\,a + 301035\,]$$

$$\mu_4 = \left[32768\,(a^2+1)^5\right]^{-1} \cdot [3675\,a^9 + 4725\,a^8 + 11340\,a^7 + 8820\,a^6 + 92610\,a^5 - 727650\,a^4 -$$
$$-1984500\,a^3 + 2407860\,a^2 + 1371195\,a - 301035\,]$$

$$\lambda_5 = \left[262144\,(a^2+1)^6\right]^{-1} \cdot [59535\,a^{11} + 72765\,a^{10} + 259875\,a^9 + 280665\,a^8 + 686070\,a^7 + 3056130\,a^6 +$$
$$+30124710\,a^5 - 98232750\,a^4 - 157827285\,a^3 + 135748305\,a^2 + 60259815\,a - 10896795\,]$$

$$\mu_5 = \left[262144\,(a^2+1)^6\right]^{-1} \cdot [-59535\,a^{11} + 72765\,a^{10} - 259875\,a^9 + 280665\,a^8 - 686070\,a^7 + 3056130\,a^6 -$$
$$-30124710\,a^5 - 98232750\,a^4 + 157827285\,a^3 + 135748305\,a^2 - 60259815\,a - 10896795\,]$$

$$\lambda_6 = \left[4194304\,(a^2+1)^7\right]^{-1} \cdot [-2401245\,a^{13} + 2837835\,a^{12} - 13243230\,a^{11} + 14864850\,a^{10} - 34189155\,a^9 +$$
$$+49054005\,a^8 + 160540380\,a^7 + 2871889020\,a^6 - 11331475155\,a^5 - 22569301755\,a^4 + 25820244450\,a^3 +$$
$$-17234307090\,a^2 - 6264182925\,a - 961319205\,]$$

$$\mu_6 = \left[4194304\,(a^2+1)^7\right]^{-1} \cdot [-2401245\,a^{13} - 2837835\,a^{12} - 13243230\,a^{11} - 14864850\,a^{10} - 34189155\,a^9 -$$
$$-49054005\,a^8 + 160540380\,a^7 - 2871889020\,a^6 - 11331475155\,a^5 + 22569301755\,a^4 + 25820244450\,a^3 -$$
$$-17234307090\,a^2 - 6264182925\,a + 961319205\,]$$

Let

$$D_n(x, a) = \frac{1}{\sqrt{\pi x}} \left[\sum_{k=0}^{n} \frac{\lambda_k}{x^k} \sin x + \sum_{k=0}^{n} \frac{\mu_k}{x^k} \cos x \right] - e^{-ax} \int_0^x e^{at} \cdot J_0(t)\,dt$$

describe the 'relative difference' between the asymptotic approximation and the true function.
With $a = 0.3$, $a = 1$ and $a = 3$ one has the following behaviour at $10 \leq x \leq 40$:
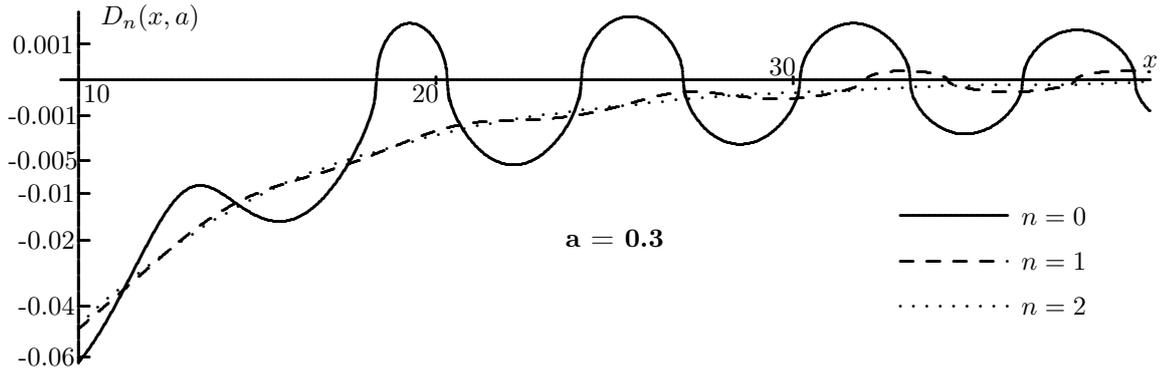
Note that there is a quadratic scale on the D_n-axis.
The first zero of $D_2(x, 0.3)$ is near $x = 48$.
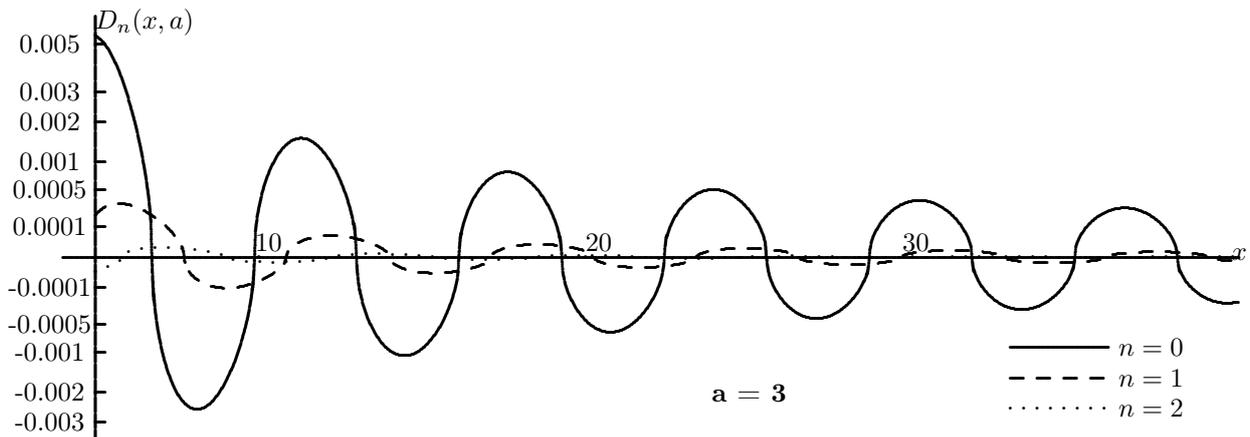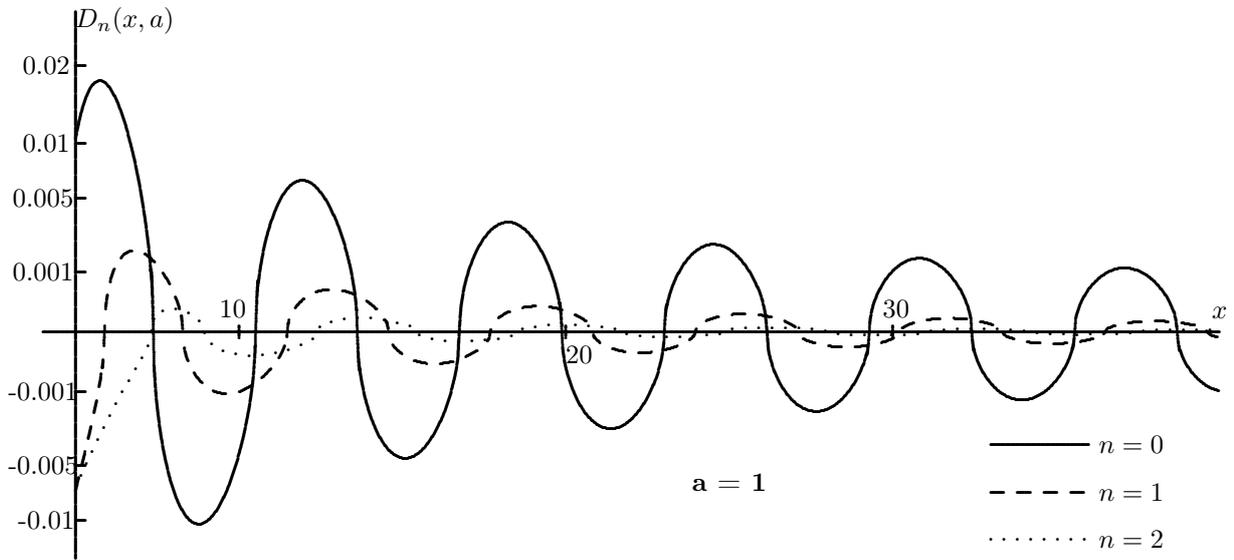
Furthermore, let $\mathfrak{H}_\nu^*(x,a)$ denote the following functions:

$$\mathfrak{H}_1^*(x,a) \;=\; \sum_{k=1}^\infty b_k^*(a)\,x^k \quad , \qquad \mathfrak{H}_0^*(x,a) \;=\; \sum_{k=1}^\infty c_k^*(a)\,x^k$$

with

$$b_1^*(a) = 1\,, \quad b_2^*(a) = 0\,, \quad b_{k+2}^*(a) = -\,\frac{a(1+2k)b_{k+1}^*(a) + (1-a^2)b_k^*(a)}{k(k+2)}\,, \quad k \geq 1$$

and

$$c_k^*(a) = -(k+1)b_{k+1}^*(a) - a\,b_k^*(a)\,.$$

Then holds with $a \in \mathbb{R}$

$$\int_0^x e^{at}\,I_0(t)\,dt \;=\; e^{ax}\,[\;\mathfrak{H}_1^*(x,a)\,I_0(x) + \;\mathfrak{H}_0^*(x,a)\,I_1(x)]\,.$$

In the case $a = 0$ one has with the Struve functions

$$\mathfrak{H}_1^*(x,0) = x - \frac{\pi x}{2}\,\mathbf{L}_1(x)\,, \qquad \mathfrak{H}_0^*(x,0) = \frac{\pi x}{2}\,\mathbf{L}_0(x)\,.$$

With $a = 1$ holds $\mathfrak{H}_1^*(x,1) = -\,\mathfrak{H}_0^*(x,1) = x$ (see page 68).

First terms of the power series:

$$\mathfrak{H}_1^*(x,a) = x - \frac{a^2-1}{3}\,x^3 + \frac{5\,a^3 - 5\,a}{24}\,x^4 - \frac{27\,a^4 - 19\,a^2 - 8}{360}\,x^5 + \frac{56\,a^5 - 7\,a^3 - 49\,a}{2880}\,x^6 -$$

$$-\frac{400\,a^6 + 291\,a^4 - 627\,a^2 - 64}{100800}\,x^7 + \frac{1080\,a^7 + 1996\,a^5 - 2227\,a^3 - 849\,a}{1612800}\,x^8 -$$

$$-\frac{9800\,a^8 + 31684\,a^6 - 18717\,a^4 - 21743\,a^2 - 1024}{101606400}\,x^9 +$$

$$+\frac{19712\,a^9 + 96184\,a^7 - 10428\,a^5 - 91289\,a^3 - 14179\,a}{1625702400}\,x^{10} -$$

$$-\frac{217728\,a^{10} + 1477352\,a^8 + 608284\,a^6 - 1686075\,a^4 - 600905\,a^2 - 16384}{160944537600}\,x^{11} +$$

$$+\frac{873600\,a^{11} + 7817888\,a^9 + 7776184\,a^7 - 9134112\,a^5 - 6750965\,a^3 - 582595\,a}{6437781504000}\,x^{12} - \ldots$$

$$\mathfrak{H}_0^*(x,a) = -ax + (a^2-1)x^2 - \frac{a^3-a}{2}\,x^3 + \frac{3\,a^4 - a^2 - 2}{18}\,x^4 - \frac{12\,a^5 + 11\,a^3 - 23\,a}{288}\,x^5 +$$

$$+\frac{60\,a^6 + 163\,a^4 - 191\,a^2 - 32}{7200}\,x^6 - \frac{40\,a^7 + 202\,a^5 - 139\,a^3 - 103\,a}{28800}\,x^7 +$$

$$+\frac{280\,a^8 + 2214\,a^6 - 391\,a^4 - 1975\,a^2 - 128}{1411200}\,x^8 -$$

$$-\frac{2240\,a^9 + 25272\,a^7 + 10844\,a^5 - 31389\,a^3 - 6967\,a}{90316800}\,x^9 +$$

$$+\frac{20160\,a^{10} + 305848\,a^8 + 351068\,a^6 - 432237\,a^4 - 236647\,a^2 - 8192}{7315660800}\,x^{10} -$$

$$-\frac{40320\,a^{11} + 789104\,a^9 + 1567792\,a^7 - 958326\,a^5 - 1294895\,a^3 - 143995\,a}{146313216000}\,x^{11} +$$

$$+\frac{443520\,a^{12} + 10857424\,a^{10} + 32019232\,a^8 - 4458746\,a^6 - 31104595\,a^4 - 7592995\,a^2 - 163840}{17703899136000}\,x^{12} - \ldots$$

$$y \;=\; -\mathfrak{H}_0^*(x,a) \;!$$
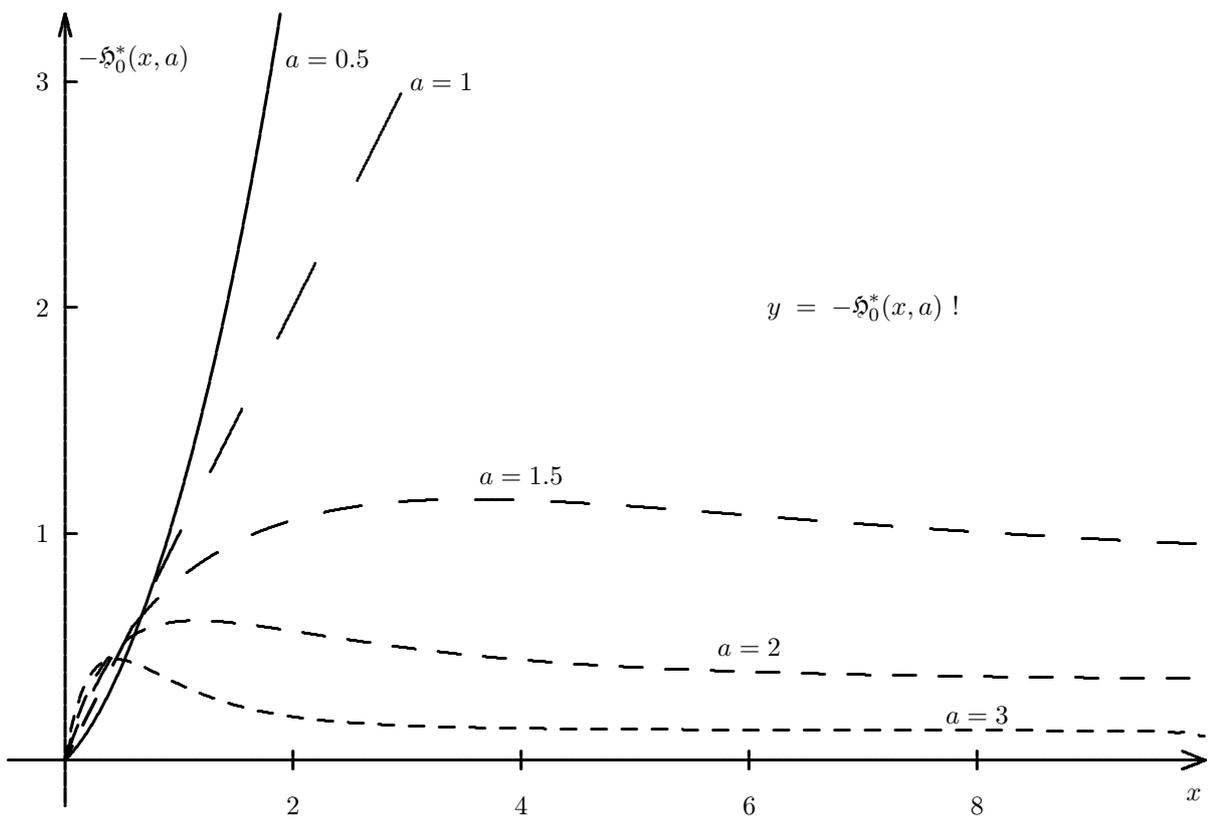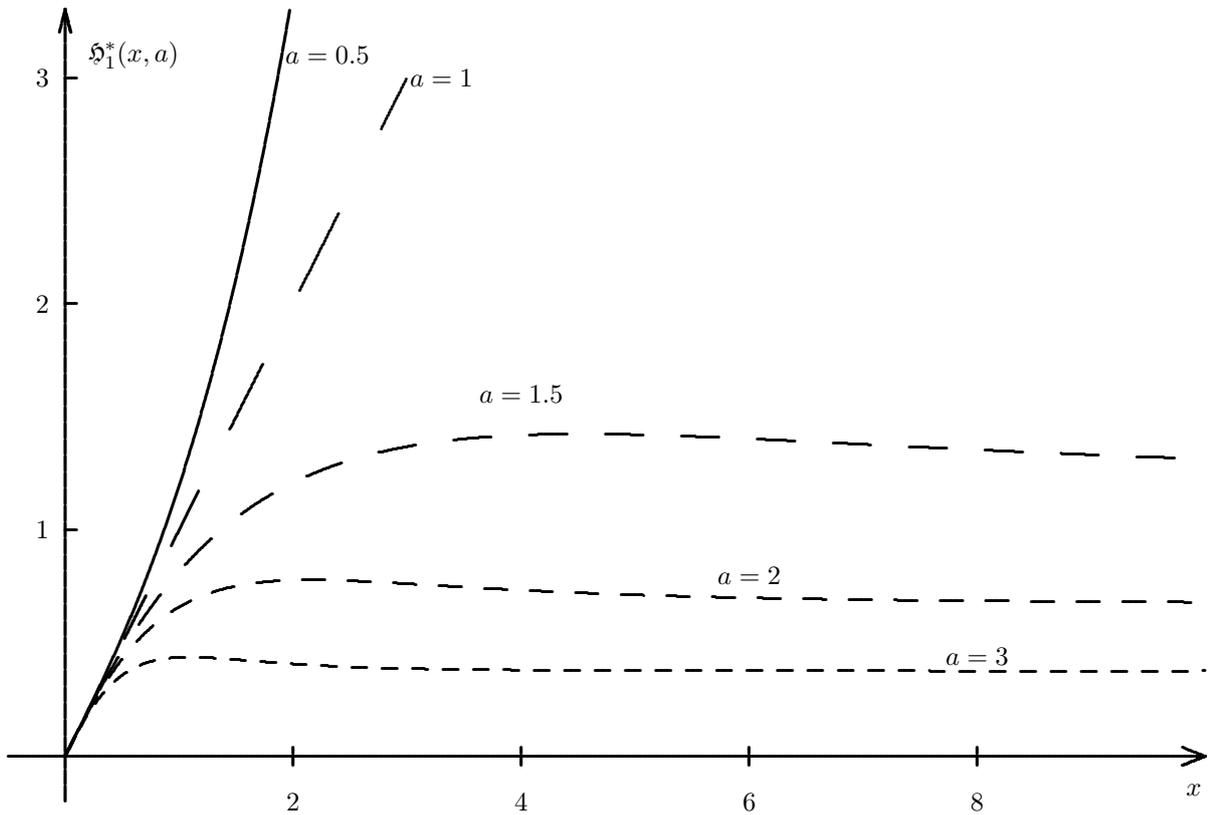
If $0 < a < 1$, then $\mathfrak{H}_1^*(x,a)$ and $-\mathfrak{H}_0^*(x,a)$ are growing rapidly with $x \to +\infty$.

Asymptotic behaviour for $x \to +\infty$ and $a > 1$:

$$\mathfrak{H}_1^*(x,a) \sim \frac{a}{a^2-1} + \frac{1}{(a^2-1)^2\,x} + \frac{3a}{(a^2-1)^3\,x^2} + \frac{12\,a^2+3}{(a^2-1)^4\,x^3} + \frac{15a\,\left(4\,a^2+3\right)}{(a^2-1)^5\,x^4} + \frac{360\,a^4+540\,a^2+45}{(a^2-1)^6\,x^5} +$$

$$+\frac{315a\,\left(8\,a^4+20\,a^2+5\right)}{(a^2-1)^7\,x^6} + \frac{20160\,a^6+75600\,a^4+37800\,a^2+1575}{(a^2-1)^8\,x^7} + \ldots$$

$$\mathfrak{H}_0^*(x,a) \sim -\frac{1}{a^2-1} - \frac{a}{(a^2-1)^2\,x} - \frac{2\,a^2+1}{(a^2-1)^3\,x^2} - \frac{3\,a\,\left(2\,a^2+3\right)}{(a^2-1)^4\,x^3} - \frac{24\,a^4+72\,a^2+9}{(a^2-1)^5\,x^4} -$$

$$-\frac{15a\,\left(8\,a^4+40\,a^2+15\right)}{(a^2-1)^6\,x^5} - \frac{720\,a^6+5400\,a^4+4050\,a^2+225}{(a^2-1)^7\,x^6} -$$

$$-\frac{315\,\left(16\,a^6+168\,a^4+210\,a^2+35\right)a}{(a^2-1)^8\,x^7} - \ldots$$

Direct asymptotic formula for the case $x \to +\infty$, $a > 0$:

$$\int_0^x e^{at} \cdot I_0(t)\,dt \sim \frac{e^{(1+a)x}}{\sqrt{2\pi x}\,(1+a)} \sum_{k=0}^{\infty} \frac{\beta_k(a)}{x^k} = \frac{e^{(1+a)x}}{\sqrt{2\pi x}\,(1+a)} \left[1 + \frac{a+5}{8(1+a)\,x} + \right.$$

$$+\frac{9\,a^2+42\,a+129}{128\,(1+a)^2\,x^2} + \frac{75\,a^3+405\,a^2+1065\,a+2655}{1024\,(1+a)^3\,x^3} +$$

$$+\frac{3675\,a^4+23100\,a^3+67410\,a^2+133980\,a+301035}{32768\,(1+a)^4\,x^4} +$$

$$+\frac{59535\,a^5+429975\,a^4+1426950\,a^3+3022110\,a^2+5120955\,a+10896795}{262144\,(1+a)^5\,x^5} +$$

$$+\frac{2401245\,a^6+19646550\,a^5+73856475\,a^4+173596500\,a^3+301964355\,a^2+465051510\,a+961319205}{4194304\,(1+a)^6\,x^6} +$$

$$\left.+\frac{135135\,(370345+181955\,a+125205\,a^2+81815\,a^3+43435\,a^4+16569\,a^5+3927\,a^6+429\,a^7)}{33554432\,(1+a)^7\,x^7} + \ldots\right]$$

Some coefficients:

k	$\beta_k(0.1)$	$\beta_k(0.3)$	$\beta_k(1)$	$\beta_k(3)$	$\beta_k(10)$
1	0.579545	0.509615	0.375000	0.250000	0.170455
2	0.860602	0.658330	0.351563	0.164063	0.093556
3	2.029155	1.339262	0.512695	0.175781	0.094505
4	6.568555	3.717857	1.009369	0.265961	0.142222
5	27.098470	13.096614	2.498188	0.526314	0.285290
6	136.064853	55.981252	7.442518	1.296183	0.715146
7	805.747313	281.633987	25.915913	3.834025	2.150314

With fixed $x > 0$ and $a \to +\infty$ holds

$$\int_0^x e^{a t}\,J_0(t)\,dt \sim e^{ax}\left[\left(\frac{1}{a} - \frac{1}{a^3} - \frac{1}{a^4 x} + \frac{x^2-3}{a^5 x^2} + \frac{2\,x^2-12}{a^6 x^3} - \frac{x^4-9\,x^2+60}{a^7 x^4} - \frac{3\,x^4-51\,x^2+360}{a^8 x^5} + \right.\right.$$

$$\left.\frac{x^6-18\,x^4+345\,x^2-2520}{a^9 x^6} + \frac{4\,x^6-132\,x^4+2700\,x^2-20160}{a^{10} x^7} + \ldots\right)\,J_0(x)+$$

$$+\left(\frac{1}{a^2} + \frac{1}{a^3 x} - \frac{x^2-2}{a^4 x^2} - \frac{2\,x^2-6}{a^5 x^3} + \frac{x^4-7\,x^2+24}{a^6 x^4} + \frac{3\,x^4-33\,x^2+120}{a^7 x^5} - \frac{x^6-15\,x^4+192\,x^2-720}{a^8 x^6} - \right.$$

$$\left.\left.-\frac{4\,x^6-96\,x^4+1320\,x^2-5040}{a^9 x^7} + \frac{x^8-26\,x^6+729\,x^4-10440\,x^2+40320}{a^{10} x^8} + \ldots\right)\,J_1(x)\right]$$

and

$$\int_0^x e^{a\,t}\, I_0(t)\, dt \ \sim\ e^{ax}\left[\left(\frac{1}{a}+\frac{1}{a^3}+\frac{1}{a^4 x}+\frac{x^2+3}{a^5 x^2}+\frac{2\,x^2+12}{a^6 x^3}+\frac{x^4+9\,x^2+60}{a^7 x^4}+\frac{3\,x^4+51\,x^2+360}{a^8 x^5}+\right.\right.$$

$$\left.+\frac{x^6+18\,x^4+345\,x^2+2520}{a^9 x^6}+\frac{4\,x^6+132\,x^4+2700\,x^2+20160}{a^{10}x^7}+\dots\right) I_0(x)-$$

$$-\left(\frac{1}{a^2}+\frac{1}{a^3 x}+\frac{x^2+2}{a^4 x^2}+\frac{2\,x^2+6}{a^5 x^3}+\frac{x^4+7\,x^2+24}{a^6 x^4}+\frac{3\,x^4+33\,x^2+120}{a^7 x^5}+\frac{x^6+15\,x^4+192\,x^2+720}{a^8 x^6}+\right.$$

$$\left.\left.+\frac{4\,x^6+96\,x^4+1320\,x^2+5040}{a^9 x^7}+\frac{x^8+26\,x^6+729\,x^4+10440\,x^2+40320}{a^{10}x^8}\right) I_1(x)\right].$$

c) The case $a < 0$:

To express this fact clearly it is written $a = -\alpha$ with $\alpha > 0$.
One has the Lipschitz integral (see [11], part I, §21, or the tables of Laplace transforms)

$$\int_0^\infty e^{a\,x}\, J_0(x)\, dx = \int_0^\infty e^{-\alpha\,x}\, J_0(x)\, dx = \frac{1}{\sqrt{1+\alpha^2}} = \frac{1}{\sqrt{1+a^2}}.$$

The representation

$$\int_0^x e^{a\,t}\, J_0(t)\, dt \ = \ e^{ax}\left[\ \mathfrak{H}_1(x,a)\, J_0(x) +\ \mathfrak{H}_0(x,a)\, J_1(x)\right]$$

keeps being true, but $\mathfrak{H}_0(x,a)$ and $\mathfrak{H}_1(x,a)$ are rapidly growing with x. For that reason other formulas are more applicable.

$$\int_0^x e^{-\alpha\,t}\, J_0(t)\, dt \ = \ \frac{1}{\sqrt{1+\alpha^2}} - e^{-\alpha\,x}\left[\frac{1}{\sqrt{1+\alpha^2}} - \sum_{k=1}^\infty \varphi_k(\alpha)\cdot x^{2k-1}\cdot\left(1+\frac{\alpha\,x}{2k}\right)\right]$$

with

$$\varphi_k(\alpha) = \frac{(-1)^{k-1}}{(2k-1)!\cdot a^2}\sum_{i=0}^\infty \frac{(-1)^i\cdot(2i+2k)!}{2^{2i+2k}\cdot[(i+k)!]^2\cdot a^{2i}}$$

Some first functions:

$$\varphi_1(\alpha) = \frac{\sqrt{1+\alpha^2}-\alpha}{\sqrt{1+\alpha^2}}$$

$$\varphi_2(\alpha) = \frac{1}{6}\cdot\frac{(2\,\alpha^2-1)\sqrt{1+\alpha^2}-2\,\alpha^3}{2\sqrt{1+\alpha^2}}$$

$$\varphi_3(\alpha) = \frac{1}{120}\cdot\frac{(8\,\alpha^4-4\,\alpha^2+3)\sqrt{1+\alpha^2}-8\,\alpha^5}{8\sqrt{1+\alpha^2}}$$

$$\varphi_4(\alpha) = \frac{1}{5040}\cdot\frac{(16\,\alpha^6-8\,\alpha^4+6\,\alpha^2-5)\sqrt{1+\alpha^2}-16\,\alpha^7}{16\sqrt{1+\alpha^2}}$$

$$\varphi_5(\alpha) = \frac{1}{362\,880}\cdot\frac{(128\,\alpha^8-64\,\alpha^6+48\,\alpha^4-40\,\alpha^2+35)\sqrt{1+\alpha^2}-128\,\alpha^9}{128\sqrt{1+\alpha^2}}$$

$$\varphi_6(\alpha) = \frac{1}{29\,916\,800}\cdot\frac{(256\,\alpha^{10}-128\,\alpha^8+96\,\alpha^6-80\,\alpha^4+70\,\alpha^2-63)\sqrt{1+\alpha^2}-256\,\alpha^{11}}{256\sqrt{1+\alpha^2}}$$

$$\varphi_7(\alpha) = \frac{1}{1\,307\,674\,368\,000}\cdot$$

$$\cdot\frac{+(1024\,\alpha^{12}-512\,\alpha^{10}+384\,\alpha^8-320\,\alpha^6+280\,\alpha^4-252\,\alpha^2+231)\sqrt{1+\alpha^2}-1024\,\alpha^{13}}{1024\sqrt{1+\alpha^2}}$$

Special values for the case $\alpha = 1$:

$$\varphi_1(1) = \frac{\sqrt{2}}{2+2\sqrt{2}} = 0.292893218813452\,,\quad \varphi_2(1) = \frac{1-\sqrt{2}}{12} = -0.034517796864425\,,$$

$$\varphi_3(1) = -\frac{1}{240}\sqrt{2} + \frac{7}{960} = 0.001399110156779 \,, \quad \varphi_4(1) = -\frac{1}{10080}\sqrt{2} + \frac{1}{8960} = -0.000028691821664 \,,$$

$$\varphi_5(1) = -\frac{1}{725760}\sqrt{2} + \frac{107}{46448640} = 0.000000355022924 \,,$$

$$\varphi_6(1) = -\frac{1}{79833600}\sqrt{2} + \frac{151}{10218700800} = -0.000000002937686 \,,$$

$$\varphi_7(1) = -\frac{1}{12454041600}\sqrt{2} + \frac{167}{1275293859840} = 0.000000000017396 \,,$$

$$\varphi_8(1) = -\frac{1}{2615348736000}\sqrt{2} + \frac{1241}{2678117105664000} = -0.000000000000077 \,,$$

Asymptotic behaviour of $\varphi_k(\alpha)$ for $\alpha \to +\infty$:

$$\frac{1}{\sqrt{1+\alpha^2}} \sim \alpha^{-1} - \frac{1}{2}\alpha^{-3} + \frac{3}{8}\alpha^{-5} - \frac{5}{16}\alpha^{-7} + \frac{35}{128}\alpha^{-9} + \dots$$

$$\varphi_1(\alpha) \sim \frac{1}{2}\alpha^{-2} - \frac{3}{8}\alpha^{-4} + \frac{5}{16}\alpha^{-6} - \frac{35}{128}\alpha^{-8} + \dots$$

$$3!\,\varphi_2(\alpha) \sim -\frac{3}{8}\alpha^{-2} + \frac{5}{16}\alpha^{-4} - \frac{35}{128}\alpha^{-6} + \frac{63}{256}\alpha^{-8} + \dots$$

$$5!\,\varphi_3(\alpha) \sim \frac{5}{16}\alpha^{-2} - \frac{35}{128}\alpha^{-4} + \frac{63}{256}\alpha^{-6} - \frac{231}{1024}\alpha^{-8} \dots$$

$$7!\,\varphi_4(\alpha) \sim -\frac{35}{128}\alpha^{-2} + \frac{63}{256}\alpha^{-4} - \frac{231}{1024}\alpha^{-6} + \frac{429}{2048}\alpha^{-8} + \dots$$

$$9!\,\varphi_5(\alpha) \sim \frac{63}{256}\alpha^{-2} - \frac{231}{1024}\alpha^{-4} + \frac{429}{2048}\alpha^{-6} - \frac{6435}{32768}\alpha^{-8} + \dots$$

$$11!\,\varphi_6(\alpha) \sim -\frac{231}{1024}\alpha^{-2} + \frac{429}{2048}\alpha^{-4} - \frac{6435}{32768}\alpha^{-6} + \frac{12155}{65536}\alpha^{-8} + \dots$$

$$13!\,\varphi_7(\alpha) \sim \frac{429}{2048}\alpha^{-2} - \frac{6435}{32768}\alpha^{-4} + \frac{12155}{65536}\alpha^{-6} - \frac{46189}{262144}\alpha^{-8} + \dots$$

Direct asymptotic formula for $x \to \infty$:

$$\int_0^x e^{-\alpha t} J_0(t)\,dt \sim \frac{1}{\sqrt{1+\alpha^2}} + \frac{e^{-\alpha x}}{\sqrt{\pi x}}\left\{\left[-\frac{\alpha-1}{1+\alpha^2} - \frac{\alpha^3 - 3\alpha^2 + 9\alpha + 5}{8(1+\alpha^2)^2\,x} + \right.\right.$$

$$+\frac{3(3\alpha^5 + 5\alpha^4 - 10\alpha^3 + 90\alpha^2 + 115\alpha - 43)}{128(1+\alpha^2)^3\,x^2} +$$

$$+\frac{15(5\alpha^7 - 7\alpha^6 + 7\alpha^5 + 35\alpha^4 - 385\alpha^3 - 805\alpha^2 + 637\alpha + 177)}{1024(1+\alpha^2)^4\,x^3} -$$

$$-\frac{105(35\alpha^9 + 45\alpha^8 + 108\alpha^7 + 84\alpha^6 + 882\alpha^5 - 6930\alpha^4 - 18900\alpha^3 + 22932\alpha^2 + 13059\alpha - 2867)}{32768(1+\alpha^2)^5\,x^4} +$$

$$-\frac{945\,s_5}{262144(1+\alpha^2)^6\,x^5} - \frac{34459425\,s_6}{17179869184(1+\alpha^2)^7\,x^6} - \frac{135135\,s_7}{33554432(1+\alpha^2)^8\,x^7} - \frac{2027025\,s_8}{2147483648(1+\alpha^2)^9\,x^8} -$$

$$\left.-\frac{34459425\,s_9}{17179869184(1+\alpha^2)^{10}\,x^9} + \frac{654729075\,s_{10}}{274877906944(1+\alpha^2)^{11}\,x^{10}} + \dots\right]\sin x +$$

$$+\left[-\frac{1+\alpha}{1+\alpha^2} + \frac{\alpha^3 + 3\alpha^2 + 9\alpha - 5}{8(1+\alpha^2)^2\,x} + \frac{3(3\alpha^5 - 5\alpha^4 - 10\alpha^3 - 90\alpha^2 + 115\alpha + 43)}{128(1+\alpha^2)^3\,x^2} - \right.$$

$$-\frac{15(5\alpha^7 + 7\alpha^6 + 7\alpha^5 - 35\alpha^4 - 385\alpha^3 + 805\alpha^2 + 637\alpha - 177)}{(1+\alpha^2)^4\,x^3} -$$

$$-\frac{105(35\alpha^9 - 45\alpha^8 + 108\alpha^7 - 84\alpha^6 + 882\alpha^5 + 6930\alpha^4 - 18900\alpha^3 - 22932\alpha^2 + 13059\alpha + 2867)}{32768(1+\alpha^2)^5\,x^4} +$$

$$+\frac{945\,c_5}{262144\,(1+\alpha^2)^6\,x^5}+\frac{10395\,c_6}{4194304\,(1+\alpha^2)^7\,x^6}-\frac{135135\,c_7}{33554432\,(1+\alpha^2)^8\,x^7}-\frac{2027025\,c_8}{2147483648\,(1+\alpha^2)^9\,x^8}+$$

$$+\frac{34459425\,c_9}{17179869184\,(1+\alpha^2)^{10}\,x^9}+\frac{654729075\;274877906944 c_{10}}{274877906944\,(1+\alpha^2)^{11}\,x^{10}}+\ldots]\cos x\Bigg\}$$

$$s_5 = 63\,\alpha^{11} - 77\,\alpha^{10} + 275\,\alpha^9 - 297\,\alpha^8 + 726\,\alpha^7 - 3234\,\alpha^6 + 31878\,\alpha^5 + 103950\,\alpha^4 - 167013\,\alpha^3 -$$
$$-143649\,\alpha^2 + 63767\,\alpha + 11531$$

$$c_5 = 63\,\alpha^{11} + 77\,\alpha^{10} + 275\,\alpha^9 + 297\,\alpha^8 + 726\,\alpha^7 + 3234\,\alpha^6 + 31878\,\alpha^5 - 103950\,\alpha^4 -$$
$$-167013\,\alpha^3 + 143649\,\alpha^2 + 63767\,\alpha - 11531$$

$$s_6 = 231\,\alpha^{13} + 273\,\alpha^{12} + 1274\,\alpha^{11} + 1430\,\alpha^{10} + 3289\,\alpha^9 + 4719\,\alpha^8 - 15444\,\alpha^7 + 276276\,\alpha^6 + 1090089\,\alpha^5 -$$
$$-2171169\,\alpha^4 - 2483910\,\alpha^3 + 1657942\,\alpha^2 + 602615\,\alpha - 92479$$

$$c_6 = 231\,\alpha^{13} - 273\,\alpha^{12} + 1274\,\alpha^{11} - 1430\,\alpha^{10} + 3289\,\alpha^9 - 4719\,\alpha^8 - 15444\,\alpha^7 - 276276\,\alpha^6 + 1090089\,\alpha^5 +$$
$$+2171169\,\alpha^4 - 2483910\,\alpha^3 - 1657942\,\alpha^2 + 602615\,\alpha + 92479$$

$$s_7 = 429\,\alpha^{15} - 495\,\alpha^{14} + 2835\,\alpha^{13} - 3185\,\alpha^{12} + 8385\,\alpha^{11} - 9867\,\alpha^{10} + 9295\,\alpha^9 + 57915\,\alpha^8 - 1151865\,\alpha^7 -$$
$$-5450445\,\alpha^6 + 13054041\,\alpha^5 + 18629325\,\alpha^4 - 16564405\,\alpha^3 - 9039225\,\alpha^2 + 2780805\,\alpha + 370345$$

$$c_7 = 429\,\alpha^{15} + 495\,\alpha^{14} + 2835\,\alpha^{13} + 3185\,\alpha^{12} + 8385\,\alpha^{11} + 9867\,\alpha^{10} + 9295\,\alpha^9 - 57915\,\alpha^8 - 1151865\,\alpha^7 +$$
$$+5450445\,\alpha^6 + 13054041\,\alpha^5 - 18629325\,\alpha^4 - 16564405\,\alpha^3 + 9039225\,\alpha^2 + 2780805\,\alpha - 370345$$

$$s_8 = 6435\,\alpha^{17} + 7293\,\alpha^{16} + 49368\,\alpha^{15} + 55080\,\alpha^{14} + 168980\,\alpha^{13} + 190060\,\alpha^{12} + 312936\,\alpha^{11} + 252824\,\alpha^{10} +$$
$$+2601170\,\alpha^9 - 39163410\,\alpha^8 - 210913560\,\alpha^7 + 591783192\,\alpha^6 + 1014047892\,\alpha^5 - 1126379540\,\alpha^4 -$$
$$-819264680\,\alpha^3 + 378189480\,\alpha^2 + 100843235\,\alpha - 11857475$$

$$c_8 = 6435\,\alpha^{17} - 7293\,\alpha^{16} + 49368\,\alpha^{15} - 55080\,\alpha^{14} + 168980\,\alpha^{13} - 190060\,\alpha^{12} + 312936\,\alpha^{11} - 252824\,\alpha^{10} +$$
$$+2601170\,\alpha^9 + 39163410\,\alpha^8 - 210913560\,\alpha^7 - 591783192\,\alpha^6 + 1014047892\,\alpha^5 + 1126379540\,\alpha^4 -$$
$$-819264680\,\alpha^3 - 378189480\,\alpha^2 + 100843235\,\alpha + 11857475$$

$$s_9 = 12155\,\alpha^{19} - 13585\,\alpha^{18} + 105963\,\alpha^{17} - 117249\,\alpha^{16} + 414732\,\alpha^{15} - 458660\,\alpha^{14} + 936700\,\alpha^{13} - 990964\,\alpha^{12} +$$
$$+1771978\,\alpha^{11} - 9884446\,\alpha^{10} + 168589850\,\alpha^9 + 1001839410\,\alpha^8 - 3209765988\,\alpha^7 - 6422303316\,\alpha^6 +$$
$$+8562147308\,\alpha^5 + 7783014460\,\alpha^4 - 4789707245\,\alpha^3 - 1916021465\,\alpha^2 + 450814995\,\alpha + 47442055$$

$$c_9 = 12155\,\alpha^{19} + 13585\,\alpha^{18} + 105963\,\alpha^{17} + 117249\,\alpha^{16} + 414732\,\alpha^{15} + 458660\,\alpha^{14} + 936700\,\alpha^{13} + 990964\,\alpha^{12} +$$
$$+1771978\,\alpha^{11} + 9884446\,\alpha^{10} + 168589850\,\alpha^9 - 1001839410\,\alpha^8 - 3209765988\,\alpha^7 + 6422303316\,\alpha^6 +$$

$$+8562147308\,\alpha^5 - 7783014460\,\alpha^4 - 4789707245\,\alpha^3 + 1916021465\,\alpha^2 + 450814995\,\alpha - 47442055$$

$$s_{10} = 46189\,\alpha^{21} + 51051\,\alpha^{20} + 450450\,\alpha^{19} + 494494\,\alpha^{18} + 1988217\,\alpha^{17} + 2177343\,\alpha^{16} + 5191256\,\alpha^{15} +$$

$$+5620200\,\alpha^{14} + 9265578\,\alpha^{13} + 12403846\,\alpha^{12} - 53260116\,\alpha^{11} + 1416154740\,\alpha^{10} + 9373133770\,\alpha^9 -$$

$$-33702542874\,\alpha^8 - 77051011752\,\alpha^7 + 119870062312\,\alpha^6 + 130763372649\,\alpha^5 - 100583852145\,\alpha^4 -$$

$$-53645367790\,\alpha^3 + 18934229790\,\alpha^2 + 3986102589\,\alpha - 379582629$$

$$c_{10} = 46189\,\alpha^{21} - 51051\,\alpha^{20} + 450450\,\alpha^{19} - 494494\,\alpha^{18} + 1988217\,\alpha^{17} - 2177343\,\alpha^{16} + 5191256\,\alpha^{15} -$$

$$-5620200\,\alpha^{14} + 9265578\,\alpha^{13} - 12403846\,\alpha^{12} - 53260116\,\alpha^{11} - 1416154740\,\alpha^{10} + 9373133770\,\alpha^9 +$$

$$+33702542874\,\alpha^8 - 77051011752\,\alpha^7 - 119870062312\,\alpha^6 + 130763372649\,\alpha^5 + 100583852145\,\alpha^4 -$$

$$-53645367790\,\alpha^3 - 18934229790\,\alpha^2 + 3986102589\,\alpha + 379582629$$

In the special case $\alpha = 1$ holds

$$\int_0^x e^{-t} J_0(t)\,dt \ \sim\ \frac{1}{\sqrt{2}} + \frac{e^{-x}}{\sqrt{\pi x}} \left[\left(\sum_{k=0}^\infty \frac{s_k^*}{x^k} \right) \sin x + \left(\sum_{k=0}^\infty \frac{c_k^*}{x^k} \right) \cos x \right]$$

with

k	s_k^*	s_k^*	c_k^*	c_k^*
0	0	0	-1	-1
1	$-\frac{3}{8}$	-0.375000000000000	$\frac{1}{4}$	0.250000000000000
2	$\frac{15}{32}$	0.468750000000000	$\frac{21}{128}$	0.164062500000000
3	$-\frac{315}{1024}$	-0.307617187500000	$-\frac{405}{512}$	-0.791015625000000
4	$-\frac{3465}{4096}$	-0.845947265625000	$\frac{59325}{32768}$	1.810455322265625
5	$\frac{1507275}{262144}$	5.749797821044922	$-\frac{284445}{131072}$	-2.170143127441406
6	$-\frac{22837815}{1048576}$	-21.77983760833740	$-\frac{38887695}{4194304}$	-9.271548986434937
7	$\frac{1422025605}{33554432}$	42.37966552376747	$\frac{1693106415}{16777216}$	100.9170064330101
8	$\frac{29462808375}{134217728}$	219.5150284096599	$-\frac{1167021130275}{2147483648}$	-543.4365618391894
9	$-\frac{56125340496225}{17179869184}$	-3266.924788257165	$\frac{11825475336675}{8589934592}$	1376.666517075500
10	$\frac{1515749532221925}{68719476736}$	22057.05870033016	$\frac{2498294907783675}{274877906944}$	9088.743928382155

The Laplace transform of $I_0(x)$ exists only in the case $\alpha > 1$:

$$*E* \qquad \int_0^\infty e^{a\,x}\,I_0(x)\,dx = \int_0^\infty e^{-\alpha\,x}\,I_0(x)\,dx = \frac{1}{\sqrt{\alpha^2 - 1}} = \frac{1}{\sqrt{a^2 - 1}}\ .$$

If $\alpha > 1$ one has

$$\int_0^x e^{-\alpha\,t}\,I_0(t)\,dt\ =\ \frac{1}{\sqrt{\alpha^2 - 1}} - e^{-\alpha\,x} \left[\frac{1}{\sqrt{\alpha^2 - 1}} + \sum_{k=1}^\infty \varphi_k^*(\alpha)\cdot x^{2k-1}\cdot\left(1 + \frac{\alpha\,x}{2k}\right) \right]$$

with

$$\varphi_k^*(\alpha) = \frac{1}{(2k-1)!\cdot a^2} \sum_{i=0}^\infty \frac{(2i+2k)!}{2^{2i+2k}\cdot[(i+k)!]^2\cdot a^{2i}}$$

This series fails to converge in the case $0 < \alpha \le 1$.

If $\alpha = 1$, then see page 68 for solutions with elementary functions.

Some first functions $\varphi_k^*(\alpha)$:

$$\varphi_1^*(\alpha) = \frac{\alpha - \sqrt{\alpha^2 - 1}}{\sqrt{\alpha^2 - 1}}$$

$$\varphi_2^*(\alpha) = \frac{1}{6} \cdot \frac{2\alpha^3 - (2\,\alpha^2 + 1)\,\sqrt{\alpha^2 - 1}}{2\,\sqrt{\alpha^2 - 1}}$$

$$\varphi_3^*(\alpha) = \frac{1}{120} \cdot \frac{8\alpha^5 - (8\,\alpha^4 + 4\,\alpha^2 + 3)\,\sqrt{\alpha^2 - 1}}{8\,\sqrt{\alpha^2 - 1}}$$

$$\varphi_4^*(\alpha) = \frac{1}{5\,040} \cdot \frac{16\alpha^7 - (16\,\alpha^6 + 8\,\alpha^4 + 6\,\alpha^2 + 5)\,\sqrt{\alpha^2 - 1}}{16\,\sqrt{\alpha^2 - 1}}$$

$$\varphi_5^*(\alpha) = \frac{1}{362\,880} \cdot \frac{128\alpha^9 - (128\,\alpha^8 + 64\,\alpha^6 + 48\,\alpha^4 + 40\,\alpha^2 + 35)\,\sqrt{\alpha^2 - 1}}{128\,\sqrt{\alpha^2 - 1}}$$

$$\varphi_6^*(\alpha) = \frac{1}{39\,916\,800} \cdot \frac{(256\,\alpha^{10} + 128\,\alpha^8 + 96\,\alpha^6 + 80\,\alpha^4 + 70\,\alpha^2 + 63)\,\sqrt{\alpha^2 - 1}}{256\,\sqrt{\alpha^2 - 1}}$$

Asymptotic behaviour of $\varphi_k^*(\alpha)$ for $\alpha \to +\infty$:

$$\frac{1}{\sqrt{\alpha^2 - 1}} \sim \alpha^{-1} + \frac{1}{2}\,\alpha^{-3} + \frac{3}{8}\,\alpha^{-5} + \frac{5}{16}\,\alpha^{-7} + \frac{35}{128}\,\alpha^{-9} + \dots$$

$$\varphi_1^*(\alpha) \sim \frac{1}{2}\,\alpha^{-2} + \frac{3}{8}\,\alpha^{-4} + \frac{5}{16}\,\alpha^{-6} + \frac{35}{128}\,\alpha^{-8} + \dots$$

$$6\,\varphi_2^*(\alpha) \sim \frac{3}{8}\,\alpha^{-2} + \frac{5}{16}\,\alpha^{-4} + \frac{35}{128}\,\alpha^{-6} + \frac{63}{256}\,\alpha^{-8} + \dots$$

$$120\,\varphi_3^*(\alpha) \sim \frac{5}{16}\,\alpha^{-2} + \frac{35}{128}\,\alpha^{-4} + \frac{63}{256}\,\alpha^{-6} + \frac{231}{1024}\,\alpha^{-8} + \dots$$

$$5\,040\,\varphi_4^*(\alpha) \sim \frac{35}{128}\,\alpha^{-2} + \frac{63}{256}\,\alpha^{-4} + \frac{231}{1024}\,\alpha^{-6} + \frac{429}{2048}\,\alpha^{-8} + \dots$$

$$362\,880\,\varphi_5^*(\alpha) \sim \frac{63}{256}\,\alpha^{-2} + \frac{231}{1024}\,\alpha^{-4} + \frac{429}{2048}\,\alpha^{-6} + \frac{6435}{32768}\,\alpha^{-8} + \dots$$

$$39\,916\,800\,\varphi_6^*(\alpha) \sim \frac{231}{1024}\,\alpha^{-2} + \frac{429}{2048}\,\alpha^{-4} + \frac{6435}{32768}\,\alpha^{-6} + \frac{12155}{65536}\,\alpha^{-8} + \dots$$

$$6\,227\,020\,800\,\varphi_7^*(\alpha) \sim \frac{429}{2048}\,\alpha^{-2} + \frac{6435}{32768}\,\alpha^{-4} + \frac{12155}{65536}\,\alpha^{-6} + \frac{46189}{262144}\,\alpha^{-8} + \dots$$

If $0 < \alpha \le 1 \iff -1 \le a < 0$ and x is not very large, then $\mathfrak{H}_\nu^*(x, a)$ may be used.
With $0 < \alpha < 1$, $x \gg 1$ one has the direct asymptotic formula

$$\int_0^x e^{-\alpha t}\,I_0(t)\,dt \sim \frac{e^{(1-\alpha)x}}{\sqrt{2\pi x}}\,\sum_{k=0}^{\infty} \frac{w_k(\alpha)}{x^k} = \frac{e^{(1-\alpha)x}}{\sqrt{2\pi x}}\,\left[\frac{1}{1-\alpha} - \frac{\alpha - 5}{8\,(1-\alpha)^2\,x} - \frac{9\,\alpha^2 - 42\,\alpha + 129}{128\,(1-\alpha)^3\,x^2} - \right.$$

$$- \frac{75\,\alpha^3 - 405\,\alpha^2 + 1065\,\alpha - 2655}{1024\,(1-\alpha)^4\,x^3} - \frac{3675\,\alpha^4 - 23100\,\alpha^3 + 67410\,\alpha^2 - 133980\,\alpha + 301035}{32768\,(1-\alpha)^5\,x^4} -$$

$$- \frac{59535\,\alpha^5 - 429975\,\alpha^4 + 1426950\,\alpha^3 - 3022110\,\alpha^2 + 5120955\,\alpha - 10896795}{262144\,(1-\alpha)^6\,x^5} -$$

$$- \frac{10395}{4194304} \cdot \frac{z_6}{(1-\alpha)^7\,x^6} - \frac{135135}{33554432} \cdot \frac{z_7}{(1-\alpha)^8\,x^7} - \frac{2027025}{2147483648} \cdot \frac{z_8}{(1-\alpha)^9\,x^8} -$$

$$\left. - \frac{34459425}{17179869184} \cdot \frac{z_9}{(1-\alpha)^{10}\,x^9} - \frac{654729075}{274877906944} \cdot \frac{z_{10}}{(1-\alpha)^{11}x^{10}} + \dots \right]$$

$$z_6 = 231\,\alpha^6 - 1890\,\alpha^5 + 7105\,\alpha^4 - 16700\,\alpha^3 + 29049\,\alpha^2 - 44738\,\alpha + 92479$$

$$z_7 = 429\,\alpha^7 - 3927\,\alpha^6 + 16569\,\alpha^5 - 43435\,\alpha^4 + 81815\,\alpha^3 - 125205\,\alpha^2 + 181955\,\alpha - 370345$$

$$z_8 = 6435\,\alpha^8 - 65208\,\alpha^7 + 305844\,\alpha^6 - 890568\,\alpha^5 + 1840370\,\alpha^4 - 2978440\,\alpha^3 + 4186740\,\alpha^2 - 5874040\,\alpha + 11857475$$

$$z_9 = 12155\,\alpha^9 - 135135\,\alpha^8 + 698412\,\alpha^7 - 2244396\,\alpha^6 + 5093802\,\alpha^5 - 8893010\,\alpha^4 + 12934780\,\alpha^3 - 17184540\,\alpha^2 +$$
$$+23605555\,\alpha - 47442055$$

$$z_{10} = 46189\,\alpha^{10} - 559130\,\alpha^9 + 3159585\,\alpha^8 - 11129976\,\alpha^7 + 27654858\,\alpha^6 - 52390044\,\alpha^5 + 80843770\,\alpha^4 -$$
$$-109020920\,\alpha^3 + 139554825\,\alpha^2 - 189306330\,\alpha + 379582629$$

The first functions $w_k(\alpha)$:

x	$w_0(\alpha)$	$w_1(\alpha)$	$w_2(\alpha)$	$w_3(\alpha)$	$w_4(\alpha)$	$w_5(\alpha)$	$w_6(\alpha)$	$w_7(\alpha)$
0.1	1.1111E+00	7.5617E-01	1.3384E+00	3.7992E+00	1.4899E+01	7.4749E+01	4.5743E+02	3.3056E+03
0.2	1.2500E+00	9.3750E-01	1.8457E+00	5.8594E+00	2.5775E+01	1.4527E+02	9.9943E+02	8.1226E+03
0.3	1.4286E+00	1.1990E+00	2.6697E+00	9.6392E+00	4.8356E+01	3.1119E+02	2.4459E+03	2.2714E+04
0.4	1.6667E+00	1.5972E+00	4.1102E+00	1.7248E+01	1.0080E+02	7.5638E+02	6.9345E+03	7.5126E+04
0.5	2.0000E+00	2.2500E+00	6.8906E+00	3.4600E+01	2.4242E+02	2.1822E+03	2.4006E+04	3.1208E+05
0.6	2.5000E+00	3.4375E+00	1.3066E+01	8.1848E+01	7.1645E+02	8.0606E+03	1.1084E+05	1.8011E+06
0.7	3.3333E+00	5.9722E+00	3.0095E+01	2.5104E+02	2.9292E+03	4.3938E+04	8.0554E+05	1.7453E+07
0.8	5.0000E+00	1.3125E+01	9.8789E+01	1.2352E+03	2.1617E+04	4.8639E+05	1.3376E+07	4.3471E+08
0.9	1.0000E+01	5.1250E+01	7.6945E+02	1.9237E+04	6.7330E+05	3.0298E+07	1.6664E+09	1.0832E+11

With fixed $x >$ and $a = -\alpha \to -\infty$ holds

$$\int_0^x e^{-\alpha t} J_0(t)\,dt \ \sim \ \frac{1}{a} - \frac{1}{2a^3} + \frac{3}{8a^5} - \frac{5}{16a^7} + \frac{35}{128a^9} - \frac{63}{256a^{11}} + \ldots =$$

$$= \frac{1}{a} - \frac{0.5}{a^3} + \frac{0.375}{a^5} - \frac{0.3125}{a^7} + \frac{0.2734375}{a^9} - \frac{0.24609375}{a^{11}} + \ldots$$

and

$$\int_0^x e^{-\alpha t} I_0(t)\,dt \ \sim \ \frac{1}{a} + \frac{1}{2a^3} + \frac{3}{8a^5} + \frac{5}{16a^7} + \frac{35}{128a^9} + \frac{63}{256a^{11}} + \ldots\ .$$

d) Integrals:

Concerning the case '$a = \pm 1$ and modified Bessel function' see page 68.

$$\int e^{ax} J_1(x)\,dx = -e^{ax} J_0(x) + a \int e^{ax} J_0(x)\,dx$$

$$\int e^{ax} I_1(x)\,dx = e^{ax} I_0(x) - a \int e^{ax} I_0(x)\,dx$$

$$\int x e^{ax} J_0(x)\,dx = e^{ax}\left[\frac{ax}{a^2+1} J_0(x) + \frac{x}{a^2+1} J_1(x)\right] - \frac{a}{a^2+1} \int e^{ax} J_0(x)\,dx$$

$$\int x e^{ax} I_0(x)\,dx = e^{ax}\left[\frac{ax}{a^2-1} I_0(x) - \frac{x}{a^2-1} I_1(x)\right] - \frac{a}{a^2-1} \int e^{ax} I_0(x)\,dx$$

$$\int x e^{ax} J_1(x)\,dx = e^{ax}\left[-\frac{x}{a^2+1} J_0(x) + \frac{ax}{1+a^2} J_1(x)\right] + \frac{1}{a^2+1} \int e^{ax} J_0(x)\,dx$$

$$\int x e^{ax} I_1(x)\,dx = e^{ax}\left[-\frac{x}{a^2-1} I_0(x) + \frac{ax}{a^2-1} I_1(x)\right] + \frac{1}{a^2-1} \int e^{ax} I_0(x)\,dx$$

$$\int x^2 e^{ax} J_0(x)\,dx = e^{ax}\left[\frac{a(a^2+1)x^2 + (-2a^2+1)x}{(a^2+1)^2} J_0(x) + \frac{(a^2+1)x^2 - 3ax}{(a^2+1)^2} J_1(x)\right] + \frac{2a^2-1}{(a^2+1)^2} \int e^{ax} J_0(x)\,dx$$

$$\int x^2 e^{ax} I_0(x)\,dx = e^{ax}\left[\frac{a(a^2-1)x^2 - (2a^2+1)x}{(a^2-1)^2} I_0(x) + \frac{-(a^2-1)x^2 + 3ax}{(a^2-1)^2} I_1(x)\right] + \frac{2\,a^2+1}{(a^2-1)^2} \int e^{ax} I_0(x)\,dx$$

$$\int x^2 e^{ax} J_1(x)\,dx = e^{ax}\left[\frac{-(a^2+1)x^2 + 3\,ax}{(a^2+1)^2} J_0(x) + \frac{a(a^2+1)x^2 + (2-a^2)x}{(a^2+1)^2} J_1(x)\right] - \frac{3a}{(a^2+1)^2} \int e^{ax} J_0(x)\,dx$$

$$\int x^2 e^{ax} I_1(x)\,dx = e^{ax}\left[\frac{-(a^2-1)x^2 + 3\,ax}{(a^2-1)^2} I_0(x) + \frac{a(a^2-1)x^2 - (a^2+2)x}{(a^2-1)^2} I_1(x)\right] - \frac{3a}{(a^2-1)^2} \int e^{ax} I_0(x)\,dx$$

$$\int x^3 e^{ax} J_0(x)\, dx = e^{ax}\left[\frac{a\left(1+a^2\right)^2 x^3 - \left(3\,a^2 - 2\right)\left(1+a^2\right)x^2 + 3\,a\left(2\,a^2 - 3\right)x}{(a^2+1)^3} J_0(x) + \right.$$

$$\left. +\frac{\left(1+a^2\right)^2 x^3 - 5\,a\left(1+a^2\right)x^2 + (11\,a^2 - 4)x}{(a^2+1)^3} J_1(x)\right] - \frac{3a(2a^2 - 3)}{(a^2+1)^3}\int e^{ax} J_0(x)\, dx$$

$$\int x^3 e^{ax} I_0(x)\, dx = e^{ax}\left[\frac{a\left(a^2-1\right)^2 x^3 + -\left(3\,a^2+2\right)\left(a^2-1\right)x^2 + 3\,a\left(2\,a^2+3\right)x}{(a^2-1)^3} I_0(x) + \right.$$

$$\left. +\frac{-\left(a^2-1\right)^2 x^3 + 5\,a\left(a^2-1\right)x^2 - (11\,a^2+4)x}{(a^2-1)^3} I_1(x)\right] - \frac{3a\left(2a^2+3\right)}{(a^2-1)^3}\int e^{ax} I_0(x)\, dx$$

$$\int x^3 e^{ax} J_1(x)\, dx = e^{ax}\left[\frac{-\left(1+a^2\right)^2 x^3 + 5\,a\left(1+a^2\right)x^2 - 3\left(4\,a^2-1\right)x}{(a^2+1)^3} J_0(x) + \right.$$

$$\left. +\frac{a\left(1+a^2\right)^2 x^3 - \left(2\,a^2-3\right)\left(1+a^2\right)x^2 + a\left(2\,a^2-13\right)x}{(a^2+1)^3} J_1(x)\right] + \frac{12a^2 - 3}{(a^2+1)^3}\int e^{ax} J_0(x)\, dx$$

$$\int x^3 e^{ax} I_1(x)\, dx = e^{ax}\left[\frac{-\left(a^2-1\right)^2 x^3 + 5\,a\left(a^2-1\right)x^2 - 3(4a^2+1)x}{(a^2-1)^3} I_0(x) + \right.$$

$$\left. +\frac{a\left(a^2-1\right)^2 x^3 - (2a^2+3)(a^2-1)x^2 + a(2a^2+13)x}{(a^2-1)^3} I_1(x)\right] - \frac{3(4a^2+1)}{(a^2-1)^3}\int e^{ax} I_0(x)\, dx$$

Let

$$\int x^n e^{ax} J_\nu(x)\, dx = e^{ax}\left[\frac{P_n^{(\nu)}}{(a^2+1)^n} J_0(x) + \frac{Q_n^{(\nu)}}{(a^2+1)^n} J_1(x)\right] + \frac{R_n^{(\nu)}}{(a^2+1)^n}\int e^{ax} J_0(x)\, dx$$

and

$$\int x^n e^{ax} I_\nu(x)\, dx = e^{ax}\left[\frac{\mathfrak{P}_n^{(\nu)}}{(a^2-1)^n} I_0(x) + \frac{\mathfrak{Q}_n^{(\nu)}}{(a^2-1)^n} I_1(x)\right] + \frac{\mathfrak{R}_n^{(\nu)}}{(a^2-1)^n}\int e^{ax} I_0(x)\, dx\ ,$$

then holds

$$P_4^{(0)} = a\left(1+a^2\right)^3 x^4 - \left(4\,a^2-3\right)\left(1+a^2\right)^2 x^3 + a\left(12\,a^2-23\right)\left(1+a^2\right)x^2 + (-24\,a^4 + 72\,a^2 - 9)x$$

$$Q_4^{(0)} = \left(1+a^2\right)^3 x^4 - 7\,a\left(1+a^2\right)^2 x^3 + \left(26\,a^2-9\right)\left(1+a^2\right)x^2 - 5\,a\left(10\,a^2-11\right)x$$

$$R_4^{(0)} = 24\,a^4 - 72\,a^2 + 9$$

$$\mathfrak{P}_4^{(0)} = a\left(a^2-1\right)^3 x^4 - \left(4\,a^2+3\right)\left(a^2-1\right)^2 x^3 + a\left(12\,a^2+23\right)\left(a^2-1\right)x^2 - (24\,a^4 + 72\,a^2 + 9)x$$

$$\mathfrak{Q}_4^{(0)} = -\left(a^2-1\right)^3 x^4 + 7a\left(a^2-1\right)^2 x^3 - \left(26\,a^2+9\right)\left(a^2-1\right)x^2 + 5\,a\left(10\,a^2+11\right)x$$

$$\mathfrak{R}_4^{(0)} = 24\,a^4 + 72\,a^2 + 9$$

$$P_4^{(1)} = -\left(1+a^2\right)^3 x^4 + 7\,a\left(1+a^2\right)^2 x^3 - \left(27\,a^2-8\right)\left(1+a^2\right)x^2 + 15\,a\left(4\,a^2-3\right)x$$

$$Q_4^{(1)} = a\left(1+a^2\right)^3 x^4 - \left(3\,a^2-4\right)\left(1+a^2\right)^2 x^3 + a\left(6\,a^2-29\right)\left(1+a^2\right)x^2 + (-6\,a^4 + 83\,a^2 - 16)x$$

$$R_4^{(1)} = -15a\left(4\,a^2-3\right)$$

$$\mathfrak{P}_4^{(1)} = -\left(a^2-1\right)^3 x^4 + 7\,a\left(a^2-1\right)^2 x^3 - \left(27\,a^2+8\right)\left(a^2-1\right)x^2 + 15\,a\left(4\,a^2+3\right)x$$

$$\mathfrak{Q}_4^{(1)} = a\left(a^2-1\right)^3 x^4 - \left(4+3\,a^2\right)\left(a^2-1\right)^2 x^3 + a\left(6\,a^2+29\right)\left(a^2-1\right)x^2 - (6\,a^4 + 83\,a^2 + 16)x$$

$$\mathfrak{R}_4^{(1)} = -15a\left(4a^2+3\right)$$

$$P_5^{(0)} = a\left(1+a^2\right)^4 x^5 - \left(5a^2-4\right)\left(a^2+1\right)^3 x^4 + a\left(20a^2-43\right)\left(a^2+1\right)^2 x^3 -$$
$$- \left(a^2+1\right)\left(60a^4-223a^2+32\right)x^2 + 15\left(8a^4-40a^2+15\right)ax$$
$$Q_5^{(0)} = \left(a^2+1\right)^4 x^5 - 9a\left(a^2+1\right)^3 x^4 + \left(47a^2-16\right)\left(a^2+1\right)^2 x^3 -$$
$$-7a\left(22a^2-23\right)\left(a^2+1\right)x^2 + \left(274a^4-607a^2+64\right)x$$
$$R_5^{(0)} = -15a\left(8a^4-40a^2+15\right)$$

$$\mathfrak{P}_5^{(0)} = a\left(a^2-1\right)^4 x^5 - \left(4+5a^2\right)\left(a^2-1\right)^3 x^4 + a\left(20a^2+43\right)\left(a^2-1\right)^2 x^3 -$$
$$- \left(60a^4+223a^2+32\right)\left(a^2-1\right)x^2 + 15a\left(8a^4+40a^2+15\right)x$$
$$\mathfrak{Q}_5^{(0)} = -\left(a^2-1\right)^4 x^5 + 9a\left(a^2-1\right)^3 x^4 - \left(16+47a^2\right)\left(a^2-1\right)^2 x^3 + 7a\left(22a^2+23\right)\left(a^2-1\right)x^2 -$$
$$-\left(274a^4+607a^2+64\right)x$$
$$\mathfrak{R}_5^{(0)} = -15a\left(8a^4+40a^2+15\right)$$

$$P_5^{(1)} = -\left(a^2+1\right)^4 x^5 + 9\left(a^2+1\right)^3 ax^4 - 3\left(16a^2-5\right)\left(a^2+1\right)^2 x^3 + 21a\left(8a^2-7\right)\left(a^2+1\right)x^2 +$$
$$+\left(-360a^4+540a^2-45\right)x$$
$$Q_5^{(1)} = a\left(a^2+1\right)^4 x^5 - \left(-5+4a^2\right)\left(a^2+1\right)^3 x^4 + 3a\left(4a^2-17\right)\left(a^2+1\right)^2 x^3 -$$
$$-3\left(a^2+1\right)\left(8a^4-82a^2+15\right)x^2 + 3a\left(8a^4-194a^2+113\right)x$$
$$R_5^{(1)} = 360a^4-540a^2+45$$

$$\mathfrak{P}_5^{(1)} = -\left(a^2-1\right)^4 x^5 + 9a\left(a^2-1\right)^3 x^4 - 3\left(16a^2+5\right)\left(a^2-1\right)^2 x^3 + 21a\left(8a^2+7\right)\left(a^2-1\right)x^2 -$$
$$-\left(360a^4+540a^2+45\right)x$$
$$\mathfrak{Q}_5^{(1)} = a\left(a^2-1\right)^4 x^5 - \left(4a^2+5\right)\left(a^2-1\right)^3 x^4 + 3a\left(4a^2+17\right)\left(a^2-1\right)^2 x^3 -$$
$$-3\left(a^2-1\right)\left(8a^4+82a^2+15\right)x^2 + 3a\left(8a^4+194a^2+113\right)x$$
$$\mathfrak{R}_5^{(1)} = 360a^4+540a^2+45$$

$$P_6^{(0)} = a\left(a^2+1\right)^5 x^6 - \left(6a^2-5\right)\left(a^2+1\right)^4 x^5 + 3a\left(10a^2-23\right)\left(a^2+1\right)^3 x^4 -$$
$$-3\left(40a^4-166a^2+25\right)\left(a^2+1\right)^2 x^3 + 9a\left(a^2+1\right)\left(40a^4-242a^2+103\right)x^2 +$$
$$+\left(-720a^6+5400a^4-4050a^2+225\right)x$$
$$Q_6^{(0)} = \left(a^2+1\right)^5 x^6 - 11a\left(a^2+1\right)^4 x^5 + \left(74a^2-25\right)\left(a^2+1\right)^3 x^4 - 9a\left(38a^2-39\right)\left(a^2+1\right)^2 x^3 +$$
$$+9\left(a^2+1\right)\left(116a^4-244a^2+25\right)x^2 - 63\left(28a^4-104a^2+33\right)ax$$
$$R_6^{(0)} = 720a^6-5400a^4+4050a^2-225$$

$$\mathfrak{P}_6^{(0)} = a\left(a^2-1\right)^5 x^6 - \left(6a^2+5\right)\left(a^2-1\right)^4 x^5 + 3a\left(10a^2+23\right)\left(a^2-1\right)^3 x^4 -$$
$$-3\left(40a^4+166a^2+25\right)\left(a^2-1\right)^2 x^3 + 9a\left(40a^4+242a^2+103\right)\left(a^2-1\right)x^2 - \left(720a^6+5400a^4+4050a^2+225\right)x$$
$$\mathfrak{Q}_6^{(0)} = -\left(a^2-1\right)^5 x^6 + 11a\left(a^2-1\right)^4 x^5 - \left(74a^2+25\right)\left(a^2-1\right)^3 x^4 + 9a\left(38a^2+39\right)\left(a^2-1\right)^2 x^3 -$$
$$-9\left(116a^4+244a^2+25\right)\left(a^2-1\right)x^2 + 63a\left(28a^4+104a^2+33\right)x$$

$$\mathfrak{R}_6^{(0)} = 720\,a^6 + 5400\,a^4 + 4050\,a^2 + 225$$

$$P_6^{(1)} = -\left(a^2+1\right)^5 x^6 + 11\,a\left(a^2+1\right)^4 x^5 - 3\left(25\,a^2-8\right)\left(a^2+1\right)^3 x^4 + 9\,a\left(40\,a^2-37\right)\left(a^2+1\right)^2 x^3 -$$
$$-3\left(a^2+1\right)\left(400\,a^4 - 691\,a^2 + 64\right)x^2 + 315\left(8\,a^4 - 20\,a^2 + 5\right)ax$$
$$Q_6^{(1)} = a\left(a^2+1\right)^5 x^6 - \left(5\,a^2-6\right)\left(a^2+1\right)^4 x^5 + a\left(20\,a^2-79\right)\left(a^2+1\right)^3 x^4 -$$
$$-3\left(20\,a^4 - 179\,a^2 + 32\right)\left(a^2+1\right)^2 x^3 + 3\,a\left(a^2+1\right)\left(40\,a^4 - 718\,a^2 + 397\right)x^2 +$$
$$+\left(-120\,a^6 + 4554\,a^4 - 5337\,a^2 + 384\right)x$$
$$R_6^{(1)} = -315\left(8\,a^4 - 20\,a^2 + 5\right)a$$

$$\mathfrak{P}_6^{(1)} = -\left(a^2-1\right)^5 x^6 + 11\,a\left(a^2-1\right)^4 x^5 - 3\left(25\,a^2+8\right)\left(a^2-1\right)^3 x^4 + 9\,a\left(40\,a^2+37\right)\left(a^2-1\right)^2 x^3 -$$
$$-3\left(400\,a^4 + 691\,a^2 + 64\right)\left(a^2-1\right)x^2 + 315\,a\left(8\,a^4 + 20\,a^2 + 5\right)x$$
$$\mathfrak{Q}_6^{(1)} = a\left(a^2-1\right)^5 x^6 - \left(5\,a^2+6\right)\left(a^2-1\right)^4 x^5 + a\left(20\,a^2+79\right)\left(a^2-1\right)^3 x^4 -$$
$$-3\left(20\,a^4 + 179\,a^2 + 32\right)\left(a^2-1\right)^2 x^3 + 3\,a\left(a^2-1\right)\left(40\,a^4 + 718\,a^2 + 397\right)x^2 -$$
$$-\left(120\,a^6 + 4554\,a^4 + 5337\,a^2 + 384\right)x$$
$$\mathfrak{R}_6^{(1)} = -315\left(8\,a^4 + 20\,a^2 + 5\right)a$$

$$P_7^{(0)} = a\left(1+a^2\right)^6 x^7 - \left(7\,a^2-6\right)\left(a^2+1\right)^5 x^6 + a\left(42\,a^2-101\right)\left(a^2+1\right)^4 x^5 -$$
$$-3\left(70\,a^4 - 311\,a^2 + 48\right)\left(a^2+1\right)^3 x^4 + 3\,a\left(280\,a^4 - 1882\,a^2 + 841\right)\left(a^2+1\right)^2 x^3 +$$
$$-9\left(a^2+1\right)\left(280\,a^6 - 2494\,a^4 + 2103\,a^2 - 128\right)x^2 + 315\left(16\,a^6 - 168\,a^4 + 210\,a^2 - 35\right)ax$$
$$Q_7^{(0)} = \left(a^2+1\right)^6 x^7 - 13\,a\left(a^2+1\right)^5 x^6 + \left(107\,a^2-36\right)\left(a^2+1\right)^4 x^5 - 11\left(58\,a^2-59\right)\left(a^2+1\right)^3 ax^4 +$$
$$+9\left(306\,a^4 - 631\,a^2 + 64\right)\left(a^2+1\right)^2 x^3 - 9\left(a^2+1\right)\left(892\,a^4 - 3144\,a^2 + 969\right)ax^2 +$$
$$+\left(13068\,a^6 - 73188\,a^4 + 46575\,a^2 - 2304\right)x$$
$$R_7^{(0)} = -315\left(16\,a^6 - 168\,a^4 + 210\,a^2 - 35\right)a$$

$$\mathfrak{P}_7^{(0)} = a\left(a^2-1\right)^6 x^7 - \left(7\,a^2+6\right)\left(a^2-1\right)^5 x^6 + a\left(101+42\,a^2\right)\left(a^2-1\right)^4 x^5 -$$
$$-3\left(70\,a^4 + 311\,a^2 + 48\right)\left(a^2-1\right)^3 x^4 + 3\,a\left(280\,a^4 + 1882\,a^2 + 841\right)\left(a^2-1\right)^2 x^3 -$$
$$-9\left(280\,a^6 + 2494\,a^4 + 2103\,a^2 + 128\right)\left(a^2-1\right)x^2 + 315\,a\left(16\,a^6 + 168\,a^4 + 210\,a^2 + 35\right)x$$
$$\mathfrak{Q}_7^{(0)} = -\left(a^2-1\right)^6 x^7 + 13\,a\left(a^2-1\right)^5 x^6 - \left(107\,a^2+36\right)\left(a^2-1\right)^4 x^5 + 11\,a\left(59+58\,a^2\right)\left(a^2-1\right)^3 x^4 -$$
$$-9\left(306\,a^4 + 631\,a^2 + 64\right)\left(a^2-1\right)^2 x^3 + 9\,a\left(892\,a^4 + 3144\,a^2 + 969\right)\left(a^2-1\right)x^2 -$$
$$-\left(13068\,a^6 + 73188\,a^4 + 46575\,a^2 + 2304\right)x$$
$$\mathfrak{R}_7^{(0)} = -315a\left(16\,a^6 + 168\,a^4 + 210\,a^2 + 35\right)$$

$$P_7^{(1)} = -\left(1+a^2\right)^6 x^7 + 13\,a\left(a^2+1\right)^5 x^6 - \left(108\,a^2-35\right)\left(a^2+1\right)^4 x^5 + 33\,a\left(20\,a^2-19\right)\left(a^2+1\right)^3 x^4 -$$
$$-3\left(1000\,a^4 - 1828\,a^2 + 175\right)\left(a^2+1\right)^2 x^3 + 9\,a\left(a^2+1\right)\left(1080\,a^4 - 3076\,a^2 + 849\right)x^2 +$$
$$+\left(-20160\,a^6 + 75600\,a^4 - 37800\,a^2 + 1575\right)x$$
$$Q_7^{(1)} = a\left(a^2+1\right)^6 x^7 - \left(6\,a^2-7\right)\left(a^2+1\right)^5 x^6 + a\left(30\,a^2-113\right)\left(a^2+1\right)^4 x^5 -$$

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$$- \left(120\,a^4 - 992\,a^2 + 175\right)\left(a^2+1\right)^3 x^4 + 9\,a\left(40\,a^4 - 624\,a^2 + 337\right)\left(a^2+1\right)^2 x^3-$$
$$-9\left(a^2+1\right)\left(80\,a^6 - 2248\,a^4 + 2502\,a^2 - 175\right)x^2 + 9\,a\left(80\,a^6 - 4408\,a^4 + 8654\,a^2 - 1873\right)x$$
$$R_7^{(1)} = 20160\,a^6 - 75600\,a^4 + 37800\,a^2 - 1575$$

$$\mathfrak{P}_7^{(1)} = -\left(a^2-1\right)^6 x^7 + 13\,a\left(a^2-1\right)^5 x^6 - \left(108\,a^2 + 35\right)\left(a^2-1\right)^4 x^5+$$
$$+33\,a\left(20\,a^2 + 19\right)\left(a^2-1\right)^3 x^4 - 3\left(1000\,a^4 + 1828\,a^2 + 175\right)\left(a^2-1\right)^2 x^3+$$
$$+9\,a\left(1080\,a^4 + 3076\,a^2 + 849\right)\left(a^2-1\right) - \left(20160\,a^6 + 75600\,a^4 + 37800\,a^2 + 1575\right)x$$
$$\mathfrak{Q}_7^{(1)} = a\left(a^2-1\right)^6 x^7 - \left(6\,a^2 + 7\right)\left(a^2-1\right)^5 x^6 + a\left(30\,a^2 + 113\right)\left(a^2-1\right)^4 x^5-$$
$$-\left(120\,a^4 + 992\,a^2 + 175\right)\left(a^2-1\right)^3 x^4 + 9\,a\left(40\,a^4 + 624\,a^2 + 337\right)\left(a^2-1\right)^2 x^3-$$
$$-9\left(80\,a^6 + 2248\,a^4 + 2502\,a^2 + 175\right)\left(a^2-1\right)x^2 + 9\,a\left(80\,a^6 + 4408\,a^4 + 8654\,a^2 + 1873\right)x$$
$$\mathfrak{R}_7^{(1)} = 20160\,a^6 + 75600\,a^4 + 37800\,a^2 + 1575$$

$$P_8^{(0)} = a\left(a^2+1\right)^7 x^8 - \left(8\,a^2 - 7\right)\left(a^2+1\right)^6 x^7 + a\left(56\,a^2 - 139\right)\left(a^2+1\right)^5 x^6-$$
$$-\left(336\,a^4 - 1564\,a^2 + 245\right)\left(a^2+1\right)^4 x^5 + 3\,a\left(560\,a^4 - 4028\,a^2 + 1847\right)\left(a^2+1\right)^3 x^4-$$
$$-3\left(2240\,a^6 - 22056\,a^4 + 19524\,a^2 - 1225\right)\left(a^2+1\right)^2 x^3+$$
$$+9\,a\left(a^2+1\right)\left(2240\,a^6 - 27512\,a^4 + 38356\,a^2 - 6967\right)x^2+$$
$$+\left(-40320\,a^8 + 564480\,a^6 - 1058400\,a^4 + 352800\,a^2 - 11025\right)x$$
$$Q_8^{(0)} = \left(a^2+1\right)^7 x^8 - 15\,a\left(a^2+1\right)^6 x^7 + \left(146\,a^2 - 49\right)\left(a^2+1\right)^5 x^6 - 13\,a\left(82\,a^2 - 83\right)\left(a^2+1\right)^4 x^5+$$
$$+\left(5944\,a^4 - 12136\,a^2 + 1225\right)\left(a^2+1\right)^3 x^4 - 99\,a\left(248\,a^4 - 856\,a^2 + 261\right)\left(a^2+1\right)^2 x^3+$$
$$+9\left(a^2+1\right)\left(7696\,a^6 - 40888\,a^4 + 25266\,a^2 - 1225\right)x^2 - 9\,a\left(12176\,a^6 - 95912\,a^4 + 101978\,a^2 - 15159\right)x$$
$$R_8^{(0)} = 40320\,a^8 - 564480\,a^6 + 1058400\,a^4 - 352800\,a^2 + 11025$$

$$\mathfrak{P}_8^{(0)} = a\left(a^2-1\right)^7 x^8 - \left(8\,a^2 + 7\right)\left(a^2-1\right)^6 x^7 + a\left(56\,a^2 + 139\right)\left(a^2-1\right)^5 x^6-$$
$$-\left(336\,a^4 + 1564\,a^2 + 245\right)\left(a^2-1\right)^4 x^5 + 3\,a\left(560\,a^4 + 4028\,a^2 + 1847\right)\left(a^2-1\right)^3 x^4-$$
$$-3\left(2240\,a^6 + 22056\,a^4 + 19524\,a^2 + 1225\right)\left(a^2-1\right)^2 x^3+$$
$$+9\,a\left(2240\,a^6 + 27512\,a^4 + 38356\,a^2 + 6967\right)\left(a^2-1\right)x^2-$$
$$-\left(40320\,a^8 + 564480\,a^6 + 1058400\,a^4 + 352800\,a^2 + 11025\right)x$$
$$\mathfrak{Q}_8^{(0)} = -\left(a^2-1\right)^7 x^8 + 15\,a\left(a^2-1\right)^6 x^7 - \left(146\,a^2 + 49\right)\left(a^2-1\right)^5 x^6 + 13\,a\left(82\,a^2 + 83\right)\left(a^2-1\right)^4 x^5-$$
$$-\left(5944\,a^4 + 12136\,a^2 + 1225\right)\left(a^2-1\right)^3 x^4 + 99\,a\left(248\,a^4 + 856\,a^2 + 261\right)\left(a^2-1\right)^2 x^3-$$
$$-9\left(7696\,a^6 + 40888\,a^4 + 25266\,a^2 + 1225\right)\left(a^2-1\right)x^2+$$
$$+9a\left(12176\,a^6 + 95912\,a^4 + 101978\,a^2 + 15159\right)x$$
$$\mathfrak{R}_8^{(0)} = 40320\,a^8 + 564480\,a^6 + 1058400\,a^4 + 352800\,a^2 + 11025$$

$$P_8^{(1)} = -\left(a^2+1\right)^7 x^8 + 15\,a\left(a^2+1\right)^6 x^7 - 3\left(49\,a^2 - 16\right)\left(a^2+1\right)^5 x^6 + 39\,a\left(28\,a^2 - 27\right)\left(a^2+1\right)^4 x^5-$$
$$-9\left(700\,a^4 - 1317\,a^2 + 128\right)\left(a^2+1\right)^3 x^4 + 99\,a\left(280\,a^4 - 844\,a^2 + 241\right)\left(a^2+1\right)^2 x^3-$$
$$-9\left(a^2+1\right)\left(9800\,a^6 - 41484\,a^4 + 22767\,a^2 - 1024\right)x^2 + 2835\,a\left(64\,a^6 - 336\,a^4 + 280\,a^2 - 35\right)x$$

$$Q_8^{(1)} = a\,(a^2+1)^7\,x^8 - (7\,a^2-8)\,(a^2+1)^6\,x^7 + 3\,a\,(14\,a^2-51)\,(a^2+1)^5\,x^6-$$

$$-3\,(70\,a^4-549\,a^2+96)\,(a^2+1)^4\,x^5 + 3\,a\,(280\,a^4-4016\,a^2+2139)\,(a^2+1)^3\,x^4-$$

$$-9\,(280\,a^6-6816\,a^4+7407\,a^2-512)\,(a^2+1)^2\,x^3 + 9\,a\,(a^2+1)\,(560\,a^6-22872\,a^4+42666\,a^2-8977)\,x^2+$$

$$+(-5040\,a^8+382248\,a^6-1130706\,a^4+490599\,a^2-18432)x$$

$$R_8^{(1)} = -2835a\,(64\,a^6-336\,a^4+280\,a^2-35)$$

$$\mathfrak{P}_8^{(1)} = -\,(a^2-1)^7\,x^8 + 15\,a\,(a^2-1)^6\,x^7 - 3\,(49\,a^2+16)\,(a^2-1)^5\,x^6 + 39\,a\,(28\,a^2+27)\,(a^2-1)^4\,x^5-$$

$$-9\,(700\,a^4+1317\,a^2+128)\,(a^2-1)^3\,x^4 + 99\,a\,(280\,a^4+844\,a^2+241)\,(a^2-1)^2\,x^3-$$

$$-9\,(9800\,a^6+41484\,a^4+22767\,a^2+1024)\,(a^2-1)\,x^2 + 2835\,a\,(64\,a^6+336\,a^4+280\,a^2+35)\,x$$

$$\mathfrak{Q}_8^{(1)} = a\,(a^2-1)^7\,x^8 - (7\,a^2+8)\,(a^2-1)^6\,x^7 + 3\,a\,(14\,a^2+51)\,(a^2-1)^5\,x^6-$$

$$-3\,(70\,a^4+549\,a^2+96)\,(a^2-1)^4\,x^5 + 3\,a\,(280\,a^4+4016\,a^2+2139)\,(a^2-1)^3\,x^4-$$

$$-9\,(280\,a^6+6816\,a^4+7407\,a^2+512)\,(a^2-1)^2\,x^3 + 9\,a\,(560\,a^6+22872\,a^4+42666\,a^2+8977)\,(a^2-1)\,x^2-$$

$$-(5040\,a^8+382248\,a^6+1130706\,a^4+490599\,a^2+18432)x$$

$$\mathfrak{R}_8^{(1)} = -2835a\,(64\,a^6+336\,a^4+280\,a^2+35)$$

Recurrence formulas: Let

$$\mathbf{J}_n^{(\nu)} = \int x^n e^{ax} J_\nu(x)\,dx\,, \qquad \mathbf{I}_n^{(\nu)} = \int x^n e^{ax} I_\nu(x)\,dx\,,$$

then holds

$$\mathbf{J}_{n+1}^{(0)} = \frac{x^{n+1}e^{ax}[aJ_0(x)+J_1(x)]-(n+1)\,a\,\mathbf{J}_n^{(0)}-n\mathbf{J}_n^{(1)}}{a^2+1}\,,$$

$$\mathbf{J}_{n+1}^{(1)} = \frac{x^{n+1}e^{ax}[aJ_1(x)-J_0(x)]+(n+1)\,\mathbf{J}_n^{(0)}-a\,n\,\mathbf{J}_n^{(1)}}{a^2+1}$$

and

$$\mathbf{I}_{n+1}^{(0)} = \frac{x^{n+1}e^{ax}[aI_0(x)-I_1(x)]-(n+1)\,a\,\mathbf{I}_n^{(0)}+n\mathbf{I}_n^{(1)}}{a^2-1}\,,$$

$$\mathbf{I}_{n+1}^{(1)} = \frac{x^{n+1}e^{ax}[aI_1(x)-I_0(x)]+(n+1)\,\mathbf{I}_n^{(0)}-a\,n\,\mathbf{I}_n^{(1)}}{a^2-1}\,.$$

e) Special Cases:

$$\int x^2 \exp\left(\frac{x}{\sqrt{2}}\right) J_0(x)\,dx = \frac{x}{3}\left[\sqrt{2}x\,J_0(x)+2(x-\sqrt{2})\,J_1(x)\right]\exp\left(\frac{x}{\sqrt{2}}\right)$$

$$\int x^2 \exp\left(-\frac{x}{\sqrt{2}}\right) J_0(x)\,dx = \frac{x}{3}\left[-\sqrt{2}x\,J_0(x)+2(x+\sqrt{2})\,J_1(x)\right]\exp\left(-\frac{x}{\sqrt{2}}\right)$$

$$\int x^3 \exp\left(\sqrt{\frac{3}{2}}x\right) J_0(x)\,dx = \frac{x}{5}\left[(\sqrt{6}x^2-2\,x)\,J_0(x)+(x^2-\sqrt{6}x+2)\,J_1(x)\right]\exp\left(\sqrt{\frac{3}{2}}x\right)$$

$$\int x^3 \exp\left(-\sqrt{\frac{3}{2}}x\right) J_0(x)\,dx = \frac{x}{5}\left[-(\sqrt{6}x^2+2\,x)\,J_0(x)+(x^2+\sqrt{6}x+2)\,J_1(x)\right]\exp\left(\sqrt{\frac{3}{2}}x\right)$$

$$\int x^3 \exp\left(\frac{x}{2}\right) J_0(x)\,dx = \frac{2x}{5}\left[-(2x^2-4x)\,J_0(x)+(x^2+4x-8)\,J_1(x)\right]\exp\left(\frac{x}{2}\right)$$

$$\int x^3 \exp\left(-\frac{x}{2}\right) J_0(x)\, dx = -\frac{2x}{5}\left[(2x^2 + 4x)\, J_0(x) + (x^2 - 4x - 8)\, J_1(x)\right] \exp\left(-\frac{x}{2}\right)$$

$$w = \pm\frac{\sqrt{6 \pm \sqrt{30}}}{2} = \pm\{0.361\,516;\ 1.693\,903\}$$

$$\int x^4\, e^{wx}\, J_0(x)\, dx = \frac{x}{5w(4w^2 - 3)}\left\{\left[(12\,w^2 - 3)x^3 - 24wx^2 + 24x\right] J_0(x) + \right.$$
$$\left. + \left[(8\,w^3 - 12\,w)x^3 + (-24\,w^2 + 12)x^2 + 48wx - 48\right] J_1(x)\right\} e^{wx}$$

$$w = \pm\frac{\sqrt{3}}{2} = \pm\, 0.866\,025$$

$$\int x^4\, e^{wx}\, J_1(x)\, dx = \frac{x}{8\,w^4 - 24\,w^2 + 3}\left\{\left[(12\,w^2 - 3)x^3 - 24wx^2 + 24x\right] J_0(x) + \right.$$
$$\left. + \left[(8\,w^3 - 12\,w)x^3 + (-24\,w^2 + 12)x^2 + 48wx - 48\right] J_1(x)\right\} e^{wx}$$

$$w = \pm\frac{\sqrt{10 \pm \sqrt{70}}}{2} = \pm\{0.639\,023;\ 2.142\,814\}$$

$$\int x^5\, e^{wx}\, J_0(x)\, dx = \frac{x}{3(8\,w^4 - 12\,w^2 + 1)}\left\{\left[(16\,w^3 - 12\,w)x^4 + (-48\,w^2 + 12)x^3 + 96wx - 96x\right] J_0(x) + \right.$$
$$\left. + \left[(8\,w^4 - 24\,w^2 + 3)x^4 + (-32\,w^3 + 48\,w)x^3 + (96\,w^2 - 48)x^2 - 192wx + 192\right] J_1(x)\right\} e^{wx}$$

$$w = \pm\frac{\sqrt{3 \pm \sqrt{7}}}{2} = \pm\{0.297\,594;\ 1.188\,039\}$$

$$\int x^5\, e^{wx}\, J_1(x)\, dx = \frac{x}{w(8\,w^4 - 40\,w^2 + 15)}\left\{\left[(16\,w^3 - 12\,w)x^4 + (-48\,w^2 + 12)x^3 + 96wx^2 - 96x\right] J_0(x) + \right.$$
$$\left. + \left[(8\,w^4 - 24\,w^2 + 3)x^4 + (-32\,w^3 + 48\,w)x^3 + (96\,w^2 - 48)x^2 - 192wx + 192\right] J_1(x)\right\} e^{wx}$$

$$16w^6 - 120w^4 + 90w^2 - 5 = 0 \implies w \in \{\pm\, 0.245\,717\,164,\ \pm\, 0.881\,375\,831,\ \pm\, 2.581\,239\,958\}$$

$$\int x^6\, e^{wx}\, J_0(x)\, dx = \frac{x}{7w(8w^4 - 20w^2 + 5)} \cdot$$
$$\cdot \left\{\left[(40\,w^4 - 60\,w^2 + 5)x^5 + (-160\,w^3 + 120\,w)x^4 + (480\,w^2 - 120)x^3 - 960wx^2 + 960x\right] J_0(x) + \right.$$
$$+ \left[(16\,w^5 - 80\,w^3 + 30\,w)x^5 + (-80\,w^4 + 240\,w^2 - 30)x^4 + (320\,w^3 - 480\,w)x^3 + \right.$$
$$\left. + (-960\,w^2 + 480)x^2 + 1920wx - 1920\right] J_1(x)\right\} e^{wx}$$

$$w = \pm\frac{\sqrt{5 \pm \sqrt{15}}}{2} = \pm\{0.530\,805,\ 1.489\,378\}$$

$$\int x^6\, e^{wx}\, J_1(x)\, dx = \frac{x}{16w^6 - 120w^4 + 90w^2 - 5}\left\{\left[(40\,w^4 - 60\,w^2 + 5)x^5 + (-160\,w^3 + 120\,w)x^4 + \right.\right.$$
$$\left. + (480\,w^2 - 120)x^3 - 960wx^2 + 960w\right] J_0(x) +$$
$$\left[(16\,w^5 - 80\,w^3 + 30\,w)x^5 + (-80\,w^4 + 240\,w^2 - 30)x^4 + (320\,w^3 - 480\,w)x^3 \right.$$
$$\left. + (-960\,w^2 + 480)x^2 + 1920x - 1920\right] J_1(x)\right\} e^{wx}$$

$$16w^6 - 168w^4 + 210w^2 - 35 = 0 \quad \Rightarrow \quad w \in \{\pm\,0.444\,060\,144,\ \pm\,1.105\,247\,947,\ \pm\,3.013\,509\,178\}$$

$$\int x^7\, e^{wx}\, J_0(x)\, dx = \frac{x}{64w^6 - 240w^4 + 120w^2 - 5} \Big\{ \big[(48\,w^5 - 120\,w^3 + 30\,w)x^6 + (-240\,w^4 + 360\,w^2 - 30)x^5 +$$

$$+\, (960\,w^3 - 720\,w)x^4 + (-2880\,w^2 + 720)x^3 + 5760wx^2 - 5760x \big]\ J_0(x) +$$

$$+\, \big[(16\,w^6 - 120\,w^4 + 90\,w^2 - 5)x^6 + (-96\,w^5 + 480\,w^3 - 180\,w)x^5 + (480\,w^4 - 1440\,w^2 + 180)x^4 +$$

$$+\, (-1920\,w^3 + 2880\,w)x^3 + (5760\,w^2 - 2880)x^2 - 11520wx + 11520 \big]\ J_1(x) \Big\}\ e^{wx}$$

$$64w^6 - 240w^4 + 120w^2 - 5 = 0 \quad \Rightarrow \quad w \in \{\pm\,0.214\,039\,849,\ \pm\,0.733\,975\,352,\ \pm\,1.779\,175\,968\}$$

$$\int x^7\, e^{wx}\, J_1(x)\, dx = \frac{x}{w(16w^6 - 168w^4 + 210w^2 - 35)} \cdot$$

$$\cdot \Big\{ \big[(48\,w^5 - 120\,w^3 + 30\,w)x^6 + (-240\,w^4 + 360\,w^2 - 30)x^5 + (960\,w^3 - 720\,w)x^4$$

$$+\, (-2880\,w^2 + 720)x^3 + 5760wx^2 - 5760 \big]\ J_0(x) +$$

$$+\, \big[(16\,w^6 - 120\,w^4 + 90\,w^2 - 5)x^6 + (-96\,w^5 + 480\,w^3 - 180\,w)x^5 + (480\,w^4 - 1440\,w^2 + 180)x^4 +$$

$$+\, (-1920\,w^3 + 2880\,w)x^3 + (5760\,w^2 - 2880)x^2 - 11520wx + 11520 \big]\ J_1(x) \Big\}\ e^{wx}$$

1.2.6. Integrals of the type $\int x^{-n-1/2} \sin x\, J_\nu(x)\, dx$ and $\int x^{-n-1/2} \cos x\, J_\nu(x)\, dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$.

The next four integrals are special cases of the general integral 1.8.2.7 from [4].

$$\int \frac{\sin x\, J_0(x)\, dx}{x^{3/2}} = \frac{(4x \cos x - 2 \sin x)J_0(x) + 4x \sin x\, J_1(x)}{\sqrt{x}}$$

$$\int \frac{\cos x\, J_0(x)\, dx}{x^{3/2}} = \frac{(-4x \sin x - 2 \cos x)J_0(x) + 4x \cos x\, J_1(x)}{\sqrt{x}}$$

$$\int \frac{\sin x\, J_1(x)\, dx}{x^{3/2}} = \frac{4x \sin x\, J_0(x) - (4x \cos x + 2 \sin x)\, J_1(x)}{3\sqrt{x}}$$

$$\int \frac{\cos x\, J_1(x)\, dx}{x^{3/2}} = \frac{4x \cos x\, J_0(x) + (4x \sin x - 2 \cos x)\, J_1(x)}{3\sqrt{x}}$$

$$\int \frac{\sin x\, J_0(x)\, dx}{x^{5/2}} = \frac{[(-32\, x^2 - 6) \sin x - 12\, x \cos x]\, J_0(x) + [4\, x \sin x + 32\, x^2 \cos x]\, J_1(x)}{9\, x^{3/2}}$$

$$\int \frac{\cos x\, J_0(x)\, dx}{x^{5/2}} = \frac{[12\, x \sin x - (32\, x^2 + 6) \cos x]\, J_0(x) + [-32\, x^2 \sin x + 4\, x \cos x]\, J_1(x)}{9\, x^{3/2}}$$

$$\int \frac{\sin x\, J_1(x)\, dx}{x^{5/2}} = \frac{[-12\, x \sin x + 32\, x^2 \cos x]\, J_0(x) + [(32\, x^2 - 6) \sin x - 4\, x \cos x]\, J_1(x)}{15\, x^{3/2}}$$

$$\int \frac{\cos x\, J_1(x)\, dx}{x^{5/2}} = \frac{[-32\, x^2 \sin x - 12\, x \cos x]\, J_0(x) + [4\, x \sin x + (32\, x^2 - 6) \cos x]\, J_1(x)}{15\, x^{3/2}}$$

$$\int \frac{\sin x\, J_0(x)\, dx}{x^{7/2}} =$$

$$= \frac{[(192\, x^2 - 90) \sin x + (-512\, x^3 - 60\, x) \cos x]\, J_0(x) + [(-512\, x^3 + 36\, x) \sin x + 64\, x^2 \cos x]\, J_1(x)}{225\, x^{5/2}}$$

$$\int \frac{\cos x\, J_0(x)\, dx}{x^{7/2}} =$$

$$= \frac{[(512\, x^3 + 60\, x) \sin x + (192\, x^2 - 90) \cos x]\, J_0(x) + [-64\, x^2 \sin x + (-512\, x^3 + 36\, x) \cos x]\, J_1(x)}{225\, x^{5/2}}$$

$$\int \frac{\sin x\, J_1(x)\, dx}{x^{7/2}} =$$

$$= \frac{[(-512\, x^3 - 60\, x) \sin x - 192\, x^2 \cos x]\, J_0(x) + [(64\, x^2 - 90) \sin x + (512\, x^3 - 36\, x) \cos x]\, J_1(x)}{315\, x^{5/2}}$$

$$\int \frac{\cos x\, J_1(x)\, dx}{x^{7/2}} =$$

$$= \frac{[192\, x^2 \sin x + (-512\, x^3 - 60\, x) \cos x]\, J_0(x) + [(-512\, x^3 + 36\, x) \sin x + (64\, x^2 - 90) \cos x]\, J_1(x)}{315\, x^{5/2}}$$

Let

$$\int \frac{\sin x\, J_\nu(x)\, dx}{x^{n+1/2}} = \frac{[P_n^{(s,\nu)}(x)\, \sin x + Q_n^{(s,\nu)}(x) \cos x]\, J_0(x) + [R_n^{(s,\nu)}(x)\, \sin x + S_n^{(s,\nu)}(x)\, \cos x]\, J_1(x)}{N_n^{(s,\nu)}\, x^{n-1/2}}$$

and

$$\int \frac{\cos x\, J_\nu(x)\, dx}{x^{n+1/2}} = \frac{[P_n^{(c,\nu)}(x)\, \sin x + Q_n^{(c,\nu)}(x) \cos x]\, J_0(x) + [R_n^{(c,\nu)}(x)\, \sin x + S_n^{(c,\nu)}(x)\, \cos x]\, J_1(x)}{N_n^{(c,\nu)}\, x^{n-1/2}} \,,$$

then holds

$$P_4^{(s,0)}(x) = 4096\,x^4 + 480\,x^2 - 1050\,, \quad Q_4^{(s,0)}(x) = 1536\,x^3 - 420\,x\,, \quad R_4^{(s,0)}(x) = -512\,x^3 + 300\,x\,,$$

$$S_4^{(s,0)}(x) = -4096\,x^4 + 288\,x^2\,, \quad N_4^{(s,0)} = 3675$$

$$P_4^{(c,0)}(x) = -1536\,x^3 + 420\,x\,, \quad Q_4^{(c,0)}(x) = 4096\,x^4 + 480\,x^2 - 1050\,, \quad R_4^{(c,0)}(x) = 4096\,x^4 - 288\,x^2\,,$$

$$S_4^{(c,0)}(x) = -512\,x^3 + 300\,x\,, \quad N_4^{(c,0)} = 3675$$

$$P_4^{(s,1)}(x) = 1536\,x^3 - 420\,x\,, \quad Q_4^{(s,1)}(x) = -4096\,x^4 - 480\,x^2\,, \quad R_4^{(s,1)}(x) = -4096\,x^4 + 288\,x^2 - 1050\,,$$

$$S_4^{(s,1)}(x) = 512\,x^3 - 300\,x\,, \quad N_4^{(s,1)} = 4725$$

$$P_4^{(c,1)}(x) = 4096\,x^4 + 480\,x^2\,, \quad Q_4^{(c,1)}(x) = 1536\,x^3 - 420\,x\,, \quad R_4^{(c,1)}(x) = -512\,x^3 + 300\,x\,,$$

$$S_4^{(c,1)}(x) = -4096\,x^4 + 288\,x^2 - 1050\,, \quad N_4^{(c,1)} = 4725$$

$$P_5^{(s,0)}(x) = -49152\,x^4 + 13440\,x^2 - 66150\,, \quad Q_5^{(s,0)}(x) = 131072\,x^5 + 15360\,x^3 - 18900\,x\,,$$

$$R_5^{(s,0)}(x) = 131072\,x^5 - 9216\,x^3 + 14700\,x\,, \quad S_5^{(s,0)}(x) = -16384\,x^4 + 9600\,x^2\,, \quad N_5^{(s,0)} = 297675$$

$$P_5^{(c,0)}(x) = -131072\,x^5 - 15360\,x^3 + 18900\,x\,, \quad Q_5^{(c,0)}(x) = -49152\,x^4 + 13440\,x^2 - 66150\,,$$

$$R_5^{(c,0)}(x) = 16384\,x^4 - 9600\,x^2\,, \quad S_5^{(c,0)}(x) = 131072\,x^5 - 9216\,x^3 + 14700\,x\,, \quad N_5^{(c,0)} = 297675$$

$$P_5^{(s,1)}(x) = 131072\,x^5 + 15360\,x^3 - 18900\,x\,, \quad Q_5^{(s,1)}(x) = 49152\,x^4 - 13440\,x^2\,,$$

$$R_5^{(s,1)}(x) = -16384\,x^4 + 9600\,x^2 - 66150\,, \quad S_5^{(s,1)}(x) = -131072\,x^5 + 9216\,x^3 - 14700\,x\,, \quad N_5^{(s,1)} = 363825$$

$$P_5^{(c,1)}(x) = -49152\,x^4 + 13440\,x^2\,, \quad Q_5^{(c,1)}(x) = 131072\,x^5 + 15360\,x^3 - 18900\,x\,,$$

$$R_5^{(c,1)}(x) = 131072\,x^5 - 9216\,x^3 + 14700\,x\,, \quad S_5^{(c,1)}(x) = -16384\,x^4 + 9600\,x^2 - 66150\,, \quad N_5^{(c,1)} = 363825$$

$$P_6^{(s,0)}(x) = -1048576\,x^6 - 122880\,x^4 + 151200\,x^2 - 1309770, \quad Q_6^{(s,0)}(x) = -393216\,x^5 + 107520\,x^3 - 291060\,x\,,$$

$$R_6^{(s,0)}(x) = 131072\,x^5 - 76800\,x^3 + 238140\,x, \quad S_6^{(s,0)}(x) = 1048576\,x^6 - 73728\,x^4 + 117600\,x^2, \quad N_6^{(s,0)} = 7203735$$

$$P_6^{(c,0)}(x) = 393216\,x^5 - 107520\,x^3 + 291060\,x\,, \quad Q_6^{(c,0)}(x) = -1048576\,x^6 - 122880\,x^4 + 151200\,x^2 - 1309770\,,$$

$$R_6^{(c,0)}(x) = -1048576\,x^6 + 73728\,x^4 - 117600\,x^2\,, \quad S_6^{(c,0)}(x) = 131072\,x^5 - 76800\,x^3 + 238140\,x\,,$$

$$N_6^{(c,0)} = 7203735$$

$$P_6^{(s,1)}(x) = -393216\,x^5 + 107520\,x^3 - 291060\,x\,, \quad Q_6^{(s,1)}(x) = 1048576\,x^6 + 122880\,x^4 - 151200\,x^2\,,$$

$$R_6^{(s,1)}(x) = 1048576\,x^6 - 73728\,x^4 + 117600\,x^2 - 1309770\,, \quad S_6^{(s,1)}(x) = -131072\,x^5 + 76800\,x^3 - 238140\,x\,,$$

$$N_6^{(s,1)} = 8513505$$

$$P_6^{(c,1)}(x) = -1048576\,x^6 - 122880\,x^4 + 151200\,x^2\,, \quad Q_6^{(c,1)}(x) = -393216\,x^5 + 107520\,x^3 - 291060\,x\,,$$

$$R_6^{(c,1)}(x) = 131072\,x^5 - 76800\,x^3 + 238140\,x\,, \quad S_6^{(c,1)}(x) = 1048576\,x^6 - 73728\,x^4 + 117600\,x^2 - 1309770\,,$$

$$N_6^{(c,1)} = 8513505$$

$$P_7^{(s,0)}(x) = 6291456\,x^6 - 1720320\,x^4 + 4656960\,x^2 - 62432370\,,$$

$$Q_7^{(s,0)}(x) = -16777216\,x^7 - 1966080\,x^5 + 2419200\,x^3 - 11351340\,x\,,$$

$$R_7^{(s,0)}(x) = -16777216\,x^7 + 1179648\,x^5 - 1881600\,x^3 + 9604980\,x\,,$$

$$S_7^{(s,0)}(x) = 2097152\,x^6 - 1228800\,x^4 + 3810240\,x^2\,, \quad N_7^{(s,0)} = 405810405$$

$$P_7^{(c,0)}(x) = 16777216\,x^7 + 1966080\,x^5 - 2419200\,x^3 + 11351340\,x\;,$$

$$Q_7^{(c,0)}(x) = 6291456\,x^6 - 1720320\,x^4 + 4656960\,x^2 - 62432370\;,$$

$$R_7^{(c,0)}(x) = -2097152\,x^6 + 1228800\,x^4 - 3810240\,x^2\;,$$

$$S_7^{(c,0)}(x) = -16777216\,x^7 + 1179648\,x^5 - 1881600\,x^3 + 9604980\,x\;,\quad N_7^{(c,0)} = 405810405$$

$$P_7^{(s,1)}(x) = -16777216\,x^7 - 1966080\,x^5 + 2419200\,x^3 - 11351340\,x\;,$$

$$Q_7^{(s,1)}(x) = -6291456\,x^6 + 1720320\,x^4 - 4656960\,x^2\;,$$

$$R_7^{(s,1)}(x) = 2097152\,x^6 - 1228800\,x^4 + 3810240\,x^2 - 62432370\;,$$

$$S_7^{(s,1)}(x) = 16777216\,x^7 - 1179648\,x^5 + 1881600\,x^3 - 9604980\,x\;,\quad N_7^{(s,1)} = 468242775$$

$$P_7^{(c,1)}(x) = 6291456\,x^6 - 1720320\,x^4 + 4656960\,x^2\;,$$

$$Q_7^{(c,1)}(x) = -16777216\,x^7 - 1966080\,x^5 + 2419200\,x^3 - 11351340\,x\;,$$

$$R_7^{(c,1)}(x) = -16777216\,x^7 + 1179648\,x^5 - 1881600\,x^3 + 9604980\,x\;,$$

$$S_7^{(c,1)}(x) = 2097152\,x^6 - 1228800\,x^4 + 3810240\,x^2 - 62432370\;,\quad N_7^{(c,1)} = 468242775$$

$$P_8^{(s,0)}(x) = 134217728\,x^8 + 15728640\,x^6 - 19353600\,x^4 + 90810720\,x^2 - 1739187450\;,$$

$$Q_8^{(s,0)}(x) = 50331648\,x^7 - 13762560\,x^5 + 37255680\,x^3 - 267567300\,x\;,$$

$$R_8^{(s,0)}(x) = -16777216\,x^7 + 9830400\,x^5 - 30481920\,x^3 + 231891660\,x\;,$$

$$S_8^{(s,0)}(x) = -134217728\,x^8 + 9437184\,x^6 - 15052800\,x^4 + 76839840\,x^2\;,\quad N_8^{(s,0)} = 13043905875$$

$$P_8^{(c,0)}(x) = -50331648\,x^7 + 13762560\,x^5 - 37255680\,x^3 + 267567300\,x\;,$$

$$Q_8^{(c,0)}(x) = 134217728\,x^8 + 15728640\,x^6 - 19353600\,x^4 + 90810720\,x^2 - 1739187450\;,$$

$$R_8^{(c,0)}(x) = 134217728\,x^8 - 9437184\,x^6 + 15052800\,x^4 - 76839840\,x^2\;,$$

$$S_8^{(c,0)}(x) = -16777216\,x^7 + 9830400\,x^5 - 30481920\,x^3 + 231891660\,x\;,\quad N_8^{(c,0)} = 13043905875$$

$$P_8^{(s,1)}(x) = 50331648\,x^7 - 13762560\,x^5 + 37255680\,x^3 - 267567300\,x\;,$$

$$Q_8^{(s,1)}(x) = -134217728\,x^8 - 15728640\,x^6 + 19353600\,x^4 - 90810720\,x^2\;,$$

$$R_8^{(s,1)}(x) = -134217728\,x^8 + 9437184\,x^6 - 15052800\,x^4 + 76839840\,x^2 - 1739187450\;,$$

$$S_8^{(s,1)}(x) = 16777216\,x^7 - 9830400\,x^5 + 30481920\,x^3 - 231891660\,x\;,\quad N_8^{(s,1)} = 14783093325$$

$$P_8^{(c,1)}(x) = 134217728\,x^8 + 15728640\,x^6 - 19353600\,x^4 + 90810720\,x^2\;,$$

$$Q_8^{(c,1)}(x) = 50331648\,x^7 - 13762560\,x^5 + 37255680\,x^3 - 267567300\,x\;,$$

$$R_8^{(c,1)}(x) = -16777216\,x^7 + 9830400\,x^5 - 30481920\,x^3 + 231891660\,x\;,$$

$$S_8^{(c,1)}(x) = -134217728\,x^8 + 9437184\,x^6 - 15052800\,x^4 + 76839840\,x^2 - 1739187450\;,$$

$$N_8^{(c,1)} = 14783093325$$

Recurrence relations:

$$\text{Let}\qquad I_n^{(s,\nu)} = \int \frac{\sin x\,J_\nu(x)\,dx}{x^{n+1/2}}\qquad\text{and}\qquad I_n^{(c,\nu)} = \int \frac{\cos x\,J_\nu(x)\,dx}{x^{n+1/2}}\;,\qquad\text{then holds}$$

$$I_{n+1}^{(s,0)} = \frac{2}{2n+1}\left[I_n^{(c,0)} - I_n^{(s,1)} - \frac{\sin x \cdot J_0(x)}{x^{n+1/2}}\right],\qquad I_{n+1}^{(s,1)} = \frac{2}{2n+3}\left[I_n^{(s,0)} + I_n^{(c,1)} - \frac{\sin x \cdot J_1(x)}{x^{n+1/2}}\right]$$

$$I_{n+1}^{(c,0)} = \frac{2}{2n+1}\left[-I_n^{(s,0)} - I_n^{(c,1)} - \frac{\cos x \cdot J_0(x)}{x^{n+1/2}}\right],\qquad I_{n+1}^{(c,1)} = \frac{2}{2n+3}\left[I_n^{(c,0)} - I_n^{(s,1)} - \frac{\cos x \cdot J_1(x)}{x^{n+1/2}}\right]$$

1.2.7. Integrals of the Type $\int x^{-n-1/2}\, e^{\pm x}\left\{\begin{array}{c} I_\nu(x) \\ K_\nu(x) \end{array}\right\}\, dx$

a) Integrals $\int x^{-n-1/2}\, e^{x}\, I_\nu(x)\, dx$:

See also [4], 1.11.2. and 1.11.? .

$$\int \frac{e^x\, I_0(x)\, dx}{x^{3/2}} = \frac{(4x-2)\, I_0(x) - 4x\, I_1(x)}{\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{3/2}} = \frac{4x\, I_0(x) - (2+4x)\, I_1(x)}{3\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, K_0(x)\, dx}{x^{3/2}} = \frac{(4x-2)\, K_0(x) + 4x\, K_1(x)}{\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{3/2}} = -\frac{4x\, K_0(x) + (2+4x)\, K_1(x)}{3\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, I_0(x)\, dx}{x^{5/2}} = \frac{(-6-12x+32x^2)\, I_0(x) - (4x+32x^2)\, I_1(x)}{9\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{5/2}} = \frac{(-12x+32x^2)\, I_0(x) - (6+4x+32x^2)\, I_1(x)}{15\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, K_0(x)\, dx}{x^{5/2}} = \frac{(-6-12x+32x^2)\, K_0(x) + (4x+32x^2)\, K_1(x)}{9\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{5/2}} = \frac{(12x-32x^2)\, K_0(x) - (6+4x+32x^2)\, K_1(x)}{15\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, I_0(x)\, dx}{x^{7/2}} = \frac{(512\, x^3 - 192\, x^2 - 60\, x - 90)\, I_0(x) - (512\, x^3 + 64\, x^2 + 36\, x)\, I_1(x)}{225\, x^{5/2}}\, e^x$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{7/2}} = \frac{(512\, x^3 - 192\, x^2 - 60\, x)\, I_0(x) - (512\, x^3 + 64\, x^2 + 36\, x + 90)\, I_1(x)}{315\, x^{5/2}}\, e^x$$

$$\int \frac{e^x\, K_0(x)\, dx}{x^{7/2}} = \frac{(512\, x^3 - 192\, x^2 - 60\, x - 90)\, K_0(x) + (512\, x^3 + 64\, x^2 + 36\, x)\, K_1(x)}{225\, x^{5/2}}\, e^x$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{7/2}} = \frac{(-512\, x^3 + 192\, x^2 + 60\, x)\, K_0(x) - (512\, x^3 + 64\, x^2 + 36\, x + 90)\, K_1(x)}{315\, x^{5/2}}\, e^x$$

Let

$$\int \frac{e^x\, I_0(x)\, dx}{x^{n+1/2}} = \frac{P_n^{(0,+)}(x)\, I_0(x) - Q_n^{(0,+)}(x)\, I_1(x)}{N_n^{(0,+)}\, x^{n-1/2}}\, e^x\ ,$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{n+1/2}} = \frac{P_n^{(1,+)}(x)\, I_0(x) - Q_n^{(1,+)}(x)\, I_1(x)}{N_n^{(1,+)}\, x^{n-1/2}}\, e^x\ ,$$

then holds[1] with the same coefficients P_n and Q_n

$$\int \frac{e^x\, K_0(x)\, dx}{x^{n+1/2}} = \frac{P_n^{(0,+)}(x)\, K_0(x) + Q_n^{(0,+)}(x)\, K_1(x)}{N_n^{(0,+)}\, x^{n-1/2}}\, e^x\ ,$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{n+1/2}} = \frac{-P_n^{(1,+)}(x)\, K_0(x) - Q_n^{(1,+)}(x)\, K_1(x)}{N_n^{(1,+)}\, x^{n-1/2}}\, e^x\ .$$

$$P_4^{(0,+)}(x) = 4096\, x^4 - 1536\, x^3 - 480\, x^2 - 420\, x - 1050\ ,$$

---

[1] Note that there are signs '-' in the numerators!

$$Q_4^{(0,+)}(x) = 4096\,x^4 + 512\,x^3 + 288\,x^2 + 300\,x \,, \quad N_4^{(0,+)} = 3675$$

$$P_4^{(1,+)}(x) = 4096\,x^4 - 1536\,x^3 - 480\,x^2 - 420\,x \,,$$

$$Q_4^{(1,+)}(x) = 4096\,x^4 + 512\,x^3 + 288\,x^2 + 300\,x + 1050 \,, \quad N_4^{(1,+)} = 4725$$

$$P_5^{(0,+)}(x) = 131072\,x^5 - 49152\,x^4 - 15360\,x^3 - 13440\,x^2 - 18900\,x - 66150 \,,$$

$$Q_5^{(0,+)}(x) = 131072\,x^5 + 16384\,x^4 + 9216\,x^3 + 9600\,x^2 + 14700\,x \,, \quad N_5^{(0,+)} = 297675$$

$$P_5^{(1,+)}(x) = 131072\,x^5 - 49152\,x^4 - 15360\,x^3 - 13440\,x^2 - 18900\,x \,,$$

$$Q_5^{(1,+)}(x) = 131072\,x^5 + 16384\,x^4 + 9216\,x^3 + 9600\,x^2 + 14700\,x + 66150 \,, \quad N_5^{(1,+)} = 363825$$

$$P_6^{(0,+)}(x) = 1048576\,x^6 - 393216\,x^5 - 122880\,x^4 - 107520\,x^3 - 151200\,x^2 - 291060\,x - 1309770$$

$$Q_6^{(0,+)}(x) = 1048576\,x^6 + 131072\,x^5 + 73728\,x^4 + 76800\,x^3 + 117600\,x^2 + 238140\,x \,,$$

$$N_6^{(0,+)} = 7203735$$

$$P_6^{(1,+)}(x) = 1048576\,x^6 - 393216\,x^5 - 122880\,x^4 - 107520\,x^3 - 151200\,x^2 - 291060\,x \,,$$

$$Q_6^{(1,+)}(x) = 1048576\,x^6 + 131072\,x^5 + 73728\,x^4 + 76800\,x^3 + 117600\,x^2 + 238140\,x + 1309770 \,,$$

$$N_6^{(1,+)} = 8513505$$

$$P_7^{(0,+)}(x) =$$
$$= 16777216\,x^7 - 6291456\,x^6 - 1966080\,x^5 - 1720320\,x^4 - 2419200\,x^3 - 4656960\,x^2 - 11351340\,x - 62432370 \,,$$

$$Q_7^{(0,+)}(x) =$$
$$= 16777216\,x^7 + 2097152\,x^6 + 1179648\,x^5 + 1228800\,x^4 + 1881600\,x^3 + 3810240\,x^2 + 9604980\,x \,,$$

$$N_7^{(0,+)} = 405810405$$

$$P_7^{(1,+)}(x) =$$
$$= 16777216\,x^7 - 6291456\,x^6 - 1966080\,x^5 - 1720320\,x^4 - 2419200\,x^3 - 4656960\,x^2 - 11351340\,x \,,$$

$$Q_7^{(1,+)}(x) =$$
$$= 16777216\,x^7 + 2097152\,x^6 + 1179648\,x^5 + 1228800\,x^4 + 1881600\,x^3 + 3810240\,x^2 + 9604980\,x + 62432370 \,,$$

$$N_7^{(1,+)} = 468242775$$

Recurrence relations:

$$\text{With} \qquad \mathbf{I}_n^{(\nu,+)} = \int \frac{e^x\,I_\nu(x)\,dx}{x^{n+1/2}} \qquad \text{and} \qquad \mathbf{K}_n^{(\nu,+)} = \int \frac{e^x\,K_\nu(x)\,dx}{x^{n+1/2}} \qquad \text{holds}$$

$$\mathbf{I}_n^{(0,+)} = \frac{2}{2n-1}\left[\mathbf{I}_{n-1}^{(0,+)} + \mathbf{I}_{n-1}^{(1,+)} - \frac{e^x\,I_0(x)}{x^{n-1/2}}\right], \qquad \mathbf{I}_n^{(1,+)} = \frac{2}{2n+1}\left[\mathbf{I}_{n-1}^{(0,+)} + \mathbf{I}_{n-1}^{(1,+)} - \frac{e^x\,I_1(x)}{x^{n-1/2}}\right],$$

$$\mathbf{K}_n^{(0,+)} = \frac{2}{2n-1}\left[\mathbf{K}_{n-1}^{(0,+)} - \mathbf{K}_{n-1}^{(1,+)} - \frac{e^x\,K_0(x)}{x^{n-1/2}}\right], \qquad \mathbf{K}_n^{(1,+)} = \frac{2}{2n+1}\left[-\mathbf{K}_{n-1}^{(0,+)} + \mathbf{K}_{n-1}^{(1,+)} - \frac{e^x\,K_1(x)}{x^{n-1/2}}\right]$$

b) Integrals $\int x^{-n-1/2}\,e^{-x}\,I_\nu(x)\,dx$:

See also [4], 1.11.2.1 and 1.11.?. .

$$\int \frac{e^{-x}\,I_0(x)\,dx}{x^{3/2}} = -\frac{(4x+2)\,I_0(x) + 4x\,I_1(x)}{\sqrt{x}}\,e^{-x}$$

$$\int \frac{e^{-x}\,I_1(x)\,dx}{x^{3/2}} = \frac{4x\,I_0(x) + (4x-2)\,I_1(x)}{3\sqrt{x}}\,e^{-x}$$

$$\int \frac{e^{-x} K_0(x)\,dx}{x^{3/2}} = \frac{-(4x+2)\,K_0(x) + 4x\,K_1(x)}{\sqrt{x}}\,e^{-x}$$

$$\int \frac{e^{-x} K_1(x)\,dx}{x^{3/2}} = \frac{-4x\,K_0(x) + (4x-2)\,K_1(x)}{3\sqrt{x}}\,e^{-x}$$

$$\int \frac{e^{-x} I_0(x)\,dx}{x^{5/2}} = \frac{(32\,x^2 + 12\,x - 6)\,I_0(x) + (32\,x^2 - 4\,x)\,I_1(x)}{9\,x^{3/2}}\,e^{-x}$$

$$\int \frac{e^{-x} I_1(x)\,dx}{x^{5/2}} = \frac{(-32\,x^2 - 12\,x)\,I_0(x) + (-32\,x^2 + 4\,x - 6)\,I_1(x)}{15\,x^{3/2}}\,e^{-x}$$

$$\int \frac{e^{-x} K_0(x)\,dx}{x^{5/2}} = \frac{(32\,x^2 + 12\,x - 6)\,K_0(x) - (32\,x^2 - 4\,x)\,K_1(x)}{9\,x^{3/2}}\,e^{-x}$$

$$\int \frac{e^{-x} K_1(x)\,dx}{x^{5/2}} = \frac{(32\,x^2 + 12\,x)\,K_0(x) + (-32\,x^2 + 4\,x - 6)\,K_1(x)}{15\,x^{3/2}}\,e^{-x}$$

$$\int \frac{e^{-x} I_0(x)\,dx}{x^{7/2}} = \frac{(-512\,x^3 - 192\,x^2 + 60\,x - 90)\,I_0(x) + (-512\,x^3 + 64\,x^2 - 36\,x)\,I_1(x)}{225\,x^{5/2}}\,e^{-x}$$

$$\int \frac{e^{-x} I_1(x)\,dx}{x^{7/2}} = \frac{(512\,x^3 + 192\,x^2 - 60\,x)\,I_0(x) + (512\,x^3 - 64\,x^2 + 36\,x - 90)\,I_1(x)}{315\,x^{5/2}}\,e^{-x}$$

$$\int \frac{e^{-x} K_0(x)\,dx}{x^{7/2}} = \frac{(-512\,x^3 - 192\,x^2 + 60\,x - 90)\,K_0(x) - (-512\,x^3 + 64\,x^2 - 36\,x)\,K_1(x)}{225\,x^{5/2}}\,e^{-x}$$

$$\int \frac{e^{-x} K_1(x)\,dx}{x^{7/2}} = \frac{-(512\,x^3 + 192\,x^2 - 60\,x)\,K_0(x) + (512\,x^3 - 64\,x^2 + 36\,x - 90)\,K_1(x)}{315\,x^{5/2}}\,e^{-x}$$

Let

$$\int \frac{e^{-x} I_0(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(0,-)}(x)\,I_0(x) + Q_n^{(0,-)}(x)\,I_1(x)}{N_n^{(0,-)}\,x^{n-1/2}}\,e^x\;,$$

$$\int \frac{e^{-x} I_1(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(1,-)}(x)\,I_0(x) + Q_n^{(1,-)}(x)\,I_1(x)}{N_n^{(1,-)}\,x^{n-1/2}}\,e^x\;,$$

then holds

$$\int \frac{e^{-x} K_0(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(0,-)}(x)\,K_0(x) - Q_n^{(0,-)}(x)\,K_1(x)}{N_n^{(0,-)}\,x^{n-1/2}}\,e^x\;,$$

$$\int \frac{e^{-x} K_1(x)\,dx}{x^{n+1/2}} = \frac{-P_n^{(1,-)}(x)\,K_0(x) + Q_n^{(1,-)}(x)\,K_1(x)}{N_n^{(1,-)}\,x^{n-1/2}}\,e^x\;.$$

$$P_4^{(0,-)}(x) = 4096\,x^4 + 1536\,x^3 - 480\,x^2 + 420\,x - 1050\;,$$

$$Q_4^{(0,-)}(x) = 4096\,x^4 - 512\,x^3 + 288\,x^2 - 300\,x\;, \quad N_4^{(0,-)} = 3675$$

$$P_4^{(1,-)}(x) = -4096\,x^4 - 1536\,x^3 + 480\,x^2 - 420\,x\;,$$

$$Q_4^{(1,-)}(x) = -4096\,x^4 + 512\,x^3 - 288\,x^2 + 300\,x - 1050\;, \quad N_4^{(1,-)} = 4725$$

$$P_5^{(0,-)}(x) = -131072\,x^5 - 49152\,x^4 + 15360\,x^3 - 13440\,x^2 + 18900\,x - 66150\;,$$

$$Q_5^{(0,-)}(x) = -131072\,x^5 + 16384\,x^4 - 9216\,x^3 + 9600\,x^2 - 14700\,x\;, \quad N_5^{(0,-)} = 297675$$

$$P_5^{(1,-)}(x) = 131072\,x^5 + 49152\,x^4 - 15360\,x^3 + 13440\,x^2 - 18900\,x\;,$$

$$Q_5^{(1,-)}(x) = 131072\,x^5 - 16384\,x^4 + 9216\,x^3 - 9600\,x^2 + 14700\,x - 66150\;, \quad N_5^{(1,-)} = 363825$$

$$P_6^{(0,-)}(x) = 1048576\,x^6 + 393216\,x^5 - 122880\,x^4 + 107520\,x^3 - 151200\,x^2 + 291060\,x - 1309770\ ,$$

$$Q_6^{(0,-)}(x) = 1048576\,x^6 - 131072\,x^5 + 73728\,x^4 - 76800\,x^3 + 117600\,x^2 - 238140\,x\ ,$$

$$N_6^{(0,-)} = 7203735$$

$$P_6^{(1,-)}(x) = -1048576\,x^6 - 393216\,x^5 + 122880\,x^4 - 107520\,x^3 + 151200\,x^2 - 291060\,x\ ,$$

$$Q_6^{(1,-)}(x) = -1048576\,x^6 + 131072\,x^5 - 73728\,x^4 + 76800\,x^3 - 117600\,x^2 + 238140\,x - 1309770\ ,$$

$$N_6^{(1,-)} = 8513505$$

$$P_7^{(0,-)}(x) =$$
$$= -16777216\,x^7 - 6291456\,x^6 + 1966080\,x^5 - 1720320\,x^4 + 2419200\,x^3 - 4656960\,x^2 + 11351340\,x - 62432370\ ,$$

$$Q_7^{(0,-)}(x) = -16777216\,x^7 + 2097152\,x^6 - 1179648\,x^5 + 1228800\,x^4 - 1881600\,x^3 + 3810240\,x^2 - 9604980\,x\ ,$$

$$N_7^{(0,-)} = 405810405$$

$$P_7^{(1,-)}(x) = 16777216\,x^7 + 6291456\,x^6 - 1966080\,x^5 + 1720320\,x^4 - 2419200\,x^3 + 4656960\,x^2 - 11351340\,x\ ,$$

$$Q_7^{(1,-)}(x) =$$
$$= 16777216\,x^7 - 2097152\,x^6 + 1179648\,x^5 - 1228800\,x^4 + 1881600\,x^3 - 3810240\,x^2 + 9604980\,x - 62432370\ ,$$

$$N_7^{(1,-)} = 468242775$$

Recurrence relations:

$$\text{With} \qquad \mathbf{I}_n^{(\nu,-)} = \int \frac{e^{-x}\,I_\nu(x)\,dx}{x^{n+1/2}} \qquad \text{and} \qquad \mathbf{K}_n^{(\nu,-)} = \int \frac{e^{-x}\,K_\nu(x)\,dx}{x^{n+1/2}} \qquad \text{holds}$$

$$\mathbf{I}_n^{(0,-)} = \frac{2}{2n-1}\left[-\mathbf{I}_{n-1}^{(0,-)} + \mathbf{I}_{n-1}^{(1,-)} - \frac{e^{-x}\,I_0(x)}{x^{n-1/2}}\right]\ , \qquad \mathbf{I}_n^{(1,-)} = \frac{2}{2n+1}\left[\mathbf{I}_{n-1}^{(0,-)} - \mathbf{I}_{n-1}^{(1,-)} - \frac{e^{-x}\,I_1(x)}{x^{n-1/2}}\right]\ ,$$

$$\mathbf{K}_n^{(0,-)} = -\frac{2}{2n-1}\left[\mathbf{K}_{n-1}^{(0,-)} + \mathbf{K}_{n-1}^{(1,-)} + \frac{e^{-x}\,K_0(x)}{x^{n-1/2}}\right]\ , \qquad \mathbf{K}_n^{(1,-)} = -\frac{2}{2n+1}\left[\mathbf{K}_{n-1}^{(0,-)} + \mathbf{K}_{n-1}^{(1,-)} + \frac{e^{-x}\,K_1(x)}{x^{n-1/2}}\right]$$

c) General formulas

Let

$$P_n^{(\nu,\pm)}(x) = \sum_{k=0}^{n} \vartheta_k^{(n,\nu,\pm)}\,x^k \qquad \text{and} \quad Q_n^{(\nu,\pm)}(x) = \sum_{k=0}^{n} \eta_k^{(n,\nu,\pm)}\,x^k\ ,$$

then holds

$$\int \frac{e^{\pm x}\,I_\nu(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(\nu,\pm)}(x)\,I_0(x)\ +\ Q_n^{(\nu,\pm)}(x)\,I_1(x)}{x^{n-1/2}}\ .$$

The integrals with $K_\nu(x)$ may be expressed as described before.

I. $\nu = 0$, e^x :

$$\vartheta_0^{(n,0,+)} = -\frac{2}{2n-1}\ , \qquad \vartheta_1^{(n,0,+)} = -\frac{2^2}{(2n-1)(2n-3)}\ ,$$

$$\vartheta_2^{(n,0,+)} = -\frac{2^5(n-1)}{(2n-1)^2(2n-3)(2n-5)} = \frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,0,+)}\ ,$$

$$\vartheta_3^{(n,0,+)} = -\frac{2^8(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)(2n-7)} = \frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,0,+)}\ ,$$

$$\vartheta_4^{(n,0,+)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n-1)^2(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = \frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,0,+)}\ , \quad \dots \ ,$$

which gives $\vartheta_0^{(n,0,+)} = -\,2/(2n-1)$ and

$$\vartheta_k^{(n,0,+)} = -\,\frac{2^{k-1}\cdot\Gamma(n-k-\frac{1}{2})\cdot\Gamma(n-k+\frac{3}{2})\cdot(n-1)!}{\Gamma^2(n+\frac{1}{2})\cdot(n-k)!}\ ,\ k > 0\ .$$

Furthermore

$$\eta_1^{(n,0,+)} = -\frac{2^2}{(2n-1)^2}\ ,\quad \eta_2^{(n,0,+)} = -\frac{2^5\,(n-1)}{(2n-1)^2(2n-3)^2} = \frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,0,+)}\ ,$$

$$\eta_3^{(n,0,+)} = -\frac{2^8\,(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)^2} = \frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_2^{(n,0,+)}\ ,\ \dots\ ,$$

which gives $\eta_0^{(n,0,+)} = 0$ and

$$\eta_k^{(n,0,+)} = -\,\frac{2^{k-1}\cdot\Gamma^2(n-k+\frac{1}{2})\cdot(n-1)!}{\Gamma^2(n+\frac{1}{2})\cdot(n-k)!}\ ,\quad k > 0\ .$$

II. $\nu = 1,\ e^x$:

$$\vartheta_0^{(n,1,+)} = 0\ ,\quad \vartheta_1^{(n,1,+)} = -\frac{2^2}{(2n+1)(2n-3)}\ ,$$

$$\vartheta_2^{(n,1,+)} = -\frac{2^5(n-1)}{(2n+1)(2n-1)(2n-3)(2n-5)} = \frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,1,+)}\ ,$$

$$\vartheta_3^{(n,1,+)} = -\frac{2^8(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)(2n-7)} = \frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,1,+)}\ ,$$

$$\vartheta_4^{(n,1,+)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = \frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,1,+)}\ ,\quad \dots\ ,$$

which gives $\vartheta_0^{(n,1,+)} = 0$ and

$$\vartheta_k^{(n,1,+)} = -\,\frac{2^{k-1}\cdot\Gamma(n-k-\frac{1}{2})\cdot\Gamma(n-k+\frac{3}{2})\cdot(n-1)!}{\Gamma(n+\frac{3}{2})\cdot\Gamma(n-\frac{1}{2})\cdot(n-k)!}\ ,\ k > 0\ .$$

Furthermore

$$\eta_0^{(n,1,+)} = -\frac{2}{2n+1}\ ,\quad \eta_1^{(n,1,+)} = -\frac{2^2}{(2n+1)(2n-1)}\ ,$$

$$\eta_2^{(n,1,+)} = -\frac{2^5\,(n-1)}{(2n+1)(2n-1)(2n-3)^2} = \frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,1,+)}\ ,$$

$$\eta_3^{(n,1,+)} = -\frac{2^8\,(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2} = \frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_1^{(n,2,+)}\ ,\ \dots\ ,$$

which gives $\eta_0^{(n,1,+)} = -2/(2n+1)$ and

$$\eta_k^{(n,0,+)} = -\,\frac{2^{k-1}\cdot\Gamma^2(n-k+\frac{1}{2})\cdot(n-1)!}{\Gamma(n+\frac{3}{2})\cdot\Gamma(n-\frac{1}{2})\cdot(n-k)!}\ ,\quad k > 1\ .$$

III. $\nu = 0,\ e^{-x}$:

$$\vartheta_0^{(n,0,-)} = -2/(2n-1)\ ,\quad \vartheta_1^{(n,0,-)} = \frac{2^2}{(2n-1)(2n-3)}\ ,$$

$$\vartheta_2^{(n,0,-)} = -\frac{2^5(n-1)}{(2n-1)^2(2n-3)(2n-5)} = -\frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,0,-)}\ ,$$

$$\vartheta_3^{(n,0,-)} = \frac{2^8(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)(2n-7)} = \frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,0,-)}\ ,$$

$$\vartheta_4^{(n,0,-)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n-1)^2(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = \frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,0,-)} \ , \quad \dots \ ,$$

which gives $\vartheta_0^{(n,0,-)} = -2/(2n-1)$ and

$$\vartheta_k^{(n,0,-)} = \frac{(-2)^{k-1}\cdot\Gamma(n-k-\frac{1}{2})\cdot\Gamma(n-k+\frac{3}{2})\cdot(n-1)!}{\Gamma^2(n+\frac{1}{2})\cdot(n-k)!} \ , \ k > 0 \ .$$

Furthermore

$$\eta_1^{(n,0,-)} = -\frac{2^2}{(2n-1)^2} \ , \quad \eta_2^{(n,0,-)} = \frac{2^5\,(n-1)}{(2n-1)^2(2n-3)^2} = -\frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,0,-)} \ ,$$

$$\eta_3^{(n,0,-)} = -\frac{2^8\,(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)^2} = -\frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_2^{(n,0,-)} \ , \ \dots \ ,$$

which gives $\eta_0^{(n,0,-)} = 0$ and

$$\eta_k^{(n,0,-)} = -\frac{(-2)^{k-1}\cdot\Gamma^2(n-k+\frac{1}{2})\cdot(n-1)!}{\Gamma^2(n+\frac{1}{2})\cdot(n-k)!} \ , \quad k > 0 \ .$$

IV. $\nu = 1$, e^{-x} :

$$\vartheta_0^{(n,1,-)} = 0 \ , \quad \vartheta_1^{(n,1,-)} = -\frac{2^2}{(2n+1)(2n-3)} \ ,$$

$$\vartheta_2^{(n,1,-)} = \frac{2^5(n-1)}{(2n+1)(2n-1)(2n-3)(2n-5)} = -\frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,1,-)} \ ,$$

$$\vartheta_3^{(n,1,-)} = \frac{2^8(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)(2n-7)} = -\frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,1,-)} \ ,$$

$$\vartheta_4^{(n,1,-)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = -\frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,1,-)} \ , \quad \dots \ ,$$

which gives $\vartheta_0^{(n,1,-)} = 0$ and

$$\vartheta_k^{(n,1,-)} = -\frac{(-2)^{k-1}\cdot\Gamma(n-k-\frac{1}{2})\cdot\Gamma(n-k+\frac{3}{2})\cdot(n-1)!}{\Gamma(n+\frac{3}{2})\cdot\Gamma(n-\frac{1}{2})\cdot(n-k)!} \ , \ k > 0 \ .$$

Furthermore

$$\eta_0^{(n,1,-)} = -\frac{2}{2n+1} \ , \quad \eta_1^{(n,1,-)} = \frac{2^2}{(2n+1)(2n-1)} \ ,$$

$$q\eta_2^{(n,1,-)} = -\frac{2^5\,(n-1)}{(2n+1)(2n-1)(2n-3)^2} = \frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,1,-)} \ ,$$

$$\eta_3^{(n,1,-)} = \frac{2^8\,(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2} = \frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_2^{(n,1,-)} \ , \ \dots \ ,$$

which gives $\eta_0^{(n,1,-)} = -2/(2n+1)$ and

$$\eta_k^{(n,1,-)} = \frac{(-2)^{k-1}\cdot\Gamma^2(n-k+\frac{1}{2})\cdot(n-1)!}{\Gamma(n+\frac{3}{2})\cdot\Gamma(n-\frac{1}{2})\cdot(n-k)!} \ .$$

When $n > 0$, then the functions $P_n^{(\nu,\pm)}(x)$ and $Q_n^{(\nu,\pm)}(x)$ are polynomials. In the case $n \le 0$ they are power series with the radius of convergence $R = +\infty$. Their coefficients may be found by the given recurrence relations.

$$P_0^{(0,+)} = 2 - \frac{4}{3}\,x + \frac{32}{15}\,x^2 - \frac{512}{315}\,x^3 + \frac{4096}{4725}\,x^4 - \frac{131072}{363825}\,x^5 + \frac{1048576}{8513505}\,x^6 - \frac{16777216}{468242775}\,x^7 + \frac{134217728}{14783093325}\,x^8 -$$
$$- \frac{8589934592}{4213181597625}\,x^9 + \frac{68719476736}{167122870039125}\,x^{10} + \dots$$

$$Q_0^{(0,+)} = -4\,x + \frac{32}{9}\,x^2 - \frac{512}{225}\,x^3 + \frac{4096}{3675}\,x^4 - \frac{131072}{297675}\,x^5 + \frac{1048576}{7203735}\,x^6 - \frac{16777216}{405810405}\,x^7 + \frac{134217728}{13043905875}\,x^8 -$$
$$- \frac{8589934592}{3769688797875}\,x^9 + \frac{68719476736}{151206406225875}\,x^{10} + \dots$$

$$P_{-1}^{(0,+)} = \frac{2}{3} - \frac{4}{15}\,x + \frac{64}{315}\,x^2 - \frac{512}{4725}\,x^3 + \frac{16384}{363825}\,x^4 - \frac{131072}{8513505}\,x^5 + \frac{2097152}{468242775}\,x^6 - \frac{16777216}{14783093325}\,x^7 +$$
$$+ \frac{1073741824}{4213181597625}\,x^8 - \frac{8589934592}{167122870039125}\,x^9 + \frac{137438953472}{14606538841419525}\,x^{10} + \dots$$

$$Q_{-1}^{(0,+)} = -\frac{4}{9}\,x + \frac{64}{225}\,x^2 - \frac{512}{3675}\,x^3 + \frac{16384}{297675}\,x^4 - \frac{131072}{7203735}\,x^5 + \frac{2097152}{405810405}\,x^6 - \frac{16777216}{13043905875}\,x^7 +$$
$$+ \frac{1073741824}{3769688797875}\,x^8 - \frac{8589934592}{151206406225875}\,x^9 + \frac{137438953472}{1333640502912175}\,x^{10} + \dots$$

$$P_{-2}^{(0,+)} = \frac{2}{5} - \frac{4}{35}\,x + \frac{32}{525}\,x^2 - \frac{1024}{40425}\,x^3 + \frac{8192}{945945}\,x^4 - \frac{131072}{52026975}\,x^5 + \frac{1048576}{1642565925}\,x^6 - \frac{67108864}{468131288625}\,x^7 +$$
$$+ \frac{536870912}{18569207782125}\,x^8 - \frac{8589934592}{1622948760157725}\,x^9 + \frac{68719476736}{77458918098436875}\,x^{10} + \dots$$

$$Q_{-2}^{(0,+)} = -\frac{4}{25}\,x + \frac{96}{1225}\,x^2 - \frac{1024}{33075}\,x^3 + \frac{8192}{800415}\,x^4 - \frac{131072}{45090045}\,x^5 + \frac{1048576}{1449322875}\,x^6 - \frac{67108864}{418854310875}\,x^7 +$$
$$+ \frac{536870912}{16800711802875}\,x^8 - \frac{8589934592}{1481822781013575}\,x^9 + \frac{68719476736}{71262204650561925}\,x^{10} + \dots$$

$$P_0^{(1,+)} = \frac{4}{3}\,x - \frac{32}{15}\,x^2 + \frac{512}{315}\,x^3 - \frac{4096}{4725}\,x^4 + \frac{131072}{363825}\,x^5 - \frac{1048576}{8513505}\,x^6 + \frac{16777216}{468242775}\,x^7 - \frac{134217728}{14783093325}\,x^8 +$$
$$+ \frac{8589934592}{4213181597625}\,x^9 - \frac{68719476736}{167122870039125}\,x^{10} + \dots$$

$$Q_0^{(1,+)} = \frac{2}{3} - \frac{4}{15}\,x + \frac{32}{245}\,x^2 - \frac{1024}{19845}\,x^3 + \frac{8192}{480249}\,x^4 - \frac{131072}{27054027}\,x^5 + \frac{1048576}{869593725}\,x^6 - \frac{67108864}{251312586525}\,x^7 +$$
$$+ \frac{536870912}{10080427081725}\,x^8 - -\frac{8589934592}{889093668608145}\,x^9 + \frac{68719476736}{42757322790337155}\,x^{10} + \dots$$

$$P_{-1}^{(1,+)} = -\frac{4}{5}\,x + \frac{64}{105}\,x^2 - \frac{512}{1575}\,x^3 + \frac{16384}{121275}\,x^4 - \frac{131072}{2837835}\,x^5 + \frac{2097152}{156080925}\,x^6 - \frac{16777216}{4927697775}\,x^7 +$$
$$+ \frac{1073741824}{1404393865875}\,x^8 - \frac{8589934592}{55707623346375}\,x^9 + \frac{137438953472}{4868846280473175}\,x^{10} + \dots$$

$$Q_{-1}^{(1,+)} = 2 - \frac{4}{3}\,x + \frac{64}{75}\,x^2 - \frac{512}{1225}\,x^3 + \frac{16384}{99225}\,x^4 - \frac{131072}{2401245}\,x^5 + \frac{2097152}{135270135}\,x^6 - \frac{16777216}{4347968625}\,x^7 +$$
$$+ \frac{1073741824}{1256562932625}\,x^8 - \frac{8589934592}{50402135408625}\,x^9 + \frac{137438953472}{4445468343040725}\,x^{10} + \dots$$

$$P_{-2}^{(1,+)} = -\frac{4}{21}\,x + \frac{32}{315}\,x^2 - \frac{1024}{24255}\,x^3 + \frac{8192}{567567}\,x^4 - \frac{131072}{31216185}\,x^5 + \frac{1048576}{985539555}\,x^6 - \frac{67108864}{280878773175}\,x^7 +$$
$$+ \frac{536870912}{11141524669275}\,x^8 - \frac{8589934592}{973769256094635}\,x^9 + \frac{68719476736}{46475350859062125}\,x^{10} + \dots$$

$$Q_{-2}^{(1,+)} = \frac{2}{3} - \frac{4}{15}\,x + \frac{32}{245}\,x^2 - \frac{1024}{19845}\,x^3 + \frac{8192}{480249}\,x^4 - \frac{131072}{27054027}\,x^5 + \frac{1048576}{869593725}\,x^6 - \frac{67108864}{251312586525}\,x^7 +$$

$$+\frac{536870912}{10080427081725}\,x^8-\frac{8589934592}{889093668608145}\,x^9+\frac{68719476736}{42757322790337155}\,x^{10}+\dots$$

$$P_0^{(0,-)}=2+\frac{4}{3}\,x+\frac{32}{15}\,x^2+\frac{512}{315}\,x^3+\frac{4096}{4725}\,x^4+\frac{131072}{363825}\,x^5+\frac{1048576}{8513505}\,x^6+\frac{16777216}{468242775}\,x^7+\frac{134217728}{14783093325}\,x^8+$$
$$+\frac{8589934592}{4213181597625}\,x^9+\frac{68719476736}{167122870039125}\,x^{10}+\dots$$

$$Q_0^{(0,-)}=-4\,x-\frac{32}{9}\,x^2-\frac{512}{225}\,x^3-\frac{4096}{3675}\,x^4-\frac{131072}{297675}\,x^5-\frac{1048576}{7203735}\,x^6-\frac{16777216}{405810405}\,x^7-\frac{134217728}{13043905875}\,x^8-$$
$$-\frac{8589934592}{3769688797875}\,x^9-\frac{68719476736}{151206406225875}\,x^{10}+\dots$$

$$P_{-1}^{(0,-)}=\frac{2}{3}+\frac{4}{15}\,x+\frac{64}{315}\,x^2+\frac{512}{4725}\,x^3+\frac{16384}{363825}\,x^4+\frac{131072}{8513505}\,x^5+\frac{2097152}{468242775}\,x^6+\frac{16777216}{14783093325}\,x^7+$$
$$+\frac{1073741824}{4213181597625}\,x^8+\frac{8589934592}{167122870039125}\,x^9+\frac{137438953472}{14606538841419525}\,x^{10}+\dots$$

$$Q_{-1}^{(0,-)}=-\frac{4}{9}\,x-\frac{64}{225}\,x^2-\frac{512}{3675}\,x^3-\frac{16384}{297675}\,x^4-\frac{131072}{7203735}\,x^5-\frac{2097152}{405810405}\,x^6-\frac{16777216}{13043905875}\,x^7-$$
$$-\frac{1073741824}{3769688797875}\,x^8-\frac{8589934592}{151206406225875}\,x^9-\frac{137438953472}{13336405029122175}\,x^{10}+\dots$$

$$P_{-2}^{(0,-)}=\frac{2}{5}+\frac{4}{35}\,x+\frac{32}{525}\,x^2+\frac{1024}{40425}\,x^3+\frac{8192}{945945}\,x^4+\frac{131072}{52026975}\,x^5+\frac{1048576}{1642565925}\,x^6+\frac{67108864}{468131288625}\,x^7+$$
$$+\frac{536870912}{18569207782125}\,x^8+\frac{8589934592}{1622948760157725}\,x^9+\frac{68719476736}{77458918098436875}\,x^{10}+\dots$$

$$Q_{-2}^{(0,-)}=-\frac{4}{25}\,x-\frac{96}{1225}\,x^2-\frac{1024}{33075}\,x^3-\frac{8192}{800415}\,x^4-\frac{131072}{45090045}\,x^5-\frac{1048576}{1449322875}\,x^6-\frac{67108864}{418854310875}\,x^7-$$
$$-\frac{536870912}{16800711802875}\,x^8-\frac{8589934592}{1481822781013575}\,x^9-\frac{68719476736}{71262204650561925}\,x^{10}+\dots$$

$$P_0^{(1,-)}=\frac{4}{3}\,x+\frac{32}{15}\,x^2+\frac{512}{315}\,x^3+\frac{4096}{4725}\,x^4+\frac{131072}{363825}\,x^5+\frac{1048576}{8513505}\,x^6+\frac{16777216}{468242775}\,x^7+\frac{134217728}{14783093325}\,x^8+$$
$$+\frac{8589934592}{4213181597625}\,x^9+\frac{68719476736}{167122870039125}\,x^{10}+\dots$$

$$Q_0^{(1,-)}=-2-4\,x-\frac{32}{9}\,x^2-\frac{512}{225}\,x^3-\frac{4096}{3675}\,x^4-\frac{131072}{297675}\,x^5-\frac{1048576}{7203735}\,x^6-\frac{16777216}{405810405}\,x^7-\frac{134217728}{13043905875}\,x^8-$$
$$-\frac{8589934592}{3769688797875}\,x^9-\frac{68719476736}{151206406225875}\,x^{10}+\dots$$

$$P_{-1}^{(1,-)}=-\frac{4}{5}\,x-\frac{64}{105}\,x^2-\frac{512}{1575}\,x^3-\frac{16384}{121275}\,x^4-\frac{131072}{2837835}\,x^5-\frac{2097152}{156080925}\,x^6-\frac{16777216}{4927697775}\,x^7-$$
$$-\frac{1073741824}{1404393865875}\,x^8-\frac{8589934592}{55707623346375}\,x^9-\frac{137438953472}{4868846280473175}\,x^{10}+\dots$$

$$Q_{-1}^{(1,-)}=2+\frac{4}{3}\,x+\frac{64}{75}\,x^2+\frac{512}{1225}\,x^3+\frac{16384}{99225}\,x^4+\frac{131072}{2401245}\,x^5+\frac{2097152}{135270135}\,x^6+\frac{16777216}{4347968625}\,x^7+$$
$$-\frac{1073741824}{1256562932625}\,x^8+\frac{8589934592}{50402135408625}\,x^9+\frac{137438953472}{4445468343040725}\,x^{10}+\dots$$

$$P_{-2}^{(1,-)}=-\frac{4}{21}\,x-\frac{32}{315}\,x^2-\frac{1024}{24255}\,x^3-\frac{8192}{567567}\,x^4-\frac{131072}{31216185}\,x^5-\frac{1048576}{985539555}\,x^6-\frac{67108864}{280878773175}\,x^7-$$
$$-\frac{536870912}{11141524669275}\,x^8-\frac{8589934592}{973769256094635}\,x^9-\frac{68719476736}{46475350859062125}\,x^{10}+\dots$$

$$Q_{-2}^{(1,-)}=\frac{2}{3}+\frac{4}{15}\,x+\frac{32}{245}\,x^2+\frac{1024}{19845}\,x^3+\frac{8192}{480249}\,x^4+\frac{131072}{27054027}\,x^5+\frac{1048576}{869593725}\,x^6+\frac{67108864}{251312586525}\,x^7+$$
$$+\frac{536870912}{10080427081725}\,x^8+\frac{8589934592}{889093668608145}\,x^9+\frac{68719476736}{42757322790337155}\,x^{10}+\dots$$

1.2.8. Integrals of the Type $\int x^{-n-1/2} \left\{ \begin{array}{c} \sinh x \\ \cosh x \end{array} \right\} \left\{ \begin{array}{c} I_\nu(x) \\ K_\nu(x) \end{array} \right\} dx$

$$\int x^{-3/2} \sinh x \cdot I_0(x)\, dx = \frac{2}{\sqrt{x}} \left[-\sinh x\, I_0(x) + 2x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) \right]$$

$$\int x^{-3/2} \sinh x \cdot I_1(x)\, dx = \frac{2}{3\sqrt{x}} \left[2x \sinh x\, I_0(x) - \sinh x\, I_1(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-3/2} \sinh x \cdot K_0(x)\, dx = \frac{2}{\sqrt{x}} \left[-\sinh x\, K_0(x) + 2x \cosh x\, K_0(x) + 2x \sinh x\, K_1(x) \right]$$

$$\int x^{-3/2} \sinh x \cdot K_1(x)\, dx = -\frac{2}{3\sqrt{x}} \left[2x \sinh x\, K_0(x) + \sinh x\, K_1(x) + 2x \cosh x\, K_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot I_0(x)\, dx = \frac{2}{\sqrt{x}} \left[2x \sinh x\, I_0(x) - \cosh x\, I_0(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot I_1(x)\, dx = \frac{2}{3\sqrt{x}} \left[2x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) - \cosh x\, I_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot K_0(x)\, dx = \frac{2}{\sqrt{x}} \left[2x \sinh x\, K_0(x) - \cosh x\, K_0(x) + 2x \cosh x\, K_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot K_1(x)\, dx = -\frac{2}{3\sqrt{x}} \left[2x \cosh x\, K_0(x) + 2x \sinh x\, K_1(x) + \cosh x\, K_1(x) \right]$$

$$\int x^{-5/2} \sinh x\, I_0(x)\, dx =$$

$$= \frac{2}{9x^{3/2}} \left[(16x^2 - 3) \sinh x\, I_0(x) - 6x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) - 16x^2 \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2} \sinh x \cdot I_1(x)\, dx =$$

$$= \frac{2}{15x^{/2}} \left[-6x \sinh x\, I_0(x) + 16x^2 \cosh x\, I_0(x) - (16x^2 + 3) \sinh x\, I_1(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2} \sinh x \cdot K_0(x)\, dx =$$

$$= \frac{2}{9x^{3/2}} \left[(16x^2 - 3) \sinh x\, K_0(x) - 6x \cosh x\, K_0(x) + 2x \sinh x\, K_1(x) + 16x^2 \cosh x\, K_1(x) \right]$$

$$\int x^{-5/2} \sinh x \cdot K_1(x)\, dx =$$

$$= \frac{2}{15x^{3/2}} \left[6x \sinh x\, K_0(x) - 16x^2 \cosh x\, K_0(x) - (16x^2 + 3) \sinh x\, K_1(x) - 2x \cosh x\, K_1(x) \right]$$

$$\int x^{-5/2} \cosh x \cdot I_0(x)\, dx =$$

$$= \frac{2}{9x^{3/2}} \left[-6x \sinh x\, I_0(x) + (16x^2 - 3) \cosh x\, I_0(x) - 16x^2 \sinh x\, I_1(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2} \cosh x \cdot I_1(x)\, dx =$$

$$= \frac{2}{15x^{3/2}} \left[16x^2 \sinh x\, I_0(x) - 6x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) - (16x^2 + 3) \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2} \cosh x \cdot K_0(x)\, dx =$$

$$= \frac{2}{9x^{3/2}} \left[-6x \sinh x\, K_0(x) + (16x^2 - 3) \cosh x\, K_0(x) + 16x^2 \sinh x\, K_1(x) + 2x \cosh x\, K_1(x) \right]$$

$$\int x^{-5/2} \cosh x \cdot K_1(x)\, dx =$$

$$= \frac{2}{15x^{3/2}} \left[-16\, x^2 \sinh x\, K_0(x) + 6\, x \cosh x\, K_0(x) - 2\, x \sinh x\, K_1(x) - \left(3 + 16\, x^2\right) \cosh x\, K_1(x) \right]$$

Let

$$\int x^{-(2n+1)/2} \sinh x \cdot I_0(x)\, dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}} \left[P_n^{(s0)}(x) \sinh x\, I_0(x) + Q_n^{(s0)}(x) \cosh x\, I_0(x) + R_n^{(s0)}(x) \sinh x\, I_1(x) + S_n^{(s0)}(x) \cosh x\, I_1(x) \right] ,$$

$$\int x^{-(2n+1)/2} \sinh x \cdot I_1(x)\, dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}} \left[P_n^{(s1)}(x) \sinh x\, I_0(x) + Q_n^{(s1)}(x) \cosh x\, I_0(x) + R_n^{(s1)}(x) \sinh x\, I_1(x) + S_n^{(s1)}(x) \cosh x\, I_1(x) \right] ,$$

$$\int x^{-(2n+1)/2} \cosh x \cdot I_0(x)\, dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}} \left[P_n^{(c0)}(x) \sinh x\, I_0(x) + Q_n^{(c0)}(x) \cosh x\, I_0(x) + R_n^{(c0)}(x) \sinh x\, I_1(x) + S_n^{(c0)}(x) \cosh x\, I_1(x) \right] ,$$

$$\int x^{-(2n+1)/2} \cosh x \cdot I_1(x)\, dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}} \left[P_n^{(c1)}(x) \sinh x\, I_0(x) + Q_n^{(c1)}(x) \cosh x\, I_0(x) + R_n^{(c1)}(x) \sinh x\, I_1(x) + S_n^{(c1)}(x) \cosh x\, I_1(x) \right] ,$$

then holds

$$\int x^{-(2n+1)/2} \sinh x \cdot K_0(x)\, dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}} \left[P_n^{(s0)}(x) \sinh x\, K_0(x) + Q_n^{(s0)}(x) \cosh x\, K_0(x) - R_n^{(s0)}(x) \sinh x\, K_1(x) - S_n^{(s0)}(x) \cosh x\, K_1(x) \right] ,$$

$$\int x^{-(2n+1)/2} \sinh x \cdot K_1(x)\, dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}} \left[- P_n^{(s1)}(x) \sinh x\, K_0(x) - Q_n^{(s1)}(x) \cosh x\, K_0(x) + R_n^{(s1)}(x) \sinh x\, K_1(x) + S_n^{(s1)}(x) \cosh x\, K_1(x) \right] ,$$

$$\int x^{-(2n+1)/2} \cosh x \cdot K_0(x)\, dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}} \left[P_n^{(c0)}(x) \sinh x\, K_0(x) + Q_n^{(c0)}(x) \cosh x\, K_0(x) - R_n^{(c0)}(x) \sinh x\, K_1(x) - S_n^{(c0)}(x) \cosh x\, K_1(x) \right] ,$$

$$\int x^{-(2n+1)/2} \cosh x \cdot K_1(x)\, dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}} \left[- P_n^{(c1)}(x) \sinh x\, K_0(x) - Q_n^{(c1)}(x) \cosh x\, K_0(x) + R_n^{(c1)}(x) \sinh x\, K_1(x) + S_n^{(c1)}(x) \cosh x\, K_1(x) \right] .$$

$M_3 = 225 ,\ P_3^{(s0)}(x) = -192\, x^2 - 90 ,\ Q_3^{(s0)}(x) = 512\, x^3 - 60\, x ,\ R_3^{(s0)}(x) = -512\, x^3 - 36\, x ,\ S_3^{(s0)}(x) = -64\, x^2$

$P_3^{(c0)}(x) = 512\, x^3 - 60\, x ,\ Q_3^{(c0)}(x) = -192\, x^2 - 90 ,\ R_3^{(c0)}(x) = -64\, x^2 ,\ S_3^{(c0)}(x) = -512\, x^3 - 36\, x$

$N_3 = 315 ,\ P_3^{(s1)}(x) = 512\, x^3 - 60\, x ,\ Q_3^{(s1)}(x) = -192\, x^2 ,\ R_3^{(s1)}(x) = -64\, x^2 - 90 ,\ S_3^{(s1)}(x) = -512\, x^3 - 36\, x$

$$P_3^{(c1)}(x) = -192\,x^2\,,\ Q_3^{(c1)}(x) = 512\,x^3 - 60\,x\,,\ R_3^{(c1)}(x) = -512\,x^3 - 36\,x\,,\ S_3^{(c1)}(x) = -64\,x^2 - 90$$

$$M_4 = 3675\,,\ P_4^{(s0)}(x) = 4096\,x^4 - 480\,x^2 - 1050\,,\ Q_4^{(s0)}(x) = -1536\,x^3 - 420\,x$$

$$R_4^{(s0)}(x) = -512\,x^3 - 300\,x\,,\ S_4^{(s0)}(x) = -4096\,x^4 - 288\,x^2$$

$$P_4^{(c0)}(x) = -1536\,x^3 - 420\,x\,,\ Q_4^{(c0)}(x) = 4096\,x^4 - 480\,x^2 - 1050\,,$$

$$R_4^{(c0)}(x) = -4096\,x^4 - 288\,x^2\,,\ S_4^{(c0)}(x) = -512\,x^3 - 300\,x$$

$$N_4 = 4725\,,\ P_4^{(s1)}(x) = -1536\,x^3 - 420\,x\,,\ Q_4^{(s1)}(x) = 4096\,x^4 - 480\,x^2\,,$$

$$R_4^{(s1)}(x) = -4096\,x^4 - 288\,x^2 - 1050\,,\ S_4^{(s1)}(x) = -512\,x^3 - 300\,x$$

$$P_4^{(c1)}(x) = 4096\,x^4 - 480\,x^2\,,\ Q_4^{(c1)}(x) = -1536\,x^3 - 420\,x\,,$$

$$R_4^{(c1)}(x) = -512\,x^3 - 300\,x\,,\ S_4^{(c1)}(x) = -4096\,x^4 - 288\,x^2 - 1050$$

$$M_5 = 297675\,,\ P_5^{(s0)}(x) = -49152\,x^4 - 13440\,x^2 - 66150\,,\ Q_5^{(s0)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,$$

$$R_5^{(s0)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x\,,\ S_5^{(s0)}(x) = -16384\,x^4 - 9600\,x^2$$

$$P_5^{(c0)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,\ Q_5^{(c0)}(x) = -49152\,x^4 - 13440\,x^2 - 66150\,,$$

$$R_5^{(c0)}(x) = -16384\,x^4 - 9600\,x^2\,,\ S_5^{(c0)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x$$

$$N_5 = 363825\,,\ P_5^{(s1)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,\ Q_5^{(s1)}(x) = -49152\,x^4 - 13440\,x^2\,,$$

$$R_5^{(s1)}(x) = -16384\,x^4 - 9600\,x^2 - 66150\,,\ S_5^{(s1)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x$$

$$P_5^{(c1)}(x) = -49152\,x^4 - 13440\,x^2\,,\ Q_5^{(c1)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,$$

$$R_5^{(c1)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x\,,\ S_5^{(c1)}(x) = -16384\,x^4 - 9600\,x^2 - 66150$$

$$M_6 = 7203735\,,$$

$$P_6^{(s0)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2 - 1309770\,,\ Q_6^{(s0)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,$$

$$R_6^{(s0)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x\,,\ S_6^{(s0)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2$$

$$P_6^{(c0)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,\ Q_6^{(c0)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2 - 1309770\,,$$

$$R_6^{(c0)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2\,,\ S_6^{(c0)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x$$

$$N_6 = 8513505\,,\ P_6^{(s1)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,\ Q_6^{(s1)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2\,,$$

$$R_6^{(s1)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2 - 1309770\,,\ S_6^{(s1)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x$$

$$P_6^{(c1)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2\,,\ Q_6^{(c1)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,$$

$$R_6^{(c1)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x\,,\ S_6^{(c1)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2 - 1309770$$

$$M_7 = 405810405\,,\ P_7^{(s0)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2 - 62432370\,,$$

$$Q_7^{(s0)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$

$$R_7^{(s0)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x\,,$$

$$S_7^{(s0)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2$$

$$P_7^{(c0)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$

$$Q_7^{(c0)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2 - 62432370\,,$$

$$R_7^{(c0)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2\,,$$

$$S_7^{(c0)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x$$

$$N_7 = 468242775\,,\ P_7^{(s1)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$

$$Q_7^{(s1)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2\,,$$

$$R_7^{(s1)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2 - 62432370\,,$$

$$S_7^{(s1)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x$$

$$P_7^{(c1)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2\,,$$

$$Q_7^{(c1)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$

$$R_7^{(c1)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x\,,$$

$$S_7^{(c1)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2 - 62432370$$

$$M_8 = 13043905875\,,\ P_8^{(s0)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2 - 1739187450\,,$$

$$Q_8^{(s0)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$

$$R_8^{(s0)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x\,,$$

$$S_8^{(s0)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2$$

$$P_8^{(c0)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$

$$Q_8^{(c0)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2 - 1739187450\,,$$

$$R_8^{(c0)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2\,,$$

$$S_8^{(c0)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x$$

$$N_8 = 14783093325\,,\ P_8^{(s1)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$

$$Q_8^{(s1)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2\,,$$

$$R_8^{(s1)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2 - 1739187450\,,$$

$$S_8^{(s1)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x$$

$$P_8^{(c1)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2\,,$$

$$Q_8^{(c1)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$

$$R_8^{(c1)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x\,,$$

$$S_8^{(c1)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2 - 1739187450$$

Recurrence Relations:

$$\int x^{-(n+1/2)} \sinh x \cdot I_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \cosh x \cdot I_0(x)\, dx -$$

$$-\frac{2x^{-n+1/2}}{(2n-1)^2}\left[(2n-1)\sinh x \cdot I_0(x) - 2x\cosh x \cdot I_0(x) + 2x\sinh x \cdot I_1(x)\right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot I_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \sinh x \cdot I_0(x)\, dx -$$

$$-\frac{2x^{-n+1/2}}{(2n-1)^2}\left[(2n-1)\cosh x \cdot I_0(x) + 2x\cosh x \cdot I_1(x) - 2x\sinh x \cdot I_0(x)\right]$$

$$\int x^{-(n+1/2)} \sinh x \cdot I_1(x)\, dx = \frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \sinh x \cdot I_0(x)\, dx +$$

$$+\frac{2x^{-n+1/2}}{4n^2-1}\left[2x\sinh x \cdot I_0(x) - 2x\cosh x \cdot I_1(x) - (2n-1)\sinh x \cdot I_1(x)\right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot I_1(x)\, dx = \frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \cosh x \cdot I_0(x)\, dx +$$

$$+\frac{2x^{-n+1/2}}{4n^2-1}\left[2x\cosh x \cdot I_0(x) - 2x\sinh x \cdot I_1(x) - (2n-1)\cosh x \cdot I_1(x)\right]$$

$$\int x^{-(n+1/2)} \sinh x \cdot K_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \cosh x \cdot K_0(x)\, dx +$$

$$+\frac{2x^{-n+1/2}}{(2n-1)^2}\left[-(2n-1)\sinh x \cdot K_0(x) + 2x\cosh x \cdot K_0(x) + 2x\sinh x \cdot K_1(x)\right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot K_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \sinh x \cdot K_0(x)\, dx +$$

$$+\frac{2x^{-n+1/2}}{(2n-1)^2}\left[-(2n-1)\cosh x \cdot K_0(x) + 2x\cosh x \cdot K_1(x) + 2x\sinh x \cdot K_0(x)\right]$$

$$\int x^{-(n+1/2)} \sinh x \cdot K_1(x)\, dx = -\frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \sinh x \cdot K_0(x)\, dx -$$

$$-\frac{2x^{-n+1/2}}{4n^2-1}\left[2x\sinh x \cdot K_0(x) + 2x\cosh x \cdot K_1(x) + (2n-1)\sinh x \cdot K_1(x)\right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot K_1(x)\, dx = -\frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \cosh x \cdot K_0(x)\, dx -$$

$$-\frac{2x^{-n+1/2}}{4n^2-1}\left[2x\cosh x \cdot K_0(x) + 2x\sinh x \cdot K_1(x) + (2n-1)\cosh x \cdot K_1(x)\right]$$

1.2.9. Integrals of the type $\int x^{2n+1} \ln x \cdot Z_0(x) \, dx$

$$\int x \ln x \cdot J_0(x) \, dx = J_0(x) + x \ln x \cdot J_1(x)$$

$$\int x \ln x \cdot I_0(x) \, dx = -I_0(x) + x \ln x \cdot I_1(x)$$

$$\int x \ln x \cdot K_0(x) \, dx = -K_0(x) - x \ln x \cdot K_1(x)$$

$$\int x^3 \ln x \cdot J_0(x) \, dx = \left(x^2 - 4 + 2x^2 \ln x\right) J_0(x) + \left[-4x + \left(x^3 - 4x\right) \ln x\right] J_1(x)$$

$$\int x^3 \ln x \cdot I_0(x) \, dx = \left(-x^2 - 4 - 2x^2 \ln x\right) I_0(x) + \left[4x + \left(x^3 + 4x\right) \ln x\right] I_1(x)$$

$$\int x^3 \ln x \cdot K_0(x) \, dx = \left(-x^2 - 4 - 2x^2 \ln x\right) K_0(x) - \left[4x + \left(x^3 + 4x\right) \ln x\right] K_1(x)$$

$$\int x^5 \ln x \cdot J_0(x) \, dx =$$

$$= \left[x^4 - 32\,x^2 + 64 + \left(4\,x^4 - 32\,x^2\right) \ln x\right] J_0(x) + \left[-8\,x^3 + 96\,x + \left(x^5 - 16\,x^3 + 64\,x\right) \ln x\right] J_1(x)$$

$$\int x^5 \ln x \cdot I_0(x) \, dx =$$

$$= \left[-x^4 - 32\,x^2 - 64 + \left(-4\,x^4 - 32\,x^2\right) \ln x\right] I_0(x) + \left[8\,x^3 + 96\,x + \left(x^5 + 16\,x^3 + 64\,x\right) \ln x\right] I_1(x)$$

$$\int x^5 \ln x \cdot K_0(x) \, dx =$$

$$= \left[-x^4 - 32\,x^2 - 64 + \left(-4\,x^4 - 32\,x^2\right) \ln x\right] K_0(x) - \left[8\,x^3 + 96\,x + \left(x^5 + 16\,x^3 + 64\,x\right) \ln x\right] K_1(x)$$

The integrals with $K_0(x)$ can be found by comparison with the integrals for $I_0(x)$.

$$\int x^7 \ln x \cdot J_0(x) \, dx = \left[x^6 - 84\,x^4 + 1536\,x^2 - 2304 + \left(6\,x^6 - 144\,x^4 + 1152\,x^2\right) \ln x\right] J_0(x) +$$

$$+ \left[-12\,x^5 + 480\,x^3 - 4224\,x + \left(x^7 - 36\,x^5 + 576\,x^3 - 2304\,x\right) \ln x\right] J_1(x)$$

$$\int x^7 \ln x \cdot I_0(x) \, dx = \left[-x^6 - 84\,x^4 - 1536\,x^2 - 2304 + \left(-6\,x^6 - 144\,x^4 - 1152\,x^2\right) \ln x\right] I_0(x) +$$

$$+ \left[12\,x^5 + 480\,x^3 + 4224\,x + \left(x^7 + 36\,x^5 + 576\,x^3 + 2304\,x\right) \ln x\right] I_1(x)$$

$$\int x^9 \ln x \cdot J_0(x) \, dx =$$

$$= \left[x^8 - 160\,x^6 + 7680\,x^4 - 116736\,x^2 + 147456 + \left(8\,x^8 - 384\,x^6 + 9216\,x^4 - 73728\,x^2\right) \ln x\right] J_0(x) +$$

$$+ \left[-16\,x^7 + 1344\,x^5 - 39936\,x^3 + 307200\,x + \left(x^9 - 64\,x^7 + 2304\,x^5 - 36864\,x^3 + 147456\,x\right) \ln x\right] J_1(x)$$

$$\int x^9 \ln x \cdot I_0(x) \, dx =$$

$$= \left[-x^8 - 160\,x^6 - 7680\,x^4 - 116736\,x^2 - 147456 + \left(-8\,x^8 - 384\,x^6 - 9216\,x^4 - 73728\,x^2\right) \ln x\right] I_0(x) +$$

$$+ \left[16\,x^7 + 1344\,x^5 + 39936\,x^3 + 307200\,x + \left(x^9 + 64\,x^7 + 2304\,x^5 + 36864\,x^3 + 147456\,x\right) \ln x\right] I_1(x)$$

Let

$$\int x^n \ln x \cdot J_0(x) \, dx = (P_n(x) + Q_n(x) \ln x) J_0(x) + (R_n(x) + S_n(x) \ln x) J_1(x) \,,$$

$$\int x^n \ln x \cdot I_0(x) \, dx = (P_n^*(x) + Q_n^*(x) \ln x) I_0(x) + (R_n^*(x) + S_n^*(x) \ln x) I_1(x) \,,$$

$$\int x^n \ln x \cdot K_0(x) \, dx = (P_n^*(x) + Q_n^*(x) \ln x) K_0(x) - (R_n^*(x) + S_n^*(x) \ln x) K_1(x) \,,$$

then holds

$$P_{11} = x^{10} - 260\,x^8 + 23680\,x^6 - 952320\,x^4 + 13148160\,x^2 - 14745600$$

$$Q_{11} = 10\,x^{10} - 800\,x^8 + 38400\,x^6 - 921600\,x^4 + 7372800\,x^2$$

$$R_{11} = -20\,x^9 + 2880\,x^7 - 180480\,x^5 + 4730880\,x^3 - 33669120\,x$$

$$S_{11} = x^{11} - 100\,x^9 + 6400\,x^7 - 230400\,x^5 + 3686400\,x^3 - 14745600\,x$$

$$P_{11}^* = -x^{10} - 260\,x^8 - 23680\,x^6 - 952320\,x^4 - 13148160\,x^2 - 14745600$$

$$Q_{11}^* = -10\,x^{10} - 800\,x^8 - 38400\,x^6 - 921600\,x^4 - 7372800\,x^2$$

$$R_{11}^* = 20\,x^9 + 2880\,x^7 + 180480\,x^5 + 4730880\,x^3 + 33669120\,x$$

$$S_{11}^* = x^{11} + 100\,x^9 + 6400\,x^7 + 230400\,x^5 + 3686400\,x^3 + 14745600\,x$$

$$P_{13} = x^{12} - 384\,x^{10} + 56640\,x^8 - 4331520\,x^6 + 159252480\,x^4 - 2070282240\,x^2 + 2123366400$$

$$Q_{13} = 12\,x^{12} - 1440\,x^{10} + 115200\,x^8 - 5529600\,x^6 + 132710400\,x^4 - 1061683200\,x^2$$

$$R_{13} = -24\,x^{11} + 5280\,x^9 - 568320\,x^7 + 31518720\,x^5 - 769720320\,x^3 + 5202247680\,x$$

$$S_{13} = x^{13} - 144\,x^{11} + 14400\,x^9 - 921600\,x^7 + 33177600\,x^5 - 530841600\,x^3 + 2123366400\,x$$

$$P_{13}^* = -x^{12} - 384\,x^{10} - 56640\,x^8 - 4331520\,x^6 - 159252480\,x^4 - 2070282240\,x^2 - 2123366400$$

$$Q_{13}^* = -12\,x^{12} - 1440\,x^{10} - 115200\,x^8 - 5529600\,x^6 - 132710400\,x^4 - 1061683200\,x^2$$

$$R_{13}^* = 24\,x^{11} + 5280\,x^9 + 568320\,x^7 + 31518720\,x^5 + 769720320\,x^3 + 5202247680\,x$$

$$S_{13}^* = x^{13} + 144\,x^{11} + 14400\,x^9 + 921600\,x^7 + 33177600\,x^5 + 530841600\,x^3 + 2123366400\,x$$

$$P_{15} = x^{14} - 532\,x^{12} + 115584\,x^{10} - 14327040\,x^8 + 1003806720\,x^6 -$$
$$-34929377280\,x^4 + 435502448640\,x^2 - 416179814400$$

$$Q_{15} = 14\,x^{14} - 2352\,x^{12} + 282240\,x^{10} - 22579200\,x^8 +$$
$$+1083801600\,x^6 - 26011238400\,x^4 + 208089907200\,x^2$$

$$R_{15} = -28\,x^{13} + 8736\,x^{11} - 1438080\,x^9 + 137195520\,x^7 -$$
$$-7106641920\,x^5 + 165728747520\,x^3 - 1079094804480\,x$$

$$S_{15} = x^{15} - 196\,x^{13} + 28224\,x^{11} - 2822400\,x^9 + 180633600\,x^7 -$$
$$-6502809600\,x^5 + 104044953600\,x^3 - 416179814400\,x$$

$$P_{15}^* = -x^{14} - 532\,x^{12} - 115584\,x^{10} - 14327040\,x^8 - 1003806720\,x^6 -$$
$$-34929377280\,x^4 - 435502448640\,x^2 - 416179814400$$

$$Q_{15}^* = -14\,x^{14} - 2352\,x^{12} - 282240\,x^{10} - 22579200\,x^8 -$$
$$-1083801600\,x^6 - 26011238400\,x^4 - 208089907200\,x^2$$

$$R_{15}^* = 28\,x^{13} + 8736\,x^{11} + 1438080\,x^9 + 137195520\,x^7 +$$
$$+7106641920\,x^5 + 165728747520\,x^3 + 1079094804480\,x$$

$$S_{15}^* = x^{15} + 196\,x^{13} + 28224\,x^{11} + 2822400\,x^9 + 180633600\,x^7 +$$
$$+6502809600\,x^5 + 104044953600\,x^3 + 416179814400\,x$$

Recurrence formulas:

$$\int x^{2n+1} \cdot \ln x \cdot J_0(x)\, dx =$$

$$= x^{2n} \ln x[2nJ_0(x) + xJ_1(x)] - 2n \int x^{2n-1} J_0(x)\, dx - \int x^{2n} J_1(x)\, dx - 4n^2 \int x^{2n-1} \cdot \ln x \cdot J_0(x)\, dx$$

$$\int x^{2n+1} \cdot \ln x \cdot I_0(x)\, dx =$$

$$= x^{2n} \ln x[xI_1(x) - 2nI_0(x)] + 2n \int x^{2n-1} I_0(x)\, dx - \int x^{2n} I_1(x)\, dx + 4n^2 \int x^{2n-1} \cdot \ln x \cdot I_0(x)\, dx$$

$$\int x^{2n+1} \cdot \ln x \cdot K_0(x)\, dx =$$

$$= -\, x^{2n} \ln x[2nJ_0(x) + xJ_1(x)] + 2n \int x^{2n-1} J_0(x)\, dx + \int x^{2n} J_1(x)\, dx + 4n^2 \int x^{2n-1} \cdot \ln x \cdot J_0(x)\, dx$$

The integrals of the type $\int x^m Z_\nu(x)\, dx$ are described before.

1.2.10. Integrals of the type $\int x^{2n}\,\ln x\cdot Z_1(x)\,dx$

$$\int \ln x\cdot J_1(x)\,dx = -\ln x\cdot J_0(x) + \int \frac{J_0(x)}{x}\,dx$$

$$\int \ln x\cdot I_1(x)\,dx = \ln x\cdot I_0(x) - \int \frac{I_0(x)}{x}\,dx$$

Concerning the integrals on the right hand side see 1.1.3, page 13.

$$\int x^2\,\ln x\cdot J_1(x)\,dx = \left(2 - x^2\ln x\right)J_0(x) + x\left(1 + 2\ln x\right)J_1(x)$$

$$\int x^2\,\ln x\cdot I_1(x)\,dx = \left(2 + x^2\ln x\right)I_0(x) - x\left(1 + 2\ln x\right)I_1(x)$$

$$\int x^4\,\ln x\cdot J_1(x)\,dx = \left[6\,x^2 - 16 + \left(-x^4 + 8\,x^2\right)\ln x\right]J_0(x) + \left[x^3 - 20\,x + \left(4\,x^3 - 16\,x\right)\ln x\right]J_1(x)$$

$$\int x^4\,\ln x\cdot I_1(x)\,dx = \left[6\,x^2 + 16 + \left(x^4 + 8\,x^2\right)\ln x\right]I_0(x) + \left[-x^3 - 20\,x + \left(-4\,x^3 - 16\,x\right)\ln x\right]I_1(x)$$

$$\int x^6\,\ln x\cdot J_1(x)\,dx = \left[10\,x^4 - 224\,x^2 + 384 + \left(-x^6 + 24\,x^4 - 192\,x^2\right)\ln x\right]J_0(x)+$$
$$+ \left[x^5 - 64\,x^3 + 640\,x + \left(6\,x^5 - 96\,x^3 + 384\,x\right)\ln x\right]J_1(x)$$

$$\int x^6\,\ln x\cdot I_1(x)\,dx = \left[10\,x^4 + 224\,x^2 + 384 + \left(x^6 + 24\,x^4 + 192\,x^2\right)\ln x\right]I_0(x)+$$
$$+ \left[-x^5 - 64\,x^3 - 640\,x + \left(-6\,x^5 - 96\,x^3 - 384\,x\right)\ln x\right]I_1(x)$$

$$\int x^8\,\ln x\cdot J_1(x)\,dx = \left[14\,x^6 - 816\,x^4 + 13440\,x^2 - 18432 + \left(-x^8 + 48\,x^6 - 1152\,x^4 + 9216\,x^2\right)\ln x\right]J_0(x)+$$
$$+ \left[x^7 - 132\,x^5 + 4416\,x^3 - 36096\,x + \left(8\,x^7 - 288\,x^5 + 4608\,x^3 - 18432\,x\right)\ln x\right]J_1(x)$$

$$\int x^8\,\ln x\cdot I_1(x)\,dx = \left[14\,x^6 + 816\,x^4 + 13440\,x^2 + 18432 + \left(x^8 + 48\,x^6 + 1152\,x^4 + 9216\,x^2\right)\ln x\right]I_0(x)+$$
$$+ \left[-x^7 - 132\,x^5 - 4416\,x^3 - 36096\,x + \left(-8\,x^7 - 288\,x^5 - 4608\,x^3 - 18432\,x\right)\ln x\right]I_1(x)$$

Let

$$\int x^n\,\ln x\cdot J_1(x)\,dx = [P_n(x) + Q_n(x)\ln x]J_0(x) + [R_n(x) + S_n(x)\ln x]J_1(x)\ ,$$

$$\int x^n\,\ln x\cdot I_1(x)\,dx = [P_n^*(x) + Q_n^*(x)\ln x]I_0(x) + [R_n^*(x) + S_n^*(x)\ln x]I_1(x)\ ,$$

then holds:

$$P_{10}(x) = 18\,x^8 - 1984\,x^6 + 86016\,x^4 - 1241088\,x^2 + 1474560$$
$$Q_{10}(x) = -x^{10} + 80\,x^8 - 3840\,x^6 + 92160\,x^4 - 737280\,x^2$$
$$R_{10}(x) = x^9 - 224\,x^7 + 15744\,x^5 - 436224\,x^3 + 3219456\,x$$
$$S_{10}(x) = 10\,x^9 - 640\,x^7 + 23040\,x^5 - 368640\,x^3 + 1474560\,x$$
$$P_{10}^*(x) = 18\,x^8 + 1984\,x^6 + 86016\,x^4 + 1241088\,x^2 + 1474560$$
$$Q_{10}^*(x) = x^{10} + 80\,x^8 + 3840\,x^6 + 92160\,x^4 + 737280\,x^2$$
$$R_{10}^*(x) = -x^9 - 224\,x^7 - 15744\,x^5 - 436224\,x^3 - 3219456\,x$$
$$S_{10}^*(x) = -10\,x^9 - 640\,x^7 - 23040\,x^5 - 368640\,x^3 - 1474560\,x$$

$$P_{12}(x) = 22\,x^{10} - 3920\,x^8 + 322560\,x^6 - 12349440\,x^4 + 165150720\,x^2 - 176947200$$
$$Q_{12}(x) = -x^{12} + 120\,x^{10} - 9600\,x^8 + 460800\,x^6 - 11059200\,x^4 + 88473600\,x^2$$

$$R_{12}(x) = x^{11} - 340\,x^9 + 40960\,x^7 - 2396160\,x^5 + 60456960\,x^3 - 418775040\,x$$

$$S_{12}(x) = 12\,x^{11} - 1200\,x^9 + 76800\,x^7 - 2764800\,x^5 + 44236800\,x^3 - 176947200\,x$$

$$P_{12}^*(x) = 22\,x^{10} + 3920\,x^8 + 322560\,x^6 + 12349440\,x^4 + 165150720\,x^2 + 176947200$$

$$Q_{12}^*(x) = x^{12} + 120\,x^{10} + 9600\,x^8 + 460800\,x^6 + 11059200\,x^4 + 88473600\,x^2$$

$$R_{12}^*(x) = -x^{11} - 340\,x^9 - 40960\,x^7 - 2396160\,x^5 - 60456960\,x^3 - 418775040\,x$$

$$S_{12}^*(x) = -12\,x^{11} - 1200\,x^9 - 76800\,x^7 - 2764800\,x^5 - 44236800\,x^3 - 176947200\,x$$

$$P_{14}(x) = 26\,x^{12} - 6816\,x^{10} + 908160\,x^8 - 66170880\,x^6 +$$
$$+ 2362245120\,x^4 - 30045634560\,x^2 + 29727129600$$

$$Q_{14}(x) = -x^{14} + 168\,x^{12} - 20160\,x^{10} + 1612800\,x^8 -$$
$$- 77414400\,x^6 + 1857945600\,x^4 - 14863564800\,x^2$$

$$R_{14}(x) = x^{13} - 480\,x^{11} + 88320\,x^9 - 8878080\,x^7 +$$
$$+ 474439680\,x^5 - 11306926080\,x^3 + 74954833920\,x$$

$$S_{14}(x) = 14\,x^{13} - 2016\,x^{11} + 201600\,x^9 - 12902400\,x^7 +$$
$$+ 464486400\,x^5 - 7431782400\,x^3 + 29727129600\,x$$

$$P_{14}^*(x) = 26\,x^{12} + 6816\,x^{10} + 908160\,x^8 + 66170880\,x^6 +$$
$$+ 2362245120\,x^4 + 30045634560\,x^2 + 29727129600$$

$$Q_{14}^*(x) = x^{14} + 168\,x^{12} + 20160\,x^{10} + 1612800\,x^8 +$$
$$+ 77414400\,x^6 + 1857945600\,x^4 + 14863564800\,x^2$$

$$R_{14}^*(x) = -x^{13} - 480\,x^{11} - 88320\,x^9 - 8878080\,x^7 -$$
$$- 474439680\,x^5 - 11306926080\,x^3 - 74954833920\,x$$

$$S_{14}^*(x) = -14\,x^{13} - 2016\,x^{11} - 201600\,x^9 - 12902400\,x^7 -$$
$$- 464486400\,x^5 - 7431782400\,x^3 - 29727129600\,x$$

Recurrence formulas:

$$\int x^{2n+2} \cdot \ln x \cdot J_1(x)\,dx = x^{2n+1} \ln x [(2n+2)J_1(x) - xJ_0(x)] +$$

$$+ \int x^{2n+1} J_0(x)\,dx - (2n+2) \int x^{2n} J_1(x)\,dx - 4n(n+1) \int x^{2n} \cdot \ln x \cdot J_1(x)\,dx$$

$$\int x^{2n+2} \cdot \ln x \cdot I_1(x)\,dx = x^{2n+1} \ln x [xI_0(x) - (2n+2)I_1(x)] -$$

$$- \int x^{2n+1} I_0(x)\,dx + (2n+2) \int x^{2n} I_1(x)\,dx + 4n(n+1) \int x^{2n} \cdot \ln x \cdot I_1(x)\,dx$$

The integrals of the type $\int x^m Z_\nu(x)\,dx$ are described before.

1.2.11. Integrals of the type $\int x^{2n+\nu} \ln x \cdot Z_\nu(x)\, dx$

a) The Functions Λ_k and Λ_k^*, $k = 0, 1$:

Let

$$\Lambda_0(x) \;=\; \sum_{k=0}^{\infty} \alpha_k\, x^{2k+1} \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} \cdot (k!)^2 \cdot (2k+1)}\, x^{2k+1} = \int_0^x J_0(t)\, dt = x\, J_0(x) + \Phi(x)\ .$$

($\Phi(x)$ and further on $\Psi(x)$ defined as on page 7)

$$\Lambda_0(x) = x - \frac{x^3}{12} + \frac{x^5}{320} - \frac{x^7}{16\,128} + \frac{x^9}{1\,327\,104} - \frac{x^{11}}{162\,201\,600} + \frac{x^{13}}{27\,603\,763\,200} - \frac{x^{15}}{6\,242\,697\,216\,000} + \ldots$$

k	α_k	$1/\alpha_k$
0	1.0000000000000000E+00	1
1	-8.3333333333333329E-02	-12
2	3.1250000000000002E-03	320
3	-6.2003968253968251E-05	-16128
4	7.5352044753086416E-07	1327104
5	-6.1651672979797980E-09	-162201600
6	3.6226944592830012E-11	27603763200
7	-1.6018716996829596E-13	-6242697216000
8	5.5211570531352012E-16	1811214552268800
9	-1.5246859958300586E-18	-655872751986278400
10	3.4486945143775135E-21	289964795614986240000
11	-6.5058017249306315E-24	-153708957370763182080000
12	1.0391211088430869E-26	96235173310390861824000000
13	-1.4232975959389203E-29	-70259375330450160400465920000
14	1.6902284962328838E-32	59163598426411660994999746560000
15	-1.7568683294176930E-35	-56919461934375356612430790656000000
16	1.6117104111016952E-38	-56919461934375356612430790656000000
17	-1.3145438350557573E-41	-7607201626392202499431885757743104000 0000
18	9.5948102742224525E-45	10422300925393111264364508167294209228 8000000
19	-6.3038564554696844E-48	-15863305376065904161187882214847045592 6784000000
20	3.7477195390749646E-51	26682893145382649050613463417794004894 3513600000000

Values of this function may be found in [1], Table 11.1 .
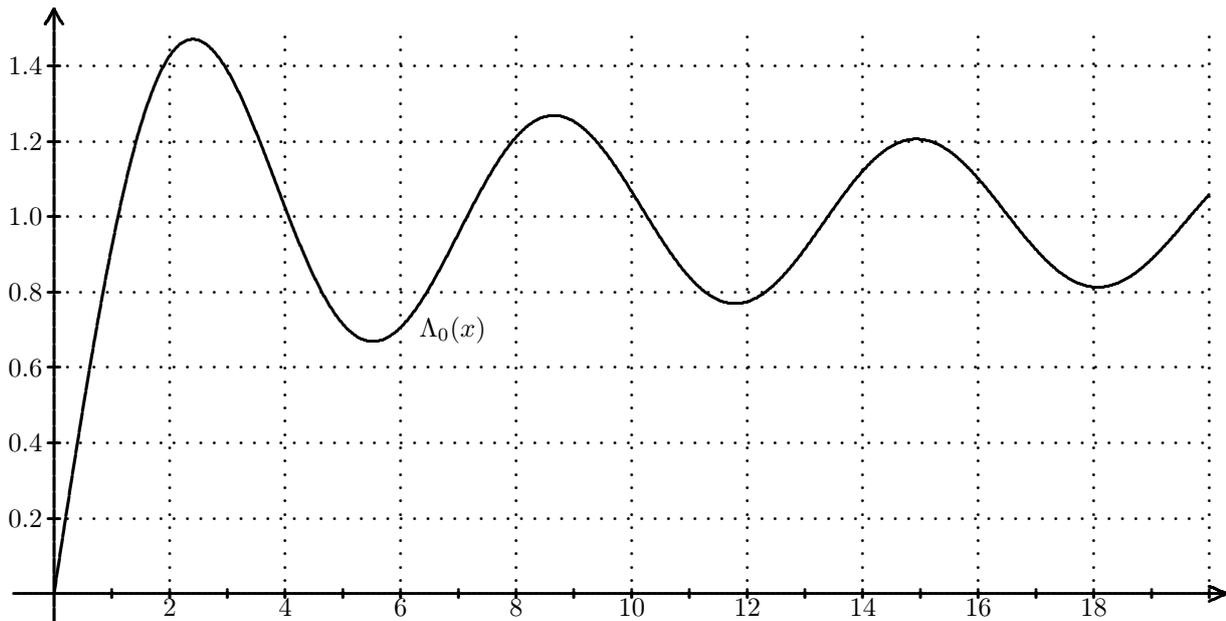
FIGURE 1 :   *Function $\Lambda_0(x)$*

Approximations with Chebyshev polynomials are given in [1], table 9.3 .

The maxima and minima of $\Lambda_0(x)$ are situated in the zeros of $J_0(x)$:

k	1	2	3	4	5	6	7	8
max	1.470300	1.268168	1.205654	1.172888	1.151982	1.137178	1.125991	1.117157
min	0.668846	0.769119	0.812831	0.838567	0.855986	0.868771	0.878666	0.886617

Asymptotic expansion:
$$\Lambda_0(x) \sim 1 + \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{\lambda_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right)$$

Recurrence relation:
$$\lambda_{k+1} = -\frac{2k+1}{16(k+1)} \left[(12k+10)\lambda_k + (4k^2-1)\lambda_{k-1}\right]$$

If $k > 1$, then up to $k \approx 30$ holds

$$\lambda_k \approx (-1)^k \, \Gamma(s_k) \quad \text{with} \quad s_k = k + \frac{1}{2} - \frac{1}{3\sqrt{k}} \ .$$

Coefficients of the asymptotic formula:

k	λ_k	λ_k	$q_k = |\lambda_k/\lambda_{k-1}|$	$|\lambda_k|/\Gamma(s_k)$
0	1	1	-	-
1	$-\dfrac{5}{8}$	-0.625	0.625	-
2	$\dfrac{129}{128}$	1.007812500	1.612500000	0.882203509
3	$-\dfrac{2655}{1024}$	-2.592773438	2.572674419	0.958684418
4	$\dfrac{301035}{32768}$	9.186859131	3.543255650	0.992044824
5	$-\dfrac{10896795}{262144}$	-41.56797409	4.524720963	1.007474317
6	$\dfrac{961319205}{4194304}$	229.1963589	5.513772656	1.014554883
7	$-\dfrac{50046571575}{33554432}$	-1491.504060	6.507538198	1.017456390
8	$\dfrac{24035398261875}{2147483648}$	11192.35450	7.504072427	1.018151550
9	$-\dfrac{1634825936118375}{17179869184}$	-95159.39374	8.502178320	1.017633944
10	$\dfrac{248523783571238175}{274877906944}$	904124.2577	9.501156135	1.016430975
11	$-\dfrac{20877210220441199625}{2199023255552}$	-9493856.042	10.50060980	1.014835786
12	$\dfrac{7683027147736313147775}{70368744177664}$	109182382.6	11.50032001	1.014835786

Roughly spoken, the item
$$\frac{\lambda_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right)$$

in the asymptotic series should not be used if $|x| < q_k$.

Let
$$d_n(x) = 1 + \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{n} \frac{\lambda_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right) - \Lambda_0(x) \ .$$

The following table gives some consecutive maxima and minima of interest of this functions:

n = 0		n = 1		n = 2		n = 3	
x	$d_n(x)$	x	$d_n(x)$	x	$d_n(x)$	x	$d_n(x)$
3.953	-5.390E-2	5.510	-9.219-3	3.936	1.020E-2	5.501	2.128E-3
7.084	2.479E-2	8.647	3.315E-3	7.074	-1.751-3	8.642	-3.501E-4
10.221	-1.475E-2	11.787	-1.592E-3	10.214	5.360E-4	11.783	9.561E-5
13.360	1.000E-2	14.927	9.002E-4	13.355	-2.197E-4	14.924	-3.469E-5
16.500	-7.337E-3	18.068	-5.647E-4	16.496	1.076E-4	18.065	1.511E-5
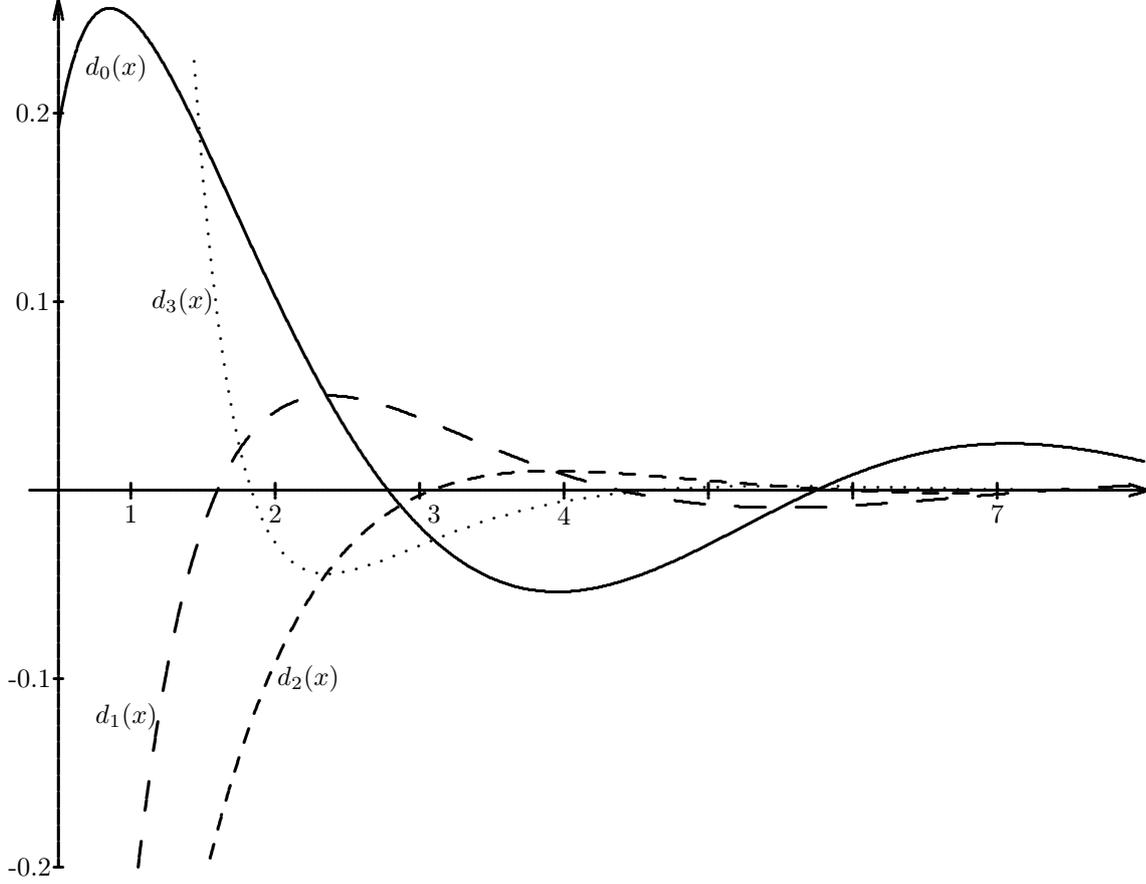
FIGURE 1 :   *Differences $d_0(x)$ ... $d_3(x)$*

Let

$$\Lambda_0^*(x) \;=\; \sum_{k=0}^{\infty} |\alpha_k|\, x^{2k+1} \;=\; \sum_{k=0}^{\infty} \frac{1}{2^{2k}\cdot(k!)^2\cdot(2k+1)}\, x^{2k+1} = \int_0^x I_0(t)\,dt = x\,I_0(x) + \Psi(x)\;.$$

Asymptotic expansion (see $\Lambda_0(x)$):

$$\Lambda_0^*(x) = \frac{e^x}{\sqrt{2\pi x}}\left[1 + \frac{5}{8x} + \frac{129}{128x^2} + \frac{2655}{1024x^3} + \ldots\right]$$

Furthermore, let

$$\Lambda_1(x) \;=\; \sum_{k=0}^{\infty} \beta_k\, x^{2k+1} \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\cdot(k!)^2\cdot(2k+1)^2}\, x^{2k+1} \;=\; \sum_{k=0}^{\infty} \frac{\alpha_k}{2k+1}\, x^{2k+1}\;.$$

$\Lambda_1(x)$ can be written as a hypergeometric function.
One has

$$\Lambda_0(x) = x\,\Lambda_1'(x)\,,\ \Lambda_1(0) = 0 \quad \Longleftrightarrow \quad \Lambda_1(x) = \int_0^x \frac{\Lambda_0(t)\,dt}{t}\;.$$

$$\Lambda_1(x) = x - \frac{x^3}{36} + \frac{x^5}{1\,600} - \frac{x^7}{112\,896} + \frac{x^9}{11\,943\,936} - \frac{x^{11}}{1\,784\,217\,600} + \frac{x^{13}}{358\,848\,921\,600} - \frac{x^{15}}{93\,640\,458\,240\,000} + \ldots$$

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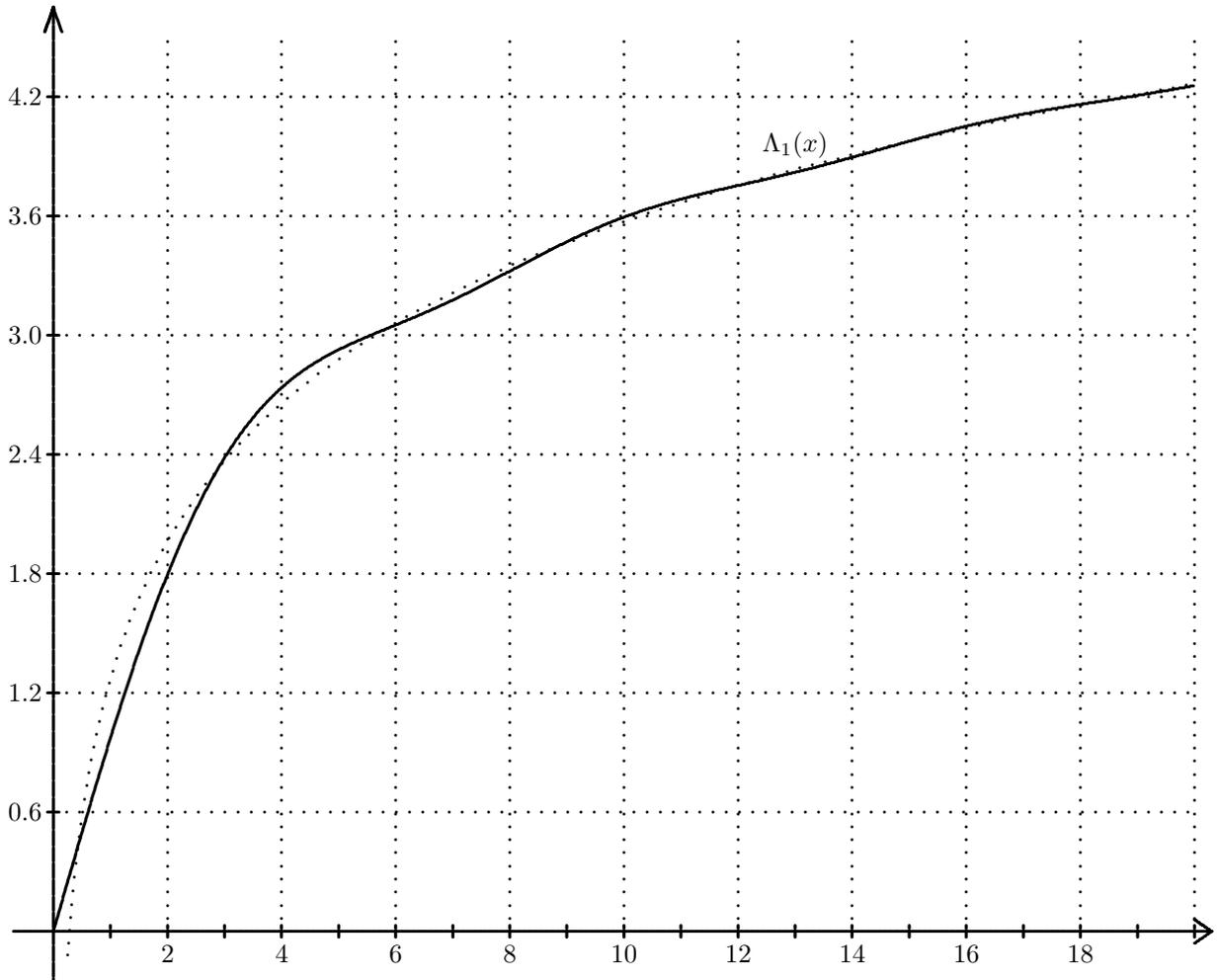
FIGURE 2 :   *Function* $\Lambda_1(x)$,   *dots:* $C + \ln 2x$

Asymptotic series with Euler's constant $\mathbf{C} = 0.577\ 215\ 664\ 901\ 533$:

$$\Lambda_1(x) \sim C + \ln 2x - \sqrt{\frac{2}{\pi x}} \sum_{k=1}^{\infty} \frac{\mu_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right)$$

k	μ_k	μ_k	$\lvert\mu_k/\mu_{k-1}\rvert$
1	1	1	1
2	$-\dfrac{17}{8}$	-2.125	2.125
3	$\dfrac{809}{128}$	6.320 312 500	2.974
4	$-\dfrac{25307}{1024}$	-24.713 867 187 500	3.910
5	$\dfrac{3945243}{32768}$	120.399 261 474 609	4.871
6	$-\dfrac{184487487}{262144}$	-703.763 912 200 928	5.845
7	$\dfrac{20148017853}{4194304}$	4 803.661 788 225 174	6.826
8	$-\dfrac{1258927642755}{33554432}$	-37 518.967 472 166	7.810
9	$\dfrac{708892035920595}{2147483648}$	330 103.578 008 988	8.798
10	$-\dfrac{55510620666083595}{17179869184}$	-3 231 143.384 827 510	9.788
11	$\dfrac{9574308055473282135}{274877906944}$	34 831 129.798 379	10.780
12	$-\dfrac{90171355132983156045}{2199023255552}$	-410 051 848.723 007	11.773

Some consecutive maxima and minima of the differences

$$\delta_n(x) \;=\; C + \ln 2x - \sqrt{\frac{2}{\pi x}} \sum_{k=1}^{n} \frac{\mu_k}{x^k} \, \sin\left(x + \frac{3-2k}{4}\,\pi\right) - \Lambda_1\left(x\right)$$

$n = 1$	x	2.4704	5.569	8.689	11.819	14.953	18.090
	$\delta_n(x)$	9.374E-2	-1.855E-2	6.824E-3	-3.309E-3	1.880E-3	-1.182E-3
$n = 2$	x	3.991	7.115	10.246	13.380	16.517	19.655
	$\delta_n(x)$	2.312E-2	-4.117E-3	1.279E-3	-5.284E-4	2.598E-4	-1.436E-4
$n = 3$	x	5.541	8.673	11.808	14.945	18.083	21.222
	$\delta_n(x)$	5.439E-3	-9.148E-4	2.525E-4	-9.214E-5	4.028E-5	-1.997E-5
$n = 4$	x	10.237	13.374	16.512	19.651	22.791	25.931
	$\delta_n(x)$	-2.043E-4	5.163E-5	-1.707E-5	6.771E-6	-3.061E-6	1.527E-6

Let

$$\Lambda_1^*(x) \;=\; \sum_{k=0}^{\infty} |\beta_k|\, x^{2k+1} \;=\; \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^{2k} \cdot (k!)^2 \cdot (2k+1)^2}\;.$$

b) Basic Integrals:

$$\int \ln x \cdot J_0(x)\, dx \;=\; \Lambda_0\left(x\right) \cdot \ln x - \Lambda_1\left(x\right) + c$$

$$\int \ln x \cdot I_0(x)\, dx \;=\; \Lambda_0^*\left(x\right) \cdot \ln x - \Lambda_1^*\left(x\right) + c$$

In particular, let

$$\int_0^x \ln t \cdot J_0(t)\, dt \;=\; F\left(x\right) \quad \text{and} \quad \int_0^x \ln t \cdot I_0(t)\, dt \;=\; F^*\left(x\right)\;.$$
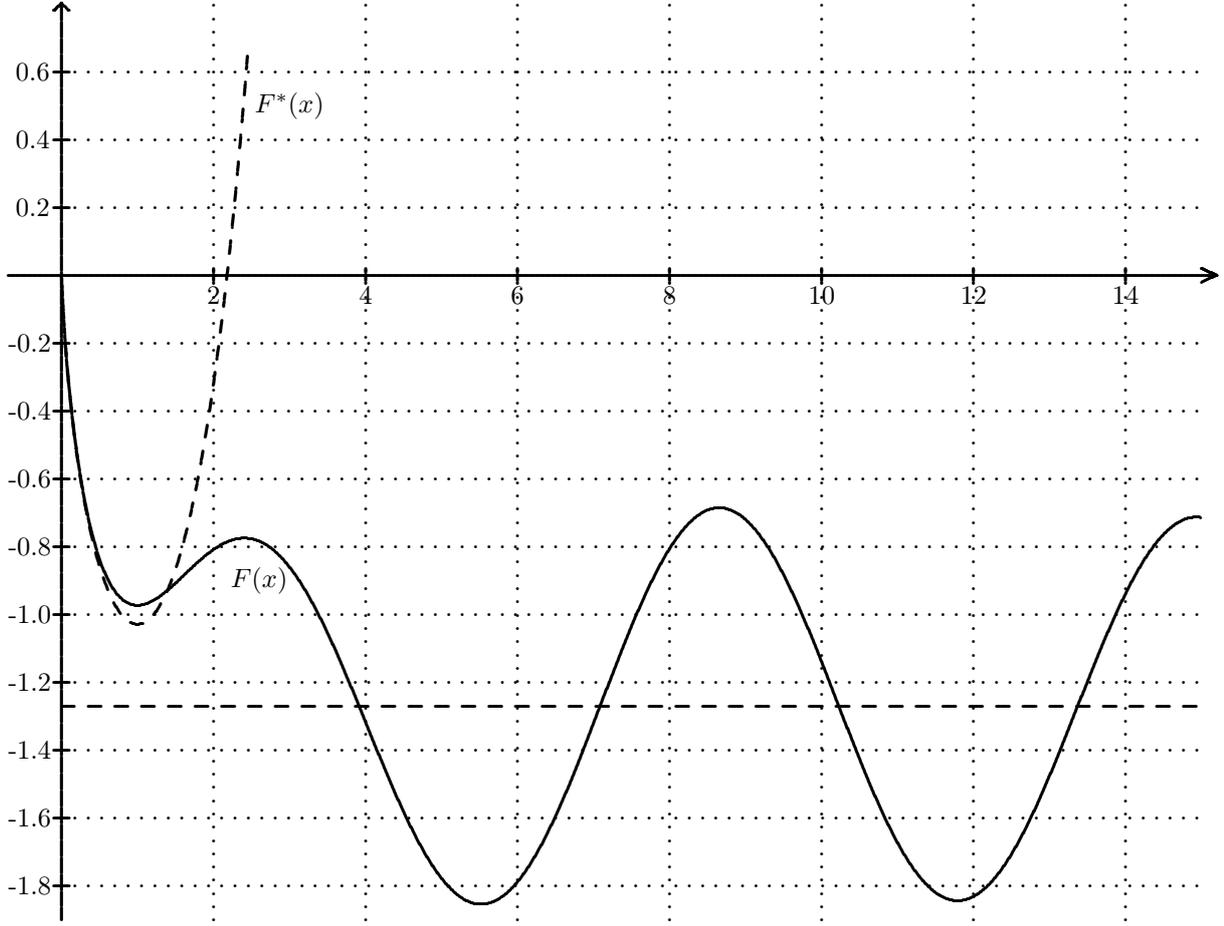
FIGURE 3 :   *Functions $F(x)$ and $F^*(x)$*

Holds ([7], 6.772) with Euler's constant $\mathbf{C} = 0.577...$

$$\lim_{x \to \infty} F(x) \; = \; -\ln 2 - \mathbf{C} \; = \; -1.270\,362\,845\,461\,478\,170 \, .$$

Asymptotic expansion:

$$F(x) \; \sim \; -\ln 2 - \mathbf{C} + \sqrt{\frac{2}{\pi x}} \left[\ln x \cdot \sin\left(x - \frac{\pi}{4}\right) + \sum_{k=1}^{\infty} \frac{\lambda_k \ln x + \mu_k}{x^k} \cdot \sin\left(x + \frac{(2k-1)\pi}{4}\right) \right] =$$

$$= -\ln 2 - \mathbf{C} + \sqrt{\frac{2}{\pi x}} \left[\ln x \cdot \sin\left(x - \frac{\pi}{4}\right) - \frac{5 \ln x + 8}{8x} \sin\left(x + \frac{\pi}{4}\right) + \frac{129 \ln x + 272}{128 x^2} \sin\left(x + \frac{3\pi}{4}\right) - \right.$$

$$\left. - \frac{2\,655 \ln x + 6\,472}{1\,024 x^3} \sin\left(x + \frac{5\pi}{4}\right) + \frac{301\,035 \ln x + 809\,824}{32\,768 x^4} \sin\left(x + \frac{7\pi}{4}\right) + \ldots \right]$$

Let

$$\Delta_n(x) = -\ln 2 - \mathbf{C} + \sqrt{\frac{2}{\pi x}} \left[\ln x \cdot \sin\left(x - \frac{\pi}{4}\right) + \sum_{k=1}^{n} \frac{\lambda_k \ln x + \mu_k}{x^k} \cdot \sin\left(x + \frac{(2k-1)\pi}{4}\right) \right] - F(x)$$
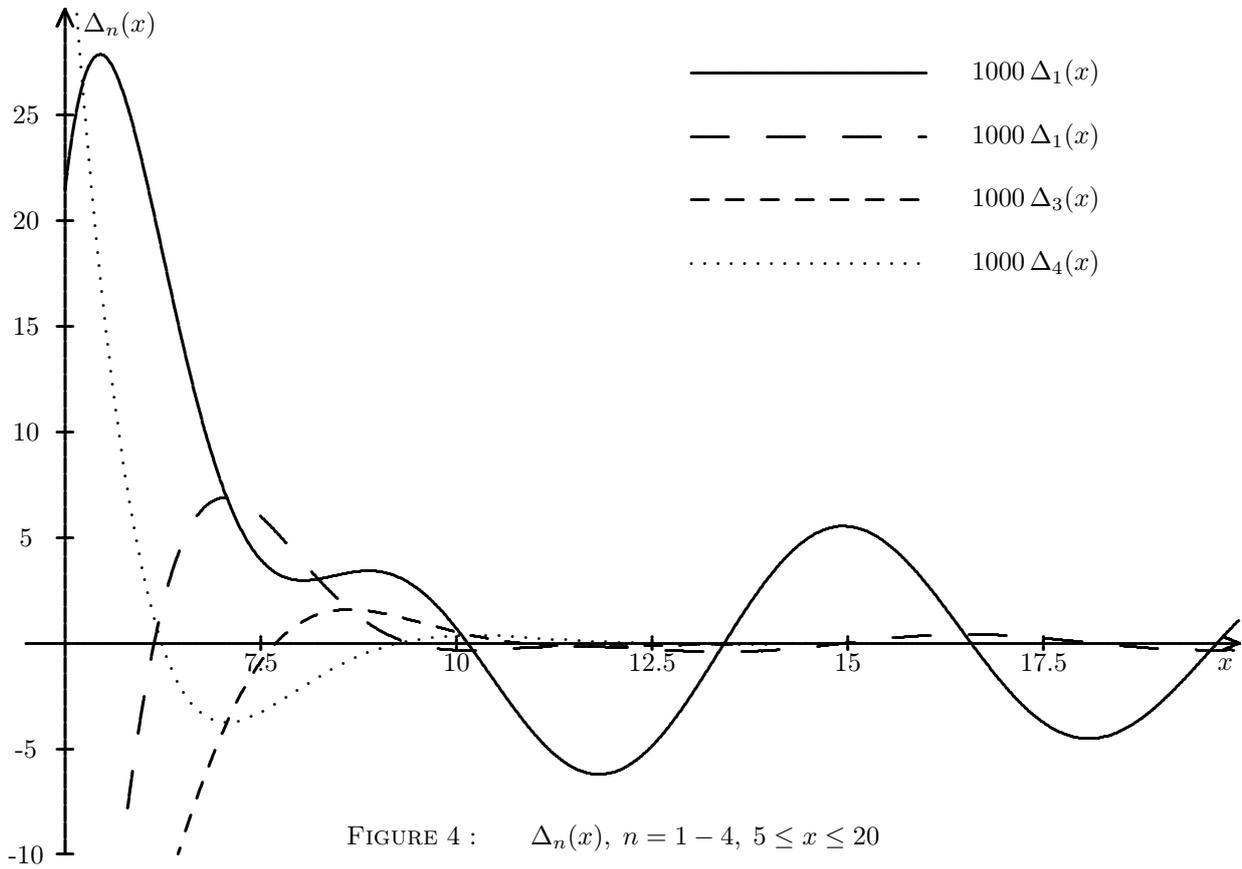
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FIGURE 4 : $\Delta_n(x)$, $n = 1-4$, $5 \le x \le 20$

Legend for Figure 4:
1000 $\Delta_1(x)$
1000 $\Delta_1(x)$
1000 $\Delta_3(x)$
1000 $\Delta_4(x)$

FIGURE 5 : $\Delta_n(x)$, $n = 2-4$, $10 \le x \le 30$

Legend for Figure 5:
$10^5 \Delta_2(x)$
$10^5 \Delta_3(x)$
$10^5 \Delta_4(x)$

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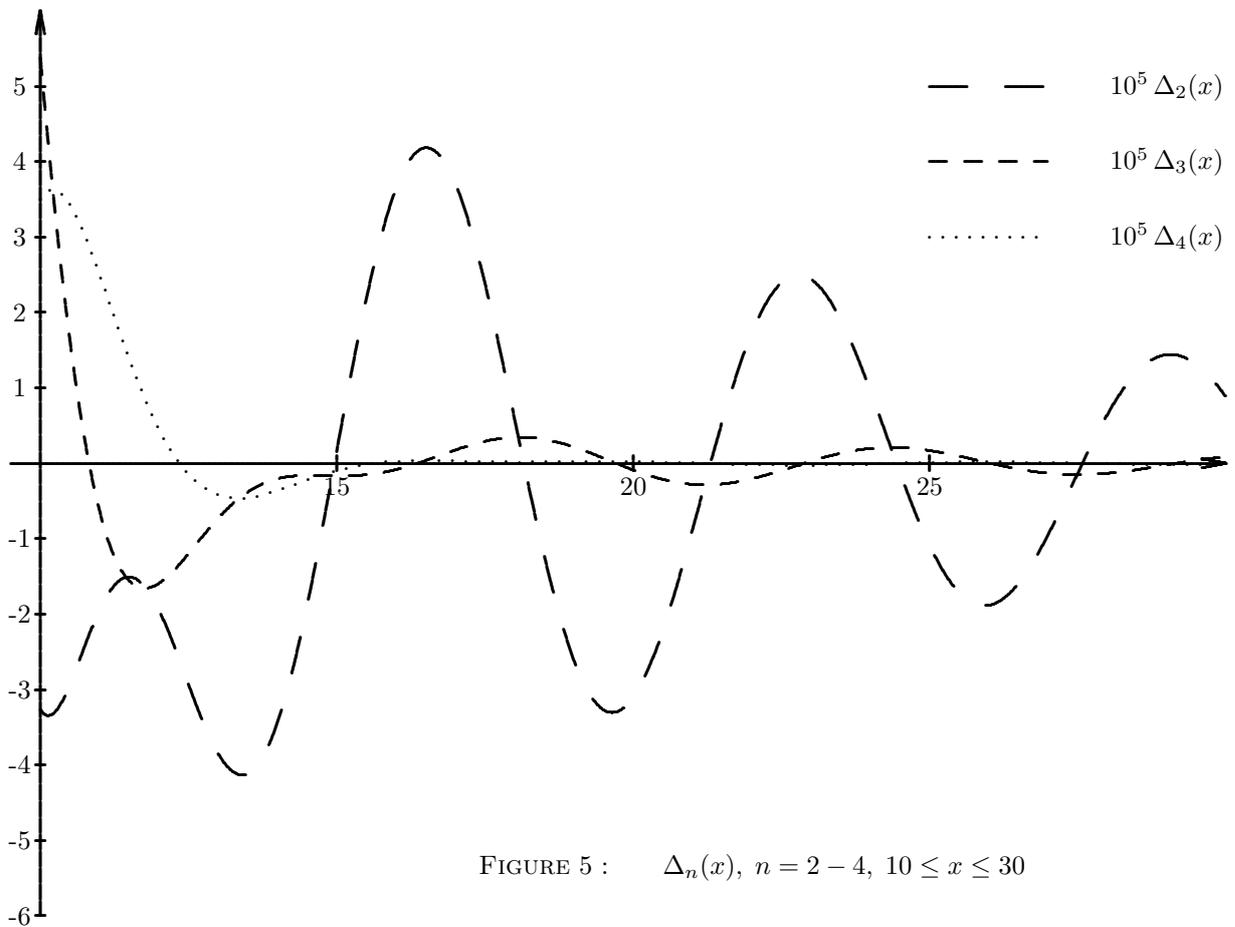
Some consecutive maxima and minima of the differences $\Delta_n(x)$:

$n=1$	x	11.822	14.944	18.079	21.217	24.356	27.496
	$\Delta_n(x)$	-6.213E-4	5.541E-4	-4.520E-4	3.644E-4	-2.959E-4	2.431E-4
$n=2$	x	7.046	10.134	13.404	16.513	19.647	22.785
	$\Delta_n(x)$	6.878E-4	-3.344E-5	-4.133E-5	4.183E-5	-3.303E-5	2.498E-5
$n=3$	x	18.083	21.215	24.354	27.494	30.634	33.775
	$\Delta_n(x)$	3.458-6	-2.863E-6	2.100E-6	-1.507E-6	1.087E-6	-7.949E-7
$n=4$	x	22.785	25.923	29.063	32.204	35.345	38.486
	$\Delta_n(x)$	-2.648E-7	1.982E-7	-1.385E-7	9.570E-8	-6.666E-8	4.709E-8

c) Integrals of $x^{2n} \ln x \cdot Z_0(x)$:

$$\int \ln x \cdot J_0(x)\, dx = [x\, J_0(x) + \Phi(x)] \cdot \ln x - \Lambda_1(x)$$

$$\int \ln x \cdot I_0(x)\, dx = [x\, I_0(x) + \Psi(x)] \cdot \ln x - \Lambda_1^*(x)$$

$$\int x^2 \ln x \cdot J_0(x)\, dx = [x^2\, J_1(x) - \Phi(x)] \cdot \ln x - x J_0(x) + \Lambda_1(x) - 2\Phi(x)$$

$$\int x^2 \ln x \cdot I_0(x)\, dx = [x^2\, I_1(x) + \Psi(x)] \cdot \ln x + x I_0(x) - \Lambda_1^*(x) + 2\Psi(x)$$

$$\int x^4 \ln x \cdot J_0(x)\, dx =$$

$$= [3\, x^3\, J_0(x) + (x^4 - 9\, x^2)\, J_1(x) + 9\Phi(x)] \cdot \ln x + (x^3 + 9\, x)\, J_0(x) - 6\, x^2\, J_1(x) - 9\Lambda_1(x) + 24\Phi(x)$$

$$\int x^4 \ln x \cdot I_0(x)\, dx =$$

$$= [-3\, x^3\, I_0(x) + (x^4 + 9\, x^2)\, I_1(x) + 9\Psi(x)] \cdot \ln x + (-x^3 + 9\, x)\, I_0(x) + 6\, x^2\, I_1(x) - 9\Lambda_1^*(x) + 24\Psi(x)$$

Let

$$\int x^n \ln x \cdot J_0(x)\, dx =$$

$$= [P_n(x)\, J_0(x) + Q_n(x)\, J_1(x) + p_n\, \Phi(x)] \cdot \ln x + R_n(x)\, J_0(x) + S_n(x)\, J_1(x) - p_n\, \Lambda_1(x) + q_n\, \Phi(x)$$

and

$$\int x^n \ln x \cdot I_0(x)\, dx =$$

$$= [P_n^*(x)\, I_0(x) + Q_n^*(x)\, I_1(x) + p_n^*\, \Psi(x)] \cdot \ln x + R_n^*(x)\, I_0(x) + S_n^*(x)\, I_1(x) - p_n^*\, \Lambda_1^*(x) + q_n^*\, \Psi(x) ,$$

then holds

$$P_6(x) = 5\, x^5 - 75\, x^3 , \quad Q_6(x) = x^6 - 25\, x^4 + 225\, x^2 , \quad R_6(x) = x^5 - 55\, x^3 - 225\, x$$

$$S_6(x) = -10\, x^4 + 240\, x^2 , \quad p_6 = -225 , \quad q_6 = -690$$

$$P_6^*(x) = -5\, x^5 - 75\, x^3 , \quad Q_6^*(x) = x^6 + 25\, x^4 + 225\, x^2 , \quad R_6^*(x) = -x^5 - 55\, x^3 + 225\, x$$

$$S_6^*(x) = 10\, x^4 + 240\, x^2 , \quad p_6^* = 225 , \quad q_6^* = 690$$

$$P_8(x) = 7\, x^7 - 245\, x^5 + 3675\, x^3 , \quad Q_8(x) = x^8 - 49\, x^6 + 1225\, x^4 - 11025\, x^2$$

$$R_8(x) = x^7 - 119\, x^5 + 3745\, x^3 + 11025\, x , \quad S_8(x) = -14\, x^6 + 840\, x^4 - 14910\, x^2 , \quad p_8 = 11025 , \quad q_8 = 36960$$

$$P_8^*(x) = -7\, x^7 - 245\, x^5 - 3675\, x^3 , \quad Q_8^*(x) = x^8 + 49\, x^6 + 1225\, x^4 + 11025\, x^2$$

$$R_8^*(x) = -x^7 - 119\, x^5 - 3745\, x^3 + 11025\, x , \quad S_8^*(x) = 14\, x^6 + 840\, x^4 + 14910\, x^2 , \quad p_8^* = 11025 , \quad q_8^* = 36960$$

$$P_{10}(x) = 9\,x^9 - 567\,x^7 + 19845\,x^5 - 297675\,x^3 \ , \quad Q_{10}(x) = x^{10} - 81\,x^8 + 3969\,x^6 - 99225\,x^4 + 893025\,x^2$$

$$R_{10}(x) = x^9 - 207\,x^7 + 14049\,x^5 - 369495\,x^3 - 893025\,x \ , \quad S_{10}(x) = -18\,x^8 + 2016\,x^6 - 90090\,x^4 + 1406160\,x^2$$

$$p_{10} = -893025 \ , \quad q_{10} = -3192210$$

$$P_{10}^*(x) = -9\,x^9 - 567\,x^7 - 19845\,x^5 - 297675\,x^3 \ , \quad Q_{10}^*(x) = x^{10} + 81\,x^8 + 3969\,x^6 + 99225\,x^4 + 893025\,x^2$$

$$R_{10}^*(x) = -x^9 - 207\,x^7 - 14049\,x^5 - 369495\,x^3 + 893025\,x \ , \quad S_{10}^*(x) = 18\,x^8 + 2016\,x^6 + 90090\,x^4 + 1406160\,x^2$$

$$p_{10}^* = 893025 \ , \quad q_{10}^* = 3192210$$

$$P_{12}(x) = 11\,x^{11} - 1089\,x^9 + 68607\,x^7 - 2401245\,x^5 + 36018675\,x^3$$

$$Q_{12}(x) = x^{12} - 121\,x^{10} + 9801\,x^8 - 480249\,x^6 + 12006225\,x^4 - 108056025\,x^2$$

$$R_{12}(x) = x^{11} - 319\,x^9 + 37521\,x^7 - 2136519\,x^5 + 51257745\,x^3 + 108056025\,x$$

$$S_{12}(x) = -22\,x^{10} + 3960\,x^8 - 331254\,x^6 + 13083840\,x^4 - 189791910\,x^2$$

$$p_{12} = 108056025 \ , \quad q_{12} = 405903960$$

$$P_{12}^*(x) = -11\,x^{11} - 1089\,x^9 - 68607\,x^7 - 2401245\,x^5 - 36018675\,x^3$$

$$Q_{12}^*(x) = x^{12} + 121\,x^{10} + 9801\,x^8 + 480249\,x^6 + 12006225\,x^4 + 108056025\,x^2$$

$$R_{12}^*(x) = -x^{11} - 319\,x^9 - 37521\,x^7 - 2136519\,x^5 - 51257745\,x^3 + 108056025\,x$$

$$Q_{12}^*(x) = 22\,x^{10} + 3960\,x^8 + 331254\,x^6 + 13083840\,x^4 + 189791910\,x^2$$

$$p_{12}^* = 108056025 \ , \quad q_{12}^* = 405903960$$

$$P_{14}(x) = 13\,x^{13} - 1859\,x^{11} + 184041\,x^9 - 11594583\,x^7 + 405810405\,x^5 - 6087156075\,x^3$$

$$Q_{14}(x) = x^{14} - 169\,x^{12} + 20449\,x^{10} - 1656369\,x^8 + 81162081\,x^6 - 2029052025\,x^4 + 18261468225\,x^2$$

$$R_{14}(x) = x^{13} - 455\,x^{11} + 82225\,x^9 - 8124831\,x^7 + 423504081\,x^5 - 9599044455\,x^3 - 18261468225\,x$$

$$S_{14}(x) = -26\,x^{12} + 6864\,x^{10} - 924066\,x^8 + 68468400\,x^6 - 2523330810\,x^4 + 34884289440\,x^2$$

$$p_{14} = -18261468225 \ , \quad q_{14} = -71407225890$$

$$P_{14}^*(x) = -13\,x^{13} - 1859\,x^{11} - 184041\,x^9 - 11594583\,x^7 - 405810405\,x^5 - 6087156075\,x^3$$

$$Q_{14}^*(x) = x^{14} + 169\,x^{12} + 20449\,x^{10} + 1656369\,x^8 + 81162081\,x^6 + 2029052025\,x^4 + 18261468225\,x^2$$

$$R_{14}^*(x) = -x^{13} - 455\,x^{11} - 82225\,x^9 - 8124831\,x^7 - 423504081\,x^5 - 9599044455\,x^3 + 18261468225\,x$$

$$S_{14}^*(x) = 26\,x^{12} + 6864\,x^{10} + 924066\,x^8 + 68468400\,x^6 + 2523330810\,x^4 + 34884289440\,x^2$$

$$p_{14}^* = 18261468225 \ , \quad q_{14}^* = 71407225890$$

$$P_{16}(x) = 15\,x^{15} - 2925\,x^{13} + 418275\,x^{11} - 41409225\,x^9 + 2608781175\,x^7 - 91307341125\,x^5 + 1369610116875\,x^3$$

$$Q_{16}(x) = x^{16} - 225\,x^{14} + 38025\,x^{12} - 4601025\,x^{10} + 372683025\,x^8 - 18261468225\,x^6 +$$
$$+ 456536705625\,x^4 - 4108830350625\,x^2$$

$$R_{16}(x) = x^{15} - 615\,x^{13} + 158145\,x^{11} - 24021855\,x^9 + 2175924465\,x^7 - 107462730375\,x^5 +$$
$$+ 2342399684625\,x^3 + 4108830350625\,x$$

$$S_{16}(x) = -30\,x^{14} + 10920\,x^{12} - 2157870\,x^{10} + 257605920\,x^8 - 17840252430\,x^6 +$$
$$+ 628620993000\,x^4 - 8396809170750\,x^2$$

$$p_{16} = 4108830350625 \ , \quad q_{16} = 16614469872000$$

$$P_{16}^*(x) = -15\,x^{15} - 2925\,x^{13} - 418275\,x^{11} - 41409225\,x^9 - 2608781175\,x^7 - 91307341125\,x^5 - 1369610116875\,x^3$$

$$Q_{16}^*(x) = x^{16} + 225\,x^{14} + 38025\,x^{12} + 4601025\,x^{10} + 372683025\,x^8 + 18261468225\,x^6 +$$
$$+ 456536705625\,x^4 + 4108830350625\,x^2$$

$$R_{16}^*(x) = -x^{15} - 615\,x^{13} - 158145\,x^{11} - 24021855\,x^9 - 2175924465\,x^7 - 107462730375\,x^5 -$$
$$- 2342399684625\,x^3 + 4108830350625\,x$$

$$S_{16}^*(x) = 30\,x^{14} + 10920\,x^{12} + 2157870\,x^{10} + 257605920\,x^8 + 17840252430\,x^6 +$$
$$+ 628620993000\,x^4 + 8396809170750\,x^2$$

$$p_{16}^* = 4108830350625\,, \quad q_{16}^* = 16614469872000$$

$$P_{18}(x) = 17\,x^{17} - 4335\,x^{15} + 845325\,x^{13} - 120881475\,x^{11} + 11967266025\,x^9 - 753937759575\,x^7 +$$
$$+ 26387821585125\,x^5 - 395817323776875\,x^3$$

$$Q_{18}(x) = x^{18} - 289\,x^{16} + 65025\,x^{14} - 10989225\,x^{12} + 1329696225\,x^{10} - 107705394225\,x^8 +$$
$$+ 5277564317025\,x^6 - 131939107925625\,x^4 + 1187451971330625\,x^2$$

$$R_{18}(x) = x^{17} - 799\,x^{15} + 277185\,x^{13} - 59925255\,x^{11} + 8350229745\,x^9 - 717540730335\,x^7 +$$
$$+ 34161178676625\,x^5 - 723520252830375\,x^3 - 1187451971330625\,x$$

$$S_{18}(x) = -34\,x^{16} + 16320\,x^{14} - 4448730\,x^{12} + 780059280\,x^{10} - 87119333730\,x^8 + 5776722871920\,x^6 -$$
$$- 197193714968250\,x^4 + 2566378082268000\,x^2$$

$$p_{18} = -1187451971330625\,, \quad q_{18} = -4941282024929250$$

$$P_{18}^*(x) = -17\,x^{17} - 4335\,x^{15} - 845325\,x^{13} - 120881475\,x^{11} - 11967266025\,x^9 - 753937759575\,x^7 -$$
$$- 26387821585125\,x^5 - 395817323776875\,x^3$$

$$Q_{18}^*(x) = x^{18} + 289\,x^{16} + 65025\,x^{14} + 10989225\,x^{12} + 1329696225\,x^{10} + 107705394225\,x^8 +$$
$$+ 5277564317025\,x^6 + 131939107925625\,x^4 + 1187451971330625\,x^2$$

$$R_{18}^*(x) = -x^{17} - 799\,x^{15} - 277185\,x^{13} - 59925255\,x^{11} - 8350229745\,x^9 - 717540730335\,x^7 -$$
$$- 34161178676625\,x^5 - 723520252830375\,x^3 + 1187451971330625\,x$$

$$S_{18}^*(x) = 34\,x^{16} + 16320\,x^{14} + 4448730\,x^{12} + 780059280\,x^{10} + 87119333730\,x^8 + 5776722871920\,x^6 +$$
$$+ 197193714968250\,x^4 + 2566378082268000\,x^2$$

$$p_{18}^* = 1187451971330625\,, \quad q_{18}^* = 4941282024929250$$

Recurrence formulas:

$$\int x^{2n+2} \cdot \ln x \cdot J_0(x)\,dx = x^{2n+1} \ln x[(2n+1)J_0(x) + xJ_1(x)] -$$

$$-(2n+1)\int x^{2n} J_0(x)\,dx - \int x^{2n+1} J_1(x)\,dx - (2n+1)^2 \int x^{2n} \cdot \ln x \cdot J_0(x)\,dx$$

$$\int x^{2n+2} \cdot \ln x \cdot I_0(x)\,dx = -x^{2n+1} \ln x[(2n+1)I_0(x) - xI_1(x)] +$$

$$+(2n+1)\int x^{2n} I_0(x)\,dx - \int x^{2n+1} I_1(x)\,dx + (2n+1)^2 \int x^{2n} \cdot \ln x \cdot I_0(x)\,dx$$

The integrals of the type $\int x^m Z_\nu(x)\,dx$ are described before.

d) Integrals of $x^{2n+1} \ln x \cdot Z_1(x)$:

$$\int x \ln x \cdot J_1(x)\, dx = \Phi(x) \cdot \ln x + x\, J_0(x) - \Lambda_1(x) + \Phi(x)$$

$$\int x \ln x \cdot I_1(x)\, dx = -\Psi(x) \cdot \ln x - x\, I_0(x) + \Lambda_1^*(x) - \Psi(x)$$

$$\int x^3 \ln x \cdot J_1(x)\, dx = [-x^3 J_0(x) + 3x^2 J_1(x) - 3\,\Phi(x)]\ln x - 3x\, J_0(x) + x^2 J_1(x) + 3\,\Lambda_1(x) - 7\,\Phi(x)$$

$$\int x^3 \ln x \cdot I_1(x)\, dx = [x^3 I_0(x) - 3x^2 I_1(x) - 3\,\Psi(x)]\ln x - 3x\, I_0(x) - x^2 I_1(x) + 3\,\Lambda_1^*(x) - 7\,\Psi(x)$$

Let

$$\int x^n \ln x \cdot J_0(x)\, dx =$$

$$= [P_n(x)\, J_0(x) + Q_n(x)\, J_1(x) + p_n\, \Phi(x)] \cdot \ln x + R_n(x)\, J_0(x) + S_n(x)\, J_1(x) - p_n\, \Lambda_1(x) + q_n\, \Phi(x)$$

and

$$\int x^n \ln x \cdot I_0(x)\, dx =$$

$$= [P_n^*(x)\, I_0(x) + Q_n^*(x)\, I_1(x) + p_n^*\, \Psi(x)] \cdot \ln x + R_n^*(x)\, I_0(x) + S_n^*(x)\, I_1(x) - p_n^*\, \Lambda_1^*(x) + q_n^*\, \Psi(x)\,,$$

then holds

$$P_5(x) = -x^5 + 15\, x^3\,, \quad Q_5(x) = 5\, x^4 - 45\, x^2\,, \quad R_5(x) = 8\, x^3 + 45\, x\,, \quad S_5(x) = x^4 - 39\, x^2\,,$$
$$p_5 = 45\,, \quad q_5 = 129$$
$$P_5^*(x) = x^5 + 15\, x^3\,, \quad Q_5^*(x) = -5\, x^4 - 45\, x^2\,, \quad R_5^*(x) = 8\, x^3 - 45\, x\,, \quad S_5^*(x) = -x^4 - 39\, x^2$$
$$p_5^* = -45\,, \quad q_5^* = -129$$

$$P_7(x) = -x^7 + 35\, x^5 - 525\, x^3\,, \quad Q_7(x) = 7\, x^6 - 175\, x^4 + 1575\, x^2$$
$$R_7(x) = 12\, x^5 - 460\, x^3 - 1575\, x\,, \quad S_7(x) = x^6 - 95\, x^4 + 1905\, x^2\,, \quad p_7 = -1575\,, \quad q_7 = -5055$$
$$P_7^*(x) = x^7 + 35\, x^5 + 525\, x^3\,, \quad Q_7^*(x) = -7\, x^6 - 175\, x^4 - 1575\, x^2$$
$$R_7^*(x) = 12\, x^5 + 460\, x^3 - 1575\, x\,, \quad S_7^*(x) = -x^6 - 95\, x^4 - 1905\, x^2\,, \quad p_7^* = -1575\,, \quad q_7^* - 5055$$

$$P_9(x) = -x^9 + 63\, x^7 - 2205\, x^5 + 33075\, x^3\,, \quad Q_9(x) = 9\, x^8 - 441\, x^6 + 11025\, x^4 - 99225\, x^2$$
$$R_9(x) = 16\, x^7 - 1316\, x^5 + 37380\, x^3 + 99225\, x\,, \quad S_9(x) = x^8 - 175\, x^6 + 8785\, x^4 - 145215\, x^2$$
$$p_9 = 99225\,, \quad q_9 = 343665$$
$$P_9^*(x) = x^9 + 63\, x^7 + 2205\, x^5 + 33075\, x^3\,, \quad Q_9^*(x) = -9\, x^8 - 441\, x^6 - 11025\, x^4 - 99225\, x^2$$
$$R_9^*(x) = 16\, x^7 + 1316\, x^5 + 37380\, x^3 - 99225\, x\,, \quad S_9^*(x) = -x^8 - 175\, x^6 - 8785\, x^4 - 145215\, x^2$$
$$p_9^* = -99225\,, \quad q_9^* = -343665$$

$$P_{11}(x) = -x^{11} + 99\, x^9 - 6237\, x^7 + 218295\, x^5 - 3274425\, x^3$$
$$Q_{11}(x) = 11\, x^{10} - 891\, x^8 + 43659\, x^6 - 1091475\, x^4 + 9823275\, x^2$$
$$R_{11}(x) = 20\, x^9 - 2844\, x^7 + 174384\, x^5 - 4362120\, x^3 - 9823275\, x$$
$$S_{11}(x) = x^{10} - 279\, x^8 + 26145\, x^6 - 1090215\, x^4 + 16360785\, x^2$$
$$p_{11} = -9823275\,, \quad q_{11} = -36007335$$

$$P_{11}^*(x) = x^{11} + 99\,x^9 + 6237\,x^7 + 218295\,x^5 + 3274425\,x^3$$

$$Q_{11}^*(x) = -11\,x^{10} - 891\,x^8 - 43659\,x^6 - 1091475\,x^4 - 9823275\,x^2$$

$$R_{11}^*(x) = 20\,x^9 + 2844\,x^7 + 174384\,x^5 + 4362120\,x^3 - 9823275\,x$$

$$R_{11}^*(x) = -x^{10} - 279\,x^8 - 26145\,x^6 - 1090215\,x^4 - 16360785\,x^2$$

$$p_{11}^* = -9823275\,, \quad q_{11}^* = -36007335$$

$$P_{13}(x) = -x^{13} + 143\,x^{11} - 14157\,x^9 + 891891\,x^7 - 31216185\,x^5 + 468242775\,x^3$$

$$Q_{13}(x) = 13\,x^{12} - 1573\,x^{10} + 127413\,x^8 - 6243237\,x^6 + 156080925\,x^4 - 1404728325\,x^2$$

$$R_{13}(x) = 24\,x^{11} - 5236\,x^9 + 556380\,x^7 - 30175992\,x^5 + 702369360\,x^3 + 1404728325\,x$$

$$S_{13}(x) = x^{12} - 407\,x^{10} + 61281\,x^8 - 4786551\,x^6 + 182096145\,x^4 - 2575350855\,x^2$$

$$p_{13} = 1404728325\,, \quad q_{13} = 5384807505$$

$$P_{13}^*(x) = x^{13} + 143\,x^{11} + 14157\,x^9 + 891891\,x^7 + 31216185\,x^5 + 468242775\,x^3$$

$$R_{13}^*(x) = -13\,x^{12} - 1573\,x^{10} - 127413\,x^8 - 6243237\,x^6 - 156080925\,x^4 - 1404728325\,x^2$$

$$R_{13}^*(x) = 24\,x^{11} + 5236\,x^9 + 556380\,x^7 + 30175992\,x^5 + 702369360\,x^3 - 1404728325\,x$$

$$S_{13}^*(x) = -x^{12} - 407\,x^{10} - 61281\,x^8 - 4786551\,x^6 - 182096145\,x^4 - 2575350855\,x^2$$

$$p_{13}^* = -1404728325\,, \quad q_{13}^* = -5384807505$$

$$P_{15}(x) = -x^{15} + 195\,x^{13} - 27885\,x^{11} + 2760615\,x^9 - 173918745\,x^7 + \\ + 6087156075\,x^5 - 91307341125\,x^3$$

$$Q_{15}(x) = 15\,x^{14} - 2535\,x^{12} + 306735\,x^{10} - 24845535\,x^8 + 1217431215\,x^6 - \\ - 30435780375\,x^4 + 273922023375\,x^2$$

$$R_{15}(x) = 28\,x^{13} - 8684\,x^{11} + 1417416\,x^9 - 133467048\,x^7 + 6758371620\,x^5 - \\ - 150072822900\,x^3 - 273922023375\,x$$

$$S_{15}(x) = x^{14} - 559\,x^{12} + 123409\,x^{10} - 15517359\,x^8 + 1108188081\,x^6 - \\ - 39879014175\,x^4 + 541525809825\,x^2$$

$$p_{15} = -273922023375\,, \quad q_{15} = -1089369856575$$

$$P_{15}^*(x) = x^{15} + 195\,x^{13} + 27885\,x^{11} + 2760615\,x^9 + 173918745\,x^7 + \\ + 6087156075\,x^5 + 91307341125\,x^3$$

$$Q_{15}^*(x) = -15\,x^{14} - 2535\,x^{12} - 306735\,x^{10} - 24845535\,x^8 - 1217431215\,x^6 - \\ - 30435780375\,x^4 - 273922023375\,x^2$$

$$R_{15}^*(x) = 28\,x^{13} + 8684\,x^{11} + 1417416\,x^9 + 133467048\,x^7 + 6758371620\,x^5 + \\ + 150072822900\,x^3 - 273922023375\,x$$

$$S_{15}^*(x) = -x^{14} - 559\,x^{12} - 123409\,x^{10} - 15517359\,x^8 - 1108188081\,x^6 - \\ - 39879014175\,x^4 - 541525809825\,x^2$$

$$p_{15}^* = -273922023375\,, \quad q_{15}^* = -1089369856575$$

$$P_{17}(x) = -x^{17} + 255\,x^{15} - 49725\,x^{13} + 7110675\,x^{11} - 703956825\,x^9 + 44349279975\,x^7 - \\ - 1552224799125\,x^5 + 23283371986875\,x^3$$

$$Q_{17}(x) = 17\,x^{16} - 3825\,x^{14} + 646425\,x^{12} - 78217425\,x^{10} + 6335611425\,x^8 - 310444959825\,x^6 +$$
$$+ 7761123995625\,x^4 - 69850115960625\,x^2$$

$$R_{17}(x) = 32\,x^{15} - 13380\,x^{13} + 3106740\,x^{11} - 449780760\,x^9 + 39599497080\,x^7 - 1918173757500\,x^5 +$$
$$+ 41190404755500\,x^3 + 69850115960625\,x$$

$$S_{17}(x) = x^{16} - 735\,x^{14} + 223665\,x^{12} - 41284815\,x^{10} + 4751983665\,x^8 - 321545759535\,x^6 +$$
$$+ 11143093586625\,x^4 - 146854586253375\,x^2$$

$$p_{17} = 69850115960625\,, \quad q_{17} = 286554818174625$$

$$P_{17}^*(x) = x^{17} + 255\,x^{15} + 49725\,x^{13} + 7110675\,x^{11} + 703956825\,x^9 + 44349279975\,x^7 + 1552224799125\,x^5 +$$
$$+ 23283371986875\,x^3$$

$$Q_{17}^*(x) = -17\,x^{16} - 3825\,x^{14} - 646425\,x^{12} - 78217425\,x^{10} - 6335611425\,x^8 - 310444959825\,x^6 -$$
$$- 7761123995625\,x^4 - 69850115960625\,x^2$$

$$R_{17}^*(x) = 32\,x^{15} + 13380\,x^{13} + 3106740\,x^{11} + 449780760\,x^9 + 39599497080\,x^7 + 1918173757500\,x^5 +$$
$$+ 41190404755500\,x^3 - 69850115960625\,x$$

$$S_{17}^*(x) = -x^{16} - 735\,x^{14} - 223665\,x^{12} - 41284815\,x^{10} - 4751983665\,x^8 - 321545759535\,x^6 -$$
$$- 11143093586625\,x^4 - 146854586253375\,x^2$$

$$p_{17}^* = -69850115960625\,, \quad q_{17}^* = -286554818174625$$

$$P_{19}(x) = -x^{19} + 323\,x^{17} - 82365\,x^{15} + 16061175\,x^{13} - 2296748025\,x^{11} + 227378054475\,x^9 -$$
$$- 14324817431925\,x^7 + 501368610117375\,x^5 - 7520529151760625\,x^3$$

$$Q_{19}(x) = 19\,x^{18} - 5491\,x^{16} + 1235475\,x^{14} - 208795275\,x^{12} + 25264228275\,x^{10} - 2046402490275\,x^8 +$$
$$+ 100273722023475\,x^6 - 2506843050586875\,x^4 + 22561587455281875\,x^2$$

$$R_{19}(x) = 36\,x^{17} - 19516\,x^{15} + 6111840\,x^{13} - 1259461320\,x^{11} + 170621631180\,x^9 - 14387211635940\,x^7 +$$
$$+ 675450216441000\,x^5 - 14142702127554000\,x^3 - 22561587455281875\,x$$

$$S_{19}(x) = x^{18} - 935\,x^{16} + 375105\,x^{14} - 95515095\,x^{12} + 16150822545\,x^{10} - 1762972735095\,x^8 +$$
$$+ 115035298883505\,x^6 - 3878619692322375\,x^4 + 49948635534422625\,x^2$$

$$p_{19} = -22561587455281875\,, \quad q_{19} = -95071810444986375$$

$$P_{19}^*(x) = x^{19} + 323\,x^{17} + 82365\,x^{15} + 16061175\,x^{13} + 2296748025\,x^{11} + 227378054475\,x^9 +$$
$$+ 14324817431925\,x^7 + 501368610117375\,x^5 + 7520529151760625\,x^3$$

$$Q_{19}^*(x) = -19\,x^{18} - 5491\,x^{16} - 1235475\,x^{14} - 208795275\,x^{12} - 25264228275\,x^{10} - 2046402490275\,x^8 -$$
$$- 100273722023475\,x^6 - 2506843050586875\,x^4 - 22561587455281875\,x^2$$

$$R_{19}^*(x) = 36\,x^{17} + 19516\,x^{15} + 6111840\,x^{13} + 1259461320\,x^{11} + 170621631180\,x^9 + 14387211635940\,x^7 +$$
$$+ 675450216441000\,x^5 + 14142702127554000\,x^3 - 22561587455281875\,x$$

$$S_{19}^*(x) = -x^{18} - 935\,x^{16} - 375105\,x^{14} - 95515095\,x^{12} - 16150822545\,x^{10} - 1762972735095\,x^8 -$$
$$- 115035298883505\,x^6 - 3878619692322375\,x^4 - 49948635534422625\,x^2$$

$$p_{19}^* = -22561587455281875\,, \quad q_{19}^* = -95071810444986375$$

Recurrence formulas:

$$\int x^{2n+1} \cdot \ln x \cdot J_1(x)\,dx = x^{2n} \ln x \left[(2n+1)J_1(x) - xJ_0(x)\right] +$$

$$+ \int x^{2n} J_0(x)\,dx - (2n+1) \int x^{2n-1} J_1(x)\,dx - (4n^2 - 1) \int x^{2n-1} \cdot \ln x \cdot J_1(x)\,dx$$

$$\int x^{2n+1} \cdot \ln x \cdot I_1(x)\,dx = -x^{2n} \ln x \left[xI_0(x) - (2n+1)I_1(x)\right] -$$

$$- \int x^{2n} I_0(x)\,dx + (2n+1) \int x^{2n-1} I_1(x)\,dx + (4n^2 - 1) \int x^{2n-1} \cdot \ln x \cdot I_1(x)\,dx$$

The integrals of the type $\int x^m Z_\nu x(x)\,dx$ are described before.

1.2.12. Integrals of the type $\int x^n e^{\pm x} \ln x \cdot Z_\nu(x)\, dx$

n = 0:

$$\int e^x \ln x\, I_0(x)\, dx = e^x \left\{ (1 - 2x)\, I_0(x) + 2x\, I_1(x) + \ln x\, [x\, I_0(x) - x\, I_1(x)] \right\}$$

$$\int e^{-x} \ln x\, I_0(x)\, dx = e^{-x} \left\{ -(1 + 2x)\, I_0(x) - 2x\, I_1(x) + \ln x\, [x\, I_0(x) + x\, I_1(x)] \right\}$$

$$\int e^x \ln x\, K_0(x)\, dx = e^x \left\{ (1 - 2x)\, K_0(x) - 2x\, K_1(x) + \ln x\, [x\, K_0(x) + x\, K_1(x)] \right\}$$

$$\int e^{-x} \ln x\, K_0(x)\, dx = e^{-x} \left\{ -(1 + 2x)\, K_0(x) + 2x\, K_1(x) + \ln x\, [x\, K_0(x) - x\, K_1(x)] \right\}$$

n = 1:

$$\int x\, e^x \ln x\, I_0(x)\, dx =$$

$$= \frac{e^x}{9} \left\{ (-2x^2 + 3x - 3)\, I_0(x) + (2x^2 - 2x)\, I_1(x) + \ln x\, [3x^2\, I_0(x) - (3x^2 - 3x)\, I_1(x)] \right\}$$

$$\int x\, e^{-x} \ln x\, I_0(x)\, dx =$$

$$= \frac{e^{-x}}{9} \left\{ -(2x^2 + 3x + 3)\, I_0(x) - (2x^2 + 2x)\, I_1(x) + \ln x\, [3x^2\, I_0(x) + (3x^2 + 3x)\, I_1(x)] \right\}$$

$$\int x\, e^x \ln x\, K_0(x)\, dx =$$

$$= \frac{e^x}{9} \left\{ (-2x^2 + 3x - 3)\, K_0(x) + (-2x^2 + 2x)\, K_1(x) + \ln x\, [3x^2\, K_0(x) + (3x^2 - 3x)\, K_1(x)] \right\}$$

$$\int x\, e^{-x} \ln x\, K_0(x)\, dx =$$

$$= \frac{e^{-x}}{9} \left\{ -(2x^2 + 3x + 3)\, K_0(x) + (2x^2 + 2x)\, K_1(x) + \ln x\, [3x^2\, K_0(x) - (3x^2 + 3x)\, K_1(x)] \right\}$$

$$\int x\, e^x \ln x\, I_1(x)\, dx =$$

$$= \frac{e^x}{9} \left\{ (2x^2 + 6x - 6)\, I_0(x) - (2x^2 + 7x)\, I_1(x) + \ln x\, [-3x^2\, I_0(x) + (3x^2 + 6x)\, I_1(x)] \right\}$$

$$\int x\, e^{-x} \ln x\, I_1(x)\, dx =$$

$$= \frac{e^{-x}}{9} \left\{ (-2x^2 + 6x + 6)\, I_0(x) - (2x^2 - 7x)\, I_1(x) + \ln x\, [3x^2\, I_0(x) + (3x^2 - 6x)\, I_1(x)] \right\}$$

$$\int x\, e^x \ln x\, K_1(x)\, dx =$$

$$= \frac{e^x}{9} \left\{ -(2x^2 + 6x - 6)\, K_0(x) - (2x^2 + 7x)\, K_1(x) + \ln x\, [3x^2\, K_0(x) + (3x^2 + 6x)\, K_1(x)] \right\}$$

$$\int x\, e^{-x} \ln x\, K_1(x)\, dx =$$

$$= \frac{e^{-x}}{9} \left\{ (2x^2 - 6x - 6)\, K_0(x) - (2x^2 - 7x)\, K_1(x) - \ln x\, [3x^2\, K_0(x) - (3x^2 - 3x)\, K_1(x)] \right\}$$

n = 2:

$$\int x^2 e^x \ln x\, I_0(x)\, dx = \frac{e^x}{225} \left\{ (-18x^3 + 13x^2 - 60x + 60)\, I_0(x) + (18x^3 - 4x^2 + 4x)\, I_1(x) + \right.$$

$$\left. + \ln x\, [(45x^3 + 30x^2)\, I_0(x) + (-45x^3 + 60x^2 - 60x)\, I_1(x)] \right\}$$

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$$\int x^2 e^{-x} \ln x\, I_0(x)\, dx = \frac{e^{-x}}{225}\left\{-(18\,x^3 + 13\,x^2 + 60\,x + 60)\,I_0(x) - (18\,x^3 + 4\,x^2 + 4\,x)\,I_1(x)+\right.$$
$$\left.+ \ln x\,[(45\,x^3 - 30\,x^2)\,I_0(x) + (45\,x^3 + 60\,x^2 + 60\,x)\,I_1(x)]\right\}$$

$$\int x^2 e^{x} \ln x\, K_0(x)\, dx = \frac{e^{x}}{225}\left\{(-18\,x^3 + 13\,x^2 - 60\,x + 60)\,K_0(x) + (-18\,x^3 + 4\,x^2 - 4\,x)\,K_1(x)+\right.$$
$$\left.+ \ln x\,[(45\,x^3 + 30\,x^2)\,K_0(x) + (45\,x^3 - 60\,x^2 + 60\,x)\,K_1(x)]\right\}$$

$$\int x^2 e^{-x} \ln x\, K_0(x)\, dx = \frac{e^{-x}}{225}\left\{-(18\,x^3 + 13\,x^2 + 60\,x + 60)\,K_0(x) + (18\,x^3 + 4\,x^2 + 4\,x)\,K_1(x)+\right.$$
$$\left.+ \ln x\,[(45\,x^3 - 30\,x^2)\,K_0(x) - (45\,x^3 + 60\,x^2 + 60\,x)\,K_1(x)]\right\}$$

$$\int x^2 e^{x} \ln x\, I_1(x)\, dx = \frac{e^{x}}{75}\left\{(6\,x^3 + 4\,x^2 - 30\,x + 30)\,I_0(x) + (-6\,x^3 - 7\,x^2 + 7\,x)\,I_1(x)+\right.$$
$$\left.+ \ln x\,[(-15\,x^3 + 15\,x^2)\,I_0(x) + (15\,x^3 + 30\,x^2 - 30\,x)\,I_1(x)]\right\}$$

$$\int x^2 e^{-x} \ln x\, I_1(x)\, dx = \frac{e^{-x}}{75}\left\{(-6\,x^3 + 4\,x^2 + 30\,x + 30)\,I_0(x) + (-6\,x^3 + 7\,x^2 + 7\,x)\,I_1(x)+\right.$$
$$\left.+ \ln x\,[(15\,x^3 + 15\,x^2)\,I_0(x) + (15\,x^3 - 30\,x^2 - 30\,x)\,I_1(x)]\right\}$$

$$\int x^2 e^{x} \ln x\, K_1(x)\, dx = \frac{e^{x}}{75}\left\{(-6\,x^3 - 4\,x^2 + 30\,x - 30)\,K_0(x) + (-6\,x^3 - 7\,x^2 + 7\,x)\,K_1(x)+\right.$$
$$\left.+ \ln x\,[(15\,x^3 - 15\,x^2)\,K_0(x) + (15\,x^3 + 30\,x^2 - 30\,x)\,K_1(x)]\right\}$$

$$\int x^2 e^{-x} \ln x\, K_1(x)\, dx = \frac{e^{-x}}{75}\left\{(6\,x^3 - 4\,x^2 - 30\,x - 30)\,K_0(x) + (-6\,x^3 + 7\,x^2 + 7\,x)\,K_1(x)+\right.$$
$$\left.+ \ln x\,[(-15\,x^3 - 15\,x^2)\,K_0(x) + (15\,x^3 - 30\,x^2 - 30\,x)\,K_1(x)]\right\}$$

n = 3:

$$\int x^3 e^{x} \ln x\, I_0(x)\, dx =$$

$$= \frac{e^{x}}{1225}\left\{(-50\,x^4 + 31\,x^3 - 171\,x^2 + 420\,x - 420)\,I_0(x) + (50\,x^4 - 6\,x^3 - 132\,x^2 + 132\,x)\,I_1(x)+\right.$$
$$\left.+ \ln x\,[(175\,x^4 + 210\,x^3 - 210\,x^2)\,I_0(x) + (-175\,x^4 + 315\,x^3 - 420\,x^2 + 420\,x)\,I_1(x)]\right\}$$

$$\int x^3 e^{-x} \ln x\, I_0(x)\, dx =$$

$$= \frac{e^{-x}}{1225}\left\{-(50\,x^4 + 31\,x^3 + 171\,x^2 + 420\,x + 420)\,I_0(x) - (50\,x^4 + 6\,x^3 - 132\,x^2 - 132\,x)\,I_1(x)+\right.$$
$$\left.+ \ln x\,[(175\,x^4 - 210\,x^3 - 210\,x^2)\,I_0(x) + (175\,x^4 + 315\,x^3 + 420\,x^2 + 420\,x)\,I_1(x)]\right\}$$

$$\int x^3 e^{x} \ln x\, K_0(x)\, dx =$$

$$= \frac{e^{x}}{1225}\left\{(-50\,x^4 + 31\,x^3 - 171\,x^2 + 420\,x - 420)\,K_0(x) + (-50\,x^4 + 6\,x^3 + 132\,x^2 - 132\,x)\,K_1(x)+\right.$$
$$\left.+ \ln x\,[(175\,x^4 + 210\,x^3 - 210\,x^2)\,K_0(x) + (175\,x^4 - 315\,x^3 + 420\,x^2 - 420\,x)\,K_1(x)]\right\}$$

$$\int x^3 e^{-x} \ln x\, K_0(x)\, dx =$$

$$= \frac{e^{-x}}{1225}\left\{-(50\,x^4 + 31\,x^3 + 171\,x^2 + 420\,x + 420)\,K_0(x) + (50\,x^4 + 6\,x^3 - 132\,x^2 - 132\,x)\,K_1(x)+\right.$$
$$\left.+ \ln x\,[(175\,x^4 - 210\,x^3 - 210\,x^2)\,K_0(x) - (175\,x^4 + 315\,x^3 + 420\,x^2 + 420\,x)\,K_1(x)]\right\}$$

$$\int x^3 e^{x} \ln x\, I_1(x)\, dx =$$

$$= \frac{e^{x}}{3675}\left\{(150\,x^4 + 54\,x^3 - 614\,x^2 + 1680\,x - 1680)\,I_0(x) + (-150\,x^4 - 129\,x^3 - 388\,x^2 + 388\,x)\,I_1(x)+\right.$$

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$$+ \ln x \left[(-525\,x^4 + 840\,x^3 - 840\,x^2)\,I_0(x) + (525\,x^4 + 1260\,x^3 - 1680\,x^2 + 1680\,x)\,I_1(x) \right] \}$$

$$\int x^3\, e^{-x}\, \ln x\, I_1(x)\, dx =$$

$$= \frac{e^{-x}}{3675} \left\{ (-150\,x^4 + 54\,x^3 + 614\,x^2 + 1680\,x + 1680)\,I_0(x) + (-150\,x^4 + 129\,x^3 - 388\,x^2 - 388\,x)\,I_1(x) + \right.$$

$$\left. + \ln x \left[(525\,x^4 + 840\,x^3 + 840\,x^2)\,I_0(x) + (525\,x^4 - 1260\,x^3 - 1680\,x^2 - 1680\,x)\,I_1(x) \right] \right\}$$

$$\int x^3\, e^{x}\, \ln x\, K_1(x)\, dx =$$

$$= \frac{e^{x}}{3675} \left\{ (-150\,x^4 - 54\,x^3 + 614\,x^2 - 1680\,x + 1680)\,K_0(x) + (-150\,x^4 - 129\,x^3 - 388\,x^2 + 388\,x)\,K_1(x) + \right.$$

$$\left. + \ln x \left[(525\,x^4 - 840\,x^3 + 840\,x^2)\,K_0(x) + (525\,x^4 + 1260\,x^3 - 1680\,x^2 + 1680\,x)\,K_1(x) \right] \right\}$$

$$\int x^3\, e^{-x}\, \ln x\, K_1(x)\, dx =$$

$$= \frac{e^{-x}}{3675} \left\{ (150\,x^4 - 54\,x^3 - 614\,x^2 - 1680\,x - 1680)\,K_0(x) + (-150\,x^4 + 129\,x^3 - 388\,x^2 - 388\,x)\,K_1(x) + \right.$$

$$\left. + \ln x \left[(-525\,x^4 - 840\,x^3 - 840\,x^2)\,K_0(x) + (525\,x^4 - 1260\,x^3 - 1680\,x^2 - 1680\,x)\,K_1(x) \right] \right\}$$

n = 4:

$$\int x^4\, e^{x}\, \ln x\, I_0(x)\, dx = \frac{e^{x}}{99225} \left\{ (-2450\,x^5 + 1425\,x^4 - 12864\,x^3 + 33024\,x^2 - 60480\,x + 60480)\,I_0(x) + \right.$$

$$+ (2450\,x^5 - 200\,x^4 - 11736\,x^3 + 35808\,x^2 - 35808\,x)\,I_1(x) + \ln x \left[(11025\,x^5 + 18900\,x^4 - 30240\,x^3 + 30240\,x^2)\,I_0(x) + \right.$$

$$\left. + (-11025\,x^5 + 25200\,x^4 - 45360\,x^3 + 60480\,x^2 - 60480\,x)\,I_1(x) \right] \}$$

$$\int x^4\, e^{-x}\, \ln x\, I_0(x)\, dx = \frac{e^{-x}}{99225} \left\{ -(2450\,x^5 + 1425\,x^4 + 12864\,x^3 + 33024\,x^2 + 60480\,x + 60480)\,I_0(x) - \right.$$

$$- (2450\,x^5 + 200\,x^4 - 11736\,x^3 - 35808\,x^2 - 35808\,x)\,I_1(x) + \ln x \left[(11025\,x^5 - 18900\,x^4 - 30240\,x^3 - 30240\,x^2)\,I_0(x) + \right.$$

$$\left. + (11025\,x^5 + 25200\,x^4 + 45360\,x^3 + 60480\,x^2 + 60480\,x)\,I_1(x) \right] \}$$

$$\int x^4\, e^{x}\, \ln x\, K_0(x)\, dx = \frac{e^{x}}{99225} \left\{ (-2450\,x^5 + 1425\,x^4 - 12864\,x^3 + 33024\,x^2 - 60480\,x + 60480)\,K_0(x) + \right.$$

$$+ (-2450\,x^5 + 200\,x^4 + 11736\,x^3 - 35808\,x^2 + 35808\,x)\,K_1(x) + \ln x \left[(11025\,x^5 + 18900\,x^4 - 30240\,x^3 + \right.$$

$$\left. + 30240\,x^2)\,K_0(x) + (11025\,x^5 - 25200\,x^4 + 45360\,x^3 - 60480\,x^2 + 60480\,x)\,K_1(x) \right] \}$$

$$\int x^4\, e^{-x}\, \ln x\, K_0(x)\, dx = \frac{e^{-x}}{99225} \left\{ -(2450\,x^5 + 1425\,x^4 + 12864\,x^3 + 33024\,x^2 + 60480\,x + 60480)\,K_0(x) + \right.$$

$$+ (2450\,x^5 + 200\,x^4 - 11736\,x^3 - 35808\,x^2 - 35808\,x)\,K_1(x) + \ln x \left[(11025\,x^5 - 18900\,x^4 - 30240\,x^3 - \right.$$

$$\left. - 30240\,x^2)\,K_0(x) - (11025\,x^5 + 25200\,x^4 + 45360\,x^3 + 60480\,x^2 + 60480\,x)\,K_1(x) \right] \}$$

$$\int x^4\, e^{x}\, \ln x\, I_1(x)\, dx = \frac{e^{x}}{19845} \left\{ (490\,x^5 + 120\,x^4 - 2838\,x^3 + 7878\,x^2 - 15120\,x + 15120)\,I_0(x) + \right.$$

$$+ (-490\,x^5 - 365\,x^4 - 2367\,x^3 + 8196\,x^2 - 8196\,x)\,I_1(x) + \ln x \left[(-2205\,x^5 + 4725\,x^4 - 7560\,x^3 + 7560\,x^2)\,I_0(x) + \right.$$

$$\left. + (2205\,x^5 + 6300\,x^4 - 11340\,x^3 + 15120\,x^2 - 15120\,x)\,I_1(x) \right] \}$$

$$\int x^4\, e^{-x}\, \ln x\, I_1(x)\, dx = \frac{e^{-x}}{19845} \left\{ (-490\,x^5 + 120\,x^4 + 2838\,x^3 + 7878\,x^2 + 15120\,x + 15120)\,I_0(x) + \right.$$

$$+ (-490\,x^5 + 365\,x^4 - 2367\,x^3 - 8196\,x^2 - 8196\,x)\,I_1(x) + \ln x \left[(2205\,x^5 + 4725\,x^4 + 7560\,x^3 + 7560\,x^2)\,I_0(x) + \right.$$

$$\left. + (2205\,x^5 - 6300\,x^4 - 11340\,x^3 - 15120\,x^2 - 15120\,x)\,I_1(x) \right] \}$$

$$\int x^4\, e^{x}\, \ln x\, K_1(x)\, dx = \frac{e^{x}}{19845} \left\{ (-490\,x^5 - 120\,x^4 + 2838\,x^3 - 7878\,x^2 + 15120\,x - 15120)\,K_0(x) + \right.$$

$$+(-490\,x^5-365\,x^4-2367\,x^3+8196\,x^2-8196\,x)\,K_1(x)+\ln x\,[(2205\,x^5-4725\,x^4+7560\,x^3-7560\,x^2)\,K_0(x)+$$
$$+(2205\,x^5+6300\,x^4-11340\,x^3+15120\,x^2-15120\,x)\,K_1(x)]\,\}$$

$$\int x^4\,e^{-x}\,\ln x\,K_1(x)\,dx=\frac{e^{-x}}{19845}\,\{(490\,x^5-120\,x^4-2838\,x^3-7878\,x^2-15120\,x-15120)\,K_0(x)+$$

$$+(-490\,x^5+365\,x^4-2367\,x^3-8196\,x^2-8196\,x)\,K_1(x)+\ln x\,[(-2205\,x^5-4725\,x^4-7560\,x^3-7560\,x^2)\,K_0(x)+$$
$$+(2205\,x^5-6300\,x^4-11340\,x^3-15120\,x^2-15120\,x)\,K_1(x)]\,\}$$

n = 5:

$$\int x^5\,e^x\,\ln x\,I_0(x)\,dx=$$

$$=\frac{e^x}{480249}\,\{(-7938\,x^6+4459\,x^5-61035\,x^4+214080\,x^3-435840\,x^2+665280\,x-665280)\,I_0(x)+$$

$$+(7938\,x^6-490\,x^5-58280\,x^4+237960\,x^3-539040\,x^2+539040\,x)\,I_1(x)+$$

$$+\ln x\,[(43659\,x^6+97020\,x^5-207900\,x^4+332640\,x^3-332640\,x^2)\,I_0(x)+$$

$$+(-43659\,x^6+121275\,x^5-277200\,x^4+498960\,x^3-665280\,x^2+665280\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,I_0(x)\,dx=$$

$$=\frac{e^{-x}}{480249}\,\{-(7938\,x^6+4459\,x^5+61035\,x^4+214080\,x^3+435840\,x^2+665280\,x+665280)\,I_0(x)-$$

$$-(7938\,x^6+490\,x^5-58280\,x^4-237960\,x^3-539040\,x^2-539040\,x)\,I_1(x)+$$

$$+\ln x\,[(43659\,x^6-97020\,x^5-207900\,x^4-332640\,x^3-332640\,x^2)\,I_0(x)+$$

$$-(43659\,x^6+121275\,x^5+277200\,x^4+498960\,x^3+665280\,x^2+665280\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^x\,\ln x\,K_0(x)\,dx=$$

$$=\frac{e^x}{480249}\,\{(-7938\,x^6+4459\,x^5-61035\,x^4+214080\,x^3-435840\,x^2+665280\,x-665280)\,K_0(x)+$$

$$+(-7938\,x^6+490\,x^5+58280\,x^4-237960\,x^3+539040\,x^2-539040\,x)\,K_1(x)+$$

$$+\ln x\,[(43659\,x^6+97020\,x^5-207900\,x^4+332640\,x^3-332640\,x^2)\,K_0(x)+$$

$$+(43659\,x^6-121275\,x^5+277200\,x^4-498960\,x^3+665280\,x^2-665280\,x)\,K_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,K_0(x)\,dx=$$

$$=\frac{e^{-x}}{480249}\,\{-(7938\,x^6+4459\,x^5+61035\,x^4+214080\,x^3+435840\,x^2+665280\,x+665280)\,K_0(x)+$$

$$+(7938\,x^6+490\,x^5-58280\,x^4-237960\,x^3-539040\,x^2-539040\,x)\,K_1(x)+$$

$$+\ln x\,[(43659\,x^6-97020\,x^5-207900\,x^4-332640\,x^3-332640\,x^2)\,K_0(x)-$$

$$-(43659\,x^6+121275\,x^5+277200\,x^4+498960\,x^3+665280\,x^2+665280\,x)\,K_1(x)]\,\}$$

$$\int x^5\,e^x\,\ln x\,I_1(x)\,dx=$$

$$=\frac{e^x}{800415}\,\{(13230\,x^6+2450\,x^5-108210\,x^4+405984\,x^3-849504\,x^2+1330560\,x-1330560)\,I_0(x)+$$

$$+(-13230\,x^6-9065\,x^5-98080\,x^4+442656\,x^3-1033728\,x^2+1033728\,x)\,I_1(x)+$$

$$+\ln x\,[(-72765\,x^6+194040\,x^5-415800\,x^4+665280\,x^3-665280\,x^2)\,I_0(x)+$$

$$+(72765\,x^6+242550\,x^5-554400\,x^4+997920\,x^3-1330560\,x^2+1330560\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,I_1(x)\,dx=$$

$$= \frac{e^{-x}}{800415} \left\{ (-13230\,x^6 + 2450\,x^5 + 108210\,x^4 + 405984\,x^3 + 849504\,x^2 + 1330560\,x + 1330560)\,I_0(x)+ \right.$$

$$+(-13230\,x^6 + 9065\,x^5 - 98080\,x^4 - 442656\,x^3 - 1033728\,x^2 - 1033728\,x)\,I_1(x)+$$

$$+ \ln x\,[(72765\,x^6 + 194040\,x^5 + 415800\,x^4 + 665280\,x^3 + 665280\,x^2)\,I_0(x)+$$

$$\left. +(72765\,x^6 - 242550\,x^5 - 554400\,x^4 - 997920\,x^3 - 1330560\,x^2 - 1330560\,x)\,I_1(x)] \right\}$$

$$\int x^5\, e^x\, \ln x\, K_1(x)\, dx =$$

$$= \frac{e^{x}}{800415} \left\{ (-13230\,x^6 - 2450\,x^5 + 108210\,x^4 - 405984\,x^3 + 849504\,x^2 - 1330560\,x + 1330560)\,K_0(x)+ \right.$$

$$+(-13230\,x^6 - 9065\,x^5 - 98080\,x^4 + 442656\,x^3 - 1033728\,x^2 + 1033728\,x)\,K_1(x)+$$

$$+ \ln x\,[(72765\,x^6 - 194040\,x^5 + 415800\,x^4 - 665280\,x^3 + 665280\,x^2)\,K_0(x)+$$

$$\left. +(72765\,x^6 + 242550\,x^5 - 554400\,x^4 + 997920\,x^3 - 1330560\,x^2 + 1330560\,x)\,K_1(x)] \right\}$$

$$\int x^5\, e^{-x}\, \ln x\, K_1(x)\, dx =$$

$$= \frac{e^{-x}}{800415} \left\{ (13230\,x^6 - 2450\,x^5 - 108210\,x^4 - 405984\,x^3 - 849504\,x^2 - 1330560\,x - 1330560)\,K_0(x)+ \right.$$

$$+(-13230\,x^6 + 9065\,x^5 - 98080\,x^4 - 442656\,x^3 - 1033728\,x^2 - 1033728\,x)\,K_1(x)+$$

$$+ \ln x\,[(-72765\,x^6 - 194040\,x^5 - 415800\,x^4 - 665280\,x^3 - 665280\,x^2)\,K_0(x)+$$

$$\left. +(72765\,x^6 - 242550\,x^5 - 554400\,x^4 - 997920\,x^3 - 1330560\,x^2 - 1330560\,x)\,K_1(x)] \right\}$$

n = 6:

$$\int x^6\, e^x\, \ln x\, I_0(x)\, dx =$$

$$= \frac{e^{x}}{9018009} \left\{ (-106722\,x^7 + 58653\,x^6 - 1137388\,x^5 + 5114220\,x^4 - 14236800\,x^3 + 25768320\,x^2 - 34594560\,x+ \right.$$

$$+34594560)\,I_0(x) + (106722\,x^7 - 5292\,x^6 - 1106420\,x^5 + 5617760\,x^4 - 17030880\,x^3 + 34239360\,x^2-$$

$$-34239360\,x)\,I_1(x) + \ln x\,[(693693\,x^7 + 1891890\,x^6 - 5045040\,x^5 + 10810800\,x^4 - 17297280\,x^3 + 17297280\,x^2)\,I_0(x)+$$

$$\left. +(-693693\,x^7 + 2270268\,x^6 - 6306300\,x^5 + 14414400\,x^4 - 25945920\,x^3 + 34594560\,x^2 - 34594560\,x)\,I_1(x)] \right\}$$

$$\int x^6\, e^{-x}\, \ln x\, I_0(x)\, dx =$$

$$= \frac{e^{-x}}{9018009} \left\{ -(106722\,x^7 + 58653\,x^6 + 1137388\,x^5 + 5114220\,x^4 + 14236800\,x^3 + 25768320\,x^2 + 34594560\,x+ \right.$$

$$+34594560)\,I_0(x) - (106722\,x^7 + 5292\,x^6 - 1106420\,x^5 - 5617760\,x^4 - 17030880\,x^3 - 34239360\,x^2-$$

$$-34239360\,x)\,I_1(x) + \ln x\,[(693693\,x^7 - 1891890\,x^6 - 5045040\,x^5 - 10810800\,x^4 - 17297280\,x^3 - 17297280\,x^2)\,I_0(x)+$$

$$\left. +(693693\,x^7 + 2270268\,x^6 + 6306300\,x^5 + 14414400\,x^4 + 25945920\,x^3 + 34594560\,x^2 + 34594560\,x)\,I_1(x)] \right\}$$

$$\int x^6\, e^x\, \ln x\, K_0(x)\, dx =$$

$$= \frac{e^{x}}{9018009} \left\{ (-106722\,x^7 + 58653\,x^6 - 1137388\,x^5 + 5114220\,x^4 - 14236800\,x^3 + 25768320\,x^2 - 34594560\,x+ \right.$$

$$+34594560)\,K_0(x) + (-106722\,x^7 + 5292\,x^6 + 1106420\,x^5 - 5617760\,x^4 + 17030880\,x^3 - 34239360\,x^2+$$

$$+34239360\,x)\,K_1(x) + \ln x\,[(693693\,x^7 + 1891890\,x^6 - 5045040\,x^5 + 10810800\,x^4 - 17297280\,x^3+$$

$$+17297280\,x^2)\,K_0(x) + (693693\,x^7 - 2270268\,x^6 + 6306300\,x^5 - 14414400\,x^4 + 25945920\,x^3 - 34594560\,x^2+$$

$$\left. +34594560\,x)\,K_1(x)] \right\}$$

$$\int x^6\, e^{-x}\, \ln x\, K_0(x)\, dx =$$

$$= \frac{e^{-x}}{9018009} \{-(106722\,x^7 + 58653\,x^6 + 1137388\,x^5 + 5114220\,x^4 + 14236800\,x^3 + 25768320\,x^2 + 34594560\,x +$$
$$+34594560)\,K_0(x) + (106722\,x^7 + 5292\,x^6 - 1106420\,x^5 - 5617760\,x^4 - 17030880\,x^3 - 34239360\,x^2 -$$
$$-34239360\,x)\,K_1(x) + \ln x\,[(693693\,x^7 - 1891890\,x^6 - 5045040\,x^5 - 10810800\,x^4 - 17297280\,x^3 -$$
$$-17297280\,x^2)\,K_0(x) - (693693\,x^7 + 2270268\,x^6 + 6306300\,x^5 + 14414400\,x^4 + 25945920\,x^3 + 34594560\,x^2 +$$
$$+34594560\,x)\,K_1(x)]\}$$

$$\int x^6\,e^x\,\ln x\,I_1(x)\,dx = \frac{e^x}{3864861}\{(45738\,x^7 + 6804\,x^6 - 508634\,x^5 + 2428410\,x^4 - 6912480\,x^3 + 12678240\,x^2 -$$
$$-17297280\,x + 17297280)\,I_0(x) + (-45738\,x^7 - 29673\,x^6 - 478135\,x^5 + 2637280\,x^4 - 8206560\,x^3 + 16707840\,x^2 -$$
$$-16707840\,x)\,I_1(x) + \ln x\,[(-297297\,x^7 + 945945\,x^6 - 2522520\,x^5 + 5405400\,x^4 - 8648640\,x^3 + 8648640\,x^2)\,I_0(x) +$$
$$+(297297\,x^7 + 1135134\,x^6 - 3153150\,x^5 + 7207200\,x^4 - 12972960\,x^3 + 17297280\,x^2 - 17297280\,x)\,I_1(x)]\}$$

$$\int x^6\,e^{-x}\,\ln x\,I_1(x)\,dx = \frac{e^{-x}}{3864861}\{(-45738\,x^7 + 6804\,x^6 + 508634\,x^5 + 2428410\,x^4 + 6912480\,x^3 + 12678240\,x^2 +$$
$$+17297280\,x + 17297280)\,I_0(x) + (-45738\,x^7 + 29673\,x^6 - 478135\,x^5 - 2637280\,x^4 - 8206560\,x^3 - 16707840\,x^2 -$$
$$-16707840\,x)\,I_1(x) + \ln x\,[(297297\,x^7 + 945945\,x^6 + 2522520\,x^5 + 5405400\,x^4 + 8648640\,x^3 + 8648640\,x^2)\,I_0(x) +$$
$$+(297297\,x^7 - 1135134\,x^6 - 3153150\,x^5 - 7207200\,x^4 - 12972960\,x^3 - 17297280\,x^2 - 17297280\,x)\,I_1(x)]\}$$

$$\int x^6\,e^x\,\ln x\,K_1(x)\,dx = \frac{e^x}{3864861}\{(-45738\,x^7 - 6804\,x^6 + 508634\,x^5 - 2428410\,x^4 + 6912480\,x^3 - 12678240\,x^2 +$$
$$+17297280\,x - 17297280)\,K_0(x) + (-45738\,x^7 - 29673\,x^6 - 478135\,x^5 + 2637280\,x^4 - 8206560\,x^3 + 16707840\,x^2 -$$
$$-16707840\,x)\,K_1(x) + \ln x\,[(297297\,x^7 - 945945\,x^6 + 2522520\,x^5 - 5405400\,x^4 + 8648640\,x^3 - 8648640\,x^2)\,K_0(x) +$$
$$+(297297\,x^7 + 1135134\,x^6 - 3153150\,x^5 + 7207200\,x^4 - 12972960\,x^3 + 17297280\,x^2 - 17297280\,x)\,K_1(x)]\}$$

$$\int x^6\,e^{-x}\,\ln x\,K_1(x)\,dx = \frac{e^{-x}}{3864861}\{(45738\,x^7 - 6804\,x^6 - 508634\,x^5 - 2428410\,x^4 - 6912480\,x^3 - 12678240\,x^2 -$$
$$-17297280\,x - 17297280)\,K_0(x) + (-45738\,x^7 + 29673\,x^6 - 478135\,x^5 - 2637280\,x^4 - 8206560\,x^3 - 16707840\,x^2 -$$
$$-16707840\,x)\,K_1(x) + \ln x\,[(-297297\,x^7 - 945945\,x^6 - 2522520\,x^5 - 5405400\,x^4 - 8648640\,x^3 - 8648640\,x^2)\,K_0(x) +$$
$$+(297297\,x^7 - 1135134\,x^6 - 3153150\,x^5 - 7207200\,x^4 - 12972960\,x^3 - 17297280\,x^2 - 17297280\,x)\,K_1(x)]\}$$

n = 7:

$$\int x^7\,e^x\,\ln x\,I_0(x)\,dx = \frac{e^x}{13803075}\{(-122694\,x^8 + 66429\,x^7 - 1734705\,x^6 + 9530780\,x^5 - 33807900\,x^4 +$$
$$+84362880\,x^3 - 142020480\,x^2 + 172972800\,x - 172972800)\,I_0(x) + (122694\,x^8 - 5082\,x^7 - 1703268\,x^6 +$$
$$+10336900\,x^5 - 39071200\,x^4 + 104922720\,x^3 - 197554560\,x^2 + 197554560)\,I_1(x) +$$
$$+\ln x\,[(920205\,x^8 + 2972970\,x^7 - 9459450\,x^6 + 25225200\,x^5 - 54054000\,x^4 + 86486400\,x^3 - 86486400\,x^2)\,I_0(x) +$$
$$+(-920205\,x^8 + 3468465\,x^7 - 11351340\,x^6 + 31531500\,x^5 - 72072000\,x^4 + 129729600\,x^3 - 172972800\,x^2 +$$
$$+172972800\,x)\,I_1(x)]\}$$

$$\int x^7\,e^{-x}\,\ln x\,I_0(x)\,dx = \frac{e^{-x}}{13803075}\{(-122694\,x^8 - 66429\,x^7 - 1734705\,x^6 - 9530780\,x^5 - 33807900\,x^4 -$$
$$-84362880\,x^3 - 142020480\,x^2 - 172972800\,x - 172972800)\,I_0(x) + (-122694\,x^8 - 5082\,x^7 + 1703268\,x^6 +$$
$$+10336900\,x^5 + 39071200\,x^4 + 104922720\,x^3 + 197554560\,x^2 + 197554560)\,I_1(x) +$$
$$+\ln x\,[(920205\,x^8 - 2972970\,x^7 - 9459450\,x^6 - 25225200\,x^5 - 54054000\,x^4 - 86486400\,x^3 - 86486400\,x^2)\,I_0(x) +$$
$$+(920205\,x^8 + 3468465\,x^7 + 11351340\,x^6 + 31531500\,x^5 + 72072000\,x^4 + 129729600\,x^3 + 172972800\,x^2 +$$
$$+172972800\,x)\,I_1(x)]\}$$

$$\int x^7\, e^x \ln x\, K_0(x)\, dx = \frac{e^x}{13803075}\left\{(-122694\, x^8 + 66429\, x^7 - 1734705\, x^6 + 9530780\, x^5 - 33807900\, x^4 + \right.$$
$$+84362880\, x^3 - 142020480\, x^2 + 172972800\, x - 172972800)\, K_0(x) + (-122694\, x^8 + 5082\, x^7 + 1703268\, x^6 -$$
$$-10336900\, x^5 + 39071200\, x^4 - 104922720\, x^3 + 197554560\, x^2 - 197554560\, x)\, K_1(x) +$$
$$+ \ln x\, [(920205\, x^8 + 2972970\, x^7 - 9459450\, x^6 + 25225200\, x^5 - 54054000\, x^4 + 86486400\, x^3 - 86486400\, x^2)\, K_0(x) +$$
$$+(920205\, x^8 - 3468465\, x^7 + 11351340\, x^6 - 31531500\, x^5 + 72072000\, x^4 - 129729600\, x^3 + 172972800\, x^2 -$$
$$\left. -172972800\, x)\, K_1(x)]\right\}$$

$$\int x^7\, e^{-x} \ln x\, K_0(x)\, dx = \frac{e^{-x}}{13803075}\left\{-(122694\, x^8 + 66429\, x^7 + 1734705\, x^6 + 9530780\, x^5 + 33807900\, x^4 + \right.$$
$$+84362880\, x^3 + 142020480\, x^2 + 172972800\, x + 172972800)\, K_0(x) + (122694\, x^8 + 5082\, x^7 - 1703268\, x^6 -$$
$$-10336900\, x^5 - 39071200\, x^4 - 104922720\, x^3 - 197554560\, x^2 - 197554560\, x)\, K_1(x) +$$
$$+ \ln x\, [(920205\, x^8 - 2972970\, x^7 - 9459450\, x^6 - 25225200\, x^5 - 54054000\, x^4 - 86486400\, x^3 - 86486400\, x^2)\, K_0(x) -$$
$$-(920205\, x^8 + 3468465\, x^7 + 11351340\, x^6 + 31531500\, x^5 + 72072000\, x^4 + 129729600\, x^3 + 172972800\, x^2 +$$
$$\left. +172972800\, x)\, K_1(x)]\right\}$$

$$\int x^7\, e^x \ln x\, I_1(x)\, dx = \frac{e^x}{96621525}\left\{(858858\, x^8 + 106722\, x^7 - 12526290\, x^6 + 72642640\, x^5 - 262741200\, x^4 + \right.$$
$$+662547840\, x^3 - 1123808640\, x^2 + 1383782400\, x - 1383782400)\, I_0(x) + (-858858\, x^8 - 536151\, x^7 - 12004524\, x^6 +$$
$$+78190700\, x^5 - 302273600\, x^4 + 820848960\, x^3 - 1555726080\, x^2 + 1555726080\, x)\, I_1(x) +$$
$$+ \ln x\, [(-6441435\, x^8 + 23783760\, x^7 - 75675600\, x^6 + 201801600\, x^5 - 432432000\, x^4 + 691891200\, x^3 -$$
$$-691891200\, x^2)\, I_0(x) + (6441435\, x^8 + 27747720\, x^7 - 90810720\, x^6 + 252252000\, x^5 - 576576000\, x^4 +$$
$$\left. -1037836800\, x^3 - 1383782400\, x^2 + 1383782400\, x)\, I_1(x)]\right\}$$

$$\int x^7\, e^{-x} \ln x\, I_1(x)\, dx = \frac{e^{-x}}{96621525}\left\{(-858858\, x^8 + 106722\, x^7 + 12526290\, x^6 + 72642640\, x^5 + 262741200\, x^4 + \right.$$
$$+662547840\, x^3 + 1123808640\, x^2 + 1383782400\, x + 1383782400)\, I_0(x) + (-858858\, x^8 + 536151\, x^7 - 12004524\, x^6 -$$
$$-78190700\, x^5 - 302273600\, x^4 - 820848960\, x^3 - 1555726080\, x^2 - 1555726080\, x)\, I_1(x) +$$
$$+ \ln x\, [(6441435\, x^8 + 23783760\, x^7 + 75675600\, x^6 + 201801600\, x^5 + 432432000\, x^4 + 691891200\, x^3 +$$
$$+691891200\, x^2)\, I_0(x) + (6441435\, x^8 - 27747720\, x^7 - 90810720\, x^6 - 252252000\, x^5 - 576576000\, x^4 -$$
$$\left. -1037836800\, x^3 - 1383782400\, x^2 - 1383782400\, x)\, I_1(x)]\right\}$$

$$\int x^7\, e^x \ln x\, K_1(x)\, dx = \frac{e^x}{96621525}\left\{(-858858\, x^8 - 106722\, x^7 + 12526290\, x^6 - 72642640\, x^5 + 262741200\, x^4 - \right.$$
$$-662547840\, x^3 + 1123808640\, x^2 - 1383782400\, x + 1383782400)\, K_0(x) + (-858858\, x^8 - 536151\, x^7 - 12004524\, x^6 +$$
$$+78190700\, x^5 - 302273600\, x^4 + 820848960\, x^3 - 1555726080\, x^2 + 1555726080\, x)\, K_1(x) +$$
$$+ \ln x\, [(6441435\, x^8 - 23783760\, x^7 + 75675600\, x^6 - 201801600\, x^5 + 432432000\, x^4 - 691891200\, x^3 +$$
$$+691891200\, x^2)\, K_0(x) + (6441435\, x^8 + 27747720\, x^7 - 90810720\, x^6 + 252252000\, x^5 - 576576000\, x^4 +$$
$$\left. +1037836800\, x^3 - 1383782400\, x^2 + 1383782400\, x)\, K_1(x)]\right\}$$

$$\int x^7\, e^{-x} \ln x\, K_1(x)\, dx = \frac{e^{-x}}{96621525}\left\{(858858\, x^8 - 106722\, x^7 - 12526290\, x^6 - 72642640\, x^5 - 262741200\, x^4 - \right.$$
$$-662547840\, x^3 - 1123808640\, x^2 - 1383782400\, x - 1383782400)\, K_0(x) + (-858858\, x^8 + 536151\, x^7 - 12004524\, x^6 -$$
$$-78190700\, x^5 - 302273600\, x^4 - 820848960\, x^3 - 1555726080\, x^2 - 1555726080\, x)\, K_1(x) +$$
$$+ \ln x\, [(-6441435\, x^8 - 23783760\, x^7 - 75675600\, x^6 - 201801600\, x^5 - 432432000\, x^4 - 691891200\, x^3 -$$
$$-691891200\, x^2)\, K_0(x) + (6441435\, x^8 - 27747720\, x^7 - 90810720\, x^6 - 252252000\, x^5 - 576576000\, x^4 -$$

$$-1037836800\,x^3 - 1383782400\,x^2 - 1383782400\,x)\,K_1(x)]\,\}$$

n = 8:

$$\int x^8\,e^x\,\ln x\,I_0(x)\,dx = \frac{e^x}{3989088675}\,\{(-27606150\,x^9 + 14784627\,x^8 - 500382432\,x^7 + 3249519840\,x^6 -$$

$$-14001917440\,x^5 + 44566771200\,x^4 - 104240855040\,x^3 + 166972323840\,x^2 - 188194406400\,x +$$

$$+188194406400)\,I_0(x) + (27606150\,x^9 - 981552\,x^8 - 493929744\,x^7 + 3487748544\,x^6 - 15787083200\,x^5 +$$

$$+52887833600\,x^4 - 132836981760\,x^3 + 239847444480\,x^2 - 239847444480\,x)\,I_1(x) +$$

$$+\ln x\,[(234652275\,x^9 + 876035160\,x^8 - 3234591360\,x^7 + 10291881600\,x^6 - 27445017600\,x^5 + 58810752000\,x^4 -$$

$$-94097203200\,x^3 + 94097203200\,x^2)\,I_0(x) + (-234652275\,x^9 + 1001183040\,x^8 - 3773689920\,x^7 +$$

$$+12350257920\,x^6 - 34306272000\,x^5 + 78414336000\,x^4 - 141145804800\,x^3 + 188194406400\,x^2 -$$

$$-188194406400\,x)\,I_1(x)]\,\}$$

$$\int x^8\,e^{-x}\,\ln x\,I_0(x)\,dx = \frac{e^{-x}}{3989088675}\,\{(-27606150\,x^9 - 14784627\,x^8 - 500382432\,x^7 - 3249519840\,x^6 -$$

$$-14001917440\,x^5 - 44566771200\,x^4 - 104240855040\,x^3 - 166972323840\,x^2 - 188194406400\,x -$$

$$-188194406400)\,I_0(x) + (-27606150\,x^9 - 981552\,x^8 + 493929744\,x^7 + 3487748544\,x^6 + 15787083200\,x^5 +$$

$$+52887833600\,x^4 + 132836981760\,x^3 + 239847444480\,x^2 + 239847444480\,x)\,I_1(x) +$$

$$+\ln x\,[(234652275\,x^9 - 876035160\,x^8 - 3234591360\,x^7 - 10291881600\,x^6 - 27445017600\,x^5 - 58810752000\,x^4 -$$

$$-94097203200\,x^3 - 94097203200\,x^2)\,I_0(x) + (234652275\,x^9 + 1001183040\,x^8 + 3773689920\,x^7 +$$

$$+12350257920\,x^6 + 34306272000\,x^5 + 78414336000\,x^4 + 141145804800\,x^3 + 188194406400\,x^2 +$$

$$+188194406400\,x)\,I_1(x)]\,\}$$

$$\int x^8\,e^x\,\ln x\,K_0(x)\,dx = \frac{e^x}{3989088675}\,\{(-27606150\,x^9 + 14784627\,x^8 - 500382432\,x^7 + 3249519840\,x^6 -$$

$$-14001917440\,x^5 + 44566771200\,x^4 - 104240855040\,x^3 + 166972323840\,x^2 - 188194406400\,x +$$

$$+188194406400)\,K_0(x) + (-27606150\,x^9 + 981552\,x^8 + 493929744\,x^7 - 3487748544\,x^6 + 15787083200\,x^5 -$$

$$-52887833600\,x^4 + 132836981760\,x^3 - 239847444480\,x^2 + 239847444480\,x)\,K_1(x) +$$

$$+\ln x\,[(234652275\,x^9 + 876035160\,x^8 - 3234591360\,x^7 + 10291881600\,x^6 - 27445017600\,x^5 + 58810752000\,x^4 -$$

$$-94097203200\,x^3 + 94097203200\,x^2)\,K_0(x) + (234652275\,x^9 - 1001183040\,x^8 + 3773689920\,x^7 - 12350257920\,x^6 +$$

$$+34306272000\,x^5 - 78414336000\,x^4 + 141145804800\,x^3 - 188194406400\,x^2 + 188194406400\,x)\,K_1(x)]\,\}$$

$$\int x^8\,e^{-x}\,\ln x\,K_0(x)\,dx = \frac{e^{-x}}{3989088675}\,\{-(27606150\,x^9 + 14784627\,x^8 + 500382432\,x^7 + 3249519840\,x^6 +$$

$$+14001917440\,x^5 + 44566771200\,x^4 + 104240855040\,x^3 + 166972323840\,x^2 + 188194406400\,x +$$

$$+188194406400)\,K_0(x) + (27606150\,x^9 + 981552\,x^8 - 493929744\,x^7 - 3487748544\,x^6 - 15787083200\,x^5 -$$

$$-52887833600\,x^4 - 132836981760\,x^3 - 239847444480\,x^2 - 239847444480\,x)\,K_1(x) +$$

$$+\ln x\,[(234652275\,x^9 - 876035160\,x^8 - 3234591360\,x^7 - 10291881600\,x^6 - 27445017600\,x^5 - 58810752000\,x^4 -$$

$$-94097203200\,x^3 - 94097203200\,x^2)\,K_0(x) - (234652275\,x^9 + 1001183040\,x^8 + 3773689920\,x^7 + 12350257920\,x^6 +$$

$$+34306272000\,x^5 + 78414336000\,x^4 + 141145804800\,x^3 + 188194406400\,x^2 + 188194406400\,x)\,K_1(x)]\,\}$$

$$\int x^8\,e^x\,\ln x\,I_1(x)\,dx = \frac{e^x}{443232075}\,\{(3067350\,x^9 + 327184\,x^8 - 56932194\,x^7 + 388322130\,x^6 - 1702592080\,x^5 +$$

$$+5468744400\,x^4 - 12866743680\,x^3 + 20708177280\,x^2 - 23524300800\,x + 23524300800)\,I_0(x) + (-3067350\,x^9 -$$

$$-1860859\,x^8 - 55189673\,x^7 + 414527148\,x^6 - 1913825900\,x^5 + 6474843200\,x^4 - 16359577920\,x^3 +$$

$$+29654204160\,x^2-29654204160\,x)\,I_1(x)+\ln x\,[(-26072475\,x^9+109504395\,x^8-404323920\,x^7+1286485200\,x^6-$$
$$-3430627200\,x^5+7351344000\,x^4-11762150400\,x^3+11762150400\,x^2)\,I_0(x)+(26072475\,x^9+125147880\,x^8-$$
$$-471711240\,x^7+1543782240\,x^6-4288284000\,x^5+9801792000\,x^4-17643225600\,x^3+23524300800\,x^2-$$
$$-23524300800\,x)\,I_1(x)]\,\}$$

$$\int x^8\,e^{-x}\,\ln x\,I_1(x)\,dx=\frac{e^{-x}}{443232075}\,\{(-3067350\,x^9+327184\,x^8+56932194\,x^7+388322130\,x^6+1702592080\,x^5+$$
$$+5468744400\,x^4+12866743680\,x^3+20708177280\,x^2+23524300800\,x+23524300800)\,I_0(x)+(-3067350\,x^9+$$
$$+1860859\,x^8-55189673\,x^7-414527148\,x^6-1913825900\,x^5-6474843200\,x^4-16359577920\,x^3-$$
$$-29654204160\,x^2-29654204160\,x)\,I_1(x)+\ln x\,[(26072475\,x^9+109504395\,x^8+404323920\,x^7+1286485200\,x^6+$$
$$+3430627200\,x^5+7351344000\,x^4+11762150400\,x^3+11762150400\,x^2)\,I_0(x)+(26072475\,x^9-125147880\,x^8-$$
$$-471711240\,x^7-1543782240\,x^6-4288284000\,x^5-9801792000\,x^4-17643225600\,x^3-23524300800\,x^2-$$
$$-23524300800\,x)\,I_1(x)]\,\}$$

$$\int x^8\,e^{x}\,\ln x\,K_1(x)\,dx=\frac{e^{x}}{443232075}\,\{(-3067350\,x^9-327184\,x^8+56932194\,x^7-388322130\,x^6+1702592080\,x^5-$$
$$-5468744400\,x^4+12866743680\,x^3-20708177280\,x^2+23524300800\,x-23524300800)\,K_0(x)+(-3067350\,x^9-$$
$$-1860859\,x^8-55189673\,x^7+414527148\,x^6-1913825900\,x^5+6474843200\,x^4-16359577920\,x^3+$$
$$+29654204160\,x^2-29654204160\,x)\,K_1(x)+\ln x\,[(26072475\,x^9-109504395\,x^8+404323920\,x^7-1286485200\,x^6+$$
$$+3430627200\,x^5-7351344000\,x^4+11762150400\,x^3-11762150400\,x^2)\,K_0(x)+(26072475\,x^9+125147880\,x^8-$$
$$-471711240\,x^7+1543782240\,x^6-4288284000\,x^5+9801792000\,x^4-17643225600\,x^3+23524300800\,x^2-$$
$$-23524300800\,x)\,K_1(x)]\,\}$$

$$\int x^8\,e^{-x}\,\ln x\,K_1(x)\,dx=\frac{e^{-x}}{443232075}\,\{(3067350\,x^9-327184\,x^8-56932194\,x^7-388322130\,x^6-1702592080\,x^5-$$
$$-5468744400\,x^4-12866743680\,x^3-20708177280\,x^2-23524300800\,x-23524300800)\,K_0(x)+(-3067350\,x^9+$$
$$+1860859\,x^8-55189673\,x^7-414527148\,x^6-1913825900\,x^5-6474843200\,x^4-16359577920\,x^3-$$
$$-29654204160\,x^2-29654204160\,x)\,K_1(x)+\ln x\,[(-26072475\,x^9-109504395\,x^8-404323920\,x^7-1286485200\,x^6-$$
$$-3430627200\,x^5-7351344000\,x^4-11762150400\,x^3-11762150400\,x^2)\,K_0(x)+(26072475\,x^9-125147880\,x^8-$$
$$-471711240\,x^7-1543782240\,x^6-4288284000\,x^5-9801792000\,x^4-17643225600\,x^3-23524300800\,x^2-$$
$$-23524300800\,x)\,K_1(x)]\,\}$$

n = 9:

$$\int x^9\,e^{x}\,\ln x\,I_0(x)\,dx=\frac{e^{x}}{53335593025}\,\{(-295488050\,x^{10}+156946075\,x^9-6682958139\,x^8+50085741024\,x^7-$$
$$-253835174880\,x^6+981076078080\,x^5-2932377638400\,x^4+6569043425280\,x^3-10144737146880\,x^2+$$
$$+10727081164800\,x-10727081164800)\,I_0(x)+(295488050\,x^{10}-9202050\,x^9-6618605136\,x^8+53311928208\,x^7-$$
$$-281136719808\,x^6+1128572222400\,x^5-3537368755200\,x^4+8512679992320\,x^3-14925933711360\,x^2+$$
$$+14925933711360\,x)\,I_1(x)+\ln x\,[(2807136475\,x^{10}+11889048600\,x^9-49934004120\,x^8+184371707520\,x^7-$$
$$-586637251200\,x^6+1564366003200\,x^5-3352212864000\,x^4+5363540582400\,x^3-5363540582400\,x^2)\,I_0(x)+$$
$$+(-2807136475\,x^{10}+13375179675\,x^9-57067433280\,x^8+215100325440\,x^7-703964701440\,x^6+$$
$$+1955457504000\,x^5-4469617152000\,x^4+8045310873600\,x^3-10727081164800\,x^2+10727081164800\,x)\,I_1(x)]\,\}$$

$$\int x^9\,e^{-x}\,\ln x\,I_0(x)\,dx=\frac{e^{-x}}{53335593025}\,\{(-295488050\,x^{10}-156946075\,x^9-6682958139\,x^8-50085741024\,x^7-$$

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$$-253835174880\,x^6 - 981076078080\,x^5 - 2932377638400\,x^4 - 6569043425280\,x^3 - 10144737146880\,x^2 -$$
$$-10727081164800\,x - 10727081164800)\,I_0(x) + (-295488050\,x^{10} - 9202050\,x^9 + 6618605136\,x^8 + 53311928208\,x^7 +$$
$$+281136719808\,x^6 + 1128572222400\,x^5 + 3537368755200\,x^4 + 8512679992320\,x^3 + 14925933711360\,x^2 +$$
$$+14925933711360\,x)\,I_1(x) + \ln x\,[(2807136475\,x^{10} - 11889048600\,x^9 - 49934004120\,x^8 - 184371707520\,x^7 -$$
$$-586637251200\,x^6 - 1564366003200\,x^5 - 3352212864000\,x^4 - 5363540582400\,x^3 - 5363540582400\,x^2)\,I_0(x) +$$
$$+(2807136475\,x^{10} + 13375179675\,x^9 + 57067433280\,x^8 + 215100325440\,x^7 + 703964701440\,x^6 + 1955457504000\,x^5 +$$
$$+4469617152000\,x^4 + 8045310873600\,x^3 + 10727081164800\,x^2 + 10727081164800\,x)\,I_1(x)]\,\}$$

$$\int x^9\,e^x\,\ln x\,K_0(x)\,dx = \frac{e^x}{53335593025}\,\{(-295488050\,x^{10} + 156946075\,x^9 - 6682958139\,x^8 + 50085741024\,x^7 -$$
$$-253835174880\,x^6 + 981076078080\,x^5 - 2932377638400\,x^4 + 6569043425280\,x^3 - 10144737146880\,x^2 +$$
$$+10727081164800\,x - 10727081164800)\,K_0(x) + (-295488050\,x^{10} + 9202050\,x^9 + 6618605136\,x^8 - 53311928208\,x^7 +$$
$$+281136719808\,x^6 - 1128572222400\,x^5 + 3537368755200\,x^4 - 8512679992320\,x^3 + 14925933711360\,x^2 -$$
$$-14925933711360\,x)\,K_1(x) + \ln x\,[(2807136475\,x^{10} + 11889048600\,x^9 - 49934004120\,x^8 + 184371707520\,x^7 -$$
$$-586637251200\,x^6 + 1564366003200\,x^5 - 3352212864000\,x^4 + 5363540582400\,x^3 - 5363540582400\,x^2)\,K_0(x) +$$
$$+(2807136475\,x^{10} - 13375179675\,x^9 + 57067433280\,x^8 - 215100325440\,x^7 + 703964701440\,x^6 - 1955457504000\,x^5 +$$
$$+4469617152000\,x^4 - 8045310873600\,x^3 + 10727081164800\,x^2 - 10727081164800\,x)\,K_1(x)]\,\}$$

$$\int x^9\,e^{-x}\,\ln x\,K_0(x)\,dx = \frac{e^{-x}}{53335593025}\,\{-(295488050\,x^{10} + 156946075\,x^9 + 6682958139\,x^8 + 50085741024\,x^7 +$$
$$+253835174880\,x^6 + 981076078080\,x^5 + 2932377638400\,x^4 + 6569043425280\,x^3 + 10144737146880\,x^2 +$$
$$+10727081164800\,x + 10727081164800)\,K_0(x) + (295488050\,x^{10} + 9202050\,x^9 - 6618605136\,x^8 - 53311928208\,x^7 -$$
$$-281136719808\,x^6 - 1128572222400\,x^5 - 3537368755200\,x^4 - 8512679992320\,x^3 - 14925933711360\,x^2 -$$
$$-14925933711360\,x)\,K_1(x) + \ln x\,[(2807136475\,x^{10} - 11889048600\,x^9 - 49934004120\,x^8 - 184371707520\,x^7 -$$
$$-586637251200\,x^6 - 1564366003200\,x^5 - 3352212864000\,x^4 - 5363540582400\,x^3 - 5363540582400\,x^2)\,K_0(x) -$$
$$-(2807136475\,x^{10} + 13375179675\,x^9 + 57067433280\,x^8 + 215100325440\,x^7 + 703964701440\,x^6 + 1955457504000\,x^5 +$$
$$+4469617152000\,x^4 + 8045310873600\,x^3 + 10727081164800\,x^2 + 10727081164800\,x)\,K_1(x)]\,\}$$

$$\int x^9\,e^x\,\ln x\,I_1(x)\,dx = \frac{e^x}{32001355815}\,\{(177292830\,x^{10} + 16563690\,x^9 - 4085423914\,x^8 + 32024777664\,x^7 -$$
$$-164877988800\,x^6 + 642462822400\,x^5 - 1930087219200\,x^4 + 4339632353280\,x^3 - 6723428167680\,x^2 +$$
$$+7151387443200\,x - 7151387443200)\,I_0(x) + (-177292830\,x^{10} - 105210105\,x^9 - 3989681696\,x^8 + 33947949728\,x^7 -$$
$$-182209926528\,x^6 + 737896611200\,x^5 - 2325137561600\,x^4 + 5615525099520\,x^3 - 9871162613760\,x^2 +$$
$$+9871162613760\,x)\,I_1(x) + \ln x\,[(-1684281885\,x^{10} + 7926032400\,x^9 - 33289336080\,x^8 + 122914471680\,x^7 -$$
$$-391091500800\,x^6 + 1042910668800\,x^5 - 2234808576000\,x^4 + 3575693721600\,x^3 - 3575693721600\,x^2)\,I_0(x) +$$
$$+(1684281885\,x^{10} + 8916786450\,x^9 - 38044955520\,x^8 + 143400216960\,x^7 - 469309800960\,x^6 + 1303638336000\,x^5 -$$
$$-2979744768000\,x^4 + 5363540582400\,x^3 - 7151387443200\,x^2 + 7151387443200\,x)\,I_1(x)]\,\}$$

$$\int x^9\,e^{-x}\,\ln x\,I_1(x)\,dx = \frac{e^{-x}}{32001355815}\,\{(-177292830\,x^{10} + 16563690\,x^9 + 4085423914\,x^8 + 32024777664\,x^7 +$$
$$+164877988800\,x^6 + 642462822400\,x^5 + 1930087219200\,x^4 + 4339632353280\,x^3 + 6723428167680\,x^2 +$$
$$+7151387443200\,x + 7151387443200)\,I_0(x) + (-177292830\,x^{10} + 105210105\,x^9 - 3989681696\,x^8 - 33947949728\,x^7 -$$
$$-182209926528\,x^6 - 737896611200\,x^5 - 2325137561600\,x^4 - 5615525099520\,x^3 - 9871162613760\,x^2 -$$
$$-9871162613760\,x)\,I_1(x) + \ln x\,[(1684281885\,x^{10} + 7926032400\,x^9 + 33289336080\,x^8 + 122914471680\,x^7 +$$
$$+391091500800\,x^6 + 1042910668800\,x^5 + 2234808576000\,x^4 + 3575693721600\,x^3 + 3575693721600\,x^2)\,I_0(x) +$$

$$+(1684281885\,x^{10}-8916786450\,x^9-38044955520\,x^8-143400216960\,x^7-469309800960\,x^6-1303638336000\,x^5-$$
$$-2979744768000\,x^4-5363540582400\,x^3-7151387443200\,x^2-7151387443200\,x)\,I_1(x)]\,\}$$

$$\int x^9\,e^x\,\ln x\,K_1(x)\,dx=\frac{e^x}{32001355815}\,\{(-177292830\,x^{10}-16563690\,x^9+4085423914\,x^8-32024777664\,x^7+$$
$$+164877988800\,x^6-642462822400\,x^5+1930087219200\,x^4-4339632353280\,x^3+6723428167680\,x^2-$$
$$-7151387443200\,x+7151387443200)\,K_0(x)+(-177292830\,x^{10}-105210105\,x^9-3989681696\,x^8+33947949728\,x^7-$$
$$-182209926528\,x^6+737896611200\,x^5-2325137561600\,x^4+5615525099520\,x^3-9871162613760\,x^2+$$
$$+9871162613760\,x)\,K_1(x)+\ln x\,[(1684281885\,x^{10}-7926032400\,x^9+33289336080\,x^8-122914471680\,x^7+$$
$$+391091500800\,x^6-1042910668800\,x^5+2234808576000\,x^4-3575693721600\,x^3+3575693721600\,x^2)\,K_0(x)+$$
$$+(1684281885\,x^{10}+8916786450\,x^9-38044955520\,x^8+143400216960\,x^7-469309800960\,x^6+1303638336000\,x^5-$$
$$-2979744768000\,x^4+5363540582400\,x^3-7151387443200\,x^2+7151387443200\,x)\,K_1(x)]\,\}$$

$$\int x^9\,e^{-x}\,\ln x\,K_1(x)\,dx=\frac{e^{-x}}{32001355815}\,\{(177292830\,x^{10}-16563690\,x^9-4085423914\,x^8-32024777664\,x^7-$$
$$-164877988800\,x^6-642462822400\,x^5-1930087219200\,x^4-4339632353280\,x^3-6723428167680\,x^2-$$
$$-7151387443200\,x-7151387443200)\,K_0(x)+(-177292830\,x^{10}+105210105\,x^9-3989681696\,x^8-33947949728\,x^7-$$
$$-182209926528\,x^6-737896611200\,x^5-2325137561600\,x^4-5615525099520\,x^3-9871162613760\,x^2-$$
$$-9871162613760\,x)\,K_1(x)+\ln x\,[(-1684281885\,x^{10}-7926032400\,x^9-33289336080\,x^8-122914471680\,x^7-$$
$$-391091500800\,x^6-1042910668800\,x^5-2234808576000\,x^4-3575693721600\,x^3-3575693721600\,x^2)\,K_0(x)+$$
$$+(1684281885\,x^{10}-8916786450\,x^9-38044955520\,x^8-143400216960\,x^7-469309800960\,x^6-1303638336000\,x^5-$$
$$-2979744768000\,x^4-5363540582400\,x^3-7151387443200\,x^2-7151387443200\,x)\,K_1(x)]\,\}$$

n = 10:

$$\int x^{10}\,e^x\,\ln x\,I_0(x)\,dx=\frac{e^x}{940839860961}\,\{(-4266847442\,x^{11}+2251618941\,x^{10}-117807097980\,x^9+$$
$$+1000787719932\,x^8-5829673272192\,x^7+26484562500480\,x^6-96176811386880\,x^5+275819194828800\,x^4-$$
$$-598998804848640\,x^3+899357077463040\,x^2-901074817843200\,x+901074817843200)\,I_0(x)+$$
$$+(4266847442\,x^{11}-118195220\,x^{10}-116928608940\,x^9+1058156244288\,x^8-6371084833344\,x^7+$$
$$+29810373836544\,x^6-112008092716800\,x^5+336471606374400\,x^4-785863855042560\,x^3+$$
$$+1348176746004480\,x^2-1348176746004480\,x)\,I_1(x)+\ln x\,[(44801898141\,x^{11}+212219517510\,x^{10}-$$
$$-998680082400\,x^9+4194456346080\,x^8-15487223431680\,x^7+49277529100800\,x^6-131406744268800\,x^5+$$
$$+281585880576000\,x^4-450537408921600\,x^3+450537408921600\,x^2)\,I_0(x)+$$
$$+(-44801898141\,x^{11}+235799463900\,x^{10}-1123515092700\,x^9+4793664395520\,x^8-18068427336960\,x^7+$$
$$+59133034920960\,x^6-164258430336000\,x^5+375447840768000\,x^4-675806113382400\,x^3+$$
$$+901074817843200\,x^2-901074817843200\,x)\,I_1(x)]\,\}$$

$$\int x^{10}\,e^{-x}\,\ln x\,I_0(x)\,dx=\frac{e^{-x}}{940839860961}\,\{(-4266847442\,x^{11}-2251618941\,x^{10}-117807097980\,x^9-$$
$$-1000787719932\,x^8-5829673272192\,x^7-26484562500480\,x^6-96176811386880\,x^5-275819194828800\,x^4-$$
$$-598998804848640\,x^3-899357077463040\,x^2-901074817843200\,x-901074817843200)\,I_0(x)+$$
$$+(-4266847442\,x^{11}-118195220\,x^{10}+116928608940\,x^9+1058156244288\,x^8+6371084833344\,x^7+$$
$$+29810373836544\,x^6+112008092716800\,x^5+336471606374400\,x^4+785863855042560\,x^3+1348176746004480\,x^2+$$
$$+1348176746004480\,x)\,I_1(x)+\ln x\,[(44801898141\,x^{11}-212219517510\,x^{10}-998680082400\,x^9-4194456346080\,x^8-$$

$$-15487223431680\,x^7 - 49277529100800\,x^6 - 131406744268800\,x^5 - 281585880576000\,x^4-$$
$$-450537408921600\,x^3 - 450537408921600\,x^2)\,I_0(x) + (44801898141\,x^{11} + 235799463900\,x^{10}+$$
$$+1123515092700\,x^9 + 4793664395520\,x^8 + 18068427336960\,x^7 + 59133034920960\,x^6 + 164258430336000\,x^5+$$
$$+375447840768000\,x^4 + 675806113382400\,x^3 + 901074817843200\,x^2 + 901074817843200\,x)\,I_1(x)]\,\}$$

$$\int x^{10}\,e^x\,\ln x\,K_0(x)\,dx = \frac{e^x}{940839860961}\,\{(-4266847442\,x^{11} + 2251618941\,x^{10} - 117807097980\,x^9+$$
$$+1000787719932\,x^8 - 5829673272192\,x^7 + 26484562500480\,x^6 - 96176811386880\,x^5 + 275819194828800\,x^4-$$
$$-598998804848640\,x^3 + 899357077463040\,x^2 - 901074817843200\,x + 901074817843200)\,K_0(x)+$$
$$+(-4266847442\,x^{11} + 118195220\,x^{10} + 116928608940\,x^9 - 1058156244288\,x^8 + 6371084833344\,x^7-$$
$$-29810373836544\,x^6 + 112008092716800\,x^5 - 336471606374400\,x^4 + 785863855042560\,x^3 - 1348176746004480\,x^2+$$
$$+1348176746004480\,x)\,K_1(x) + \ln x\,[(44801898141\,x^{11} + 212219517510\,x^{10} - 998680082400\,x^9+$$
$$+4194456346080\,x^8 - 15487223431680\,x^7 + 49277529100800\,x^6 - 131406744268800\,x^5 + 281585880576000\,x^4-$$
$$-450537408921600\,x^3 + 450537408921600\,x^2)\,K_0(x) + (44801898141\,x^{11} - 235799463900\,x^{10}+$$
$$+1123515092700\,x^9 - 4793664395520\,x^8 + 18068427336960\,x^7 - 59133034920960\,x^6 + 164258430336000\,x^5-$$
$$-375447840768000\,x^4 + 675806113382400\,x^3 - 901074817843200\,x^2 + 901074817843200\,x)\,K_1(x)]\,\}$$

$$\int x^{10}\,e^{-x}\,\ln x\,K_0(x)\,dx = \frac{e^{-x}}{940839860961}\,\{-(4266847442\,x^{11} + 2251618941\,x^{10} + 117807097980\,x^9+$$
$$+1000787719932\,x^8 + 5829673272192\,x^7 + 26484562500480\,x^6 + 96176811386880\,x^5 + 275819194828800\,x^4+$$
$$+598998804848640\,x^3 + 899357077463040\,x^2 + 901074817843200\,x + 901074817843200)\,K_0(x)+$$
$$+(4266847442\,x^{11} + 118195220\,x^{10} - 116928608940\,x^9 - 1058156244288\,x^8 - 6371084833344\,x^7-$$
$$-29810373836544\,x^6 - 112008092716800\,x^5 - 336471606374400\,x^4 - 785863855042560\,x^3-$$
$$-1348176746004480\,x^2 - 1348176746004480\,x)\,K_1(x) + \ln x\,[(44801898141\,x^{11} - 212219517510\,x^{10}-$$
$$-998680082400\,x^9 - 4194456346080\,x^8 - 15487223431680\,x^7 - 49277529100800\,x^6 - 131406744268800\,x^5-$$
$$-281585880576000\,x^4 - 450537408921600\,x^3 - 450537408921600\,x^2)\,K_0(x) - (44801898141\,x^{11}+$$
$$+235799463900\,x^{10} + 1123515092700\,x^9 + 4793664395520\,x^8 + 18068427336960\,x^7 + 59133034920960\,x^6+$$
$$+164258430336000\,x^5 + 375447840768000\,x^4 + 675806113382400\,x^3 + 901074817843200\,x^2+$$
$$+901074817843200\,x)\,K_1(x)]\,\}$$

$$\int x^{10}\,e^x\,\ln x\,I_1(x)\,dx = \frac{e^x}{427654482255}\,\{(1939476110\,x^{11} + 161175300\,x^{10} - 54364094070\,x^9+$$
$$+481328149302\,x^8 - 2844440165952\,x^7 + 13018292481600\,x^6 - 47491102310400\,x^5 + 136629661593600\,x^4-$$
$$-297451505111040\,x^3 + 447630641418240\,x^2 - 450537408921600\,x + 450537408921600)\,I_0(x)+$$
$$+(-1939476110\,x^{11} - 1130913355\,x^{10} - 53357417685\,x^9 + 507288738528\,x^8 - 3103413201504\,x^7+$$
$$+14636400395904\,x^6 - 55257417129600\,x^5 + 166529222092800\,x^4 - 389860081551360\,x^3 + 669992578375680\,x^2-$$
$$-669992578375680\,x)\,I_1(x) + \ln x\,[(-20364499155\,x^{11} + 106109758755\,x^{10} - 499340041200\,x^9+$$
$$+2097228173040\,x^8 - 7743611715840\,x^7 + 24638764550400\,x^6 - 65703372134400\,x^5 + 140792940288000\,x^4-$$
$$-225268704460800\,x^3 + 225268704460800\,x^2)\,I_0(x) + (20364499155\,x^{11} + 117899731950\,x^{10} - 561757546350\,x^9+$$
$$+2396832197760\,x^8 - 9034213668480\,x^7 + 29566517460480\,x^6 - 82129215168000\,x^5 + 187723920384000\,x^4-$$
$$-337903056691200\,x^3 + 450537408921600\,x^2 - 450537408921600\,x)\,I_1(x)]\,\}$$

$$\int x^{10}\,e^{-x}\,\ln x\,I_1(x)\,dx = \frac{e^{-x}}{427654482255}\,\{(-1939476110\,x^{11} + 161175300\,x^{10} + 54364094070\,x^9+$$
$$+481328149302\,x^8 + 2844440165952\,x^7 + 13018292481600\,x^6 + 47491102310400\,x^5 + 136629661593600\,x^4+$$

$$+297451505111040\,x^3 + 447630641418240\,x^2 + 450537408921600\,x + 450537408921600)\,I_0(x)+$$

$$+(-1939476110\,x^{11} + 1130913355\,x^{10} - 53357417685\,x^9 - 507288738528\,x^8 - 3103413201504\,x^7-$$

$$-14636400395904\,x^6 - 55257417129600\,x^5 - 166529222092800\,x^4 - 389860081551360\,x^3 - 669992578375680\,x^2-$$

$$-669992578375680\,x)\,I_1(x) + \ln x\,[(20364499155\,x^{11} + 106109758755\,x^{10} + 499340041200\,x^9+$$

$$+2097228173040\,x^8 + 7743611715840\,x^7 + 24638764550400\,x^6 + 65703372134400\,x^5 + 140792940288000\,x^4+$$

$$+225268704460800\,x^3 + 225268704460800\,x^2)\,I_0(x) + (20364499155\,x^{11} - 117899731950\,x^{10} - 561757546350\,x^9-$$

$$-2396832197760\,x^8 - 9034213668480\,x^7 - 29566517460480\,x^6 - 82129215168000\,x^5 - 187723920384000\,x^4-$$

$$-337903056691200\,x^3 - 450537408921600\,x^2 - 450537408921600\,x)\,I_1(x)]\,\}$$

$$\int x^{10}\,e^x\,\ln x\,K_1(x)\,dx = \frac{e^x}{427654482255}\,\{(-1939476110\,x^{11} - 161175300\,x^{10} + 54364094070\,x^9-$$

$$-481328149302\,x^8 + 2844440165952\,x^7 - 13018292481600\,x^6 + 47491102310400\,x^5 - 136629661593600\,x^4+$$

$$+297451505111040\,x^3 - 447630641418240\,x^2 + 450537408921600\,x - 450537408921600)\,K_0(x)+$$

$$+(-1939476110\,x^{11} - 1130913355\,x^{10} - 53357417685\,x^9 + 507288738528\,x^8 - 3103413201504\,x^7+$$

$$+14636400395904\,x^6 - 55257417129600\,x^5 + 166529222092800\,x^4 - 389860081551360\,x^3 + 669992578375680\,x^2-$$

$$-669992578375680\,x)\,K_1(x) + \ln x\,[+(20364499155\,x^{11} - 106109758755\,x^{10} + 499340041200\,x^9-$$

$$-2097228173040\,x^8 + 7743611715840\,x^7 - 24638764550400\,x^6 + 65703372134400\,x^5 - 140792940288000\,x^4+$$

$$+225268704460800\,x^3 - 225268704460800\,x^2)\,K_0(x) + (20364499155\,x^{11} + 117899731950\,x^{10} - 561757546350\,x^9+$$

$$+2396832197760\,x^8 - 9034213668480\,x^7 + 29566517460480\,x^6 - 82129215168000\,x^5 + 187723920384000\,x^4-$$

$$-337903056691200\,x^3 + 450537408921600\,x^2 - 450537408921600\,x)\,K_1(x)]\,\}$$

$$\int x^{10}\,e^{-x}\,\ln x\,K_1(x)\,dx = \frac{e^{-x}}{427654482255}\,\{(1939476110\,x^{11} - 161175300\,x^{10} - 54364094070\,x^9-$$

$$-481328149302\,x^8 - 2844440165952\,x^7 - 13018292481600\,x^6 - 47491102310400\,x^5 - 136629661593600\,x^4-$$

$$-297451505111040\,x^3 - 447630641418240\,x^2 - 450537408921600\,x - 450537408921600)\,K_0(x)+$$

$$+(-1939476110\,x^{11} + 1130913355\,x^{10} - 53357417685\,x^9 - 507288738528\,x^8 - 3103413201504\,x^7-$$

$$-14636400395904\,x^6 - 55257417129600\,x^5 - 166529222092800\,x^4 - 389860081551360\,x^3 - 669992578375680\,x^2-$$

$$-669992578375680\,x)\,K_1(x) + \ln x\,[(-20364499155\,x^{11} - 106109758755\,x^{10} - 499340041200\,x^9-$$

$$-2097228173040\,x^8 - 7743611715840\,x^7 - 24638764550400\,x^6 - 65703372134400\,x^5 - 140792940288000\,x^4-$$

$$-225268704460800\,x^3 - 225268704460800\,x^2)\,K_0(x) + (20364499155\,x^{11} - 117899731950\,x^{10} - 561757546350\,x^9-$$

$$-2396832197760\,x^8 - 9034213668480\,x^7 - 29566517460480\,x^6 - 82129215168000\,x^5 - 187723920384000\,x^4-$$

$$-337903056691200\,x^3 - 450537408921600\,x^2 - 450537408921600\,x)\,K_1(x)]\,\}$$

<u>Recurrence relations:</u>

About the recurrence relations for the integrals $\int x^n e^{\pm x} I_\nu(x)\,dx$ and $\int x^n e^{\pm x} K_\nu(x)\,dx$ see the pages 69 and 72.

$$\int x^{n+1} e^x \ln(x) I_0(x)\,dx = \frac{x^{n+1} e^x}{2n+3} \{[(n+1+x) I_0(x) - x I_1(x)] \ln(x) - I_0(x)\} -$$

$$-\frac{(n+1)^2}{2n+3} \int x^n \ln x\, e^x I_0(x)\,dx + \frac{2}{2n+3} \int x^{n+1} e^x I_1(x)\,dx$$

$$\int x^{n+1} e^x \ln(x) K_0(x)\,dx = \frac{x^{n+1} e^x}{2n+3} \{[(n+1+x) K_0(x) + x K_1(x)] \ln(x) - K_0(x)\} -$$

$$-\frac{(n+1)^2}{2n+3} \int x^n \ln x\, e^x K_0(x)\,dx - \frac{2}{2n+3} \int x^{n+1} e^x K_1(x)\,dx$$

$$\int x^{n+1} e^{-x} \ln(x) I_0(x)\,dx = \frac{x^{n+1} e^{-x}}{2n+3} \{[-(n+1-x) I_0(x) + x I_1(x)] \ln(x) + I_0(x)\} +$$

$$+\frac{(n+1)^2}{2n+3} \int x^n \ln x\, e^x I_0(x)\,dx - \frac{2}{2n+3} \int x^{n+1} e^x I_1(x)\,dx$$

$$\int x^{n+1} e^{-x} \ln(x) K_0(x)\,dx = \frac{x^{n+1} e^{-x}}{2n+3} \{[(x-n-1) K_0(x) - x K_1(x)] \ln(x) + K_0(x)\} +$$

$$+\frac{(n+1)^2}{2n+3} \int x^n \ln x\, e^x K_0(x)\,dx + \frac{2}{2n+3} \int x^{n+1} e^x K_1(x)\,dx$$

The following recurrence relations for the integrals $\int x^{n+1} e^{\pm x} Z_1(x)\,dx$ refer to $\int x^n e^{\pm x} Z_0(x)\,dx$ instead of $\int x^n e^{\pm x} Z_1(x)\,dx$.

$$\int x^{n+1} e^x \ln(x) I_1(x)\,dx = \frac{x^{n+1} e^x}{2n+3} \{[(n+2-x) I_0(x) + x I_1(x)] \ln(x) - \frac{n+2}{n+1} I_0(x)\} -$$

$$-\frac{(n+1)(n+2)}{2n+3} \int x^n \ln x\, e^x I_0(x)\,dx + \frac{1}{n+1} \int x^{n+1} e^x I_0(x)\,dx + \frac{1}{(n+1)(2n+3)} \int x^{n+1} e^x I_1(x)\,dx$$

$$\int x^{n+1} e^{-x} \ln(x) I_1(x)\,dx = \frac{x^{n+1} e^{-x}}{2n+3} \{[(n+2+x) I_0(x) + x I_1(x)] \ln(x) - \frac{n+2}{n+1} I_0(x)\} -$$

$$-\frac{(n+1)(n+2)}{2n+3} \int x^n \ln x\, e^x I_0(x)\,dx - \frac{1}{n+1} \int x^{n+1} e^x I_0(x)\,dx - \frac{1}{(n+1)(2n+3)} \int x^{n+1} e^x I_1(x)\,dx$$

$$\int x^{n+1} e^x \ln(x) K_1(x)\,dx = \frac{x^{n+1} e^x}{2n+3} \{[(x-n-2) K_0(x) + x K_1(x)] \ln(x) + \frac{n+2}{n+1} K_0(x)\} +$$

$$+\frac{(n+1)(n+2)}{2n+3} \int x^n \ln x\, e^x K_0(x)\,dx - \frac{1}{n+1} \int x^{n+1} e^x K_0(x)\,dx + \frac{1}{(n+1)(2n+3)} \int x^{n+1} e^x K_1(x)\,dx$$

$$\int x^{n+1} e^{-x} \ln(x) K_1(x)\,dx = \frac{x^{n+1} e^{-x}}{2n+3} \{[-(n+2+x) K_0(x) + x K_1(x)] \ln(x) + \frac{n+2}{n+1} K_0(x)\} +$$

$$+\frac{(n+1)(n+2)}{2n+3} \int x^n \ln x\, e^x K_0(x)\,dx + \frac{1}{n+1} \int x^{n+1} e^x K_0(x)\,dx + \frac{1}{(n+1)(2n+3)} \int x^{n+1} e^x K_1(x)\,dx$$

1.2.13. $\int x^n e^{-x^2} J_\nu(\alpha x)\,dx$

a) The Case $\alpha = 1$, Basic Integrals:

Some improper integrals: From [13], 4.14. (34) and (35) one has (or [14], 8.2.(21); see also [7], 6.643 and 9.235; or [4], 2.12.9.1.-3.)

$$\int_0^\infty e^{-x^2} J_0(x)\,dx = \frac{\sqrt{\pi}\,e^{-1/8}}{2} I_0\left(\frac{1}{8}\right) = 0.78515\,05503$$

$$\int_0^\infty x\,e^{-x^2} J_0(x)\,dx = \frac{e^{-1/4}}{2} = 0.38940\,03915$$

$$\int_0^\infty x^2\,e^{-x^2} J_0(x)\,dx = \frac{\sqrt{\pi}\,e^{-1/8}}{16}\left[3I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] = 0.30055\,34957$$

$$\int_0^\infty x^3\,e^{-x^2} J_0(x)\,dx = \frac{3e^{-1/4}}{8} = 0.29205\,02937$$

$$\int_0^\infty x^4\,e^{-x^2} J_0(x)\,dx = \frac{\sqrt{\pi}\,e^{-1/8}}{64}\left[13I_0\left(\frac{1}{8}\right) + 7I_1\left(\frac{1}{8}\right)\right] = 0.32968\,09799$$

$$\int_0^\infty x^5\,e^{-x^2} J_0(x)\,dx = \frac{17e^{-1/4}}{32} = 0.41373\,79160$$

$$\int_0^\infty x^6\,e^{-x^2} J_0(x)\,dx = \frac{\sqrt{\pi}\,e^{-1/8}}{256}\left[87I_0\left(\frac{1}{8}\right) + 69I_1\left(\frac{1}{8}\right)\right] = 0.56005\,83093$$

$$\int_0^\infty \frac{e^{-x^2} J_1(x)\,dx}{x} = \frac{\sqrt{\pi}\,e^{-1/8}}{4}\left[I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] = 0.41706\,34325$$

$$\int_0^\infty e^{-x^2} J_1(x)\,dx = 1 - e^{-1/4} = 0.22119\,92169$$

$$\int_0^\infty x\,e^{-x^2} J_1(x)\,dx = \frac{\sqrt{\pi}\,e^{-1/8}}{8}\left[I_0\left(\frac{1}{8}\right) - I_1\left(\frac{1}{8}\right)\right] = 0.18404\,35589$$

$$\int_0^\infty x^2\,e^{-x^2} J_1(x)\,dx = \frac{e^{-1/4}}{4} = 0.19470\,01958$$

$$\int_0^\infty x^3\,e^{-x^2} J_1(x)\,dx = \frac{\sqrt{\pi}\,e^{-1/8}}{32}\left[5I_0\left(\frac{1}{8}\right) - I_1\left(\frac{1}{8}\right)\right] = 0.24229\,85273$$

$$\int_0^\infty x^4\,e^{-x^2} J_1(x)\,dx = \frac{7e^{-1/4}}{16} = 0.34072\,53426$$

$$\int_0^\infty x^5\,e^{-x^2} J_1(x)\,dx = \frac{\sqrt{\pi}\,e^{-1/8}}{128}\left[43I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] = 0.52828\,82809$$

$$\int_0^\infty x^6\,e^{-x^2} J_1(x)\,dx = \frac{73e^{-1/4}}{64} = 0.88831\,96432$$

Let

$$F_\nu(x) = \int_0^x e^{-t^2} J_\nu(t)\,dt = \nu + e^{-x^2}\left[P_\nu(x)\,J_0(x) + Q_\nu(x)\,J_1(x)\right], \quad \nu = 0,\,1,$$

with

$$P_\nu(x) = \sum_{k=0}^\infty a_k^{(\nu)}\,x^{2k+1-\nu} = \sum_{k=0}^\infty \frac{\alpha_k^{(\nu)}}{\gamma_k^{(\nu)}}\,x^{2k+1-\nu} \quad \text{and} \quad Q_\nu(x) = \sum_{k=0}^\infty b_k^{(\nu)}\,x^{2k+\nu} = \sum_{k=0}^\infty \frac{\beta_k^{(\nu)}}{\delta_k^{(\nu)}}\,x^{2k+\nu}.$$

Furthermore, let

$$F_-(x) = \int_0^x \frac{e^{-t^2} J_1(t)\,dt}{t} = e^{-x^2}\left[P_-(x)\,J_0(x) + Q_-(x)\,J_1(x)\right]$$

with

$$P_-(x) = \sum_{k=0}^{\infty} a_k^{(-)} x^{2k+1} = \sum_{k=0}^{\infty} \frac{\alpha_k^{(-)}}{\gamma_k^{(-)}} x^{2k+1} \quad \text{and} \quad Q_-(x) = \sum_{k=0}^{\infty} b_k^{(-)} x^{2k+\nu} = \sum_{k=0}^{\infty} \frac{\beta_k^{(-)}}{\delta_k^{(-)}} x^{2k} .$$

The minimum and maximum values of $F_s(x)$ are located in the zeros $x_k^{(\nu)}$ of $J_\nu(x)$, $0 < x_k^{(\nu)} < x_{k+1}^{(\nu)}$.
Let $F_s(x_k^{(\nu)}) = \Delta_k^{(s)} + \lim_{x\to\infty} F_s(x)$.
The following table shows, that $F_s(x)$ must not be be computed for large values of x.

s	Value	$k=1$	$k=2$	$k=3$	$k=4$
0	x_k	2.4048	5.5201	8.6537	11.7915
	$\Delta_k^{(0)}$	$5.1225 \cdot 10^{-5}$	$-1.5215 \cdot 10^{-16}$	$2.6395 \cdot 10^{-36}$	$-1.6974 \cdot 10^{-64}$
1	x_k	3.8317	7.0156	10.1735	13.3237
	$\Delta_k^{(1)}$	$2.5200 \cdot 10^{-9}$	$-6.1486 \cdot 10^{-25}$	$6.6365 \cdot 10^{-49}$	$-2.4330 \cdot 10^{-81}$
-	$\Delta_k^{(-)}$	$6.2084 \cdot 10^{-10}$	$-8.5976 \cdot 10^{-26}$	$6.4624 \cdot 10^{-50}$	$-1.8160 \cdot 10^{-82}$

Remark: In any case, if $F_s(x)$ is written as

$$F_s(x) = e^{-x^2} \psi_s(x) + \lim_{x\to\infty} F_s(x) ,$$

then one has

x	1.52	2.15	2.63	3.03	3.39	3.72	4.01	4.29	4.55	4.80
$\exp(-x^2)$	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}

.

The influence of $\psi_s(x)$ vanishes soon.

I) $s = 0$:
With $k \geq 1$ holds

$$a_{k+1}^{(0)} = \frac{(8k+3)a_k^{(0)} - 4a_{k-1}^{(0)}}{(2k+1)(2k+3)} , \quad b_{k+1}^{(0)} = \frac{(8k-1)b_k^{(0)} - 4b_{k-1}^{(0)}}{(2k+1)^2} .$$

$$\int_0^x e^{-t^2} J_0(t)\, dt =$$

$$= e^{-x^2} \left[\left(x + \frac{x^3}{3} - \frac{x^5}{45} - \frac{79}{1575} x^7 - \dots \right) J_0(x) + \left(x^2 + \frac{7}{9} x^4 + \frac{23}{75} x^6 + \dots \right) J_1(x) \right]$$

k	$\alpha_k^{(0)}$	$\gamma_k^{(0)}$	$a_k^{(0)}$
0	1	1	1.00000 00000
1	1	3	0.33333 33333
2	-1	45	-0.02222 22222
3	-79	1575	-0.05015 87302
4	-1993	99225	-0.02008 56639
5	-7121	1403325	-0.00507 43769
6	-1354193	1404728325	-0.00096 40248
7	-40551359	273922023375	-0.00014 80398
8	-1336259641	69850115960625	-0.00001 91304
9	-48167009767	22561587455281875	-0.00000 21349
10	-1886078276353	9002073394657468125	-0.00000 02095
11	-79669949349167	4348001449619557104375	-0.00000 00183
12	-515151737265743	35715726193303504785 9375	-0.00000 00014
13	-9145224759056621	888193717175574000597 65625	-0.00000 00001

k	$\beta_k^{(0)}$	$\delta_k^{(0)}$	$b_k^{(0)}$
0	0	1	0.00000 00000
1	1	1	1.00000 00000
2	7	9	0.77777 77778
3	23	75	0.3066666667
4	887	11025	0.0804535147
5	13973	893025	0.0156468184
6	85853	36018675	0.0023835691
7	5342341	18261468225	0.0002925472
8	119718871	4108830350625	0.0000291370
9	33755333	14659900880625	0.0000023026
10	5066536837	38970014695486875	0.0000001300
11	26744808373	11120208311047460625	0.0000000024
12	-19585169733827	33334677780416604466875	-0.0000000006
13	-594894329175841	5682047348934648488671875	-0.0000000001

One has $a_0^{(0)}, a_1^{(0)} > 0$, $a_2^{(0)}, \ldots, a_{22}^{(0)} < 0$, but $a_{23}^{(0)} > 0$. Holds $b_{11}^{(0)} \cdot b_{12}^{(0)} < 0$ and $b_{41}^{(0)} \cdot b_{42}^{(0)} < 0$.

First positive zeros of $P_0(x)$: 1.5091, 4.8104, 7.9822 .

Maxima: $P_0(1.1915) = 1.3870$, $P_0(7.9191) = 3.6414 \cdot 10^{26}$, mimimum: $P_0(4.7057) = -1.1187 \cdot 10^9$.

First positive zeros of $Q_0(x)$: 3.3521, 6.4769, 9.6124.

Maxima: $Q_0(3.2008) = 9162.0$, $Q_0(9.5602) = 9.5630 \cdot 10^{38}$, minimum: $Q_0(6.3994) = -1.4373 \cdot 10^{17}$

Approximation:

$$F_0(x) \approx \tilde{F}_0(x) = 0.7851505503 \operatorname{erf}(x) + e^{-x^2} \sum_{k=0}^{5} c_k^{(0)} x^{2k+1}$$

k	$c_k^{(0)}$	k	$c_k^{(0)}$	k	$c_k^{(0)}$
0	$1.14052\ 47597 \cdot 10^{-1}$	1	$-7.29834\ 93536 \cdot 10^{-3}$	2	$2.05660\ 25858 \cdot 10^{-4}$
3	$-3.24389\ 43744 \cdot 10^{-6}$	4	$3.26550\ 30992 \cdot 10^{-8}$	5	$-2.27888\ 93576 \cdot 10^{-10}$

$$10^{10} \left[\tilde{F}_0(x) - F_0(x) \right]$$
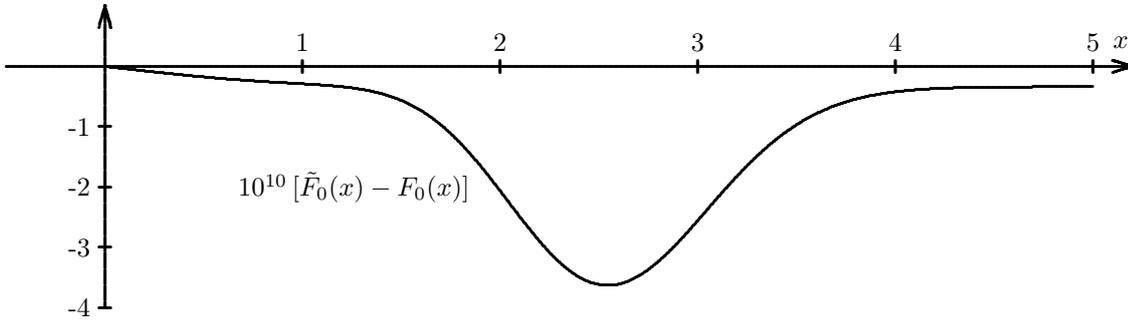
Asymptotic expansion:

$$\int_0^x e^{-t^2} J_0(t)\, dx \quad \sim \quad \frac{\sqrt{\pi}\, e^{-1/8}}{2} I_0\left(\frac{1}{8}\right) +$$

$$+ \frac{\sqrt{2} e^{-x^2}}{\sqrt{\pi x}} \left[\left(-\frac{1}{2x} + \frac{129}{256x^3} - \frac{76203}{65536x^5} + \ldots \right) \sin\left(x + \frac{\pi}{4}\right) + \left(-\frac{3}{16x^2} + \frac{921}{2048x^4} - \frac{775773}{524288x^6} + \ldots \right) \cos\left(x + \frac{\pi}{4}\right) \right]$$

See the remark on page 150.

Let

$$\varphi_1(x) = e^{x^2} \left[\int_0^x e^{-t^2} J_0(t)\, dt - \frac{\sqrt{\pi}\, e^{-1/8}}{2} I_0\left(\frac{1}{8}\right) \right] , \quad \varphi_2(x) = -\frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{1}{2x} \sin\left(x + \frac{\pi}{4}\right) ,$$

$$\varphi_3(x) = -\frac{\sqrt{2}}{\sqrt{\pi x}} \left[\frac{1}{2x} \sin\left(x + \frac{\pi}{4}\right) + \frac{3}{16x^2} \cos\left(x + \frac{\pi}{4}\right) \right] .$$

The following figure shows that $\varphi_1(x) \approx \varphi_2(x)$ if $x > 2$. From this

$$e^{x^2} \left[\int_0^x e^{-t^2} J_0(t)\, dt - \frac{\sqrt{\pi}\, e^{-1/8}}{2} I_0\left(\frac{1}{8}\right) \right] \approx -\frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{1}{2x} \sin\left(x + \frac{\pi}{4}\right)$$

and therefore holds

$$\int_0^x e^{-t^2} J_0(t)\, dt \approx \frac{\sqrt{\pi}\, e^{-1/8}}{2} I_0\left(\frac{1}{8}\right) - \frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{e^{-x^2}}{2x} \sin\left(x + \frac{\pi}{4}\right)\ .$$

It can be seen that $\varphi_3(x)$ is better than $\varphi_2(x)$ if $x > 4$.
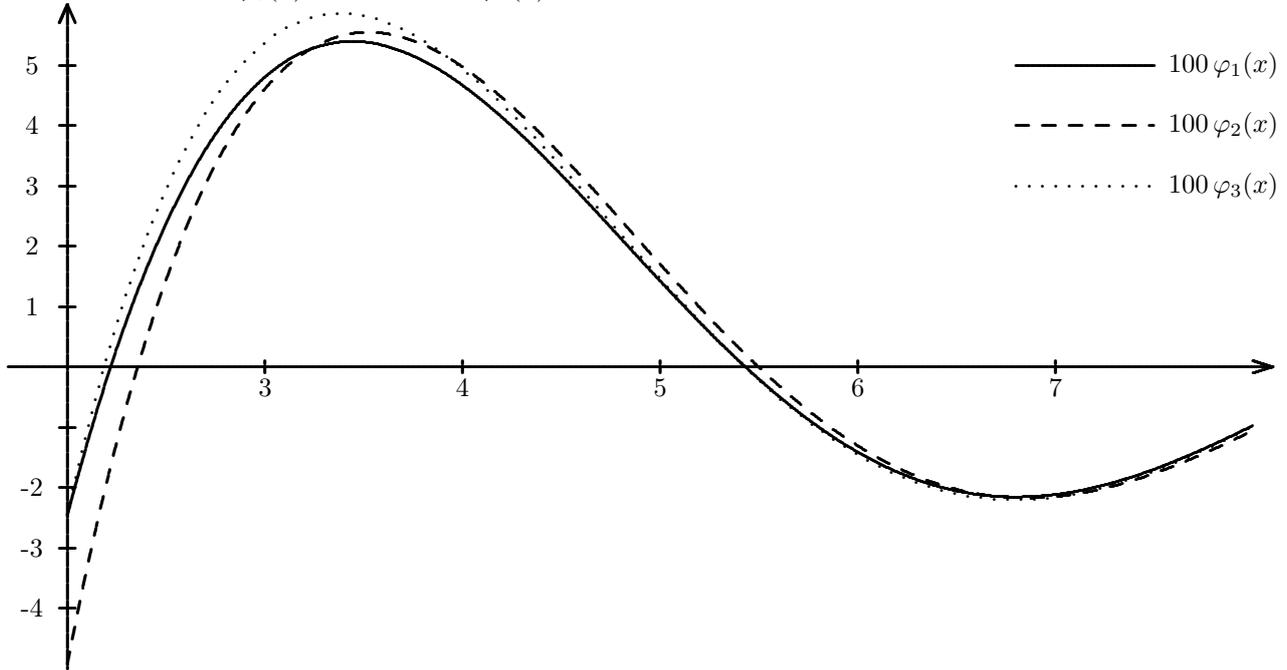
II) $s = 1$:
With $k \geq 1$ holds

$$a_{k+1}^{(1)} = \frac{(8k-1)a_k^{(1)} - 4a_{k-1}^{(1)}}{4k(k+1)}\ , \quad b_{k+1}^{(1)} = \frac{(8k+3)b_k^{(1)} - 4b_{k-1}^{(1)}}{4(k+1)^2}\ .$$

$$\int_0^x e^{-t^2} J_1(t)\, dt =$$

$$= 1 + e^{-x^2}\left[\left(-1 - x^2 - \frac{3}{8}x^3 - \frac{13}{192}x^5 - \dots\right) J_0(x) + \left(-\frac{x^3}{2} - \frac{11}{32}x^5 - \frac{145}{1152}x^7 - \dots\right) J_1(x)\right]$$

k	$\alpha_k^{(1)}$	$\gamma_k^{(1)}$	$a_k^{(1)}$
0	-1	1	-1.00000 00000
1	-1	1	-1.00000 00000
2	-3	8	-0.37500 00000
3	-13	192	-0.06770 83333
4	-11	9216	-0.00119 35764
5	431	147456	0.00292 29058
6	17513	17694720	0.00098 97303
7	88033	424673280	0.00020 72958
8	3160567	95126814720	0.00003 32248
9	17176879	3913788948480	0.00000 43888
10	4895935679	9862748150169600	0.00000 04964
11	213635978321	4339609186074624000	0.00000 00492
12	9969483318887	2291313650247401472000	0.00000 00044
13	495901729080313	142977971775437851 8528000	0.00000 00003

k	$\beta_k^{(1)}$	$\delta_k^{(1)}$	$b_k^{(1)}$
0	0	1	0.00000 00000
1	-1	2	-0.50000 00000
2	-11	32	-0.34375 00000
3	-145	1152	-0.12586 80556
4	-259	8192	-0.03161 62109
5	-8893	1474560	-0.00603 09516
6	-195919	212336640	-0.00092 26811
7	-231881	1981808640	-0.00011 70047
8	-19100009	1522029035520	-0.00001 25490
9	-567362171	493137407508480	-0.00000 11505
10	-5932850387	65751654334464000	-0.00000 00902
11	-569500272763	95471402093641728000	-0.00000 00060
12	-17366529773737	5499152760593763532800	-0.00000 00003

One has $a_4^{(1)} \cdot a_5^{(1)} < 0$, $a_{27}^{(1)} \cdot a_{28}^{(1)} < 0$ and $b_{13}^{(1)} \cdot b_{14}^{(1)} < 0$, $b_{47}^{(1)} \cdot b_{48}^{(1)} < 0$.

First positive zeros of $P_1(x)$: 2.0140, 5.2565 and 8.4241.

Minima: $P_1(1.7541) = -7.0905$ and $P_1(8.3646) = -4.1348 \cdot 10^{29}$, maximum: $P_1(5.1608) = 7.8803 \cdot 10^{10}$

First positive zeros of $Q_1(x)$: 3.7880, 6.9155 and 10.0515.

Minima: $Q_1(3.6545) = -1.5968 \cdot 10^5$ and $Q_1(10.0017) = -4.3429 \cdot 10^{42}$,

maximum: $Q_1(6.8429) = 4.0769 \cdot 10^{19}$

Approximation:

$$F_1(x) \approx \tilde{F}_1(x) = 0.22119\,92169 + e^{-x^2} \left[-0.22119\,92169 + \sum_{k=1}^{6} c_k^{(1)} x^{2k} \right]$$

k	$c_k^{(1)}$	k	$c_k^{(1)}$	k	$c_k^{(1)}$
1	$2.88007\,83071 \cdot 10^{-2}$	2	$-1.22460\,84642 \cdot 10^{-3}$	3	$2.58249\,56345 \cdot 10^{-5}$
4	$-3.25444\,94146 \cdot 10^{-7}$	5	$2.72785\,19850 \cdot 10^{-9}$	6	$-1.63082\,82205 \cdot 10^{-11}$

$10^{11} \left[\tilde{F}_1(x) - F_1(x) \right]$
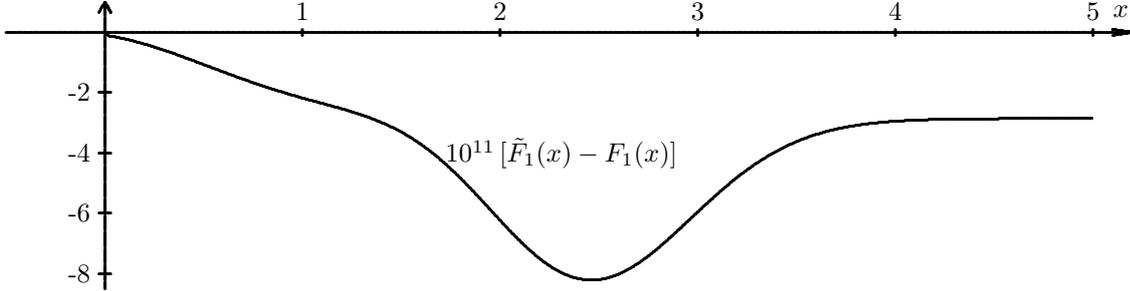
Asymptotic expansion:

$$\int_0^x e^{-t^2} J_1(t)\,dt \quad \sim \quad 1 - e^{-1/4} +$$

$$+ \frac{\sqrt{2} e^{-x^2}}{\sqrt{\pi x}} \left[\left(\frac{1}{2x} - \frac{137}{256 x^3} + \frac{85019}{65536 x^5} + \ldots \right) \cos\left(x + \frac{\pi}{4} \right) + \left(-\frac{7}{16 x^2} + \frac{1773}{2048 x^4} - \frac{1434089}{524288 x^6} + \ldots \right) \sin\left(x + \frac{\pi}{4} \right) \right]$$

See the remark on page 150.

Let

$$\varphi_1(x) = e^{x^2} \left[\int_0^x e^{-t^2} J_1(t)\,dt - 1 + e^{-1/4} \right] , \quad \varphi_2(x) = \frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{1}{2x} \cos\left(x + \frac{\pi}{4} \right) ,$$

$$\varphi_3(x) = \frac{\sqrt{2}}{\sqrt{\pi x}} \left[\frac{1}{2x} \cos\left(x + \frac{\pi}{4} \right) - \frac{7}{16 x^2} \sin\left(x + \frac{\pi}{4} \right) \right] .$$
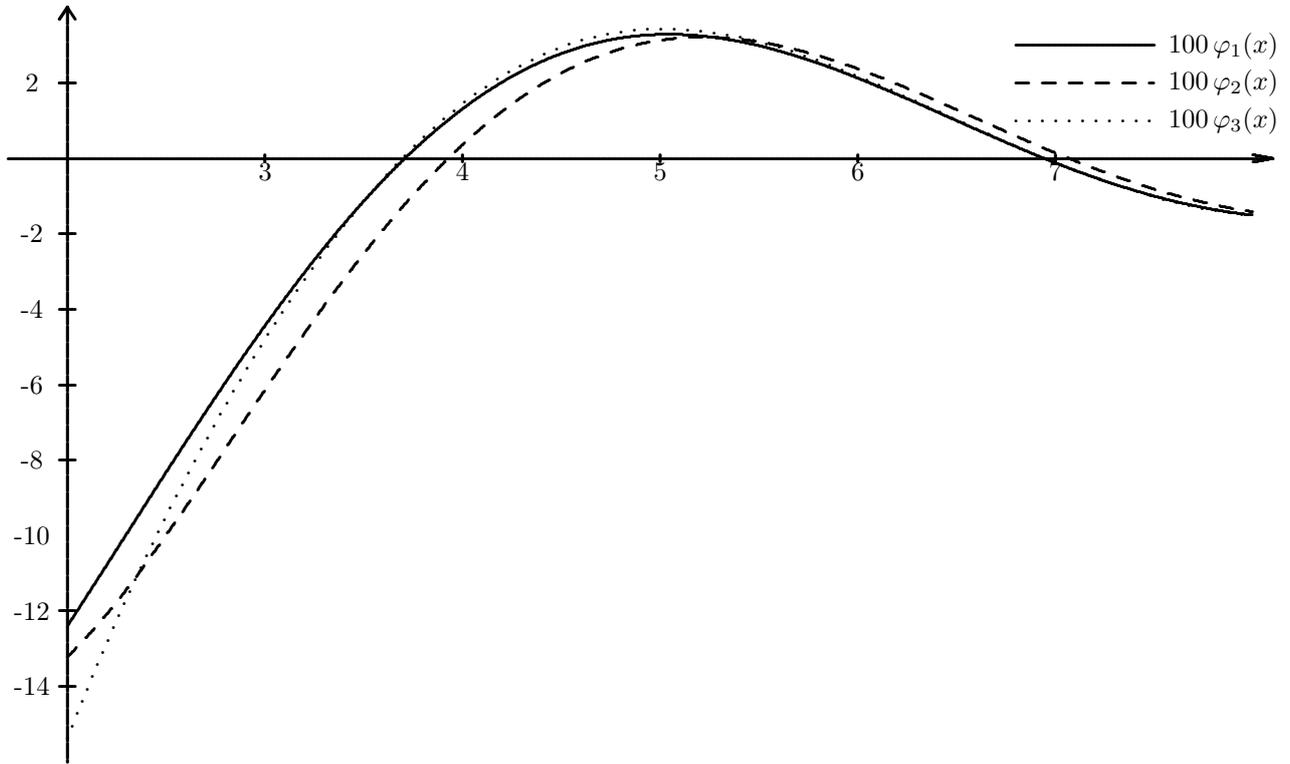
III) $s = -$:

With $k \geq 1$ holds

$$a_{k+1}^{(-)} = \frac{(8k+3)a_k^{(-)} - 4a_{k-1}^{(-)}}{(2k+1)(2k+3)} \ , \quad b_{k+1}^{(-)} = \frac{(8k-1)b_k^{(-)} - 4b_{k-1}^{(-)}}{(2k+1)^2} \ .$$

$$\int_0^x \frac{e^{-t^2} J_0(t)\, dt}{t} =$$

$$= e^{-x^2} \left[\left(x + x^3 + \frac{7x^5}{15} + \frac{73}{525} x^7 - \ldots \right) J_0(x) + \left(-1 - x^2 - \frac{x^4}{3} - \frac{x^6}{25} + \ldots \right) J_1(x) \right]$$

k	$\alpha_k^{(0)}$	$\gamma_k^{(0)}$	$a_k^{(0)}$
0	1	1	1.00000 00000
1	1	1	1.00000 00000
2	7	15	0.46666 66667
3	73	525	0.13904 76190
4	991	33075	0.02996 22071
5	2327	467775	0.00497 46139
6	307991	468242775	0.00065 77592
7	6390233	91307341125	0.00006 99860
8	136790767	23283371986875	0.00000 58750
9	2646943729	7520529151760625	0.00000 03520
10	21787108711	3000691131552489375	0.00000 00726
11	-2416192168471	1449333816539852368125	-0.00000 00017
12	-37423740194359	1190524206443450015953125	-0.00000 00003

k	$\beta_k^{(-)}$	$\delta_k^{(-)}$	$b_k^{(-)}$
0	-1	1	-1.00000 00000
1	-1	1	-1.00000 00000
2	-1	3	-0.33333 33333
3	-1	25	-0.0400 00000
4	31	3675	0.00843 53741
5	1549	297675	0.00520 36617
6	16789	12006225	0.00139 83579
7	1617533	6087156075	0.00026 57289
8	54916223	1369610116875	0.00004 00962
9	24740029	4886633626875	0.00000 50628
10	7163341181	12990004898495625	0.00000 05145
11	195955925549	3706736103682486875	0.00000 00529
12	50273780536949	11111559260138868155625	0.00000 00045
13	661736445380167	18940157829782161622890625	0.00000 00003

One has $a_{10}^{(-)} \cdot a_{11}^{(-)} < 0$, $a_{41}^{(-)} \cdot a_{42}^{(-)} < 0$ and $b_3^{(-)} \cdot b_4^{(-)} < 0$, $b_{25}^{(-)} \cdot b_{26}^{(-)} < 0$, $b_{65}^{(-)} \cdot b_{66}^{(-)} < 0$.

First positive zeros of $P_-(x)$: 3.2793, 6.4739 and 9.6341.

Maxima: $P_-(3.1244) = 4757.3$ and $P_-(9.5821) = 1.2302 \cdot 10^{39}$, minimum: $P_-(6.3963) = -1.1684 \cdot 10^{17}$

First positive zeros of $Q_-(x)$: 1.8812, 4.9850 and 8.1173.

Minima: $Q_-(1.6008) = -4.7895$ and $Q_-(8.0555) = -2.7233 \cdot 10^{27}$, maximum: $Q_-(4.8841) = 5.2197 \cdot 10^9$

Approximation:

$$F_-(x) \approx \tilde{F}_-(x) = 0.41706\ 34325 \operatorname{erf}(x) + e^{-x^2} \sum_{k=0}^{5} c_k^{(-)} x^{2k+1}$$

k	$c_k^{(-)}$	k	$c_k^{(-)}$	k	$c_k^{(-)}$
0	$2.93943\ 11364 \cdot 10^{-2}$	1	$-1.23712\ 57573 \cdot 10^{-3}$	2	$2.59830\ 30409 \cdot 10^{-5}$
3	$-3.26773\ 05781 \cdot 10^{-7}$	4	$2.73580\ 96839 \cdot 10^{-9}$	5	$-1.63439\ 98979 \cdot 10^{-11}$

$10^{11}\left[\tilde{F}_1(x) - F_1(x)\right]$
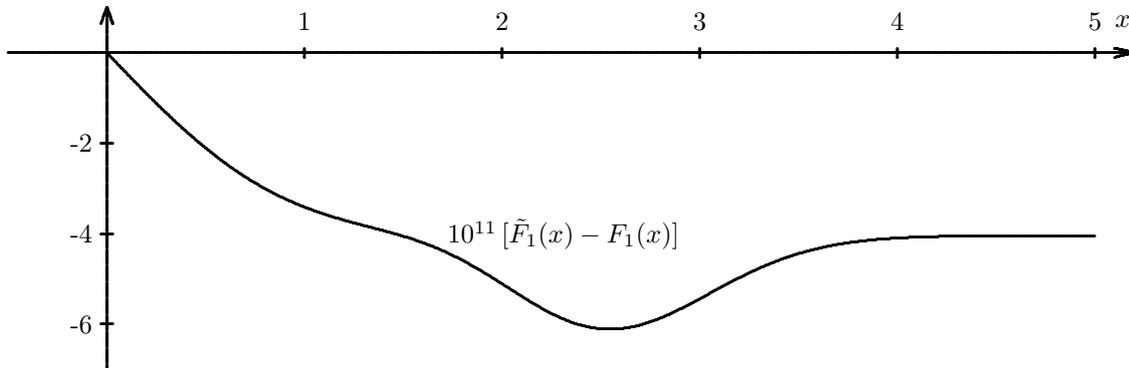
Asymptotic expansion:

$$\int_0^x \frac{e^{-t^2} J_1(t)\,dt}{t} \sim \frac{\sqrt{\pi}\,e^{-1/8}}{4}\left[I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] +$$

$$+\sqrt{\frac{2}{\pi x}}\,e^{-x^2}\left[\left(\frac{1}{2x^2} - \frac{201}{256x^4} + \frac{150683}{65536x^6} - \dots\right)\cos\left(x + \frac{\pi}{4}\right) + \left(-\frac{7}{16x^3} + \frac{2477}{2048x^5} - \frac{2419305}{524288x^7} + \dots\right)\sin\left(x + \frac{\pi}{4}\right)\right]$$

See the remark on page 150.

Let

$$\varphi 1 - (x) = e^{x^2}\left[\int_0^x \frac{e^{-t^2} J_1(t)\,dt}{t} - \frac{\sqrt{\pi}\,e^{-1/8}}{4}\right], \quad \varphi_2(x) = \frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{1}{2x^2}\cos\left(x + \frac{\pi}{4}\right),$$

$$\varphi_3(x) = \frac{\sqrt{2}}{\sqrt{\pi x}}\left[\frac{1}{2x^2}\cos\left(x + \frac{\pi}{4}\right) - \frac{7}{16x^3}\sin\left(x + \frac{\pi}{4}\right)\right].$$

$$100\,\varphi_1(x)$$

$$100\,\varphi_2(x)$$

$$100\,\varphi_3(x)$$

b) Integrals $(\alpha = 1)$

Let
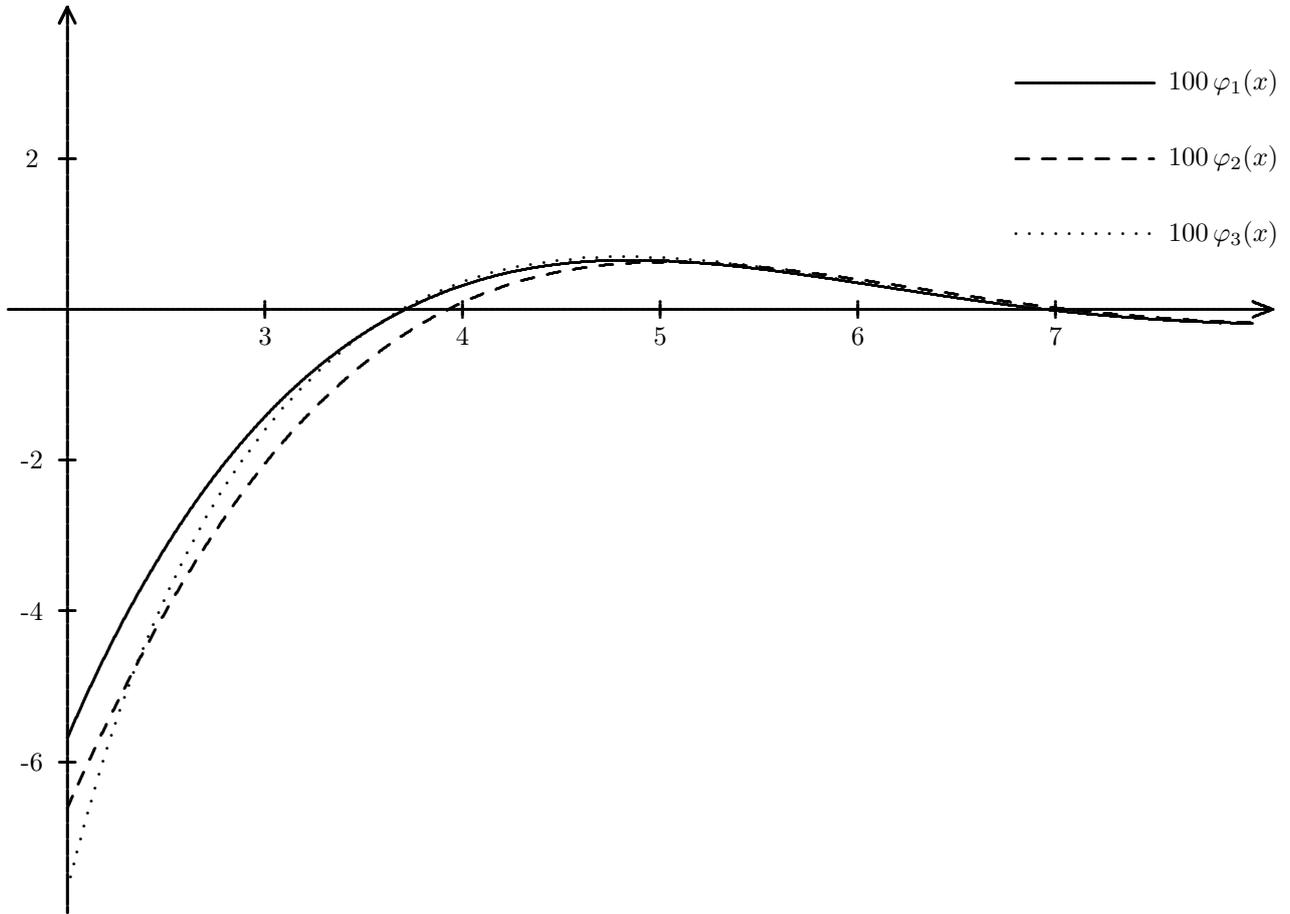
$$\mathcal{J}_\nu \;=\; \int e^{-x^2} J_\nu(x)\,dx\,, \qquad \nu = 0,1 \quad \text{and} \quad \mathcal{J}_- \;=\; \int \frac{e^{-x^2} J_\nu(x)\,dx}{x}\,.$$

$$\int x e^{-x^2} J_0(x)\,dx = -\frac{1}{2}\,e^{-x^2} J_0(x) - \frac{1}{2}\,\mathcal{J}_1$$

$$\int x e^{-x^2} J_1(x)\,dx = -\frac{1}{2}\,e^{-x^2} J_1(x) + \frac{1}{2}\,\mathcal{J}_0 - \frac{1}{2}\,\mathcal{J}_-$$

$$\int x^2 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{4}\,[-2x J_0(x) + J_1(x)] + \frac{1}{4}\,\mathcal{J}_0 + \frac{1}{4}\,\mathcal{J}_-$$

$$\int x^2 e^{-x^2} J_1(x)\,dx = \frac{e^{-x^2}}{4}\,[-J_0(x) - 2x J_1(x)] - \frac{1}{4}\,\mathcal{J}_1$$

$$\int x^3 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{8}\,\left[-(4x^2 + 3)J_0(x) + 2x J_1(x)\right] - \frac{3}{8}\,\mathcal{J}_1$$

$$\int x^3 e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{8}\,\left[2x J_0(x) + (4x^2 + 1)J_1(x)\right] + \frac{3}{8}\,\mathcal{J}_0 - \frac{1}{8}\,\mathcal{J}_-$$

$$\int x^4 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{16}\,\left[-(8x^3 + 10x)J_0(x) + (4x^2 + 7)J_1(x)\right] + \frac{3}{16}\,\mathcal{J}_0 + \frac{7}{16}\,\mathcal{J}_-$$

$$\int x^4 e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{16}\,\left[(4x^2 + 7)J_0(x) + (8x^3 + 6x)J_1(x)\right] - \frac{7}{16}\,\mathcal{J}_1$$

$$\int x^5 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{32}\,\left[-(16x^4 + 28x^2 + 17)J_0(x) + (8x^3 + 22x)J_1(x)\right] - \frac{17}{32}\,\mathcal{J}_1$$

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$$\int x^5 e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{32}\left[(8x^3 + 22x)J_0(x) + (16x^4 + 20x^2 - 1)J_1(x)\right] + \frac{21}{32}\,\mathcal{J}_0 + \frac{1}{32}\,\mathcal{J}_-$$

$$\int x^6 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{64}\left[-(32x^5 + 72x^3 + 78x)J_0(x) + (16x^4 + 66x^2 + 69)J_1(x)\right] + \frac{9}{64}\,\mathcal{J}_0 + \frac{69}{64}\,\mathcal{J}_-$$

$$\int x^6 e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{64}\left[(16x^4 + 60x^2 + 73)J_0(x) + (32x^5 + 56x^3 + 26x)J_1(x)\right] - \frac{73}{64}\,\mathcal{J}_1$$

$$\int x^7 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{128}\left[-(64x^6 + 176x^4 + 276x^2 + 131)J_0(x) + (32x^5 + 152x^3 + 290x)J_1(x)\right] - \frac{131}{128}\,\mathcal{J}_1$$

$$\int x^7 e^{-x^2} J_1(x)\,dx =$$

$$= -\frac{e^{-x^2}}{128}\left[(32xu + 152x^3 + 298x)J_0(x) + (64x^6 + 144x^4 + 140x^2 + 79)J_1(x)\right] + \frac{219}{128}\,\mathcal{J}_0 + \frac{79}{128}\,\mathcal{J}_1$$

$$\int x^8 e^{-x^2} J_0(x)\,dx =$$

$$= \frac{e^{-x^2}}{256}\left[-(128\,x^7 + 416\,x^5 + 856\,x^3 + 794\,x)J_0(x) + (64\,x^6 + 368\,x^4 + 980\,x^2 + 887)J_1(x)\right] - \frac{93}{256}\,\mathcal{J}_0 + \frac{887}{256}\,\mathcal{J}_-$$

$$\int x^8 e^{-x^2} J_1(x)\,dx =$$

$$= -\frac{e^{-x^2}}{256}\left[(64\,x^6 + 368\,x^4 + 996\,x^2 + 1007)J_0(x) + (128\,x^7 + 352\,x^5 + 520\,x^3 + 22\,x)J_1(x)\right] - \frac{1007}{256}\,\mathcal{J}_1$$

$$\int x^9 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{512}\left[-(256\,x^8 + 960\,x^6 + 2448\,x^4 + 3420\,x^2 + 1089)J_0(x)\right.$$

$$\left.+(128\,x^7 + 864\,x^5 + 2952\,x^3 + 4662\,x)J_1(x)\right] - \frac{1089}{512}\,\mathcal{J}_1$$

$$\int x^9 e^{-x^2} J_0(x)\,dx = -\frac{e^{-x^2}}{512}\left[(128\,x^7 + 864\,x^5 + 2984\,x^3 + 4966\,x)J_0(x)+\right.$$

$$\left.+(256\,x^8 + 832\,x^6 + 1648\,x^4 + 980\,x^2 - 1993)J_1(x)\right] + \frac{2973}{512}\,\mathcal{J}_0 + \frac{1993}{512}\,\mathcal{J}_-$$

$$\int x^{10} e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{1024}\left[-(512\,x^9 + 2176\,x^7 + 6624\,x^5 + 12424\,x^3 + 9326\,x)J_0(x)+\right.$$

$$\left.+(256\,x^8 + 1984\,x^6 + 8272\,x^4 + 18620\,x^2 + 13973)J_1(x)\right] - \frac{4647}{1025}\,\mathcal{J}_0 + \frac{13973}{1024}\,\mathcal{J}_-$$

$$\int x^{10} e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{1024}\left[(256\,x^8 + 1984\,x^6 + 8336\,x^4 + 19356\,x^2 + 17201)J_0(x)+\right.$$

$$\left.+(512\,x^9 + 1920\,x^7 + 4768\,x^5 + 5368\,x^3 - 4310\,x)J_1(x)\right] - \frac{17201}{1024}\,\mathcal{J}_1$$

Recurrence relations:

$$\int x^{2n+2}\, e^{-x^2} J_0(x)\,dx =$$

$$= \frac{x^{2n}\, e^{-x^2}}{4}\left[J_1(x) - 2xJ_0(x)\right] + \frac{4n+1}{4}\int x^{2n}\, e^{-x^2} J_0(x)\,dx - \frac{2n-1}{4}\int x^{2n-1}\, e^{-x^2} J_1(x)\,dx$$

$$\int x^{2n+1}\, e^{-x^2} J_0(x)\,dx = -\frac{x^{2n}\, e^{-x^2}}{2}\, J_0(x) + n\int x^{2n-1}\, e^{-x^2} J_0(x)\,dx - \frac{1}{2}\int x^{2n}\, e^{-x^2} J_1(x)\,dx$$

$$\int x^{2n+2}\, e^{-x^2} J_1(x)\,dx =$$

$$= -\frac{x^{2n}\,e^{-x^2}}{2}\,[\,J_0(x) + 2xJ_1(x)\,] + \frac{n}{2}\int x^{2n-1}\,e^{-x^2}\,J_0(x)\,dx + \frac{4n-1}{4}\int x^{2n}\,e^{-x^2}\,J_1(x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_1(x)\,dx = -\frac{x^{2n}\,e^{-x^2}}{2}\,J_1(x) + \frac{1}{2}\int x^{2n}\,e^{-x^2}\,J_0(x)\,dx + \frac{2n-1}{2}\int x^{2n-1}\,e^{-x^2}\,J_1(x)\,dx$$

c) General Case $\alpha \neq 1$, Basic Integrals

Some improper integrals:

$$\int_0^\infty e^{-x^2}\,J_0(\alpha x)\,dx = \frac{\sqrt{\pi}}{2}\,e^{-\alpha^2/8}\,I_0\left(\frac{\alpha^2}{8}\right)$$

$$\int_0^\infty e^{-x^2}\,J_1(\alpha x)\,dx = \frac{1 - e^{-\alpha^2/4}}{\alpha}$$

$$\int_0^\infty \frac{e^{-x^2}\,J_1(\alpha x)\,dx}{x} = \frac{\sqrt{\pi}}{4}\,e^{-\alpha^2/8}\left[I_0\left(\frac{\alpha^2}{8}\right) + I_1\left(\frac{\alpha^2}{8}\right)\right]$$

Let

$$F_\nu(x;\alpha) = \int_0^x e^{-t^2}\,J_\nu(\alpha t)\,dt = \frac{\nu}{\alpha} + e^{-x^2}\left[\,P_\nu(x;\alpha)\,J_0(\alpha x) + Q_\nu(x;\alpha)\,J_1(\alpha x)\,\right]\,, \quad \nu = 0,\,1,$$

with

$$P_\nu(x;\alpha) = \sum_{k=0}^\infty a_k^{(\nu;\alpha)}\,x^{2k+1-\nu} = \sum_{k=0}^\infty \frac{\alpha_k^{(\nu;\alpha)}}{\gamma_k^{(\nu)}}\,x^{2k+1-\nu} \quad \text{and} \quad Q_\nu(x) = \sum_{k=0}^\infty b_k^{(\nu;\alpha)}\,x^{2k+\nu} = \sum_{k=0}^\infty \frac{\beta_k^{(\nu;\alpha)}}{\delta_k^{(\nu)}}\,x^{2k+\nu}\,.$$

Furthermore, let

$$F_-(x;\alpha) = \int_0^x \frac{e^{-t^2}\,J_1(\alpha t)\,dt}{t} = e^{-x^2}\left[\,P_-(x;\alpha)\,J_0(\alpha x) + Q_-(x;\alpha)\,J_1(\alpha x)\right]$$

with

$$P_-(x) = \sum_{k=0}^\infty a_k^{(-;\alpha)}\,x^{2k+1} = \sum_{k=0}^\infty \frac{\alpha_k^{(-;\alpha)}}{\gamma_k^{(-)}}\,x^{2k+1} \quad \text{and} \quad Q_-(x;\alpha) = \sum_{k=0}^\infty b_k^{(-;\alpha)}\,x^{2k+\nu} = \sum_{k=0}^\infty \frac{\beta_k^{(-;\alpha)}}{\delta_k^{(-)}}\,x^{2k}\,.$$

I) $s = 0$:

$$F_0(x;\alpha) =$$

$$= e^{-x^2}\left[\left(x + \frac{2-\alpha^2}{3}\,x^3 + \frac{\alpha^4 - 14\alpha^2 + 12}{45}\,x^5 + \frac{-\alpha^6 + 34\alpha^4 - 232\alpha^2 + 120}{1575}\,x^7 + \ldots\right)J_0(\alpha x) +\right.$$

$$\left. + \left(\alpha x^2 + \frac{8\alpha - \alpha^3}{9}\,x^4 + \frac{\alpha^5 - 24\alpha^3 + 92\alpha}{225}\,x^6 \ldots\right)J_1(\alpha x)\right]$$

Recurrence relations:

$$a_{k+1}^{(0;\alpha)} = \frac{(4k+2-\alpha^2)a_k^{(0;\alpha)} - 2\alpha b_k^{(0;\alpha)}}{(2k+1)(2k+3)}\,, \quad b_{k+1}^{(0;\alpha)} = \frac{2b_k^{(0;\alpha)} + \alpha a_k^{(0;\alpha)}}{2k+1}$$

Approximation:

$$F_0(x;\alpha) = \frac{\sqrt{\pi}}{2}\,e^{-\alpha^2/8}\,I_0\left(\frac{\alpha^2}{8}\right)\text{erf}(x) + e^{-x^2}\sum_{k=0}^5 c_k^{(0;\alpha)}\,x^{2k+1}$$

$$c_0^{(0;\alpha)} = -\frac{77}{1006632960}\,\alpha^{12} + \frac{21}{10485760}\,\alpha^{10} - \frac{35}{786432}\,\alpha^8 + \frac{5}{6144}\,\alpha^6 - \frac{3}{256}\,\alpha^4 + \frac{\alpha^2}{8} =$$

$$= -0.00000\,00765\,\alpha^{12} + 0.00000\,20027\,\alpha^{10} - 0.00004\,45048\,\alpha^8 + 0.00081\,38021\,\alpha^6 -$$

$$-0.01171\,87500\,\alpha^4 + 0.12500\,00000\,\alpha^2$$

$$c_1^{(0;\alpha)} = -0.00000\,00510\,\alpha^{12} + 0.00000\,13351\,\alpha^{10} - 0.00002\,96699\,\alpha^8 + 0.00054\,25347\,\alpha^6 - 0.00781\,25000\,\alpha^4$$

$$c_2^{(0;\alpha)} = -0.00000\,00204\,\alpha^{12} + 0.00000\,05341\,\alpha^{10} - 0.00001\,18679\,\alpha^8 + 0.00021\,70139\,\alpha^6$$

$$c_3^{(0;\alpha)} = -0.00000\,00058\,\alpha^{12} + 0.00000\,01526\,\alpha^{10} - 0.00000\,33908\,\alpha^8$$

$$c_4^{(0;\alpha)} = -0.00000\,00013\,\alpha^{12} + 0.00000\,00339\,\alpha^{10}$$

$$c_5^{(0;\alpha)} = -0.00000\,00002\,\alpha^{12}$$

In the case of small values of $|x|$ and $\alpha >> 1$ the following approximation may be used ($\Phi(x)$ defined as on page 7):

$$F_0(x;\alpha) = \sigma^{(0)}(\alpha)\,\Phi(\alpha x) + \left[\sum_{k=0}^{7}\varphi_k^{(0)}(\alpha)\,x^{2k+1}\right]J_0(\alpha x) + \left[\sum_{k=0}^{7}\psi_k^{(0)}(\alpha)\,x^{2k+2}\right]J_1(\alpha x)$$

$$\sigma^{(0)}(\alpha) = 1 + \frac{1}{\alpha^2} + \frac{9}{2\,\alpha^4} + \frac{75}{2\,\alpha^6} + \frac{3675}{8\,\alpha^8} + \frac{59535}{8\,\alpha^{10}} + \frac{2401245}{16\,\alpha^{12}} + \frac{57972915}{16\,\alpha^{14}} + \frac{13043905875}{128\,\alpha^{16}} + \frac{418854310875}{128\,\alpha^{18}} +$$

$$+\frac{30241281245175}{256\,\alpha^{20}} + \frac{1212400457192925}{256\,\alpha^{22}} + \frac{213786613951685775}{1024\,\alpha^{24}} + \frac{10278202593831046875}{1024\,\alpha^{26}} +$$

$$+\frac{1070401384414690453125}{2048\,\alpha^{28}} + \frac{60013837619516978071875}{2048\,\alpha^{30}} =$$

$$= 1+0.04\left(\frac{5}{\alpha}\right)^2+0.0072\left(\frac{5}{\alpha}\right)^4+0.0024\left(\frac{5}{\alpha}\right)^6+0.00117\,6\left(\frac{5}{\alpha}\right)^8+0.00076\,2048\left(\frac{5}{\alpha}\right)^{10}+0.00061\,47187\left(\frac{5}{\alpha}\right)^{12}+$$

$$+0.00059\,36426\left(\frac{5}{\alpha}\right)^{14}+0.00066\,78479808\left(\frac{5}{\alpha}\right)^{16}+0.00085\,78136287\left(\frac{5}{\alpha}\right)^{-18}+0.00123\,86829\left(\frac{5}{\alpha}\right)^{20}+$$

$$+0.00198\,63969\left(\frac{5}{\alpha}\right)^{22}+0.00350\,26799\left(\frac{5}{\alpha}\right)^{24}+0.00673\,5922852\left(\frac{5}{\alpha}\right)^{26}+0.01402\,996503\left(\frac{5}{\alpha}\right)^{28}+$$

$$+0.03146\,45349\left(\frac{5}{\alpha}\right)^{30}$$

Let

$$\varphi_k^{(0)}(\alpha) = \sum_{j=0}^{7}s_j^{(k,0)}\left(\frac{5}{\alpha}\right)^{2k} \quad \text{and} \quad \psi_k^{(0)}(\alpha) = \sum_{j=0}^{7}t_j^{(k,0)}\left(\frac{5}{\alpha}\right)^{2k+1}$$

$s_j^{(k,0)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	1	0	0	0	0	0	0	0
1	0	0.0600000000	-0.0333333333	0.0116666667	-0.0030000000	0.0006111111	-0.0001031746	0.0000148810
2	0	0.0200000000	-0.0163333333	0.0075600000	-0.0024200000	0.0005901587	-0.0001160714	0.0000191138
3	0	0.0098000000	-0.0105840000	0.0060984000	-0.0023370286	0.0006639286	-0.0001490873	0.0000276003
4	0	0.0063504000	-0.0085377600	0.0058893120	-0.0026291571	0.0008527794	-0.0002152821	0.0000442608
5	0	0.0051226560	-0.0082450368	0.0066254760	-0.0033770063	0.0012314134	-0.0003452341	0.0000780465
6	0	0.0049470221	-0.0092756664	0.0085100558	-0.0048763971	0.0019747393	-0.0006087629	0.0001500895
7	0	0.0055653998	-0.0119140782	0.0122885206	-0.0078199677	0.0034821237	-0.0011706978	0.0003126149

$t_j^{(k,0)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	-0.2000000000	0.1000000000	-0.0333333333	0.0083333333	-0.0016666667	0.0002777778	-0.0000396825	0.0000049603
1	-0.0360000000	0.0333333333	-0.0163333333	0.0054000000	-0.0013444444	0.0002682540	-0.0000446429	0.0000063713
2	-0.0120000000	0.0163333333	-0.0105840000	0.0043560000	-0.0012983492	0.0003017857	-0.0000573413	0.0000092001
3	-0.0058800000	0.0105840000	-0.0085377600	0.0042066514	-0.0014606429	0.0003876270	-0.0000828008	0.0000147536
4	-0.0038102400	0.0085377600	-0.0082450368	0.0047324829	-0.0018761146	0.0005597334	-0.0001327824	0.0000260155
5	-0.0030735936	0.0082450368	-0.0092756664	0.0060786113	-0.0027091095	0.0008976088	-0.0002341396	0.0000500298
6	-0.0029682132	0.0092756664	-0.0119140782	0.0087775147	-0.0043444265	0.0015827835	-0.0004502684	0.0001042050
7	-0.0033392399	0.0119140782	-0.0172039289	0.0140759418	-0.0076606720	0.0030438144	-0.0009378448	0.0002336970

Asymptotic expansion:

$$F_0(x;\alpha) \quad \sim \quad \frac{\sqrt{\pi}}{2}\,e^{-\alpha^2/8}\,I_0\left(\frac{\alpha^2}{8}\right) +$$

$$+\frac{2\,e^{-x^2}}{\sqrt{\pi\alpha x}}\left[\left(\frac{1}{2x}-\frac{32\alpha^4+120\alpha^2-15}{256x^3}-\frac{2048\alpha^8+32256\alpha^6+60480\alpha^4-5040\alpha^2-4725}{65536x^5}+\ldots\right)\cos\left(x+\frac{\pi}{4}\right)+\right.$$

$$+\left(-\frac{4\alpha^2+3}{16x^2}+\frac{128\alpha^6+1120\alpha^4+420\alpha^2+105}{2048x^4}-\right.$$

$$\left.\left.\frac{8192\alpha^{10}+202752\alpha^8+887040\alpha^6+221760\alpha^4+41580\alpha^2+72765}{524288x^6}+\ldots\right)\sin\left(x+\frac{\pi}{4}\right)\right]$$

II) $s=1$:

$$\int_0^x e^{-t^2}\,J_1(\alpha t)\,dt =$$

$$=\frac{1}{\alpha}+\frac{e^{-x^2}}{\alpha}\left[\left(-1-x^2+\frac{\alpha^2-4}{8}\,x^4-\frac{\alpha^4-20\,\alpha^2+32}{192}\,x^6+\frac{\alpha^6-44\,\alpha^4+416\,\alpha^2-384}{9216}\,x^8+\ldots\right)J_0(\alpha x)+\right.$$

$$\left.+\left(-\frac{\alpha}{2}\,x^3+\frac{\alpha^3-12\,\alpha}{32}\,x^5-\frac{\alpha^5-32\,\alpha^3+176\,\alpha}{1152}\,x^7+\frac{\alpha^7-60\,\alpha^5+928\,\alpha^3-3200\,\alpha}{73728}\,x^9+\ldots\right)J_1(\alpha x)\right]$$

Recurrence relations:

$$a_{k+1}^{(1;\alpha)}=\frac{2a_k^{(1;\alpha)}-\alpha b_k^{(1;\alpha)}}{2k+2}\ ,\quad b_{k+1}^{(1;\alpha)}=\frac{2\alpha a_k^{(1;\alpha)}+(4k+4-\alpha^2)b_k^{(1;\alpha)}}{(2k+2)^2}$$

Approximation:

$$F_1(x;\alpha)=c_0^{(1;\alpha)}\left(1-e^{-x^2}\right)+e^{-x^2}\sum_{k=1}^{5}c_k^{(1;\alpha)}\,x^{2k}$$

$$c_0^{(1;\alpha)}=-0.00000\,03391\,\alpha^{11}+0.00000\,81380\,\alpha^9-0.00016\,27604\,\alpha^7+0.00260\,41667\,\alpha^5-0.03125\,\alpha^3+0.25\,\alpha$$

$$c_1^{(1;\alpha)}=0.00000\,03391\,\alpha^{11}-0.00000\,81380\,\alpha^9+0.00016\,27604\,\alpha^7-0.00260\,41667\,\alpha^5+0.03125\,\alpha^3$$

$$c_2^{(1;\alpha)}=0.00000\,01695\,\alpha^{11}-0.00000\,40690\,\alpha^9+0.00008\,13802\,\alpha^7-0.00130\,20833\,\alpha^5$$

$$c_3^{(1;\alpha)}=0.00000\,00565\,\alpha^{11}-0.00000\,13563\,\alpha^9+0.00002\,71267\,\alpha^7$$

$$c_4^{(1;\alpha)}=0.00000\,00141\,\alpha^{11}-0.00000\,03391\,\alpha^9\ ,\quad c_5^{(1;\alpha)}=0.00000\,00028\,\alpha^{11}$$

In the case of small values of $|x|$ and $\alpha >> 1$ the following approximation may be used:

$$F_1(x;\alpha)=\left[\sum_{k=1}^{7}\varphi_k^{(1)}(\alpha)\,x^{2k}\right]J_0(\alpha x)+\left[\sum_{k=0}^{7}\psi_k^{(1)}(\alpha)\,x^{2k+1}\right]J_1(\alpha x)$$

Let

$$\varphi_k^{(1)}(\alpha)=\sum_{j=0}^{7}s_j^{(k,1)}\left(\frac{5}{\alpha}\right)^{2k+1}\quad\text{and}\quad\psi_k^{(1)}(\alpha)=\sum_{j=0}^{7}t_j^{(k,1)}\left(\frac{5}{\alpha}\right)^{2k+2}$$

$s_j^{(k,1)}$:

j	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7
0	0.2000000000	-0.1000000000	0.0333333333	-0.0083333333	0.0016666667	-.0002777778	0.0000396825
1	0.0320000000	-0.0320000000	0.0160000000	-0.0053333333	0.0013333333	-0.0002666667	0.0000444444
2	0.0102400000	-0.0153600000	0.0102400000	-0.0042666667	0.0012800000	-0.0002986667	0.0000568889
3	0.0049152000	-0.0098304000	0.0081920000	-0.0040960000	0.0014336000	-0.0003822933	0.0000819200
4	0.0031457280	-0.0078643200	0.0078643200	-0.0045875200	0.0018350080	-0.0005505024	0.0001310720
5	0.0025165824	-0.0075497472	0.0088080384	-0.0058720256	0.0026424115	-0.0008808038	0.0002306867
6	0.0024159191	-0.0084557169	0.0112742892	-0.0084557169	0.0042278584	-0.0015502148	0.0004429185
7	0.0027058294	-0.0108233176	0.0162349764	-0.0135291470	0.0074410308	-0.0029764123	0.0009212705

$t_j^{(k,1)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	-0.0800000000	0.0800000000	-0.0400000000	0.0133333333	-0.0033333333	0.0006666667	-0.0001111111	0.0000158730
1	-0.0128000000	0.0256000000	-0.0192000000	0.0085333333	-0.0026666667	0.0006400000	-0.0001244444	0.0000203175
2	-0.0040960000	0.0122880000	-0.0122880000	0.0068266667	-0.0025600000	0.0007168000	-0.0001592889	0.0000292571
3	-0.0019660800	0.0078643200	-0.0098304000	0.0065536000	-0.0028672000	0.0009175040	-0.0002293760	0.0000468114
4	-0.0012582912	0.0062914560	-0.0094371840	0.0073400320	-0.0036700160	0.0013212058	-0.0003670016	0.0000823881
5	-0.0010066330	0.0060397978	-0.0105696461	0.0093952410	-0.0052848230	0.0021139292	-0.0006459228	0.0001581852
6	-0.0009663676	0.0067645735	-0.0135291470	0.0135291470	-0.0084557169	0.0037205154	-0.0012401718	0.0003290252
7	-0.0010823318	0.0086586541	-0.0194819717	0.0216466352	-0.0148820617	0.0071433896	-0.0025795574	0.0007370164

Asymptotic expansion:

$$F_1(x;\alpha) \sim \frac{1 - e^{-\alpha^2/4}}{\alpha} + \sqrt{\frac{2}{\pi\alpha x}}\, e^{-x^2}.$$

$$\cdot \left[\left(\frac{1}{2} - \frac{32\,\alpha^4 + 88\,\alpha^2 - 15}{256\,\alpha^2 x^2} + \frac{2048\,\alpha^8 + 26112\,\alpha^6 + 38976\,\alpha^4 - 4080\,\alpha^2 - 4725}{65536\,\alpha^4 x^4} + \ldots \right) \cos\left(x + \frac{\pi}{4}\right) - \right.$$

$$- \left(\frac{4\,\alpha^2 + 3}{16\,\alpha x} - \frac{128\,\alpha^6 + 864\,\alpha^4 + 324\,\alpha^2 + 105}{2048\,\alpha^3 x^3} + \right.$$

$$\left. \left. + \frac{8192\,\alpha^{10} + 169984\,\alpha^8 + 598272\,\alpha^6 + 149568\,\alpha^4 + 34860\,\alpha^2 + 72765}{524288\,\alpha^5 x^5} + \ldots \right) \sin\left(x + \frac{\pi}{4}\right) \right]$$

<u>III) $s = -$:</u>

$$F_-(x;\alpha) =$$

$$= e^{-x^2} \left[\left(\alpha x + \frac{4\alpha - \alpha^3}{3} x^3 + \frac{\alpha^5 - 16\alpha^3 + 36\alpha}{45} x^5 + \frac{-\alpha^7 + 36\alpha^4 - 296\alpha^3 + 480}{1575} x^7 + \ldots \right) J_0(\alpha x) + \right.$$

$$\left. + \left(-1 + (\alpha^2 - 2)x^2 + \frac{-\alpha^4 + 10\alpha^2 - 12}{9} x^4 + \frac{\alpha^6 - 26\alpha^4 + 136\alpha^2 - 120}{225} x^6 \ldots \right) J_1(\alpha x) \right]$$

Recurrence relations:

$$a_{k+1}^{(-;\alpha)} = \frac{(4k + 2 - \alpha^2)a_k^{(-;\alpha)} - 2\alpha b_k^{(-;\alpha)}}{(2k+1)(2k+3)} \;,\quad b_{k+1}^{(-;\alpha)} = \frac{2b_k^{(-;\alpha)} + \alpha a_k^{(-;\alpha)}}{2k+1}$$

Approximation:

$$F_-(x;\alpha) = c_0^{(-;\alpha)} \operatorname{erf}(x) + e^{-x^2} \sum_{k=1}^{5} c_k^{(-;\alpha)} x^{2k-1}$$

$$c_0^{(-;\alpha)} = -0.00000\,01479\,\alpha^{11} + 0.00000\,39441\,\alpha^9 - 0.00009\,01517\,\alpha^7 + 0.00173\,09120\,\alpha^5 -$$

$$- 0.02769\,45914\,\alpha^3 + 0.44311\,34627\,\alpha$$

$$c_1^{(-;\alpha)} = 0.00000\,016689\,\alpha^{11} - 0.00000\,44505\,\alpha^9 + 0.00010\,17253\,\alpha^7 - 0.00195\,3125\,\alpha^5 + 0.03125\,\alpha^3$$

$$c_2^{(-;\alpha)} = 0.00000\,01113\,\alpha^{11} - 0.00000\,29670\,\alpha^9 + 0.00006\,78168\,\alpha^7 - 0.00130\,20833\,\alpha^5$$

$$c_3^{(-;\alpha)} = 0.00000\,00445\,\alpha^{11} - 0.00000\,11868\,\alpha^9 + 0.00002\,71267\,\alpha^7$$

$$c_4^{(-;\alpha)} = 0.00000\,00127\,\alpha^{11} - 0.00000\,03391\,\alpha^9 \;,\quad c_5^{(-;\alpha)} = 0.00000\,00028\,\alpha^{11}$$

In the case of small values of $|x|$ and $\alpha \gg 1$ the following approximation may be used ($\Phi(x)$ defined as on page 7):

$$F_-(x;\alpha) = \sigma^{(-)}(\alpha)\,\Phi(\alpha x) + \left[\sum_{k=0}^{7} \varphi_k^{(-)}(\alpha)\, x^{2k+1}\right] J_0(\alpha x) + \left[\sum_{k=0}^{7} \psi_k^{(-)}(\alpha)\, x^{2k}\right] J_1(\alpha x)$$

Let

$$\varphi_k^{(0)}(\alpha) = \sum_{j=0}^{7} s_j^{(k,0)} \left(\frac{5}{\alpha}\right)^{2k-1} \quad \text{and} \quad \psi_k^{(-)}(\alpha) = \sum_{j=0}^{7} t_j^{(k,-)} \left(\frac{5}{\alpha}\right)^{2k}$$

$s_j^{(k,-)}$:

j	k = 0	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7
0	5	0	0	0	0	0	0	0
1	0	-0.1000000000	0.0333333333	-0.0083333333	0.0016666667	-0.0002777778	0.0000396825	-0.0000049603
2	0	-0.0200000000	0.0116666667	-0.0042000000	0.0011000000	-0.0002269841	0.0000386905	-0.0000056217
3	0	-0.0070000000	0.0058800000	-0.0027720000	0.0008988571	-0.0002213095	0.0000438492	-0.0000072632
4	0	-0.0035280000	0.0038808000	-0.0022651200	0.0008763857	-0.0002508175	0.0000566532	-0.0000105383
5	0	-0.0023284800	0.0031711680	-0.0022084920	0.0009932371	-0.0003240562	0.0000821986	-0.0000169666
6	0	-0.0019027008	0.0030918888	-0.0025029576	0.0012832624	-0.0004701760	0.0001323398	-0.0000300179
7	0	-0.0018551333	0.0035041406	-0.0032338212	0.0018618971	-0.0007569834	0.0002341396	-0.0000578917

$t_j^{(k,-)}$:

j	k = 0	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7
0	1	0	0	0	0	0	0	0
1	0	0.0600000000	-0.0333333333	0.0116666667	-0.0030000000	0.0006111111	-0.0001031746	0.0000148810
2	0	0.0120000000	-0.0116666667	0.0058800000	-0.0019800000	0.0004993651	-0.0001005952	0.0000168651
3	0	0.0042000000	-0.0058800000	0.0038808000	-0.0016179429	0.0004868810	-0.0001140079	0.0000217897
4	0	0.0021168000	-0.0038808000	0.0031711680	-0.0015774943	0.0005517984	-0.0001472983	0.0000316148
5	0	0.0013970880	-0.0031711680	0.0030918888	-0.0017878269	0.0007129235	-0.0002137164	0.0000508999
6	0	0.0011416205	-0.0030918888	0.0035041406	-0.0023098723	0.0010343873	-0.0003440834	0.0000900537
7	0	0.0011130800	-0.0035041406	0.0045273497	-0.0033514147	0.0016653635	-0.0006087629	0.0001736750

Asymptotic expansion:

$$F_-(x;\alpha) \sim \frac{\sqrt{\pi}}{4}\, e^{-\alpha^2/8}\left[I_0\left(\frac{\alpha^2}{8}\right) + I_1\left(\frac{\alpha^2}{8}\right)\right] + \sqrt{\frac{2}{\pi\alpha x}}\, e^{-x^2}.$$

$$\cdot\left[\left(\frac{1}{2x} - \frac{-32\,\alpha^4 - 152\,\alpha^2 + 15}{256\,\alpha^2 x^3} + \frac{2048\,\alpha^8 + 38400\,\alpha^6 + 86080\,\alpha^4 - 6000\,\alpha^2 - 4725}{65536\,\alpha^4 x^5} + \dots\right)\cos\left(x + \frac{\pi}{4}\right) -\right.$$

$$-\left(\frac{4\,\alpha^2 + 3}{16\,\alpha x^2} - \frac{128\,\alpha^6 + 1376\,\alpha^4 + 516\,\alpha^2 + 105}{2048\,\alpha^3 x^4} +\right.$$

$$\left.\left. + \frac{8192\,\alpha^{10} + 235520\,\alpha^8 + 1224960\,\alpha^6 + 306240\,\alpha^4 + 48300\,\alpha^2 + 72765}{524288\,\alpha^5 x^6} + \dots\right)\sin\left(x + \frac{\pi}{4}\right)\right]$$

d) Integrals ($\alpha \neq 1$)

$$\int x\, e^{-x^2} J_0(\alpha x)\, dx = -\frac{e^{-x^2}}{2} J_0(\alpha x) - \frac{\alpha}{2}\int e^{-x^2} J_1(\alpha x)\, dx$$

$$\int x\, e^{-x^2} J_1(\alpha x)\, dx = -\frac{e^{-x^2}}{2} J_1(\alpha x) - \frac{\alpha}{2}\int e^{-x^2} J_0(\alpha x)\, dx - \frac{1}{2}\int \frac{e^{-x^2} J_1(\alpha x)}{x}\, dx$$

Let

$$\int x^{2n+\nu}\, e^{-x^2} J_\nu(\alpha x)\, dx =$$

$$= e^{-x^2}\left[A_{n;\nu}(x;\alpha)\, J_0(\alpha x) + B_{n;\nu}(x;\alpha)\, J_1(\alpha x)\right] + P_{n;\nu}\int e^{-x^2} J_0(\alpha x)\, dx + Q_{n;\nu}\int \frac{e^{-x^2} J_1(\alpha x)\, dx}{x}$$

and

$$\int x^{2n+1-\nu}\, e^{-x^2} J_\nu(\alpha x)\, dx = e^{-x^2}\left[C_{n;\nu}(x;\alpha)\, J_0(\alpha x) + D_{n;\nu}(x;\alpha)\, J_1(\alpha x)\right] + R_{n;\nu}\int e^{-x^2} J_1(\alpha x)\, dx .$$

$$A_{1;0}(x;\alpha) = -\frac{x}{2}, \quad B_{1;0}(x;\alpha) = \frac{\alpha}{4}, \quad P_{1;0}(\alpha) = \frac{-\alpha^2 + 2}{4}, \quad Q_{1;0}(x;\alpha) = \frac{\alpha}{4}$$

$$A_{1;1}(x;\alpha) = -\frac{\alpha x}{8}, \quad B_{1;1}(x;\alpha) = \frac{-4x^2 + \alpha^2 - 2}{8}, \quad P_{1;1}(\alpha) = \frac{-\alpha^3 + 4\alpha}{8}, \quad Q_{1;1}(x;\alpha) = \frac{\alpha^2 - 2}{8}$$

$$C_{1;0}(x;\alpha) = \frac{-4x^2 + \alpha^2 - 4}{8}, \quad D_{1;0}(x;\alpha) = \frac{\alpha x}{4}, \quad R_{1;0}(\alpha) = \frac{\alpha^3 - 4\alpha}{8}$$

$$C_{1;1}(x;\alpha) = -\frac{\alpha}{4} \ , \quad D_{1;1}(x;\alpha) = -\frac{x}{2} \ , \quad R_{1;1}(\alpha) = -\frac{\alpha^2}{4}$$

$$A_{2;0}(x;\alpha) = \frac{-4\,x^3 + (\alpha^2 - 6)\,x}{8} \ , \quad B_{2;0}(x;\alpha) = \frac{4\,\alpha\,x^2 - \alpha^3 + 8\,\alpha}{16} \ ,$$

$$P_{2;0}(\alpha) = \frac{\alpha^4 - 10\,\alpha^2 + 12}{16} \ , \quad Q_{2;0}(x;\alpha) = \frac{-\alpha^3 + 8\,\alpha}{16}$$

$$A_{2;1}(x;\alpha) = \frac{-4\,\alpha\,x^3 + (\alpha^3 - 12\,\alpha)\,x}{16} \ , \quad B_{2;1}(x;\alpha) = \frac{-12 - 16\,x^4 + (4\,\alpha^2 - 24)\,x^2 - \alpha^4 + 14\,\alpha^2}{32} \ ,$$

$$P_{2;1}(\alpha) = \frac{\alpha^5 - 16\,\alpha^3 + 36\,\alpha}{32} \ , \quad Q_{2;1}(x;\alpha) = \frac{-\alpha^4 + 14\,\alpha^2 - 12}{32}$$

$$C_{2;0}(x;\alpha) = \frac{-16\,x^4 + (4\,\alpha^2 - 32)\,x^2 - \alpha^4 + 16\,\alpha^2 - 32}{32} \ , \quad D_{2;0}(x;\alpha) = \frac{4\,\alpha\,x^3 + (-\alpha^3 + 12\,\alpha)\,x}{16} \ ,$$

$$R_{2;0}(\alpha) = \frac{-\alpha^5 + 16\,\alpha^3 - 32\,\alpha}{32}$$

$$C_{2;1}(x;\alpha) = \frac{-4\,\alpha\,x^2 + \alpha^3 - 8\,\alpha}{16} \ , \quad D_{2;1}(x;\alpha) = \frac{-4\,x^3 + (\alpha^2 - 4)\,x}{8} \ , \quad R_{2;1}(\alpha) = \frac{\alpha^4 - 8\,\alpha^2}{16}$$

$$A_{3;0}(x;\alpha) = \frac{-16\,x^5 + (4\,\alpha^2 - 40)x^3 - (\alpha^4 - 22\,\alpha^2 + 60)x}{32} \ ,$$

$$B_{3;0}(x;\alpha) = \frac{16\,\alpha\,x^4 + (64\,\alpha - 4\,\alpha^3)x^2 + \alpha^5 - 24\,\alpha^3 + 92\,\alpha}{64} \ ,$$

$$P_{3;0}(\alpha) = \frac{-\alpha^6 + 26\,\alpha^4 - 136\,\alpha^2 + 120}{64} \ , \quad Q_{3;0}(x;\alpha) = \frac{\alpha^5 - 24\,\alpha^3 + 92\,\alpha}{64}$$

$$A_{3;1}(x;\alpha) = \frac{-16\,\alpha\,x^5 + (4\,\alpha^3 - 80\,\alpha)\,x^3 + (-\alpha^5 + 32\,\alpha^3 - 180\,\alpha)\,x}{64} \ ,$$

$$B_{3;1}(x;\alpha) = \frac{-64\,x^6 + (16\,\alpha^2 - 160)\,x^4 + (-4\,\alpha^4 + 104\,\alpha^2 - 240)\,x^2 + \alpha^6 - 34\,\alpha^4 + 232\,\alpha^2 - 120}{128} \ ,$$

$$P_{3;1}(\alpha) = \frac{-\alpha^7 + 36\,\alpha^5 - 296\,\alpha^3 + 480\,\alpha}{128} \ , \quad Q_{3;1}(x;\alpha) = \frac{\alpha^6 - 34\,\alpha^4 + 232\,\alpha^2 - 120}{128}$$

$$C_{3;0}(x;\alpha) = \frac{-64\,x^6 + (16\,\alpha^2 - 192)\,x^4 + (-4\,\alpha^4 + 112\,\alpha^2 - 384)\,x^2 + \alpha^6 - 36\,\alpha^4 + 288\,\alpha^2 - 384}{128} \ ,$$

$$D_{3;0}(x;\alpha) = \frac{16\,\alpha\,x^5 + (-4\,\alpha^3 + 80\,\alpha)\,x^3 + (\alpha^5 - 32\,\alpha^3 + 176\,\alpha)\,x}{54} \ ,$$

$$R_{3;0}(\alpha) = \frac{\alpha^7 - 36\,\alpha^5 + 288\,\alpha^3 - 384\,\alpha}{128}$$

$$C_{3;1}(x;\alpha) = \frac{-16\,\alpha\,x^4 + (4\,\alpha^3 - 64\,\alpha)\,x^2 - \alpha^5 + 24\,\alpha^3 - 96\,\alpha}{64} \ ,$$

$$D_{3;1}(x;\alpha) = \frac{-16\,x^5 + (4\,\alpha^2 - 32)\,x^3 + (-\alpha^4 + 20\,\alpha^2 - 32)\,x}{32} \ ,$$

$$R_{3;1}(\alpha) = \frac{-\alpha^6 + 24\,\alpha^4 - 96\,\alpha^2}{64}$$

$$A_{4;0}(x;\alpha) = \frac{-64\,x^7 + (16\,\alpha^2 - 224)\,x^5 + (-4\,\alpha^4 + 136\,\alpha^2 - 560)\,x^3 + (\alpha^6 - 46\,\alpha^4 + 488\,\alpha^2 - 840)\,x}{128} \ ,$$

$$B_{4;0}(x;\alpha) = \frac{64\,\alpha\,x^6 + (-16\,\alpha^3 + 384\,\alpha)\,x^4 + (4\,\alpha^5 - 160\,\alpha^3 + 1136\,\alpha)\,x^2 - \alpha^7 + 48\,\alpha^5 - 568\,\alpha^3 + 1408\,\alpha}{256} \ ,$$

$$P_{4;0}(\alpha) = \frac{\alpha^8 - 50\,\alpha^6 + 660\,\alpha^4 - 2384\,\alpha^2 + 1680}{256} \ , \quad Q_{4;0}(x;\alpha) = \frac{-\alpha^7 + 48\,\alpha^5 - 568\,\alpha^3 + 1408\,\alpha}{256}$$

$$A_{4;1}(x;\alpha) = \frac{-64\,\alpha\,x^7 + (16\,\alpha^3 - 448\,\alpha)\,x^5 + (-4\,\alpha^5 + 192\,\alpha^3 - 1680\,\alpha)\,x^3 + (\alpha^7 - 60\,\alpha^5 + 936\,\alpha^3 - 3360\,\alpha)\,x}{256} \ ,$$

$$B_{4;1}(x;\alpha) = \frac{1}{512}\left[-256\,x^8 + \left(64\,\alpha^2 - 896\right)x^6 + \left(-16\,\alpha^4 + 608\,\alpha^2 - 2240\right)x^4 + \right.$$
$$\left. + \left(4\,\alpha^6 - 216\,\alpha^4 + 2592\,\alpha^2 - 3360\right)x^2 - \alpha^8 + 62\,\alpha^6 - 1044\,\alpha^4 + 4656\,\alpha^2 - 1680\right],$$

$$P_{4;1}(\alpha) = \frac{\alpha^9 - 64\,\alpha^7 + 1164\,\alpha^5 - 6528\,\alpha^3 + 8400\,\alpha}{512},$$

$$Q_{4;1}(x;\alpha) = \frac{-\alpha^8 + 62\,\alpha^6 - 1044\,\alpha^4 + 4656\,\alpha^2 - 1680}{512}$$

$$C_{4;0}(x;\alpha) = \frac{1}{512}\left[-256\,x^8 + \left(64\,\alpha^2 - 1024\right)x^6 + \left(-16\,\alpha^4 + 640\,\alpha^2 - 3072\right)x^4 + \right.$$
$$\left. + \left(4\,\alpha^6 - 224\,\alpha^4 + 2944\,\alpha^2 - 6144\right)x^2 - \alpha^8 + 64\,\alpha^6 - 1152\,\alpha^4 + 6144\,\alpha^2 - 6144\right],$$

$$D_{4;0}(x;\alpha) = \frac{64\,\alpha\,x^7 + \left(-16\,\alpha^3 + 448\,\alpha\right)x^5 + \left(4\,\alpha^5 - 192\,\alpha^3 + 1664\,\alpha\right)x^3 + \left(-\alpha^7 + 60\,\alpha^5 - 928\,\alpha^3 + 3200\,\alpha\right)x}{256},$$

$$R_{4;0}(\alpha) = \frac{-\alpha^9 + 64\,\alpha^7 - 1152\,\alpha^5 + 6144\,\alpha^3 - 6144\,\alpha}{512}$$

$$C_{4;1}(x;\alpha) = \frac{-64\,\alpha\,x^6 + \left(16\,\alpha^3 - 384\,\alpha\right)x^4 + \left(-4\,\alpha^5 + 160\,\alpha^3 - 1152\,\alpha\right)x^2 + \alpha^7 - 48\,\alpha^5 + 576\,\alpha^3 - 1536\,\alpha}{256},$$

$$D_{4;1}(x;\alpha) = \frac{-64\,x^7 + \left(16\,\alpha^2 - 192\right)x^5 + \left(-4\,\alpha^4 + 128\,\alpha^2 - 384\right)x^3 + \left(\alpha^6 - 44\,\alpha^4 + 416\,\alpha^2 - 384\right)x}{128},$$

$$R_{4;1}(\alpha) = \frac{\alpha^8 - 48\,\alpha^6 + 576\,\alpha^4 - 1536\,\alpha^2}{256}$$

$$A_{5;0}(x;\alpha) = \frac{1}{512}\left[-256\,x^9 + \left(64\,\alpha^2 - 1152\right)x^7 + \left(-16\,\alpha^4 + 736\,\alpha^2 - 4032\right)x^5 + \right.$$
$$\left. + \left(4\,\alpha^6 - 264\,\alpha^4 + 4128\,\alpha^2 - 10080\right)x^3 + \left(-\alpha^8 + 78\,\alpha^6 - 1764\,\alpha^4 + 12144\,\alpha^2 - 15120\right)x\right],$$

$$B_{5;0}(x;\alpha) = \frac{1}{1024}\left[256\,\alpha\,x^8 + \left(-64\,\alpha^3 + 2048\,\alpha\right)x^6 + \left(16\,\alpha^5 - 896\,\alpha^3 + 9152\,\alpha\right)x^4 + \right.$$
$$\left. + \left(-4\,\alpha^7 + 288\,\alpha^5 - 5472\,\alpha^3 + 23808\,\alpha\right)x^2 + \alpha^9 - 80\,\alpha^7 + 1908\,\alpha^5 - 14880\,\alpha^3 + 27024\,\alpha\right],$$

$$P_{5;0}(\alpha) = \frac{-\alpha^{10} + 82\,\alpha^8 - 2064\,\alpha^6 + 18408\,\alpha^4 - 51312\,\alpha^2 + 30240}{1024},$$

$$Q_{5;0}(x;\alpha) = \frac{\alpha^9 - 80\,\alpha^7 + 1908\,\alpha^5 - 14880\,\alpha^3 + 27024\,\alpha}{1024}$$

$$A_{5;1}(x;\alpha) = \frac{1}{1024}\left[-256\,\alpha\,x^9 + \left(64\,\alpha^3 - 2304\,\alpha\right)x^7 + \left(-16\,\alpha^5 + 1024\,\alpha^3 - 12096\,\alpha\right)x^5 + \right.$$
$$\left. + \left(4\,\alpha^7 - 336\,\alpha^5 + 7584\,\alpha^3 - 40320\,\alpha\right)x^3 + \left(-\alpha^9 + 96\,\alpha^7 - 2844\,\alpha^5 + 28992\,\alpha^3 - 75600\,\alpha\right)x\right],$$

$$B_{5;1}(x;\alpha) = \frac{1}{2048}\left[-1024\,x^{10} + \left(256\,\alpha^2 - 4608\right)x^8 + \left(-64\,\alpha^4 + 3200\,\alpha^2 - 16128\right)x^6 + \right.$$
$$+ \left(16\,\alpha^6 - 1184\,\alpha^4 + 20096\,\alpha^2 - 40320\right)x^4 + \left(-4\,\alpha^8 + 360\,\alpha^6 - 9360\,\alpha^4 + 70464\,\alpha^2 - 60480\right)x^2 +$$
$$\left. + \alpha^{10} - 98\,\alpha^8 + 3024\,\alpha^6 - 33672\,\alpha^4 + 110832\,\alpha^2 - 30240\right],$$

$$P_{5;1}(\alpha) = \frac{-\alpha^{11} + 100\,\alpha^9 - 3216\,\alpha^7 + 39360\,\alpha^5 - 168816\,\alpha^3 + 181440\,\alpha}{2048},$$

$$Q_{5;1}(x;\alpha) = \frac{\alpha^{10} - 98\,\alpha^8 + 3024\,\alpha^6 - 33672\,\alpha^4 + 110832\,\alpha^2 - 30240}{2048}$$

$$C_{5;0}(x;\alpha) = \frac{1}{2048}\left[-1024\,x^{10} + \left(256\,\alpha^2 - 5120\right)x^8 + \left(-64\,\alpha^4 + 3328\,\alpha^2 - 20480\right)x^6 + \right.$$
$$+ \left(16\,\alpha^6 - 1216\,\alpha^4 + 22016\,\alpha^2 - 61440\right)x^4 + \left(-4\,\alpha^8 + 368\,\alpha^6 - 9984\,\alpha^4 + 83456\,\alpha^2 - 122880\right)x^2 +$$
$$\left. + \alpha^{10} - 100\,\alpha^8 + 3200\,\alpha^6 - 38400\,\alpha^4 + 153600\,\alpha^2 - 122880\right],$$

$$D_{5;0}(x;\alpha) = \frac{1}{1024}\left[256\,\alpha\,x^9 + \left(-64\,\alpha^3 + 2304\,\alpha\right)x^7 + \left(16\,\alpha^5 - 1024\,\alpha^3 + 12032\,\alpha\right)x^5 + \right.$$

$$+ \left(-4\,\alpha^7 + 336\,\alpha^5 - 7552\,\alpha^3 + 39424\,\alpha\right) x^3 + \left(\alpha^9 - 96\,\alpha^7 + 2832\,\alpha^5 - 28416\,\alpha^3 + 70144\,\alpha\right) x\Big] \,,$$

$$R_{5;0}(\alpha) = \frac{\alpha^{11} - 100\,\alpha^9 + 3200\,\alpha^7 - 38400\,\alpha^5 + 153600\,\alpha^3 - 122880\,\alpha}{2048}$$

$$C_{5;1}(x;\alpha) = \frac{1}{1024}\Big[-256\,\alpha\,x^8 + \left(64\,\alpha^3 - 2048\,\alpha\right) x^6 + \left(-16\,\alpha^5 + 896\,\alpha^3 - 9216\,\alpha\right) x^4 +$$
$$+ \left(4\,\alpha^7 - 288\,\alpha^5 + 5504\,\alpha^3 - 24576\,\alpha\right) x^2 - \alpha^9 + 80\,\alpha^7 - 1920\,\alpha^5 + 15360\,\alpha^3 - 30720\,\alpha\Big] \,,$$

$$D_{5;1}(x;\alpha) = \frac{1}{512}\Big[-256\,x^9 + \left(64\,\alpha^2 - 1024\right) x^7 + \left(-16\,\alpha^4 + 704\,\alpha^2 - 3072\right) x^5 +$$
$$+ \left(4\,\alpha^6 - 256\,\alpha^4 + 3712\,\alpha^2 - 6144\right) x^3 + \left(-\alpha^8 + 76\,\alpha^6 - 1632\,\alpha^4 + 9856\,\alpha^2 - 6144\right) x\Big] \,,$$

$$R_{5;1}(\alpha) = \frac{-\alpha^{10} + 80\,\alpha^8 - 1920\,\alpha^6 + 15360\,\alpha^4 - 30720\,\alpha^2}{1024}$$

$$A_{6;0}(x;\alpha) = \frac{1}{2048}\Big[-1024\,x^{11} + \left(256\,\alpha^2 - 5632\right) x^9 + \left(-64\,\alpha^4 + 3712\,\alpha^2 - 25344\right) x^7 +$$
$$+ \left(16\,\alpha^6 - 1376\,\alpha^4 + 28288\,\alpha^2 - 88704\right) x^5 + \left(-4\,\alpha^8 + 424\,\alpha^6 - 13392\,\alpha^4 + 131136\,\alpha^2 - 221760\right) x^3 +$$
$$+ \left(\alpha^{10} - 118\,\alpha^8 + 4560\,\alpha^6 - 67800\,\alpha^4 + 342768\,\alpha^2 - 332640\right) x\Big] \,,$$

$$B_{6;0}(x;\alpha) = \frac{1}{4096}\Big[1024\,\alpha\,x^{10} + \left(-256\,\alpha^3 + 10240\,\alpha\right) x^8 + \left(64\,\alpha^5 - 4608\,\alpha^3 + 61184\,\alpha\right) x^6 +$$
$$+ \left(-16\,\alpha^7 + 1536\,\alpha^5 - 39808\,\alpha^3 + 241664\,\alpha\right) x^4 + \left(4\,\alpha^9 - 448\,\alpha^7 + 15696\,\alpha^5 - 190848\,\alpha^3 + 584256\,\alpha\right) x^2 -$$
$$- \alpha^{11} + 120\,\alpha^9 - 4784\,\alpha^7 + 75648\,\alpha^5 - 438192\,\alpha^3 + 624768\,\alpha\Big] \,,$$

$$P_{6;0}(\alpha) = \frac{\alpha^{12} - 122\,\alpha^{10} + 5020\,\alpha^8 - 84768\,\alpha^6 + 573792\,\alpha^4 - 1310304\,\alpha^2 + 665280}{4096} \,,$$

$$Q_{6;0}(x;\alpha) = \frac{-\alpha^{11} + 120\,\alpha^9 - 4784\,\alpha^7 + 75648\,\alpha^5 - 438192\,\alpha^3 + 624768\,\alpha}{4096}$$

$$A_{6;1}(x;\alpha) = \frac{1}{4096}\Big[-1024\,\alpha\,x^{11} + \left(256\,\alpha^3 - 11264\,\alpha\right) x^9 + \left(-64\,\alpha^5 + 5120\,\alpha^3 - 76032\,\alpha\right) x^7 +$$
$$+ \left(16\,\alpha^7 - 1728\,\alpha^5 + 50816\,\alpha^3 - 354816\,\alpha\right) x^5 + \left(-4\,\alpha^9 + 512\,\alpha^7 - 20784\,\alpha^5 + 297984\,\alpha^3 - 1108800\,\alpha\right) x^3 +$$
$$+ \left(\alpha^{11} - 140\,\alpha^9 + 6672\,\alpha^7 - 130368\,\alpha^5 + 980592\,\alpha^3 - 1995840\,\alpha\right) x\Big] \,,$$

$$B_{6;1}(x;\alpha) = \frac{1}{8192}\Big[-4096\,x^{12} + \left(1024\,\alpha^2 - 22528\right) x^{10} + \left(-256\,\alpha^4 + 15872\,\alpha^2 - 101376\right) x^8 +$$
$$+ \left(64\,\alpha^6 - 6016\,\alpha^4 + 131584\,\alpha^2 - 354816\right) x^6 + \left(-16\,\alpha^8 + 1888\,\alpha^6 - 65856\,\alpha^4 + 683776\,\alpha^2 - 887040\right) x^4 +$$
$$+ \left(4\,\alpha^{10} - 536\,\alpha^8 + 23616\,\alpha^6 - 396768\,\alpha^4 + 2134464\,\alpha^2 - 1330560\right) x^2 -$$
$$- \alpha^{12} + 142\,\alpha^{10} - 6940\,\alpha^8 + 142176\,\alpha^6 - 1178976\,\alpha^4 + 3063072\,\alpha^2 - 665280\Big] \,,$$

$$P_{6;1}(\alpha) = \frac{\alpha^{13} - 144\,\alpha^{11} + 7220\,\alpha^9 - 155520\,\alpha^7 + 1439712\,\alpha^5 - 5024256\,\alpha^3 + 4656960\,\alpha}{8192} \,,$$

$$Q_{6;1}(x;\alpha) = \frac{-\alpha^{12} + 142\,\alpha^{10} - 6940\,\alpha^8 + 142176\,\alpha^6 - 1178976\,\alpha^4 + 3063072\,\alpha^2 - 665280}{8192}$$

$$C_{6;0}(x;\alpha) = \frac{1}{8192}\Big[-4096\,x^{12} + \left(1024\,\alpha^2 - 24576\right) x^{10} + \left(-256\,\alpha^4 + 16384\,\alpha^2 - 122880\right) x^8 +$$
$$+ \left(64\,\alpha^6 - 6144\,\alpha^4 + 141312\,\alpha^2 - 491520\right) x^6 + \left(-16\,\alpha^8 + 1920\,\alpha^6 - 69120\,\alpha^4 + 774144\,\alpha^2 - 1474560\right) x^4 +$$
$$+ \left(4\,\alpha^{10} - 544\,\alpha^8 + 24576\,\alpha^6 - 433152\,\alpha^4 + 2617344\,\alpha^2 - 2949120\right) x^2 -$$
$$- \alpha^{12} + 144\,\alpha^{10} - 7200\,\alpha^8 + 153600\,\alpha^6 - 1382400\,\alpha^4 + 4423680\,\alpha^2 - 2949120\Big] \,,$$

$$D_{6;0}(x;\alpha) = \frac{1}{4096}\Big[1024\,\alpha\,x^{11} + \left(-256\,\alpha^3 + 11264\,\alpha\right) x^9 + \left(64\,\alpha^5 - 5120\,\alpha^3 + 75776\,\alpha\right) x^7 +$$
$$+ \left(-16\,\alpha^7 + 1728\,\alpha^5 - 50688\,\alpha^3 + 350208\,\alpha\right) x^5 + \left(4\,\alpha^9 - 512\,\alpha^7 + 20736\,\alpha^5 - 294912\,\alpha^3 + 1069056\,\alpha\right) x^3 +$$
$$+ \left(-\alpha^{11} + 140\,\alpha^9 - 6656\,\alpha^7 + 129024\,\alpha^5 - 949248\,\alpha^3 + 1806336\,\alpha\right) x\Big] \,,$$

$$R_{6;0}(\alpha) = \frac{-\alpha^{13} + 144\,\alpha^{11} - 7200\,\alpha^9 + 153600\,\alpha^7 - 1382400\,\alpha^5 + 4423680\,\alpha^3 - 2949120\,\alpha}{8192}$$

$$C_{6;1}(x;\alpha) = \frac{1}{4096}\left[-1024\,\alpha\,x^{10} + \left(256\,\alpha^3 - 10240\,\alpha\right)x^8 + \left(-64\,\alpha^5 + 4608\,\alpha^3 - 61440\,\alpha\right)x^6 + \right.$$

$$+ \left(16\,\alpha^7 - 1536\,\alpha^5 + 39936\,\alpha^3 - 245760\,\alpha\right)x^4 + \left(-4\,\alpha^9 + 448\,\alpha^7 - 15744\,\alpha^5 + 193536\,\alpha^3 - 614400\,\alpha\right)x^2 +$$

$$\left. + \alpha^{11} - 120\,\alpha^9 + 4800\,\alpha^7 - 76800\,\alpha^5 + 460800\,\alpha^3 - 737280\,\alpha\right],$$

$$D_{6;1}(x;\alpha) = \frac{1}{2048}\left[-1024\,x^{11} + \left(256\,\alpha^2 - 5120\right)x^9 + \left(-64\,\alpha^4 + 3584\,\alpha^2 - 20480\right)x^7 + \right.$$

$$+ \left(16\,\alpha^6 - 1344\,\alpha^4 + 26112\,\alpha^2 - 61440\right)x^5 + \left(-4\,\alpha^8 + 416\,\alpha^6 - 12672\,\alpha^4 + 113664\,\alpha^2 - 122880\right)x^3 +$$

$$\left. + \left(\alpha^{10} - 116\,\alpha^8 + 4352\,\alpha^6 - 61056\,\alpha^4 + 267264\,\alpha^2 - 122880\right)x\right],$$

$$R_{6;1}(\alpha) = \frac{\alpha^{12} - 120\,\alpha^{10} + 4800\,\alpha^8 - 76800\,\alpha^6 + 460800\,\alpha^4 - 737280\,\alpha^2}{4096}$$

<u>Recurrence relations:</u>

$$\int x^{2n+2}\,e^{-x^2}\,J_0(\alpha x)\,dx =$$

$$= \frac{x^{2n}\,e^{-x^2}}{4}\left[\alpha J_1(\alpha x) - 2x J_0(\alpha x)\right] + \frac{4n+2-\alpha}{4}\int x^{2n}\,e^{-x^2}\,J_0(\alpha x)\,dx - \frac{(2n-1)\alpha}{4}\int x^{2n-1}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_0(\alpha x)\,dx = -\frac{x^{2n}\,e^{-x^2}}{2}\,J_0(\alpha x) + n\int x^{2n-1}\,e^{-x^2}\,J_0(\alpha x)\,dx - \frac{\alpha}{2}\int x^{2n}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

$$\int x^{2n+2}\,e^{-x^2}\,J_1(\alpha x)\,dx =$$

$$= -\frac{x^{2n}\,e^{-x^2}}{4}\left[\alpha J_0(\alpha x) + 2x J_1(\alpha x)\right] + \frac{\alpha n}{2}\int x^{2n-1}\,e^{-x^2}\,J_0(\alpha x)\,dx + \frac{4n-\alpha^2}{4}\int x^{2n}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_1(\alpha x)\,dx = -\frac{x^{2n}\,e^{-x^2}}{2}\,J_1(\alpha x) + \frac{\alpha}{2}\int x^{2n}\,e^{-x^2}\,J_0(\alpha x)\,dx + \frac{2n-1}{2}\int x^{2n-1}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

e) Special Cases: $\nu = 0$

$$\int x^3\,e^{-x^2}\,J_0(2x)\,dx = \frac{x}{2}\,e^{-x^2}\left[J_1(2x) - x J_0(2x)\right]$$

$$\int x^3\,e^{x^2}\,I_0(2x)\,dx = \frac{x}{2}\,e^{x^2}\left[x I_0(2x) - I_1(2x)\right]$$

$$\int x^3\,e^{x^2}\,K_0(2x)\,dx = \frac{x}{2}\,e^{x^2}\left[x K_0(2x) + K_1(2x)\right]$$

$$2\sqrt{2+\sqrt{2}} = 3.69551\,81300\,, \qquad 2\sqrt{2-\sqrt{2}} = 1.53073\,37295$$

$$\int x^5\,e^{-x^2}\,J_0\left(2\sqrt{2+\sqrt{2}}\,x\right)\,dx =$$

$$= \frac{x}{2}\,e^{-x^2}\left[x(\sqrt{2}-x^2)\,J_0\left((2\sqrt{2+\sqrt{2}}\,x\right) + \sqrt{2+\sqrt{2}}\,(x^2+1-\sqrt{2})\,J_1\left(2\sqrt{2+\sqrt{2}}\,x\right)\right]$$

$$\int x^5\,e^{x^2}\,I_0\left(2\sqrt{2+\sqrt{2}}\,x\right)\,dx =$$

$$= \frac{x}{2}\,e^{x^2}\left[x(\sqrt{2}+x^2)\,I_0\left((2\sqrt{2+\sqrt{2}}\,x\right) - \sqrt{2+\sqrt{2}}\,(x^2-1+\sqrt{2})\,I_1\left(2\sqrt{2+\sqrt{2}}\,x\right)\right]$$

$$\int x^5\,e^{x^2}\,K_0\left(2\sqrt{2+\sqrt{2}}\,x\right)\,dx =$$

$$= \frac{x}{2} e^{x^2} \left[x(\sqrt{2} + x^2) K_0 \left((2\sqrt{2 + \sqrt{2}}\, x \right) + \sqrt{2 + \sqrt{2}}\, (x^2 - 1 + \sqrt{2}) K_1 \left(2\sqrt{2 + \sqrt{2}}\, x \right) \right]$$

$$\int x^5 e^{-x^2} J_0 \left(2\sqrt{2 - \sqrt{2}}\, x \right) dx =$$

$$= \frac{x}{2} e^{-x^2} \left[-x(\sqrt{2} + x^2) J_0 \left((2\sqrt{2 - \sqrt{2}}\, x \right) + \sqrt{2 - \sqrt{2}}\, (x^2 + 1 + \sqrt{2}) J_1 \left(2\sqrt{2 - \sqrt{2}}\, x \right) \right]$$

$$\int x^5 e^{x^2} I_0 \left(2\sqrt{2 - \sqrt{2}}\, x \right) dx =$$

$$= \frac{x}{2} e^{x^2} \left[x(x^2 - \sqrt{2}) I_0 \left((2\sqrt{2 - \sqrt{2}}\, x \right) - \sqrt{2 - \sqrt{2}}\, (x^2 - 1 - \sqrt{2}) I_1 \left(2\sqrt{2 - \sqrt{2}}\, x \right) \right]$$

$$\int x^5 e^{x^2} K_0 \left(2\sqrt{2 - \sqrt{2}}\, x \right) dx =$$

$$= \frac{x}{2} e^{x^2} \left[x(x^2 - \sqrt{2}) K_0 \left((2\sqrt{2 - \sqrt{2}}\, x \right) + \sqrt{2 - \sqrt{2}}\, (x^2 - 1 - \sqrt{2}) K_1 \left(2\sqrt{2 - \sqrt{2}}\, x \right) \right]$$

f) Special Cases: $\nu = 1$

$$\int x^4 e^{-x^2} J_1(2\sqrt{2}x)\, dx = \frac{x}{\sqrt{2}} e^{-x^2} [(1 - x^2) J_1(2\sqrt{2}x) - x\sqrt{2}\, J_0(2\sqrt{2}x)]$$

$$\int x^4 e^{x^2} I_1(2\sqrt{2}x)\, dx = \frac{x}{\sqrt{2}} e^{x^2} [(1 + x^2) I_1(2\sqrt{2}x) - x\sqrt{2}\, I_0(2\sqrt{2}x)]$$

$$\int x^4 e^{x^2} K_1(2\sqrt{2}x)\, dx = \frac{x}{\sqrt{2}} e^{x^2} [(1 + x^2) K_1(2\sqrt{2}x) + x\sqrt{2}\, K_0(2\sqrt{2}x)]$$

$$2\sqrt{3 + \sqrt{3}} = 4.35065\ 54943\,, \qquad 2\sqrt{3 - \sqrt{3}} = 2.25206\ 50012$$

$$\int x^6 e^{-x^2} J_1 \left(2\sqrt{3 + \sqrt{3}}\, x \right) dx =$$

$$= \frac{x}{2} e^{-x^2} \left[(-x^4 + (1 + \sqrt{3})x^2 - \sqrt{3} + 1) J_1 \left((2\sqrt{3 + \sqrt{3}}\, x \right) - \sqrt{3 + \sqrt{3}}\, x\, (x^2 + 1 - \sqrt{3}) J_0 \left(2\sqrt{3 + \sqrt{3}}\, x \right) \right]$$

$$\int x^6 e^{x^2} I_1 \left(2\sqrt{3 + \sqrt{3}}\, x \right) dx =$$

$$= \frac{x}{2} e^{x^2} \left[(x^4 + (1 + \sqrt{3})x^2 + \sqrt{3} - 1) I_1 \left((2\sqrt{3 + \sqrt{3}}\, x \right) - \sqrt{3 + \sqrt{3}}\, x\, (x^2 - 1 + \sqrt{3}) I_0 \left(2\sqrt{3 + \sqrt{3}}\, x \right) \right]$$

$$\int x^6 e^{x^2} K_1 \left(2\sqrt{3 + \sqrt{3}}\, x \right) dx =$$

$$= \frac{x}{2} e^{x^2} \left[(x^4 + (1 + \sqrt{3})x^2 + \sqrt{3} - 1) K_1 \left((2\sqrt{3 + \sqrt{3}}\, x \right) + \sqrt{3 + \sqrt{3}}\, x\, (x^2 - 1 + \sqrt{3}) K_0 \left(2\sqrt{3 + \sqrt{3}}\, x \right) \right]$$

$$\int x^6 e^{-x^2} J_1 \left(2\sqrt{3 - \sqrt{3}}\, x \right) dx =$$

$$= \frac{x}{2} e^{-x^2} \left[(-x^4 + (1 - \sqrt{3})x^2 + \sqrt{3} + 1) J_1 \left((2\sqrt{3 - \sqrt{3}}\, x \right) - \sqrt{3 - \sqrt{3}}\, x\, (x^2 + 1 + \sqrt{3}) J_0 \left(2\sqrt{3 - \sqrt{3}}\, x \right) \right]$$

$$\int x^6 e^{x^2} I_1 \left(2\sqrt{3 - \sqrt{3}}\, x \right) dx =$$

$$= \frac{x}{2} e^{x^2} \left[(x^4 + (1 - \sqrt{3})x^2 - \sqrt{3} - 1) I_1 \left((2\sqrt{3 - \sqrt{3}} \, x \right) - \sqrt{3 - \sqrt{3}} \, x \, (x^2 - 1 - \sqrt{3}) I_0 \left(2\sqrt{3 - \sqrt{3}} \, x \right) \right]$$

$$\int x^6 e^{x^2} K_1 \left(2\sqrt{3 - \sqrt{3}} \, x \right) dx =$$

$$= \frac{x}{2} e^{x^2} \left[(x^4 + (1 - \sqrt{3})x^2 - \sqrt{3} - 1) K_1 \left((2\sqrt{3 - \sqrt{3}} \, x \right) + \sqrt{3 - \sqrt{3}} \, x \, (x^2 - 1 - \sqrt{3}) K_0 \left(2\sqrt{3 - \sqrt{3}} \, x \right) \right]$$

g) Integrals of $x^n \, e^{-(x+\beta)^2} J_\nu(\alpha x)$ or $x^n \, e^{-x^2} J_\nu(\alpha x + \gamma)$

With $z = x + \beta$ follows

$$\int x^n \, e^{-(x+\beta)^2} J_\nu(\alpha x) \, dx = \int (z - \beta)^n \, e^{-z^2} J_\nu(\alpha z - \alpha \beta) \, dx \ .$$

About the integrals on the right side see page 175. To describe the left integrals three basic integrals must be defined:

$$\int e^{-(x+\beta)^2} J_0(\alpha x) \, dx = F_0(x; \alpha, \beta) = e^{-(x+\beta)^2} \left[\left(\sum_{k=0}^{\infty} p_k^{(0)}(\alpha, \beta) \, x^k \right) J_0(\alpha x) + \left(\sum_{k=0}^{\infty} q_k^{(0)}(\alpha, \beta) \, x^k \right) J_1(\alpha x) \right]$$

$$\int e^{-(x+\beta)^2} J_1(\alpha x) \, dx = F_1(x; \alpha, \beta) =$$

$$= e^{-(x+\beta)^2} \left[\left(\sum_{k=0}^{\infty} p_k^{(1)}(\alpha, \beta) \, x^k \right) J_0(\alpha x) + \left(\sum_{k=0}^{\infty} q_k^{(1)}(\alpha, \beta) \, x^k \right) J_1(\alpha x) \right] - p_0 \, e^{-\beta^2}$$

$$\int \frac{e^{-(x+\beta)^2} J_1(\alpha x) \, dx}{x} = F_-(x; \alpha, \beta) = e^{-(x+\beta)^2} \left[\left(\sum_{k=0}^{\infty} p_k^{(-)}(\alpha, \beta) \, x^k \right) J_0(\alpha x) + \left(\sum_{k=0}^{\infty} q_k^{(-)}(\alpha, \beta) \, x^k \right) J_1(\alpha x) \right]$$

I) $F_0(x; \alpha, \beta)$:

$$p_0^{(0)} = 0 \, , \ p_1^{(0)} = 1 \, , \ q_0^{(0)} = 0 \, , \ q_1^{(0)} = 0 \, ,$$

$$p_{n+1}^{(0)} = \frac{2p_{n-1}^{(0)}(\alpha, \beta) + 2\beta \, p_n^{(0)}(\alpha, \beta) - \alpha \, q_n^{(0)}(\alpha, \beta)}{n + 1} \, , \quad q_{n+1}^{(0)}(\alpha, \beta) = \frac{2q_{n-1}^{(0)}(\alpha, \beta) + 2\beta \, q_n^{(0)}(\alpha, \beta) + \alpha \, p_n^{(0)}(\alpha, \beta)}{n}$$

$$F_0(x; \alpha, \beta) = e^{-(x+\beta)^2} \left[\left(x + \beta x^2 + \frac{2\beta^2 - \alpha^2 + 2}{3} \, x^3 + \frac{\beta}{24}(8\beta^2 - 13\alpha^2 + 20) \, x^4 + \ldots \right) J_0(\alpha x) + \right.$$

$$\left. + \left(\alpha \, x^2 + \frac{3\alpha\beta}{2} \, x^3 + \frac{\alpha}{9}(11\beta^2 - \alpha^2 + 8) \, x^4 + \frac{\alpha\beta \, (200\beta^2 - 55\alpha^2 + 404)}{288} \, x^5 + \ldots \right) J_1(\alpha x) \right]$$

About the using of this formula see page ??, concerning $a_k^{(0;\alpha)}$ and $b_k^{(0;\alpha)}$.
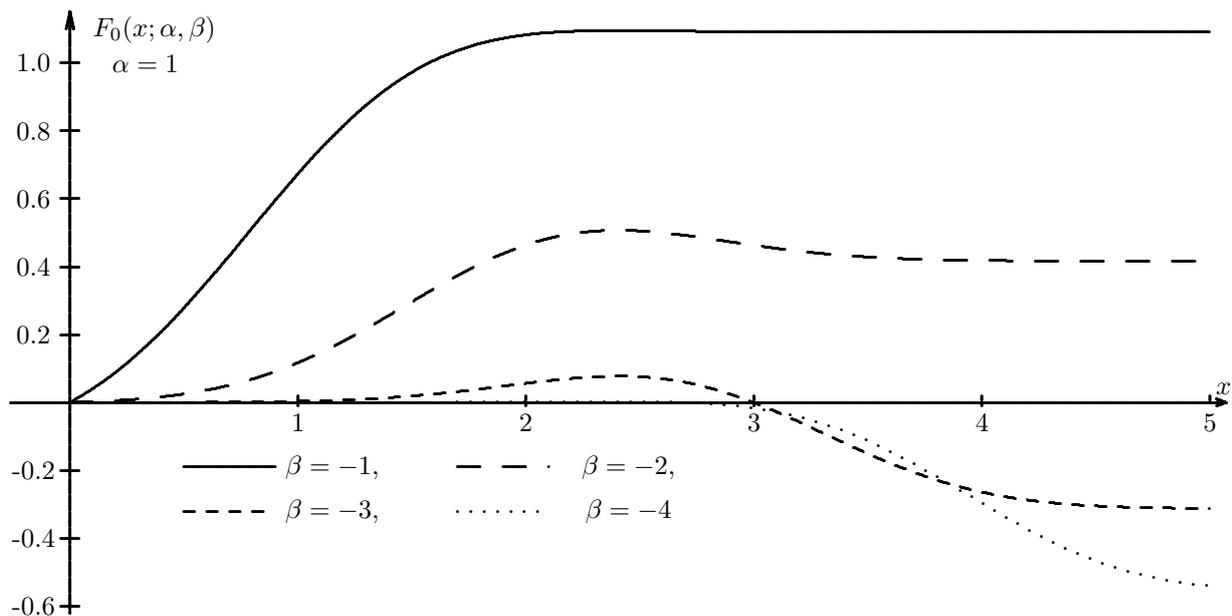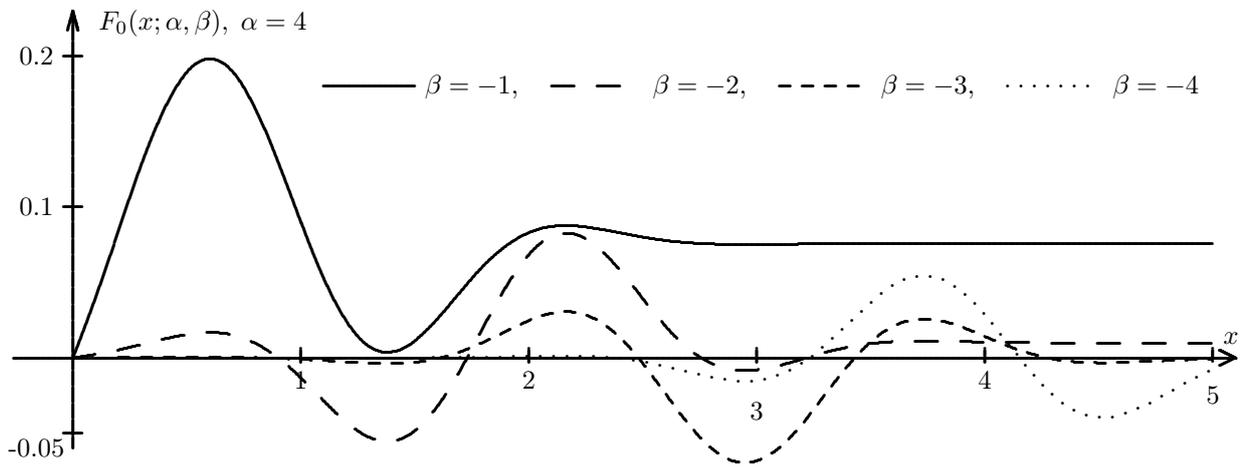
Let

$$U_0(\alpha, \beta) \quad = \quad \int_0^\infty e^{-(x-\beta)^2} J_0(\alpha x)\, dx \ .$$

The following pictures show some of this functions:
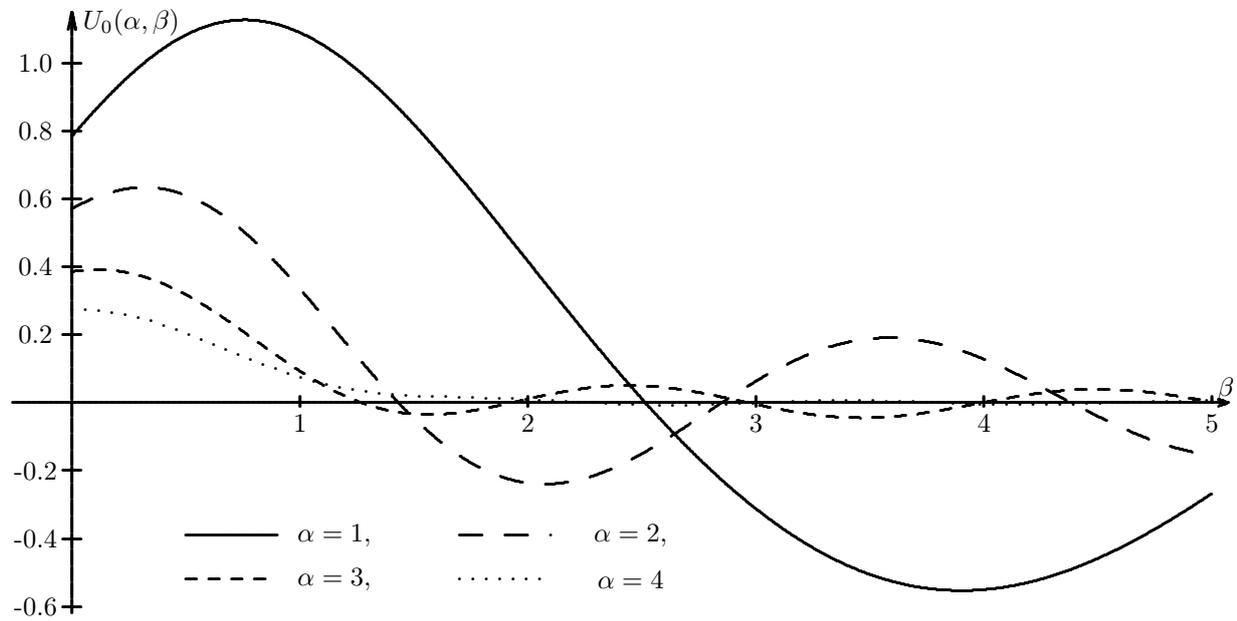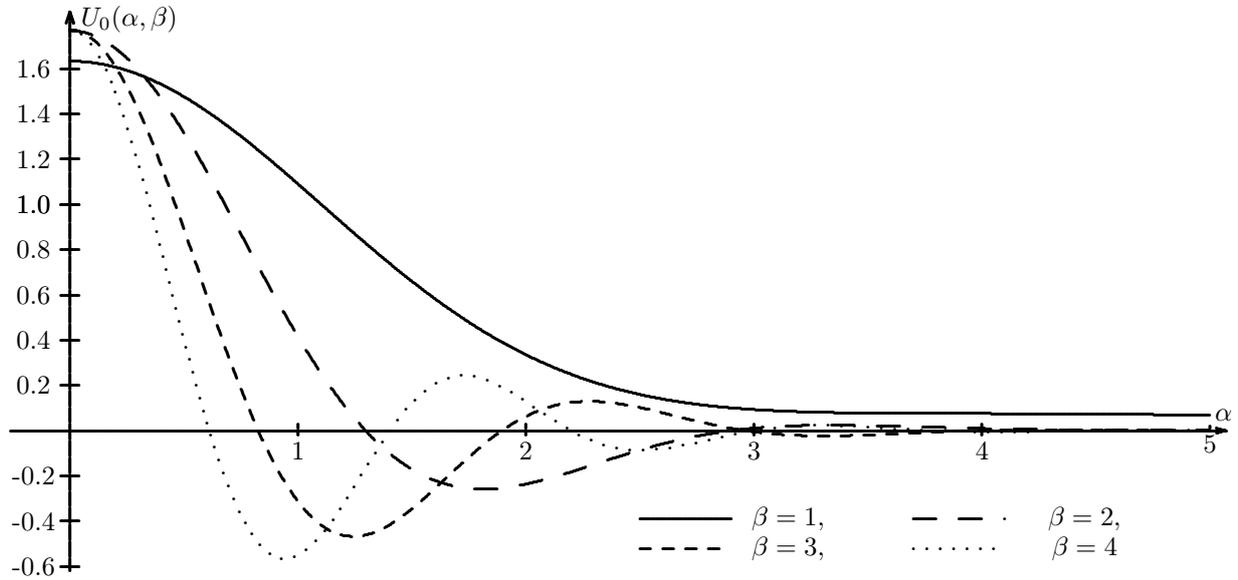
II) $F_1(x; \alpha, \beta)$:

$$p_0^{(1)} = -\frac{1}{\alpha}\,,\ p_1^{(1)} = -\frac{2\beta}{\alpha}\,,\ q_0^{(1)} = 0\,,\ q_1^{(1)} = 0\,,$$

$$p_{n+1}^{(1)} = \frac{2p_{n-1}^{(1)}(\alpha, \beta) + 2\beta\, p_n^{(1)}(\alpha, \beta) - \alpha\, q_n^{(1)}(\alpha, \beta)}{n+1}\,,$$

$$q_{n+1}^{(1)}(\alpha, \beta) = \frac{2q_{n-1}^{(1)}(\alpha, \beta) + 2\beta\, q_n^{(1)}(\alpha, \beta) + \alpha\, p_n^{(1)}(\alpha, \beta)}{n}$$

$$F_1(x; \alpha, \beta) = \frac{e^{-\beta^2}}{\alpha} -$$

$$-e^{-(x+\beta)^2}\left[\left(\frac{1}{\alpha} + \frac{2\beta x}{\alpha} + \frac{2\beta^2+1}{\alpha}x^2 + \frac{4\beta^3 + 2\beta(3-2\alpha^2)}{3\alpha}x^3 + \frac{16\beta^4 + \beta^2(48 - 26\alpha^2) + 12 - 3\alpha^2}{24\alpha}x^4 + \ldots\right) J_0(\alpha x) +$$

$$+\left(2\beta\, x^2 + \frac{6\beta^2+1}{2}\,x^3 + \frac{22\beta^3 + \beta(21 - 2\alpha^2)}{9}\,x^4 + \frac{400\beta^4 + \beta^2\,(912 - 110\alpha^2) + 108 - 9\alpha^2}{288}\,x^5 + \ldots\right) J_1(\alpha x)\right]$$

About the using of this formula see page **??**, concerning $a_k^{(0;\alpha)}$ and $b_k^{(0;\alpha)}$.
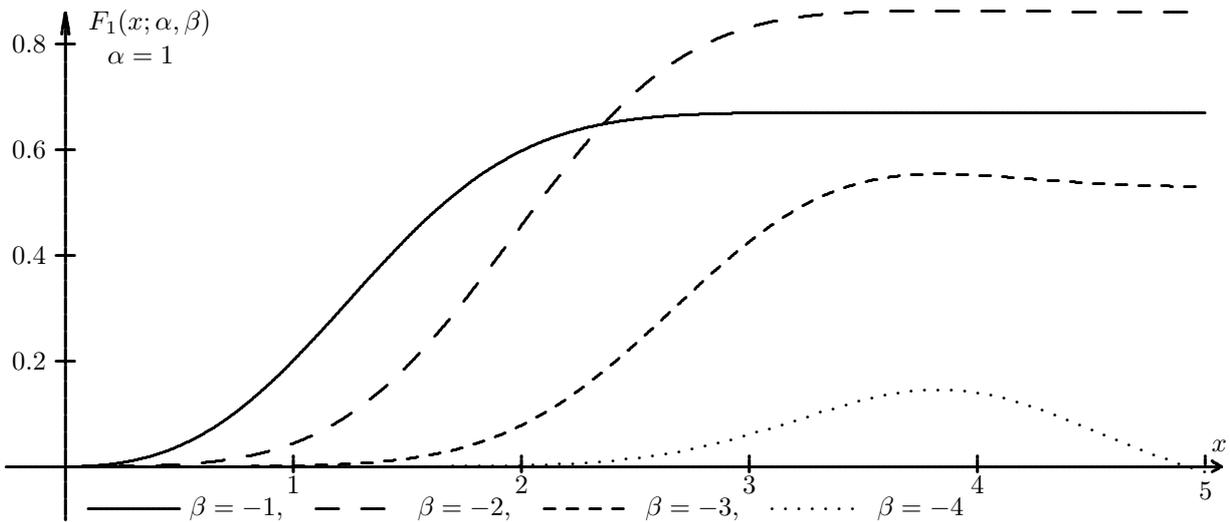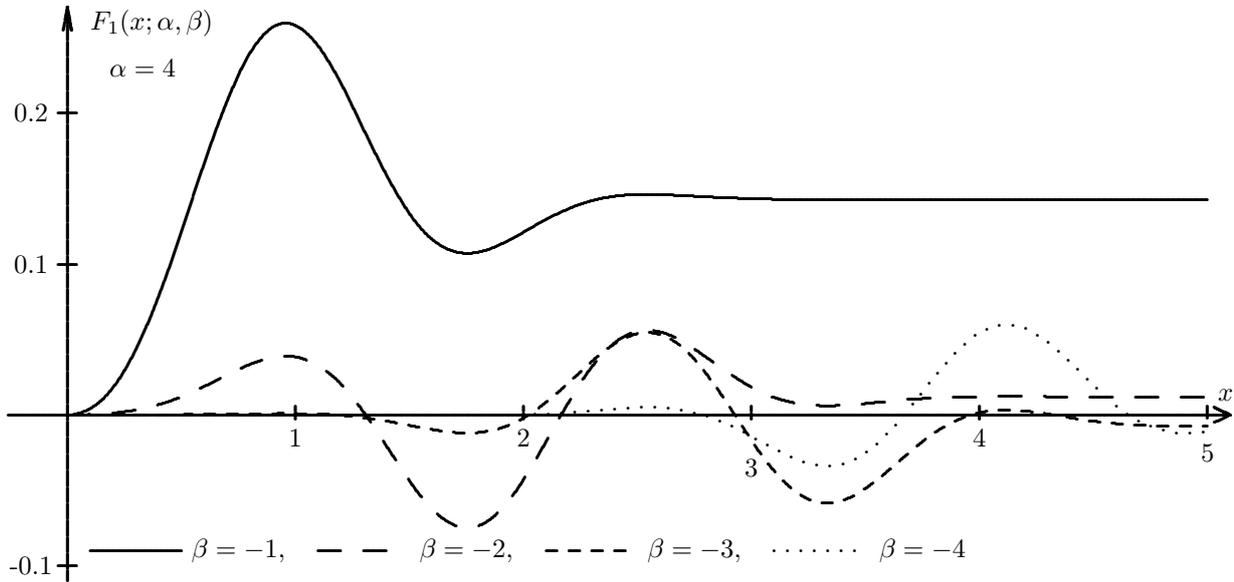
Let

$$U_1(\alpha, \beta) \quad = \quad \int_0^\infty e^{-(x-\beta)^2} J_1(\alpha x)\, dx \ .$$

The following pictures show some of this functions:

$$\beta = 1, \quad \text{-- -- --} \quad \beta = 2,$$
$$\text{-- -- --} \quad \beta = 3, \quad \cdots\cdots \quad \beta = 4$$
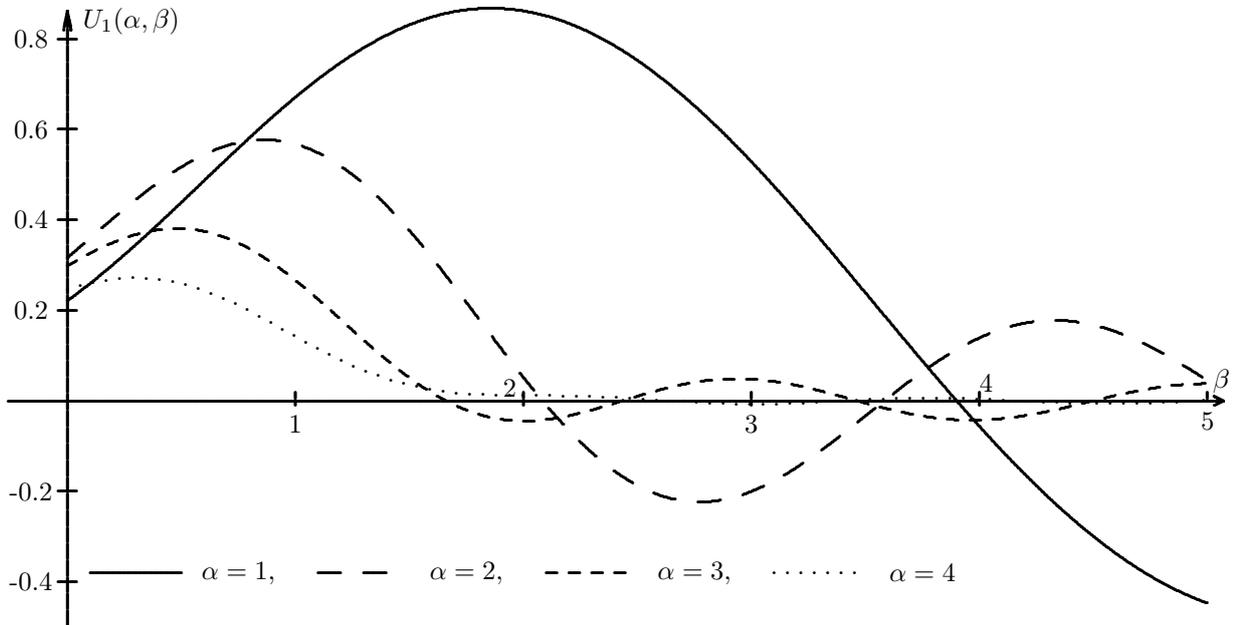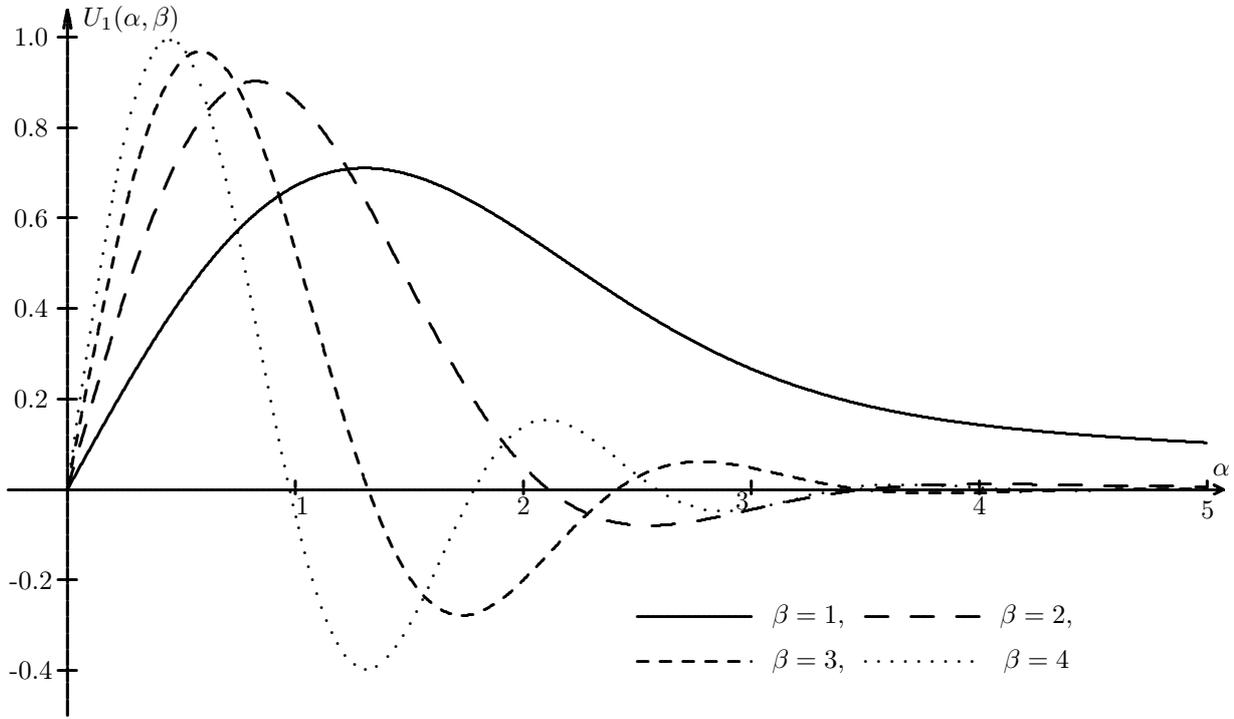
III) $F_-(x; \alpha, \beta)$:

$$p_0^{(-)} = \frac{2\beta}{\alpha}, \; p_1^{(-)} = \frac{\alpha^2 + 4\beta^2}{\alpha}, \; q_0^{(-)} = -1, \; q_1^{(-)} = 0,$$

$$p_{n+1}^{(-)} = \frac{2p_{n-1}^{(-)}(\alpha, \beta) + 2\beta \, p_n^{(-)}(\alpha, \beta) - \alpha \, q_n^{(-)}(\alpha, \beta)}{n+1},$$

$$q_{n+1}^{(-)}(\alpha, \beta) = \frac{2q_{n-1}^{(-)}(\alpha, \beta) + 2\beta \, q_n^{(-)}(\alpha, \beta) + \alpha \, p_n^{(-)}(\alpha, \beta)}{n}$$

$$F_-(x; \alpha, \beta) =$$

$$= e^{-(x+\beta)^2} \left[\left(\frac{2\beta}{\alpha} + \frac{\alpha^2 + 4\beta^2}{\alpha} x + \frac{\beta(4\beta^2 + \alpha^2 + 2)}{\alpha} x^2 + \frac{8\beta^3 + \beta^2(12 - 2\alpha^2) + \alpha^2(4 - \alpha^2)}{\alpha} x^3 + \ldots \right) J_0(\alpha x) + \right.$$

$$\left. \left(-x + (4\beta^2 + \alpha^2 - 2) x^2 + \frac{\beta(12\beta^2 + 3\alpha^2 - 2)}{2} x^3 + \frac{44\beta^4 + \beta^2(30 + 7\alpha^2) - \alpha^4 + 10\alpha^2 - 12}{9} x^4 + \ldots \right) J_1(\alpha x) \right] -$$

$$- \frac{2\beta \, e^{-\beta^2}}{\alpha}$$

About the using of this formula see page **??**, concerning $a_k^{(0;\alpha)}$ and $b_k^{(0;\alpha)}$.

$$\beta = -1, \quad \text{-- --} \quad \beta = -2, \quad \text{-- -- --} \quad \beta = -3, \quad \cdots\cdots \quad \beta = -4$$

$\beta = -1.5,$ — — $\beta = -2,$ – – – – $\beta = -3,$ · · · · · · · · $\beta = -4$
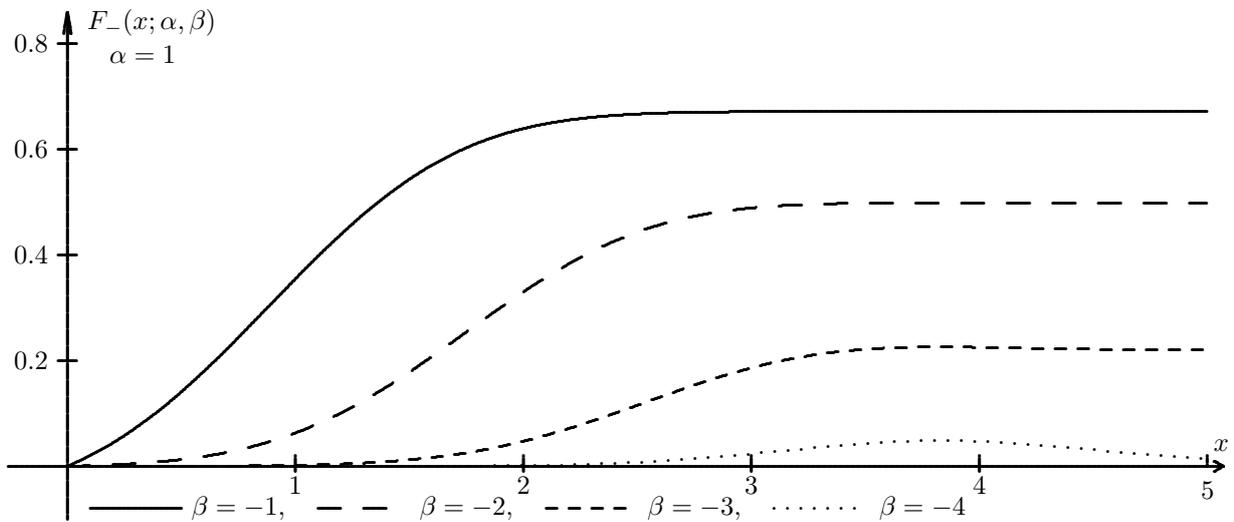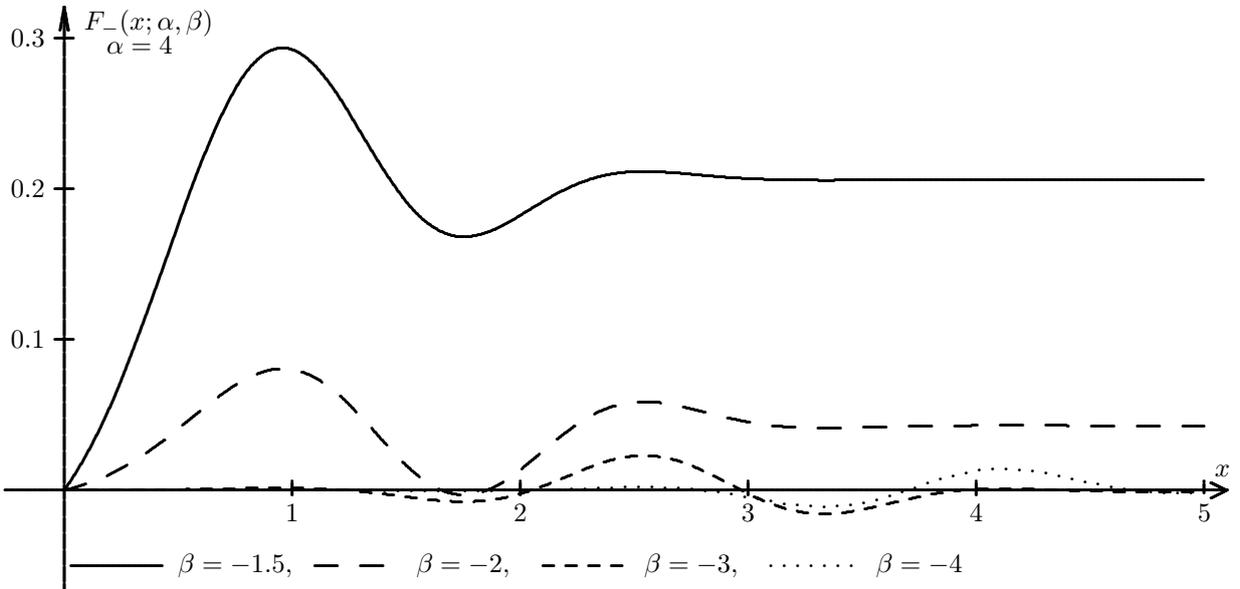
Let

$$U_-(\alpha, \beta) \quad = \quad \int_0^\infty \frac{e^{-(x-\beta)^2} J_1(\alpha x) \, dx}{x} \ .$$

The following pictures show some of this functions:

$\alpha = 1,$ — — $\alpha = 2,$ – – – – $\alpha = 3,$ · · · · · · · · $\alpha = 4$

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$$U_-(\alpha,\beta)$$

$$\text{——} \quad \beta = 1, \qquad \text{— —} \quad \beta = 2, \qquad \text{- - - -} \quad \beta = 3, \qquad \cdots\cdots \quad \beta = 4$$
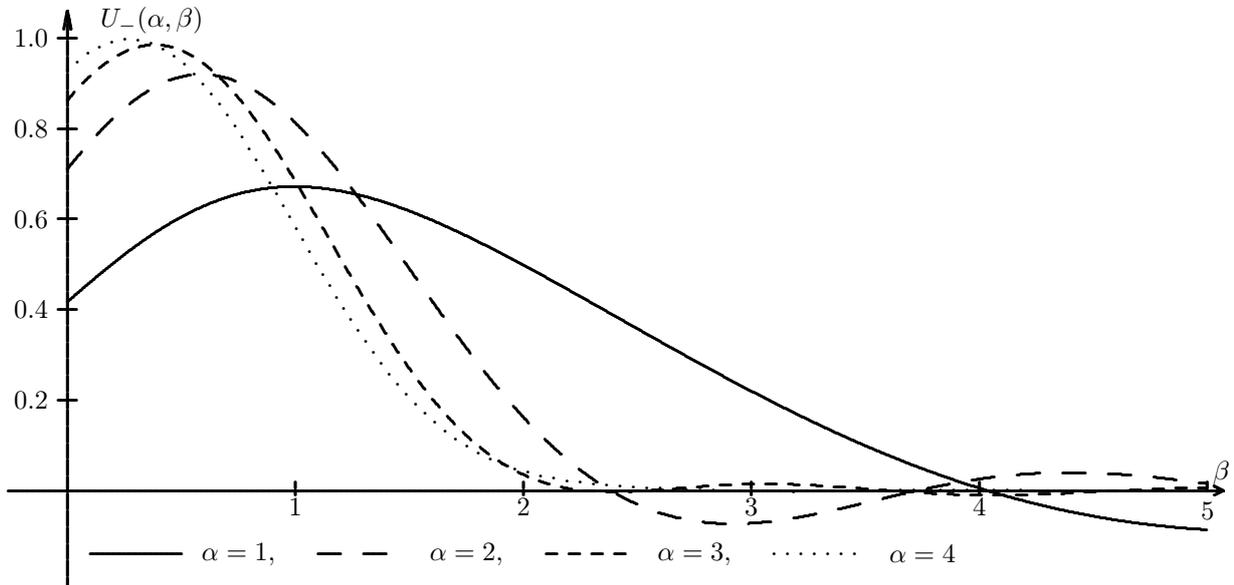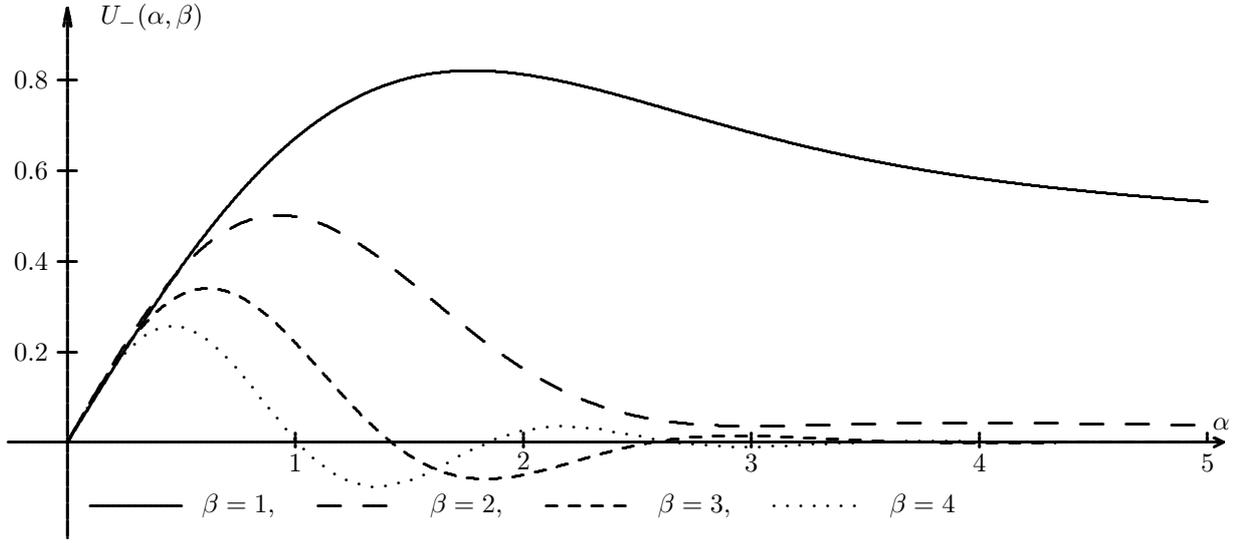
IV) Integrals:

$$\int x\, e^{-(x+\beta)^2} J_0(\alpha x)\, dx = -\frac{e^{-(x+\beta)^2} J_0(\alpha x)}{2} - \beta\, F_0(x;\alpha,\beta) - \frac{\alpha}{2}\, F_1(x;\alpha,\beta)$$

$$\int x\, e^{-(x+\beta)^2} J_1(\alpha x)\, dx = -\frac{e^{-(x+\beta)^2} J_1(\alpha x)}{2} + \frac{\alpha}{2}\, F_0(x;\alpha,\beta) - \beta\, F_1(x;\alpha,\beta) - \frac{1}{2}\, F_-(x;\alpha,\beta)$$

$$\int x^2\, e^{-(x+\beta)^2} J_0(\alpha x)\, dx =$$

$$= e^{-(x+\beta)^2} \left[\frac{\beta-x}{2}\, J_0(\alpha x) + \frac{\alpha}{4}\, J_1(\alpha x)\right] + \frac{2+4\beta^2-\alpha^2}{4}\, F_0(x;\alpha,\beta) + \alpha\beta\, F_1(x;\alpha,\beta) + \frac{\alpha}{4}\, F_-(x;\alpha,\beta)$$

$$\int x^2\, e^{-(x+\beta)^2} J_1(\alpha x)\, dx = e^{-(x+\beta)^2} \left[-\frac{\alpha}{4}\, J_0(\alpha x) + \frac{\beta-x}{2}\, J_1(\alpha x)\right] -$$

$$-4\alpha\beta\, F_0(x;\alpha,\beta) + \frac{4\beta^2-\alpha^2}{4}\, F_1(x;\alpha,\beta) + \frac{\beta}{2}\, F_-(x;\alpha,\beta)$$

$$\int x^3\, e^{-(x+\beta)^2} J_0(\alpha x)\, dx = e^{-(x+\beta)^2} \left[\frac{\alpha^2-4\beta^2-4+4\beta x-4x^2}{8}\, J_0(\alpha x) + \frac{\alpha(x-2\beta)}{4}\, J_1(\alpha x)\right] -$$

$$-\frac{\beta(4\beta^2+6-3\alpha^2)}{4}\, F_0(x;\alpha,\beta) - \frac{\alpha(12\beta^2+4-\alpha^2)}{8}\, F_1(x;\alpha,\beta) - \frac{\alpha\beta}{2}\, F_-(x;\alpha,\beta)$$

$$\int x^3\, e^{-(x+\beta)^2} J_1(\alpha x)\, dx = e^{-(x+\beta)^2} \left[\frac{\alpha(2\beta-x)}{4}\, J_0(\alpha x) - \frac{4\beta^2+2-\alpha^2-4\beta x+4x^2}{8}\, J_1(\alpha x)\right] +$$

$$+\frac{\alpha(12\beta^2+4-\alpha^2)}{8}\, F_0(x;\alpha,\beta) - \frac{\beta(4\beta^2+2-3\alpha^2)}{4}\, F_1(x;\alpha,\beta) - \frac{4\beta^2+2-\alpha^2}{8}\, F_-(x;\alpha,\beta)$$

$$\int x^4\, e^{-(x+\beta)^2} J_0(\alpha x)\, dx = e^{-(x+\beta)^2} \left[\frac{-4x^3+4\beta x^2+x(\alpha^2-4\beta^2-6)+\beta(4\beta^2-3\alpha^2+10)}{8}\, J_0(\alpha x)+\right.$$

$$\left.+\frac{\alpha[4x(x-2\beta+12\beta^2+8-\alpha^2)}{16}\, J_1(\alpha x)\right] + \frac{16\beta^4+24\beta^2(2-\alpha^2)+\alpha^4-10\alpha^2+12}{16}\, F_0(x;\alpha,\beta)+$$

$$+\frac{\alpha\beta(8\beta^2-2\alpha^2+9)}{4}\, F_1(x;\alpha,\beta) + \frac{\alpha(12\beta^2+8-\alpha^2)}{16}\, F_-(x;\alpha,\beta)$$

$$\int x^4\, e^{-(x+\beta)^2} J_1(\alpha x)\, dx = e^{-(x+\beta)^2} \left[\frac{\alpha[-4x^2+8\beta x-12\beta^2-8+\alpha^2]}{16}\, J_0(\alpha x)+\right.$$

$$\left.+\frac{-4x^3+4\beta x^2+x(\alpha^2-4\beta^2-4)+\beta(4\beta^2+6-3\alpha^2)}{8}\, J_1(\alpha x)\right] + \frac{\alpha\beta(2\alpha^2-8\beta^2-9)}{4}\, F_0(x;\alpha,\beta)+$$

$$+\frac{\alpha^4 - 24\alpha^2\beta^2 + 16\beta^4 - 8\alpha^2 + 24\beta^2}{16}\, F_1(x;\alpha,\beta) + \frac{\beta(4\beta^2 + 6 - 3\alpha^2)}{8}\, F_-(x;\alpha,\beta)$$

Recurrence relations:

$$\text{Let} \qquad I_m^{(\nu)} \quad = \quad \int x^m\, e^{-(x+\beta)^2}\, J_\nu(\alpha x)\, dx\ .$$

$$I_{2n+1}^{(0)} = -\frac{x^{2n}\, e^{-(x+\beta)^2}\, J_0(\alpha x)}{2} - \beta I_{2n}^{(0)} + n\, I_{2n-1}^{(0)} - \frac{\alpha}{2}\, I_{2n}^{(1)}$$

$$I_{2n+1}^{(1)} = -\frac{x^{2n}\, e^{-(x+\beta)^2}\, J_1(\alpha x)}{2} + \frac{\alpha}{2}\, I_{2n}^{(0)} - \beta I_{2n}^{(1)} + \frac{2n-1}{2}\, I_{2n-1}^{(1)}$$

$$I_{2n+2}^{(0)} = -\frac{x^{2n+1}\, e^{-(x+\beta)^2}\, J_0(\alpha x)}{2} + \frac{2n+1}{2}\, I_{2n}^{(0)} - \frac{\alpha}{2}\, I_{2n+1}^{(1)} - \beta\, I_{2n+1}^{(0)}$$

$$I_{2n+2}^{(1)} = -\frac{x^{2n+1}\, e^{-(x+\beta)^2}\, J_1(\alpha x)}{2} + n\, I_{2n}^{(1)} - \beta I_{2n+1}^{(1)} + \frac{\alpha}{2}\, I_{2n+1}^{(0)}$$

<u>Integrals of the type</u> $\int x^n\, e^{-x^2}\, J_\nu(\alpha x + \gamma)\, dx$:

$$\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx = F_0\left(x - \frac{\gamma}{\alpha};\alpha,-\frac{\gamma}{\alpha}\right) , \quad \int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx = F_1\left(x - \frac{\gamma}{\alpha};\alpha,-\frac{\gamma}{\alpha}\right)$$

$$\text{The integral} \quad \int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{x} \quad \text{cannot be expressed by} \quad F_-\ .$$

$$\int_0^y \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{x} \quad \text{is not defined if} \quad J_1(\gamma) \neq 0\ .$$

$$\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma} = \frac{1}{\alpha} \int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{x + \gamma/\alpha} = \frac{1}{\alpha}\, F_-\left(x - \frac{\gamma}{\alpha};\alpha,-\frac{\gamma}{\alpha}\right)$$

Integrals:

$$\int x\, e^{-x^2}\, J_0(\alpha x + \gamma)\, dx = -\frac{e^{-x^2}\, J_0(\alpha x + \gamma)}{2} - \frac{\alpha}{2} \int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx$$

$$\int x\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx =$$

$$= -\frac{e^{-x^2}\, J_1(\alpha x + \gamma)}{2} + \frac{\alpha}{2} \int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx - \frac{\alpha}{2} \int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int \frac{x\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma} = \frac{1}{\alpha} \int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx - \frac{\gamma}{\alpha} \int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int x^2\, e^{-x^2}\, J_0(\alpha x + \gamma)\, dx =$$

$$= \frac{e^{-x^2}}{4}\left[a\, J_1(\alpha x + \gamma) - 2x\, J_0(\alpha x + \gamma)\right] + \frac{2-\alpha^2}{4} \int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx + \frac{\alpha^2}{4} \int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int x^2\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx =$$

$$= -\frac{e^{-x^2}}{4}\left[\alpha J_0(\alpha x + \gamma) + 2x\, J_1(\alpha x + \gamma)\right] - \frac{\alpha^2}{4} \int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx + \frac{\gamma}{2} \int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int \frac{x^2\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma} =$$

$$= -\frac{e^{-x^2}\, J_1(\alpha x + \gamma)}{2\alpha} + \frac{1}{2} \int e^{-x^2}\, J_0(\alpha x+\gamma)\, dx - \frac{\gamma}{\alpha^2} \int e^{-x^2}\, J_1(\alpha x+\gamma)\, dx + \frac{2\gamma^2-\alpha^2}{2\alpha^2} \int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int x^3\, e^{-x^2}\, J_0(\alpha x + \gamma)\, dx = \frac{e^{-x^2}}{8}\left[(\alpha^2 - 4 - 4x^2)\, J_0(\alpha x + \gamma) + 2\alpha x\, J_1(\alpha x + \gamma)\right] +$$

$$+\frac{\alpha(\alpha^2 - 4)}{8}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx - \frac{\alpha\gamma}{4}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int x^3\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx = \frac{e^{-x^2}}{8}\left[(\alpha^2 - 2 - 4x^2)\, J_1(\alpha x + \gamma) - 2\alpha x\, J_0(\alpha x + \gamma)\right] +$$

$$+\frac{\alpha(4 - \alpha^2)}{8}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx + \frac{\gamma}{2\alpha}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx + \frac{\alpha^4 - 2\alpha^2 - 4\gamma^2}{8\alpha}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int \frac{x^3\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma} = -\frac{e^{-x^2}}{4\alpha^2}\left[\alpha^2\, J_0(\alpha x + \gamma) + (2\alpha x - 2\gamma)\, J_1(\alpha x + \gamma)\right] -$$

$$-\frac{\gamma}{2\alpha}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx + \frac{4\gamma^2 - \alpha^4}{4\alpha^3}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx - \frac{\gamma(\gamma^2 - \alpha^2)}{\alpha^3}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int x^4\, e^{-x^2}\, J_0(\alpha x + \gamma)\, dx = \frac{e^{-x^2}}{16}\left[2(\alpha^2 - 6 - 4x^2)x\, J_0(\alpha x + \gamma) + \alpha(4x^2 - \alpha^2 + 8)\, J_1(\alpha x + \gamma)\right] +$$

$$+\frac{\alpha^4 - 10\alpha^2 + 12}{16}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx - \frac{\gamma}{4}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx + \frac{\alpha^2(8 - \alpha^2) + 4\gamma^2}{16}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int x^4\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx = \frac{e^{-x^2}}{16\alpha}\left\{[\alpha^4 - 4\alpha^2(2 + x^2)]\, J_0(\alpha x + \gamma) - [8\alpha x^3 + 2\alpha(4 - \alpha^2)x + 4\gamma]\, J_1(\alpha x + \gamma)\right\} +$$

$$+\frac{\gamma}{4}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx - \frac{\alpha^4(8 - \alpha^2) + 8\gamma^2}{16\alpha^2}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx + \frac{\gamma(2\alpha^2 - \alpha^4 + 4\gamma^2)}{8\alpha^2}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int \frac{x^4\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma} = \frac{e^{-x^2}}{8\alpha^3}\left[2\alpha^2(\gamma - \alpha x)\, J_0(\alpha x + \gamma) - (2\alpha^2 - \alpha^4 + 4\gamma^2 - 4\alpha\gamma x + 4\alpha^2 x^2)\, J_1(\alpha x + \gamma)\right] +$$

$$+\frac{4\alpha^2 - \alpha^4 + 4\gamma^2}{8\alpha^2}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx + \frac{\gamma(\alpha^4 + 2\alpha^2 - 4\gamma^2)}{4\alpha^4}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx +$$

$$+\frac{\alpha^6 - 2\alpha^4 - 12\alpha^2\gamma^2 + 8\gamma^4}{8\alpha^4}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int x^5\, e^{-x^2}\, J_0(\alpha x + \gamma)\, dx = \frac{e^{-x^2}}{32}\left\{2[4\alpha x^3 + \alpha(12 - \alpha^2)x + 2\gamma]\, J_1(\alpha x + \gamma) -\right.$$

$$\left. -[16x^4 + (32 - 4\alpha^2)x^2 + \alpha^4 - 16\alpha^2 + 32]\, J_0(\alpha x + \gamma)\right\} - \frac{\alpha\gamma}{8}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx -$$

$$-\frac{\alpha^6 - 16\alpha^4 + 32\alpha^2 - 8\gamma^2}{32\alpha}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx + \frac{\gamma(\alpha^4 - 10\alpha^2 - 4\gamma^2)}{16\alpha}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int x^5\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx = -\frac{e^{-x^2}}{32\,\alpha^2}\left\{2\alpha^2[4\alpha x^3 + \alpha(12 - \alpha^2)x + 2\gamma]\, J_0(\alpha x + \gamma) +\right.$$

$$\left. +[16\alpha^2 x^4 + 4\alpha^2(6 - \alpha^2)x^2 + 8\alpha\gamma x + \alpha^6 - 14\alpha^4 + 12\alpha^2 - 8\gamma^2]\, J_0(\alpha x + \gamma)\right\} +$$

$$+\frac{\alpha^6 - 16\alpha^4 + 36\alpha^2 - 8\gamma^2}{32\,\alpha}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx - \frac{\gamma(\alpha^4 - 3\alpha^2 - 2\gamma^2)}{4\,\alpha^3}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx -$$

$$-\frac{\alpha^8 - 14\alpha^6 + 4\alpha^4(3 - \gamma^2) + 8\gamma^2(\alpha^2 + 2\gamma^2)}{32\,\alpha^3}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma} -$$

$$\int \frac{x^5\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma} = -\frac{e^{-x^2}}{16\,\alpha^4}\left\{\alpha^2[4\alpha^2 x^2 - 4\alpha\gamma x - \alpha^4 + 8\alpha^2 + 4\gamma^2]\, J_0(\alpha x + \gamma) + \right.$$

$$\left. +[8\alpha^3 x^3 - 8\alpha^2\gamma x^2 - \alpha(2\alpha^4 - 8\alpha^2 - 8\gamma^2)x + 2\gamma(\alpha^4 - 4\gamma^2)]\, J_1(\alpha x + \gamma)\right\} +$$

$$+\frac{\gamma(\alpha^4 - 2\alpha^2 - 4\gamma^2)}{8\alpha^3}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx + \frac{\alpha^8 - 8\alpha^6 - 4\alpha^4\gamma^2 - 16\alpha^2\gamma^2 + 16\gamma^4}{16\alpha^5}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx -$$

$$-\frac{\gamma(\alpha^6 - 2\alpha^4 - 8\alpha^2\gamma^2 + 4\gamma^4)}{4\alpha^5}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int x^6\, e^{-x^2}\, J_0(\alpha x + \gamma)\, dx =$$

$$= \frac{e^{-x^2}}{64\alpha}\left\{[16\alpha^2 x^4 + 4\alpha^2(16 - \alpha^2)x^2 + 8\alpha\gamma x + \alpha^6 - 24\alpha^4 + 92\alpha^2 - 8\gamma^2]\, J_1(\alpha x + \gamma) - \right.$$

$$\left. -\alpha[32x^5 + 8(10 - \alpha^2)x^3 + 2(\alpha^4 - 22\alpha^2 + 60)x - 4\alpha\gamma]\, J_0(\alpha x + \gamma)\right\} -$$

$$-\frac{\alpha^6 - 26\alpha^4 + 136\alpha^2 - 120 - 8\gamma^2}{64}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx + \frac{\gamma(\alpha^4 - 8\alpha^2 - 2\gamma^2)}{8\alpha^2}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx +$$

$$+\frac{\alpha^8 - 24\alpha^6 + 4\alpha^4(23 - \gamma^2) + 48\alpha^2\gamma^2 + 16\gamma^4}{64\alpha^2}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int x^6\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx =$$

$$= -\frac{e^{-x^2}}{64\,\alpha^3}\left\{\alpha^2\,[16\alpha^2 x^4 + 4\alpha^2(16 - \alpha^2)x^2 + 8\alpha\gamma x + \alpha^6 - 24\alpha^4 + 96\alpha^2 - 8\gamma^2]\, J_0(\alpha x + \gamma) + \right.$$

$$\left. +2[16\alpha^3 x^5 + 4\alpha^3(8 - \alpha^2)x^3 + 8\alpha^2\gamma x^2 + \alpha(\alpha^6 - 20\alpha^4 + 32\alpha^2 - 8\gamma^2)x - 4\gamma(\alpha^4 - 5\alpha^2 - 2\gamma^2)]\, J_1(\alpha x + \gamma)\right\} -$$

$$-\frac{\gamma(\alpha^4 - 6\alpha^2 - 2\gamma^2)}{8\,\alpha^2}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx -$$

$$-\frac{\alpha^{10} - 24\alpha^8 + 96\alpha^6 - 16\gamma^2(\alpha^4 - 3\alpha^2 - 2\gamma^2)}{64\,\alpha^4}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx +$$

$$+\frac{\gamma[\alpha^8 - 16\alpha^6 + 12\alpha^4 - 4\gamma^2(\alpha^4 - 2\alpha^2 - 4\gamma^2)]}{32\,\alpha^4}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

$$\int \frac{x^6\, e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma} =$$

$$= -\frac{e^{-x^2}}{64\,\alpha^5}\left\{4\alpha^2\,[4\alpha^3 x^3 - 4\alpha^2\gamma x^2 + \alpha(4\gamma^2 - \alpha^4 + 12\alpha^2)x + \gamma(\alpha^4 - 6\alpha^2 - 4\gamma^2)]\, J_0(\alpha x + \gamma) + \right.$$

$$+[32\alpha^4 x^4 - 32\alpha^3\gamma x^3 - 8\alpha^2(\alpha^4 - 6\alpha^2 - 4\gamma^2)x^2 + 8\alpha\gamma(\alpha^4 - 2\alpha^2 - 4\gamma^2) + 2\alpha^8 - 28\alpha^6 +$$

$$\left. +8\alpha^4(3 - \gamma^2) - 16\alpha^2\gamma^2 + 32\gamma^4]\, J_1(\alpha x + \gamma)\right\} +$$

$$+\frac{\alpha^4(\alpha^4 - 16\alpha^2 + 36 - 4\gamma^2) + 16\gamma^4}{32\,\alpha^4}\int e^{-x^2}\, J_0(\alpha x + \gamma)\, dx -$$

$$-\frac{\gamma(\alpha^8 - 4\alpha^6 - 4\alpha^4(3 + \gamma^2) + 8\gamma^2(2\gamma^2 - 3\alpha^2))}{16\,\alpha^6}\int e^{-x^2}\, J_1(\alpha x + \gamma)\, dx -$$

$$-\frac{\alpha^{10} - 14\alpha^8 + 12\alpha^6(1 - \gamma^2) + 24\alpha^4\gamma^2 + 80\alpha^2\gamma^4 - 32\gamma^6}{32\,\alpha^6}\int \frac{e^{-x^2}\, J_1(\alpha x + \gamma)\, dx}{\alpha x + \gamma}$$

h) $I_\nu(\alpha x)$; Special Case: $\alpha = 1$

About the following improper integrals see also [4], 2.15.5, or [7], 6.618.4 .

$$\lambda_0 = \int_0^\infty \exp(-x^2)\, I_0(x)\, dx = \frac{\sqrt{\pi}}{2}\, e^{1/8}\, I_0\left(\frac{1}{8}\right) = 1.00815\ 32626$$

$$\lambda_1 = \int_0^\infty \exp(-x^2)\, I_1(x)\, dx = e^{1/4} - 1 = 0.28402\ 54167$$

$$\lambda_- = \int_0^\infty \frac{\exp(-x^2)\, I_1(x)\, dx}{x} = \frac{\sqrt{\pi}}{4}\, e^{1/8} \left[I_0\left(\frac{1}{8}\right) - I_1\left(\frac{1}{8}\right) \right] = 0.47263\ 32148$$

Let

$$\tilde{F}_\nu(x) = \int_0^x e^{-t^2}\, I_\nu(t)\, dt = \sum_{k=0}^\infty a_k^{(\nu)}\, x^{2k+1+\nu} \quad \text{and} \quad \tilde{F}_-(x) = \int_0^x \frac{e^{-t^2}\, I_\nu(t)\, dt}{t} = \sum_{k=0}^\infty a_k^{(-)}\, x^{2k+1}\ .$$
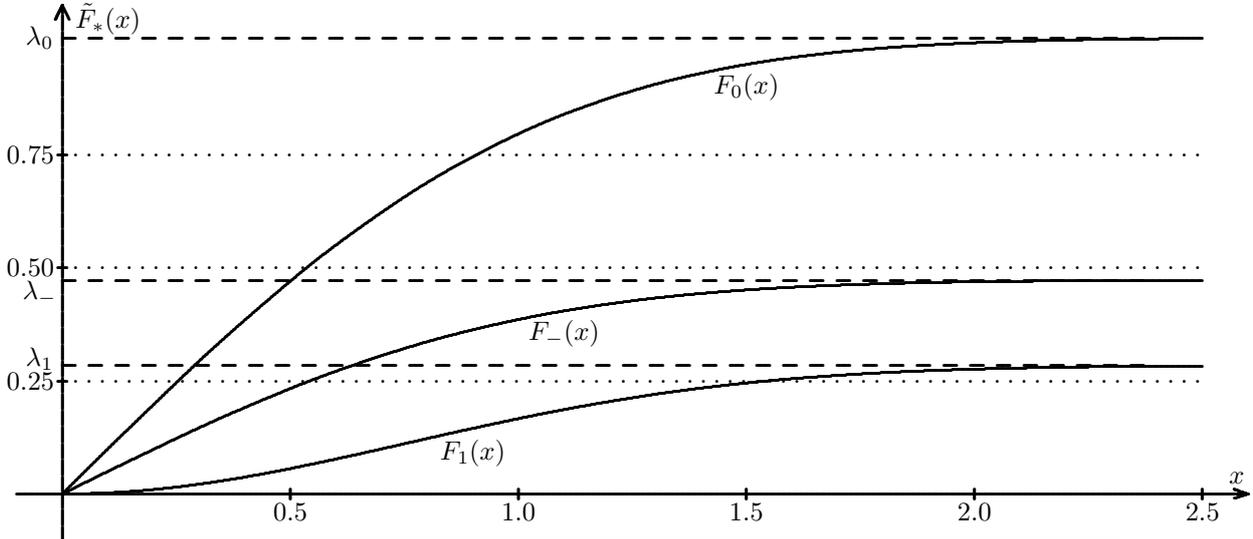
k	$a_k^{(0)}$	$a_k^{(0)}$
0	1	$1.00000\ 00000\ 00000\text{E}+00$
1	$-\frac{1}{4}$	$-2.50000\ 00000\ 00000\text{E}-01$
2	$\frac{17}{320}$	$5.31250\ 00000\ 00000\text{E}-02$
3	$-\frac{131}{16128}$	$-8.12251\ 98412\ 69841\text{E}-03$
4	$\frac{121}{147456}$	$8.20583\ 76736\ 11111\text{E}-04$
5	$-\frac{4579}{162201600}$	$-2.82303\ 01057\ 44949\text{E}-05$
6	$\frac{-238703}{27603763200}$	$-8.64748\ 03551\ 42302\text{E}-06$
7	$\frac{4937119}{2080899072000}$	$2.37258\ 93612\ 20110\text{E}-06$
8	$-\frac{686726911}{1811214552268800}$	$-3.79152\ 712824\ 5399\text{E}-07$
9	$\frac{30843873469}{655872751986278400}$	$4.70272\ 21935\ 33859\text{E}-08$
10	$-\frac{474430811173}{96654931871662080000}$	$-4.90850\ 08078\ 31997\text{E}-09$
11	$\frac{68783074431677}{153708957370763182080000}$	$4.47489\ 04428\ 36362\text{E}-10$
12	$-\frac{3503765621989823}{96235173310390861824000000}$	$-3.64083\ 68182\ 48353\text{E}-11$
13	$\frac{20932954787378741}{7806597258938906711162880000}$	$2.68144\ 41802\ 29685\text{E}-12$
14	$-\frac{10684253218385246191}{59163598426411660994999746560000}$	$-1.80588\ 29250\ 68265\text{E}-13$
15	$\frac{57988854926847793271}{517449653948668782948253696000000}$	$1.12066\ 66094\ 81936\text{E}-14$
16	$-\frac{13331869709946059336021}{20681961910606581189755497611264000000}$	$-6.44613\ 39632\ 91420\text{E}-16$
17	$\frac{26266789484247471454380 77}{76072016263922024994318857577431040000000}$	$3.45288\ 46183\ 22491\text{E}-17$
18	$-\frac{1637897065587891982787358 1}{9474819023084646603967734697540190208000000}$	$-1.72868\ 42752\ 32366\text{E}-18$
19	$\frac{2523155270965161599945213 59}{311045203452272630611527102251902854758400000 0}$	$8.11186\ 04079\ 43308\text{E}-20$
20	$-\frac{953899962106037194572915500479}{2668289314538264905061346341779400489435136000000000 0}$	$-3.57494\ 95263\ 07664\text{E}-21$

k	$a_k^{(1)}$	$a_k^{(1)}$
0	$\frac{1}{4}$	$2.50000\ 00000\ 00000\text{E}{-}01$
1	$-\frac{7}{64}$	$-1.09375\ 00000\ 00000\text{E}{-}01$
2	$\frac{73}{2304}$	$3.16840\ 27777\ 77778\text{E}{-}02$
3	$-\frac{1007}{147456}$	$-6.82915\ 58159\ 72222\text{E}{-}03$
4	$\frac{17201}{14745600}$	$1.16651\ 74696\ 18056\text{E}{-}03$
5	$-\frac{348599}{2123366400}$	$-1.64172\ 79655\ 55073\text{E}{-}04$
6	$\frac{8127673}{416179814400}$	$1.95292\ 34044\ 46595\text{E}{-}05$
7	$-\frac{212763487}{106542032486400}$	$-1.99699\ 10657\ 29472\text{E}{-}06$
8	$\frac{874529239}{4931374075084800}$	$1.77339\ 87032\ 50800\text{E}{-}07$
9	$-\frac{189537494759}{13807847410237440000}$	$-1.37267\ 95287\ 32862\text{E}{-}08$
10	$\frac{6158207356841}{6682998146554920960000}$	$9.21473\ 74902\ 61837\text{E}{-}10$
11	$-\frac{202178049269327}{3849406932415634472960000}$	$-5.25218\ 69685\ 11883\text{E}{-}11$
12	$\frac{6200780742937873}{2602199086312968903720960000}$	$2.38290\ 02075\ 79313\text{E}{-}12$
13	$-\frac{134044005546360727}{204012408366936762051 7232640000}$	$-6.57038\ 49397\ 86072\text{E}{-}14$
14	$-\frac{3177788907789045479}{1836111675302430858465509376000000}$	$-1.73071\ 65737\ 97519\text{E}{-}15$
15	$\frac{2415582328462590247}{5481569549590930609529683968000000}$	$4.40673\ 47985\ 08315\text{E}{-}16$
16	$-\frac{9302247240424895 9110207}{217348617896920071412339593078374 4000000}$	$-4.27987\ 41167\ 22742\text{E}{-}17$
17	$\frac{8952207071913676364932153}{281683808794408412550392112629573 2224000000}$	$3.17810\ 49504\ 50877\text{E}{-}18$
18	$-\frac{8247275863924528163817244 07}{4067514198991257477227662106371037 331456000000}$	$-2.02759\ 60845\ 98758\text{E}{-}19$
19	$\frac{75547388447748738204731644241}{65080227183860119635642593701936597 30329600000000}$	$1.16083\ 47376\ 28028\text{E}{-}20$
20	$-\frac{6997691037925936250951447039759}{1148015207523292510372735352902161576 4301414400000000}$	$-6.09546\ 89381\ 01116\text{E}{-}22$

k	$a_k^{(-)}$	$a_k^{(-)}$
0	$\frac{1}{2}$	$5.00000\ 00000\ 00000\text{E}{-}01$
1	$-\frac{7}{48}$	$-1.45833\ 33333\ 33333\text{E}{-}01$
2	$\frac{73}{1920}$	$3.80208\ 33333\ 33333\text{E}{-}02$
3	$-\frac{1007}{129024}$	$-7.80474\ 95039\ 68254\text{E}{-}03$
4	$\frac{17201}{13271040}$	$1.29613\ 05217\ 97840\text{E}{-}03$
5	$-\frac{348599}{1946419200}$	$-1.79097\ 59624\ 23716\text{E}{-}04$
6	$\frac{8127673}{386452684800}$	$2.10314\ 82817\ 11718\text{E}{-}05$
7	$-\frac{212763487}{99883155456000}$	$-2.13012\ 38034\ 44770\text{E}{-}06$
8	$\frac{874529239}{4657408848691200}$	$1.87771\ 62740\ 30259\text{E}{-}07$
9	$-\frac{189537494759}{13117455039725568000}$	$-1.44492\ 58197\ 18802\text{E}{-}08$
10	$\frac{6158207356841}{6379225503529697280000}$	$9.65353\ 45136\ 07639\text{E}{-}10$
11	$-\frac{202178049269327}{3689014976898316369920000}$	$-5.48054\ 29236\ 64574\text{E}{-}11$
12	$\frac{6200780742937873}{2502114506070162407424000000}$	$2.47821\ 62158\ 82486\text{E}{-}12$
13	$-\frac{134044005546360727}{1967262509252604491213045760000}$	$-6.81373\ 25301\ 48519\text{E}{-}14$
14	$-\frac{3177788907789045479}{17749079527923498298499923968000}$	$-1.79039\ 64556\ 52606\text{E}{-}15$
15	$\frac{2415582328462590247}{53102705011662140279818813440000 00}$	$4.54888\ 75339\ 44067\text{E}{-}16$
16	$-\frac{9302247240424895 9110207}{210956011488187128135506075634892 8000000}$	$-4.40956\ 72717\ 74947\text{E}{-}17$
17	$\frac{8952207071913676364932153}{273859258550119289979547887278751 7440000000}$	$3.26890\ 79490\ 35188\text{E}{-}18$

k	$a_k^{(-)}$	$a_k^{(-)}$
18	$-\dfrac{8247275863924528163817224407}{3960474351649382280458513103571799506944000000}$	$-2.08239\ 59787\ 77103\text{E}-19$
19	$\dfrac{7554738844774873820473164424 1}{6345322150426361664475152885938818237071360000000}$	$1.19059\ 97309\ 00542\text{E}-20$
20	$-\dfrac{170675391168925274413449927799}{2733369541722125024696988935481337086738432000000 00}$	$-6.24413\ 89122\ 01143\text{E}-22$

Asymptotic formulas:

$$\int_x^\infty e^{-t^2} I_0(t)\, dt \ \sim\ \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\left[1 + \frac{5}{8x} - \frac{47}{128x^2} - \frac{913}{1024x^3} + \frac{10123}{32768x^4} + \frac{625915}{262144x^5} + \dots\right] = \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\sum_{k=0}^\infty \frac{c_k^{(0)}}{x^k}$$

Let

$$\varphi_n^{(0)}(x) = \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\sum_{k=0}^n \frac{c_k^{(0)}}{x^k} \quad\text{and}\quad q_n^{(0)}(x) = \left[\varphi_n^{(0)}(x)\ \bigg/ \int_x^\infty e^{-t^2} I_0(t)\, dt\right] - 1\ .$$

Value	$x = 1.5$	$x = 2.0$	$x = 2.5$	$x = 3.0$	$x = 3.5$	$x = 4.0$	$x = 5.0$	$x = 6.0$
Exactly	5.90956E-02	1.11275E-02	1.37491E-03	1.09079E-04	5.47423E-06	1.72020E-07	4.06147E-09	1.38485E-15
$\varphi_0^{(0)}(x)$ $q_0^{(0)}(x)$	5.1289E-02 -0.1321	9.5443E-03 -0.1423	1.1868E-03 7 -0.1368	9.5155E-05 -0.1277	4.8273E-06 -0.1182	1.5320E-07 -0.1094	3.6774E-11 -0.0946	1.2700E-15 -0.0829
$\varphi_1^{(0)}(x)$ $q_1^{(0)}(x)$	7.2659E-02 0.2295	1.2527E-02 0.1258	1.4835E-03 0.0790	1.1498E-04 0.0541	5.6893E-06 0.0393	1.7714E-07 0.0297	4.1370E-11 0.0186	1.4023E-15 0.0126
$\varphi_2^{(0)}(x)$ $q_2^{(0)}(x)$	6.4289E-02 0.08788	1.1651E-02 0.04703	1.4137E-03 0.02824	1.1110E-04 0.01850	5.5446E-06 0.01285	1.7362E-07 0.00930	4.0830E-11 0.00530	1.3894E-15 0.00328
$\varphi_3^{(0)}(x)$ $q_3^{(0)}(x)$	5.0740E-02 -0.14140	1.0587E-02 -0.04856	1.3460E-03 -0.02101	1.0795E-04 -0.01031	5.4442E-06 -0.00549	1.7149E-07 -0.00310	4.0568E-11 -0.00116	1.38414E-15 -0.000508
$\varphi_4^{(0)}(x)$ $q_4^{(0)}(x)$	5.38695E-02 -0.088435	1.07714E-02 -0.032002	1.35540E-03 -0.014184	1.08317E-04 -0.006984	5.45414E-06 -0.003670	1.71671E-07 -0.002028	4.05860E-11 -0.000708	1.38445E-15 -0.000289
$\varphi_5^{(0)}(x)$ $q_5^{(0)}(x)$	6.99960E-02 0.184454	1.14835E-02 0.031997	1.38442E-03 0.006920	1.09252E-04 0.001588	5.47608E-06 0.000339	1.72029E-07 0.000048	4.06141E-11 -0.000016	1.38484E-15 -0.000008

$$\int_x^\infty e^{-t^2} I_1(t)\, dt \ \sim\ \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\left[1 + \frac{1}{8x} - \frac{103}{128x^2} - \frac{677}{1024x^3} + \frac{30587}{32768x^4} + \frac{439535}{262144x^5} + \dots\right] = \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\sum_{k=0}^\infty \frac{c_k^{(1)}}{x^k}$$

Let

$$\varphi_n^{(1)}(x) = \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\sum_{k=0}^n \frac{c_k^{(1)}}{x^k} \quad\text{and}\quad q_n^{(1)}(x) = \left[\varphi_n^{(1)}(x)\ \bigg/ \int_x^\infty e^{-t^2} I_1(t)\, dt\right] - 1\ .$$

Value	$x = 1.5$	$x = 2.0$	$x = 2.5$	$x = 3.0$	$x = 3.5$	$x = 4.0$	$x = 5.0$	$x = 6.0$
Exactly	3.87808E-02	8.13674E-03	1.07802E-03	8.95917E-05	4.64340E-06	1.49352E-07	3.63793E-11	1.26530E-15
$\varphi_0^{(1)}(x)$ $q_0^{(1)}(x)$	5.12888E-02 0.3225	9.54434E-03 0.1730	1.18677E-03 0.1009	9.51549E-05 0.0621	4.82727E-06 0.0396	1.53199E-07 0.0258	3.67735E-11 0.0108	1.27004E-15 0.00375
$\varphi_1^{(1)}(x)$ $q_1^{(1)}(x)$	5.55628E-02 0.4327	1.01409E-02 0.2463	1.24611E-03 0.1559	9.91197E-05 0.1063	4.99967E-06 0.0767	1.57987E-07 0.0578	3.76929E-11 0.0361	1.29650E-15 0.0247
$\varphi_2^{(1)}(x)$ $q_2^{(1)}(x)$	3.72200E-02 -0.0402	8.22081E-03 0.0103	1.09331E-03 0.0142	9.06119E-05 0.0114	4.68257E-06 0.00844	1.50282E-07 0.00622	3.65092E-11 0.00357	1.26811E-15 0.00222
$\varphi_3^{(1)}(x)$ $q_3^{(1)}(x)$	2.71730E-02 -0.2993	7.43205E-03 -0.08661	1.04309E-03 -0.03240	8.82819E-05 -0.0146	4.60814E-06 -0.00759	1.48699E-07 -0.00437	3.63147E-11 -0.00177	1.26423E-15 -0.000852
$\varphi_4^{(1)}(x)$ $q_4^{(1)}(x)$	3.66298E-02 -0.0555	7.98887E-03 -0.0182	1.07145E-03 -0.00609	8.93785E-05 -0.00238	4.63816E-06 -0.00113	1.49258E-07 -0.000632	3.63697E-11 -0.000264	1.26514E-15 -0.000129
$\varphi_5^{(1)}(x)$ $q_5^{(1)}(x)$	4.79543E-02 0.2365	8.48896E-03 0.0433	1.09183E-03 0.0128	9.00351E-05 0.00495	4.65357E-06 0.00219	1.49509E-07 0.00105	3.63894E-11 0.000278	1.26542E-15 0.000088

The values of $\varphi_1^{(1)}(x)$ seem to be worse than such of $\varphi_0^{(1)}(x)$, but for $x > x^* = 7.0135\ldots$ one has

$$\varphi_0^{(1)}(x) \; < \; \int_x^\infty e^{-t^2}\, I_1(t)\, dt$$

and therefore $c_1^{(1)} = 1/8 > 0$ is true. Holds $\varphi_0^{(1)}(x^*) = 5.18\,\text{E-}21$.

$$\int_x^\infty \frac{e^{-t^2}\, I_1(t)\, dt}{t} \;\sim\; \frac{e^{-x^2+x}}{\sqrt{8\pi x^5}}\left[1 + \frac{1}{8x} - \frac{167}{128x^2} - \frac{997}{1024x^3} + \frac{75515}{32768x^4} + \frac{931183}{262144x^5} + \ldots\right] = \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}} \sum_{k=0}^\infty \frac{c_k^{(-)}}{x^k}$$

Let
$$\varphi_n^{(-)}(x) = \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}} \sum_{k=0}^n \frac{c_k^{(\cdot)}}{x^k} \quad \text{and} \quad q_n^{(-)}(x) = \left[\varphi_n^{(1)}(x) \bigg/ \int_x^\infty \frac{e^{-t^2}\, I_1(t)\, dt}{t}\right] - 1\ .$$

Value	$x = 1.5$	$x = 2.0$	$x = 2.5$	$x = 3.0$	$x = 3.5$	$x = 4.0$	$x = 5.0$	$x = 6.0$
Exactly	2.17978E-02	3.64967E-03	4.00293E-04	2.83007E-05	1.27407E-06	3.61817E-08	7.12851E-12	2.07890E-16
$\varphi_0^{(-)}(x)$	3.41925E-02	4.77217E-03	4.74707E-04	3.17183E-05	1.37922E-06	3.82998E-08	7.35471E-12	2.11674E-16
$q_0^{(-)}(x)$	0.5686	0.3076	0.1859	0.1208	0.0825	0.0585	0.0317	0.0182
$\varphi_1^{(-)}(x)$	3.70419E-02	5.07043E-03	4.98442E-04	3.30399E-05	1.42848E-06	3.94967E-08	7.53858E-12	2.16084E-16
$q_1^{(-)}(x)$	0.6993	0.3893	0.2452	0.1675	0.1212	0.0916	0.0575	0.0394
$\varphi_2^{(-)}(x)$	1.72150E-02	3.51388E-03	3.99347E-04	2.84419E-05	1.28158E-06	3.63736E-08	7.15475E-12	2.08413E-16
$q_2^{(-)}(x)$	-0.2102	-0.0372	-0.00236	0.00499	0.00590	0.00530	0.00368	0.00251
$\varphi_3^{(-)}(x)$	7.35100E-03	2.93309E-03	3.69767E-04	2.72981E-05	1.25026E-06	3.57909E-08	7.09747E-12	2.07458E-16
$q_3^{(-)}(x)$	-0.6628	-0.1963	-0.0763	-0.0354	-0.0187	-0.0108	-0.00436	-0.00208
$\varphi_4^{(-)}(x)$	2.29160E-02	3.62044E-03	3.97773E-04	2.82005E-05	1.27144E-06	3.61357E-08	7.12458E-12	2.07835E-16
$q_4^{(-)}(x)$	0.0513	-0.00801	-0.00629	-0.00354	-0.00206	-0.00127	-0.000551	-0.000266
$\varphi_5^{(-)}(x)$	3.89105E-02	4.15018E-03	4.15040E-04	2.86642E-05	1.28077E-06	3.62686E-08	7.13294E-12	2.07931E-16
$q_5^{(-)}(x)$	0.7851	0.1371	0.0368	0.0128	0.00526	0.00240	0.000622	0.000199

Integrals:

$$\int x\, e^{-x^2}\, I_0(x)\, dx = -\frac{e^{-x^2}\, I_0(x)}{2} + \frac{1}{2}\int e^{-x^2}\, I_1(x)\, dx$$

$$\int x\, e^{-x^2}\, I_1(x)\, dx = -\frac{e^{-x^2}\, I_1(x)}{2} + \frac{1}{2}\int e^{-x^2}\, I_0(x)\, dx - \frac{1}{2}\int \frac{e^{-x^2}\, I_1(x)\, dx}{x}$$

$$\int x^2\, e^{-x^2}\, I_0(x)\, dx =$$

$$= -\frac{e^{-x^2}}{4}\, [2x\, I_0(x) + I_1(x)] + \frac{3}{4}\int e^{-x^2}\, I_0(x)\, dx - \frac{1}{4}\int \frac{e^{-x^2}\, I_1(x)\, dx}{x}$$

$$\int x^2\, e^{-x^2}\, I_1(x)\, dx = -\frac{e^{-x^2}}{4}\, [2x\, I_1(x) + I_0(x)] + \frac{1}{4}\int e^{-x^2}\, I_1(x)\, dx$$

Recurrence relations:

$$\int x^{2n+1}\, e^{-x^2}\, I_0(x)\, dx = -\frac{x^{2n} e^{-x^2}}{2}\, I_0(x) + \frac{1}{2}\int x^{2n}\, e^{-x^2}\, I_1(x)\, dx + n\int x^{2n-1}\, e^{-x^2}\, I_0(x)\, dx$$

$$\int x^{2n+1}\, e^{-x^2}\, I_1(x)\, dx = -\frac{x^{2n} e^{-x^2}}{2}\, I_1(x) + \frac{1}{2}\int x^{2n}\, e^{-x^2}\, I_0(x)\, dx + \frac{2n-1}{2}\int x^{2n-1}\, e^{-x^2}\, I_1(x)\, dx$$

$$\int x^{2n+2}\, e^{-x^2}\, I_0(x)\, dx =$$

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$$= -\frac{x^{2n}e^{-x^2}}{4}\left[I_1(x) + 2xI_0(x)\right] + \frac{4n+3}{4}\int x^{2n}\,e^{-x^2}\,I_0(x)\,dx + \frac{2n-1}{4}\int x^{2n-1}\,e^{-x^2}\,I_1(x)\,dx$$

$$\int x^{2n+2}\,e^{-x^2}\,I_1(x)\,dx =$$

$$= -\frac{x^{2n}e^{-x^2}}{4}\left[I_0(x) + 2xI_1(x)\right] + \frac{4n+1}{4}\int x^{2n}\,e^{-x^2}\,I_1(x)\,dx + \frac{n}{2}\int x^{2n-1}\,e^{-x^2}\,I_0(x)\,dx$$

i) General case

About the following improper integrals see also [4], 2.15.5, or [7], 6.618.4 .

$$\int_0^\infty \exp(-x^2)\,I_0(\alpha x)\,dx = \frac{\sqrt{\pi}}{2}\,e^{\alpha^2/8}\,I_0\left(\frac{\alpha^2}{8}\right)$$

$$\int_0^\infty \exp(-x^2)\,I_1(\alpha x)\,dx = \frac{1}{\alpha}\left[e^{\alpha^2/4} - 1\right]$$

$$\int_0^\infty \frac{\exp(-x^2)\,I_1(\alpha x)\,dx}{x} = \frac{\sqrt{\pi}\,\alpha^2}{4}\,e^{\alpha^2/8}\left[I_0\left(\frac{\alpha^2}{8}\right) - I_1\left(\frac{\alpha^2}{8}\right)\right]$$

Let

$$\tilde{F}_\nu(x;\alpha) = \int_0^x e^{-t^2}\,I_\nu(\alpha t)\,dt = \sum_{k=0}^\infty a_k^{(\nu)}(\alpha)\,x^{2k+1+\nu} \quad \text{and} \quad \tilde{F}_-(x;\alpha) = \int_0^x \frac{e^{-t^2}\,I_\nu(\alpha t)\,dt}{t} = \sum_{k=0}^\infty a_k^{(-)}(\alpha)\,x^{2k+1}$$

and $a_k^{(*)}(\alpha) = b_k^{(*)}(\alpha)/n_k^{(*)}$.

k	$n_k^{(0)}$, $b_k^{(0)}(\alpha)$
0	1 1
1	12 $\alpha^2 - 4$
2	320 $\alpha^4 - 16\,\alpha^2 + 32$
3	16128 $\alpha^6 - 36\,\alpha^4 + 288\,\alpha^2 - 384$
4	13 27104 $\alpha^8 - 64\,\alpha^6 + 1152\,\alpha^4 - 6144\,\alpha^2 + 6144$
5	1622 01600 $\alpha^{10} - 100\,\alpha^8 + 3200\,\alpha^6 - 38400\,\alpha^4 + 153600\,\alpha^2 - 122880$
6	2 76037 63200 $\alpha^{12} - 144\,\alpha^{10} + 7200\,\alpha^8 - 153600\,\alpha^6 + 1382400\,\alpha^4 - 4423680\,\alpha^2 + 2949120$
7	624 26972 16000 $\alpha^{14} - 196\,\alpha^{12} + 14112\,\alpha^{10} - 470400\,\alpha^8 + 7526400\,\alpha^6 - 54190080\,\alpha^4 + 144506880\,\alpha^2 - 82575360$

k	$n_k^{(1)}$, $b_k^{(1)}(\alpha)$
0	4 α
1	64 $\alpha^3 - 8\,\alpha$
2	2304 $\alpha^5 - 24\,\alpha^3 + 96\,\alpha$
3	1 47456 $\alpha^7 - 48\,\alpha^5 + 576\,\alpha^3 - 1536\,\alpha$
4	147 45600 $\alpha^9 - 80\,\alpha^7 + 1920\,\alpha^5 - 15360\,\alpha^3 + 30720\,\alpha$

k	$n_k^{(1)}$, $b_k^{(1)}(\alpha)$
5	$21233\,66400$ $\alpha^{11} - 120\,\alpha^9 + 4800\,\alpha^7 - 76800\,\alpha^5 + 460800\,\alpha^3 - 737280\,\alpha$
6	$41\,61798\,14400$ $\alpha^{13} - 168\,\alpha^{11} + 10080\,\alpha^9 - 268800\,\alpha^7 + 3225600\,\alpha^5 - 15482880\,\alpha^3 + 20643840\,\alpha$
7	$10654\,20324\,86400$ $\alpha^{15} - 224\,\alpha^{13} + 18816\,\alpha^{11} - 752640\,\alpha^9 + 15052800\,\alpha^7 - 144506880\,\alpha^5 + 578027520\,\alpha^3 - 660602880\,\alpha$

k	$n_k^{(-)}$, $b_k^{(-)}(\alpha)$
0	2 α
1	48 $\alpha^3 - 8\,\alpha$
2	1920 $\alpha^5 - 24\,\alpha^3 + 96\,\alpha$
3	$1\,29024$ $\alpha^7 - 48\,\alpha^5 + 576\,\alpha^3 - 1536\,\alpha$
4	$132\,71040$ $\alpha^9 - 80\,\alpha^7 + 1920\,\alpha^5 - 15360\,\alpha^3 + 30720\,\alpha$
5	$19464\,19200$ $\alpha^{11} - 120\,\alpha^9 + 4800\,\alpha^7 - 76800\,\alpha^5 + 460800\,\alpha^3 - 737280\,\alpha$
6	$38\,64526\,84800$ $\alpha^{13} - 168\,\alpha^{11} + 10080\,\alpha^9 - 268800\,\alpha^7 + 3225600\,\alpha^5 - 15482880\,\alpha^3 + 20643840\,\alpha$
7	$9988\,31554\,56000$ $\alpha^{15} - 224\,\alpha^{13} + 18816\,\alpha^{11} - 752640\,\alpha^9 + 15052800\,\alpha^7 - 144506880\,\alpha^5 + 578027520\,\alpha^3 - 660602880\,\alpha$

Asymptotic formulas: They are based on the asymptotic formulas for $I_\nu(x)$. So they are inaccurate for small values of α. At least they should not be used for $x < 2/\alpha$, roughly spoken.

$$\int_x^\infty e^{-t^2} I_0(\alpha t)\,dt \;\sim\; \frac{e^{-x^2+\alpha x}}{\sqrt{8\pi x^3}} \left[1 + \frac{4\alpha^2+1}{8\alpha x} + \frac{32\,\alpha^4 - 88\,\alpha^2 + 9}{128\alpha^2 x^2} + \frac{128\,\alpha^6 - 992\,\alpha^4 - 124\,\alpha^2 + 75}{1024\alpha^3 x^3} + \right.$$
$$+ \frac{2048\,\alpha^8 - 30208\,\alpha^6 + 37440\,\alpha^4 - 2832\,\alpha^2 + 3675}{32768\alpha^4 x^4} +$$
$$\left. + \frac{8192\,\alpha^{10} - 194560\,\alpha^8 + 721152\,\alpha^6 + 60096\,\alpha^4 - 28500\,\alpha^2 + 59535}{262144\alpha^5 x^5} + \ldots \right]$$

$$\int_x^\infty e^{-t^2} I_1(\alpha t)\,dt \;\sim\; \frac{e^{-x^2+\alpha x}}{\sqrt{8\pi x^3}} \left[1 + \frac{4\alpha^2-3}{8\alpha x} + \frac{32\,\alpha^4 - 120\,\alpha^2 - 15}{128\alpha^2 x^2} + \frac{128\,\alpha^6 - 1120\,\alpha^4 + 420\,\alpha^2 - 105}{1024\alpha^3 x^3} + \right.$$
$$+ \frac{2048\,\alpha^8 - 32256\,\alpha^6 + 60480\,\alpha^4 + 5040\,\alpha^2 - 4725}{32768\alpha^4 x^4} +$$
$$\left. + \frac{8192\,\alpha^{10} - 202752\,\alpha^8 + 887040\,\alpha^6 - 221760\,\alpha^4 + 41580\,\alpha^2 - 72765}{262144\alpha^5 x^5} + \ldots \right]$$

$$\int_x^\infty \frac{e^{-t^2} I_1(\alpha t)}{t}\,dt \;\sim\; \frac{e^{-x^2+\alpha x}}{\sqrt{8\pi\alpha x^5}} \left[1 + \frac{4\alpha^2-3}{8\alpha x} + \frac{32\,\alpha^4 - 184\,\alpha^2 - 15}{128\alpha^2 x^2} + \frac{128\,\alpha^6 - 1632\,\alpha^4 + 612\,\alpha^2 - 105}{1024\alpha^3 x^3} + \right.$$
$$+ \frac{2048\,\alpha^8 - 44544\,\alpha^6 + 115776\,\alpha^4 + 6960\,\alpha^2 - 4725}{32768\alpha^4 x^4} +$$
$$\left. + \frac{8192\,\alpha^{10} - 268288\,\alpha^8 + 1612032\,\alpha^6 - 403008\,\alpha^4 + 55020\,\alpha^2 - 72765}{262144\alpha^5 x^5} + \ldots \right]$$

Approximations for $\alpha < 1$:

$$\tilde{F}_0(x;\alpha) \approx \frac{\sqrt{\pi}}{2} \left[1 + \frac{\alpha^2}{8} + \frac{3\,\alpha^4}{256} + \frac{5\,\alpha^6}{6144} + \frac{35\,\alpha^8}{786432} + \frac{21\,\alpha^{10}}{10485760} + \frac{77\,\alpha^{12}}{1006632960} + \frac{143\,\alpha^{14}}{56371445760} + \right.$$

$$+\frac{143}{1924145348608}\,\alpha^{16}+\frac{2431\,\alpha^{18}}{1246846185897984}+\frac{46189\,\alpha^{20}}{997476948718387200}\Bigg]\operatorname{erf}(x)-$$

$$-\frac{\alpha^2 x}{8}\left[1+\frac{3\,\alpha^2}{32}+\frac{5\alpha^4}{768}+\frac{35\alpha^6}{98304}+\frac{21\alpha^8}{1310720}+\frac{77\alpha^{10}}{125829120}+\frac{143\alpha^{12}}{7046430720}+\frac{143\alpha^{14}}{240518168576}+\right.$$

$$\left.+\frac{2431\alpha^{16}}{155855773237248}+\frac{46189\alpha^{18}}{124684618589798400}\right]-\frac{\alpha^4 x^3}{128}\left[1+\frac{5\alpha^2}{72}+\frac{35\alpha^4}{9216}+\frac{7\alpha^6}{40960}+\frac{77\alpha^8}{11796480}+\right.$$

$$\left.+\frac{143\alpha^{10}}{660602880}+\frac{143\alpha^{12}}{22548578304}+\frac{2431\alpha^{14}}{14611478740992}+\frac{2431\alpha^{14}}{14611478740992}+\frac{4199\alpha^{16}}{48983243017420800}\right]-$$

$$-\frac{\alpha^6 x^5}{4608}\left[1+\frac{7\alpha^2}{128}+\frac{63\alpha^4}{25600}+\frac{77\alpha^6}{819200}+\frac{143\alpha^8}{45875200}+\frac{429\alpha^{10}}{4697620480}+\frac{2431\alpha^{12}}{1014686023680}+\frac{46189\alpha^{14}}{811748818944000}\right]-$$

$$-\frac{\alpha^8 x^7}{294912}\left[1+\frac{9\alpha^2}{200}+\frac{11\alpha^4}{6400}+\frac{143\alpha^6}{2508800}+\frac{429\alpha^8}{256901120}+\frac{2431\alpha^{10}}{55490641920}+\frac{46189\alpha^{12}}{44392513536000}\right]-$$

$$-\frac{\alpha^{10} x^9}{294912}\left[1+\frac{11\alpha^2}{288}+\frac{143\alpha^4}{112896}+\frac{715\alpha^6}{19267584}+\frac{12155\alpha^8}{12485394432}+\frac{46189\alpha^{10}}{1997663109120}\right]-$$

$$-\frac{\alpha^{12} x^{11}}{4246732800}\left[1+\frac{13\alpha^2}{392}+\frac{195\alpha^4}{200704}+\frac{1105\alpha^6}{43352064}+\frac{4199\alpha^8}{6936330240}\right]-$$

$$-\frac{\alpha^{14} x^{13}}{832359628800}\left[1+\frac{15\,\alpha^2}{512}+\frac{85\alpha^4}{110592}+\frac{323\alpha^6}{17694720}\right]-\frac{\alpha^{16} x^{15}}{213084064972800}\left[1+\frac{17\alpha^2}{648}+\frac{323\alpha^4}{518400}\right]-$$

$$-\frac{\alpha^{18} x^{17}}{69039237051187200}\left[1+\frac{19\alpha^2}{800}\right]-\frac{\alpha^{20} x^{19}}{69039237051187200}$$

$$\tilde{F}_1(x;\alpha)\approx\frac{\alpha(1-e^{-x^2})}{4}\left[1+\frac{\alpha^2}{8}+\frac{\alpha^4}{96}+\frac{\alpha^6}{1536}+\frac{1\alpha^8}{30720}+\frac{\alpha^{10}}{737280}+\frac{\alpha^{12}}{20643840}+\frac{\alpha^{14}}{660602880}+\right.$$

$$\left.+\frac{\alpha^{16}}{23781703680}+\frac{\alpha^{18}}{951268147200}\right]-\frac{\alpha^3 x^2}{32}\left[1+\frac{\alpha^2}{12}+\frac{\alpha^4}{192}+\frac{\alpha^6}{3840}+\right.$$

$$\left.+\frac{\alpha^8}{92160}+\frac{\alpha^{10}}{2580480}+\frac{\alpha^{12}}{82575360}++\frac{\alpha^{14}}{2972712960}+\frac{\alpha^{16}}{118908518400}+\frac{\alpha^{18}}{5231974809600}\right]-$$

$$-\frac{\alpha^5 x^4}{768}\left[1+\frac{\alpha^2}{16}+\frac{\alpha^4}{320}+\frac{\alpha^6}{7680}+\frac{\alpha^8}{215040}+\frac{\alpha^{10}}{6881280}+\frac{\alpha^{12}}{247726080}+\frac{\alpha^{14}}{9909043200}+\frac{\alpha^{16}}{435997900800}\right]-$$

$$-\frac{\alpha^7 x^6}{36864}\left[1+\frac{\alpha^2}{20}+\frac{\alpha^4}{480}+\frac{\alpha^6}{13440}+\frac{\alpha^8}{430080}+\frac{\alpha^{10}}{15482880}+\frac{\alpha^{12}}{619315200}+\frac{\alpha^{14}}{27249868800}\right]-$$

$$-\frac{\alpha^9 x^8}{2949120}\left[1+\frac{\alpha^2}{24}+\frac{\alpha^4}{672}+\frac{\alpha^6}{21504}+\frac{\alpha^8}{774144}+\frac{\alpha^{10}}{30965760}+\frac{\alpha^{12}}{1362493440}\right]-$$

$$-\frac{\alpha^{11} x^{10}}{353894400}\left[1+\frac{\alpha^2}{28}+\frac{\alpha^4}{896}+\frac{\alpha^6}{32256}+\frac{\alpha^8}{1290240}+\frac{\alpha^{10}}{56770560}\right]-$$

$$-\frac{\alpha^{13} x^{12}}{59454259200}\left[1+\frac{\alpha^2}{32}+\frac{\alpha^4}{1152}+\frac{\alpha^6}{46080}+\frac{\alpha^8}{2027520}\right]-\frac{\alpha^{15} x^{14}}{13317754060800}\left[1+\frac{\alpha^2}{36}+\frac{\alpha^4}{1440}+\frac{\alpha^6}{63360}\right]-$$

$$-\frac{\alpha^{17} x^{16}}{3835513169510400}\left[1+\frac{\alpha^2}{40}+\frac{\alpha^4}{1760}\right]-\frac{\alpha^{19} x^{18}}{1380784741023744000}\left[1+\frac{\alpha^2}{44}\right]-\frac{\alpha^{21} x^{20}}{607545286050447360000}$$

$$\tilde{F}_-(x;\alpha)\approx\frac{\sqrt{\pi}\,\alpha}{4}\left[1+\frac{\alpha^2}{16}+\frac{\alpha^4}{256}+\frac{5\alpha^6}{24576}+\frac{7\alpha^8}{786432}+\frac{7\alpha^{10}}{20971520}+\frac{11\alpha^{12}}{1006632960}+\frac{143\,\alpha^{14}}{450971566080}+\right.$$

$$\left.+\frac{143\,\alpha^{16}}{17317308137472}+\frac{2431\alpha^{18}}{12468461858979840}+\frac{4199\,\alpha^{20}}{997476948718387200}\right]\operatorname{erf}(x)-\frac{\alpha^3 x}{32}\left[1+\frac{\alpha^2}{16}+\frac{5\alpha^4}{1536}+\right.$$

$$\left.+\frac{7\alpha^6}{49152}+\frac{7\alpha^8}{1310720}+\frac{11\alpha^{10}}{62914560}+\frac{143\alpha^{12}}{28185722880}+\frac{143\alpha^{14}}{1082331758592}+\frac{2431\alpha^{16}}{779278866186240}\right]-$$

$$-\frac{\alpha^5 x^3}{32}\left[1+\frac{5\alpha^2}{96}+\frac{7\alpha^4}{3072}+\frac{7\alpha^6}{81920}+\frac{11\alpha^8}{3932160}+\frac{143\alpha^{10}}{1761607680}+\frac{143\alpha^{12}}{67645734912}+\frac{2431\alpha^{14}}{48704929136640}\right]-$$

$$-\frac{\alpha^7 x^5}{36864}\left[1+\frac{7\alpha^2}{160}+\frac{21\alpha^4}{12800}+\frac{11\alpha^6}{204800}+\frac{143\alpha^8}{91750400}+\frac{143\alpha^8}{91750400}+\frac{143\alpha^{10}}{3523215360}+\frac{2431\alpha^{12}}{2536715059200}\right]-$$

$$-\frac{\alpha^9 x^7}{2949120}\left[1+\frac{3\alpha^2}{80}+\frac{11\alpha^4}{8960}+\frac{143\alpha^6}{4014080}+\frac{143\alpha^8}{154140672}+\frac{2431\alpha^{10}}{110981283840}\right]-$$

$$-\frac{\alpha^{11} x^9}{353894400}\left[1+\frac{11\alpha^2}{336}+\frac{143\alpha^4}{150528}+\frac{715\alpha^6}{28901376}+\frac{2431\alpha^8}{4161798144}\right]-$$

$$-\frac{\alpha^{13} x^{11}}{59454259200}\left[1+\frac{13\alpha^2}{448}+\frac{65\alpha^4}{86016}+\frac{221\alpha^6}{12386304}\right]-\frac{\alpha^{15} x^{13}}{13317754060800}\left[1+\frac{5\alpha^2}{192}+\frac{17\alpha^4}{27648}\right]-$$

$$-\frac{\alpha^{17} x^{15}}{3835513169510400}\left[1+\frac{17\alpha^2}{720}\right]-\frac{\alpha^{19} x^{17}}{1380784741023744000}$$

Integrals:

$$\int x\,e^{-x^2}\,I_0(\alpha x)\,dx = -\frac{e^{-x^2}\,I_0(\alpha x)}{2}+\frac{\alpha}{2}\int e^{-x^2}\,I_1(\alpha x)\,dx$$

$$\int x\,e^{-x^2}\,I_1(\alpha x)\,dx = -\frac{e^{-x^2}\,I_1(\alpha x)}{2}+\frac{\alpha}{2}\int e^{-x^2}\,I_0(\alpha x)\,dx-\frac{1}{2}\int\frac{e^{-x^2}\,I_1(\alpha x)\,dx}{x}$$

$$\int x^2\,e^{-x^2}\,I_0(\alpha x)\,dx =$$

$$=-\frac{e^{-x^2}}{4}\left[2x\,I_0(\alpha x)+\alpha\,I_1(\alpha x)\right]+\frac{\alpha^2+2}{4}\int e^{-x^2}\,I_0(\alpha x)\,dx-\frac{\alpha}{4}\int\frac{e^{-x^2}\,I_1(\alpha x)\,dx}{x}$$

$$\int x^2\,e^{-x^2}\,I_1(\alpha x)\,dx = -\frac{e^{-x^2}}{4}\left[2x\,I_1(\alpha x)+\alpha\,I_0(\alpha x)\right]+\frac{\alpha^2}{4}\int e^{-x^2}\,I_1(\alpha x)\,dx$$

Recurrence relations:

$$\int x^{2n+1}\,e^{-x^2}\,I_0(\alpha x)\,dx = -\frac{x^{2n}e^{-x^2}}{2}\,I_0(\alpha x)+\frac{\alpha}{2}\int x^{2n}\,e^{-x^2}\,I_1(\alpha x)\,dx+n\int x^{2n-1}\,e^{-x^2}\,I_0(\alpha x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,I_1(\alpha x)\,dx = -\frac{x^{2n}e^{-x^2}}{2}\,I_1(\alpha x)+\frac{\alpha}{2}\int x^{2n}\,e^{-x^2}\,I_0(\alpha x)\,dx+\frac{2n-1}{2}\int x^{2n-1}\,e^{-x^2}\,I_1(\alpha x)\,dx$$

$$\int x^{2n+2}\,e^{-x^2}\,I_0(\alpha x)\,dx =$$

$$=-\frac{x^{2n}e^{-x^2}}{4}\left[\alpha I_1(\alpha x)+2xI_0(\alpha x)\right]+\frac{4n+2+\alpha^2}{4}\int x^{2n}\,e^{-x^2}\,I_1(\alpha x)\,dx+\frac{(2n-1)\alpha}{2}\int x^{2n-1}\,e^{-x^2}\,I_1(\alpha x)\,dx$$

$$\int x^{2n+2}\,e^{-x^2}\,I_1(\alpha x)\,dx =$$

$$=-\frac{x^{2n}e^{-x^2}}{4}\left[\alpha I_0(\alpha x)+2xI_1(\alpha x)\right]+\frac{4n+\alpha^2}{4}\int x^{2n}\,e^{-x^2}\,I_0(\alpha x)\,dx+\frac{n\alpha}{2}\int x^{2n-1}\,e^{-x^2}\,I_0(\alpha x)\,dx$$

1.3.12. Integrals with Orthogonal Polynomials $F_n(x)$: $\int F_n(x) \cdot Z_\nu(x)\,dx$

No simple recurrence relations were found.

Let $p_n(x)$ denote some system of orthogonal polynomials with degree$[p_n(x)] = n$. Furthermore, let

$$\int p_n(x)\, Z_\nu(x)\,dx = \sum_{k=0}^{n+1} \left[\lambda_k\, p_k(x)\, Z_0(x) + \mu_k\, p_k(x)\, Z_1(x)\,\right] + \gamma\, \Xi(x)\,,$$

where $\Xi(x)$ denotes one of the functions of the type $\Phi(x)$ or $\Psi(x)$, defined as on page 7 and depending from $Z_\nu(x)$.

In the following the non-zero values of λ_k, μ_k and γ are given for $2 \le n \le 10$, if $p_1(x) = x$, otherwise for $1 \le n \le 10$.

a) Legendre Polynomials $P_n(x)$:

$$\int_{-1}^{1} P_n^2(x)\,dx \quad = \quad \frac{2}{2n+1}$$

First polynomials:

$$P_0(x) = 1\,,\ P_1(x) = x\,,\ P_2(x) = \frac{3x^2-1}{2}\,,\ P_3(x) = \frac{5x^3-3x}{2}\,,\ P_4(x) = \frac{35x^4-30x^2+3}{8}$$

$Z_\nu(x) \;=\; J_0(x)$:

n	λ_k	μ_k	γ
2	$\lambda_1 = -1/2$	$\mu_0 = 1/2\,,\ \mu_2 = 1$	-2
3	$\lambda_0 = 5/3\,,\ \lambda_2 = 10/3$	$\mu_1 = -10\,,\ \mu_3 = 1$	0
4	$\lambda_1 = 33/4\,,\ \lambda_3 = 21/4$	$\mu_0 = -27/2\,,\ \mu_2 = -105/4\,,\ \mu_4 = 1$	$87/2$
5	$\lambda_0 = -1253/15\,,\ \lambda_2 = -485/3\,,\ \lambda_4 = 36/5$	$\mu_1 = 2317/5\,,\ \mu_3 = -252/5\,,\ \mu_5 = 1$	0
6	$\lambda_1 = -1309/2\,,\ \lambda_3 = -1274/3\,,\ \lambda_5 = 55/6$	$\mu_0 = 1070\,,\ \mu_2 = 4155/2\,,\ \mu_4 = -165/2\,,$ $\mu_6 = 1$	$-6865/2$
7	$\lambda_0 = 349554/35\,,\ \lambda_2 = 135195/7\,,$ $\lambda_4 = -30519/35\,,\ \lambda_6 = 78/7$	$\mu_1 = -1937538/35\,,\ \mu_3 = 30129/5\,,$ $\mu_5 = -858/7\,,\ \mu_7 = 1$	0
8	$\lambda_1 = 878489/8\,,\ \lambda_3 = 3419941/48\,,$ $\lambda_5 = -37235/24\,,\ \lambda_7 = 105/8\,,$	$\mu_0 = -2872345/16\,,\ \mu_2 = -5576835/16\,,$ $\mu_4 = 13845\,,\ \mu_6 = -1365/8\,,\ \mu_8 = 1$	$4607015/8$
9	$\lambda_0 = -704613008/315\,,$ $\lambda_2 = -272518010/63\,,$ $\lambda_4 = 6835407/35\,,$ $\lambda_6 = -158197/63\,,\ \lambda_8 = 136/9$	$\mu_1 = 1301857612/105\,,$ $\mu_3 = -20244146/15\,,$ $\mu_5 = 576565/21\,,$ $\mu_7 = -680/3\,,\ \mu_9 = 1$	0
10	$\lambda_1 = -253102035/8\,,$ $\lambda_3 = -20527528\,,$ $\lambda_5 = 8939843/20\,,$ $\lambda_7 = -15195/4\,,$ $\lambda_9 = 171/10\,,$	$\mu_0 = 413776575/8\,,$ $\mu_2 = 200843175/2\,,$ $\mu_4 = -7977789/2\,,$ $\mu_6 = 983229/20\,,$ $\mu_8 = -2907/10\,,\ \mu_{10} = 1$	-165916233

$Z_\nu(x) \;=\; I_0(x)$:

n	λ_k	μ_k	γ
2	$\lambda_1 = -1/2$	$\mu_0 = 1/2\,,\ \mu_2 = 1$	-1
3	$\lambda_0 = -5/3\,,\ \lambda_2 = -10/3$	$\mu_1 = 10\,,\ \mu_3 = 1$	0
4	$\lambda_1 = -15/2\,,\ \lambda_3 = -21/4$	$\mu_0 = 51/4\,,\ \mu_2 = 105/4\,,\ \mu_4 = 1$	36
5	$\lambda_0 = -1267/15\,,\ \lambda_2 = -523/3\,,\ \lambda_4 = -36/5$	$\mu_1 = 2723/5\,,\ \mu_3 = 252/5\,,\ \mu_5 = 1$	0

n	λ_k	μ_k	γ
6	$\lambda_1 = -1291/2$, $\lambda_3 = -5299/12$, $\lambda_5 = -55/6$	$\mu_0 = 4385/4$, $\mu_2 = 9015/4$, $\mu_4 = 165/2$, $\mu_6 = 1$	$6155/2$
7	$\lambda_0 = -371268/35$, $\lambda_2 = -153147/7$, $\lambda_4 = -31257/35$, $\lambda_6 = -78/7$	$\mu_1 = 2392146/35$, $\mu_3 = 31647/5$, $\mu_5 = 858/7$, $\mu_7 = 1$	0
8	$\lambda_1 = -1790753/16$, $\lambda_3 = -918799/12$, $\lambda_5 = -4730/3$, $\lambda_7 = -105/8$,	$\mu_0 = 3041255/16$, $\mu_2 = 3126255/8$, $\mu_4 = 114465/8$, $\mu_6 = 1365/8$, $\mu_8 = 1$	$1067255/2$
9	$\lambda_0 = -766234942/315$, $\lambda_2 = -316070848/63$, $\lambda_4 = -7167687/35$, $\lambda_6 = -160043/63$, $\lambda_8 = -136/9$	$\mu_1 = 1645668398/105$, $\mu_3 = 21771386/15$, $\mu_5 = 590315/21$, $\mu_7 = 680/3$, $\mu_9 = 1$	0
10	$\lambda_1 = -525156393/16$, $\lambda_3 = -179631375/8$, $\lambda_5 = -4623729/10$, $\lambda_7 = -30657/8$, $\lambda_9 = -171/10$,	$\mu_0 = 891878697/16$, $\mu_2 = 114600720$, $\mu_4 = 33567993/8$, $\mu_6 = 2001597/40$, $\mu_8 = 2907/10$, $\mu_{10} = 1$	156491622

$Z_\nu(x) = K_0(x)$:

n	λ_k	μ_k	γ
2	$\lambda_1 = -1/2$	$\mu_0 = -1/2$, $\mu_2 = -1$	1
3	$\lambda_0 = -5/3$, $\lambda_2 = -10/3$	$\mu_1 = -10$, $\mu_3 = -1$	0
4	$\lambda_1 = -15/2$, $\lambda_3 = -21/4$	$\mu_0 = -51/4$, $\mu_2 = -105/4$, $\mu_4 = -1$	36
5	$\lambda_0 = -1267/15$, $\lambda_2 = -523/3$, $\lambda_4 = -36/5$	$\mu_1 = -2723/5$, $\mu_3 = -252/5$, $\mu_5 = -1$	0
6	$\lambda_1 = -1291/2$, $\lambda_3 = -5299/12$, $\lambda_5 = -55/6$	$\mu_0 = -4385/4$, $\mu_2 = -9015/4$, $\mu_4 = -165/2$, $\mu_6 = -1$	$6155/2$
7	$\lambda_0 = -371268/35$, $\lambda_2 = -153147/7$, $\lambda_4 = -31257/35$, $\lambda_6 = -78/7$	$\mu_1 = -2392146/35$, $\mu_3 = -31647/5$, $\mu_5 = -858/7$, $\mu_7 = -1$	0
8	$\lambda_1 = -1790753/16$, $\lambda_3 = -918799/12$, $\lambda_5 = -4730/3$, $\lambda_7 = -105/8$,	$\mu_0 = -3041255/16$, $\mu_2 = -3126255/8$, $\mu_4 = -114465/8$, $\mu_6 = -1365/8$, $\mu_8 = -1$	$1067255/2$
9	$\lambda_0 = -766234942/315$, $\lambda_2 = -316070848/63$, $\lambda_4 = -7167687/35$, $\lambda_6 = -160043/63$, $\lambda_8 = -136/9$	$\mu_1 = -1645668398/105$, $\mu_3 = -21771386/15$, $\mu_5 = -590315/21$, $\mu_7 = -680/3$, $\mu_9 = -1$	0
10	$\lambda_1 = -525156393/16$, $\lambda_3 = -179631375/8$, $\lambda_5 = -4623729/10$, $\lambda_7 = -30657/8$, $\lambda_9 = -171/10$,	$\mu_0 = -891878697/16$, $\mu_2 = -114600720$, $\mu_4 = -33567993/8$, $\mu_6 = -2001597/40$, $\mu_8 = -2907/10$, $\mu_{10} = -1$	156491622

$Z_\nu(x) = J_1(x)$:

n	λ_k	μ_k	γ
2	$\lambda_2 = -1$,	$\mu_1 = 3$,	0
3	$\lambda_1 = -3/2$, $\lambda_3 = -1$	$\mu_0 = 5/2$, $\mu_2 = 5$	-9
4	$\lambda_0 = 35/3$, $\lambda_2 = 70/3$, $\lambda_4 = -1$	$\mu_1 = -67$, $\mu_3 = 7$	0
5	$\lambda_1 = 291/4$, $\lambda_3 = 189/4$, $\lambda_5 = -1$	$\mu_0 = -119$, $\mu_2 = -925/4$, $\mu_4 = 9$	$765/2$
6	$\lambda_0 = -4536/5$, $\lambda_2 = -1755$, $\lambda_4 = 396/5$, $\lambda_6 = -1$,	$\mu_1 = 25152/5$, $\mu_3 = -2737/5$, $\mu_5 = 11$	0
7	$\lambda_1 = -33743/4$, $\lambda_3 = -65681/12$, $\lambda_5 = 715/6$, $\lambda_7 = -1$	$\mu_0 = 13791$, $\mu_2 = 107105/4$, $\mu_4 = -2127/2$, $\mu_6 = 13$	-44240

n	λ_k	μ_k	γ
8	$\lambda_0 = 5211558/35$, $\lambda_2 = 2015640/7$, $\lambda_4 = -455013/35$, $\lambda_6 = 1170/7$, $\lambda_8 = -1$	$\mu_1 = -28887006/35$, $\mu_3 = 449198/5$, $\mu_5 = -12793/7$, $\mu_7 = 15$,	0
9	$\lambda_1 = 14866827/8$, $\lambda_3 = 19292091/16$, $\lambda_5 = -210045/8$, $\lambda_7 = 1785/8$, $\lambda_9 = -1$	$\mu_0 = -48609209/16$, $\mu_2 = -94377775/16$, $\mu_4 = 468603/2$, $\mu_6 = -23101/8$, $\mu_8 = 17$	77965335/8
10	$\lambda_0 = -2668148626/63$, $\lambda_2 = -5159701430/63$, $\lambda_4 = 25883544/7$, $\lambda_6 = -2995213/63$, $\lambda_8 = 2584/9$, $\lambda_{10} = -1$	$\mu_1 = 4929726722/21$, $\mu_3 = -76658236/3$, $\mu_5 = 10916356/21$, $\mu_7 = -12875/3$, $\mu_9 = 19$,	0

$Z_\nu(x) \ = \ I_1(x)$:

n	λ_k	μ_k	γ
2	$\lambda_2 = 1$,	$\mu_1 = -3$,	0
3	$\lambda_1 = 3/2$, $\lambda_3 = 1$	$\mu_0 = -5/2$, $\mu_2 = -5$	-6
4	$\lambda_0 = 35/3$, $\lambda_2 = 70/3$, $\lambda_4 = 1$	$\mu_1 = -73$, $\mu_3 = -7$	0
5	$\lambda_1 = 69$, $\lambda_3 = 189/4$, $\lambda_5 = 1$	$\mu_0 = -469/4$, $\mu_2 = -965/4$, $\mu_4 = -9$	-330
6	$\lambda_0 = 4704/5$, $\lambda_2 = 1941$, $\lambda_4 = 396/5$, $\lambda_6 = 1$,	$\mu_1 = -30318/5$, $\mu_3 = -2807/5$, $\mu_5 = -11$	0
7	$\lambda_1 = 16921/2$, $\lambda_3 = 34727/6$, $\lambda_5 = 715/6$, $\lambda_7 = 1$	$\mu_0 = -28737/2$, $\mu_2 = -29540$, $\mu_4 = -2163/2$, $\mu_6 = -13$	$-80675/2$
8	$\lambda_0 = 5601948/35$, $\lambda_2 = 5601948/35$, $\lambda_4 = 471627/35$, $\lambda_6 = 1170/7$, $\lambda_8 = 1$	$\mu_1 = -36094416/35$, $\mu_3 = -477512/5$, $\mu_5 = -12947/7$, $\mu_7 = -15$,	0
9	$\lambda_1 = 30578169/16$, $\lambda_3 = 5229679/4$, $\lambda_5 = 53845/2$, $\lambda_7 = 1785/8$, $\lambda_9 = 1$	$\mu_0 = -51931231/16$, $\mu_2 = -53382655/8$, $\mu_4 = -1954557/8$, $\mu_6 = -23309/8$, $\mu_8 = -17$	-9112005
10	$\lambda_0 = 2921776286/63$, $\lambda_2 = 2921776286/63$, $\lambda_4 = 2921776286/63$, $\lambda_6 = 3051347/63$, $\lambda_8 = 2584/9$, $\lambda_{10} = 1$	$\mu_1 = -6275196562/21$, $\mu_3 = -83017774/3$, $\mu_5 = -11254826/21$, $\mu_7 = -12965/3$, $\mu_9 = -19$,	0

$Z_\nu(x) \ = \ K_1(x)$:

n	λ_k	μ_k	γ
2	$\lambda_2 = -1$,	$\mu_1 = -3$,	0
3	$\lambda_1 = -3/2$, $\lambda_3 = -1$	$\mu_0 = -5/2$, $\mu_2 = -5$	6
4	$\lambda_0 = -35/3$, $\lambda_2 = -70/3$, $\lambda_4 = -1$	$\mu_1 = -73$, $\mu_3 = -7$	0
5	$\lambda_1 = -69$, $\lambda_3 = -189/4$, $\lambda_5 = -1$	$\mu_0 = -469/4$, $\mu_2 = -965/4$, $\mu_4 = -9$	330
6	$\lambda_0 = -4704/5$, $\lambda_2 = -1941$, $\lambda_4 = -396/5$, $\lambda_6 = -1$,	$\mu_1 = -30318/5$, $\mu_3 = -2807/5$, $\mu_5 = -11$	0
7	$\lambda_1 = -16921/2$, $\lambda_3 = -34727/6$, $\lambda_5 = -715/6$, $\lambda_7 = -1$	$\mu_0 = -28737/2$, $\mu_2 = -29540$, $\mu_4 = -2163/2$, $\mu_6 = -13$	$80675/2$
8	$\lambda_0 = -5601948/35$, $\lambda_2 = -2310792/7$, $\lambda_4 = -471627/35$, $\lambda_6 = -1170/7$, $\lambda_8 = -1$	$\mu_1 = -36094416/35$, $\mu_3 = -477512/5$, $\mu_5 = -477512/5$, $\mu_7 = -15$,	0

n	λ_k	μ_k	γ
9	$\lambda_1 = -30578169/16$, $\lambda_3 = -5229679/4$, $\lambda_5 = -5229679/4$, $\lambda_7 = -1785/8$, $\lambda_9 = -1$	$\mu_0 = -51931231/16$, $\mu_2 = -53382655/8$, $\mu_4 = -1954557/8$, $\mu_6 = -23309/8$, $\mu_8 = -17$	$-23309/8$
10	$\lambda_0 = -2921776286/63$, $\lambda_2 = -606143240/63$, $\lambda_4 = -27331536/7$, $\lambda_6 = -3051347/63$, $\lambda_8 = -2584/9$, $\lambda_{10} = -1$	$\mu_1 = -6275196562/21$, $\mu_3 = -83017774/3$, $\mu_5 = -11254826/21$, $\mu_7 = -12965/3$, $\mu_9 = -19$,	0

b) Chebyshev Polynomials of the First Kind $T_n(x)$:

$$\int_{-1}^{1} \frac{T_n^2(x)\,dx}{\sqrt{1-x^2}} \;=\; \begin{cases} \pi & , \quad n = 0, \\ \dfrac{\pi}{2} & , \quad n > 0 \end{cases}$$

First polynomials:

$$T_0(x) = 1\,,\; T_1(x) = x\,,\; T_2(x) = 2x^2 - 1\,,\; T_3(x) = 4x^3 - 3x\,,\; T_4(x) = 8x^4 - 8x^2 + 1$$

$Z_\nu(x) \;=\; J_0(x)$:

n	λ_k	μ_k	γ
2	$\lambda_1 = -1$,	$\mu_0 = 1$, $\mu_2 = 1$	-3
3	$\lambda_0 = 4$, $\lambda_2 = 4$	$\mu_1 = -16$, $\mu_3 = 1$	0
4	$\lambda_1 = 19$, $\lambda_3 = 6$	$\mu_0 = -37$, $\mu_2 = -36$, $\mu_4 = 1$	81
5	$\lambda_0 = -252$, $\lambda_2 = -244$, $\lambda_4 = 8$	$\mu_1 = 912$, $\mu_3 = -64$, $\mu_5 = 1$	0
6	$\lambda_1 = -1809$, $\lambda_3 = -586$, $\lambda_5 = 10$	$\mu_0 = 3517$, $\mu_2 = 3416$, $\mu_4 = -100$, $\mu_6 = 1$	-7651
7	$\lambda_0 = 35208$, $\lambda_2 = 34060$, $\lambda_4 = -1136$, $\lambda_6 = 12$	$\mu_1 = -127296$, $\mu_3 = 8944$, $\mu_5 = -144$, $\mu_7 = 1$	0
8	$\lambda_1 = 347651$, $\lambda_3 = 112614$, $\lambda_5 = -1942$, $\lambda_7 = 14$	$\mu_0 = -675881$, $\mu_2 = -656460$, $\mu_4 = 19224$, $\mu_6 = -196$, $\mu_8 = 1$	1470273
9	$\lambda_0 = -8890744$, $\lambda_2 = -8600812$, $\lambda_4 = 286864$, $\lambda_6 = -3052$, $\lambda_8 = 16$	$\mu_1 = 32144704$, $\mu_3 = -2258544$, $\mu_5 = 36368$, $\mu_7 = -256$, $\mu_9 = 1$	0
10	$\lambda_1 = -111510593$, $\lambda_3 = -36121434$, $\lambda_5 = 622906$, $\lambda_7 = -4514$, $\lambda_9 = 18$	$\mu_0 = 216791801$, $\mu_2 = 210562416$, $\mu_4 = -6166188$, $\mu_6 = 62872$, $\mu_8 = -324$, $\mu_{10} = 1$	-471596451

$Z_\nu(x) \;=\; I_0(x)$:

n	λ_k	μ_k	γ
2	$\lambda_1 = -1$,	$\mu_0 = 1$, $\mu_2 = 1$	1
3	$\lambda_0 = -4$, $\lambda_2 = -4$	$\mu_1 = 16$, $\mu_3 = 1$	0
4	$\lambda_1 = -17$, $\lambda_3 = -6$	$\mu_0 = 35$, $\mu_2 = 36$, $\mu_4 = 1$	65
5	$\lambda_0 = -260$, $\lambda_2 = -268$, $\lambda_4 = -8$	$\mu_1 = 1136$, $\mu_3 = 64$, $\mu_5 = 1$	0
6	$\lambda_1 = -1793$, $\lambda_3 = -614$, $\lambda_5 = -10$	$\mu_0 = 3685$, $\mu_2 = 3784$, $\mu_4 = 100$, $\mu_6 = 1$	6785
7	$\lambda_0 = -38536$, $\lambda_2 = -39692$, $\lambda_4 = -1168$, $\lambda_6 = -12$	$\mu_1 = 168256$, $\mu_3 = 9488$, $\mu_5 = 144$, $\mu_7 = 1$	0

n	λ_k	μ_k	γ
8	$\lambda_1 = -358049$, $\lambda_3 = -122614$, $\lambda_5 = -1978$, $\lambda_7 = -14$	$\mu_0 = -735879$, $\mu_2 = 755660$, $\mu_4 = 19976$, $\mu_6 = 196$, $\mu_8 = 1$	1355009
9	$\lambda_0 = -9996680$, $\lambda_2 = -10296596$, $\lambda_4 = -302992$, $\lambda_6 = -3052$, $\lambda_8 = -16$	$\mu_1 = 43647680$, $\mu_3 = 43647680$, $\mu_5 = 43647680$, $\mu_7 = 256$, $\mu_9 = 1$	0
10	$\lambda_1 = -117155329$, $\lambda_3 = -40119878$, $\lambda_5 = -40119878$, $\lambda_7 = -4558$, $\lambda_9 = -18$	$\mu_0 = 240783081$, $\mu_2 = 247255504$, $\mu_4 = 6536236$, $\mu_6 = 64136$, $\mu_8 = 324$, $\mu_{10} = 1$	443365249

$Z_\nu(x) = K_0(x):$

n	λ_k	μ_k	γ
2	$\lambda_1 = -1$,	$\mu_0 = -1$, $\mu_2 = -1$	1
3	$\lambda_0 = -4$, $\lambda_2 = -4$	$\mu_1 = -16$, $\mu_3 = -1$	0
4	$\lambda_1 = -17$, $\lambda_3 = -6$	$\mu_0 = -35$, $\mu_2 = -36$, $\mu_4 = -1$	65
5	$\lambda_0 = -260$, $\lambda_2 = -268$, $\lambda_4 = -8$	$\mu_1 = -1136$, $\mu_3 = -64$, $\mu_5 = -1$	0
6	$\lambda_1 = -1793$, $\lambda_3 = -614$, $\lambda_5 = -10$	$\mu_0 = -3685$, $\mu_2 = -3784$, $\mu_4 = -100$, $\mu_6 = -1$	6785
7	$\lambda_0 = -38536$, $\lambda_2 = -39692$, $\lambda_4 = -1186$, $\lambda_6 = -12$	$\mu_1 = -168256$, $\mu_3 = -9488$, $\mu_5 = -144$, $\mu_7 = -1$	0
8	$\lambda_1 = -358049$, $\lambda_3 = -122614$, $\lambda_5 = -1978$, $\lambda_7 = -14$	$\mu_0 = -735879$, $\mu_2 = -755660$, $\mu_4 = -19976$, $\mu_6 = -196$, $\mu_8 = -1$	1470273
9	$\lambda_0 = -9996680$, $\lambda_2 = -10296596$, $\lambda_4 = -302992$, $\lambda_6 = -3092$, $\lambda_8 = -16$	$\mu_1 = -43647680$, $\mu_3 = -2461296$, $\mu_5 = -37360$, $\mu_7 = -256$, $\mu_9 = -1$	0
10	$\lambda_1 = -117155329$, $\lambda_3 = -40119878$, $\lambda_5 = -647210$, $\lambda_7 = -4558$, $\lambda_9 = -18$	$\mu_0 = -240783081$, $\mu_2 = -247255504$, $\mu_4 = -6536236$, $\mu_6 = -64136$, $\mu_8 = -324$, $\mu_{10} = -1$	443365249

$Z_\nu(x) = J_1(x):$

n	λ_k	μ_k	γ
2	$\lambda_2 = -1$,	$\mu_1 = 4$,	0
3	$\lambda_1 = -3$, $\lambda_3 = -1$	$\mu_0 = 6$, $\mu_2 = 6$	-15
4	$\lambda_0 = 32$, $\lambda_2 = 32$, $\lambda_4 = -1$	$\mu_1 = -120$, $\mu_3 = 8$	0
5	$\lambda_1 = 185$, $\lambda_3 = 60$, $\lambda_5 = -1$	$\mu_0 = -360$, $\mu_2 = -350$, $\mu_4 = 10$	785
6	$\lambda_0 = -2976$, $\lambda_2 = -2880$, $\lambda_4 = 96$, $\lambda_6 = -1$	$\mu_1 = 10764$, $\mu_3 = -756$, $\mu_5 = 12$	0
7	$\lambda_1 = -25067$, $\lambda_3 = -8120$, $\lambda_5 = 140$, $\lambda_7 = -1$,	$\mu_0 = 48734$, $\mu_2 = 47334$, $\mu_4 = -1386$, $\mu_6 = 14$,	-106015
8	$\lambda_0 = 559360$, $\lambda_2 = 541120$, $\lambda_4 = -18048$, $\lambda_6 = 192$, $\lambda_8 = -1$	$\mu_1 = -2022384$, $\mu_3 = 142096$, $\mu_5 = -2288$, $\mu_7 = 16$	0
9	$\lambda_1 = 6225489$, $\lambda_3 = 2016612$, $\lambda_5 = -34776$, $\lambda_7 = 252$, $\lambda_9 = -1$	$\mu_0 = -12103200$, $\mu_2 = -11755422$, $\mu_4 = 344250$, $\mu_6 = -3510$, $\mu_8 = 18$	26328609
10	$\lambda_0 = -177115680$, $\lambda_2 = -171339840$, $\lambda_4 = 5714720$, $\lambda_6 = -60800$, $\lambda_8 = 320$, $\lambda_{10} = -1$,	$\mu_1 = 640366100$, $\mu_3 = -44993260$, $\mu_5 = 724500$, $\mu_7 = -5100$, $\mu_9 = 20$	0

$Z_\nu(x) = I_1(x)$:

n	λ_k	μ_k	γ
2	$\lambda_2 = 1,$	$\mu_1 = -4,$	0
3	$\lambda_1 = 3,\ \lambda_3 = 1$	$\mu_0 = -6,\ \mu_2 = -6$	-9
4	$\lambda_0 = 32,\ \lambda_2 = 32,\ \lambda_4 = 1$	$\mu_1 = -136,\ \mu_3 = -8$	0
5	$\lambda_1 = 175,\ \lambda_3 = 60,\ \lambda_5 = 1$	$\mu_0 = -360,\ \mu_2 = -370,\ \mu_4 = -10$	-665
6	$\lambda_0 = 3168,\ \lambda_2 = 3264,\ \lambda_4 = 96,\ \lambda_6 = 1$	$\mu_1 = -13836,\ \mu_3 = -780,\ \mu_5 = -12$	0
7	$\lambda_1 = 25347,\ \lambda_3 = 8680,\ \lambda_5 = 140,$ $\lambda_7 = 1,$	$\mu_0 = -52094,\ \mu_2 = -53494,$ $\mu_4 = -1414,\ \mu_6 = -14,$	-95921
8	$\lambda_0 = 620800,\ \lambda_2 = 639424,$ $\lambda_4 = 18816,\ \lambda_6 = 192,\ \lambda_8 = 1$	$\mu_1 = -2710544,\ \mu_3 = -152848,$ $\mu_5 = -2320,\ \mu_7 = -16$	0
9	$\lambda_1 = 6477471,\ \lambda_3 = 2218212,$ $\lambda_5 = 35784,\ \lambda_7 = 252,$ $\lambda_9 = 1$	$\mu_0 = -13312800,$ $\mu_2 = -13670658,\ \mu_4 = -361386,$ $\mu_6 = -3546,\ \mu_8 = -18$	-24513489
10	$\lambda_0 = 200709600,\ \lambda_2 = 206731200,$ $\lambda_4 = 6083360,\ \lambda_6 = 62080,$ $\lambda_8 = 320,\ \lambda_{10} = 1,$	$\mu_1 = 62080,$ $\mu_3 = -49416980,$ $\mu_5 = -750100,\ \mu_7 = -5140,$ $\mu_9 = -20$	0

$Z_\nu(x) = K_1(x)$:

n	λ_k	μ_k	γ
2	$\lambda_2 = -1,$	$\mu_1 = -4,$	0
3	$\lambda_1 = -3,\ \lambda_3 = -1$	$\mu_0 = -6,\ \mu_2 = -6$	9
4	$\lambda_0 = -32,\ \lambda_2 = -32,\ \lambda_4 = -1$	$\mu_1 = -136,\ \mu_3 = -8$	0
5	$\lambda_1 = -175,\ \lambda_3 = -60,\ \lambda_5 = -1$	$\mu_0 = -360,\ \mu_2 = -370,\ \mu_4 = -10$	665
6	$\lambda_0 = -3168,\ \lambda_2 = -3264,\ \lambda_4 = -96,\ \lambda_6 = -1$	$\mu_1 = -13836,\ \mu_3 = -780,\ \mu_5 = -12$	0
7	$\lambda_1 = -25347,\ \lambda_3 = -8680,\ \lambda_5 = -140,$ $\lambda_7 = -1,$	$\mu_0 = -52094,\ \mu_2 = -53494,$ $\mu_4 = -1414,\ \mu_6 = -14,$	95921
8	$\lambda_0 = -620800,\ \lambda_2 = -639424,$ $\lambda_4 = -18816,\ \lambda_6 = -192,\ \lambda_8 = -1$	$\mu_1 = -2710544,\ \mu_3 = -152848,$ $\mu_5 = -152848,\ \mu_7 = -16$	0
9	$\lambda_1 = -6477471,\ \lambda_3 = -2218212,$ $\lambda_5 = -35784,\ \lambda_7 = -252,$ $\lambda_9 = -1$	$\mu_0 = -13312800,$ $\mu_2 = -13670658,\ \mu_4 = -361386,$ $\mu_6 = --3546,\ \mu_8 = -18$	24513489
10	$\lambda_0 = -200709600,\ \lambda_2 = -206731200,$ $\lambda_4 = -6083360,\ \lambda_6 = -62080,$ $\lambda_8 = -320,\ \lambda_{10} = -1,$	$\mu_1 = -876341780,$ $\mu_3 = -49416980,$ $\mu_5 = -750100,\ \mu_7 = -5140,$ $\mu_9 = -20$	0

b) Chebyshev Polynomials of the Second Kind $U_n(x)$:

$$\int_{-1}^{1} \sqrt{1 - x^2}\, U_n^2(x)\, dx \quad = \quad \frac{\pi}{2}$$

First polynomials:

$$U_0(x) = 1,\ U_1(x) = 2x,\ U_2(x) = 4x^2 - 1,\ U_3(x) = 8x^3 - 4x,\ U_4(x) = 16x^4 - 12x^2 + 1$$

$Z_\nu(x) \;=\; J_0(x)$:

n	λ_k	μ_k	γ
1	$-$	$\mu_1 = 1\,,$	0
2	$\lambda_1 = -1/2\,,$	$\mu_0 = 1\,,\ \mu_2 = 1$	-5
3	$\lambda_0 = 4\,,\ \lambda_2 = 4$	$\mu_1 = -16\,,\ \mu_3 = 1$	0
4	$\lambda_1 = 25/2\,,\ \lambda_3 = 6$	$\mu_0 = -37\,,\ \mu_2 = -36\,,\ \mu_4 = 1$	157
5	$\lambda_0 = -256\,,\ \lambda_2 = -248\,,\ \lambda_4 = 8$	$\mu_1 = 960\,,\ \mu_3 = -64\,,\ \mu_5 = 1\,,$	0
6	$\lambda_1 = -2421/2\,,\ \lambda_3 = -590\,,\ \lambda_5 = 10$	$\mu_0 = 3581\,,\ \mu_2 = 3480\,,\ \mu_4 = -100$ $\mu_6 = 1\,,$	-15145
7	$\lambda_0 = 36100\,,\ \lambda_2 = 34948\,,$ $\lambda_4 = -1140\,,\ \lambda_6 = 12$	$\mu_1 = -135280\,,\ \mu_3 = 9024$ $\mu_5 = -144\,,\ \mu_7 = 1$	0
8	$\lambda_1 = 467653/2\,,\ \lambda_3 = 113966$ $\lambda_5 = -1946\,,\ \lambda_7 = 14$	$\mu_0 = -691721\,,\ \mu_2 = -672204\,,$ $\mu_4 = 19320\,,\ \mu_6 = -196\,,\ \mu_8 = 1$	2925401
9	$\lambda_0 = -9144576\,,\ \lambda_2 = -8852728\,,$ $\lambda_4 = 288776\,,\ \lambda_6 = -3056\,,\lambda_8 = 16$	$\mu_1 = 34267968\,,\ \mu_3 = -2285888\,,$ $\mu_5 = 36480\,,\ \mu_7 = -256\,,\mu_9 = 1$	0
10	$\lambda_1 = -150310665/2\,,\ \lambda_3 = -36630374\,,$ $\lambda_5 = 625474\,,\ \lambda_7 = -4518\,,$ $\lambda_9 = 18$	$\mu_0 = 222329465\,,\ \mu_2 = 216056400\,,$ $\mu_4 = -6209740\,,\ \mu_6 = 63000\,,$ $\mu_8 = -324\,,\ \mu_{10} = 1$	-940267501

$Z_\nu(x) \;=\; I_0(x)$:

n	λ_k	μ_k	γ
1	$-$	$\mu_1 = 1\,,$	0
2	$\lambda_1 = -1/2\,,$	$\mu_0 = 1\,,\ \mu_2 = 1$	3
3	$\lambda_0 = -4\,,\ \lambda_2 = -4$	$\mu_1 = 16\,,\ \mu_3 = 1$	0
4	$\lambda_1 = -23/2\,,\ \lambda_3 = -6$	$\mu_0 = 35\,,\ \mu_2 = 36\,,\ \mu_4 = 1$	133
5	$\lambda_0 = -256\,,\ \lambda_2 = -264\,,\ \lambda_4 = -8$	$\mu_1 = 1880\,,\ \mu_3 = 64\,,\ \mu_5 = 1\,,$	0
6	$\lambda_1 = -2381/2\,,\ \lambda_3 = -610\,,\ \lambda_5 = -10$	$\mu_0 = 3621\,,\ \mu_2 = 3720\,,\ \mu_4 = 100$ $\mu_6 = 1\,,$	13703
7	$\lambda_0 = -37636\,,\ \lambda_2 = -38788\,,$ $\lambda_4 = -1164\,,\ \lambda_6 = -12$	$\mu_1 = 159856\,,\ \mu_3 = 9408$ $\mu_5 = 144\,,\ \mu_7 = 1$	0
8	$\lambda_1 = -473251/2\,,\ \lambda_3 = -121246$ $\lambda_5 = -1974\,,\ \lambda_7 = -14$	$\mu_0 = 719719\,,\ \mu_2 = 739404\,,$ $\mu_4 = 19880\,,\ \mu_6 = 196\,,\ \mu_8 = 1$	2723721
9	$\lambda_0 = -9734400\,,\ \lambda_2 = -10032392\,,$ $\lambda_4 = -10032392\,,\ \lambda_6 = -3088\,,\lambda_8 = -16$	$\mu_1 = 41346240\,,\ \mu_3 = 2433344\,,$ $\mu_5 = 37248\,,\ \mu_7 = 256\,,\mu_9 = 1$	0
10	$\lambda_1 = -154544153/2\,,\ \lambda_3 = -39593914\,,$ $\lambda_5 = -644626\,,\ \lambda_7 = -4554\,,$ $\lambda_9 = -18$	$\mu_0 = 235030377\,,\ \mu_2 = 241458672\,,$ $\mu_4 = 6491980\,,\ \mu_6 = 64008\,,$ $\mu_8 = 324\,,\ \mu_{10} = 1$	889454219

$Z_\nu(x) \;=\; K_0(x)$:

n	λ_k	μ_k	γ
1	$-$	$\mu_1 = -1\,,$	0
2	$\lambda_1 = -1/2\,,$	$\mu_0 = -1\,,\ \mu_2 = -1$	3
3	$\lambda_0 = -4\,,\ \lambda_2 = -4$	$\mu_1 = -16\,,\ \mu_3 = -1$	0
4	$\lambda_1 = -23/2\,,\ \lambda_3 = -6$	$\mu_0 = -35\,,\ \mu_2 = -36\,,\ \mu_4 = -1$	133
5	$\lambda_0 = -256\,,\ \lambda_2 = -264\,,\ \lambda_4 = -8$	$\mu_1 = -1088\,,\ \mu_3 = -64\,,\ \mu_5 = -1\,,$	0
6	$\lambda_1 = -2381/2\,,\ \lambda_3 = -610\,,\ \lambda_5 = -10$	$\mu_0 = -3621\,,\ \mu_2 = -3720\,,\ \mu_4 = -100$ $\mu_6 = -1\,,$	13703

n	λ_k	μ_k	γ
7	$\lambda_0 = -37636$, $\lambda_2 = -38788$, $\lambda_4 = -1164$, $\lambda_6 = -12$	$\mu_1 = -159856$, $\mu_3 = -9408$ $\mu_5 = -144$, $\mu_7 = -1$	0
8	$\lambda_1 = -473251/2$, $\lambda_3 = -121246$ $\lambda_5 = -1974$, $\lambda_7 = -14$	$\mu_0 = -719719$, $\mu_2 = -739404$, $\mu_4 = -19880$, $\mu_6 = -196$, $\mu_8 = -1$	2723721
9	$\lambda_0 = 2723721$, $\lambda_2 = -10032392$, $\lambda_4 = -301064$, $\lambda_6 = -3088$, $\lambda_8 = -16$	$\mu_1 = -41346240$, $\mu_3 = -2433344$, $\mu_5 = -37248$, $\mu_7 = -256$, $\mu_9 = -1$	0
10	$\lambda_1 = -154544153/2$, $\lambda_3 = -39593914$, $\lambda_5 = -644626$, $\lambda_7 = -4554$, $\lambda_9 = -18$	$\mu_0 = -235030377$, $\mu_2 = -241458672$, $\mu_4 = -6491980$, $\mu_6 = -64008$, $\mu_8 = -324$, $\mu_{10} = -1$	889454219

$Z_\nu(x) = J_1(x):$

n	λ_k	μ_k	γ
1	$-$	$-$	2
2	$\lambda_2 = -1$	$\mu_1 = 4$	0
3	$\lambda_1 = -2$, $\lambda_3 = -1$	$\mu_0 = 6$, $\mu_2 = 6$	-28
4	$\lambda_0 = 32$, $\lambda_2 = 32$, $\lambda_4 = -1$	$\mu_1 = -124$, $\mu_3 = 8$	0
5	$\lambda_1 = 123$, $\lambda_3 = 60$, $\lambda_5 = -1$	$\mu_0 = -364$, $\mu_2 = -354$, $\mu_4 = 10$	1542
6	$\lambda_0 = -3040$, $\lambda_2 = -2944$, $\lambda_4 = 96$, $\lambda_6 = -1$	$\mu_1 = 11396$, $\mu_3 = -760$, $\mu_5 = 12$	0
7	$\lambda_1 = -16824$, $\lambda_3 = -8200$, $\lambda_5 = 140$, $\lambda_7 = -1$	$\mu_0 = 49770$, $\mu_2 = 48366$, $\mu_4 = -1390$, $\mu_6 = 14$	-210488
8	$\lambda_0 = 574560$, $\lambda_2 = 574560$, $\lambda_4 = -18144$, $\lambda_6 = 192$, $\lambda_8 = -1$	$\mu_1 = -2153084$, $\mu_3 = 143624$, $\mu_5 = -2292$, $\mu_7 = 16$	0
9	$\lambda_1 = 4192053$, $\lambda_3 = 2043188$, $\lambda_5 = -34888$, $\lambda_7 = 252$, $\lambda_9 = -1$	$\mu_0 = -12401208$, $\mu_2 = -12051306$, $\mu_4 = 346370$, $\mu_6 = -3514$, $\mu_8 = 18$	52446730
10	$\lambda_0 = -182316960$, $\lambda_2 = -176498336$, $\lambda_4 = 5757376$, $\lambda_6 = -60928$, $\lambda_8 = 320$, $\lambda_{10} = -1$	$\mu_1 = 683206276$, $\mu_3 = -45574136$, $\mu_5 = 727308$, $\mu_7 = -5104$, $\mu_9 = 20$	0

$Z_\nu(x) = I_1(x):$

n	λ_k	μ_k	γ
1	$-$	$-$	-2
2	$\lambda_2 = 1$	$\mu_1 = -4$	0
3	$\lambda_1 = 2$, $\lambda_3 = 1$	$\mu_0 = -6$, $\mu_2 = -6$	-20
4	$\lambda_0 = 32$, $\lambda_2 = 32$, $\lambda_4 = 1$	$\mu_1 = -132$, $\mu_3 = -8$	0
5	$\lambda_1 = 117$, $\lambda_3 = 60$, $\lambda_5 = 1$	$\mu_0 = -356$, $\mu_2 = -366$, $\mu_4 = -10$	-1350
6	$\lambda_0 = 3104$, $\lambda_2 = 3200$, $\lambda_4 = 96$, $\lambda_6 = 1$	$\mu_1 = -13188$, $\mu_3 = -776$, $\mu_5 = -12$	0
7	$\lambda_1 = 16784$, $\lambda_3 = 8600$, $\lambda_5 = 140$, $\lambda_7 = 1$	$\mu_0 = -51050$, $\mu_2 = -51050$, $\mu_4 = -1410$, $\mu_6 = -14$	-193192
8	$\lambda_0 = 605280$, $\lambda_2 = 623808$, $\lambda_4 = 18720$, $\lambda_6 = 192$, $\lambda_8 = 1$	$\mu_1 = -2570884$, $\mu_3 = -2570884$, $\mu_5 = -2316$, $\mu_7 = -16$	0
9	$\lambda_1 = 4276043$, $\lambda_3 = 2191028$, $\lambda_5 = 35672$, $\lambda_7 = 252$, $\lambda_9 = 1$	$\mu_0 = -13005992$, $\mu_2 = -13361718$, $\mu_4 = -359250$, $\mu_6 = -359250$, $\mu_8 = -18$	-49220170
10	$\lambda_0 = 195293280$, $\lambda_2 = 201271648$, $\lambda_4 = 6040000$, $\lambda_6 = 61952$, $\lambda_8 = 320$, $\lambda_{10} = 1$	$\mu_1 = -829495684$, $\mu_3 = -48818184$, $\mu_5 = -747276$, $\mu_7 = -5136$, $\mu_9 = -20$	0

$Z_\nu(x) \; = \; K_1(x)$:

n	λ_k	μ_k	γ
1	–	–	2
2	$\lambda_2 = -1$	$\mu_1 = -4$	0
3	$\lambda_1 = -2\,,\ \lambda_3 = -1$	$\mu_0 = -6\,,\ \mu_2 = -6$	20
4	$\lambda_0 = -32\,,\ \lambda_2 = -32\,,\ \lambda_4 = -1$	$\mu_1 = -132\,,\ \mu_3 = -8$	0
5	$\lambda_1 = -117\,,\ \lambda_3 = -60\,,\ \lambda_5 = -1$	$\mu_0 = -356\,,\ \mu_2 = -366\,,\ \mu_4 = -10$	1350
6	$\lambda_0 = -3104\,,\ \lambda_2 = -3200\,,\ \lambda_4 = -96\,,\ \lambda_6 = -1$	$\mu_1 = -13188\,,\ \mu_3 = -776\,,\ \mu_5 = -12$	0
7	$\lambda_1 = -16784\,,\ \lambda_3 = -8600\,,\ \lambda_5 = -140\,,$ $\lambda_7 = -1$	$\mu_0 = -51050\,,\ \mu_2 = -52446\,,$ $\mu_4 = -1410\,,\ \mu_6 = -14$	-52446
8	$\lambda_0 = -605280\,,\ \lambda_2 = -623808\,,\ \lambda_4 = -623808\,,$ $\lambda_6 = -192\,,\ \lambda_8 = -1$	$\mu_1 = -623808\,,\ \mu_3 = -151304\,,$ $\mu_5 = -2316\,,\ \mu_7 = -16$	0
9	$\lambda_1 = -4276043\,,\ \lambda_3 = -2191028\,,$ $\lambda_5 = -35672\,,\ \lambda_7 = -252\,,\ \lambda_9 = -1$	$\mu_0 = -13005992\,,\ \mu_2 = -13361718\,,$ $\mu_4 = -359250\,,\ \mu_6 = -3542\,,\ \mu_8 = -18$	49220170
10	$\lambda_0 = -195293280\,,\ \lambda_2 = -195293280\,,$ $\lambda_4 = -6040000\,,\ \lambda_6 = -61952\,,$ $\lambda_8 = -320\,,\ \lambda_{10} = -1$	$\mu_1 = -829495684\,,\ \mu_3 = -48818184\,,$ $\mu_5 = -48818184\,,\ \mu_7 = -5136\,,\ \mu_9 = -20$	0

d) Ultraspherical (Gegenbauer) Polynomials $C_n^{(\alpha)}(x)$:

$$\int_{-1}^{1} (1 - x^2)^{\alpha - 1/2}\, [C_n^{(\alpha)}(x)]^2\, dx \;\; = \;\; \frac{\pi\, 2^{1-2\alpha}\, \Gamma(n + 2\alpha)}{n!\,(n + \alpha)\,[\Gamma(\alpha)]^2}\,, \quad \alpha > -1/2,\ \alpha \neq 0$$

First polynomials:

$$C_0^{(\alpha)}(x) = 1\,,\ C_1^{(\alpha)}(x) = 2\alpha x\,,\ C_2^{(\alpha)}(x) = 2\alpha(\alpha + 1)x^2 - \alpha\,,\ C_3^{(\alpha)}(x) = \frac{\alpha(\alpha+1)[4(\alpha+2)\,x^2 - 6]\,x}{3}\,,$$

$$C_4^{(\alpha)}(x) = \frac{\alpha(\alpha+1)[4(\alpha+2)(\alpha+3)\,x^4 - 12(\alpha+2)\,x^2 + 3]}{6}$$

$Z_\nu(x) \; = \; J_0(x)$:

n	λ_k	μ_k	γ
1	–	$\mu_1 = 1$	0
2	$\lambda_1 = -1/2\,,$	$\mu_0 = \alpha\,,\ \mu_2 = 1$	$-\alpha(2\alpha + 3)$
3	$\lambda_0 = 4\alpha(\alpha+2)/3\,,\ \lambda_2 = (4\alpha+8)/3$	$\mu_1 = -8(\alpha+1)(\alpha+2)/3\,,\ \mu_3 = 1$	0
4	$\lambda_1 = (6\alpha + 19)(\alpha + 1)/4\,,$ $\lambda_3 = 3(\alpha + 3)/2\,,$	$\mu_0 = -\alpha(6\alpha^2 + 31\alpha + 37)/2\,,$ $\mu_2 = -3(\alpha+2)(\alpha+3)\,,\ \mu_4 = 1$	$\alpha(\alpha+1)\cdot$ $\cdot(12\alpha^2 + 64\alpha + 81)/2$
5	$\lambda_0 = -8\alpha(\alpha+3)(8\alpha^2 + 49\alpha + 63)/15\,,$ $\lambda_2 = -4(\alpha+2)(16\alpha^2 + 111\alpha + 183)/15\,,$ $\lambda_4 = (8\alpha + 32)/15$	$\mu_1 = 8(\alpha+1)(\alpha+3)\cdot$ $\cdot(16\alpha^2 + 95\alpha + 114)/15\,,$ $\mu_3 = -16(\alpha+3)(\alpha+4)/5\,,$ $\mu_5 = 1$	0
6	$\lambda_1 = -(\alpha+1)(60\alpha^3 + 728\alpha^2 +$ $+2857a + 3618)/12\,,$ $\lambda_3 = -(\alpha+3)\cdot$ $\cdot(30\alpha^2 + 269\alpha + 586)/6\,,$ $\lambda_5 = (5\alpha + 25)/3\,,$	$\mu_0 = \alpha(60\alpha^4 + 848\alpha^3 + 4303\alpha^2 +$ $+9241\alpha + 7034)/6\,,$ $\mu_2 = (\alpha+2)(\alpha+4)\cdot$ $\cdot(30\alpha^2 + 239\alpha + 427)/3\,,$ $\mu_4 = -10(\alpha+4)(\alpha+5)/3\,,\ \mu_6 = 1$	$-\alpha(\alpha+1)(\alpha+2)\cdot$ $\cdot(120\alpha^3 + 1476\alpha^2 +$ $+5898\alpha + 7651)/6$

n	$\lambda_k,\ \mu_k,\ \gamma$
7	$\lambda_0 = 4\,\alpha\,(\alpha+4)\left(384\,\alpha^4 + 6184\,\alpha^3 + 35060\,\alpha^2 + 81882\,\alpha + 66015\right)/105$
	$\lambda_2 = 4\,(\alpha+2)\left(384\,\alpha^4 + 6880\,\alpha^3 + 44878\,\alpha^2 + 125928\,\alpha + 127725\right)/105$
	$\lambda_4 = -4\,(\alpha+4)\left(48\,\alpha^2 + 527\,\alpha + 1420\right)/35\,,\ \lambda_6 = 12(aaa+6)/7$
	$\mu_1 = -8\,(\alpha+4)\,(\alpha+1)\left(384\,\alpha^4 + 6112\,\alpha^3 + 34046\,\alpha^2 + 77343\,\alpha + 59670\right)/105$
	$\mu_3 = 8\,(\alpha+3)\,(\alpha+5)\left(48\,\alpha^2 + 479\,\alpha + 1118\right)/35$
	$\mu_5 = -24\,(\alpha+5)\,(\alpha+6)/7\,,\ \mu_7 = 1$
	$\gamma = 0$
8	$\lambda_1 = (\alpha+1)\left(840\,\alpha^5 + 21100\,\alpha^4 + 207050\,\alpha^3 + 990585\,\alpha^2 + 2306355\,\alpha + 2085906\right)/48$
	$\lambda_3 = (\alpha+3)\left(420\,\alpha^4 + 9220\,\alpha^3 + 74459\,\alpha^2 + 261855\,\alpha + 337842\right)/24$
	$\lambda_5 = -(\alpha+5)\left(70\,\alpha^2 + 909\,\alpha + 2913\right)/12\,,\ \lambda_7 = 7(\alpha+7)/4$
	$\mu_0 = -\alpha\,(22780\,\alpha^5 + 249110\,\alpha^4 + 1401593\,\alpha^3 + 4262368\,\alpha^2 + 840\,\alpha^6 + 6609327\,\alpha +$ $+\,4055286)/24$
	$\mu_2 = -(\alpha+2)\,(\alpha+5)\left(420\,\alpha^4 + 8380\,\alpha^3 + 60079\,\alpha^2 + 182319\,\alpha + 196938\right)/12$
	$\mu_4 = (\alpha+4)\,(\alpha+6)\left(70\,\alpha^2 + 839\,\alpha + 2403\right)/6\,,\ \mu_6 = -7\,(\alpha+6)\,(\alpha+7)/2$
	$\gamma = \alpha\,(\alpha+3)\,(\alpha+2)\,(\alpha+1)\left(1680\,\alpha^4 + 37440\,\alpha^3 + 307992\,\alpha^2 + 1108016\,\alpha + 1470273\right)/24$
9	$\lambda_0 = -16\,\alpha\,(\alpha+5)\,(3072\,\alpha^6 + 92448\,\alpha^5 + 1111896\,\alpha^4 + 6804152\,\alpha^3 + 22211724\,\alpha^2 +$ $+\,36455425\,\alpha + 23338203)/945$
	$\lambda_2 = -4\,(\alpha+2)\,(12288\,\alpha^6 + 404352\,\alpha^5 + 5416248\,\alpha^4 + 37733036\,\alpha^3 + 143879274\,\alpha^2 +$ $+\,283935817\,\alpha + 225771315)/945$
	$\lambda_4 = 8\,(\alpha+4)\left(768\,\alpha^4 + 19944\,\alpha^3 + 191615\,\alpha^2 + 806754\,\alpha + 1255030\right)/315$
	$\lambda_6 = -4\,(\alpha+6)\left(96\,\alpha^2 + 1439\,\alpha + 5341\right)/63\,,\ \lambda_8 = 16(\alpha+8)/9$
	$\mu_1 = 16\,(\alpha+1)\,(\alpha+5)\,(6144\,\alpha^6 + 183744\,\alpha^5 + 2191452\,\alpha^4 + 13256458\,\alpha^3 + 42576930\,\alpha^2 +$ $+\,68258003\,\alpha + 42189924)/945$
	$\mu_3 = -8\,(\alpha+3)\,(\alpha+6)\left(1536\,\alpha^4 + 36816\,\alpha^3 + 321406\,\alpha^2 + 1207917\,\alpha + 1646855\right)/315$
	$\mu_5 = 8\,(\alpha+5)\,(\alpha+7)\left(96\,\alpha^2 + 1343\,\alpha + 4546\right)/63$
	$\mu_7 = -32\,(\alpha+7)\,(\alpha+8)/9\,,\ \mu_9 = 1$
	$\gamma = 0$
10	$\lambda_1 = -(\alpha+1)\,(15120\,\alpha^7 + 636720\,\alpha^6 + 11259512\,\alpha^5 + 108253632\,\alpha^4 + 610299361\,\alpha^3 +$ $+\,2014357569\,\alpha^2 + 3597563754\,\alpha + 2676254232)/240$
	$\lambda_3 = -(\alpha+3)\,(7560\,\alpha^6 + 294420\,\alpha^5 + 4699778\,\alpha^4 + 39326439\,\alpha^3 + 181752061\,\alpha^2 +$ $+\,439373754\,\alpha + 433457208)/120$
	$\lambda_5 = (\alpha+5)\left(1260\,\alpha^4 + 37772\,\alpha^3 + 420377\,\alpha^2 + 2057895\,\alpha + 3737436\right)/60$
	$\lambda_7 = (\alpha+5)\left(1260\,\alpha^4 + 37772\,\alpha^3 + 420377\,\alpha^2 + 2057895\,\alpha + 3737436\right)/20\,,$
	$\lambda_9 = 9(\alpha+9)/5$
	$\mu_0 = \alpha\,(15120\,\alpha^8 + 666960\,\alpha^7 + 12530432\,\alpha^6 + 130674180\,\alpha^5 + 825228251\,\alpha^4 +$ $+\,3221684250\,\alpha^3 + 7564578353\,\alpha^2 + 9721155030\,\alpha + 5203003224)/120$
	$\mu_2 = (\alpha+2)\,(\alpha+6)(7560\,\alpha^6 + 271740\,\alpha^5 + 3950078\,\alpha^4 + 29637161\,\alpha^3 + 120690694\,\alpha^2 +$ $+\,252184351\,\alpha + 210562416)/60$
	$\mu_4 = -(\alpha+4)\,(\alpha+7)\left(1260\,\alpha^4 + 35252\,\alpha^3 + 362095\,\alpha^2 + 1616046\,\alpha + 2642652\right)/30$
	$\mu_6 = (\alpha+6)\,(\alpha+8)\left(126\,\alpha^2 + 2015\,\alpha + 7859\right)/10$
	$\mu_8 = -18\,(\alpha+8)\,(\alpha+9)/5\,,\ \mu_{10} = 1$
	$\gamma = -\alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\,(\alpha+4)\,(30240\,\alpha^5 + 1066800\,\alpha^4 + 14886000\,\alpha^3 +$ $+\,102678120\,\alpha^2 + 350009890\,\alpha + 471596451)/120$

$$Z_\nu(x) \,=\, I_0(x),\ K_0(x)\,:\qquad \text{Let}\qquad s \,=\, \begin{cases} 1 & ,\quad Z_\nu(x) = I_\nu(x) \\ -1 & ,\quad Z_\nu(x) = K_\nu(x) \end{cases}.$$

n	λ_k	μ_k	γ
1	$-$	$\mu_1 = s$	0
2	$\lambda_1 = -1/2\,,$	$\mu_0 = s\alpha,\ \mu_2 = s$	$\alpha(2\alpha+1)$
3	$\lambda_0 = -4\alpha(\alpha+2)/3,\ \lambda_2 = -(4\alpha+8)/3$	$\mu_1 = 8s(\alpha+1)(\alpha+2)/3,\ \mu_3 = s$	0
4	$\lambda_1 = -(6\alpha+17)(\alpha+1)/4,$ $\lambda_3 = -3(\alpha+3)/2,$	$\mu_0 = s\alpha\,(2\,\alpha+5)\,(3\,\alpha+7)/2,$ $\mu_2 = 3s(\alpha+2)(\alpha+3),\ \mu_4 = s$	$\alpha(\alpha+1)\cdot$ $\cdot(2\,\alpha+5)(6\,\alpha+13)/2$

n	$\lambda_k,\ \mu_k,\ \gamma$
5	$\lambda_0 = -8\,\alpha\,(\alpha+3)\,(8\,\alpha^2+47\,\alpha+65)/15$ $\lambda_2 = -4\,(\alpha+2)(16\,\alpha^2+113\,\alpha+201)/15,\ \lambda_4 = -(8\alpha+32)/15$ $\mu_1 = 8s(\alpha+1)(\alpha+3)(16\alpha^2+97\alpha+142)/15$ $\mu_3 = 16s(\alpha+3)(\alpha+4)/5,\ \mu_5 = s$ $\gamma = 0$
6	$\lambda_1 = -(\alpha+1)(60\alpha^3+712\alpha^2+2785\alpha+3586)/12$ $\lambda_3 = -(\alpha+3)\,(30\,\alpha^2+271\,\alpha+614)/6$ $\lambda_5 = -(5\alpha+25)/3$ $\mu_0 = s\alpha(60\alpha^4+832\alpha^3+4219\alpha^2+9245\alpha+7370)/6$ $\mu_2 = s(\alpha+2)(\alpha+4)(30\alpha^2+271\alpha+614)/6$ $\mu_4 = 10s(\alpha+4)(\alpha+5)/3,\ \mu_6 = s$ $\gamma = \alpha(\alpha+1)(\alpha+2)(120\alpha^3+1404\alpha^2+5394\alpha+6785)/6$
7	$\lambda_0 = -4\,\alpha\,(\alpha+4)\,(384\,\alpha^4+6104\,\alpha^3+34836\,\alpha^2+84010\,\alpha+72255)/105$ $\lambda_2 = -4\,(\alpha+2)\,(384\,\alpha^4+6944\,\alpha^3+46510\,\alpha^2+136712\,\alpha+148845)/105$ $\lambda_4 = -4\,(\alpha+4)\,(48\,\alpha^2+529\,\alpha+1460)/35,\ \lambda_6 = -12(\alpha+6)/7$ $\mu_1 = 8s\,(\alpha+1)\,(\alpha+4)\,(2\alpha+5)\,(192\,\alpha^3+2608\,\alpha^2+11399\,\alpha+15774)/105$ $\mu_3 = 8s\,(\alpha+3)\,(\alpha+5)\,(48\,\alpha^2+481\,\alpha+1186)/35$ $\mu_5 = 24s\,(\alpha+5)\,(\alpha+6)/7,\ \mu_7 = s$ $\gamma = 0$
8	$\lambda_1 = -\,(\alpha+1)\,(840\,\alpha^5+20900\,\alpha^4+204570\,\alpha^3+983583\,\alpha^2+2320825\,\alpha+2148294)/48$ $\lambda_3 = -\,(\alpha+3)\,(420\,\alpha^4+9260\,\alpha^3+75899\,\alpha^2+274055\,\alpha+367842)/24$ $\lambda_5 = -\,(\alpha+5)\,(70\,\alpha^2+911\,\alpha+2967)/12,\ \lambda_7 = -7(\alpha+7)/4$ $\mu_0 = s\alpha\,(22580\,\alpha^5+246510\,\alpha^4+1395791\,\alpha^3+840\,\alpha^6+4312956\,\alpha^2+6879305\,\alpha+$ $\quad +4415274)/24$ $\mu_2 = s\,(\alpha+2)\,(\alpha+5)\,(420\,\alpha^4+8420\,\alpha^3+61719\,\alpha^2+195679\,\alpha+226698)/12$ $\mu_4 = s\,(\alpha+4)\,(\alpha+6)\,(70\,\alpha^2+841\,\alpha+2497)/6,\ \mu_6 = 7s(\alpha+6)(\alpha+7)/2,\ \mu_8 = s$ $\gamma = \alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\,(1680\,\alpha^4+36480\,\alpha^3+293592\,\alpha^2+1036960\,\alpha+1355009)/24$
9	$\lambda_0 = -16\,\alpha\,(\alpha+5)(3072\,\alpha^6+91872\,\alpha^5+1106136\,\alpha^4+6835828\,\alpha^3+$ $\quad +22787712\,\alpha^2+38757095\,\alpha+26241285)/945$ $\lambda_2 = -4\,(2\,\alpha+7)\,(\alpha+2)\,(6144\,\alpha^5+181824\,\alpha^4+2123580\,\alpha^3+$ $\quad +12223112\,\alpha^2+34636535\,\alpha+38612235)/945$ $\lambda_4 = -8\,(\alpha+4)\,(768\,\alpha^4+19992\,\alpha^3+193919\,\alpha^2+830610\,\alpha+1325590)\,/315$ $\lambda_6 = -4\,(\alpha+6)\,(96\,\alpha^2+1441\,\alpha+5411)/63,\ \lambda_8 = -16(\alpha+8)/9$ $\mu_1 = 16s\,(2\,\alpha+5)\,(\alpha+5)\,(\alpha+1)(3072\,\alpha^5+84768\,\alpha^4+910302\,\alpha^3+$ $\quad +4735516\,\alpha^2+11880401\,\alpha+11457516)/945$ $\mu_3 = 8s\,(\alpha+6)\,(\alpha+3)\,(1536\,\alpha^4+36912\,\alpha^3+326782\,\alpha^2+1261965\,\alpha+1794695)\,/315$ $\mu_5 = 8s\,(\alpha+7)\,(\alpha+5)\,(96\,\alpha^2+1345\,\alpha+4670)\,/63$ $\mu_7 = 32s\,(\alpha+8)\,(\alpha+7)\,/9,\ \mu_9 = s$ $\gamma = 0$
10	$\lambda_1 = -\,(\alpha+1)\,(15120\,\alpha^7+633360\,\alpha^6+11178872\,\alpha^5+107665608\,\alpha^4+610500641\,\alpha^3+$ $\quad +2036115369\,\alpha^2+3694812314\,\alpha+2811727896)/240$ $\lambda_3 = -\,(\alpha+3)\,(7560\,\alpha^6+295260\,\alpha^5+4750178\,\alpha^4+40279841\,\alpha^3+189816079\,\alpha^2+$ $\quad 471229966\,\alpha+481438536)/120$ $\lambda_5 = -\,(\alpha+5)\,(1260\,\alpha^4+37828\,\alpha^3+423821\,\alpha^2+2100091\,\alpha+3883260)\,/60$ $\lambda_7 = -\,(\alpha+7)\,(126\,\alpha^2+2143\,\alpha+9116)\,/20,\ \lambda_9 = -9(\alpha+9)/5$ $\mu_0 = s\alpha\,(15120\,\alpha^8+663600\,\alpha^7+12448112\,\alpha^6+130121436\,\alpha^5+827403511\,\alpha^4+$ $\quad +3270380570\,\alpha^3+7829197173\,\alpha^2+10354621774\,\alpha+5778793944)/120$ $\mu_2 = s\,(\alpha+6)\,(\alpha+2)\,(7560\,\alpha^6+272580\,\alpha^5+4002998\,\alpha^4+30600643\,\alpha^3+$ $\quad +128278422\,\alpha^2+279464213\,\alpha+247255504)/60$ $\mu_4 = s\,(\alpha+7)\,(\alpha+4)\,(1260\,\alpha^4+35308\,\alpha^3+366183\,\alpha^2+1664990\,\alpha+2801244)\,/30$ $\mu_6 = s\,(\alpha+8)\,(\alpha+6)\,(126\,\alpha^2+2017\,\alpha+8017)\,/10$ $\mu_8 = 18s\,(\alpha+9)\,(\alpha+8)\,/5,\ \mu_{10} = s$ $\gamma = \alpha\,(\alpha+4)\,(\alpha+3)\,(\alpha+2)\,(\alpha+1)\,(30240\,\alpha^5+1050000\,\alpha^4+14449200\,\alpha^3+$ $\quad +98461080\,\alpha^2+332098450\,\alpha+443365249)/120$

$Z_\nu(x) = J_1(x):$

n	λ_k	μ_k	γ
1	$-$	$-$	2α
2	$\lambda_2 = -1,$	$\mu_1 = 2(\alpha+1),$	0
3	$\lambda_1 = -(\alpha+1),\ \lambda_3 = -1$	$\mu_0 = 2\alpha(\alpha+2),\ \mu_2 = 2(\alpha+2)$	$-2\alpha(\alpha+1)(2\alpha+5)$
4	$\lambda_0 = 8\alpha(\alpha+2)(\alpha+3)/3,$ $\lambda_2 = 8(\alpha+2)(\alpha+3)/3,\ \lambda_4 = -1$	$\mu_1 = -2(\alpha+1)(8\alpha^2+40\alpha+45)/3,$ $\mu_3 = 2(\alpha+3)$	0
5	$\lambda_1 = (\alpha+1)(6\alpha^2+43\alpha+74)/2,$ $\lambda_3 = 3(\alpha+3)(\alpha+4),\ \lambda_5 = -1$	$\mu_0 = -\alpha(\alpha+3)(6\alpha^2+37\alpha+48),$ $\mu_2 = -2(\alpha+2)(3\alpha^2+21\alpha+35),$ $\mu_4 = 2(\alpha+4)$	$6\alpha(\alpha+1)(\alpha+2)\cdot$ $\cdot(12\alpha^2+88\alpha+157)$
6	$\lambda_0 = 8\alpha(\alpha+3)(\alpha+4)\cdot$ $\cdot(16\alpha^2+114\alpha+155)/15$ $\lambda_2 = -8(\alpha+2)(\alpha+4)\cdot$ $\cdot(16\alpha^2+127\alpha+225)/15,$ $\lambda_4 = 16(\alpha+4)(\alpha+5)/5,\ \lambda_6 = -1$ $\lambda_6 = -1$	$\mu_1 = 2(\alpha+1)(128\alpha^4+1784\alpha^3+$ $+8872\alpha^2+18496\alpha+13455)/15,$ $\mu_3 = -2(\alpha+3)(4\alpha+15)(4\alpha+21),$ $\mu_5 = 2(\alpha+5)$	0

n	
7	$\lambda_1 = -(\alpha+1)\left(60\alpha^4+1088\alpha^3+7207\alpha^2+20631\alpha+21486\right)/6$ $\lambda_3 = -(\alpha+3)(\alpha+5)\left(30\alpha^2+299\alpha+696\right)/3$ $\lambda_5 = 10(\alpha+5)(\alpha+6)/3,\ \lambda_7 = -1$ $\mu_0 = \alpha(\alpha+4)\left(60\alpha^4+968\alpha^3+5501\alpha^2+12890\alpha+10443\right)/3$ $\mu_2 = 2(\alpha+2)\left(30\alpha^4+539\alpha^3+3528\alpha^2+9943\alpha+10143\right)/3$ $\mu_4 = -2(\alpha+4)\left(10\alpha^2+110\alpha+297\right)/3,\ \mu_6 = 2(\alpha+6)$ $\gamma = -\alpha(\alpha+1)(\alpha+2)(\alpha+3)\left(120\alpha^3+1836\alpha^2+9210\alpha+15145\right)/3$
8	$\lambda_0 = 16\alpha(\alpha+4)(\alpha+5)\left(192\alpha^4+3476\alpha^3+21738\alpha^2+54394\alpha+45885\right)/105$ $\lambda_2 = 8(\alpha+2)(\alpha+5)\left(384\alpha^4+7648\alpha^3+54686\alpha^2+165307\alpha+177555\right)/105$ $\lambda_4 = -8(\alpha+4)(\alpha+6)\left(48\alpha^2+575\alpha+1645\right)/35$ $\lambda_6 = 24(\alpha+6)(\alpha+7)/7,\ \lambda_8 = -1$ $\mu_1 = -2(\alpha+1)(3072\alpha^6+82688\alpha^5+895344\alpha^4+4971392\alpha^3+14847744\alpha^2+$ $+22446320\alpha+13271895)/105$ $\mu_3 = 2(\alpha+3)\left(384\alpha^4+8440\alpha^3+68256\alpha^2+240440\alpha+310835\right)/35$ $\mu_4 = -2(\alpha+5)\left(24\alpha^2+312\alpha+1001\right)/7,\ \mu_7 = 2(\alpha+7)$ $\gamma = 0$
9	$\lambda_1 = (\alpha+1)(840\alpha^6+27820\alpha^5+375610\alpha^4+2642633\alpha^3+10202207\alpha^2+$ $+20454222\alpha+16601304)/24$ $\lambda_3 = (\alpha+3)(\alpha+6)\left(420\alpha^4+10060\alpha^3+87739\alpha^2+329297\alpha+448136\right)/12$ $\lambda_5 = -(\alpha+5)(\alpha+7)\left(70\alpha^2+979\alpha+3312\right)/6,\ \lambda_7 = 7(\alpha+7)(\alpha+8)/2$ $\mu_0 = -\alpha(\alpha+5)(840\alpha^6+25300\alpha^5+304610\alpha^4+1866591\alpha^3+6104665\alpha^2+$ $+10045370\alpha+6455040)/12$ $\mu_2 = -(\alpha+2)(420\alpha^6+13840\alpha^5+185699\alpha^4+1296390\alpha^3+4956133\alpha^2+$ $+9813182\alpha+7836948)/6$ $\mu_4 = (\alpha+4)\left(70\alpha^4+1819\alpha^3+17489\alpha^2+73694\alpha+114750\right)/3$ $\mu_6 = -(\alpha+6)\left(7\alpha^2+105\alpha+390\right),\ \mu_8 = 2(\alpha+8)$ $\gamma = \alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(1680\alpha^4+44160\alpha^3+430392\alpha^2+$ $+1843040\alpha+2925401)/12$
10	$\lambda_0 = -16\alpha(\alpha+5)(\alpha+6)(6144\alpha^6+203328\alpha^5+2666160\alpha^4+17587276\alpha^3+61053750\alpha^2+$ $+105127268\alpha+69739299)/945$ $\lambda_2 = -16(\alpha+2)(\alpha+6)(6144\alpha^6+220608\alpha^5+3202332\alpha^4+23982586\alpha^3+97424616\alpha^2+$ $+202902629\alpha+168662655)/945$ $\lambda_4 = 8(\alpha+4)(\alpha+7)\left(1536\alpha^4+42960\alpha^3+441070\alpha^2+1967321\alpha+3214530\right)/315$ $\lambda_6 = -8(\alpha+6)(\alpha+8)\left(96\alpha^2+1535\alpha+5985\right)/63$ $\lambda_8 = 32(\alpha+9)(\alpha+8)/9,\ \lambda_{10} = -1$ $\mu_1 = 2(\alpha+1)(98304\alpha^8+4316160\alpha^7+80617920\alpha^6+834540064\alpha^5+5220464816\alpha^4+$ $+20129267600\alpha^3+46486591360\alpha^2+58394288256\alpha+30257298225)/945$

n	λ_k	μ_k	γ
		$\mu_3 = -2\,(\alpha+3)\,(12288\,\alpha^6 + 478848\,\alpha^5 + 7649264\,\alpha^4 + 64060608\,\alpha^3 + 296357968\,\alpha^2 + {}$ $+717278784\,\alpha + 708643845)/315$ $\mu_5 = 2\,(\alpha+5)\,\left(768\,\alpha^4 + 23032\,\alpha^3 + 256440\,\alpha^2 + 1255952\,\alpha + 2282175\right)/63$ $\mu_7 = -2\,(\alpha+7)\,\left(32\,\alpha^2 + 544\,\alpha + 2295\right)/9,\ \mu_9 = 2(\alpha+9)$ $\gamma = 0$	

$Z_\nu(x) = I_1(x),\ K_1(x)$ with s defined as on page 195:

n	λ_k	μ_k	γ
1	$-$	$-$	$-2s\alpha$
2	$\lambda_2 = s\,,$	$\mu_1 = -2(\alpha+1)\,,$	0
3	$\lambda_1 = s(\alpha+1)\,,\ \lambda_3 = s$	$\mu_0 = -2\alpha(\alpha+2)\,,\ \mu_2 = -2(\alpha+2)$	$-2s\alpha(\alpha+1)(2\alpha+3)$
4	$\lambda_0 = 8s\alpha(\alpha+2)(\alpha+3)/3\,,$ $\lambda_2 = 8s(\alpha+2)(\alpha+3)/3\,,\ \lambda_4 = s$	$\mu_1 = -2(\alpha+1)(8\alpha^2 + 40\alpha + 45)/3\,,$ $\mu_3 = 2(\alpha+3)$	0
5	$\lambda_1 = s(\alpha+1)(2\,\alpha+7)(3\,\alpha+10)/2\,,$ $\lambda_3 = 3s(\alpha+3)(\alpha+4)\,,\ \lambda_5 = s$	$\mu_0 = -\alpha(\alpha+3)(6\alpha^2 + 35\alpha + 48)\,,$ $\mu_2 = -2(\alpha+2)(3\alpha^2 + 21\alpha + 37)\,,$ $\mu_4 = -2(\alpha+4)$	$-s\alpha(\alpha+1)(\alpha+2)\cdot$ $\cdot(2\,\alpha+7)(6\,\alpha+19)$
6	$\lambda_0 = 8s\alpha(\alpha+3)(\alpha+4)\cdot$ $\cdot(16\alpha^2 + 110\alpha + 165)/15$ $\lambda_2 = 8s(\alpha+2)(\alpha+4)\cdot$ $\cdot(16\alpha^2 + 129\alpha + 255)/15\,,$ $\lambda_4 = 16s(\alpha+4)(\alpha+5)/5\,,\ \lambda_6 = s$	$\mu_1 = -2(\alpha+1)(2\alpha+5)\cdot$ $\cdot(64\alpha^3 + 740\alpha^2 + 2802\alpha + 3459)/15\,,$ $\mu_3 = -2(\alpha+3)\cdot$ $\cdot(16\alpha^2 + 144\alpha + 325)/5\,,$ $\mu_5 = -2(\alpha+5)$	0
7	$\lambda_1 = s(\alpha+1)(60\,\alpha^4 + 1072\,\alpha^3 + 7075\,\alpha^2 + 20419\,\alpha + 21726)/6$ $\lambda_3 = s(\alpha+3)(\alpha+5)\left(30\,\alpha^2 + 301\,\alpha + 744\right)/3$ $\lambda_5 = 10s(\alpha+5)(\alpha+6)/3\,,\ \lambda_7 = s$ $\mu_0 = -\alpha\,(\alpha+4)\left(60\,\alpha^4 + 952\,\alpha^3 + 5421\,\alpha^2 + 13034\,\alpha + 11163\right)/3$ $\mu_2 = -2\,(\alpha+2)\left(30\,\alpha^4 + 541\,\alpha^3 + 3612\,\alpha^2 + 10577\,\alpha + 11463\right)/3$ $\mu_4 = -2\,(\alpha+4)\left(10\,\alpha^2 + 110\,\alpha + 303\right)/3,\ \mu_6 = -2(\alpha+6)$ $\gamma = -s\alpha\,(\alpha+1)(\alpha+2)\,(\alpha+3)\left(120\,\alpha^3 + 1764\,\alpha^2 + 8562\,\alpha + 13703\right)/3$		
8	$\lambda_0 = 16s\,\alpha\,(\alpha+4)\,(\alpha+5)\left(192\,\alpha^4 + 3436\,\alpha^3 + 21738\,\alpha^2 + 54394\,\alpha + 45885\right)/105$ $\lambda_2 = 8s\,(\alpha+5)(\alpha+2)(2\alpha+7)\left(192\,\alpha^3 + 3184\,\alpha^2 + 17191\,\alpha + 29973\right)/105$ $\lambda_4 = 8s\,(\alpha+4)(\alpha+6)(48\,\alpha^2 + 577\,\alpha + 1715)/35$ $\lambda_6 = 24s\,(\alpha+7)(\alpha+6)\,,\ \lambda_8 = s$ $\mu_1 = -2\,(\alpha+1)\,(2\,\alpha+5)(1536\,\alpha^5 + 37760\,\alpha^4 + 364152\,\alpha^3 + 1718676\,\alpha^2 + {}$ $+3961102\,\alpha + 3557589)/105$ $\mu_3 = -2\,(\alpha+3)\left(384\,\alpha^4 + 8456\,\alpha^3 + 69216\,\alpha^2 + 249544\,\alpha + 334355\right)/35$ $\mu_4 = -2\,(\alpha+5)\left(24\,\alpha^2 + 312\,\alpha + 1015\right)/7,\ \mu_7 = -2(\alpha+7)$ $\gamma = 0$		
9	$\lambda_1 = s(\alpha+1)(840\,\alpha^6 + 27620\,\alpha^5 + 372010\,\alpha^4 + 2624431\,\alpha^3 + 10217789\,\alpha^2 + {}$ $+20796570\,\alpha + 17273256)/24$ $\lambda_3 = s(\alpha+3)(\alpha+6)(420\,\alpha^4 + 10100\,\alpha^3 + 89499\,\alpha^2 + 346057\,\alpha + 492936)/12$ $\lambda_5 = s(\alpha+5)(\alpha+7)(70\,\alpha^2 + 981\,\alpha + 3408)/6$ $\lambda_7 = 7s\,(\alpha+7)\,(\alpha+8)\,/2,\ \lambda_9 = s$ $\mu_0 = -\alpha\,(\alpha+5)(840\,\alpha^6 + 25100\,\alpha^5 + 301890\,\alpha^4 + 1863189\,\alpha^3 + 6200255\,\alpha^2 + {}$ $+10520550\,\alpha + 7100160)/12$ $\mu_2 = -\,(\alpha+2)\,(420\,\alpha^6 + 13880\,\alpha^5 + 188099\,\alpha^4 + 1336990\,\alpha^3 + 5253733\,\alpha^2 + {}$ $+10816542\,\alpha + 9113772)/6$ $\mu_4 = -(\alpha+4)(70\,\alpha^4 + 1821\,\alpha^3 + 17651\,\alpha^2 + 75546\,\alpha + 120462)/3$ $\mu_6 = -(\alpha+6)(7\,\alpha^2 + 105\,\alpha + 394),\ \mu_8 = -2(\alpha+8)$ $\gamma = -s\alpha\,(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(1680\,\alpha^4 + {}$ $+43200\,\alpha^3 + 413112\,\alpha^2 + 1740304\,\alpha + 2723721)/12$		

n	λ_k	μ_k	γ
10	$\lambda_0 = 16s\,\alpha\,(\alpha+5)(\alpha+6)(6144\,\alpha^6 + 202176\,\alpha^5 + 2654640\,\alpha^4 + 17692100\,\alpha^3 + 62786346\,\alpha^2 +$ $+112260364\,\alpha + 79029405)/945$ $\lambda_2 = 16s\,(\alpha+2)(\alpha+6)(2\alpha+7)(3072\,\alpha^5 + 100128\,\alpha^4 + 1280094\,\alpha^3 + 8005750\,\alpha^2 +$ $+24436771\,\alpha + 29071575)/945$ $\lambda_4 = 8s\,(\alpha+4)(\alpha+7)(1536\,\alpha^4 + 43056\,\alpha^3 + 446734\,\alpha^2 + 2032409\,\alpha + 3421890)/315$ $\lambda_6 = 8s\,(\alpha+8)(\alpha+6)\left(96\,\alpha^2 + 1537\,\alpha + 6111\right)/63$ $\lambda_8 = 32s\,(\alpha+9)(\alpha+8)/9,\ \lambda_{10} = s$ $\mu_1 = -2\,(\alpha+1)(2\alpha+5)(49152\,\alpha^7 + 2044416\,\alpha^6 + 35778528\,\alpha^5 + 341048704\,\alpha^4 +$ $1909536808\,\alpha^3 + +6269569684\,\alpha^2 + 11156022222\,\alpha + 8281429821)/915$ $\mu_3 = -2\,(\alpha+3)(12288\,\alpha^6 + 479616\,\alpha^5 + 7710704\,\alpha^4 + 65331648\,\alpha^3 + 307586128\,\alpha^2 +$ $+762778176\,\alpha + 778317435)/315$ $\mu_5 = -2\,(\alpha+5)(768\,\alpha^4 + 23048\,\alpha^3 + 258120\,\alpha^2 + 1278448\,\alpha + 2362815)/63$ $\mu_7 = -2\,(\alpha+7)\left(32\,\alpha^2 + 544\,\alpha + 231^3\right)/9,\ \mu_9 = -2(\alpha+9)$ $\gamma = 0$		

d) Jacobi Polynomials $C_n^{(\alpha,\beta)}(x)$:

$$\int_{-1}^{1} (1-x)^\alpha\,(1+x)^\beta\,[C_n^{(\alpha,\beta)}(x)]^2\,dx \;=\; \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}\;,\quad \alpha,\beta > -1$$

First polynomials:

$$C_0^{(\alpha,\beta)}(x) = 1\,,\ C_1^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta)\,x + \alpha - \beta}{2}\,,$$

$$C_2^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+3)(\alpha+\beta+4)}{8}\,x^2 + \frac{(\alpha+\beta+3)(\alpha-\beta)}{4}\,x + \frac{(\alpha-\beta-1)(\alpha-\beta)}{8}\,,$$

$$C_3^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+4)(\alpha+\beta+5)(\alpha+\beta+6)}{48}\,x^3 +$$

$$+\frac{(\alpha+\beta+4)(\alpha+\beta+5)(\alpha-\beta)}{16}\,x^2 + \frac{(\alpha+\beta+4)\left[(\alpha-\beta)^2 - (\alpha+\beta) - 6\right]}{16}\,x + \frac{(\alpha-\beta)\left[(\alpha-\beta)^2 - 3(\alpha+\beta) - 16\right]}{48}\,,$$

$$C_4^{(\alpha,\beta)}(x) =$$

$$= \frac{(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)}{384}\,x^4 + \frac{(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha-\beta)}{96}\,x^3 +$$

$$+\frac{(\alpha+\beta+5)(\alpha+\beta+6)[(\alpha-\beta)(\alpha-\beta-1) - 8]}{64}\,x^2 + \frac{(\alpha+\beta+5)(\alpha-\beta)[(\alpha-\beta)(\alpha-\beta-3) - 22]}{96}\,x +$$

$$+\frac{(\alpha-\beta)^4 - 6(\alpha^3 - \alpha^2\beta - \alpha\beta^2 + \beta^3) - (37\alpha^2 - 86\alpha\beta + 37\beta^2) + 42(\alpha+\beta) + 144}{384}$$

$Z_\nu(x) \;=\; J_0(x):$

n	λ_k	μ_k	γ
1	$\lambda_0 = -(\alpha-\beta)^2/[2(\alpha+\beta+2)],\ \lambda_1 = (\alpha-\beta)/(\alpha+\beta+2)$	$\mu_0 = -(\alpha-\beta)/2,\ \mu_1 = 1$	$(\alpha-\beta)/2$
2	$\lambda_0 = -(\alpha-\beta)\,[(\alpha-\beta)^2 - \alpha - \beta - 4]/[8(\alpha+\beta+2)]\,,\ \lambda_1 = [(\alpha-\beta)^2 - \alpha - \beta - 4]/[4(\alpha+\beta+2)]$ $\mu_0 = -[(\alpha-\beta)^2 - \alpha - \beta - 44]/8,\ \mu_2 = 1$ $\gamma = -(\alpha+2)(\beta+2)/2$		
3	$\lambda_0 = [(\alpha+\beta)(\alpha-\beta)^2(\alpha^2 + 6\,\alpha\,\beta + \beta^2) + (32\,\alpha^4 + 20\,\alpha^3\,\beta - 72\alpha^2\,\beta^2 + 20\alpha\,\beta^3 + 32\,\beta^4) +$ $+(\alpha+\beta)(207\,\alpha^2 - 278\,\alpha\,\beta + 207\,\beta^2) + (496\,\alpha^2 - 160\,\alpha\,\beta + 496\,\beta^2) + 536\,(\alpha+\beta) +$ $+480]/[48\,(\alpha+\beta+2)(\alpha+\beta+3)]$ $\lambda_1 = -(\alpha-\beta)[((3\alpha+\beta)(\alpha+3\beta) + 47(\alpha+\beta) + 136]\,/\,[24(\alpha+\beta+2)]$ $\lambda_2 = (\alpha+\beta+5)(\alpha+\beta+6)\,/\,[3(\alpha+\beta+3)]$ $\mu_0 = (\alpha-\beta)[(\alpha+\beta)(3\alpha+\beta)(\alpha+3\beta) + (61\alpha^2 + 130\alpha\beta + 61\beta^2) + 318(\alpha+\beta) +$ $+512]\,/\,[48(\alpha+\beta+2)]$ $\mu_1 = -(\alpha+\beta+4)(\alpha+\beta+5)(\alpha+\beta+6)\,/\,[6(\alpha+\beta+2)],\ \mu_3 = 1$ $\gamma = -(\alpha-\beta)[(\alpha^2 + 4\alpha\beta + \beta^2) + 15(\alpha+\beta) + 38]\,/\,24$		

n	λ_k	μ_k	γ
4	$\lambda_0 = (\alpha - \beta)\,[(\alpha^2 - \beta^2)^2(\alpha^2 + 3\,\beta\,\alpha + \beta^2) + 2\,(\alpha + \beta)(12\,\alpha^4 + 3\,\alpha^3\,\beta - 32\alpha^2\,\beta^2 + 3\,\alpha\beta^3 + 12\,\beta^4) +$ $\quad + (170\,\alpha^4 - 57\,\alpha^3\,\beta - 502\,\alpha^2\beta^2 - 57\,\alpha\beta^3 + 170\,\beta^4) + 2\,(\alpha + \beta)(137\,\alpha^2 - 855\,\alpha\,\beta + 137\,\beta^2) -$ $\quad - (1491\,\alpha^2 + 5142\,\alpha\,\beta + 1491\,\beta^2) - 6322\,(\alpha + \beta) - 7152]\,/\,[96(\alpha + \beta + 2)(\alpha + \beta + 3)(\alpha + \beta + 4)]$ $\lambda_1 = -[(\alpha + \beta)(\alpha - \beta)^2(3\,\alpha^2 + 8\,\alpha\,\beta + 3\,\beta^2) + (46\,\alpha^4 - 10\,\alpha^3\,\beta - 144\,\alpha^2\beta^2 - 10\,\alpha\beta^3 + 46\,\beta^4) +$ $\quad + 3\,(\alpha + \beta)(39\,\alpha^2 - 230\,\alpha\,\beta + 39\,\beta^2) - (814\,\alpha^2\,\alpha + 2620\,\beta + 814\,\beta^2) - 4296(\alpha + \beta) - 6336]\,/$ $\quad /\,[96(\alpha + \beta + 2)(\alpha + \beta + 4)]$ $\lambda_2 = -(\alpha - \beta)(\alpha + \beta + 5)(\alpha + \beta + 7)(\alpha + \beta + 24)\,/\,[48(\alpha + \beta + 3)(\alpha + \beta + 4)]$ $\lambda_3 = 3(\alpha + \beta + 7)(\alpha + \beta + 8)\,/\,[8(\alpha + \beta + 4)]$	$\mu_0 = [(\alpha^2 - \beta^2)^2(3\,\alpha^2 + 8\,\alpha\,\beta + 3\,\beta^2)(\alpha^2 - \beta^2)^2 + (\alpha + \beta)(47\,\alpha^4 - 8\,\alpha^3\,\beta - 150\,\alpha^2\beta^2 - 8\,\alpha\beta^3 + 47\,\beta^4) +$ $\quad + (63\,\alpha^4 - 498\,\alpha^3\,\beta - 1170\,\alpha^2\beta^2 - 498\,\alpha\beta^3 + 63\,\beta^4) - (\alpha + \beta)(1823\,\alpha^2 + 1922\,\alpha\,\beta + 1823\,\beta^2) -$ $\quad - (9642\,\alpha^2 + 9684\,\alpha\,\beta + 9642\,\beta^2) - 17640(\alpha + \beta) - 15552]\,/\,[192(\alpha + \beta + 2)(\alpha + \beta + 3)]$ $\mu_1 = (\alpha - \beta)(\alpha + \beta + 5)(\alpha + \beta + 6)(\alpha + \beta + 7)(\alpha + \beta + 40)\,/\,[96(\alpha + \beta + 2)(\alpha + \beta + 4)]$ $\mu_2 = -3(\alpha + \beta + 5)(\alpha + \beta + 6)(\alpha + \beta + 7)(\alpha + \beta + 8)\,/\,[16(\alpha + \beta + 3)(\alpha + \beta + 4)]\,,\ \mu_4 = 1$	$\gamma = [(\alpha^2 + 4\,\alpha\,\beta + \beta^2)^2 + 6\,(\alpha + \beta)(7\,\alpha^2 + 26\,\alpha\,\beta + 7\,\beta^2) + (539\,\alpha^2 + 1298\,\alpha\,\beta + 539\,\beta^2) +$ $\quad + 2586\,(\alpha + \beta) + 4176]\,/\,96$
5	$\lambda_0 = -[(\alpha + \beta)^3(\alpha - \beta)^2(6\,\alpha^4 + 31\,\alpha^3\,\beta + 54\,\alpha^2\beta^2 + 31\,\alpha\beta^3 + 6\,\beta^4) +$ $\quad + (\alpha + \beta)^2(347\,\alpha^6 + 947\,\alpha^5\,\beta - 171\,\alpha^4\beta^2 - 1734\,\alpha^3\beta^3 - 171\,\alpha^2\beta^4 + 947\,\alpha\beta^5 + 347\,\beta^6) +$ $\quad + (\alpha + \beta)(8108\,\alpha^6 + 20327\,\alpha^5\,\beta - 1360\,\alpha^4\beta^2 - 27910\,\alpha^3\beta^3 - 1360\,\alpha^2\beta^4 + 20327\,\alpha\beta^5 + 8108\,\beta^6) +$ $\quad + (101076\,\alpha^6 + 250891\,\alpha^5\,\beta + 42780\,\alpha^4\beta^2 - 215030\,\alpha^3\beta^3 + 42780\,\alpha^2\beta^4 + 250891\,\alpha\beta^5 + 101076\,\beta^6) +$ $\quad + 6\,(\alpha + \beta)(123960\,\alpha^4 + 78295\,\alpha^3\,\beta - 112814\,\alpha^2\beta^2 + 78295\,\alpha\beta^3 + 123960\,\beta^4) +$ $\quad + (3347073\,\alpha^4 + 3270528\,\alpha^3\,\beta - 286530\,\alpha^2\beta^2 + 3270528\,\alpha\beta^3 + 3347073\,\beta^4) +$ $\quad + 6\,(\alpha + \beta)(1533317\,\alpha^2 - 594074\,\alpha\,\beta + 1533317\,\beta^2) + (15202864\,\alpha^2 + 10520288\,\alpha\,\beta + 15202864\,\beta^2) +$ $\quad + 15425344(\alpha + \beta) + 9623040]\,/\,[960(\alpha + \beta + 2)(\alpha + \beta + 3)(\alpha + \beta + 4)(\alpha + \beta + 5)]$ $\lambda_1 = (\alpha - \beta)\,[(\alpha + \beta)^2\,(27\,\alpha^4 + 126\,\alpha^3\,\beta + 206\,\alpha^2\beta^2 + 126\,\alpha\beta^3 + 27\,\beta^4) +$ $\quad + (\alpha + \beta)(1341\,\alpha^4 + 5816\,\alpha^3\,\beta + 9022\,\alpha^2\beta^2 + 5816\,\alpha\beta^3 + 1341\,\beta^4) +$ $\quad + (26279\,\alpha^4 + 109098\,\alpha^3\,\beta + 165798\,\alpha^2\beta^2 + 109098\,\alpha\beta^3 + 26279\,\beta^4) +$ $\quad + (\alpha + \beta)(264131\,\alpha^2 + 544130\,\alpha\,\beta + 264131\,\beta^2) + (1452462\,\alpha^2 + 2931644\,\alpha\,\beta + 1452462\,\beta^2) +$ $\quad + 4170640\,(\alpha + \beta) + 4892320]\,/\,[960(\alpha + \beta + 2)(\alpha + \beta + 4)(\alpha + \beta + 5)]$ $\lambda_2 = -(\alpha + \beta + 8)\,[(\alpha + \beta)^2(33\,\alpha^2 + 62\,\alpha\,\beta + 33\,\beta^2) + 16\,(\alpha + \beta)\,(64\,\alpha^2 + 127\,\alpha\,\beta + 64\,\beta^2) +$ $\quad + (11823\,\alpha^2 + 24306\,\alpha\,\beta + 11823\,\beta^2) + 61520\,(\alpha + \beta) + 116400]\,/\,[480(\alpha + \beta + 3)(\alpha + \beta + 4)]$ $\lambda_3 = -(\alpha - \beta)\,(\alpha + \beta + 7)(\alpha + \beta + 9)(\alpha + \beta + 40)\,/\,[80(\alpha + \beta + 4)(\alpha + \beta + 5)]$ $\lambda_4 = 2(\alpha + \beta + 9)(\alpha + \beta + 10)\,/\,[5(\alpha + \beta + 5)]$	$\mu_0 = -(\alpha - \beta)[(\alpha + \beta)^3(27\,\alpha^4 + 126\,\alpha^3\,\beta + 206\,\alpha^2\beta^2 + 126\,\alpha\beta^3 + 27\,\beta^4) +$ $\quad + 3\,(\alpha + \beta)^2\,(469\,\alpha^4 + 2024\,\alpha^3\,\beta + 3134\,\alpha^2\beta^2 + 2024\,\alpha\beta^3 + 469\,\beta^4) +$ $\quad + (\alpha + \beta)(29203\,\alpha^4 + 120818\,\alpha^3\,\beta + 183438\,\alpha^2\beta^2 + 120818\,\alpha\beta^3 + 29203\,\beta^4) +$ $\quad + (318057\,\alpha^4 + 1293792\,\alpha^3\,\beta + 1951662\,\alpha^2\beta^2 + 1293792\,\alpha\beta^3 + 318057\,\beta^4) +$ $\quad + 2\,(\alpha + \beta)(993929\,\alpha^2 + 2033642\,\alpha\,\beta + 993929\,\beta^2) + (7184328\,\alpha^2 + 14574096\,\alpha\,\beta + 7184328\,\beta^2) +$ $\quad + 13918672\,(\alpha + \beta) + 11039808]\,/\,[1920(\alpha + \beta + 2)(\alpha + \beta + 3)(\alpha + \beta + 4)]$ $\mu_1 = (\alpha + \beta + 6)(\alpha + \beta + 7)(\alpha + \beta + 8)[(\alpha + \beta)^2(33\,\alpha^2 + 62\,\alpha\,\beta + 33\,\beta^2) +$ $\quad + 2\,(\alpha + \beta)(443\,\alpha^2 + 898\,\alpha\,\beta + 443\,\beta^2) + (8285\,\alpha^2 + 18710\,\alpha\,\beta + 8285\,\beta^2) + 36296\,(\alpha + \beta) +$ $\quad + 52960]\,/\,[960(\alpha + \beta + 2)(\alpha + \beta + 4)(\alpha + \beta + 5)]$ $\mu_2 = (\alpha - \beta)(\alpha + \beta + 7)(\alpha + \beta + 8)(\alpha + \beta + 9)(\alpha + \beta + 70)\,/\,[160(\alpha + \beta + 3)(\alpha + \beta + 4)]$ $\mu_3 = -(\alpha + \beta + 7)(\alpha + \beta + 8)(\alpha + \beta + 9)(\alpha + \beta + 10)\,/\,[5(\alpha + \beta + 4)(\alpha + \beta + 5)]\,,\ \mu_5 = 1$	$\gamma = (\alpha - \beta)[(9\,\alpha^4 + 44\,\alpha^3\,\beta + 74\,\alpha^2\beta^2 + 44\,\alpha\beta^3 + 9\,\beta^4) + 10\,(\alpha + \beta)(31\,\alpha^2 + 76\,\alpha\,\beta + 31\,\beta^2) +$ $\quad + (3815\,\alpha^2 + 8130\,\beta\,\alpha + 3815\,\beta^2) + 19850\,(\alpha + \beta) + 37216]\,/\,960$

$Z_\nu(x)\ =\ I_0(x)\,,\ K_0(x)$ with s defined as on page 195:

n	$\lambda_k\,,\ \mu_k\,,\ \gamma$
1	$\lambda_0 = -(\alpha - \beta)^2/[2(\alpha + \beta + 2)]\,,\ \lambda_1 = (\alpha - \beta)/(\alpha + \beta + 2)$ $\mu_0 = -s(\alpha - \beta)/2\,,\ \mu_1 = s$ $\gamma = (\alpha - \beta)/2$
2	$\lambda_0 = -(\alpha - \beta)[(\alpha - \beta)^2 - (\alpha + \beta) - 4]\,/\,[8(\alpha + \beta + 2)]$ $\lambda_1 = [(\alpha - \beta)^2 - (\alpha + \beta) - 4]\,/\,[4(\alpha + \beta + 2)]$ $\mu_0 = -s[(\alpha - \beta)^2 - (\alpha + \beta) - 4]\,/\,8\,,\ \mu_2 = s$ $\gamma = [(\alpha^2 + \beta^2) + 3\,(\alpha + 3\beta) + 4]\,/\,4$

n	λ_k , μ_k , γ
3	$\lambda_0 = -[(\alpha+\beta)(\alpha-\beta)^2(3\alpha^2+2\alpha\beta+3\beta^2) + (32\alpha^4-4\alpha^3\beta-24\alpha^2\beta^2-4\alpha\beta^3+32\beta^4) + $ $\quad + (\alpha+\beta)(157\alpha^2-178\alpha\beta+157\beta^2) + (400\alpha^2+32\alpha\beta+400\beta^2) + 536(\alpha+\beta) + 480] / $ $\qquad / [48(\alpha+\beta+2)(\alpha+\beta+3)]$ $\lambda_1 = (\alpha-\beta)[(5\alpha^2+6\alpha\beta+5\beta^2)+41(\alpha+\beta)+104]/[24(\alpha+\beta+2)]$ $\lambda_2 = -(\alpha+\beta+5)(\alpha+\beta+6)/[3(\alpha+\beta+3)]$ $\mu_0 = -s(\alpha-\beta)[(\alpha+\beta)(5\alpha^2+6\alpha\beta+5\beta^2)+(59\alpha^2+110\alpha\beta+59\beta^2)+274(\alpha+\beta)+448]/$ $\qquad / [48(\alpha+\beta+2)]$ $\mu_1 = s(\alpha+\beta+4)(\alpha+\beta+5)(\alpha+\beta+6)/[6(\alpha+\beta+2)]\,, \; \mu_3=s$ $\gamma = (\alpha-\beta)[(\alpha^2+\alpha\beta+\beta^2)+6(\alpha+\beta)+11]/12$
4	$\lambda_0 = -(\alpha-\beta)[(\alpha^2-\beta^2)^2(3\alpha^2+4\alpha\beta+3\beta^2)+(\alpha+\beta)(49\alpha^4-16\alpha^3\beta-74\alpha^2\beta^2-16\alpha\beta^3+49\beta^4)+$ $\quad\quad +(273\alpha^4-150\alpha^3\beta-750\alpha^2\beta^2-150\alpha\beta^3+273\beta^4)+259(\alpha+\beta)(\alpha^2-10\alpha\beta+\beta^2)-$ $\quad\quad -(2988\alpha^2+8376\alpha\beta+2988\beta^2)-11132(\alpha+\beta)-12576]/[192(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_1 = (\alpha+\beta)(\alpha-\beta)^2(2\alpha^2+3\alpha\beta+2\beta^2)+(22\alpha^4-13\alpha^3\beta-54\alpha^2\beta^2-13\alpha\beta^3+22\beta^4)+$ $\quad\quad +2(\alpha+\beta)(14\alpha^2-139\alpha\beta+14\beta^2)-(460\alpha^2+1096\alpha\beta+460\beta^2)-1992(\alpha+\beta)-2880]/$ $\quad\quad / [48(\alpha+\beta+2)(\alpha+\beta+4)]$ $\lambda_2 = (\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+7)(\alpha+\beta+24)/[48(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_3 = -3(\alpha+\beta+7)(\alpha+\beta+8)/[8(\alpha+\beta+4)]$ $\mu_0 = -s[(\alpha^2-\beta^2)^2(2\alpha^2+3\alpha\beta+2\beta^2)+(\alpha+\beta)(23\alpha^4-14\alpha^3\beta-54\alpha^2\beta^2-14\alpha\beta^3+23\beta^4)+$ $\quad\quad +(\alpha^4-255\alpha^3\beta-488\alpha^2\beta^2-255\alpha\beta^3+\beta^4)-(\alpha+\beta)(1001\alpha^2+668\alpha\beta+1001\beta^2)-$ $\quad\quad -(4755\alpha^2+4230\alpha\beta+4755\beta^2)-8334(\alpha+\beta)-7344]/[96(\alpha+\beta+2)(\alpha+\beta+3)]$ $\mu_1 = -s(\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+40)/[96(\alpha+\beta+2)(\alpha+\beta+4)]$ $\mu_2 = 3s(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)/[16(\alpha+\beta+3)(\alpha+\beta+4)]\,, \; \mu_4=s$ $\gamma = [(\alpha^2+\alpha\beta+\beta^2)^2+3(\alpha+\beta)(6\alpha^2+7\alpha\beta+6\beta^2)+(143\alpha^2+251\alpha\beta+143\beta^2)+$ $\quad\quad +558(\alpha+\beta)+864]/24$
5	$\lambda_0 = -[(\alpha-\beta)^2(\alpha+\beta)^3(21\alpha^4+62\alpha^3\beta+90\alpha^2\beta^2+62\alpha\beta^3+21\beta^4)+$ $\quad\quad +2(\alpha+\beta)^2(454\alpha^6+691\alpha^5\beta-278\alpha^4\beta^2-1222\alpha^3\beta^3-278\alpha^2\beta^4+691\alpha\beta^5+454\beta^6)+$ $\quad\quad +2(\alpha+\beta)(8879\alpha^6+17123\alpha^5\beta-2195\alpha^4\beta^2-21630\alpha^3\beta^3-2195\alpha^2\beta^4+17123\alpha\beta^5+8879\beta^6)+$ $\quad\quad +(201004\alpha^6+452234\alpha^5\beta+82100\alpha^4\beta^2-340180\alpha^3\beta^3+82100\alpha^2\beta^4+452234\alpha\beta^5+201004\beta^6)+$ $\quad\quad +(\alpha+\beta)(1417457\alpha^4+811660\alpha^3\beta-1023546\alpha^2\beta^2+811660\alpha\beta^3+1417457\beta^4)+$ $\quad\quad +(6324712\alpha^4+6310656\alpha^3\beta+161968\alpha^2\beta^2+6310656\alpha\beta^3+6324712\beta^4)+$ $\quad\quad +4(\alpha+\beta)(4400183\alpha^2-1485118\alpha\beta+4400183\beta^2)+$ $\quad\quad +(29794976\alpha^2+21700672\alpha\beta+29794976\beta^2)+30911872(\alpha+\beta)+19461120]/$ $\quad\quad / [1920(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\lambda_1 = (\alpha-\beta)[(\alpha+\beta)^2(19\alpha^4+63\alpha^3\beta+92\alpha^2\beta^2+63\alpha\beta^3+19\beta^4)+$ $\quad\quad +(\alpha+\beta)(769\alpha^4+2834\alpha^3\beta+4166\alpha^2\beta^2+2834\alpha\beta^3+769\beta^4)+$ $\quad\quad +(13513\alpha^4+52321\alpha^3\beta+77696\alpha^2\beta^2+52321\alpha\beta^3+13513\beta^4)+$ $\quad\quad +(\alpha+\beta)(129267\alpha^2+252440\alpha\beta+129267\beta^2)+(699524\alpha^2+1388888\alpha\beta+699524\beta^2)+$ $\quad\quad +2013308(\alpha+\beta)+2402480)/[480(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\lambda_2 = -(\alpha+\beta+8)[(\alpha+\beta)^2(31\alpha^2+66\alpha\beta+31\beta^2)+16(\alpha+\beta)(64\alpha^2+129\alpha\beta+64\beta^2)+$ $\quad\quad +(12433\alpha^2+24206\alpha\beta+12433\beta^2)+64432(\alpha+\beta)+125520]/[480(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_3 = (\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+9)(\alpha+\beta+40)/[80(\alpha+\beta+5)(\alpha+\beta+4)]$ $\lambda_4 = -2(\alpha+\beta+9)(\alpha+\beta+10)/[5(\alpha+\beta+5)]$ $\mu_0 = -s(\alpha-\beta)[(\alpha+\beta)^3(19\alpha^4+63\alpha^3\beta+92\alpha^2\beta^2+63\alpha\beta^3+19\beta^4)+$ $\quad\quad +2(\alpha+\beta)^2(400\alpha^4+1481\alpha^3\beta+2180\alpha^2\beta^2+1481\alpha\beta^3+400\beta^4)+$ $\quad\quad +(\alpha+\beta)(14998\alpha^4+58225\alpha^3\beta+86558\alpha^2\beta^2+58225\alpha\beta^3+14998\beta^4)+$ $\quad\quad +2(1147\alpha^2+2246\alpha\beta+1147\beta^2)(69\alpha^2+137\alpha\beta+69\beta^2)+$ $\quad\quad +(\alpha+\beta)(995351\alpha^2+1949558\alpha\beta+995351\beta^2)+(3675970\alpha^2+7253060\alpha\beta+3675970\beta^2)+$ $\quad\quad +7327952(\alpha+\beta)+6104544]/[960(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_1 = s(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)[(\alpha+\beta)^2(31\alpha^2+66\alpha\beta+31\beta^2)+$ $\quad\quad +6(\alpha+\beta)(151\alpha^2+298\alpha\beta+151\beta^2)+(9699\alpha^2+17258\alpha\beta+9699\beta^2)+$ $\quad\quad +39864(\alpha+\beta)+62240]/[960(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\mu_2 = -s(\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+70)/[160(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_3 = s(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+10)/[5(\alpha+\beta+4)(\alpha+\beta+5)]\,, \; \mu_5=s$ $\gamma = (\alpha-\beta)[(7\alpha^4+22\alpha^3\beta+32\alpha^2\beta^2+22\alpha\beta^3+7\beta^4)+60(\alpha+\beta)(3\alpha^2+5\alpha\beta+3\beta^2)+$ $\quad\quad +(1845\alpha^2+3520\alpha\beta+1845\beta^2)+8700(\alpha+\beta)+15668]/480$

$Z_\nu(x) \;=\; J_1(x):$

n	λ_k	μ_k	γ
1	$\lambda_0 = -(\alpha-\beta)/2$	$-$	$(\alpha+\beta+2)/2$
2	$\lambda_0 = -(\alpha-\beta)^2(\alpha+\beta+3)/$ $/[4(\alpha+\beta+2)]$ $\lambda_1 = (\alpha-\beta)(\alpha+\beta+3)/$ $/[2(\alpha+\beta+2)]\,,\ \lambda_2 = -1$	$\mu_0 = -(\alpha-\beta)(\alpha+\beta+3)(\alpha+\beta+4)/$ $/[4(\alpha+\beta+2)]$ $\mu_1 = (\alpha+\beta+3)(\alpha+\beta+4)/[2(\alpha+\beta+2]$	$(\alpha-\beta)(\alpha+\beta+3)/4$

n	
3	$\lambda_0 = -(\alpha-\beta)(\alpha+\beta+4)\left[(\alpha-\beta)^2-\alpha-\beta-6\right]/\left[16(\alpha+\beta+2)\right]$ $\lambda_1 = (\alpha+\beta+4)[(\alpha-\beta)^2-\alpha-\beta-6]/\left[8(\alpha+\beta+2)\right],\ \lambda_3 = -1$ $\mu_0 = -(\alpha+\beta+4)(\alpha+\beta+5)\left[(\alpha+\beta)(\alpha-\beta)^2-(\alpha+\beta)^2-8(\alpha+\beta)-12\right]/$ $\quad / \left[16(\alpha+\beta+2)(\alpha+\beta+3)\right]$ $\mu_1 = -(\alpha-\beta)(\alpha+\beta+5)/[2(\alpha+\beta+2)]\,,\ \mu_2 = (\alpha+\beta+5)(\alpha+\beta+6)/[2(\alpha+\beta+3)]$ $\gamma = -(\alpha+3)(\beta+3)(\alpha+\beta+4)/4$
4	$\lambda_0 = (\alpha+\beta+5)\left[(\alpha+\beta)(\alpha-\beta)^2(\alpha^2+6\,\alpha\beta+\beta^2)+(44\,\alpha^4+20\,\alpha^3\beta-96\,\alpha^2\beta^2+20\,\alpha\beta^3+44\,\beta^4)+\right.$ $\quad +(\alpha+\beta)(369\,\alpha^2-554\,\alpha\beta+369\,\beta^2)+(1114\,\alpha^2-724\,\alpha\beta+1114\,\beta^2)+1256\,(\alpha+\beta)+1344]\,/$ $\quad /\left[96\,(\alpha+\beta+2)(\alpha+\beta+3)\right]$ $\lambda_1 = -(\alpha-\beta)(\alpha+\beta+5)[(\alpha+\beta)(3\,\alpha+\beta)(\alpha+3\,\beta))+(83\,\alpha^2+182\,\alpha\beta+83\,\beta^2)+618\,(\alpha+\beta)+$ $\quad +1432]\,/\,[48\,(\alpha+\beta+2)(\alpha+\beta+4)]$ $\lambda_2 = (\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)\,/\,[6\,(\alpha+\beta+3)(\alpha+\beta+4)]\,,\ \lambda_4 = -1$ $\mu_0 = (\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+6)[(\alpha+\beta)^2(3\,\alpha+\beta)(\alpha+3\,\beta)+4\,(\alpha+\beta)(22\,\alpha^2+47\,\alpha\beta+22\,\beta^2)+$ $\quad +(715\,\alpha^2+1462\,\alpha\beta+715\,\beta^2)+2354\,(\alpha+\beta)+2696]\,/\,[96(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_1 = -(\alpha+\beta+6)[(\alpha+\beta)^4+24\,(\alpha+\beta)^3+(205\,\alpha^2+428\,\alpha\beta+205\,\beta^2)+786\,(\alpha+\beta)+1072]\,/$ $\quad /\,[12(\alpha+\beta+2)(\alpha+\beta+4)]$ $\mu_2 = -(\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+7)\,/\,[2(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_3 = (\alpha+\beta+7)(\alpha+\beta+8)\,/\,[2(\alpha+\beta+4)$ $\gamma = -(\alpha-\beta)(\alpha+\beta+5)[(\alpha^2+4\,\alpha\beta+\beta^2)+21\,(\alpha+\beta)+74]\,/\,48$
5	$\lambda_0 = (\alpha-\beta)(\alpha+\beta+6)\,[(\alpha^2-\beta^2)^2(\alpha^2+3\,\alpha\beta+\beta^2)+$ $\quad +2\,(\alpha+\beta)(15\,\alpha^4+3\,\alpha^3\beta-38\,\alpha^2\beta^2+3\,\alpha\beta^3+15\,\beta^4)+$ $\quad +(248\,\alpha^4-81\,\beta\alpha^3-706\,\beta^2\alpha^2-81\,\beta^3\alpha+248\,\beta^4)+2\,(\alpha+\beta)(164\,\alpha^2-1341\,\alpha\beta+164\,\beta^2)-$ $\quad -(4089\,\alpha^2+9810\,\alpha\beta+4089\,\beta^2)-17302\,(\alpha+\beta)-22896]\,/$ $\quad /\,[192(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_1 = -(\alpha+\beta+6)[(\alpha^2-\beta^2)^2(3\,\alpha^2+8\,\alpha\beta+3\,\beta^2)+(\alpha+\beta)(73\,\alpha^4-218\,\alpha^2\beta^2+73\,\beta^4)+$ $\quad +(419\,\alpha^4-650\,\alpha^3\beta-2298\,\alpha^2\beta^2-650\,\alpha\beta^3+419\,\beta^4)-(\alpha+\beta)(1489\,\alpha^2+7390\,\alpha\beta+1489\,\beta^2)-$ $\quad -(21014\,\alpha^2+34028\,\alpha\beta+21014\,\beta^2)-67752\,(\alpha+\beta)-93120]\,/$ $\quad /\,[192(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\lambda_2 = -(\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+42)\,/\,[96(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_3 = 3\,(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+10)\,/\,[16(\alpha+\beta+4)(\alpha+\beta+5)]\,,\ \lambda_5 = -1$ $\mu_0 = (\alpha+\beta+6)(\alpha+\beta+7)\,[(\alpha+\beta)^3(\alpha-\beta)^2(3\,\alpha^2+8\,\alpha\beta+3\,\beta^2)+$ $\quad +(\alpha+\beta)^2(65\,\alpha^4-4\,\alpha^3\beta-194\,\alpha^2\beta^2-4\,\alpha\beta^3+65\,\beta^4)+$ $\quad +(\alpha+\beta)(175\,\alpha^4-646\,\alpha^3\beta-1818\,\alpha^2\beta^2-646\,\alpha\beta^3+175\,\beta^4)-$ $\quad -(\alpha+\beta)^2(4073\,\alpha^2+2390\,\alpha\beta+4073\,\beta^2)-2\,(\alpha+\beta)(17681\,\alpha^2+5330\,\alpha\beta+17681\,\beta^2)-$ $\quad -(112392\,\alpha^2+107664\,\beta\alpha+112392\,\beta^2)-168576\,(\alpha+\beta)-130560]\,/$ $\quad /\,[384(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\mu_1 = (\alpha-\beta)(\alpha+\beta+7)\,[(\alpha+\beta)^5+86\,(\alpha+\beta)^4+1913\,(\alpha+\beta)^3+$ $\quad +(17968\,\alpha^2+36128\,\alpha\beta+17968\,\beta^2)+77004(\alpha+\beta)+$ $\quad +123168]\,/\,[192(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\mu_2 = -(\alpha+\beta+8)[3(\alpha+\beta)^4+96(\alpha+\beta)^3+(1121\alpha^2+2290\alpha\beta+1121\beta^2)+5840(\alpha+\beta)+11100]\,/$ $\quad /\,[32(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_3 = -(\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+9)/[2(\alpha+\beta+4)(\alpha+\beta+5)]$ $\mu_4 = (\alpha+\beta+9)(\alpha+\beta+10)/[2(\alpha+\beta+5)]$ $\gamma = (\alpha+\beta+6)\,[(\alpha^2+4\,\alpha\beta+\beta^2)^2+6\,(\alpha+\beta)(9\,\alpha^2+34\,\alpha\beta+9\,\beta^2)+$ $\quad +(911\,\alpha^2+2210\,\alpha\beta+911\,\beta^2)+5754\,(\alpha+\beta)+12240]/192$

$Z_\nu(x) = I_1(x)$, $K_1(x)$ with s defined as on page 195:

n	λ_k, μ_k, γ
1	$\lambda_0 = s(\alpha - \beta)/2$ $\gamma = -s(\alpha + \beta + 2)/2$
2	$\lambda_0 = s(\alpha - \beta)^2(\alpha + \beta + 3)/[4(\alpha + \beta + 2)]$ $\lambda_1 = -s(\alpha - \beta)(\alpha + \beta + 3)/[2(\alpha + \beta + 2)]$, $\lambda_2 = s$ $\mu_0 = (\alpha - \beta)(\alpha + \beta + 3)(\alpha + \beta + 4)/[4(\alpha + \beta + 2)]$ $\mu_1 = -(\alpha + \beta + 3)(\alpha + \beta + 4)/[2(\alpha + \beta + 2)]$ $\gamma = -s(\alpha - \beta)(\alpha + \beta + 3)/4$
3	$\lambda_0 = s(\alpha - \beta)(\alpha + \beta + 4)\left[(\alpha - \beta)^2 - (\alpha + \beta) - 6\right]/[16(\alpha + \beta + 2)]$ $\lambda_1 = -s(\alpha + \beta + 4)\left[(\alpha - \beta)^2 - (\alpha + \beta) - 6\right]/[8(\alpha + \beta + 2)]$, $\lambda_3 = s$ $\mu_0 = (\alpha + \beta + 4)(\alpha + \beta + 5)\left[(\alpha + \beta)(\alpha - \beta)^2 - (\alpha + \beta)^2 - 8(\alpha + \beta) - 12\right]/[16(\alpha + \beta + 2)(\alpha + \beta + 3)]$ $\mu_1 = (\alpha - \beta)(\alpha + \beta + 5)/[2(\alpha + \beta + 2)]$, $\mu_2 = -(\alpha + \beta + 5)(\alpha + \beta + 6)/[2(\alpha + \beta + 3)]$ $\gamma = -s(\alpha + \beta + 4)\left[(\alpha^2 + \beta^2) + 5(\alpha + \beta) + 12\right]/8$
4	$\lambda_0 = s(\alpha + \beta + 5)\left[(\alpha + \beta)(\alpha - \beta)^2(3\alpha^2 + 2\alpha\beta + 3\beta^2) + (44\alpha^4 - 4\alpha^3\beta - 48\alpha^2\beta^2 - 4\alpha\beta^3 + 44\beta^4) + \right.$ $\quad +(\alpha + \beta)(307\alpha^2 - 430\alpha\beta + 307\beta^2) + (982\alpha^2 - 460\alpha\beta + 982\beta^2) + 1256(\alpha + \beta) + 1344]/$ $\quad /[96(\alpha + \beta + 2)(\alpha + \beta + 3)]$ $\lambda_1 = -s(\alpha - \beta)(\alpha + \beta + 5)\left[(\alpha + \beta)(5\alpha^2 + 6\alpha\beta + 5\beta^2) + (85\alpha^2 + 154\alpha\beta + 85\beta^2) + 550(\alpha + \beta) + \right.$ $\quad +1256]/[48(\alpha + \beta + 2)(\alpha + \beta + 4)]$ $\lambda_2 = s(\alpha + \beta + 5)(\alpha + \beta + 6)(\alpha + \beta + 7)(\alpha + \beta + 8)/[6(\alpha + \beta + 3)(\alpha + \beta + 4)]$, $\lambda_4 = s$ $\mu_0 = (\alpha - \beta)(\alpha + \beta + 5)(\alpha + \beta + 6)\left[(\alpha + \beta)^2(5\alpha^2 + 6\alpha\beta + 5\beta^2) + \right.$ $\quad +4(\alpha + \beta)(22\alpha^2 + 41\alpha\beta + 22\beta^2) + (669\alpha^2 + 1306\alpha\beta + 669\beta^2) + 2222(\alpha + \beta) + 2680]/$ $\quad /[96(\alpha + \beta + 2)(\alpha + \beta + 3)(\alpha + \beta + 4)]$ $\mu_1 = -(\alpha + \beta + 6)\left[(\alpha + \beta)^4 + 24(\alpha + \beta)^3 + (217\alpha^2 + 416\alpha\beta + 217\beta^2) + 822(\alpha + \beta) + 1168\right]/$ $\quad /[12(\alpha + \beta + 2)(\alpha + \beta + 4)]$ $\mu_2 = (\alpha - \beta)(\alpha + \beta + 5)(\alpha + \beta + 7)/[2(\alpha + \beta + 3)(\alpha + \beta + 4)]$ $\mu_3 = -(\alpha + \beta + 7)(\alpha + \beta + 8)/[2(\alpha + \beta + 4)]$ $\gamma = -s(\alpha - \beta)(\alpha + \beta + 5)\left[(\alpha^2 + \alpha\beta + \beta^2) + 9(\alpha + \beta) + 26\right]/24$
5	$\lambda_0 = s(\alpha - \beta)(\alpha + \beta + 6)\left[(\alpha^2 - \beta^2)^2(3\alpha^2 + 4\alpha\beta + 3\beta^2) + \right.$ $\quad +(\alpha + \beta)(61\alpha^4 - 16\alpha^3\beta - 98\alpha^2\beta^2 - 16\alpha\beta^3 + 61\beta^4) +$ $\quad +(417\alpha^4 - 198\alpha^3\beta - 1134\alpha^2\beta^2 - 198\alpha\beta^3 + 417\beta^4) + (\alpha + \beta)(295\alpha^2 - 4342\alpha\beta + 295\beta^2) -$ $\quad -(8148\alpha^2 + 17064\alpha\beta + 8148\beta^2) - 32276(\alpha + \beta) - 42912]/$ $\quad /[384(\alpha + \beta + 2)(\alpha + \beta + 3)(\alpha + \beta + 4)]$ $\lambda_1 = -s(\alpha + \beta + 6)\left[(\alpha^2 - \beta^2)^2(2\alpha^2 + 3\alpha\beta + 2\beta^2) + \right.$ $\quad +2(\alpha + \beta)(19\alpha^4 - 9\alpha^3\beta - 38\alpha^2\beta^2 - 9\alpha\beta^3 + 19\beta^4) +$ $\quad +(168\alpha^4 - 359\alpha^3\beta - 974\alpha^2\beta^2 - 359\alpha\beta^3 + 168\beta^4) - 2(\alpha + \beta)(499\alpha^2 + 1513\alpha\beta + 499\beta^2) -$ $\quad -(10634\alpha^2 + 15188\beta\alpha + 10634\beta^2) - 32256(\alpha + \beta) - 44160]/$ $\quad /[96(\alpha + \beta + 2)(\alpha + \beta + 4)(\alpha + \beta + 5)]$ $\lambda_2 = -s(\alpha - \beta)(\alpha + \beta + 7)(\alpha + \beta + 8)(\alpha + \beta + 9)(\alpha + \beta + 42)/[96(\alpha + \beta + 3)(\alpha + \beta + 4)]$ $\lambda_3 = 3s(\alpha + \beta + 7)(\alpha + \beta + 8)(\alpha + \beta + 9)(\alpha + \beta + 10)/[16(\alpha + \beta + 4)(\alpha + \beta + 5)]$, $\lambda_5 = s$ $\mu_0 = (\alpha + \beta + 6)(\alpha + \beta + 7)\left[(\alpha - \beta)^2(\alpha + \beta)^3(2\alpha^2 + 3\alpha\beta + 2\beta^2) + \right.$ $\quad +(\alpha + \beta)^2(33\alpha^4 - 16\alpha^3\beta - 70\alpha^2\beta^2 - 16\alpha\beta^3 + 33\beta^4) +$ $\quad +(\alpha + \beta)(53\alpha^4 - 361\alpha^3\beta - 740\alpha^2\beta^2 - 361\alpha\beta^3 + 53\beta^4) -$ $\quad -3(\alpha + \beta)^2(749\alpha^2 + 208\alpha\beta + 749\beta^2) - (\alpha + \beta)(18043\alpha^2 + 3286\alpha\beta + 18043\beta^2) -$ $\quad -(55926\alpha^2 + 49452\alpha\beta + 55926\beta^2) - 82392(\alpha + \beta) - 64320]/$ $\quad /[192(\alpha + \beta + 2)(\alpha + \beta + 3)(\alpha + \beta + 4)(\alpha + \beta + 5)]$ $\mu_1 = (\alpha - \beta)(\alpha + \beta + 7)\left[(\alpha + \beta)^5 + 86(\alpha + \beta)^4 + 1913(\alpha + \beta)^3 + \right.$ $\quad +(18160\alpha^2 + 36128\alpha\beta + 18160\beta^2) + 78348(\alpha + \beta) + 127392]/$ $\quad /[192(\alpha + \beta + 2)(\alpha + \beta + 4)(\alpha + \beta + 5)]$ $\mu_2 = -(\alpha + \beta + 8)\left[3(\alpha + \beta)^4 + 96(\alpha + \beta)^3 + (1153\alpha^2 + 2258\alpha\beta + 1153\beta^2) + \right.$ $\quad +5968(\alpha + \beta) + 11580]/[32(\alpha + \beta + 3)(\alpha + \beta + 4)]$ $\mu_3 = (\alpha - \beta)(\alpha + \beta + 7)(\alpha + \beta + 9)/[2(\alpha + \beta + 4)(\alpha + \beta + 5)]$ $\mu_4 = (\alpha + \beta + 9)(\alpha + \beta + 10)/[2(\alpha + \beta + 5)]$ $\gamma = -s(\alpha + \beta + 6)\left[(\alpha^2 + \alpha\beta + \beta^2)^2 + 3(\alpha + \beta)(8\alpha^2 + 9\alpha\beta + 8\beta^2) + \right.$ $\quad +(251\alpha^2 + 431\alpha\beta + 251\beta^2) + 1284(\alpha + \beta) + 2640]/48$

f) Laguerre Polynomials $L_n(x)$:

$$\int_0^\infty e^{-x} L_n^2(x)\, dx \quad = \quad 1$$

First polynomials:

$$L_0(x) = 1\,, \ L_1(x) = -x + 1\,, \ L_2(x) = \frac{x^2}{2} - 2x + 1\,, \ L_3(x) = -\frac{x^3}{6} + \frac{3x^2}{2} - 3x + 1\,,$$

$$L_4(x) = \frac{x^4}{24} - \frac{2x^3}{3} + 3x^2 - 4x + 1$$

$Z_\nu(x) \ = \ J_0(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = 1\,, \ \lambda_1 = -1$	$\mu_0 = -1\,, \ \mu_1 = 1$	1
2	$\lambda_0 = 1\,, \ \lambda_1 = -1$	$\mu_0 = -1\,, \ \mu_2 = 1$	$1/2$
3	$\lambda_0 = 1/3\,, \ \lambda_1 = 1/3\,, \ \lambda_2 = -2/3$	$\mu_0 = -1/3\,, \ \mu_1 = -2/3\,, \ \mu_3 = 1$	$-1/2$
4	$\lambda_0 = -11/12\,, \ \lambda_1 = 25/12\,, \ \lambda_2 = -5/12\,,$ $\lambda_3 = -3/4$	$\mu_0 = 11/12\,, \ \mu_1 = -7/6\,, \ \mu_2 = -3/4\,,$ $\mu_4 = 1$	$-13/8$
5	$\lambda_0 = -131/60\,, \ \lambda_1 = 193/60\,, \ \lambda_2 = 19/60\,,$ $\lambda_3 = -11/20\,, \ \lambda_4 = -4/5$	$\mu_0 = 131/60 \ \mu_1 = -31/30\,,$ $\mu_2 = -27/20\,, \ \mu_3 = -4/5\,, \ \mu_5 = 1$	$-17/8$
6	$\lambda_0 = -99/40\,, \ \lambda_1 = 97/40\,, \ \lambda_2 = 51/40\,,$ $\lambda_3 = 29/120\,, \ \lambda_4 = -19/30\,, \ \lambda_5 = -5/6$	$\mu_0 = 99/40\,, \ \mu_1 = 1/20\,, \ \mu_2 = -49/40\,,$ $\mu_3 = -22/15\,, \ \mu_4 = -5/6\,, \ \mu_6 = 1$	$-19/16$
7	$\lambda_0 = -239/280\,, \ \lambda_1 = -283/280\,, \ \lambda_2 = 73/40\,,$ $\lambda_3 = 167/120\,, \ \lambda_4 = 41/210\,, \ \lambda_5 = -29/42\,,$ $\lambda_6 = -6/7$	$\mu_0 = 239/280\,, \ \mu_1 = 261/140\,,$ $\mu_2 = 11/280\,, \ \mu_3 = -142/105\,,$ $\mu_4 = -65/42\,, \ \mu_5 = -6/7\,, \ \mu_7 = 1$	$23/16$
8	$\lambda_0 = 6117/2240\,, \ \lambda_1 = -13751/2240\,,$ $\lambda_2 = 381/320\,, \ \lambda_3 = 2099/960\,,$ $\lambda_4 = 2477/1680\,, \ \lambda_5 = 55/336\,,$ $\lambda_6 = -41/56\,, \ \lambda_7 = -7/8$	$\mu_0 = -6117/2240\,, \ \mu_1 = 3817/1120\,,$ $\mu_2 = 4967/2240\,, \ \mu_3 = 13/420\,,$ $\mu_4 = -485/336\,, \ \mu_5 = -45/28\,,$ $\mu_6 = -7/8\,, \ \mu_8 = 1$	$611/128$
9	$\lambda_0 = 134581/20160\,,$ $\lambda_1 = -197543/20160\,,$ $\lambda_2 = -2867/2880\,, \ \lambda_3 = 4769/2880\,,$ $\lambda_4 = 12287/5040\,, \ \lambda_5 = 1549/1008\,,$ $\lambda_6 = 71/504\,, \ \lambda_7 = -55/72\,,$ $\lambda_8 = -8/9$	$\mu_0 = -134581/20160\,,$ $\mu_1 = 31481/10080\,,$ $\mu_2 = 27677/6720\,, \ \mu_3 = 3103/1260\,,$ $\mu_4 = 25/1008\,, \ \mu_5 = -127/84\,,$ $\mu_6 = -119/72\,, \ \mu_7 = -8/9\,,$ $\mu_9 = 1$	$827/128$
10	$\lambda_0 = 313217/40320\,,$ $\lambda_1 = -305131/40320\,,$ $\lambda_2 = -23239/5760\,,$ $\lambda_3 = -4547/5760\,,$ $\lambda_4 = 19939/10080\,,$ $\lambda_5 = 26461/10080\,,$ $\lambda_6 = 7991/5040\,, \ \lambda_7 = 89/720\,,$ $\lambda_8 = -71/90\,, \ \lambda_9 = -9/10$	$\mu_0 = -313217/40320\,,$ $\mu_1 = -4043/20160\,,$ $\mu_2 = 51529/13440\,,$ $\mu_3 = 11651/2520\,,$ $\mu_4 = 5333/2016\,,$ $\mu_5 = 17/840\,,$ $\mu_6 = -1127/720\,, \ \mu_7 = -76/45\,,$ $\mu_8 = -9/10\,, \ \mu_{10} = 1$	$943/256$

$Z_\nu(x) \ = \ I_0(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = 1\,, \ \lambda_1 = -1$	$\mu_0 = -1\,, \ \mu_1 = 1$	1
2	$\lambda_0 = 1\,, \ \lambda_1 = -1$	$\mu_0 = -1\,, \ \mu_2 = 1$	$3/2$
3	$\lambda_0 = 5/3\,, \ \lambda_1 = -7/3\,, \ \lambda_2 = 2/3$	$\mu_0 = -5/3\,, \ \mu_1 = 2/3\,, \ \mu_3 = 1$	$5/2$
4	$\lambda_0 = 35/12\,, \ \lambda_1 = -49/12\,, \ \lambda_2 = 5/12\,,$ $\lambda_3 = 3/4$	$\mu_0 = -35/12\,, \ \mu_1 = 7/6\,, \ \mu_2 = 3/4\,,$ $\mu_4 = 1$	$35/8$

n	λ_k	μ_k	γ
5	$\lambda_0 = 21/4$, $\lambda_1 = -147/20$, $\lambda_2 = 3/4$, $\lambda_3 = 11/20$, $\lambda_4 = 4/5$	$\mu_0 = -21/4$ $\mu_1 = 21/10$, $\mu_2 = 27/20$, $\mu_3 = 4/5$, $\mu_5 = 1$	$63/8$
6	$\lambda_0 = 77/8$, $\lambda_1 = -539/40$, $\lambda_2 = 11/8$, $\lambda_3 = 121/120$, $\lambda_4 = 19/30$, $\lambda_5 = 5/6$	$\mu_0 = -77/8$, $\mu_1 = 77/20$, $\mu_2 = 99/40$, $\mu_3 = 22/15$, $\mu_4 = 5/6$, $\mu_6 = 1$	$231/16$
7	$\lambda_0 = 143/8$, $\lambda_1 = -1001/40$, $\lambda_2 = 143/56$, $\lambda_3 = 1573/840$, $\lambda_4 = 247/210$, $\lambda_5 = 29/42$, $\lambda_6 = 6/7$	$\mu_0 = -143/8$, $\mu_1 = 143/20$, $\mu_2 = 1287/280$, $\mu_3 = 286/105$, $\mu_4 = 65/42$, $\mu_5 = 6/7$, $\mu_7 = 1$	$429/16$
8	$\lambda_0 = 429/16$, $\lambda_1 = -3003/64$, $\lambda_2 = 2145/448$, $\lambda_3 = 1573/448$, $\lambda_4 = 247/112$, $\lambda_5 = 145/112$, $\lambda_6 = 41/56$, $\lambda_7 = 7/8$	$\mu_0 = -2145/64$, $\mu_1 = 429/32$, $\mu_2 = 3861/448$, $\mu_3 = 143/28$, $\mu_4 = 325/112$, $\mu_5 = 45/28$, $\mu_6 = 7/8$, $\mu_8 = 1$	$6435/128$
9	$\lambda_0 = 12155/192$, $\lambda_1 = -17017/192$, $\lambda_2 = 12155/1344$, $\lambda_3 = 26741/4032$, $\lambda_4 = 4199/1008$, $\lambda_5 = 2465/1008$, $\lambda_6 = 697/504$, $\lambda_7 = 55/72$, $\lambda_8 = 8/9$	$\mu_0 = -12155/192$, $\mu_1 = 2431/96$, $\mu_2 = 7293/448$, $\mu_3 = 2431/252$, $\mu_4 = 5525/1008$, $\mu_5 = 85/28$, $\mu_6 = 119/72$, $\mu_7 = 8/9$, $\mu_9 = 1$	$12155/128$
10	$\lambda_0 = 46189/384$, $\lambda_1 = -323323/1920$, $\lambda_2 = 46189/2688$, $\lambda_3 = 508079/40320$, $\lambda_4 = 79781/10080$, $\lambda_5 = 9367/2016$, $\lambda_6 = 13243/5040$, $\lambda_7 = 209/144$, $\lambda_8 = 71/90$, $\lambda_9 = 9/10$	$\mu_0 = -46189/384$, $\mu_1 = 46189/960$, $\mu_2 = 138567/4480$, $\mu_3 = 46189/2520$, $\mu_4 = 20995/2016$, $\mu_5 = 323/56$, $\mu_6 = 2261/720$, $\mu_7 = 76/45$, $\mu_8 = 9/10$, $\mu_{10} = 1$	$943/256$

$Z_\nu(x) = K_0(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = 1$, $\lambda_1 = -1$	$\mu_0 = 1$, $\mu_1 = -1$	1
2	$\lambda_0 = 1$, $\lambda_1 = -1$	$\mu_0 = 1$, $\mu_2 = -1$	$3/2$
3	$\lambda_0 = 5/3$, $\lambda_1 = -7/3$, $\lambda_2 = 2/3$	$\mu_0 = 5/3$, $\mu_1 = -2/3$, $\mu_3 = -1$	$5/2$
4	$\lambda_0 = 35/12$, $\lambda_1 = -49/12$, $\lambda_2 = 5/12$, $\lambda_3 = 3/4$	$\mu_0 = 35/12$, $\mu_1 = -7/6$, $\mu_2 = -3/4$, $\mu_4 = -1$	$35/8$
5	$\lambda_0 = 21/4$, $\lambda_1 = -147/20$, $\lambda_2 = 3/4$, $\lambda_3 = 11/20$, $\lambda_4 = 4/5$	$\mu_0 = 21/4$ $\mu_1 = -21/10$, $\mu_2 = -27/20$, $\mu_3 = -4/5$, $\mu_5 = -1$	$63/8$
6	$\lambda_0 = 77/8$, $\lambda_1 = -539/40$, $\lambda_2 = 11/8$, $\lambda_3 = 121/120$, $\lambda_4 = 19/30$, $\lambda_5 = 5/6$	$\mu_0 = 77/8$, $\mu_1 = -77/20$, $\mu_2 = -99/40$, $\mu_3 = -22/15$, $\mu_4 = -5/6$, $\mu_6 = -1$	$231/16$
7	$\lambda_0 = 143/8$, $\lambda_1 = -1001/40$, $\lambda_2 = 143/56$, $\lambda_3 = 1573/840$, $\lambda_4 = 247/210$, $\lambda_5 = 29/42$, $\lambda_6 = 6/7$	$\mu_0 = 143/8$, $\mu_1 = -143/20$, $\mu_2 = -1287/280$, $\mu_3 = -286/105$, $\mu_4 = -65/42$, $\mu_5 = -6/7$, $\mu_7 = -1$	$429/16$
8	$\lambda_0 = 429/16$, $\lambda_1 = -3003/64$, $\lambda_2 = 2145/448$, $\lambda_3 = 1573/448$, $\lambda_4 = 247/112$, $\lambda_5 = 145/112$, $\lambda_6 = 41/56$, $\lambda_7 = 7/8$	$\mu_0 = 2145/64$, $\mu_1 = -429/32$, $\mu_2 = -3861/448$, $\mu_3 = -143/28$, $\mu_4 = -325/112$, $\mu_5 = -45/28$, $\mu_6 = -7/8$, $\mu_8 = -1$	$6435/128$
9	$\lambda_0 = 12155/192$, $\lambda_1 = -17017/192$, $\lambda_2 = 12155/1344$, $\lambda_3 = 26741/4032$, $\lambda_4 = 4199/1008$, $\lambda_5 = 2465/1008$, $\lambda_6 = 697/504$, $\lambda_7 = 55/72$, $\lambda_8 = 8/9$	$\mu_0 = 12155/192$, $\mu_1 = -2431/96$, $\mu_2 = -7293/448$, $\mu_3 = -2431/252$, $\mu_4 = -5525/1008$, $\mu_5 = -85/28$, $\mu_6 = -119/72$, $\mu_7 = -8/9$, $\mu_9 = -1$	$12155/128$

n	λ_k	μ_k	γ
10	$\lambda_0 = 46189/384$, $\lambda_1 = -323323/1920$, $\lambda_2 = 46189/2688$, $\lambda_3 = 508079/40320$, $\lambda_4 = 79781/10080$, $\lambda_5 = -9367/2016$, $\lambda_6 = 13243/5040$, $\lambda_7 = -209/144$, $\lambda_8 = 71/90$, $\lambda_9 = 9/10$	$\mu_0 = 46189/384$, $\mu_1 = -46189/960$, $\mu_2 = -138567/4480$, $\mu_3 = -46189/2520$, $\mu_4 = -20995/2016$, $\mu_5 = 323/56$, $\mu_6 = -2261/720$, $\mu_7 = -76/45$, $\mu_8 = -9/10$, $\mu_{10} = -1$	$943/256$

$Z_\nu(x) \,=\, J_1(x):$

n	λ_k	μ_k	γ
1	$\lambda_0 = -1$,	$-$	-1
2	$\lambda_0 = -2$, $\lambda_1 = 2$, $\lambda_2 = -1$	$\mu_0 = 1$, $\mu_1 = -1$	-2
3	$\lambda_0 = -3$, $\lambda_1 = 3$, $\lambda_3 = -1$	$\mu_0 = 2$, $\mu_1 = -1$, $\mu_2 = -1$	$-5/2$
4	$\lambda_0 = -10/3$, $\lambda_1 = 8/3$, $\lambda_2 = 2/3$, $\lambda_4 = -1$	$\mu_0 = 7/3$, $\mu_1 = -1/3$, $\mu_2 = -1$, $\mu_3 = -1$	-2
5	$\lambda_0 = -29/12$, $\lambda_1 = 7/12$, $\lambda_2 = 13/12$, $\lambda_3 = 3/4$, $\lambda_5 = -1$	$\mu_0 = 17/12$, $\mu_1 = 5/6$, $\mu_2 = -1/4$, $\mu_3 = -1$ $\mu_4 = -1$	$-3/8$
6	$\lambda_0 = -7/30$, $\lambda_1 = -79/30$, $\lambda_2 = 23/30$, $\lambda_3 = 13/10$, $\lambda_4 = 4/5$, $\lambda_6 = -1$	$\mu_0 = -23/30$, $\mu_1 = 28/15$, $\mu_2 = 11/10$, $\mu_3 = -1/5$, $\mu_4 = -1$, $\mu_5 = -1$,	$7/4$
7	$\lambda_0 = 269/120$, $\lambda_1 = -607/120$, $\lambda_2 = -61/120$, $\lambda_3 = 127/120$, $\lambda_4 = 43/30$, $\lambda_5 = 5/6$, $\lambda_7 = -1$	$\mu_0 = -389/120$, $\mu_1 = 109/60$, $\mu_2 = 93/40$, $\mu_3 = 19/15$, $\mu_4 = -1/6$, $\mu_5 = -1$, $\mu_6 = -1$	$47/16$
8	$\lambda_0 = 65/21$, $\lambda_1 = -85/21$, $\lambda_2 = -7/3$, $\lambda_3 = -1/3$, $\lambda_4 = 26/21$, $\lambda_5 = 32/21$, $\lambda_6 = 6/7$, $\lambda_8 = -1$	$\mu_0 = -86/21$, $\mu_1 = -1/21$, $\mu_2 = 16/7$, $\mu_3 = 55/21$, $\mu_4 = 29/21$, $\mu_5 = -1/7$, $\mu_6 = -1$, $\mu_7 = -1$	$3/2$
9	$\lambda_0 = 2449/6720$, $\lambda_1 = 14053/6720$, $\lambda_2 = -3383/960$, $\lambda_3 = -2419/960$, $\lambda_4 = -397/1680$, $\lambda_5 = 457/336$, $\lambda_6 = 89/56$, $\lambda_7 = 7/8$, $\lambda_9 = -1$	$\mu_0 = -9169/6720$, $\mu_1 = -11611/3360$, $\mu_2 = 153/2240$, $\mu_3 = 1087/420$, $\mu_4 = 949/336$, $\mu_5 = 41/28$, $\mu_6 = -1/8$, $\mu_7 = -1$, $\mu_8 = -1$	$-419/128$
10	$\lambda_0 = -63617/10080$, $\lambda_1 = 119851/10080$, $\lambda_2 = -3641/1440$, $\lambda_3 = -6013/1440$, $\lambda_4 = -6739/2520$, $\lambda_5 = -89/504$, $\lambda_6 = 365/252$, $\lambda_7 = 59/36$, $\lambda_8 = 8/9$, $\lambda_{10} = -1$	$\mu_0 = 53537/10080$, $\mu_1 = -33157/5040$, $\mu_2 = -13609/3360$, $\mu_3 = 79/630$, $\mu_4 = 1411/504$, $\mu_5 = 125/42$, $\mu_6 = 55/36$, $\mu_7 = -1/9$, $\mu_8 = -1$, $\mu_9 = -1$	$-623/64$

$Z_\nu(x) \,=\, I_1(x):$

n	λ_k	μ_k	γ
1	$\lambda_0 = 1$,	$-$	1
2	$\lambda_0 = 2$, $\lambda_1 = -2$, $\lambda_2 = 1$	$\mu_0 = -1$, $\mu_1 = 1$	2
3	$\lambda_0 = 3$, $\lambda_1 = -3$, $\lambda_3 = 1$	$\mu_0 = -2$, $\mu_1 = 1$, $\mu_2 = 1$	$7/2$
4	$\lambda_0 = 14/3$, $\lambda_1 = -16/3$, $\lambda_2 = 2/3$, $\lambda_4 = 1$	$\mu_0 = -11/3$, $\mu_1 = 5/3$, $\mu_2 = 1$, $\mu_3 = 1$	6
5	$\lambda_0 = 91/12$, $\lambda_1 = -113/12$, $\lambda_2 = 13/12$, $\lambda_3 = 3/4$, $\lambda_5 = 1$	$\mu_0 = -79/12$, $\mu_1 = 17/6$, $\mu_2 = 7/4$, $\mu_3 = 1$ $\mu_4 = 1$	$83/8$
6	$\lambda_0 = 77/6$, $\lambda_1 = -503/30$, $\lambda_2 = 11/6$, $\lambda_3 = 13/10$, $\lambda_4 = 4/5$, $\lambda_6 = 1$	$\mu_0 = -71/6$, $\mu_1 = 74/15$, $\mu_2 = 31/10$, $\mu_3 = 9/5$, $\mu_4 = 1$, $\mu_5 = 1$,	$73/4$

n	λ_k	μ_k	γ
7	$\lambda_0 = 539/24$, $\lambda_1 = -3629/120$, $\lambda_2 = 77/24$, $\lambda_3 = 277/120$, $\lambda_4 = 43/30$, $\lambda_5 = 5/6$, $\lambda_7 = -1$	$\mu_0 = 277/120$, $\mu_1 = 527/60$, $\mu_2 = 223/40$, $\mu_3 = 49/15$, $\mu_4 = 11/6$, $\mu_5 = 1$, $\mu_6 = 1$	$523/15$
8	$\lambda_0 = 121/3$, $\lambda_1 = -829/15$, $\lambda_2 = 121/21$, $\lambda_3 = 439/105$, $\lambda_4 = 274/105$, $\lambda_5 = 32/21$, $\lambda_6 = 6/7$, $\lambda_8 = 1$	$\mu_0 = -118/3$, $\mu_1 = 239/15$, $\mu_2 = 356/35$, $\mu_3 = 629/105$, $\mu_4 = 71/21$, $\mu_5 = 13/7$, $\mu_6 = 1$, $\mu_7 = 1$	$119/2$
9	$\lambda_0 = 14179/192$, $\lambda_1 = 14179/192$, $\lambda_2 = 14179/1344$, $\lambda_3 = 51691/6720$, $\lambda_4 = 8089/1680$, $\lambda_5 = 947/336$, $\lambda_6 = 89/56$, $\lambda_7 = 7/8$, $\lambda_9 = 1$	$\mu_0 = -13987/192$, $\mu_1 = 14083/480$, $\mu_2 = 42089/2240$, $\mu_3 = 4661/420$, $\mu_4 = 2111/336$, $\mu_5 = 97/28$, $\mu_6 = 15/8$, $\mu_7 = 1$, $\mu_8 = 1$	$14051/128$
10	$\lambda_0 = 4389/32$, $\lambda_1 = -30531/160$, $\lambda_2 = 627/32$, $\lambda_3 = 20627/1440$, $\lambda_4 = 3233/360$, $\lambda_5 = 379/72$, $\lambda_6 = 107/36$, $\lambda_7 = 59/36$, $\lambda_8 = 8/9$, $\lambda_{10} = 1$	$\mu_0 = -4357/32$, $\mu_1 = 4373/80$, $\mu_2 = 5611/160$, $\mu_3 = 1867/90$, $\mu_4 = 1867/90$, $\mu_5 = 13/2$, $\mu_6 = 127/36$, $\mu_7 = 17/9$, $\mu_8 = 1$, $\mu_9 = 1$	$13103/64$

$Z_\nu(x) = K_1(x) :$

n	λ_k	μ_k	γ
1	$\lambda_0 = -1$,	$-$	-1
2	$\lambda_0 = -2$, $\lambda_1 = 2$, $\lambda_2 = -1$	$\mu_0 = -1$, $\mu_1 = 1$	-2
3	$\lambda_0 = -3$, $\lambda_1 = 3$, $\lambda_3 = -1$	$\mu_0 = -2$, $\mu_1 = 1$, $\mu_2 = 1$	$-7/2$
4	$\lambda_0 = -14/3$, $\lambda_1 = 16/3$, $\lambda_2 = -2/3$, $\lambda_4 = -1$	$\mu_0 = -11/3$, $\mu_1 = 5/3$, $\mu_2 = 1$, $\mu_3 = 1$	-6
5	$\lambda_0 = -91/12$, $\lambda_1 = 113/12$, $\lambda_2 = -13/12$, $\lambda_3 = -3/4$, $\lambda_5 = 1$	$\mu_0 = -79/12$, $\mu_1 = 17/6$, $\mu_2 = 7/4$, $\mu_3 = 1$ $\mu_4 = 1$	$-83/8$
6	$\lambda_0 = -77/6$, $\lambda_1 = 503/30$, $\lambda_2 = -11/6$, $\lambda_3 = -13/10$, $\lambda_4 = -4/5$, $\lambda_6 = -1$	$\mu_0 = -71/6$, $\mu_1 = 74/15$, $\mu_2 = 31/10$, $\mu_3 = 9/5$, $\mu_4 = 1$, $\mu_5 = 1$,	$-73/4$
7	$\lambda_0 = -539/24$, $\lambda_1 = 3629/120$, $\lambda_2 = -77/24$, $\lambda_3 = -277/120$, $\lambda_4 = -43/30$, $\lambda_5 = -5/6$, $\lambda_7 = -1$	$\mu_0 = -277/120$, $\mu_1 = 527/60$, $\mu_2 = 223/40$, $\mu_3 = 49/15$, $\mu_4 = 11/6$, $\mu_5 = 1$, $\mu_6 = 1$	$-523/15$
8	$\lambda_0 = -121/3$, $\lambda_1 = 829/15$, $\lambda_2 = -121/21$, $\lambda_3 = -439/105$, $\lambda_4 = -274/105$, $\lambda_5 = -32/21$, $\lambda_6 = -6/7$, $\lambda_8 = -1$	$\mu_0 = -118/3$, $\mu_1 = 239/15$, $\mu_2 = 356/35$, $\mu_3 = 629/105$, $\mu_4 = 71/21$, $\mu_5 = 13/7$, $\mu_6 = 1$, $\mu_7 = 1$	$-119/2$
9	$\lambda_0 = 14179/192$, $\lambda_1 = 14179/192$, $\lambda_2 = -14179/1344$, $\lambda_3 = 51691/6720$, $\lambda_4 = -8089/1680$, $\lambda_5 = -947/336$, $\lambda_6 = -89/56$, $\lambda_7 = -7/8$, $\lambda_9 = -1$	$\mu_0 = -13987/192$, $\mu_1 = 14083/480$, $\mu_2 = 42089/2240$, $\mu_3 = 4661/420$, $\mu_4 = 2111/336$, $\mu_5 = 97/28$, $\mu_6 = 15/8$, $\mu_7 = 1$, $\mu_8 = 1$	$-14051/128$
10	$\lambda_0 = -4389/32$, $\lambda_1 = 30531/160$, $\lambda_2 = -627/32$, $\lambda_3 = -20627/1440$, $\lambda_4 = -3233/360$, $\lambda_5 = -379/72$, $\lambda_6 = -107/36$, $\lambda_7 = -59/36$, $\lambda_8 = -8/9$, $\lambda_{10} = -1$	$\mu_0 = -4357/32$, $\mu_1 = 4373/80$, $\mu_2 = 5611/160$, $\mu_3 = 1867/90$, $\mu_4 = 1867/90$, $\mu_5 = 13/2$, $\mu_6 = 127/36$, $\mu_7 = 17/9$, $\mu_8 = 1$, $\mu_9 = 1$	$-13103/64$

g) Generalized Laguerre Polynomials $L_n^{(\alpha)}(x)$:

$$\int_0^\infty x^\alpha\, e^{-x}\, [L_n^{(\alpha)}(x)]^2 \, dx \quad = \quad \frac{\Gamma(n+\alpha+1)}{n!}\,, \quad \alpha > -1$$

First polynomials:

$$L_0^{(\alpha)}(x) = 1\,, \quad L_1^{(\alpha)}(x) = -x + 1 + \alpha\,, \quad L_2^{(\alpha)}(x) = \frac{x^2}{2} - (2+\alpha)x + \frac{(\alpha+1)(\alpha+2)}{2}\,,$$

$$L_3^{(\alpha)}(x) = -\frac{x^3}{6} + \frac{3+\alpha}{2}\,x^2 - \frac{\alpha^2 + 5\alpha + 6}{2}\,x + \frac{\alpha^3 + 6\alpha + 11\alpha + 6}{6}\,,$$

$$L_4^{(\alpha)}(x) = \frac{x^4}{24} - \frac{\alpha+4}{6}\,x^3 + \frac{\alpha^2 + 7\alpha + 12}{4}\,x^2 - \frac{\alpha^3 + 9\alpha^2 + 26\alpha + 24}{6}\,x + \frac{\alpha^4 + 10\alpha^3 + 35\alpha^2 + 50\alpha + 24}{24}$$

$Z_\nu(x) = J_0(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = (\alpha+1)^2\,,\ \lambda_1 = -(\alpha+1)$	$\mu_0 = -(\alpha+1)\,,\ \mu_1 = 1$	$\alpha + 1$
2	$\lambda_0 = (\alpha+1)^2(\alpha+2)/2\,,$ $\lambda_1 = -(\alpha+1)(\alpha+2)/2$	$\mu_0 = -(\alpha+1)(\alpha+2)/2\,,\ \mu_2 = 1$	$(\alpha^2 + 3\alpha + 1)/2$
3	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^2+4\alpha+1)/6\,,$ $\lambda_1 = -(\alpha+2)(\alpha^2+4\alpha-1)/6\,,$ $\lambda_2 = -2/3$	$\mu_0 = -(\alpha+1)(\alpha^2+5\alpha+2)/6\,,$ $\mu_1 = -2/3\,,\ \mu_3 = 1$	$(\alpha+3)(\alpha^2+3\alpha-1)/6$
4	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^3 + 8\alpha^2 +$ $+14\alpha - 11)/24\,,$ $\lambda_1 = -(\alpha+2)(\alpha^3 + 8\alpha^2 + 12\alpha$ $-25)/24\,,$ $\lambda_2 = (\alpha-5)/12\,,\ \lambda_3 = -3/4$	$\mu_0 = -(\alpha+1)(\alpha^3 + 9\alpha^2 + 19\alpha -$ $-22)/24\,,$ $\mu_1 = (\alpha-14)/12\,,$ $\mu_2 = -3/4\,,\ \mu_4 = 1$	$(\alpha^4 + 10\alpha^3 + 29\alpha^2 +$ $+8\alpha - 39)/24$
5	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^4 + 13\alpha^3 + 50\alpha^2 + 19\alpha - 131)/120$ $\lambda_1 = -(\alpha+2)(\alpha^4 + 13\alpha^3 + 48\alpha^2 - 5\alpha - 193)/120$ $\lambda_2 = (\alpha^2 + 12\alpha + 19)/60\,,\ \lambda_3 = (\alpha - 11)/20\,,\ \lambda_4 = -4/5$ $\mu_0 = -(\alpha+1)(\alpha^4 + 14\alpha^3 + 60\alpha^2 + 29\alpha - 262)/120$ $\mu_1 = (\alpha^2 + 15\alpha - 62)/60\,,\ \mu_2 = (\alpha - 27)/20\,,\ \mu_3 = -4/5\,,\ \mu_5 = 1$ $\gamma = (\alpha+5)(\alpha^4 + 10\alpha^3 + 25\alpha^2 - 20\alpha - 51)/120$		
6	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^5 + 19\alpha^4 + 123\alpha^3 + 239\alpha^2 - 367\alpha - 891)/720$ $\lambda_1 = -(\alpha+2)(\alpha^5 + 19\alpha^4 + 121\alpha^3 + 203\alpha^2 - 563\alpha - 873)/720$ $\lambda_2 = (\alpha^3 + 18\alpha^2 + 86\alpha + 459)/360\,,\ \lambda_3 = (\alpha^2 + 15\alpha + 29)/120\,,\ \lambda_4 = (\alpha - 19)/30\,,\ \lambda_5 = -5/6$ $\mu_0 = -(\alpha+1)(\alpha^5 + 20\alpha^4 + 139\alpha^3 + 304\alpha^2 - 503\alpha - 1782)/720$ $\mu_1 = (\alpha^3 + 21\alpha^2 + 143\alpha + 18)/360\,,\ \mu_2 = (\alpha^2 + 19\alpha - 147)/120\,,\ \mu_3 = (\alpha - 44)/30\,,$ $\mu_4 = -5/6\,,\ \mu_6 = 1$ $\gamma = (\alpha^6 + 21\alpha^5 + 160\alpha^4 + 465\alpha^3 - 26\alpha^2 - 1881\alpha - 855)/720$		
7	$\lambda_0 = (\alpha+1)(\alpha+2)((\alpha^6 + 26\alpha^5 + 250\alpha^4 + 962\alpha^3 + 286\alpha^2 - 5374\alpha - 2151)/5040$ $\lambda_1 = -(\alpha+2)(\alpha^6 + 26\alpha^5 + 248\alpha^4 + 912\alpha^3 - 150\alpha^2 - 6464\alpha + 2547)/5040$ $\lambda_2 = (\alpha^4 + 25\alpha^3 + 206\alpha^2 + 209\alpha + 4599)/2520\,,\ \lambda_3 = (aa^3 + 22\alpha^2 + 128\alpha + 1169)/840$ $\lambda_4 = (aa^2 + 18\alpha + 41)/210\,,\ \lambda_5 = (\alpha - 29)/42\,,\ \lambda_6 = -6/7$ $\mu_0 = -(\alpha+1)(\alpha^6 + 27\alpha^5 + 273\alpha^4 + 1133\alpha^3 + 485\alpha^2 - 7817\alpha - 4302)/5040$ $\mu_1 = (\alpha^4 + 28\alpha^3 + 284\alpha^2 + 869\alpha + 4698)/2520\,,\ \mu_2 = (\alpha^3 + 26\alpha^2 + 220\alpha + 33)/840$ $\mu_3 = (\alpha^2 + 23\alpha - 284)/210\,,\ \mu_4 = (\alpha - 65)/42\,,\ \mu_5 = -6/7\,,\ \mu_7 = 1$ $\gamma = (\alpha+7)(\alpha^6 + 21\alpha^5 + 154\alpha^4 + 357\alpha^3 - 560\alpha^2 - 1953\alpha + 1035)/5040$		

n	λ_k	μ_k	γ
8	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^7 + 34\alpha^6 + 451\alpha^5 + 2745\alpha^4 + 5609\alpha^3 - 12319\alpha^2 - 39970\alpha + 55053)/40320$ $\lambda_1 = -(\alpha+2)(\alpha^7 + 34\alpha^6 + 449\alpha^5 + 2679\alpha^4 + 4787\alpha^3 - 16477\alpha^2 - 39764\alpha + 123759)/40320$ $\lambda_2 = (\alpha^5 + 33\alpha^4 + 399\alpha^3 + 1647\alpha^2 - 5839\alpha + 24003)/20160$ $\lambda_3 = (\alpha^4 + 30\alpha^3 + 297\alpha^2 + 324\alpha + 14693)/6720, \; \lambda_4 = (\alpha^3 + 26\alpha^2 + 178\alpha + 2477)/1680$ $\lambda_5 = (\alpha^2 + 21\alpha + 55)/336, \; \lambda_6 = (\alpha - 41)/56, \; \lambda_7 = -7/8$ $\mu_0 = -(\alpha+1)(\alpha^7 + 35\alpha^6 + 482\alpha^5 + 3093\alpha^4 + 6980\alpha^3 - 14885\alpha^2 - 63653\alpha + 110106)/40320$ $\mu_1 = (\alpha^5 + 36\alpha^4 + 501\alpha^3 + 2910\alpha^2 - 1111\alpha + 68706)/20160$ $\mu_2 = (\alpha^4 + 34\alpha^3 + 421\alpha^2 + 1576\alpha + 14901)/6720, \; \mu_3 = (\alpha^3 + 31\alpha^2 + 313\alpha + 52)/1680$ $\mu_4 = (\alpha^2 + 27\alpha - 485)/336, \; \mu_5 = (\alpha - 90)/56, \; \mu_6 = -7/8, \; \mu_8 = 1$ $\gamma = (\alpha^8 + 36\alpha^7 + 518\alpha^6 + 3612\alpha^5 + 10619\alpha^4 - 4116\alpha^3 - 70358\alpha^2 - 792\alpha + 192465)/40320$		
9	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^8 + 43\alpha^7 + 749\alpha^6 + 6484\alpha^5 + 25514\alpha^4 + 7954\alpha^3 - 186617\alpha^2 + 35451\alpha +$ $+1211229)/362880$ $\lambda_1 = -(\alpha+2)(\alpha^8 + 43\alpha^7 + 747\alpha^6 + 6400\alpha^5 + 24114\alpha^4 - 2978\alpha^3 - 214969\alpha^2 + 152811\alpha+$ $+1777887)/362880$ $\lambda_2 = (\alpha^6 + 42\alpha^5 + 688\alpha^4 + 4926\alpha^3 + 4648\alpha^2 - 128556\alpha - 180621)/181440$ $\lambda_3 = (\alpha^5 + 39\alpha^4 + 559\alpha^3 + 2709\alpha^2 - 12647\alpha + 100149)/60480$ $\lambda_4 = (\alpha^4 + 35\alpha^3 + 404\alpha^2 + 463\alpha + 36861)/15120, \; \lambda_5 = (\alpha^3 + 30\alpha^2 + 236\alpha + 4647)/3024$ $\lambda_6 = (\alpha^2 + 24\alpha + 71)/504, \; \lambda_7 = (\alpha - 55)/72, \; \lambda_8 = -8/9$ $\mu_0 = -(\alpha+1)(\alpha^8 + 44\alpha^7 + 789\alpha^6 + 7103\alpha^5 + 29721\alpha^4 + 13895\alpha^3 - 248986\alpha^2 - 67707\alpha+$ $+2422458)/362880$ $\mu_1 = (\alpha^6 + 45\alpha^5 + 817\alpha^4 + 7083\alpha^3 + 19783\alpha^2 - 135621\alpha + 566658)/181440$ $\mu_2 = (\alpha^5 + 43\alpha^4 + 719\alpha^3 + 5045\alpha^2 - 2355\alpha + 249093)/60480$ $\mu_3 = (\alpha^4 + 40\alpha^3 + 584\alpha^2 + 2573\alpha + 37236)/15120, \; \mu_4 = (\alpha^3 + 36\alpha^2 + 422\alpha + 75)/3024$ $\mu_5 = (\alpha^2 + 31\alpha - 762)/504, \; \mu_6 = (\alpha - 119)/72, \; \mu_7 = -8/9, \; \mu_9 = 1$ $\gamma = (\alpha+9)(\alpha^8 + 36\alpha^7 + 510\alpha^6 + 3348\alpha^5 + 7563\alpha^4 - 16092\alpha^3 - 53686\alpha^2 + 151056\alpha+$ $+260505)/362880$		
10	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^9 + 53\alpha^8 + 1170\alpha^7 + 13524\alpha^6 + 81543\alpha^5 + 183237\alpha^4 - 359590\alpha^3 -$ $-1034264\alpha^2 + 7302861\alpha + 14094765)/3628800$ $\lambda_1 = -(\alpha+2)(\alpha^9 + 53\alpha^8 + 1168\alpha^7 + 13420\alpha^6 + 79321\alpha^5 + 159187\alpha^4 - 480648\alpha^3-$ $-1063930\alpha^2 + 9274833\alpha + 13730895)/3628800$ $\lambda_2 = (\alpha^7 + 52\alpha^6 + 1099^5\alpha + 11365\alpha^4 + 45709\alpha^3 - 144707\alpha^2 - 1469934\alpha - 7320285)/1814400$ $\lambda_3 = (\alpha^6 + 49\alpha^5 + 940\alpha^4 + 7885\alpha^3 + 7594\alpha^2 - 304709\alpha - 477435)/604800$ $\lambda_4 = (\alpha^5 + 45\alpha^4 + 745\alpha^3 + 4125\alpha^2 - 24191\alpha + 299085)/151200$ $\lambda_5 = (\alpha^4 + 40\alpha^3 + 527\alpha^2 + 626\alpha + 79383)/30240$ $\lambda_6 = (\alpha^3 + 34\alpha^2 + 302\alpha + 7991)/5040, \; \lambda_7 = (\alpha^2 + 27\alpha + 89)/720$ $\lambda_8 = (\alpha - 71)/90, \; \lambda_9 = -9/10$ $\mu_0 = -(1+\alpha)(\alpha^9 + 54\alpha^8 + 1220\alpha^7 + 14534\alpha^6 + 91517\alpha^5 + 223796\alpha^4 - 415640\alpha^3-$ $-1771714\alpha^2 + 8970777\alpha + 28189530)/3628800$ $\mu_1 = (\alpha^7 + 55\alpha^6 + 1258\alpha^5 + 14785\alpha^4 + 81064\alpha^3 - 26045\alpha^2 - 2439873\alpha - 363870)/1814400$ $\mu_2 = (\alpha^6 + 53\alpha^5 + 1140\alpha^4 + 11785\alpha^3 + 39554\alpha^2 - 323313\alpha + 2318805)/604800$ $\mu_3 = (\alpha^5 + 50\alpha^4 + 975\alpha^3 + 7990\alpha^2 - 4651\alpha + 699060)/151200$ $\mu_4 = (\alpha^4 + 46\alpha^3 + 773\alpha^2 + 3908\alpha + 79995)/30240$ $\mu_5 = (\alpha^3 + 41\alpha^2 + 547\alpha + 102)/5040, \; \mu_6 = (\alpha^2 + 35\alpha - 1127)/720$ $\mu_7 = (\alpha - 152)/90, \; \mu_8 = -9/10, \; \mu_{10} = 1$ $\gamma = (\alpha^{10} + 55\alpha^9 + 1275\alpha^8 + 15810\alpha^7 + 107373\alpha^6 + 331905\alpha^5 - 81775\alpha^4 - 1939510\alpha^3+$ $+5912451\alpha^2 + 30976515\alpha + 13367025)/3628800$		

$Z_\nu(x) = I_0(x)$, $K_0(x)$ with s defined as on page 195:

n	λ_k	μ_k	γ
1	$\lambda_0 = (\alpha+1)^2$, $\lambda_1 = -(\alpha+1)$	$\mu_0 = -s(\alpha+1)$, $\mu_1 = s$	$\alpha+1$
2	$\lambda_0 = (\alpha+1)^2(\alpha+2)/2$, $\lambda_1 = -(\alpha+1)(\alpha+2)/2$	$\mu_0 = -s(\alpha+1)(\alpha+2)/2$, $\mu_2 = s$	$(\alpha^2+3\alpha+3)/2$
3	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^2+4\alpha+5)/6$ $\lambda_1 = -(\alpha+2)\left(\alpha^2+4\alpha+7\right)/6$, $\lambda_2 = 2/3$	$\mu_0 = -s(\alpha+1)(\alpha^2+5\alpha+10)/6$ $\mu_1 = 2s/3$, $\mu_3 = s$	$(\alpha+3)\cdot$ $\cdot(\alpha^2+3\alpha+5)/6$
4	$\lambda_0 = (\alpha+1)(\alpha+2)\cdot$ $\cdot(\alpha^3+8\alpha^2+24\alpha+35)/24$, $\lambda_1 = -(\alpha+2)\cdot$ $\cdot(\alpha^3+8\alpha^2+26\alpha+49)/24$, $\lambda_2 = -(\alpha-5)/12$, $\lambda_3 = 3/4$	$\mu_0 = -s(\alpha+1)\cdot$ $\cdot(\alpha^3+9\alpha^2+33\alpha+70)/24$, $\mu_1 = -s(\alpha-14)/12$, $\mu_2 = 3s/4$, $\mu_4 = s$	$(\alpha^4+10\alpha^3+41\alpha^2+$ $+92\alpha+105)/24$

| 5 | $\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^4+13\alpha^3+68\alpha^2+195\alpha+315)/120$ $\lambda_1 = -(\alpha+2)(\alpha^4+13\alpha^3+70\alpha^2+219\alpha+441)/120$ $\lambda_2 = -(\alpha+15)(\alpha-3)/60$, $\lambda_3 = -(\alpha-11)/20$, $\lambda_4 = 4/5$ $\mu_0 = -s(\alpha+1)(\alpha^4+14\alpha^3+82\alpha^2+279\alpha+630)/120$ $\mu_1 = -s(\alpha+21)(\alpha-6)/60$, $\mu_2 = -s(\alpha-27)/20$, $\mu_3 = 4s/5$, $\mu_5 = s$ $\gamma = (\alpha+5)(\alpha^4+10\alpha^3+45\alpha^2+120\alpha+189)/120$ |

| 6 | $\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^5+19\alpha^4+151\alpha^3+683\alpha^2+2005\alpha+3465)/720$ $\lambda_1 = -(\alpha+2)(\alpha^5+19\alpha^4+153\alpha^3+719\alpha^2+2285\alpha+4851)/720$ $\lambda_2 = -(\alpha^3+18\alpha^2+152\alpha-495)/360$, $\lambda_3 = -(\alpha^2+15\alpha-121)/120$ $\lambda_4 = -(\alpha-19)/30$, $\lambda_5 = 5/6$ $\mu_0 = -s(\alpha+1)(\alpha^2+11\alpha+45)(\alpha^3+9\alpha^2+27\alpha+154)/720$ $\mu_1 = -s(\alpha^3+21\alpha^2+209\alpha-1386)/360$, $\mu_2 = -s(\alpha^2+19\alpha-297)/120$ $\mu_3 = -s(\alpha-44)/30$, $\mu_4 = 5s/6$, $\mu_6 = s$ $\gamma = (\alpha^6+21\alpha^5+190\alpha^4+1005\alpha^3+3544\alpha^2+8379\alpha+10395)/720$ |

| 7 | $\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^6+26\alpha^5+290\alpha^4+1878\alpha^3+8154\alpha^2+24910\alpha+45045)/5040$ $\lambda_1 = -(\alpha+2)(\alpha^6+26\alpha^5+292\alpha^4+1928\alpha^3+8698\alpha^2+28520\alpha+63063)/5040$ $\lambda_2 = -(\alpha^4+25\alpha^3+284\alpha^2+2141\alpha-6435)/2520$, $\lambda_3 = -(\alpha^3+22\alpha^2+230\alpha+1573)/840$ $\lambda_4 = -(\alpha^2+18\alpha-247)/210$, $\lambda_5 = -(\alpha-29)/42$, $\lambda_6 = 6/7$ $\mu_0 = -s(\alpha+1)(\alpha^6+27\alpha^5+317\alpha^4+2197\alpha^3+10413\alpha^2+36227\alpha+90090)/5040$ $\mu_1 = -s(\alpha^4+28\alpha^3+362\alpha^2+3107\alpha-18018)/2520$ $\mu_2 = -s(\alpha^3+26\alpha^2+322\alpha-3861)/840$, $\mu_3 = -s(\alpha^2+23\alpha-572)/210$ $\mu_4 = -s(\alpha-65)/42$, $\mu_5 = 6s/7$, $\mu_7 = s$ $\gamma = (\alpha+7)(\alpha^6+21\alpha^5+196\alpha^4+1113\alpha^3+4438\alpha^2+12411\alpha+19305)/5040$ |

| 8 | $\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^7+34\alpha^6+505\alpha^5+4415\alpha^4+26167\alpha^3+114075\alpha^2+$ $+362100\alpha+675675)/40320$ $\lambda_1 = -(\alpha+2)(\alpha^7+34\alpha^6+507\alpha^5+4481\alpha^4+27125\alpha^3+122457\alpha^2+414654\alpha+$ $+945945)/40320$ $\lambda_2 = -(\alpha^5+33\alpha^4+491\alpha^3+4623\alpha^2+33765\alpha-96525)/20160$ $\lambda_3 = -(\alpha^4+30\alpha^3+413\alpha^2+3816\alpha-23595)/6720$ $\lambda_4 = -(\alpha^3+26\alpha^2+324\alpha-3705)/1680$, $\lambda_5 = -(\alpha^2+21\alpha-435)/336$ $\lambda_6 = -(\alpha-41)/56$, $\lambda_7 = 7/8$ $\mu_0 = -s(\alpha+1)(\alpha^7+35\alpha^6+540\alpha^5+4957\alpha^4+31202\alpha^3+146691\alpha^2+$ $+525345\alpha+1351350)/40320$ $\mu_1 = -s(\alpha^5+36\alpha^4+593\alpha^3+6234\alpha^2+50721\alpha-270270)/20160$ $\mu_2 = -s(\alpha^4+34\alpha^3+537\alpha^2+5652\alpha-57915)/6720$ $\mu_3 = -s(\alpha^3+31\alpha^2+459\alpha-8580)/1680$, $\mu_4 = -s(\alpha^2+27\alpha-975)/336$ $\mu_5 = -s(\alpha-90)/56$, $\mu_6 = 7s/8$, $\mu_8 = s$ $\gamma = (\alpha^8+36\alpha^7+574\alpha^6+5460\alpha^5+35539\alpha^4+171444\alpha^3+622866\alpha^2+$ $+1563120\alpha+2027025)/40320$ |

n	$\lambda_k \, , \ \mu_k \, , \ \gamma$
9	$\lambda_0 = (\alpha + 1)(\alpha + 2)(\alpha^8 + 43\,\alpha^7 + 819\,\alpha^6 + 9280\,\alpha^5 + 71694\,\alpha^4 + 412426\,\alpha^3 + 1836935\,\alpha^2 +$ $+6018255\,\alpha + 11486475)/362880$ $\lambda_1 = -(\alpha + 2)(\alpha^8 + 43\,\alpha^7 + 821\,\alpha^6 + 9364\,\alpha^5 + 73262\,\alpha^4 + 430054\,\alpha^3 + 1975999\,\alpha^2 +$ $+6880887\,\alpha + 16081065)/362880$ $\lambda_2 = -(\alpha^6 + 42\,\alpha^5 + 796\,\alpha^4 + 9354\,\alpha^3 + 81004\,\alpha^2 + 593640\,\alpha - 1640925)/181440$ $\lambda_3 = -(\alpha^5 + 39\,\alpha^4 + 691\,\alpha^3 + 7821\,\alpha^2 + 69373\,\alpha - 401115)/60480$ $\lambda_4 = -(\alpha^4 + 35\,\alpha^3 + 566\,\alpha^2 + 6187\,\alpha - 62985)/15120$ $\lambda_5 = -(\alpha^3 + 30\,\alpha^2 + 434\,\alpha - 7395)/3024 \, , \ \lambda_6 = -(\alpha + 41)(\alpha - 17)/504$ $\lambda_7 = -(\alpha - 55)/72 \, , \ \lambda_8 = 8/9$ $\mu_0 = -s(\alpha + 1)(\alpha^8 + 44\,\alpha^7 + 863\,\alpha^6 + 10145\,\alpha^5 + 81935\,\alpha^4 + 496493\,\alpha^3 + 2363428\,\alpha^2 +$ $+8696295\,\alpha + 22972950)/362880$ $\mu_1 = -s(\alpha^6 + 45\,\alpha^5 + 925\,\alpha^4 + 11907\,\alpha^3 + 113419\,\alpha^2 + 913203\,\alpha - 4594590)/181440$ $\mu_2 = -s(\alpha^5 + 43\,\alpha^4 + 851\,\alpha^3 - 10805\,\alpha^2 + 106521\,\alpha - 984555)/60480$ $\mu_3 = -s(\alpha^4 + 40\,\alpha^3 + 746\,\alpha^2 + 9287\,\alpha - 145860)/15120$ $\mu_4 = -(\alpha^3 + 36\,\alpha^2 + 620\,\alpha - 16575)/3024 \, , \ \mu_5 = -s(\alpha^2 + 31\,\alpha - 1530)/504$ $\mu_6 = -s(\alpha - 119)/72 \, , \ \mu_7 = 8s/9 \, , \ \mu_9 = s$ $\gamma = (\alpha + 9)(\alpha^8 + 36\,\alpha^7 + 582\,\alpha^6 + 5724\,\alpha^5 + 39603\,\alpha^4 + 209628\,\alpha^3 + 859202\,\alpha^2 +$ $+2485800\,\alpha + 3828825)/362880$
10	$\lambda_0 = (\alpha + 1)(\alpha + 2)(\alpha^9 + 53\,\alpha^8 + 1258\,\alpha^7 + 17920\,\alpha^6 + 174799\,\alpha^5 + 1273357\,\alpha^4 +$ $+7327962\,\alpha^3 + 33481160\,\alpha^2 + 112434165\,\alpha + 218243025)/3628800$ $\lambda_1 = -(\alpha + 2)(\alpha^9 + 53\,\alpha^8 + 1260\,\alpha^7 + 18024\,\alpha^6 + 177225\,\alpha^5 + 1307547\,\alpha^4 + 7663160\,\alpha^3 +$ $+36009166\,\alpha^2 + 128276829\,\alpha + 305540235)/3628800$ $\lambda_2 = -(\alpha^7 + 52\,\alpha^6 + 1225\,\alpha^5 + 17755\,\alpha^4 + 184579\,\alpha^3 + 1546663\,\alpha^2 + 11552400\,\alpha -$ $-31177575)/1814400$ $\lambda_3 = -(\alpha^6 + 49\,\alpha^5 + 1090\,\alpha^4 + 15145\,\alpha^3 + 156484\,\alpha^2 + 1382191\,\alpha - 7621185)/604800$ $\lambda_4 = -(\alpha^5 + 45\,\alpha^4 + 925\,\alpha^3 + 12225\,\alpha^2 + 127369\,\alpha - 1196715)/151200$ $\lambda_5 = -(\alpha^4 + 40\,\alpha^3 + 743\,\alpha^2 + 9374\,\alpha - 140505)/30240$ $\lambda_6 = -(\alpha^3 + 34\,\alpha^2 + 560\,\alpha - 13243)/5040 \, , \ \lambda_7 = -(\alpha^2 + 27\,\alpha - 1045)/720$ $\lambda_8 = -(\alpha - 71)/90 \, , \ \lambda_9 = 9/10$ $\mu_0 = -s(\alpha + 1)(\alpha^9 + 54\,\alpha^8 + 1312\,\alpha^7 + 19234\,\alpha^6 + 194149\,\alpha^5 + 1470616\,\alpha^4 + 8850888\,\alpha^3 +$ $+42972346\,\alpha^2 + 161767125\,\alpha + 436486050)/3628800$ $\mu_1 = -s(\alpha^7 + 55\,\alpha^6 + 1384\,\alpha^5 + 21625\,\alpha^4 + 243874\,\alpha^3 + 2222155\,\alpha^2 + 18079641\,\alpha -$ $-87297210)/1814400$ $\mu_2 = -s(\alpha^6 + 53\,\alpha^5 + 1290\,\alpha^4 + 19765\,\alpha^3 + 225164\,\alpha^2 + 2175747\,\alpha - 18706545)/604800$ $\mu_3 = -s(\alpha^5 + 50\,\alpha^4 + 1155\,\alpha^3 + 17170\,\alpha^2 + 198389\,\alpha - 2771340)/151200$ $\mu_4 = -s(\alpha^4 + 46\,\alpha^3 + 989\,\alpha^2 + 14204\,\alpha - 314925)/30240$ $\mu_5 = -s(\alpha^3 + 41\,\alpha^2 + 805\,\alpha - 29070)/5040$ $\mu_6 = -s(\alpha^2 + 35\,\alpha - 2261)/720 \, , \ \mu_7 = -s(\alpha - 152)/90 \, , \ \mu_8 = 9s/10 \, , \ \mu_{10} = s$ $\gamma = (\alpha^{10} + 55\,\alpha^9 + 1365\,\alpha^8 + 20490\,\alpha^7 + 211953\,\alpha^6 + 1642305\,\alpha^5 + 10071935\,\alpha^4 +$ $+49660010\,\alpha^3 + 188887671\,\alpha^2 + 492444315\,\alpha + 654729075)/3628800$

$Z_\nu(x) = J_1(x):$

n	λ_k	μ_k	γ
1	$\lambda_0 = -(\alpha+1)$	$-$	-1
2	$\lambda_0 = -(\alpha+1)(\alpha+2)$, $\lambda_1 = \alpha+2$, $\lambda_2 = -1$	$\mu_0 = \alpha+1$, $\mu_1 = -1$	$-(\alpha+2)$
3	$\lambda_0 = -(\alpha+1)(\alpha+2)(\alpha+3)/2$, $\lambda_1 = (\alpha+2)(\alpha+3)/2$, $\lambda_3 = -1$	$\mu_0 = (\alpha+1)(\alpha+4)/2$, $\mu_1 = -1$, $\mu_2 = -1$	$-(\alpha^2+5\alpha+5)/2$
4	$\lambda_0 = -(\alpha+1)(\alpha+2)^2(\alpha+5)/6$, $\lambda_1 = (\alpha+2)(\alpha^2+7\alpha+8)/6$, $\lambda_2 = 2/3$, $\lambda_4 = -1$	$\mu_0 = (\alpha+1)(\alpha^2+8\alpha+14)/6$, $\mu_1 = -1/3$, $\mu_2 = -1$, $\mu_3 = -1$	$-(\alpha+4) \cdot$ $\cdot (\alpha^2+8\alpha+14)/6$

5	$\lambda_0 = -(\alpha+1)(\alpha+2)(\alpha^3+12\alpha^2+42\alpha+29)/24$, $\lambda_1 = (\alpha+2)(\alpha^3+12\alpha^2+40\alpha+7)/24$ $\lambda_2 = -(\alpha-13)/12$, $\lambda_3 = 3/4$, $\lambda_5 = -1$ $\mu_0 = (\alpha+1)(\alpha^3+13\alpha^2+51\alpha+34)/24$, $\mu_1 = -(\alpha-10)/12$ $\mu_2 = -1/4$, $\mu_3 = -1$, $\mu_4 = -1$ $\gamma = -(\alpha^4+14\alpha^3+65\alpha^2+100\alpha+9)/24$
6	$\lambda_0 = -(\alpha+1)(\alpha+2)(\alpha^4+18\alpha^3+110\alpha^2+229\alpha+14)/120$ $\lambda_1 = (\alpha+2)(\alpha^4+18\alpha^3+108\alpha^2+195\alpha-158)/120$, $\lambda_2 = -(\alpha^2+17\alpha-46)/60$ $\lambda_3 = -(\alpha-26)/20$, $\lambda_4 = 4/5$, $\lambda_6 = -1$ $\mu_0 = (\alpha+1)(\alpha^4+19\alpha^3+125\alpha^2+284\alpha-92)/120$, $\mu_1 = -(\alpha^2+20\alpha-112)/60$ $\mu_2 = -(\alpha-22)/20$, $\mu_3 = -1/5$, $\mu_4 = -1$, $\mu_5 = -1$ $\gamma = -(\alpha+6)(\alpha^4+14\alpha^3+61\alpha^2+64\alpha-35)/120$
7	$\lambda_0 = -(\alpha+1)(\alpha+2)(\alpha^5+25\alpha^4+231\alpha^3+899\alpha^2+1007\alpha-807)/720$ $\lambda_1 = (\alpha+2)(\alpha^5+25\alpha^4+229\alpha^3+851\alpha^2+607\alpha-1821)/720$ $\lambda_2 = -(\alpha^3+24\alpha^2+188\alpha+183)/360$, $\lambda_3 = -(\alpha^2+21\alpha-127)/120$ $\lambda_4 = -(\alpha-43)/30$, $\lambda_5 = 5/6$, $\lambda_7 = -1$ $\mu_0 = (\alpha+1)(\alpha^5+26\alpha^4+253\alpha^3+1054\alpha^2+1201\alpha-2334)/720$ $\mu_1 = -(\alpha^3+27\alpha^2+263-654)/360$, $\mu_2 = -(\alpha^2+25\alpha-279)/120$ $\mu_3 = -(\alpha-38)/30$, $\mu_4 = -1/6$, $\mu_5 = -1$, $\mu_6 = -1$ $\gamma = -(\alpha^6+27\alpha^5+280\alpha^4+1335\alpha^3+2554\alpha^2+213\alpha-2115)/720$
8	$\lambda_0 = -(\alpha+1)(\alpha+2)(\alpha^6+33\alpha^5+425\alpha^4+2579\alpha^3+6579\alpha^2+1675\alpha-7800)/5040$ $\lambda_1 = (\alpha+2)(\alpha^6+33\alpha^5+423\alpha^4+2515\alpha^3+5807\alpha^2-2215\alpha-10200)/5040$ $\lambda_2 = -(\alpha^4+32\alpha^3+374\alpha^2+1525\alpha+5880)/2520$, $\lambda_3 = -(\alpha^3+29\alpha^2+275\alpha+280)/840$ $\lambda_4 = -(\alpha^2+25\alpha-260)/210$, $\lambda_5 = -(\alpha-64)/42$, $\lambda_6 = 6/7$, $\lambda_8 = -1$ $\mu_0 = (\alpha+1)(\alpha^6+34\alpha^5+455\alpha^4+2904\alpha^3+7863\alpha^2+590\alpha-20640)/5040$ $\mu_1 = -(\alpha^4+35\alpha^3+473\alpha^2+2710\alpha+120)/2520$, $\mu_2 = -(\alpha^3+33\alpha^2+395\alpha-1920)/840$ $\mu_3 = -(\alpha^2+30\alpha-550)/210$, $\mu_4 = -(\alpha-58)/42$, $\mu_5 = -1/7$, $\mu_6 = -1$, $\mu_7 = -1$ $\gamma = -(\alpha+8)(\alpha^6+27\alpha^5+274\alpha^4+1203\alpha^3+1660\alpha^2-1275\alpha-945)/5040$
9	$\lambda_0 = -(1+\alpha)(\alpha+2)(\alpha^7+42\alpha^6+715\alpha^5+6145\alpha^4+26241\alpha^3+40313\alpha^2-26570\alpha-$ $-7347)/40320$ $\lambda_1 = (\alpha+2)(\alpha^7+42\alpha^6+713\alpha^5+6063\alpha^4+24907\alpha^3+29979\alpha^2-57484\alpha+42159)/40320$ $\lambda_2 = -(\alpha^5+41\alpha^4+655\alpha^3+4639\alpha^2+6361\alpha+71043)/20160$ $\lambda_3 = -(\alpha^4+38\alpha^3+529\alpha^2+2524\alpha+16933)/6720$, $\lambda_4 = -(\alpha^3+34\alpha^2+378\alpha+397)/1680$ $\lambda_5 = -(\alpha^2+29\alpha-457)/336$, $\lambda_6 = -(\alpha-89)/56$, $\lambda_7 = 7/8$, $\lambda_9 = -1$ $\mu_0 = (\alpha+1)(\alpha^7+43\alpha^6+754\alpha^5+6733\alpha^4+30212\alpha^3+48019\alpha^2-58933\alpha-55014)/40320$ $\mu_1 = -(\alpha^5+44\alpha^4+781\alpha^3+6694\alpha^2+20569\alpha+69666)/20160$ $\mu_2 = -(\alpha^4+42\alpha^3+685\alpha^2+4736\alpha-459)/6720$, $\mu_3 = -(\alpha^3+39\alpha^2+553\alpha-4348)/1680$ $\mu_4 = -(\alpha^2+35\alpha-949)/336$, $\mu_5 = -(\alpha-82)/56$, $\mu_6 = -1/8$, $\mu_7 = -1$, $\mu_8 = -1$ $\gamma = -(\alpha^8+44\alpha^7+798\alpha^6+7532\alpha^5+37779\alpha^4+86156\alpha^3+25682\alpha^2-89952\alpha+$ $+131985)/40320$

n	λ_k , μ_k , γ
10	$\lambda_0 = -(\alpha+1)(\alpha+2)(\alpha^8+52\alpha^7+1127\alpha^6+12919\alpha^5+80819\alpha^4+244123\alpha^3+176200\alpha^2-$ $-203679\alpha+1145106)/362880$ $\lambda_1 = (\alpha+2)(\alpha^8+52\alpha^7+1125\alpha^6+12817\alpha^5+78681\alpha^4+221185\alpha^3+54842\alpha^2-364545\alpha+$ $+2157318)/362880$ $\lambda_2 = -(\alpha^6+51\alpha^5+1057\alpha^4+10821\alpha^3+46399\alpha2-71307\alpha+458766)/181440$ $\lambda_3 = -(\alpha^5+48\alpha^4+901\alpha^3+7470\alpha^2+10069\alpha+252546)/60480$ $\lambda_4 = -(\alpha^4+44\alpha^3+710^2\alpha^2+3865\alpha+40434)/15120$, $\lambda_5 = -(\alpha^3+39\alpha^2+497\alpha+534)/3024$ $\lambda_6 = -(\alpha^2+33\alpha-730)/504$, $\lambda_7 = -(\alpha-118)/72$, $\lambda_8 = 8/9$, $\lambda_{10} = -1$ $\mu_0 = (\alpha+1)(\alpha^8+53\alpha^7+1176\alpha^6+13889\alpha^5+90318\alpha^4+285803\alpha^3+183185\alpha^2-598104\alpha+$ $+1927332)/362880$ $\mu_1 = -(\alpha^6+54\alpha^5+1213\alpha^4+14112\alpha^3+80029\alpha^2+49500\alpha+1193652)/181440$ $\mu_2 = -(\alpha^5+52\alpha^4+1097\alpha^3+11210\alpha^2+40269\alpha+244962)/60480$ $\mu_3 = -(\alpha^4+49\alpha^3+935\alpha^2+7550\alpha-1896)/15120$, $\mu_4 = -(\alpha^3+45\alpha^2+737\alpha-8466)/3024$ $\mu_5 = -(aa^2+40\alpha-1500)/504$, $\mu_6 = -(\alpha-110)/72$, $\mu_7 = -1/9$, $\mu_8 = -1$, $\mu_9 = -1$ $\gamma = -(\alpha+10)(\alpha^8+44\alpha^7+790\alpha^6+7220\alpha^5+33283\alpha^4+59156\alpha^3-14670\alpha^2+45720\alpha+$ $+353241)/362880$

$Z_\nu(x) = I_1(x)$, $K_1(x)$ with s defined as on page 195:

n	λ_k	μ_k	γ
1	$\lambda_0 = s(\alpha+1)$	$-$	s
2	$\lambda_0 = s(\alpha+1)(\alpha+2)$, $\lambda_1 = -s(\alpha+2)$, $\lambda_2 = s$	$\mu_0 = -(\alpha+1)$, $\mu_1 = 1$	$s(\alpha+2)$
3	$\lambda_0 = s(\alpha+1)(\alpha+2)(\alpha+3)/2$, $\lambda_1 = -s(\alpha+2)(\alpha+3)/2$, $\lambda_3 = s$	$\mu_0 = -(\alpha+1)(\alpha+4)/2$, $\mu_1 = 1$, $\mu_2 = 1$	$s(\alpha^2+5\alpha+7)/2$
4	$\lambda_0 = s(\alpha+1)(\alpha+2)(\alpha^2+7\alpha+14)/6$ $\lambda_1 = -s(\alpha+2)(\alpha^2+7\alpha+16)/6$, $\lambda_2 = 2s/3$, $\lambda_4 = s$	$\mu_0 = -(\alpha+1)(\alpha^2+8\alpha+22)/6$, $\mu_1 = 5/3$, $\mu_2 = 1$, $\mu_3 = 1$	$s(\alpha+4)\cdot$ $\cdot(\alpha^2+5\alpha+9)/6$
5	$\lambda_0 = s(\alpha+1)(\alpha+2)(\alpha^3+12\alpha^2+52\alpha+91)/24$, $\lambda_1 = -s(\alpha+2)(\alpha^3+12\alpha^2+54\alpha+113)/24$ $\lambda_2 = -s(\alpha-13)/12$, $\lambda_3 = 3s/4$, $\lambda_5 = s$ $\mu_0 = -(\alpha+1)(\alpha^3+13\alpha^2+65\alpha+158)/24$, $\mu_1 = -(\alpha-34)/12$ $\mu_2 = 7/4$, $\mu_3 = 1$, $\mu_4 = 1$ $\gamma = s(\alpha^4+14\alpha^3+77\alpha^2+208\alpha+249)/24$		
6	$\lambda_0 = s(\alpha+1)(\alpha+2)(\alpha^4+18\alpha^3+128\alpha^2+455\alpha+770)/120$ $\lambda_1 = -s(\alpha+2)(\alpha^4+18\alpha^3+130\alpha^2+489\alpha+1006)/120$ $\lambda_2 = -s(\alpha+22)(\alpha-5)/60$, $\lambda_3 = -s(\alpha-26)/20$, $\lambda_4 = 4s/5$, $\lambda_6 = s$ $\mu_0 = -(\alpha+1)(\alpha^4+19\alpha^3+147\alpha^2+604\alpha+1420)/120$, $\mu_1 = -(\alpha^2+20\alpha-296)/60$ $\mu_2 = -(\alpha-62)/20$, $\mu_3 = 9/5$, $\mu_4 = 1$, $\mu_5 = 1$ $\gamma = s(\alpha+6)(\alpha^4+14\alpha^3+81\alpha^2+244\alpha+365)/120$		
7	$\lambda_0 = s(\alpha+1)(\alpha+2)(\alpha^5+25\alpha^4+259\alpha^3+1451\alpha^2+4735\alpha+8085)/720$ $\lambda_1 = -s(\alpha+2)(\alpha^2+13\alpha+57)(\alpha^3+12\alpha^2+48\alpha+191)/720$ $\lambda_2 = -s(\alpha^3+24\alpha^2+254\alpha-1155)/360$, $\lambda_3 = -s(\alpha^2+21\alpha-277)/120$ $\lambda_4 = -s(\alpha-43)/30$, $\lambda_5 = 5s/6$, $\lambda_7 = s$ $\mu_0 = -(\alpha+1)(\alpha^5+26\alpha^4+285\alpha^3+1738\alpha^2+6533\alpha+15450)/720$ $\mu_1 = -(\alpha-6)(\alpha^2+33\alpha+527)/360$, $\mu_2 = -(\alpha^2+25\alpha-669)/120$ $\mu_3 = -(\alpha-98)/30$, $\mu_4 = 11/6$, $\mu_5 = 1$, $\mu_6 = 1$ $\gamma = s(\alpha^6+27\alpha^5+310\alpha^4+1995\alpha^3+7924\alpha^2+19353\alpha+23535)/720$		
8	$\lambda_0 = s(\alpha+1)(\alpha+2)(\alpha^6+33\alpha^5+465\alpha^4+3691\alpha^3+18311\alpha^2+58055\alpha+101640)/5040$ $\lambda_1 = -s(\alpha+2)(\alpha^6+33\alpha^5+467\alpha^4+3755\alpha^3+19191\alpha^2+65053\alpha+139272)/5040$ $\lambda_2 = -s(\alpha^4+32\alpha^3+452\alpha^2+3919\alpha-14520)/2520$, $\lambda_3 = -s(\alpha^3+29\alpha^2+377\alpha-3512)/840$ $\lambda_4 = -s(\alpha^2+25\alpha-548)/210$, $\lambda_5 = -s(\alpha-64)/42$, $\lambda_6 = 6s/7$, $\lambda_8 = s$ $\mu_0 = -(\alpha+1)(\alpha^6+34\alpha^5+499\alpha^4+4192\alpha^3+22579\alpha^2+81958\alpha+198240)/5040$ $\mu_1 = -(\alpha^4+35\alpha^3+551\alpha^2+5410\alpha-40152)/2520$, $\mu_2 = -(\alpha^3+33\alpha^2+497\alpha-8544)/840$ $\mu_3 = -(\alpha^2+30\alpha-1258)/210$, $\mu_4 = -(\alpha-142)/42$, $\mu_5 = 13/7$, $\mu_6 = 1$, $\mu_7 = 1$ $\gamma = s(\alpha+8)(\alpha^6+27\alpha^5+316\alpha^4+2127\alpha^3+9178\alpha^2+25521\alpha+37485)/5040$		

n	λ_k , μ_k , γ
9	$\lambda_0 = s(\alpha + 1)(\alpha + 2)(\alpha^7 + 42\,\alpha^6 + 769\,\alpha^5 + 8135\,\alpha^4 + 55695\,\alpha^3 + 260563\,\alpha^2 + 826540\,\alpha + \\ +1488795)/40320$ $\lambda_1 = -s(\alpha + 2)(\alpha^7 + 42\,\alpha^6 + 771\,\alpha^5 + 8217\,\alpha^4 + 57165\,\alpha^3 + 275985\,\alpha^2 + 935078\,\alpha + \\ +2060121)/40320$ $\lambda_2 = -s(\alpha^5 + 41\,\alpha^4 + 747\,\alpha^3 + 8239\,\alpha^2 + 65117\,\alpha - 212685)/20160$ $\lambda_3 = -s(\alpha^4 + 38\,\alpha^3 + 645\,\alpha^2 + 6832\,\alpha - 51691)/6720$ $\lambda_4 = -s(\alpha^3 + 34\,\alpha^2 + 524\,\alpha - 8089)/1680$, $\lambda_5 = -s(\alpha^2 + 29\,\alpha - 947)/336$ $\lambda_6 = -s(\alpha - 89)/56$, $\lambda_7 = 7s/8$, $\lambda_9 = s$ $\mu_0 = -(\alpha + 1)(\alpha^7 + 43\,\alpha^6 + 812\,\alpha^5 + 8949\,\alpha^4 + 64738\,\alpha^3 + 327323\,\alpha^2 + 1181009\,\alpha + \\ +2937270)/40320$ $\mu_1 = -(\alpha^5 + 44\,\alpha^4 + 873\,\alpha^3 + 10642\,\alpha^2 + 94001\,\alpha + 591486)/20160$ $\mu_2 = -(\alpha^4 + 42\,\alpha^3 + 801\,\alpha^2 + 9628\,\alpha - 126267)/6720$ $\mu_3 = -(\alpha^3 + 39\,\alpha^2 + 699\,\alpha - 18644)/1680$, $\mu_4 = -(\alpha^2 + 35\,\alpha - 2111)/336$ $\mu_5 = -(\alpha - 194)/56$, $\mu_6 = 15/8$, $\mu_7 = 1$, $\mu_8 = 1$ $\gamma = s(\alpha^8 + 44\,\alpha^7 + 854\,\alpha^6 + 9716\,\alpha^5 + 72779\,\alpha^4 + 380996\,\alpha^3 + 1414426\,\alpha^2 + 3496344\,\alpha + \\ +4426065)/40320$
10	$\lambda_0 = s(\alpha + 1)(\alpha + 2)(\alpha^8 + 52\,\alpha^7 + 1197\,\alpha^6 + 16201\,\alpha^5 + 144909\,\alpha^4 + 913681\,\alpha^3 + 4182002\,\alpha^2 + \\ +13457115\,\alpha + 24885630)/362880$ $\lambda_1 = -s(\alpha + 2)(\alpha^8 + 52\,\alpha^7 + 1199\,\alpha^6 + 16303\,\alpha^5 + 147215\,\alpha^4 + 944539\,\alpha^3 + 4459864\,\alpha^2 + \\ +15296589\,\alpha + 34622154)/362880$ $\lambda_2 = -s(\alpha^6 + 51\,\alpha^5 + 1165\,\alpha^4 + 16077\,\alpha^3 + 155155\,\alpha^2 + 1179693\,\alpha - 3555090)/181440$ $\lambda_3 = -s(\alpha^5 + 48\,\alpha^4 + 1033\,\alpha^3 + 13626\,\alpha^2 + 130861\,\alpha - 866334)/60480$ $\lambda_4 = -s(\alpha^4 + 44\,\alpha^3 + 872\,\alpha^2 + 10903\,\alpha - 135786)/15120$ $\lambda_5 = -s(\alpha^3 + 39\,\alpha^2 + 695\,\alpha - 15918)/3024$, $\lambda_6 = -s(\alpha^2 + 33\,\alpha - 1498)/506$ $\lambda_7 = -s(\alpha - 118)/72$, $\lambda_8 = 8s/9$, $\lambda_{10} = s$ $\mu_0 = -(\alpha + 1)(\alpha^8 + 53\,\alpha^7 + 1250\,\alpha^6 + 17453\,\alpha^5 + 162476\,\alpha^4 + 1079135\,\alpha^3 + 5309335\,\alpha^2 + \\ +19325376\,\alpha + 49408380)/362880$ $\mu_1 = -(\alpha^6 + 54\,\alpha^5 + 1321\,\alpha^4 + 19764\,\alpha^3 + 209197\,\alpha^2 + 1759212\,\alpha - 9917964)/181440$ $\mu_2 = -(\alpha^5 + 52\,\alpha^4 + 1229\,\alpha^3 + 18014\,\alpha^2 + 193173\,\alpha - 2120958)/60480$ $\mu_3 = -(\alpha^4 + 49\,\alpha^3 + 1097\,\alpha^2 + 15578\,\alpha - 313656)/15120$ $\mu_4 = -(\alpha^3 + 45\,\alpha^2 + 935\,\alpha - 35574)/3024$, $\mu_5 = -(\alpha^2 + 40\,\alpha - 3276)/504$ $\mu_6 = -(\alpha - 254)/72$, $\mu_7 = 17/9$, $\mu_8 = 1$, $\mu_9 = 1$ $\gamma = s(\alpha + 10)(\alpha^8 + 44\,\alpha^7 + 862\,\alpha^6 + 10028\,\alpha^5 + 78283\,\alpha^4 + 438236\,\alpha^3 + 1792458\,\alpha^2 + \\ +5023872\,\alpha + 7429401)/362880$

g) Hermite Polynomials $H_n(x)$:

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\,dx \quad = \quad \sqrt{\pi} \cdot 2^n\, n! \quad = \quad \sqrt{\pi} \cdot (2n)!!$$

First polynomials:

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x, \quad H_4(x) = 16x^4 - 48x^2 + 12$$

$Z_\nu(x) = J_0(x):$

n	λ_k	μ_k	γ
1	$-$	$\mu_1 = 1$,	0
2	$\lambda_1 = -1$,	$\mu_0 = 2$, $\mu_2 = 1$	-6
3	$\lambda_0 = 8$, $\lambda_2 = 4$	$\mu_1 = -16$, $\mu_3 = 1$	0
4	$\lambda_1 = 42$, $\lambda_3 = 6$	$\mu_0 = -84$, $\mu_2 = -36$, $\mu_4 = 1$	204
5	$\lambda_0 = -576$, $\lambda_2 = -240$, $\lambda_4 = 8$	$\mu_1 = 960$, $\mu_3 = -64$, $\mu_5 = 1$	0
6	$\lambda_1 = -4140$, $\lambda_3 = -580$, $\lambda_5 = 10$	$\mu_0 = 8280$, $\mu_2 = 3480$, $\mu_4 = -100$, $\mu_6 = 1$	-19560

n	λ_k	μ_k	γ
7	$\lambda_0 = 82176$, $\lambda_2 = 33600$, $\lambda_4 = -1128$, $\lambda_6 = 12$	$\mu_1 = -134400$, $\mu_3 = 9024$, $\mu_5 = -144$, $\mu_7 = 1$	0
8	$\lambda_1 = 798840$, $\lambda_3 = 111720$, $\lambda_5 = -1932$, $\lambda_7 = 14$	$\mu_0 = -1597680$, $\mu_2 = -670320$, $\mu_4 = 19320$, $\mu_6 = -196$, $\mu_8 = 1$	3764880
9	$\lambda_0 = -20803584$, $\lambda_2 = -8494080$, $\lambda_4 = 285312$, $\lambda_6 = -3040$, $\lambda_8 = 16$	$\mu_1 = 33976320$, $\mu_3 = -2282496$, $\mu_5 = 36480$, $\mu_7 = -256$, $\mu_9 = 1$	0
10	$\lambda_1 = -256510800$, $\lambda_3 = -35869680$, $\lambda_5 = 620424$, $\lambda_7 = -4500$, $\lambda_9 = 18$	$\mu_0 = 513021600$, $\mu_2 = 215218080$, $\mu_4 = -6204240$, $\mu_6 = 63000$, $\mu_8 = -324$, $\mu_{10} = 1$	-1208723040

$Z_\nu(x) = I_0(x):$

n	λ_k	μ_k	γ
1	$-$	$\mu_1 = 1$,	0
2	$\lambda_1 = -1$,	$\mu_0 = 2$, $\mu_2 = 1$	2
3	$\lambda_0 = -8$, $\lambda_2 = -4$	$\mu_1 = 16$, $\mu_3 = 1$	0
4	$\lambda_1 = -30$, $\lambda_3 = -6$	$\mu_0 = 60$, $\mu_2 = 36$, $\mu_4 = 1$	108
5	$\lambda_0 = -448$, $\lambda_2 = -272$, $\lambda_4 = -8$	$\mu_1 = 1088$, $\mu_3 = 64$, $\mu_5 = 1$	0
6	$\lambda_1 = -3180$, $\lambda_3 = -620$, $\lambda_5 = -10$	$\mu_0 = 6360$, $\mu_2 = 3720$, $\mu_4 = 100$, $\mu_6 = 1$	10680
7	$\lambda_0 = -66816$, $\lambda_2 = -66816$, $\lambda_4 = -1176$, $\lambda_6 = -12$	$\mu_1 = 158976$, $\mu_3 = 9408$, $\mu_5 = 144$, $\mu_7 = 1$	0
8	$\lambda_1 = -629160$, $\lambda_3 = -122920$, $\lambda_5 = -1988$, $\lambda_7 = -14$	$\mu_0 = 1258320$, $\mu_2 = 737520$, $\mu_4 = 19880$, $\mu_6 = 196$, $\mu_8 = 1$	2125200
9	$\lambda_0 = -17240064$, $\lambda_2 = -17240064$, $\lambda_4 = -303744$, $\lambda_6 = -3104$, $\lambda_8 = -16$	$\mu_1 = 41078784$, $\mu_3 = 2429952$, $\mu_5 = 37248$, $\mu_7 = 256$, $\mu_9 = 1$	0
10	$\lambda_1 = -205344720$, $\lambda_3 = -40113360$, $\lambda_5 = -648648$, $\lambda_7 = -4572$, $\lambda_9 = -18$	$\mu_0 = 410689440$, $\mu_2 = 240680160$, $\mu_4 = 6486480$, $\mu_6 = 64008$, $\mu_8 = 324$, $\mu_{10} = 1$	693372960

$Z_\nu(x) = K_0(x):$

n	λ_k	μ_k	γ
1	$-$	$\mu_1 = -1$,	0
2	$\lambda_1 = -1$,	$\mu_0 = -2$, $\mu_2 = -1$	2
3	$\lambda_0 = -8$, $\lambda_2 = -4$	$\mu_1 = -16$, $\mu_3 = -1$	0
4	$\lambda_1 = -30$, $\lambda_3 = -6$	$\mu_0 = -60$, $\mu_2 = -36$, $\mu_4 = -1$	108
5	$\lambda_0 = -448$, $\lambda_2 = -272$, $\lambda_4 = -8$	$\mu_1 = -1088$, $\mu_3 = -64$, $\mu_5 = -1$	0
6	$\lambda_1 = -3180$, $\lambda_3 = -620$, $\lambda_5 = -10$	$\mu_0 = -6360$, $\mu_2 = -3720$, $\mu_4 = -100$, $\mu_6 = -1$	10680
7	$\lambda_0 = -66816$, $\lambda_2 = -66816$, $\lambda_4 = -1176$, $\lambda_6 = -12$	$\mu_1 = -158976$, $\mu_3 = -9408$, $\mu_5 = -144$, $\mu_7 = -1$	0
8	$\lambda_1 = -629160$, $\lambda_3 = -122920$, $\lambda_5 = -1988$, $\lambda_7 = -14$	$\mu_0 = -1258320$, $\mu_2 = -737520$, $\mu_4 = -19880$, $\mu_6 = -196$, $\mu_8 = -1$	2125200
9	$\lambda_0 = -17240064$, $\lambda_2 = -17240064$, $\lambda_4 = -303744$, $\lambda_6 = -3104$, $\lambda_8 = -16$	$\mu_1 = -41078784$, $\mu_3 = -2429952$, $\mu_5 = -37248$, $\mu_7 = 2 - 56$, $\mu_9 = -1$	0
10	$\lambda_1 = -205344720$, $\lambda_3 = -40113360$, $\lambda_5 = -648648$, $\lambda_7 = -4572$, $\lambda_9 = -18$	$\mu_0 = -410689440$, $\mu_2 = -240680160$, $\mu_4 = -6486480$, $\mu_6 = -64008$, $\mu_8 = -324$, $\mu_{10} = -1$	693372960

$Z_\nu(x) \; = \; J_1(x):$

n	λ_k	μ_k	γ
1	$-$	$-$	2
2	$\lambda_2 = -1$	$\mu_1 = 4$	0
3	$\lambda_1 = -6,\ \lambda_3 = -1$	$\mu_0 = 12,\ \mu_2 = 6$	-36
4	$\lambda_0 = 64,\ \lambda_2 = 32,\ \lambda_4 = -1$	$\mu_1 = -128,\ \mu_3 = 8$	0
5	$\lambda_1 = 420,\ \lambda_3 = 60,\ \lambda_5 = -1$	$\mu_0 = -840,\ \mu_2 = -360,\ \mu_4 = 10$	2040
6	$\lambda_0 = -6912,\ \lambda_2 = -2880,\ \lambda_4 = 96,$ $\lambda_6 = -1$	$\mu_1 = 11520,\ \mu_3 = -768,\ \mu_5 = 12$	0
7	$\lambda_1 = -57960,\ \lambda_3 = -8120,$ $\lambda_5 = 140,\ \lambda_7 = -1$	$\mu_0 = 115920,\ \mu_2 = 48720,$ $\mu_4 = -1400,\ \mu_6 = 14$	-273840
8	$\lambda_0 = 1314816,\ \lambda_2 = 537600,$ $\lambda_4 = -18048,\ \lambda_6 = 192,\ \lambda_8 = -1$	$\mu_1 = -2150400,\ \mu_3 = 144384,$ $\mu_5 = -2304,\ \mu_7 = 16$	0
9	$\lambda_1 = 14379120,\ \lambda_3 = 2010960,$ $\lambda_5 = -34776,\ \lambda_7 = 252,\ \lambda_9 = -1$	$\mu_0 = -28758240,\ \mu_2 = -12065760,$ $\mu_4 = 347760,\ \mu_6 = -3528,\ \mu_8 = 18$	67767840
10	$\lambda_0 = -416071680,\ \lambda_2 = -169881600,$ $\lambda_4 = 5706240,\ \lambda_6 = -60800,$ $\lambda_8 = 320,\ \lambda_{10} = -1$	$\mu_1 = 679526400,\ \mu_3 = -45649920,$ $\mu_5 = 729600,\ \mu_7 = -5120,$ $\mu_9 = 20$	0

$Z_\nu(x) \; = \; I_1(x):$

n	λ_k	μ_k	γ
1	$-$	$-$	-2
2	$\lambda_2 = 1$	$\mu_1 = -4$	0
3	$\lambda_1 = 6,\ \lambda_3 = 1$	$\mu_0 = -12,\ \mu_2 = -6$	-12
4	$\lambda_0 = 64,\ \lambda_2 = 32,\ \lambda_4 = 1$	$\mu_1 = -128,\ \mu_3 = -8$	0
5	$\lambda_1 = 300,\ \lambda_3 = 60,\ \lambda_5 = 1$	$\mu_0 = -600,\ \mu_2 = -360,\ \mu_4 = -10$	-1080
6	$\lambda_0 = 5376,\ \lambda_2 = 3264,\ \lambda_4 = 96,$ $\lambda_6 = 1$	$\mu_1 = -13056,\ \mu_3 = -768,\ \mu_5 = -12$	0
7	$\lambda_1 = 44520,\ \lambda_3 = 8680,$ $\lambda_5 = 140,\ \lambda_7 = 1$	$\mu_0 = 8680,\ \mu_2 = -52080,$ $\mu_4 = -1400,\ \mu_6 = -14$	-149520
8	$\lambda_0 = 1069056,\ \lambda_2 = 635904,$ $\lambda_4 = 18816,\ \lambda_6 = 192,\ \lambda_8 = 1$	$\mu_1 = -2543616,\ \mu_3 = -150528,$ $\mu_5 = -2304,\ \mu_7 = -16$	0
9	$\lambda_1 = 11324880,\ \lambda_3 = 2212560,$ $\lambda_5 = 35784,\ \lambda_7 = 252,\ \lambda_9 = 1$	$\mu_0 = -22649760,\ \mu_2 = -13275360,$ $\mu_4 = -357840,\ \mu_6 = -3528,\ \mu_8 = -18$	-38253600
10	$\lambda_0 = 344801280,\ \lambda_2 = 205393920,$ $\lambda_4 = 6074880,\ \lambda_6 = 62080,$ $\lambda_8 = 320,\ \lambda_{10} = 1$	$\mu_1 = -821575680,\ \mu_3 = -48599040,$ $\mu_5 = -744960,\ \mu_7 = -5120,$ $\mu_9 = -20$	0

$Z_\nu(x) \,=\, K_1(x)\,:$

n	λ_k	μ_k	γ
1	$-$	$-$	2
2	$\lambda_2 = -1$	$\mu_1 = -4$	0
3	$\lambda_1 = -6\,,\ \lambda_3 = -1$	$\mu_0 = -12\,,\ \mu_2 = -6$	12
4	$\lambda_0 = -64\,,\ \lambda_2 = -32\,,\ \lambda_4 = -1$	$\mu_1 = -128\,,\ \mu_3 = -8$	0
5	$\lambda_1 = -300\,,\ \lambda_3 = -60\,,\ \lambda_5 = -1$	$\mu_0 = -600\,,\ \mu_2 = -360\,,\ \mu_4 = -10$	1080
6	$\lambda_0 = -5376\,,\ \lambda_2 = -3264\,,\ \lambda_4 = -96\,,$ $\lambda_6 = 1$	$\mu_1 = -13056\,,\ \mu_3 = -768\,,\ \mu_5 = -12$	0
7	$\lambda_1 = -44520\,,\ \lambda_3 = -8680\,,$ $\lambda_5 = -140\,,\ \lambda_7 = -1$	$\mu_0 = -8680\,,\ \mu_2 = -52080\,,$ $\mu_4 = -1400\,,\ \mu_6 = -14$	149520
8	$\lambda_0 = -1069056\,,\ \lambda_2 = -635904\,,$ $\lambda_4 = -18816\,,\ \lambda_6 = -192\,,\ \lambda_8 = -1$	$\mu_1 = -2543616\,,\ \mu_3 = -150528\,,$ $\mu_5 = -2304\,,\ \mu_7 = -16$	0
9	$\lambda_1 = -11324880\,,\ \lambda_3 = -2212560\,,$ $\lambda_5 = -35784\,,\ \lambda_7 = -252\,,\ \lambda_9 = -1$	$\mu_0 = -22649760\,,\ \mu_2 = -13275360\,,$ $\mu_4 = -357840\,,\ \mu_6 = -3528\,,\ \mu_8 = -18$	38253600
10	$\lambda_0 = -344801280\,,\ \lambda_2 = -205393920\,,$ $\lambda_4 = -6074880\,,\ \lambda_6 = -62080\,,$ $\lambda_8 = -320\,,\ \lambda_{10} = -1$	$\mu_1 = -821575680\,,\ \mu_3 = -48599040\,,$ $\mu_5 = -744960\,,\ \mu_7 = -5120\,,$ $\mu_9 = -20$	0

1.3.2. Integrals of the type $\int x^n \operatorname{Ei}(x) \cdot Z_\nu(x)\,dx$

About $\operatorname{Ei}(x)$ see [1], 5.1., or [7], 8.2. In [4], page 657, is no reference to the fact, that the integral should be used as principal value.

$$\int x\,\operatorname{Ei}(x)\,I_0(x)\,dx = x\operatorname{Ei}(x)I_1(x) + e^x\left[(x-1)I_0(x) - xI_1(x)\right]$$

$$\int x\,\operatorname{Ei}(x)\,K_0(x)\,dx = -x\operatorname{Ei}(x)K_1(x) + e^x\left[(x-1)K_0(x) + xK_1(x)\right]$$

$$\int x^2\,\operatorname{Ei}(x)\,I_1(x)\,dx = \operatorname{Ei}(x)[x^2 I_0(x) - 2xI_1(x)] + \frac{e^x}{3}\left[(-x^2 - 6\,x + 6)I_0(x) + (x^2 + 5\,x)I_1(x)\right]$$

$$\int x^2\,\operatorname{Ei}(x)\,K_1(x)\,dx = -\operatorname{Ei}(x)[x^2 K_0(x) + 2xK_1(x)] + \frac{e^x}{3}\left[(x^2 + 6\,x - 6)K_0(x) + (x^2 + 5\,x)K_1(x)\right]$$

$$\int x^3\,\operatorname{Ei}(x)\,I_0(x)\,dx = \operatorname{Ei}(x)[-2x^2 I_0(x) + (x^3 + 4\,x)I_1(x)] +$$

$$+\frac{e^x}{15}\left[(3\,x^3 + 7\,x^2 + 60\,x - 60)I_0(x) - (3\,x^3 + 16\,x^2 + 44\,x)I_1(x)\right]$$

$$\int x^3\,\operatorname{Ei}(x)\,K_0(x)\,dx = -\operatorname{Ei}(x)[2x^2 K_0(x) + (x^3 + 4x)K_1(x)] +$$

$$+\frac{e^x}{15}\left[(3\,x^3 + 7\,x^2 + 60\,x - 60)K_0(x) + (3\,x^3 + 16\,x^2 + 44\,x)K_1(x)\right]$$

$$\int x^4\,\operatorname{Ei}(x)\,I_1(x)\,dx = \operatorname{Ei}(x)\left[(x^4 + 8\,x^2)I_0(x) - (16\,x + 4\,x^3)I_1(x)\right] +$$

$$+\frac{e^x}{105}\left[-(15\,x^4 + 102\,x^3 + 178\,x^2 + 1680\,x - 1680)I_0(x) + (15\,x^4 + 57\,x^3 + 484\,x^2 + 1196\,x)I_1(x)\right]$$

$$\int x^4\,\operatorname{Ei}(x)\,K_1(x)\,dx = -\operatorname{Ei}(x)\left[(x^4 + 8\,x^2)K_0(x) + (4\,x^3 + 16\,x)K_1(x)\right] +$$

$$+\frac{e^x}{105}\left[(15\,x^4 + 102\,x^3 + 178\,x^2 + 1680\,x - 1680)K_0(x) + (15\,x^4 + 57\,x^3 + 484\,x^2 + 1196\,x)K_1(x)\right]$$

$$\int x^5\,\operatorname{Ei}(x)\,I_0(x)\,dx = \operatorname{Ei}(x)\left[-(4\,x^4 + 32\,x^2)I_0(x) + (x^5 + 16\,x^3 + 64\,x)I_1(x)\right] +$$

$$+\frac{e^x}{45}\left[(5\,x^5 + 15\,x^4 + 192\,x^3 + 288\,x^2 + 2880\,x - 2880)\,I_0(x) - (5\,x^5 + 40\,x^4 + 72\,x^3 + 864\,x^2 + 2016\,x)I_1(x)\right]$$

$$\int x^5\,\operatorname{Ei}(x)\,K_0(x)\,dx = -\operatorname{Ei}(x)\left[(4\,x^4 + 32\,x^2)K_0(x) + (x^5 + 16\,x^3 + 64\,x)K_1(x)\right] +$$

$$+\frac{e^x}{45}\left[(5\,x^5 + 15\,x^4 + 192\,x^3 + 288\,x^2 + 2880\,x - 2880)K_0(x) + (5\,x^5 + 40\,x^4 + 72\,x^3 + 864\,x^2 + 2016\,x)K_1(x)\right]$$

$$\int x^6\,\operatorname{Ei}(x)\,I_1(x)\,dx = \operatorname{Ei}(x)\left[(x^6 + 24\,x^4 + 192\,x^2)I_0(x) - (384\,x + 96\,x^3 + 6\,x^5)I_1(x)\right] +$$

$$+\frac{e^x}{3465}\left[-(315\,x^6 + 3010\,x^5 + 5430\,x^4 + 91104\,x^3 + 130656\,x^2 + 1330560\,x - 1330560)I_0(x) +\right.$$

$$\left.+(315\,x^6 + 1435\,x^5 + 20480\,x^4 + 29664\,x^3 + 403968\,x^2 + 926592\,x)I_1(x)\right]$$

$$\int x^6\,\operatorname{Ei}(x)\,K_1(x)\,dx = -\operatorname{Ei}(x)\left[(x^6 + 24\,x^4 + 192\,x^2)K_0(x) + (6\,x^5 + 96\,x^3 + 384\,x)K_1(x)\right] +$$

$$+\frac{e^x}{3465}\left[(315\,x^6 + 3010\,x^5 + 5430\,x^4 + 91104\,x^3 + 130656\,x^2 + 1330560\,x - 1330560)K_0(x)+\right.$$
$$\left.+(315\,x^6 + 1435\,x^5 + 20480\,x^4 + 29664\,x^3 + 403968\,x^2 + 926592\,x)K_1(x)\right]$$

$$\int x^7\,\mathrm{Ei}(x)\,I_0(x)\,dx = \mathrm{Ei}(x)\left[-(6\,x^6 + 144\,x^4 + 1152\,x^2)I_0(x) + (x^7 + 36\,x^5 + 576\,x^3 + 2304\,x)I_1(x)\right]+$$
$$+\frac{e^x}{15015}\left[(1155\,x^7 + 4515\,x^6 + 88060\,x^5 + 120180\,x^4 + 2402304\,x^3 + 3363456\,x^2 + 34594560\,x - 34594560)I_0(x)+\right.$$
$$\left.-(1155\,x^7 + 12600\,x^6 + 25060\,x^5 + 560480\,x^4 + 720864\,x^3 + 10570368\,x^2 + 24024192\,x)I_1(x)\right]$$

$$\int x^7\,\mathrm{Ei}(x)\,K_0(x)\,dx = -\,\mathrm{Ei}(x)\left[(6\,x^6 + 144\,x^4 + 1152\,x^2)K_0(x) + (x^7 + 36\,x^5 + 576\,x^3 + 2304\,x)K_1(x)\right]+$$
$$+\frac{e^x}{15015}\left[(1155\,x^7 + 4515\,x^6 + 88060\,x^5 + 120180\,x^4 + 2402304\,x^3 + 3363456\,x^2 + 34594560\,x - 34594560)K_0(x)+\right.$$
$$\left.+ (1155\,x^7 + 12600\,x^6 + 25060\,x^5 + 560480\,x^4 + 720864\,x^3 + 10570368\,x^2 + 24024192\,x)K_1(x)\right]$$

$$\int x^8\,\mathrm{Ei}(x)\,I_1(x)\,dx = \mathrm{Ei}(x)[(x^8 + 48\,x^6 + 1152\,x^4 + 9216\,x^2)I_0(x) - (8\,x^7 + 288\,x^5 + 4608\,x^3 + 18432\,x)I_1(x)]+$$
$$+\frac{e^x}{15015}\left[-(1001\,x^8 + 12474\,x^7 + 25830\,x^6 + 731920\,x^5 + 902640\,x^4 + 19312512\,x^3 + 26813568\,x^2 +\right.$$
$$+276756480\,x - 276756480)I_0(x) + (1001\,x^8 + 5467\,x^7 + 113148\,x^6 + 166180\,x^5 + 4562240\,x^4 + 5625792\,x^3+$$
$$\left.+84751104\,x^2 + 192005376\,x)I_1(x)\right]$$

$$\int x^8\,\mathrm{Ei}(x)\,K_1(x)\,dx = -\,\mathrm{Ei}(x)[(x^8 + 48\,x^6 + 1152\,x^4 + 9216\,x^2)K_0(x) + (8\,x^7 + 288\,x^5 + 4608\,x^3 + 18432\,x)K_1(x)]+$$
$$+\frac{e^x}{15015}\left[(1001\,x^8 + 12474\,x^7 + 25830\,x^6 + 731920\,x^5 + 902640\,x^4 + 19312512\,x^3 + 26813568\,x^2 +\right.$$
$$+276756480\,x - 276756480)K_0(x) + (1001\,x^8 + 5467\,x^7 + 113148\,x^6 + 166180\,x^5 + 4562240\,x^4 + 5625792\,x^3+$$
$$\left.+84751104\,x^2 + 192005376\,x)K_1(x)\right]$$

$$\int x^9\,\mathrm{Ei}(x)\,I_0(x)\,dx =$$
$$= \mathrm{Ei}(x)\left[-(8\,x^8 + 384\,x^6 + 9216\,x^4 + 73728\,x^2)I_0(x) + (x^9 + 64\,x^7 + 2304\,x^5 + 36864\,x^3 + 147456\,x)I_1(x)\right]+$$
$$+\frac{e^x}{23205}\left[(1365\,x^9 + 6643\,x^8 + 175392\,x^7 + 252000\,x^6 + 9228800\,x^5 + 10775040\,x^4 + 239388672\,x^3 +\right.$$
$$+330897408\,x^2 + 3421716480\,x - 3421716480)I_0(x) - (1365\,x^9 + 18928\,x^8 + 42896\,x^7 + 1479744\,x^6+$$
$$\left.+1830080\,x^5 + 56919040\,x^4 + 68631552\,x^3 + 1049063424\,x^2 + 2372653056\,x)I_1(x)\right]$$

$$\int x^9\,\mathrm{Ei}(x)\,K_0(x)\,dx =$$
$$= -\,\mathrm{Ei}(x)\left[(8\,x^8 + 384\,x^6 + 9216\,x^4 + 73728\,x^2)K_0(x) + (x^9 + 64\,x^7 + 2304\,x^5 + 36864\,x^3 + 147456\,x)K_1(x)\right]+$$
$$+\frac{e^x}{23205}\left[(1365\,x^9 + 6643\,x^8 + 175392\,x^7 + 252000\,x^6 + 9228800\,x^5 + 10775040\,x^4 + 239388672\,x^3 +\right.$$
$$+330897408\,x^2 + 3421716480\,x - 3421716480)K_0(x) + (1365\,x^9 + 18928\,x^8 + 42896\,x^7 + 1479744\,x^6+$$
$$\left.+1830080\,x^5 + 56919040\,x^4 + 68631552\,x^3 + 1049063424\,x^2 + 2372653056\,x)K_1(x)\right]$$

<u>Recurrence Relations:</u>   Let $\mathfrak{E}_\nu^{(m)}(x) = \int x^m\,\mathrm{Ei}(x)\,I_\nu(x)\,dx$ and $\tilde{\mathfrak{E}}_\nu^{(m)}(x) = \int x^m\,\mathrm{Ei}(x)\,K_\nu(x)\,dx$.

$$\mathfrak{E}_0^{(2n+1)}(x) = \frac{1}{4n+1}\left\{x^{2n}\left[2n(2n+1)\,I_0(x) + (4n+1)x\,I_1(x)\right]\mathrm{Ei}(x)+\right.$$

$$+x^{2n}\left[(x-2n-1)\,I_0(x)-xI_1(x)\right]e^x-4n^2(2n+1)\,\mathfrak{E}_0^{(2n-1)}(x)-4n(3n+1)\,\mathfrak{E}_1^{(2n)}(x)\Big\}$$

$$\tilde{\mathfrak{E}}_0^{(2n+1)}(x)=\frac{1}{4n+1}\,\Big\{x^{2n}\,\left[2n(2n+1)\,K_0(x)-(4n+1)x\,K_1(x)\right]\mathrm{Ei}(x)+$$

$$+x^{2n}\left[(x-2n-1)\,K_0(x)+xK_1(x)\right]e^x-4n^2(2n+1)\,\tilde{\mathfrak{E}}_0^{(2n-1)}(x)+4n(3n+1)\,\tilde{\mathfrak{E}}_1^{(2n)}(x)\Big\}$$

$$\mathfrak{E}_1^{(2n+2)}(x)=\frac{1}{4n+3}\,\Big\{x^{2n}\,\left[(4n+3)x^2\,I_0(x)+2n(2n+1)x\,I_1(x)\right]\mathrm{Ei}(x)-$$

$$-x^{2n+1}\left[x\,I_0(x)+(x-2n-1)\,K_1(x)\right]e^x-4n^2(2n+1)\,\mathfrak{E}_1^{(2n)}(x)-4n(3n+1)\,\mathfrak{E}_0^{(2n+1)}(x)\Big\}$$

$$\tilde{\mathfrak{E}}_1^{(2n+2)}(x)=\frac{1}{4n+3}\,\Big\{x^{2n}\,\left[(4n+3)x^2\,K_0(x)+2n(2n+1)x\,K_1(x)\right]\mathrm{Ei}(x)-$$

$$-x^{2n+1}\left[x\,K_0(x)+(x-2n-1)\,K_1(x)\right]e^x-4n^2(2n+1)\,\tilde{\mathfrak{E}}_1^{(2n)}(x)-4n(3n+1)\,\tilde{\mathfrak{E}}_0^{(2n+1)}(x)\Big\}$$

1.3.3. Integrals of the type $\int x^n \operatorname{Si}(x) \cdot J_\nu(x)\,dx$ **and** $\int x^n \operatorname{Ci}(x) \cdot J_\nu(x)\,dx$

Let
$$\operatorname{Si}(x) = \int_0^x \frac{\sin t\,dt}{t} \quad \text{and} \quad \operatorname{Ci}(x) = C + \ln x + \int_0^x \frac{\cos t - 1}{t}\,dt \ .$$

About C see page 125.

(In [4], p. 656, the function $\operatorname{ci}(x)$ is defined by some integral which fails to converge.)

$$\int x \operatorname{Si}(x)\, J_0(x)\,dx = x\operatorname{Si}(x)\, J_1(x) + \sin x\, J_0(x) - x[\sin x\, J_1(x) + \cos x\, J_0(x)]$$

$$\int x \operatorname{Ci}(x)\, J_0(x)\,dx = x\operatorname{Ci}(x)\, J_1(x) + \cos x\, J_0(x) + x[\sin x\, J_0(x) - \cos x\, J_1(x)]$$

$$\int x^2 \operatorname{Si}(x)\, J_1(x)\,dx = \frac{1}{3}\left[-3\,x^2\operatorname{Si}(x)\, J_0(x) + 6\,x\operatorname{Si}(x)\, J_1(x) + \left(x^2 + 6\right)\sin x\, J_0(x) - 5\,x\sin x\, J_1(x) - \right.$$
$$\left. -6\,x\cos x\, J_0(x) - x^2\cos x\, J_1(x)\right]$$

$$\int x^2 \operatorname{Ci}(x)\, J_1(x)\,dx = \frac{1}{3}\left[-3\,x^2\operatorname{Ci}(x)\, J_0(x) + 6\,x\operatorname{Ci}(x)\, J_1(x) + 6\,x\sin x\, J_0(x) + x^2\sin x\, J_1(x) + \right.$$
$$\left. + \left(x^2 + 6\right)\cos x\, J_0(x) - 5\,x\cos x\, J_1(x)\right]$$

$$\int x^3 \operatorname{Si}(x)\, J_0(x)\,dx = \frac{1}{15}\left[30\,x^2\operatorname{Si}(x)\, J_0(x) + \left(15\,x^3 - 60\,x\right)\operatorname{Si}(x)\, J_1(x) + \left(-7\,x^2 - 60\right)\sin x\, J_0(x) + \right.$$
$$\left. + \left(-3\,x^3 + 44\,x\right)\sin x\, J_1(x) + \left(-3\,x^3 + 60\,x\right)\cos x\, J_0(x) + 16\,x^2\cos x\, J_1(x)\right]$$

$$\int x^3 \operatorname{Ci}(x)\, J_0(x)\,dx = \frac{1}{15}\left[30\,x^2\operatorname{Ci}(x)\, J_0(x) + \left(15\,x^3 - 60\,x\right)\operatorname{Ci}(x)\, J_1(x) + \left(3\,x^3 - 60\,x\right)\sin x\, J_0(x) - \right.$$
$$\left. -16\,x^2\sin x\, J_1(x) + \left(-7\,x^2 - 60\right)\cos x\, J_0(x) + \left(-3\,x^3 + 44\,x\right)\cos x\, J_1(x)\right]$$

$$\int x^4 \operatorname{Si}(x)\, J_1(x)\,dx = \frac{1}{105}\left[\left(-105\,x^4 + 840\,x^2\right)\operatorname{Si}(x)\, J_0(x) + \left(420\,x^3 - 1680\,x\right)\operatorname{Si}(x)\, J_1(x) + \right.$$
$$+ \left(15\,x^4 - 178\,x^2 - 1680\right)\sin x\, J_0(x) + \left(-57\,x^3 + 1196\,x\right)\sin x\, J_1(x) + \left(-102\,x^3 + 1680\,x\right)\cos x\, J_0(x) +$$
$$\left. + \left(-15\,x^4 + 484\,x^2\right)\cos x\, J_1(x)\right]$$

$$\int x^4 \operatorname{Ci}(x)\, J_1(x)\,dx = \frac{1}{105}\left[\left(-105\,x^4 + 840\,x^2\right)\operatorname{Ci}(x)\, J_0(x) + \left(420\,x^3 - 1680\,x\right)\operatorname{Ci}(x)\, J_1(x) + \right.$$
$$+ \left(102\,x^3 - 1680\,x\right)\sin x\, J_0(x) + \left(15\,x^4 - 484\,x^2\right)\sin x\, J_1(x) + \left(15\,x^4 - 178\,x^2 - 1680\right)\cos x\, J_0(x) +$$
$$\left. + \left(-57\,x^3 + 1196\,x\right)\cos x\, J_1(x)\right]$$

$$\int x^5 \operatorname{Si}(x)\, J_0(x)\,dx = \frac{1}{45}\left[\left(180\,x^4 - 1440\,x^2\right)\operatorname{Si}(x)\, J_0(x) + \left(45\,x^5 - 720\,x^3 + 2880\,x\right)\operatorname{Si}(x)\, J_1(x) + \right.$$
$$+ \left(-15\,x^4 + 288\,x^2 + 2880\right)\sin x\, J_0(x) + \left(-5\,x^5 + 72\,x^3 - 2016\,x\right)\sin x\, J_1(x) +$$
$$\left. + \left(-5\,x^5 + 192\,x^3 - 2880\,x\right)\cos x\, J_0(x) + \left(40\,x^4 - 864\,x^2\right)\cos x\, J_1(x)\right]$$

$$\int x^5 \operatorname{Ci}(x) J_0(x)\,dx = \frac{1}{45}\Big[\left(180\,x^4 - 1440\,x^2\right)\operatorname{Ci}(x) J_0(x) + \left(45\,x^5 - 720\,x^3 + 2880\,x\right)\operatorname{Ci}(x) J_1(x)+$$

$$+\left(5\,x^5 - 192\,x^3 + 2880\,x\right)\sin x\, J_0(x) + \left(-40\,x^4 + 864\,x^2\right)\sin x\, J_1(x)+$$

$$+\left(-15\,x^4 + 288\,x^2 + 2880\right)\cos x\, J_0(x) + \left(-5\,x^5 + 72\,x^3 - 2016\,x\right)\cos x\, J_1(x)\Big]$$

$$\int x^6 \operatorname{Si}(x) J_1(x)\,dx = \frac{1}{3465}\Big[\left(-3465\,x^6 + 83160\,x^4 - 665280\,x^2\right)\operatorname{Si}(x) J_0(x)+$$

$$+\left(20790\,x^5 - 332640\,x^3 + 1330560\,x\right)\operatorname{Si}(x) J_1(x)+$$

$$+\left(315\,x^6 - 5430\,x^4 + 130656\,x^2 + 1330560\right)\sin x\, J_0(x) + \left(-1435\,x^5 + 29664\,x^3 - 926592\,x\right)\sin x\, J_1(x)+$$

$$+\left(-3010\,x^5 + 91104\,x^3 - 1330560\,x\right)\cos x\, J_0(x) + \left(-315\,x^6 + 20480\,x^4 - 403968\,x^2\right)\cos x\, J_1(x)\Big]$$

$$\int x^6 \operatorname{Ci}(x) J_1(x)\,dx = \frac{1}{3465}\Big[\left(-3465\,x^6 + 83160\,x^4 - 665280\,x^2\right)\operatorname{Ci}(x) J_1(x)+$$

$$+\left(20790\,x^5 - 332640\,x^3 + 1330560\,x\right)\operatorname{Ci}(x) J_0(x) + \left(3010\,x^5 - 91104\,x^3 + 1330560\,x\right)\sin x\, J_0(x)+$$

$$+\left(315\,x^6 - 20480\,x^4 + 403968\,x^2\right)\sin x\, J_1(x) + \left(315\,x^6 - 5430\,x^4 + 130656\,x^2 + 1330560\right)\cos x\, J_0(x)+$$

$$+\left(-1435\,x^5 + 29664\,x^3 - 926592\,x\right)\cos x\, J_1(x)\Big]$$

$$\int x^7 \operatorname{Si}(x) J_0(x)\,dx = \frac{1}{15015}\Big[\left(90090\,x^6 - 2162160\,x^4 + 17297280\,x^2\right)\operatorname{Si}(x) J_0(x)+$$

$$+\left(15015\,x^7 - 540540\,x^5 + 8648640\,x^3 - 34594560\,x\right)\operatorname{Si}(x) J_1(x)+$$

$$+\left(-4515\,x^6 + 120180\,x^4 - 3363456\,x^2 - 34594560\right)\sin x\, J_0(x)+$$

$$+\left(-1155\,x^7 + 25060\,x^5 - 720864\,x^3 + 24024192\,x\right)\sin x\, J_1(x)+$$

$$+\left(-1155\,x^7 + 88060\,x^5 - 2402304\,x^3 + 34594560\,x\right)\cos x\, J_0(x)+$$

$$+\left(12600\,x^6 - 560480\,x^4 + 10570368\,x^2\right)\cos x\, J_1(x)\Big]$$

$$\int x^7 \operatorname{Ci}(x) J_0(x)\,dx = \frac{1}{15015}\Big[\left(90090\,x^6 - 2162160\,x^4 + 17297280\,x^2\right)\operatorname{Ci}(x) J_0(x)+$$

$$+\left(15015\,x^7 - 540540\,x^5 + 8648640\,x^3 - 34594560\,x\right)\operatorname{Ci}(x) J_1(x)+$$

$$+\left(1155\,x^7 - 88060\,x^5 + 2402304\,x^3 - 34594560\,x\right)\sin x\, J_0(x)+$$

$$+\left(-12600\,x^6 + 560480\,x^4 - 10570368\,x^2\right)\sin x\, J_1(x)+$$

$$+\left(-4515\,x^6 + 120180\,x^4 - 3363456\,x^2 - 34594560\right)\cos x\, J_0(x)+$$

$$+\left(-1155\,x^7 + 25060\,x^5 - 720864\,x^3 + 24024192\,x\right)\cos x\, J_1(x)\Big]$$

$$\int x^8 \operatorname{Si}(x) J_1(x)\,dx = \frac{1}{15015}\Big[\left(-15015\,x^8 + 720720\,x^6 - 17297280\,x^4 + 138378240\,x^2\right)\operatorname{Si}(x) J_0(x)+$$

$$+\left(120120\,x^7 - 4324320\,x^5 + 69189120\,x^3 - 276756480\,x\right)\operatorname{Si}(x) J_1(x)+$$

$$+\left(1001\,x^8 - 25830\,x^6 + 902640\,x^4 - 26813568\,x^2 - 276756480\right)\sin x\, J_0(x)+$$

$$+\left(-5467\,x^7 + 166180\,x^5 - 5625792\,x^3 + 192005376\,x\right)\sin x\, J_1(x)+$$

$$+\left(-12474\,x^7 + 731920\,x^5 - 19312512\,x^3 + 276756480\,x\right)\cos x\, J_0(x)+$$

$$+\left(-1001\,x^8 + 113148\,x^6 - 4562240\,x^4 + 84751104\,x^2\right)\cos x\, J_1(x)\Big]$$

$$\int x^8 \operatorname{Ci}(x) J_1(x)\,dx = \frac{1}{15015}\,\Big[\left(-15015\,x^8 + 720720\,x^6 - 17297280\,x^4 + 138378240\,x^2\right)\operatorname{Ci}(x)J_0(x)+$$

$$+\left(120120\,x^7 - 4324320\,x^5 + 69189120\,x^3 - 276756480\,x\right)\operatorname{Ci}(x)J_1(x)+$$

$$+\left(12474\,x^7 - 731920\,x^5 + 19312512\,x^3 - 276756480\,x\right)\sin x\,J_0(x)+$$

$$+\left(1001\,x^8 - 113148\,x^6 + 4562240\,x^4 - 84751104\,x^2\right)\sin x\,J_1(x)+$$

$$+\left(1001\,x^8 - 25830\,x^6 + 902640\,x^4 - 26813568\,x^2 - 276756480\right)\cos x\,J_0(x)+$$

$$+\left(-5467\,x^7 + 166180\,x^5 - 5625792\,x^3 + 192005376\,x\right)\cos x\,J_1(x)\Big]$$

$$\int x^9 \operatorname{Si}(x) J_0(x)\,dx = \frac{1}{23205}\,\Big[\left(185640\,x^8 - 8910720\,x^6 + 213857280\,x^4 - 1710858240\,x^2\right)\operatorname{Si}(x)J_0(x)+$$

$$+\left(23205\,x^9 - 1485120\,x^7 + 53464320\,x^5 - 855429120\,x^3 + 3421716480\,x\right)\operatorname{Si}(x)J_1(x)+$$

$$+\left(-6643\,x^8 + 252000\,x^6 - 10775040\,x^4 + 330897408\,x^2 + 3421716480\right)\sin x\,J_0(x)+$$

$$+\left(-1365\,x^9 + 42896\,x^7 - 1830080\,x^5 + 68631552\,x^3 - 2372653056\,x\right)\sin x\,J_1(x)+$$

$$+\left(-1365\,x^9 + 175392\,x^7 - 9228800\,x^5 + 239388672\,x^3 - 3421716480\,x\right)\cos x\,J_0(x)+$$

$$+\left(18928\,x^8 - 1479744\,x^6 + 56919040\,x^4 - 1049063424\,x^2\right)\cos x\,J_1(x)\Big]$$

$$\int x^9 \operatorname{Ci}(x) J_0(x)\,dx = \frac{1}{23205}\,\Big[\left(185640\,x^8 - 8910720\,x^6 + 213857280\,x^4 - 1710858240\,x^2\right)\operatorname{Ci}(x)J_0(x)+$$

$$+\left(23205\,x^9 - 1485120\,x^7 + 53464320\,x^5 - 855429120\,x^3 + 3421716480\,x\right)\operatorname{Ci}(x)J_1(x)+$$

$$+\left(1365\,x^9 - 175392\,x^7 + 9228800\,x^5 - 239388672\,x^3 + 3421716480\,x\right)\sin x\,J_0(x)+$$

$$+\left(-18928\,x^8 + 1479744\,x^6 - 56919040\,x^4 + 1049063424\,x^2\right)\sin x\,J_1(x)+$$

$$+\left(-6643\,x^8 + 252000\,x^6 - 10775040\,x^4 + 330897408\,x^2 + 3421716480\right)\cos x\,J_0(x)+$$

$$+\left(-1365\,x^9 + 42896\,x^7 - 1830080\,x^5 + 68631552\,x^3 - 2372653056\,x\right)\cos x\,J_1(x)\Big]$$

<u>Recurrence Relations:</u>   Let $\mathfrak{S}_\nu^{(m)}(x) = \int x^m \operatorname{Si}(x) J_\nu(x)\,dx$ and $\mathfrak{L}_\nu^{(m)}(x) = \int x^m \operatorname{Ci}(x) J_\nu(x)\,dx$.

$$\mathfrak{S}_0^{(2n+1)}(x) = \frac{x^{2n}}{4n+1}\,\{[(4n+1)x\,J_1(x) - 2n(2n+1)\,J_0(x)]\,\operatorname{Si}(x)+$$

$$+\,[(2n+1)\,J_0(x) - xJ_1(x)]\sin x - xJ_0(x)\,\cos x\} + \frac{4n^2(2n+1)}{4n+1}\,\mathfrak{S}_0^{(2n-1)}(x) - \frac{4n(3n+1)}{4n+1}\,\mathfrak{S}_1^{(2n)}(x)$$

$$\mathfrak{L}_0^{(2n+1)}(x) = \frac{x^{2n}}{4n+1}\,\{[(4n+1)x\,J_1(x) - 2n(2n+1)\,J_0(x)]\,\operatorname{Ci}(x)+$$

$$+\,[(2n+1)\,J_0(x) - xJ_1(x)]\cos x + xJ_0(x)\,\sin x\} + \frac{4n^2(2n+1)}{4n+1}\,\mathfrak{L}_0^{(2n-1)}(x) - \frac{4n(3n+1)}{4n+1}\,\mathfrak{L}_1^{(2n)}(x)$$

$$\mathfrak{S}_1^{(2n+2)}(x) = \frac{x^{2n}}{4n+3}\,\{ - [(4n+3)x^2 J_0(x) + 2n(2n+1)x\,J_1(x)]\,\operatorname{Si}(x)+$$

$$+\,[x^2 J_0(x) + (2n+1)x\,J_1(x)]\sin x - x^2 J_1(x)\,\cos x\} + \frac{4n^2(2n+1)}{4n+3}\,\mathfrak{S}_1^{(2n)}(x) + \frac{2(6n^2 + 8n + 3)}{4n+4}\,\mathfrak{S}_0^{(2n+1)}(x)$$

$$\mathfrak{L}_1^{(2n+2)}(x) = \frac{x^{2n}}{4n+3}\,\{ - [(4n+3)x^2 J_0(x) + 2n(2n+1)x\,J_1(x)]\,\operatorname{Ci}(x)+$$

$$+\,[x^2 J_0(x) + (2n+1)x\,J_1(x)]\cos x + x^2 J_1(x)\,\sin x\} + \frac{4n^2(2n+1)}{4n+3}\,\mathfrak{L}_1^{(2n)}(x) + \frac{2(6n^2 + 8n + 3)}{4n+4}\,\mathfrak{L}_0^{(2n+1)}(x)$$

1.3.4. $\int x^n \, \mathrm{erf}(x) \, J_\nu(\alpha x) \, dx$ and $\int x^n \, \mathrm{erf}(x) \, I_\nu(\alpha x) \, dx$

a) The Case $\alpha = 1, \ J_\nu(x)$

About the basic integrals

$$F_0(x) = \int_0^x e^{-t^2} J_0(t) \, dt \quad \text{and} \quad F_-(x) = \int_0^x \frac{e^{-t^2} J_1(t) \, dt}{t}$$

see page 148 and following.

$$\int \mathrm{erf}(x) \, J_1(x) \, dx = -\mathrm{erf}(x) \, J_0(x) + \frac{2}{\sqrt{\pi}} \int e^{-x^2} J_0(x) \, dx$$

$$\int x \, \mathrm{erf}(x) \, J_0(x) \, dx = \frac{e^{-x^2}}{\sqrt{\pi}} J_1(x) + x \, \mathrm{erf}(x) \, J_1(x) - \frac{1}{\sqrt{\pi}} \int e^{-x^2} J_0(x) \, dx + \frac{1}{\sqrt{\pi}} \int \frac{e^{-x^2} J_1(x) \, dx}{x}$$

$$\int x^2 \, \mathrm{erf}(x) \, J_1(x) \, dx = \frac{e^{-x^2}}{2\sqrt{\pi}} \left[-2x \, J_0(x) + 5 J_1(x) \right] + \mathrm{erf}(x) \left[-x^2 J_0(x) + 2x \, J_1(x) \right] -$$

$$- \frac{3}{2\sqrt{\pi}} \int e^{-x^2} J_0(x) \, dx + \frac{5}{2\sqrt{\pi}} \int \frac{e^{-x^2} J_1(x) \, dx}{x}$$

$$\int x^3 \, \mathrm{erf}(x) \, J_0(x) \, dx = \frac{e^{-x^2}}{4\sqrt{\pi}} \left[10x \, J_0(x) + (4x^2 - 19) \, J_1(x) \right] + \mathrm{erf}(x) \left[2x^2 J_0(x) + (x^3 - 4x) J_1(x) \right] +$$

$$+ \frac{9}{4\sqrt{\pi}} \int e^{-x^2} J_0(x) \, dx - \frac{19}{4\sqrt{\pi}} \int \frac{e^{-x^2} J_1(x) \, dx}{x}$$

$$\int x^4 \, \mathrm{erf}(x) \, J_1(x) \, dx = \frac{e^{-x^2}}{8\sqrt{\pi}} \left[(-8\,x^3 + 70\,x) \, J_0(x) + (36\,x^2 - 145) \, J_1(x) \right] +$$

$$+ \mathrm{erf}(x) \left[(-x^4 + 8\,x^2) \, J_0(x) + (4\,x^3 - 16\,x) \, J_1(x) \right] + \frac{75}{8\sqrt{\pi}} \int e^{-x^2} J_0(x) \, dx - \frac{145}{8\sqrt{\pi}} \int \frac{e^{-x^2} J_1(x) \, dx}{x}$$

$$\int x^5 \, \mathrm{erf}(x) \, J_0(x) \, dx = \frac{e^{-x^2}}{16\sqrt{\pi}} \left[(72\,x^3 - 538\,x) \, J_0(x) + (16\,x^4 - 268\,x^2 + 1159) \, J_1(x) \right] +$$

$$+ \mathrm{erf}(x) \left[(4\,x^4 - 32\,x^2) \, J_0(x) + (x^5 - 16\,x^3 + 64\,x) \, J_1(x) \right] - \frac{621}{16\sqrt{\pi}} \int e^{-x^2} J_0(x) \, dx + \frac{1159}{16\sqrt{\pi}} \int \frac{e^{-x^2} J_1(x) \, dx}{x}$$

$$\int x^6 \, \mathrm{erf}(x) \, J_1(x) \, dx = \frac{e^{-x^2}}{32\sqrt{\pi}} \left[(-32\,x^5 + 792\,x^3 - 6534\,x) \, J_0(x) + (208\,x^4 - 3156\,x^2 + 13977) \, J_1(x) \right] +$$

$$+ \mathrm{erf}(x) \left[(-x^6 + 24\,x^4 - 192\,x^2) \, J_0(x) + (6\,x^5 - 96\,x^3 + 384\,x) \, J_1(x) \right] -$$

$$- \frac{7443}{32\sqrt{\pi}} \int e^{-x^2} J_0(x) \, dx + \frac{13977}{32\sqrt{\pi}} \int \frac{e^{-x^2} J_1(x) \, dx}{x}$$

$$\int x^7 \, \mathrm{erf}(x) \, J_0(x) \, dx =$$

$$= \frac{e^{-x^2}}{64\sqrt{\pi}} \left[(416\,x^5 - 9352\,x^3 + 78706\,x) \, J_0(x) + (64\,x^6 - 2352\,x^4 + 38012\,x^2 - 167803) \, J_1(x) \right] +$$

$$+ \mathrm{erf}(x) \left[(6\,x^6 - 144\,x^4 + 1152\,x^2) \, J_0(x) + (x^7 - 36\,x^5 + 576\,x^3 - 2304\,x) \, J_1(x) \right] +$$

$$+ \frac{89097}{64\sqrt{\pi}} \int e^{-x^2} J_0(x) \, dx - \frac{167803}{64\sqrt{\pi}} \int \frac{e^{-x^2} J_1(x) \, dx}{x}$$

$$\int x^8 \, \mathrm{erf}(x) \, J_1(x) \, dx =$$

$$= \frac{e^{-x^2}}{128\sqrt{\pi}} \left[(-128\,x^7 + 6240\,x^5 - 150488\,x^3 + 1258502\,x) \, J_0(x) + (1088\,x^6 - 37264\,x^4 + 609172\,x^2 - 2683961) \, J_1(x) \right] +$$

$$+ \operatorname{erf}(x) \left[(-x^8 + 48\,x^6 - 1152\,x^4 + 9216\,x^2)\,J_0(x) + (8\,x^7 - 288\,x^5 + 4608\,x^3 - 18432\,x)\,J_1(x) \right] +$$

$$+ \frac{1425459}{128\sqrt{\pi}} \int e^{-x^2}\,J_0(x)\,dx - \frac{2683961}{128\sqrt{\pi}} \int \frac{e^{-x^2}\,J_1(x)\,dx}{x}$$

$$\int x^9\,\operatorname{erf}(x)\,J_0(x)\,dx = \frac{e^{-x^2}}{256\sqrt{\pi}} \left[(2176\,x^7 - 98976\,x^5 + 2410792\,x^3 - 20131066\,x)\,J_0(x) + \right.$$

$$\left. + (256\,x^8 - 16576\,x^6 + 597872\,x^4 - 9745772\,x^2 + 42941383)\,J_1(x) \right] +$$

$$+ \operatorname{erf}(x) \left[(8\,x^8 - 384\,x^6 + 9216\,x^4 - 73728\,x^2)\,J_0(x) + (x^9 - 64\,x^7 + 2304\,x^5 - 36864\,x^3 + 147456\,x)\,J_1(x) \right] -$$

$$- \frac{22810317}{256\sqrt{\pi}} \int e^{-x^2}\,J_0(x)\,dx + \frac{42941383}{256\sqrt{\pi}} \int \frac{e^{-x^2}\,J_1(x)\,dx}{x}$$

<u>Recurrence relations:</u>

$$\int x^{2n+2}\,\operatorname{erf}(x)\,J_1(x)\,dx = \frac{4n+1}{2} \int x^{2n}\,\operatorname{erf}(x)\,J_1(x)\,dx + \frac{4n+5}{2} \int x^{2n+1}\,\operatorname{erf}(x)\,J_0(x)\,dx -$$

$$- n(2n+1) \int x^{2n-1}\,\operatorname{erf}(x)\,J_0(x)\,dx - \frac{x^{2n+1}\,e^{-x^2}}{\sqrt{\pi}}\,J_0(x) + \frac{x^{2n}}{2} \left[(2n+1-2x^2)\,J_0(x) - x\,J_1(x) \right]\,\operatorname{erf}(x)$$

$$\int x^{2n+1}\,\operatorname{erf}(x)\,J_0(x)\,dx = -\frac{4n+1}{2} \int x^{2n}\,\operatorname{erf}(x)\,J_1(x)\,dx + (2n-1)(n-1) \int x^{2n-2}\,\operatorname{erf}(x)\,J_1(x)\,dx +$$

$$+ \frac{4n-1}{2} \int x^{2n-1}\,\operatorname{erf}(x)\,J_0(x)\,dx + \frac{x^{2n}\,e^{-x^2}\,J_1(x)}{\sqrt{\pi}} - \frac{x^{2n-1}}{2} \left\{ [x\,J_0(x) + (2n-1-x^2)\,J_1(x)]\,\operatorname{erf}(x) \right\}$$

b) The General Case, $J_\nu(x)$

About the basic integrals

$$F_0(x) = \int_0^x e^{-t^2}\,J_0(\alpha t)\,dt \quad \text{and} \quad F_-(x) = \int_0^x \frac{e^{-t^2}\,J_1(\alpha t)\,dt}{t}$$

see page 158 and following.

$$\int \operatorname{erf}(x)\,J_1(\alpha x)\,dx = -\frac{\operatorname{erf}(x)\,J_0(\alpha x)}{\alpha} + \frac{2}{\alpha\sqrt{\pi}} \int e^{-x^2}\,J_0(\alpha x)\,dx$$

$$\int x\,\operatorname{erf}(x)\,J_0(\alpha x)\,dx = \frac{e^{-x^2}}{\sqrt{\pi}\,\alpha}\,J_1(\alpha x) + \frac{x\,\operatorname{erf}(x)}{\alpha}\,J_1(\alpha x) - \frac{1}{\sqrt{\pi}} \int e^{-x^2}\,J_0(\alpha x)\,dx + \frac{1}{\sqrt{\pi}\,\alpha} \int \frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^2\,\operatorname{erf}(x)\,J_1(\alpha x)\,dx = \frac{e^{-x^2}}{2\sqrt{\pi}\,\alpha^2} \left[-2\alpha x\,J_0(\alpha x) + (\alpha^2+4)\,J_1(\alpha x) \right] + \frac{\operatorname{erf}(x)}{\alpha^2} \left[-\alpha x^2\,J_0(\alpha x) + 2x\,J_1(\alpha x) \right] -$$

$$- \frac{\alpha^2+2}{2\sqrt{\pi}\,\alpha} \int e^{-x^2}\,J_0(\alpha x)\,dx + \frac{\alpha^2+4}{2\sqrt{\pi}\,\alpha^2} \int \frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^3\,\operatorname{erf}(x)\,J_0(\alpha x)\,dx = \frac{e^{-x^2}}{4\sqrt{\pi}\,\alpha^3} \left\{ 2\alpha(\alpha^2+4)x\,J_0(\alpha x) + [4\alpha^2 x^2 - (\alpha^4 + 2\,\alpha^2 + 16)]\,J_1(\alpha x) \right\} +$$

$$+ \frac{\operatorname{erf}(x)}{\alpha^3} \left[2\alpha x^2\,J_0(\alpha x) + (\alpha^2 x^3 - 4x)\,J_1(\alpha x) \right] + \frac{\alpha^4+8}{4\sqrt{\pi}\,\alpha^2} \int e^{-x^2}\,J_0(\alpha x)\,dx - \frac{\alpha^4 + 2\,\alpha^2 + 16}{4\sqrt{\pi}\,\alpha^3} \int \frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^4\,\operatorname{erf}(x)\,J_1(\alpha x)\,dx =$$

$$= \frac{e^{-x^2}}{8\sqrt{\pi}\,\alpha^4} \left\{ [-8\alpha^3 x^3 + 2(\alpha^5 + 2\,\alpha^3 + 32\alpha)x]\,J_0(\alpha x) + [(4\,\alpha^4 + 32\,\alpha^2)x^2 - \alpha^6 - 16\,\alpha^2 - 128]\,J_1(\alpha x) \right\} +$$

$$+ \frac{\operatorname{erf}(x)}{\alpha^4} \left[(-\alpha^3 x^4 + 8\,\alpha x^2)\,J_0(\alpha x) + (4\,\alpha^2 x^3 - 16\,x)\,J_1(\alpha x) \right] +$$

$$+\frac{\alpha^6 - 2\,\alpha^4 + 12\,\alpha^2 + 64}{8\sqrt{\pi}\,\alpha^3} \int e^{-x^2} J_0(\alpha x)\,dx - \frac{\alpha^6 + 16\,\alpha^2 + 128}{8\sqrt{\pi}\,\alpha^4} \int \frac{e^{-x^2} J_1(\alpha x)}{x}\,dx$$

$$\int x^5 \mathrm{erf}(x)\, J_0(\alpha x)\,dx = \frac{e^{-x^2}}{16\sqrt{\pi}\,\alpha^5} \left\{ \left[(8\,\alpha^5 + 64\,\alpha^3)x^3 - (2\,\alpha^7 - 8\,\alpha^5 + 32\,\alpha^3 + 512\,\alpha)x \right] J_0(\alpha x) + \right.$$

$$+ \left[16\alpha^4 x^4 - (4\,\alpha^6 + 8\,\alpha^4 + 256\,\alpha^2)x^2 + \alpha^8 - 6\,\alpha^6 + 12\,\alpha^4 + 128\,\alpha^2 + 1024 \right] J_1(\alpha x) \right\} +$$

$$+ \frac{\mathrm{erf}(x)}{\alpha^5} \left[(4\,\alpha^3 x^4 - 32\,\alpha\, x^2) J_0(\alpha x) + (\alpha^4 x^5 - 16\,\alpha^2 x^3 + 64\,x) J_1(\alpha x) \right] -$$

$$- \frac{\alpha^8 - 8\,\alpha^6 + 20\,\alpha^4 + 96\,\alpha^2 + 512}{16\sqrt{\pi}\,\alpha^4} \int e^{-x^2} J_0(\alpha x)\,dx +$$

$$+ \frac{\alpha^8 - 6\,\alpha^6 + 12\,\alpha^4 + 128\,\alpha^2 + 1024}{16\sqrt{\pi}\,\alpha^5} \int \frac{e^{-x^2} J_1(\alpha x)}{x}\,dx$$

$$\int x^6\, \mathrm{erf}(x)\, J_1(\alpha x)\,dx =$$

$$= \frac{e^{-x^2}}{32\sqrt{\pi}\,\alpha^6} \left\{ \left[-32\alpha^5 x^5 + (8\,\alpha^7 + 16\,\alpha^5 + 768\,\alpha^3)x^3 - (2\,\alpha^9 - 20\,\alpha^7 + 24\,\alpha^5 + 384\,\alpha^3 + 6144\,\alpha)x \right] J_0(\alpha x) + \right.$$

$$+ \left[(16\,\alpha^6 + 192\,\alpha^4)x^4 - (4\,\alpha^8 - 16\,\alpha^6 + 96\,\alpha^4 + 3072\,\alpha^2)x^2 + \alpha^{10} - 12\,\alpha^8 + 20\,\alpha^6 + 144\,\alpha^4 + 1536\,\alpha^2 + 12288 \right] J_1(\alpha x) \right\} +$$

$$+ \frac{\mathrm{erf}(x)}{\alpha^6} \left[-(\alpha^5 x^6 - 24\,\alpha^3 x^4 + 192\,\alpha\, x^2) J_0(\alpha x) + (6\,\alpha^4 x^5 - 96\,\alpha^2 x^3 + 384\,x) J_1(\alpha x) \right] -$$

$$- \frac{\alpha^{10} - 14\,\alpha^8 + 40\,\alpha^6 + 120\,\alpha^4 + 1152\,\alpha^2 + 6144}{32\sqrt{\pi}\,\alpha^5} \int e^{-x^2} J_0(\alpha x)\,dx +$$

$$+ \frac{\alpha^{10} - 12\,\alpha^8 + 20\,\alpha^6 + 144\,\alpha^4 + 1536\,\alpha^2 + 12288}{32\sqrt{\pi}\,\alpha^6} \int \frac{e^{-x^2} J_1(\alpha x)}{x}\,dx$$

$$\int x^7\, \mathrm{erf}(x)\, J_0(\alpha x)\,dx = \frac{e^{-x^2}}{64\sqrt{\pi}\,\alpha^7} \left\{ \left[(32\,\alpha^7 + 384\,\alpha^5)x^5 - (8\,\alpha^9 - 64\,\alpha^7 + 192\,\alpha^5 + 9216\,\alpha^3)x^3 + \right. \right.$$

$$+ (2\,\alpha^{11} - 40\,\alpha^9 + 120\,\alpha^7 + 288\,\alpha^5 + 4608\,\alpha^3 + 73728\,\alpha)x \right] J_0(\alpha x) + \left[64\alpha^6 x^6 - (16\,\alpha^8 + 32\,\alpha^6 + 2304\,\alpha^4)x^4 + \right.$$

$$+ (4\,\alpha^{10} - 56\,\alpha^8 + 48\,\alpha^6 + 1152\,\alpha^4 + 36864\,\alpha^2)x^2 - \alpha^{12} + 22\,\alpha^{10} - 88\,\alpha^8 - 120\,\alpha^6 - 1728\,\alpha^4 - 18432\,\alpha^2 - 147456 \right] J_1(\alpha x) \right\} +$$

$$+ \frac{\mathrm{erf}(x)}{\alpha^7} \left[(6\,\alpha^5 x^6 - 144\,\alpha^3 x^4 + 1152\,\alpha\, x^2) J_0(\alpha x) + (\alpha^6 x^7 - 36\,\alpha^4 x^5 + 576\,\alpha^2 x^3 - 2304\,x) J_1(\alpha x) \right] +$$

$$+ \frac{\alpha^{12} - 24\,\alpha^{10} + 128\,\alpha^8 + 1440\,\alpha^4 + 13824\,\alpha^2 + 73728}{64\sqrt{\pi}\,\alpha^6} \int e^{-x^2} J_0(\alpha x)\,dx -$$

$$- \frac{\alpha^{12} - 22\,\alpha^{10} + 88\,\alpha^8 + 120\,\alpha^6 + 1728\,\alpha^4 + 18432\,\alpha^2 + 147456}{64\sqrt{\pi}\,\alpha^7} \int \frac{e^{-x^2} J_1(\alpha x)}{x}\,dx$$

$$\int x^8\, \mathrm{erf}(x)\, J_1(\alpha x)\,dx =$$

$$= \frac{e^{-x^2}}{128\sqrt{\pi}\,\alpha^8} \left\{ \left[-128\alpha^7 x^7 + (32\,\alpha^9 + 64\,\alpha^7 + 6144\,\alpha^5)x^5 - (8\,\alpha^{11} - 144\,\alpha^9 + 96\,\alpha^7 + 3072\,\alpha^5 + 147456\,\alpha^3)x^3 + \right. \right.$$

$$+ (2\,\alpha^{13} - 60\,\alpha^{11} + 336\,\alpha^9 + 240\,\alpha^7 + 4608\,\alpha^5 + 73728\,\alpha^3 + 1179648\,\alpha)x \right] J_0(\alpha x) +$$

$$+ \left[(64\,\alpha^8 + 1024\,\alpha^6)x^6 - (16\,\alpha^{10} - 128\,\alpha^8 + 512\,\alpha^6 + 36864\,\alpha^4)x^4 + \right.$$

$$+ (4\,\alpha^{12} - 96\,\alpha^{10} + 240\,\alpha^8 + 768\,\alpha^6 + 18432\,\alpha^4 + 589824\,\alpha^2)x^2 -$$

$$- \alpha^{14} + 32\,\alpha^{12} - 216\,\alpha^{10} - 1920\,\alpha^6 - 27648\,\alpha^4 - 294912\,\alpha^2 - 2359296 \right] J_1(\alpha x) \right\} +$$

$$+ \frac{\mathrm{erf}(x)}{\alpha^8} \left[-(\alpha^7 x^8 - 48\,\alpha^5 x^6 + 1152\,\alpha^3 x^4 - 9216\,\alpha\, x^2) J_0(\alpha x) + (8\,\alpha^6 x^7 - 288\,\alpha^4 x^5 + 4608\,\alpha^2 x^3 - 18432\,x) J_1(\alpha x) \right] +$$

$$+ \frac{\alpha^{14} - 34\,\alpha^{12} + 276\,\alpha^{10} - 336\,\alpha^8 + 1680\,\alpha^6 + 23040\,\alpha^4 + 221184\,\alpha^2 + 1179648}{128\sqrt{\pi}\,\alpha^7} \int e^{-x^2} J_0(\alpha x)\,dx -$$

$$-\frac{\alpha^{14} - 32\,\alpha^{12} + 216\,\alpha^{10} + 1920\,\alpha^{6} + 27648\,\alpha^{4} + 294912\,\alpha^{2} + 2359296}{128\sqrt{\pi}\,\alpha^{8}} \int \frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^9 \operatorname{erf}(x)\,J_0(\alpha x)\,dx = \frac{e^{-x^2}}{256\sqrt{\pi}\,\alpha^9}\left\{\left[(128\,\alpha^9 + 2048\,\alpha^7)x^7 - (32\,\alpha^{11} - 384\,\alpha^9 + 1024\,\alpha^7 + 98304\,\alpha^5)x^5 + \right.\right.$$

$$+ (8\,\alpha^{13} - 256\,\alpha^{11} + 1056\,\alpha^9 + 1536\,\alpha^7 + 49152\,\alpha^5 + 2359296\,\alpha^3)x^3 -$$

$$- (2\,\alpha^{15} - 88\,\alpha^{13} + 912\,\alpha^{11} - 1344\,\alpha^9 + 3840\,\alpha^7 + 73728\,\alpha^5 + 1179648\,\alpha^3 + 18874368\,\alpha)x]\,J_0(\alpha x) +$$

$$+ [256\alpha^8 x^8 - (64\,\alpha^{10} + 128\,\alpha^8 + 16384\,\alpha^6)x^6 + (16\,\alpha^{12} - 352\,\alpha^{10} + 192\,\alpha^8 + 8192\,\alpha^6 + 589824\,\alpha^4)x^4 -$$

$$- (4\,\alpha^{14} - 152\,\alpha^{12} + 1056\,\alpha^{10} + 480\,\alpha^8 + 12288\,\alpha^6 + 294912\,\alpha^4 + 9437184\,\alpha^2)x^2 +$$

$$+ \alpha^{16} - 46\,\alpha^{14} + 532\,\alpha^{12} - 1200\,\alpha^{10} + 1680\,\alpha^8 + 30720\,\alpha^6 + 442368\,\alpha^4 + 4718592\,\alpha^2 + 37748736]\,J_1(\alpha x)\right\} +$$

$$+ \frac{\operatorname{erf}(x)}{\alpha^9}\left[(8\,\alpha^7 x^8 - 384\,\alpha^5 x^6 + 9216\,\alpha^3 x^4 - 73728\,\alpha\,x^2)\,J_0(\alpha x) + \right.$$

$$+ (\alpha^8 x^9 - 64\,\alpha^6 x^7 + 2304\,\alpha^4 x^5 - 36864\,\alpha^2 x^3 + 147456\,x)\,J_1(\alpha x)] -$$

$$- \frac{\alpha^{16} - 48\,\alpha^{14} + 620\,\alpha^{12} - 2112\,\alpha^{10} + 3024\,\alpha^8 + 26880\,\alpha^6 + 368640\,\alpha^4 + 3538944\,\alpha^2 + 18874368}{256\sqrt{\pi}\,\alpha^8}\,.$$

$$\cdot \int e^{-x^2}\,J_0(\alpha x)\,dx +$$

$$+ \frac{\alpha^{16} - 46\,\alpha^{14} + 532\,\alpha^{12} - 1200\,\alpha^{10} + 1680\,\alpha^8 + 30720\,\alpha^6 + 442368\,\alpha^4 + 4718592\,\alpha^2 + 37748736}{256\sqrt{\pi}\,\alpha^9}\,.$$

$$\cdot \int \frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

Recurrence relations:

$$\int x^{2n+2}\operatorname{erf}(x)\,J_1(\alpha x)\,dx = \frac{\alpha^2 + 4n + 4}{2\alpha}\int x^{2n+1}\operatorname{erf}(x)\,J_0(\alpha x)\,dx - \frac{n(2n+1)}{\alpha}\int x^{2n-1}\operatorname{erf}(x)\,J_0(\alpha x)\,dx +$$

$$+ \frac{4n+1}{2}\int x^{2n}\operatorname{erf}(x)\,J_1(\alpha x)\,dx - \frac{x^{2n+1}\,e^{-x^2}}{\alpha\sqrt{\pi}}\,J_0(\alpha x) + \frac{x^{2n}}{2\alpha}\left[(2n+1-2x^2)\,J_0(\alpha x) - \alpha x\,J_1(\alpha x)\right]\operatorname{erf}(x)$$

$$\int x^{2n+1}\operatorname{erf}(x)\,J_0(\alpha x)\,dx = -\frac{4n+\alpha^2}{2\alpha}\int x^{2n}\operatorname{erf}(x)\,J_1(\alpha x)\,dx + \frac{(2n-1)(n-1)}{\alpha}\int x^{2n-2}\operatorname{erf}(x)\,J_1(\alpha x)\,dx +$$

$$+ \frac{4n-1}{2}\int x^{2n-1}\operatorname{erf}(x)\,J_0(\alpha x)\,dx + \frac{x^{2n}\,e^{-x^2}\,J_1(\alpha x)}{\sqrt{\pi}\,\alpha} - \frac{x^{2n-1}\operatorname{erf}(x)}{2\alpha}\left[\alpha x\,J_0(\alpha x) + (2n-1-2x^2)\,J_1(\alpha x)\right]$$

c) The General Case, $I_\nu(x)$

About the basic integrals

$$\tilde{F}_0(x) = \int_0^x e^{-t^2}\,I_0(\alpha t)\,dt \quad \text{and} \quad \tilde{F}_-(x) = \int_0^x \frac{e^{-t^2}\,I_1(\alpha t)\,dt}{t}$$

see page 182 and following.

$$\int \operatorname{erf}(x)\,I_1(\alpha x)\,dx = \frac{\operatorname{erf}(x)\,I_0(\alpha x)}{\alpha} - \frac{2}{\alpha\sqrt{\pi}}\int e^{-x^2}\,I_0(\alpha x)\,dx$$

$$\int x\operatorname{erf}(x)\,I_0(\alpha x)\,dx = \frac{e^{-x^2}}{\sqrt{\pi}\,\alpha}\,I_1(\alpha x) + \frac{x\operatorname{erf}(x)}{\alpha}\,I_1(\alpha x) - \frac{1}{\sqrt{\pi}}\int e^{-x^2}\,I_0(\alpha x)\,dx + \frac{1}{\sqrt{\pi}\,\alpha}\int \frac{e^{-x^2}\,I_1(\alpha x)}{x}\,dx$$

$$\int x^2\operatorname{erf}(x)\,I_1(\alpha x)\,dx = \frac{e^{-x^2}}{2\sqrt{\pi}\,\alpha^2}\left[2\alpha x\,I_0(\alpha x) + (\alpha^2 - 4)\,I_1(\alpha x)\right] + \frac{\operatorname{erf}(x)}{\alpha^2}\left[\alpha x^2\,I_0(\alpha x) - 2x\,I_1(\alpha x)\right] -$$

$$-\frac{\alpha^2 - 2}{2\sqrt{\pi}\,\alpha}\int e^{-x^2}\,I_0(\alpha x)\,dx + \frac{\alpha^2 - 4}{2\sqrt{\pi}\,\alpha^2}\int \frac{e^{-x^2}\,I_1(\alpha x)}{x}\,dx$$

$$\int x^3\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx = \frac{e^{-x^2}}{4\sqrt{\pi}\,\alpha^3}\left\{2\alpha(\alpha^2-4)x\,I_0(\alpha x) + [4\alpha^2 x^2 + \alpha^4 - 2\alpha^2 + 16]\,I_1(\alpha x)\right\}+$$

$$+\frac{\mathrm{erf}(x)}{\alpha^3}\left[2\alpha x^2\,I_0(\alpha x) - (\alpha^2 x^3 + 4x)\,I_1(\alpha x)\right] - \frac{\alpha^4 + 8}{4\sqrt{\pi}\,\alpha^2}\int e^{-x^2}\,I_0(\alpha x)\,dx + \frac{\alpha^4 - 2\alpha^2 + 16}{4\sqrt{\pi}\,\alpha^3}\int \frac{e^{-x^2}\,I_1(\alpha x)}{x}\,dx$$

$$\int x^4\mathrm{erf}(x)\,I_1(\alpha x)\,dx =$$

$$= \frac{e^{-x^2}}{8\sqrt{\pi}\,\alpha^4}\left\{[8\alpha^3 x^3 + 2(\alpha^5 - 2\alpha^3 + 32\alpha)x]\,I_0(\alpha x) + [(4\alpha^4 - 32\alpha^2)x^2 + \alpha^6 + 16\alpha^2 - 128]\,I_1(\alpha x)\right\}+$$

$$+\frac{\mathrm{erf}(x)}{\alpha^4}\left[(\alpha^3 x^4 + 8\alpha x^2)\,I_0(\alpha x) - (4\alpha^2 x^3 + 16x)\,I_1(\alpha x)\right]-$$

$$-\frac{\alpha^6 + 2\alpha^4 + 12\alpha^2 - 64}{8\sqrt{\pi}\,\alpha^3}\int e^{-x^2}\,I_0(\alpha x)\,dx + \frac{\alpha^6 + 16\alpha^2 - 128}{8\sqrt{\pi}\,\alpha^4}\int \frac{e^{-x^2}\,I_1(\alpha x)}{x}\,dx$$

$$\int x^5\mathrm{erf}(x)\,I_0(\alpha x)\,dx = \frac{e^{-x^2}}{16\sqrt{\pi}\,\alpha^5}\left\{[(8\alpha^5 - 64\alpha^3)x^3 + (2\alpha^7 + 8\alpha^5 + 32\alpha^3 - 512\alpha)x]\,I_0(\alpha x)+\right.$$

$$+[16\alpha^4 x^4 + (4\alpha^6 - 8\alpha^4 + 256\alpha^2)x^2 + \alpha^8 + 6\alpha^6 + 12\alpha^4 - 128\alpha^2 + 1024]\,I_1(\alpha x)\}+$$

$$-\frac{\mathrm{erf}(x)}{\alpha^5}\left[(4\alpha^3 x^4 + 32\alpha x^2)\,I_0(\alpha x) - (\alpha^4 x^5 + 16\alpha^2 x^3 + 64x)\,I_1(\alpha x)\right]-$$

$$-\frac{\alpha^8 + 8\alpha^6 + 20\alpha^4 - 96\alpha^2 + 512}{16\sqrt{\pi}\,\alpha^4}\int e^{-x^2}\,I_0(\alpha x)\,dx+$$

$$+\frac{\alpha^8 + 6\alpha^6 + 12\alpha^4 - 128\alpha^2 + 1024}{16\sqrt{\pi}\,\alpha^5}\int \frac{e^{-x^2}\,I_1(\alpha x)}{x}\,dx$$

Recurrence relations:

$$\int x^{2n+2}\,\mathrm{erf}(x)\,I_1(\alpha x)\,dx = \frac{\alpha^2 - 4n - 4}{2\alpha}\int x^{2n+1}\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx + \frac{n(2n+1)}{\alpha}\int x^{2n-1}\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx+$$

$$+\frac{4n+1}{2}\int x^{2n}\,\mathrm{erf}(x)\,I_1(\alpha x)\,dx + \frac{x^{2n+1}\,e^{-x^2}}{\alpha\sqrt{\pi}}\,I_0(\alpha x) - \frac{x^{2n}}{2\alpha}\left[(2n+1 - 2x^2)\,I_0(\alpha x) + \alpha x\,I_1(\alpha x)\right]\,\mathrm{erf}(x)$$

$$\int x^{2n+1}\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx = -\frac{4n - \alpha^2}{2\alpha}\int x^{2n}\,\mathrm{erf}(x)\,I_1(\alpha x)\,dx + \frac{(2n-1)(n-1)}{\alpha}\int x^{2n-2}\,\mathrm{erf}(x)\,I_1(\alpha x)\,dx+$$

$$+\frac{4n-1}{2}\int x^{2n-1}\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx + \frac{x^{2n}\,e^{-x^2}\,I_1(\alpha x)}{\sqrt{\pi}\,\alpha} + \frac{x^{2n-1}\,\mathrm{erf}(x)}{2\alpha}\left[-\alpha x\,I_0(\alpha x) + (2n-1 + 2x^2)\,I_1(\alpha x)\right]$$

1.3.4. Struve Functions

a) Integrals

$$\int \frac{J_0(x)\,\mathbf{H}_1(x)\,dx}{x} = x[J_0(x)\mathbf{H}_0(x) + J_1(x)\mathbf{H}_1(x)] - J_0(x)\mathbf{H}_1(x) - \frac{2xJ_1(x)}{\pi}$$

$$\int \frac{I_0(x)\,\mathbf{L}_1(x)\,dx}{x} = x[I_0(x)\mathbf{L}_0(x) - I_1(x)\mathbf{L}_1(x)] - I_0(x)\mathbf{L}_1(x) - \frac{2xI_1(x)}{\pi}$$

$$\int \frac{K_0(x)\,\mathbf{L}_1(x)\,dx}{x} = x[K_0(x)\mathbf{L}_0(x) + K_1(x)\mathbf{L}_1(x)] - K_0(x)\mathbf{L}_1(x) + \frac{2xK_1(x)}{\pi}$$

$$\int \frac{J_1(x)\,\mathbf{H}_0(x)\,dx}{x} = x[J_0(x)\mathbf{H}_0(x) + J_1(x)\mathbf{H}_1(x)] - J_1(x)\mathbf{H}_0(x) - \frac{2xJ_1(x) + 2J_0(x)}{\pi}$$

$$\int \frac{I_1(x)\,\mathbf{L}_0(x)\,dx}{x} = x[I_0(x)\mathbf{L}_0(x) - I_1(x)\mathbf{L}_1(x)] - I_1(x)\mathbf{L}_0(x) + \frac{2I_0(x) - 2xI_1(x)}{\pi}$$

$$\int \frac{K_1(x)\,\mathbf{L}_0(x)\,dx}{x} = -x[K_0(x)\mathbf{L}_0(x) + K_1(x)\mathbf{L}_1(x)] - K_1(x)\mathbf{L}_0(x) - \frac{2K_0(x) + 2xK_1(x)}{\pi}$$

$$\int \frac{J_1(x)\,\mathbf{H}_1(x)\,dx}{x} = \frac{1}{2}\left\{x[J_1(x)\mathbf{H}_0(x) - J_0(x)\mathbf{H}_1(x)] - J_0(x)\mathbf{H}_0(x) - J_1(x)\mathbf{H}_1(x)\right\} + \frac{xJ_0(x)}{\pi}$$

$$\int \frac{I_1(x)\,\mathbf{L}_1(x)\,dx}{x} = \frac{1}{2}\left\{x[I_1(x)\mathbf{L}_0(x) - I_0(x)\mathbf{L}_1(x)] + I_0(x)\mathbf{L}_0(x) - I_1(x)\mathbf{L}_1(x)\right\} - \frac{xI_0(x)}{\pi}$$

$$\int \frac{K_1(x)\,\mathbf{L}_1(x)\,dx}{x} = \frac{1}{2}\left\{x[K_1(x)\mathbf{L}_0(x) + K_0(x)\mathbf{L}_1(x)] - K_0(x)\mathbf{L}_0(x) - K_1(x)\mathbf{L}_1(x)\right\} + \frac{xK_0(x)}{\pi}$$

$$\int x\,J_0(x)\,\mathbf{H}_0(x)\,dx = \frac{x^2\,[J_0(x)\mathbf{H}_0(x) + J_1(x)\mathbf{H}_1(x)] + x[J_1(x)\mathbf{H}_0(x) - J_0(x)\mathbf{H}_1(x)]}{2} - \frac{x^2 J_1(x)}{\pi}$$

$$\int x\,I_0(x)\,\mathbf{L}_0(x)\,dx = \frac{x^2\,[I_0(x)\mathbf{L}_0(x) - I_1(x)\mathbf{L}_1(x)] + x[I_1(x)\mathbf{L}_0(x) - I_0(x)\mathbf{L}_1(x)]}{2} - \frac{x^2 I_1(x)}{\pi}$$

$$\int x\,K_0(x)\,\mathbf{L}_0(x)\,dx = \frac{x^2\,[K_0(x)\mathbf{L}_0(x) + K_1(x)\mathbf{L}_1(x)] - x[K_1(x)\mathbf{L}_0(x) + K_0(x)lK_1(x)]}{2} + \frac{x^2 K_1(x)}{\pi}$$

$$\int x\,J_1(x)\,\mathbf{H}_1(x)\,dx = \frac{x^2\,[J_0(x)\mathbf{H}_0(x) + J_1(x)\mathbf{H}_1(x)] + x[J_1(x)\mathbf{H}_0(x) - 3J_0(x)\mathbf{H}_1(x)]}{2} - \frac{x^2 J_1(x)}{\pi}$$

$$\int x\,I_1(x)\,\mathbf{L}_1(x)\,dx = \frac{x^2\,[I_1(x)\mathbf{L}_1(x) - I_0(x)\mathbf{L}_0(x)] + x[3I_0(x)\mathbf{L}_1(x) - I_1(x)\mathbf{L}_0(x)]}{2} + \frac{x^2 I_1(x)}{\pi}$$

$$\int x\,K_1(x)\,\mathbf{L}_1(x)\,dx = \frac{x^2\,[K_1(x)\mathbf{L}_1(x) + K_0(x)\mathbf{L}_1(x)] - x[3K_0(x)\mathbf{L}_1(x) + K_1(x)\mathbf{L}_0(x)]}{2} + \frac{x^2 K_1(x)}{\pi}$$

$$\int x^2\,J_0(x)\,\mathbf{H}_1(x)\,dx = \frac{(x^3 + 3x)\,[J_0(x)\mathbf{H}_1(x) - J_1(x)\mathbf{H}_0(x)]}{4} + \frac{x^2\,J_1(x)\mathbf{H}_1(x)}{2} + \frac{3x^2 J_1(x) - x^3 J_0(x)}{2\pi}$$

$$\int x^2\,I_0(x)\,\mathbf{L}_1(x)\,dx = \frac{(x^3 - 3x)\,[I_0(x)\mathbf{L}_1(x) - I_1(x)\mathbf{L}_0(x)]}{4} + \frac{x^2\,I_1(x)\mathbf{L}_1(x)}{2} - \frac{3x^2 I_1(x) - x^3 I_0(x)}{2\pi}$$

$$\int x^2\,K_0(x)\,\mathbf{L}_1(x)\,dx = \frac{(x^3 - 3x)\,[K_0(x)\mathbf{L}_1(x) + K_1(x)\mathbf{L}_0(x)]}{4} - \frac{x^2\,K_1(x)\mathbf{L}_1(x)}{2} + \frac{3x^2 K_1(x) + x^3 K_0(x)}{2\pi}$$

$$\int x^2\,J_1(x)\,\mathbf{H}_0(x)\,dx = \frac{(x^3 + 3x)\,[J_1(x)\mathbf{H}_0(x) - J_0(x)\mathbf{H}_1(x)]}{4} + \frac{x^2\,J_1(x)\mathbf{H}_1(x)}{2} + \frac{x^3 J_0(x) - 3x^2 J_1(x)}{2\pi}$$

$$\int x^2\,I_1(x)\,\mathbf{L}_0(x)\,dx = \frac{(x^3 - 3x)\,[I_1(x)\mathbf{L}_0(x) - I_0(x)\mathbf{L}_1(x)]}{4} + \frac{x^2\,I_1(x)\mathbf{L}_1(x)}{2} - \frac{x^3 I_0(x) - 3x^2 I_1(x)}{2\pi}$$

$$\int x^2\,K_1(x)\,\mathbf{L}_0(x)\,dx = \frac{(x^3 - 3x)\,[K_1(x)\mathbf{L}_0(x) + K_0(x)\mathbf{L}_1(x)]}{4} + \frac{x^2\,K_1(x)\mathbf{L}_1(x)}{2} + \frac{x^3 K_0(x) + 3x^2 K_1(x)}{2\pi}$$

$$\int x^3\,J_0(x)\,\mathbf{H}_0(x)\,dx =$$

$$= \frac{1}{6} \left[x^4 J_0(x)\mathbf{H}_0(x) + x(x^2+6)\, J_0(x)\mathbf{H}_1(x) + x(x^2-6)\, J_1(x)\mathbf{H}_0(x) + x^2(x^2-2)\, J_1(x)\mathbf{H}_1(x) \right] -$$

$$- \frac{2x^3\, J_0(x) + x^2(x^2-6)\, J_1(x)}{3\pi}$$

$$\int x^3\, I_0(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{1}{6} \left[x^4 I_0(x)\mathbf{L}_0(x) + x(x^2-6)\, I_0(x)\mathbf{L}_1(x) + x(x^2+6)\, I_1(x)\mathbf{L}_0(x) - x^2(x^2+2)\, I_1(x)\mathbf{L}_1(x) \right] +$$

$$+ \frac{2x^3\, I_0(x) - x^2(x^2+6)\, I_1(x)}{3\pi}$$

$$\int x^3\, K_0(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{1}{6} \left[x^4 K_0(x)\mathbf{L}_0(x) + x(x^2-6)\, K_0(x)\mathbf{L}_1(x) - x(x^2+6)\, K_1(x)\mathbf{L}_0(x) + x^2(x^2+2)\, K_1(x)\mathbf{L}_1(x) \right] +$$

$$+ \frac{2x^3\, K_0(x) + x^2(x^2+6)\, K_1(x)}{3\pi}$$

$$\int x^3\, J_1(x)\, \mathbf{H}_1(x)\, dx =$$

$$= \frac{1}{6} \left[x^4 J_0(x)\mathbf{H}_0(x) - x(2x^2-15)\, J_0(x)\mathbf{H}_1(x) - x(2x^2+15)\, J_1(x)\mathbf{H}_0(x) + x^2(x^2+4)\, J_1(x)\mathbf{H}_1(x) \right] -$$

$$- \frac{5x^3\, J_0(x) + x^2(x^2-15)\, J_1(x)}{3\pi}$$

$$\int x^3\, I_1(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{1}{6} \left[-x^4 I_0(x)\mathbf{L}_0(x) + x(2x^2+15)\, I_0(x)\mathbf{L}_1(x) + x(2x^2-15)\, I_1(x)\mathbf{L}_0(x) + x^2(x^2-4)\, I_1(x)\mathbf{L}_1(x) \right] +$$

$$+ \frac{x^2(x^2+15)\, I_1(x) - 5x^3\, I_0(x)}{3\pi}$$

$$\int x^3\, K_1(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{1}{6} \left[x^4 K_0(x)\mathbf{L}_0(x) - x(2x^2+15)\, K_0(x)\mathbf{L}_1(x) + x(2x^2-15)\, K_1(x)\mathbf{L}_0(x) + x^2(x^2-4)\, K_1(x)\mathbf{L}_1(x) \right] +$$

$$+ \frac{x^2(x^2+15)\, K_1(x) + 5x^3\, K_0(x)}{3\pi}$$

$$\int x^4\, J_0(x)\, \mathbf{H}_1(x)\, dx =$$

$$= \frac{1}{24} \left[-4x^4\, J_0(x)\mathbf{H}_0(x) + x(3x^4+8x^2-195)\, J_0(x)\mathbf{H}_1(x) - x(3x^4-8x^2-195)\, J_1(x)\mathbf{H}_0(x) + \right.$$

$$\left. + 8x^2(x^2-2)\, J_1(x)\mathbf{H}_1(x) \right] + \frac{x^3(65-3x^2)\, J_0(x) + x^2(19x^2-195)\, J_1(x)}{12\pi}$$

$$\int x^4\, I_0(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{1}{24} \left[4x^4\, I_0(x)\mathbf{L}_0(x) + x(3x^4-8x^2-195)\, I_0(x)\mathbf{L}_1(x) - x(3x^4+8x^2-195)\, I_1(x)\mathbf{L}_0(x) + \right.$$

$$\left. + 8x^2(x^2+2)\, I_1(x)\mathbf{L}_1(x) \right] + \frac{x^3(3x^2+65)\, I_0(x) - x^2(19x^2+195)\, I_1(x)}{12\pi}$$

$$\int x^4\, K_0(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{1}{24} \left[4x^4\, K_0(x)\mathbf{L}_0(x) + x(3x^4-8x^2-195)\, K_0(x)\mathbf{L}_1(x) + x(3x^4+8x^2-195)\, K_1(x)\mathbf{L}_0(x) - \right.$$

$$-8x^2(x^2+2)\,K_1(x)\mathbf{L}_1(x)\big] + \frac{x^3(3x^2+65)\,K_0(x) + x^2(19x^2+195)\,K_1(x)}{12\pi}$$

$$\int x^4 \, J_1(x)\,\mathbf{H}_0(x)\,dx =$$

$$= \frac{1}{24}\,\big[-4x^4\,J_0(x)\mathbf{H}_0(x) - x(3x^4 - 8x^2 - 75)\,J_0(x)\mathbf{H}_1(x) + x(3x^4 + 8x^2 - 75)\,J_1(x)\mathbf{H}_0(x)+$$

$$+8x^2(x^2-2)\,J_1(x)\mathbf{H}_1(x)\big] + \frac{x^3(3x^2-25)\,J_0(x) - x^2(11x^2-75)\,J_1(x)}{12\pi}$$

$$\int x^4 \, I_1(x)\,\mathbf{L}_0(x)\,dx =$$

$$= \frac{1}{24}\,\big[4x^4\,I_0(x)\mathbf{L}_0(x) - x(3x^4 + 8x^2 - 75)\,I_0(x)\mathbf{L}_1(x) + x(3x^4 - 8x^2 - 75)\,I_1(x)\mathbf{L}_0(x)+$$

$$+8x^2(x^2+2)\,I_1(x)\mathbf{L}_1(x)\big] - \frac{x^3(3x^2+25)\,I_0(x) - x^2(11x^2+75)\,I_1(x)}{12\pi}$$

$$\int x^4 \, K_1(x)\,\mathbf{L}_0(x)\,dx =$$

$$= \frac{1}{24}\,\big[-4x^4\,K_0(x)\mathbf{L}_0(x) + x(3x^4 + 8x^2 - 75)\,K_0(x)\mathbf{L}_1(x) + x(3x^4 - 8x^2 - 75)\,K_1(x)\mathbf{L}_0(x)+$$

$$+8x^2(x^2+2)\,K_1(x)\mathbf{L}_1(x)\big] + \frac{x^3(3x^2+25)\,K_0(x) + x^2(11x^2+75)\,K_1(x)}{12\pi}$$

$$\int x^5 \, J_0(x)\,\mathbf{H}_0(x)\,dx =$$

$$= \frac{1}{30}\,\big[x^4(3x^2+8)\,J_0(x)\mathbf{H}_0(x) + x(6x^4 - 16x^2 - 285)\,J_0(x)\mathbf{H}_1(x) + x(6x^4 - 16x^2 + 285)\,J_1(x)\mathbf{H}_0(x)+$$

$$+x^2(3x^4 - 16x^2 + 32)\,J_1(x)\mathbf{H}_1(x)\big] - \frac{x^3(9x^2-95)\,J_0(x) + x^2(3x^4 - 37x^2 + 285)\,J_1(x)}{15\pi}$$

$$\int x^5 \, I_0(x)\,\mathbf{L}_0(x)\,dx =$$

$$= \frac{1}{30}\,\big[x^4(3x^2-8)\,I_0(x)\mathbf{L}_0(x) + x(6x^4 + 16x^2 - 285)\,I_0(x)\mathbf{L}_1(x) + x(6x^4 + 16x^2 + 285)\,I_1(x)\mathbf{L}_0(x)-$$

$$-x^2(3x^4 + 16x^2 + 285)\,I_1(x)\mathbf{L}_1(x)\big] + \frac{x^3(9x^2+95)\,I_0(x) - x^2(3x^4 + 37x^2 + 285)\,I_1(x)}{15\pi}$$

$$\int x^5 \, K_0(x)\,\mathbf{L}_0(x)\,dx =$$

$$= \frac{1}{30}\,\big[x^4(3x^2-8)\,K_0(x)\mathbf{L}_0(x) + x(6x^4 + 16x^2 - 285)\,K_0(x)\mathbf{L}_1(x) - x(6x^4 + 16x^2 + 285)\,K_1(x)\mathbf{L}_0(x)+$$

$$+x^2(3x^4 + 16x^2 + 32)\,K_1(x)\mathbf{L}_1(x)\big] + \frac{x^3(9x^2+95)\,K_0(x) + x^2(3x^4 + 37x^2 + 285)\,K_1(x)}{15\pi}$$

$$\int x^5 \, J_1(x)\,\mathbf{H}_1(x)\,dx =$$

$$= \frac{1}{30}\,\big[3x^4(x^2-4)\,J_0(x)\mathbf{H}_0(x) - 3x(3x^4 - 8x^2 + 420)\,J_0(x)\mathbf{H}_1(x)-$$

$$-3x(3x^4 - 8x^2 - 420)\,J_1(x)\mathbf{H}_0(x)+$$

$$+3x^2(x^4 + 8x^2 - 16)\,J_1(x)\mathbf{H}_1(x)\big] - \frac{4x^3(2x^2-35)\,J_0(x) + x^2(x^2-30)(x^2-14)\,J_1(x)}{5\pi}$$

$$\int x^5 \, I_1(x)\,\mathbf{L}_1(x)\,dx =$$

$$= \frac{1}{30}\,\big[-3x^4(x^2+4)\,I_0(x)\mathbf{L}_0(x) + 3x(3x^4 + 8x^2 + 420)\,I_0(x)\mathbf{L}_1(x)+$$

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$$+3x(3x^4 + 8x^2 - 420)\, I_1(x)\mathbf{L}_0(x)+$$

$$+3x^2(x^4 - 8x^2 - 16)\, I_1(x)\mathbf{L}_1(x)\big] - \frac{4x^3(2x^2 + 35)\, I_0(x) - x^2(x^2 + 30)(x^2 + 14)\, I_1(x)}{5\,\pi}$$

$$\int x^5\, K_1(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{1}{30}\,\big[3x^4(x^2 + 4)\, K_0(x)\mathbf{L}_0(x) - 3x(3x^4 + 8x^2 + 420)\, K_0(x)\mathbf{L}_1(x)+$$

$$+3x(3x^4 + 8x^2 - 420)\, K_1(x)\mathbf{L}_0(x)+$$

$$+3x^2(x^4 - 8x^2 - 16)\, K_1(x)\mathbf{L}_1(x)\big] + \frac{4x^3(2x^2 + 35)\, K_0(x) + x^2(x^2 + 30)(x^2 + 14)\, K_1(x)}{5\,\pi}$$

$$\int x^6\, J_0(x)\, \mathbf{H}_1(x)\, dx =$$

$$= \frac{1}{60}\,\big[-12x^4(x^2 - 4)\, J_0(x)\mathbf{H}_0(x) + x(5x^6 + 36x^4 - 96x^2 + 12915)\, J_0(x)\mathbf{H}_1(x)-$$

$$-x(5x^6 - 36x^4 + 96x^2 + 12915)\, J_1(x)\mathbf{H}_0(x) + 6x^2(3x^4 - 16x^2 + 32)\, J_1(x)\mathbf{H}_1(x)\big] -$$

$$- \frac{x^3(5x^4 - 271x^2 + 4305)\, J_0(x) - x^2(47x^4 - 1403x^2 + 12915)\, J_1(x)}{30\pi}$$

$$\int x^6\, I_0(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{1}{60}\,\big[12x^4(x^2 + 4)\, I_0(x)\mathbf{L}_0(x) + x(5x^6 - 36x^4 - 96x^2 - 12915)\, I_0(x)\mathbf{L}_1(x)-$$

$$-x(5x^6 + 36x^4 + 96x^2 - 12915)\, I_1(x)\mathbf{L}_0(x) + 6x^2(3x^4 + 16x^2 + 32)\, I_1(x)\mathbf{L}_1(x)\big] +$$

$$+ \frac{x^3(5x^4 + 271x^2 + 4305)\, I_0(x) - x^2(47x^4 + 1403x^2 + 12915)\, I_1(x)}{30\pi}$$

$$\int x^6\, K_0(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{1}{60}\,\big[12x^4(x^2 + 4)\, K_0(x)\mathbf{L}_0(x) + x(5x^6 - 36x^4 - 96x^2 - 12915)\, K_0(x)\mathbf{L}_1(x)+$$

$$+x(5x^6 + 36x^4 + 96x^2 - 12915)\, K_1(x)\mathbf{L}_0(x) - 6x^2(3x^4 + 16x^2 + 32)\, K_1(x)\mathbf{L}_1(x)\big] +$$

$$+ \frac{x^3(5x^4 + 271x^2 + 4305)\, K_0(x) + x^2(47x^4 + 1403x^2 + 12915)\, K_1(x)}{30\pi}$$

$$\int x^6\, J_1(x)\, \mathbf{H}_0(x)\, dx =$$

$$= \frac{1}{60}\,\big[-12x^4(x^2 - 4)\, J_0(x)\mathbf{H}_0(x) - x(5x^6 - 36x^4 + 96x^2 + 2835)\, J_0(x)\mathbf{H}_1(x)+$$

$$+x(5x^6 + 36x^4 - 96x^2 + 2835)\, J_1(x)\mathbf{H}_0(x) + 6x^2(3x^4 - 16x^2 + 32)\, J_1(x)\mathbf{H}_1(x)\big] +$$

$$+ \frac{x^3(5x^4 - 79x^2 + 945)\, J_0(x) - x^2(23x^4 - 347x^2 + 2835)\, J_1(x)}{30\pi}$$

$$\int x^6\, I_1(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{1}{60}\,\big[12x^4(x^2 + 4)\, I_0(x)\mathbf{L}_0(x) - x(5x^6 + 36x^4 + 96x^2 - 2835)\, I_0(x)\mathbf{L}_1(x)+$$

$$+x(5x^6 - 36x^4 - 96x^2 - 2835)\, I_1(x)\mathbf{L}_0(x) + 6x^2(3x^4 + 16x^2 + 32)\, I_1(x)\mathbf{L}_1(x)\big] -$$

$$- \frac{x^3(5x^4 + 79x^2 + 945)\, I_0(x) - x^2(23x^4 + 347x^2 + 2835)\, I_1(x)}{30\pi}$$

$$\int x^6\, K_1(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{1}{60} \left[-12x^4(x^2+4)\, K_0(x)\mathbf{L}_0(x) + x(5x^6+36x^4+96x^2-2835)\, K_0(x)\mathbf{L}_1(x)+ \right.$$

$$\left. +x(5x^6-36x^4-96x^2-2835)\, K_1(x)\mathbf{L}_0(x) + 6x^2(3x^4+16x^2+32)\, K_1(x)\mathbf{L}_1(x) \right] +$$

$$+ \frac{x^3(5x^4+79x^2+945)\, K_0(x) + x^2(23x^4+347x^2+2835)\, K_1(x)}{30\pi}$$

$$\int x^7 J_0(x)\, \mathbf{H}_0(x)\, dx =$$

$$= \frac{1}{70} \left[x^4(5x^4+36x^2-144)\, J_0(x)\mathbf{H}_0(x) + 3x(5x^6-36x^4+96x^2+5460)\, J_0(x)\mathbf{H}_1(x)+ \right.$$

$$\left. +3x(5x^6-36x^4+96x^2-5460)\, J_1(x)\mathbf{H}_0(x) + x^2(5x^6-54x^4+288x^2-576)\, J_1(x)\mathbf{H}_1(x) \right] -$$

$$- \frac{4x^3(5x^4-103x^2+1365)\, J_0(x) + x^2(5x^6-104x^4+1916x^2-16380)\, J_1(x)}{35\pi}$$

$$\int x^7 I_0(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{1}{70} \left[x^4(5x^4-36x^2-144)\, I_0(x)\mathbf{L}_0(x) + 3x(5x^6+36x^4+96x^2-5460)\, I_0(x)\mathbf{L}_1(x)+ \right.$$

$$\left. +3x(5x^6+36x^4+96x^2+5460)\, I_1(x)\mathbf{L}_0(x) - x^2(5x^6+54x^4+288x^2+576)\, I_1(x)\mathbf{L}_1(x) \right] +$$

$$+ \frac{4x^3(5x^4+103x^2+1365)\, I_0(x) - x^2(5x^6+104x^4+1916x^2+16380)\, I_1(x)}{35\pi}$$

$$\int x^7 K_0(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{1}{70} \left[x^4(5x^4-36x^2-144)\, K_0(x)\mathbf{L}_0(x) + 3x(5x^6+36x^4+96x^2-5460)\, K_0(x)\mathbf{L}_1(x)- \right.$$

$$\left. -3x(5x^6+36x^4+96x^2+5460)\, K_1(x)\mathbf{L}_0(x) + x^2(5x^6+54x^4+288x^2+576)\, K_1(x)\mathbf{L}_1(x) \right] +$$

$$+ \frac{4x^3(5x^4+103x^2+1365)\, K_0(x) + x^2(5x^6+104x^4+1916x^2+16380)\, K_1(x)}{35\pi}$$

$$\int x^7 J_1(x)\, \mathbf{H}_1(x)\, dx =$$

$$= \frac{1}{70} \left[x^4(5x^4-48x^2+192)\, J_0(x)\mathbf{H}_0(x) - x(20x^6-144x^4+384x^2-106785)\, J_0(x)\mathbf{H}_1(x)- \right.$$

$$\left. -x(20x^6-144x^4+384x^2+106785)\, J_1(x)\mathbf{H}_0(x) + x^2(5x^6+72x^4-384x^2+768)\, J_1(x)\mathbf{H}_1(x) \right] -$$

$$- \frac{x^3(55x^4-2309x^2+35595)\, J_0(x) + x^2(5x^6-433x^4+11737x^2-106785)\, J_1(x)}{35\pi}$$

$$\int x^7 I_1(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{1}{70} \left[-x^4(5x^4+48x^2+192)\, I_0(x)\mathbf{L}_0(x) + x(20x^6+144x^4+384x^2+106785)\, I_0(x)\mathbf{L}_1(x)+ \right.$$

$$\left. +x(20x^6+144x^4+384x^2-106785)\, I_1(x)\mathbf{L}_0(x) + x^2(5x^6-72x^4-384x^2-768)\, I_1(x)\mathbf{L}_1(x) \right] -$$

$$- \frac{x^3(55x^4+2309x^2+35595)\, I_0(x) - x^2(5x^6+433x^4+11737x^2+106785)\, I_1(x)}{35\pi}$$

$$\int x^7 K_1(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{1}{70} \left[x^4(5x^4+48x^2+192)\, K_0(x)\mathbf{L}_0(x) - x(20x^6+144x^4+384x^2+106785)\, K_0(x)\mathbf{L}_1(x)+ \right.$$

$$\left. +x(20x^6+144x^4+384x^2-106785)\, K_1(x)\mathbf{L}_0(x) + x^2(5x^6-72x^4-384x^2-768)\, K_1(x)\mathbf{L}_1(x) \right] +$$

$$+ \frac{x^3(55x^4+2309x^2+35595)\, K_0(x) + x^2(5x^6+433x^4+11737x^2+106785)\, K_1(x)}{35\pi}$$

b) Recurrence relations:

I) Functions of the First Kind, $J_\nu(x)$:

$$\int x^{2n+1} J_0(x)\, \mathbf{H}_0(x)\, dx =$$

$$= \frac{x^{2n+1}\left[xJ_0(x)\mathbf{H}_0(x) + 2n\, J_0(x)\mathbf{H}_1(x) + x\, J_1(x)\mathbf{H}_1(x)\right]}{2(2n+1)} -$$

$$- \frac{2n^2}{2n+1}\int x^{2n} J_0(x)\mathbf{H}_1(x)\, dx - \frac{1}{(2n+1)\pi}\int x^{2n+2} J_0(x)\, dx$$

$$\int x^{2n+1} J_1(x)\, \mathbf{H}_1(x)\, dx =$$

$$= \frac{x^{2n+1}\left[x\, J_0(x)\mathbf{H}_0(x) - 2(n+1)\, J_0(x)\mathbf{H}_1(x) + x\, J_1(x)\mathbf{H}_1(x)\right]}{2(2n+1)} +$$

$$+ \frac{2n(n+1)}{2n+1}\int x^{2n} J_0(x)\mathbf{H}_1(x)\, dx - \frac{1}{(2n+1)\pi}\int x^{2n+2} J_0(x)\, dx$$

$$\int x^{2n+2} J_0(x)\, \mathbf{H}_1(x)\, dx =$$

$$= \frac{x^{2n+2}\left[x\, J_0(x)\mathbf{H}_1(x) - x\, J_1(x)\mathbf{H}_0(x) + 2(n+1)\, J_1(x)\mathbf{H}_1(x)\right]}{4(n+1)} -$$

$$- n\int x^{2n+1} J_1(x)\mathbf{H}_1(x)\, dx + \frac{1}{2(n+1)\pi}\int x^{2n+3} J_1(x)\, dx$$

$$\int x^{2n+2} J_1(x)\, \mathbf{H}_0(x)\, dx =$$

$$= \frac{x^{2n+2}\left[x\, J_1(x)\mathbf{H}_0(x) - x\, J_0(x)\mathbf{H}_1(x) + 2(n+1)\, J_1(x)\mathbf{H}_1(x)\right]}{4(n+1)} -$$

$$- n\int x^{2n+1} J_1(x)\mathbf{H}_1(x)\, dx - \frac{1}{2(n+1)\pi}\int x^{2n+3} J_1(x)\, dx$$

II) Functions of the Second Kind, $I_\nu(x)$:

$$\int x^{2n+1} I_0(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{x^{2n+1}\left[xI_0(x)\mathbf{L}_0(x) + 2n\, I_0(x)\mathbf{L}_1(x) - x\, I_1(x)\mathbf{L}_1(x)\right]}{2(2n+1)} -$$

$$- \frac{2n^2}{2n+1}\int x^{2n} I_0(x)\mathbf{L}_1(x)\, dx - \frac{1}{(2n+1)\pi}\int x^{2n+2} I_0(x)\, dx$$

$$\int x^{2n+1} I_1(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{x^{2n+1}\left[-x\, I_0(x)\mathbf{L}_0(x) + 2(n+1)\, I_0(x)\mathbf{L}_1(x) + x\, I_1(x)\mathbf{L}_1(x)\right]}{2(2n+1)} -$$

$$- \frac{2n(n+1)}{2n+1}\int x^{2n} I_0(x)\mathbf{L}_1(x)\, dx + \frac{1}{(2n+1)\pi}\int x^{2n+2} I_0(x)\, dx$$

$$\int x^{2n+2} I_0(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{x^{2n+2}\left[x\, I_0(x)\mathbf{L}_1(x) - x\, I_1(x)\mathbf{L}_0(x) + 2(n+1)\, I_1(x)\mathbf{L}_1(x)\right]}{4(n+1)} -$$

$$- n\int x^{2n+1} I_1(x)\mathbf{L}_1(x)\, dx + \frac{1}{2(n+1)\pi}\int x^{2n+3} I_1(x)\, dx$$

$$\int x^{2n+2} I_1(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{x^{2n+2}\left[x\, I_1(x)\mathbf{L}_0(x) - x\, I_0(x)\mathbf{L}_1(x) + 2(n+1)\, I_1(x)\mathbf{L}_1(x)\right]}{4(n+1)} -$$

$$- n \int x^{2n+1} I_1(x)\mathbf{H}_1(x)\, dx - \frac{1}{2(n+1)\pi} \int x^{2n+3} I_1(x)\, dx$$

III) Functions of the Second Kind, $K_\nu(x)$:

$$\int x^{2n+1} K_0(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{x^{2n+1}\left[x K_0(x)\mathbf{L}_0(x) + 2n\, K_0(x)\mathbf{L}_1(x) + x\, K_1(x)\mathbf{L}_1(x)\right]}{2(2n+1)} -$$

$$- \frac{2n^2}{2n+1} \int x^{2n} K_0(x)\mathbf{L}_1(x)\, dx - \frac{1}{(2n+1)\pi} \int x^{2n+2} K_0(x)\, dx$$

$$\int x^{2n+1} K_1(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{x^{2n+1}\left[x\, K_0(x)\mathbf{L}_0(x) - 2(n+1)\, K_0(x)\mathbf{L}_1(x) + x\, K_1(x)\mathbf{L}_1(x)\right]}{2(2n+1)} +$$

$$+ \frac{2n(n+1)}{2n+1} \int x^{2n} K_0(x)\mathbf{L}_1(x)\, dx - \frac{1}{(2n+1)\pi} \int x^{2n+2} \mathbf{K}_0(x)\, dx$$

$$\int x^{2n+2} K_0(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{x^{2n+2}\left[x\, K_0(x)\mathbf{L}_1(x) + x\, K_1(x)\mathbf{L}_0(x) - 2(n+1)\, K_1(x)\mathbf{L}_1(x)\right]}{4(2n+1)} +$$

$$+ n \int x^{2n+1} K_1(x)\mathbf{L}_1(x)\, dx - \frac{1}{2(n+1)\pi} \int x^{2n+3} K_1(x)\, dx$$

$$\int x^{2n+2} K_1(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{x^{2n+2}\left[x\, K_0(x)\mathbf{L}_1(x) + x\, K_1(x)\mathbf{L}_0(x) + 2(n+1)\, K_1(x)\mathbf{L}_1(x)\right]}{4(2n+1)} -$$

$$- n \int x^{2n+1} K_1(x)\mathbf{L}_1(x)\, dx - \frac{1}{2(n+1)\pi} \int x^{2n+3} K_1(x)\, dx$$

c) Some Special Cases:

$$\int x^2 J_0(x)\, \mathbf{H}_1(\sqrt{3}\, x)\, dx =$$

$$= \frac{-\sqrt{3}\, x^2 J_0(x)\mathbf{H}_0(\sqrt{3}\, x) + 3x\, J_0(x)\mathbf{H}_1(\sqrt{3}\, x) - \sqrt{3}\, x\, J_1(x)\mathbf{H}_0(\sqrt{3}\, x) - x^2 J_1(x)\mathbf{H}_1(\sqrt{3}\, x)}{2} + \frac{3x^2 J_1(x)}{\pi}$$

$$\int x^2 I_0(x)\, \mathbf{L}_1(\sqrt{3}\, x)\, dx =$$

$$= \frac{\sqrt{3}\, x^2 I_0(x)\mathbf{L}_0(\sqrt{3}\, x) - 3x\, I_0(x)\mathbf{L}_1(\sqrt{3}\, x) + \sqrt{3}\, x\, I_1(x)\mathbf{L}_0(\sqrt{3}\, x) - x^2 I_1(x)\mathbf{L}_1(\sqrt{3}\, x)}{2} - \frac{3x^2 I_1(x)}{\pi}$$

$$\int x^2 K_0(x)\, \mathbf{L}_1(\sqrt{3}\, x)\, dx =$$

$$= \frac{\sqrt{3}\, x^2 K_0(x)\mathbf{L}_0(\sqrt{3}\, x) - 3x\, K_0(x)\mathbf{L}_1(\sqrt{3}\, x) - \sqrt{3}\, x\, K_1(x)\mathbf{L}_0(\sqrt{3}\, x) + x^2 K_1(x)\mathbf{L}_1(\sqrt{3}\, x)}{2} + \frac{3x^2 K_1(x)}{\pi}$$

$$\int x^2 J_1(x)\, \mathbf{H}_0(\sqrt{3}\, x)\, dx =$$

$$= \frac{x^2 J_0(x)\mathbf{H}_0(\sqrt{3}\,x) - \sqrt{3}\,x\, J_0(x)\mathbf{H}_1(\sqrt{3}\,x) + x\, J_1(x)\mathbf{H}_0(\sqrt{3}\,x) + \sqrt{3}\,x^2 J_1(x)\mathbf{H}_1(\sqrt{3}\,x)}{2} - \frac{\sqrt{3}\,x^2 J_1(x)}{\pi}$$

$$\int x^2 I_1(x)\,\mathbf{L}_0(\sqrt{3}\,x)\,dx =$$

$$= \frac{-x^2 I_0(x)\mathbf{L}_0(\sqrt{3}\,x) + \sqrt{3}\,x\, I_0(x)\mathbf{L}_1(\sqrt{3}\,x) - x\, I_1(x)\mathbf{L}_0(\sqrt{3}\,x) + \sqrt{3}\,x^2 I_1(x)\mathbf{L}_1(\sqrt{3}\,x)}{2} + \frac{\sqrt{3}\,x^2 I_1(x)}{\pi}$$

$$\int x^2 K_1(x)\,\mathbf{L}_0(\sqrt{3}\,x)\,dx =$$

$$= \frac{x^2 K_0(x)\mathbf{L}_0(\sqrt{3}\,x) - \sqrt{3}\,x\, K_0(x)\mathbf{L}_1(\sqrt{3}\,x) - x\, I_1(x)\mathbf{L}_0(\sqrt{3}\,x) + \sqrt{3}\,x^2 K_1(x)\mathbf{L}_1(\sqrt{3}\,x)}{2} + \frac{\sqrt{3}\,x^2 K_1(x)}{\pi}$$

$$\int x^3 J_0(x)\,\mathbf{H}_0((\sqrt{2}+i)x)\,dx =$$

$$= -\frac{(1+\sqrt{2}\,i)x^2}{2}\, J_0(x)\,\mathbf{H}_0((\sqrt{2}+i)x) + \frac{x\,[(\sqrt{2}-2i)x^2 + 3\sqrt{2}]}{4}\, J_0(x)\,\mathbf{H}_1((\sqrt{2}+i)x) +$$

$$+ \frac{x\,[\sqrt{2}\,ix^2 - 2 + \sqrt{2}\,i]}{4}\, J_1(x)\,\mathbf{H}_0((\sqrt{2}+i)x) - \frac{(\sqrt{2}+i)\,x^2}{2}\, J_1(x)\,\mathbf{H}_1((\sqrt{2}+i)x) -$$

$$- \frac{(\sqrt{2}-2i)\,x^3 J_0(x)}{2\pi} + \frac{3\sqrt{2}\,x^2 J_1(x)}{2\pi}$$

$$\int x^3 I_0(x)\,\mathbf{L}_0((\sqrt{2}+i)x)\,dx =$$

$$= \frac{(1+\sqrt{2}\,i)x^2}{2}\, I_0(x)\,\mathbf{L}_0((\sqrt{2}+i)x) + \frac{x\,[(\sqrt{2}-2i)x^2 - 3\sqrt{2}]}{4}\, I_0(x)\,\mathbf{L}_1((\sqrt{2}+i)x) +$$

$$+ \frac{x\,[\sqrt{2}\,ix^2 + (2-\sqrt{2}\,i)]}{4}\, I_1(x)\,\mathbf{L}_0((\sqrt{2}+i)x) - \frac{(\sqrt{2}+i)\,x^2}{2}\, I_1(x)\,\mathbf{L}_1((\sqrt{2}+i)x) +$$

$$+ \frac{(\sqrt{2}-2i)\,x^3 I_0(x)}{2\pi} - \frac{3\sqrt{2}\,x^2 I_1(x)}{2\pi}$$

$$\int x^3 K_0(x)\,\mathbf{L}_0((\sqrt{2}+i)x)\,dx =$$

$$= \frac{(1+\sqrt{2}\,i)x^2}{2}\, K_0(x)\,\mathbf{L}_0((\sqrt{2}+i)x) + \frac{x\,[(\sqrt{2}-2i)x^2 - 3\sqrt{2}]}{4}\, K_0(x)\,\mathbf{L}_1((\sqrt{2}+i)x) -$$

$$- \frac{x\,[\sqrt{2}\,ix^2 + (2-\sqrt{2}\,i)]}{4}\, K_1(x)\,\mathbf{L}_0((\sqrt{2}+i)x) + \frac{(\sqrt{2}+i)\,x^2}{2}\, K_1(x)\,\mathbf{L}_1((\sqrt{2}+i)x) +$$

$$+ \frac{(\sqrt{2}-2i)\,x^3 K_0(x)}{2\pi} + \frac{3\sqrt{2}\,x^2 K_1(x)}{2\pi}$$

$$\int x^3 J_0(x)\,\mathbf{H}_0((\sqrt{2}-i)x)\,dx =$$

$$= -\frac{(1-\sqrt{2}\,i)x^2}{2}\, J_0(x)\,\mathbf{H}_0((\sqrt{2}-i)x) + \frac{x\,[(2+\sqrt{2}\,i)x^2 + 3\sqrt{2}]}{4}\, J_0(x)\,\mathbf{H}_1((\sqrt{2}-i)x) -$$

$$- \frac{x\,[\sqrt{2}\,ix^2 + (2+\sqrt{2}\,i)]}{4}\, J_1(x)\,\mathbf{H}_0((\sqrt{2}-i)x) - \frac{(\sqrt{2}-i)\,x^2}{2}\, J_1(x)\,\mathbf{H}_1((\sqrt{2}-i)x) -$$

$$- \frac{(\sqrt{2}+2i)\,x^3 J_0(x)}{2\pi} + \frac{3\sqrt{2}\,x^2 J_1(x)}{2\pi}$$

$$\int x^3 I_0(x)\,\mathbf{L}_0((\sqrt{2}-i)x)\,dx =$$

$$= \frac{(1-\sqrt{2}\,i)x^2}{2}\, I_0(x)\,\mathbf{L}_0((\sqrt{2}-i)x) + \frac{x\,[(\sqrt{2}+2i)x^2 - 3\sqrt{2}]}{4}\, I_0(x)\,\mathbf{L}_1((\sqrt{2}-i)x) +$$

$$+ \frac{x\,[-\sqrt{2}\,ix^2 + (2+\sqrt{2}\,i)]}{4}\, I_1(x)\,\mathbf{L}_0((\sqrt{2}-i)x) - \frac{(\sqrt{2}-i)\,x^2}{2}\, I_1(x)\,\mathbf{L}_1((\sqrt{2}-i)x) +$$

$$+\frac{(\sqrt{2}+2i)\,x^3\,I_0(x)}{2\pi}-\frac{3\sqrt{2}\,x^2\,I_1(x)}{2\pi}$$

$$\int x^3\,K_0(x)\,\mathbf{L}_0((\sqrt{2}-i)x)\,dx=$$

$$=\frac{(1-\sqrt{2}\,i)x^2}{2}\,K_0(x)\,\mathbf{L}_0((\sqrt{2}-i)x)+\frac{x\,[(\sqrt{2}+2i)x^2-3\sqrt{2}]}{4}\,K_0(x)\,\mathbf{L}_1((\sqrt{2}-i)x)+$$

$$-\frac{x\,[\sqrt{2}\,ix^2-(2+\sqrt{2}\,i)]}{4}\,K_1(x)\,\mathbf{L}_0((\sqrt{2}-i)x)+\frac{(\sqrt{2}-i)\,x^2}{2}\,K_1(x)\,\mathbf{L}_1((\sqrt{2}-i)x)+$$

$$+\frac{(\sqrt{2}+2i)\,x^3\,K_0(x)}{2\pi}+\frac{3\sqrt{2}\,x^2\,K_1(x)}{2\pi}$$

Let $\lambda=[\sqrt{18\sqrt{5}+30}+\sqrt{18\sqrt{5}-30}\,i]/6=1.39691\,34984+0.53357\,34771\,i$, then holds

$$\int x^3\,J_1(x)\,\mathbf{H}_1(\lambda x)\,dx=$$

$$=\frac{([\sqrt{5}-3)\sqrt{18\sqrt{5}+30}-(\sqrt{5}+3)\sqrt{18\sqrt{5}-30}\,i]\,x^3}{8}\,J_0(x)\,\mathbf{H}_0(\lambda x)+\frac{(1-\sqrt{5}\,i)x^3+3(5+\sqrt{5}\,i)x}{4}\,J_0(x)\,\mathbf{H}_1(\lambda x)+$$

$$+\frac{[(1-\sqrt{5})\sqrt{18\sqrt{5}+30}+(1+\sqrt{5})\sqrt{18\sqrt{5}-30}\,i]x^3-[(3+\sqrt{5})\sqrt{18\sqrt{5}+30}-(3-\sqrt{5})\sqrt{18\sqrt{5}-30}\,i]x}{16}\,J_1(x)\,\mathbf{H}_0(\lambda x)-$$

$$-\frac{(1+2\sqrt{5}\,i)x^2}{2}\,J_1(x)\,\mathbf{H}_1(\lambda x)-\frac{(5-\sqrt{5}\,i)x^3}{2\pi}\,J_0(x)+\frac{3(5+\sqrt{5}\,i)x^2}{2\pi}\,J_1(x)$$

$$\int x^3\,I_1(x)\,\mathbf{L}_1(\lambda x)\,dx=$$

$$=\frac{(\sqrt{5}-3)\sqrt{18\sqrt{5}+30}-(\sqrt{5}+3)\sqrt{18\sqrt{5}-30}\,i}{8}\,x^2\,I_0(x)\,\mathbf{L}_0(\lambda x)+\frac{(-1+\sqrt{5}\,i)x^3+3(5+\sqrt{5}\,i)x}{4}\,I_0(x)\,\mathbf{L}_1(\lambda x)+$$

$$+\frac{[(\sqrt{5}-1)\sqrt{18\sqrt{5}+30}-(1+\sqrt{5})\sqrt{18\sqrt{5}-30}\,i]x^3-[(3+\sqrt{5})\sqrt{18\sqrt{5}+30}+(3-\sqrt{5})\sqrt{18\sqrt{5}-30}\,i]x}{16}\,I_1(x)\,\mathbf{L}_0(\lambda x)+$$

$$+\frac{1+2\sqrt{5}\,i}{2}\,x^2\,I_1(x)\,\mathbf{L}_1(\lambda x)-\frac{5-\sqrt{5}\,i}{2\pi}\,I_0(x)+\frac{3(5+\sqrt{5}\,i)}{2\pi}\,I_1(x)$$

$$\int x^3\,K_1(x)\,\mathbf{L}_1(\lambda x)\,dx=$$

$$=\frac{(3-\sqrt{5})\sqrt{18\sqrt{5}+30}+(\sqrt{5}+3)\sqrt{18\sqrt{5}-30}\,i}{8}\,x^2\,K_0(x)\,\mathbf{L}_0(\lambda x)+\frac{(1-\sqrt{5}\,i)x^3-3(5+\sqrt{5}\,i)x}{4}\,K_0(x)\,\mathbf{L}_1(\lambda x)-$$

$$-\frac{[(1-\sqrt{5})\sqrt{18\sqrt{5}+30}+(1+\sqrt{5})\sqrt{18\sqrt{5}-30}\,i]x^3+[(3+\sqrt{5})\sqrt{18\sqrt{5}+30}+(3-\sqrt{5})\sqrt{18\sqrt{5}-30}\,i]x}{16}\,K_1(x)\,\mathbf{L}_0(\lambda x)+$$

$$+\frac{1+2\sqrt{5}\,i}{2}\,x^2\,K_1(x)\,\mathbf{L}_1(\lambda x)+\frac{5-\sqrt{5}\,i}{2\pi}\,K_0(x)+\frac{3(5+\sqrt{5}\,i)}{2\pi}\,K_1(x)$$

With $\mu=\bar{\lambda}=[\sqrt{18\sqrt{5}+30}-\sqrt{18\sqrt{5}-30}\,i]/6$ holds

$$\int x^3\,J_1(x)\,\mathbf{H}_1(\mu x)\,dx=$$

$$=\frac{(\sqrt{5}-3)\sqrt{18\sqrt{5}+30}+(\sqrt{5}+3)\sqrt{18\sqrt{5}-30}\,i}{8}\,x^2\,J_0(x)\,\mathbf{H}_0(\mu x)+\frac{(1+\sqrt{5}\,i)x^3+3(5-\sqrt{5}\,i)x}{4}\,J_0(x)\,\mathbf{H}_1(\mu x)-$$

$$-\frac{[(\sqrt{5}-1)\sqrt{18\sqrt{5}+30}+(1+\sqrt{5}\,i)\sqrt{18\sqrt{5}-30}\,i]x^3+[(3+\sqrt{5})\sqrt{18\sqrt{5}+30}+(\sqrt{5}-3)\sqrt{18\sqrt{5}-30}\,i]x}{16}\,J_1(x)\,\mathbf{H}_0(\mu x)-$$

$$-\frac{1-2\sqrt{5}\,i}{2}\,x^2\,J_1(x)\,\mathbf{H}_1(\lambda x)-\frac{5+\sqrt{5}\,i}{2\pi}\,x^2\,J_0(x)+\frac{3(5-\sqrt{5}\,i)}{2\pi}\,x^3\,J_1(x)$$

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$$\int x^3 \, I_1(x) \, \mathbf{L}_1(\mu x) \, dx =$$

$$= \frac{(\sqrt{5}-3)\sqrt{18\sqrt{5}+30}+(\sqrt{5}+3)\sqrt{18\sqrt{5}-30}\,i}{8} \, x^2 \, I_0(x) \, \mathbf{L}_0(\mu x) - \frac{(1+\sqrt{5}\,i)x^3 - 3(5-\sqrt{5}\,i)x}{4} \, I_0(x) \, \mathbf{L}_1(\mu x) +$$

$$+\frac{[(\sqrt{5}-1)\sqrt{18\sqrt{5}+30}+(\sqrt{5}+1)\sqrt{18\sqrt{5}-30}\,i]x^3 - [(3+\sqrt{5})\sqrt{18\sqrt{5}+30}-(3-\sqrt{5})\sqrt{18\sqrt{5}-30}\,i]x}{16} \, I_1(x) \, \mathbf{L}_0(\mu x) +$$

$$+\frac{1-2\sqrt{5}\,i}{2} \, x^2 \, I_1(x) \, \mathbf{L}_1(\mu x) - \frac{5+\sqrt{5}\,i}{2\pi} \, x^2 \, I_0(x) + \frac{3(5-\sqrt{5}\,i)}{2\pi} \, x^3 \, I_1(x)$$

$$\int x^3 \, K_1(x) \, \mathbf{L}_1(\mu x) \, dx =$$

$$= \frac{(3-\sqrt{5})\sqrt{18\sqrt{5}+30}-(\sqrt{5}+3)\sqrt{18\sqrt{5}-30}\,i}{8} \, x^2 \, K_0(x) \, \mathbf{L}_0(\mu x) + \frac{(1+\sqrt{5}\,i)x^3 - 3(5-\sqrt{5}\,i)x}{4} \, K_0(x) \, \mathbf{L}_1(\mu x) +$$

$$+\frac{[(\sqrt{5}-1)\sqrt{18\sqrt{5}+30}+(1+\sqrt{5})\sqrt{18\sqrt{5}-30}\,i]x^3 - [(3+\sqrt{5})\sqrt{18\sqrt{5}+30}-(3-\sqrt{5})\sqrt{18\sqrt{5}-30}\,i]x}{16} \, K_1(x) \, \mathbf{L}_0(\mu x) +$$

$$+\frac{1-2\sqrt{5}\,i}{2} \, x^2 \, I_1(x) \, \mathbf{L}_1(\mu x) + \frac{5+\sqrt{5}\,i}{2\pi} \, x^2 \, K_0(x) + \frac{3(5-\sqrt{5}\,i)}{2\pi} \, x^3 \, K_1(x)$$

$$\int x^4 \, J_0(x) \, \mathbf{H}_1(\sqrt{3}\,x) \, dx =$$

$$= \frac{\sqrt{3}\,x^2(10-x^2)}{2} \, J_0(x) \, \mathbf{H}_0(\sqrt{3}\,x) + \frac{x(7x^2-30)}{2} \, J_0(x) \, \mathbf{H}_1(\sqrt{3}\,x) +$$

$$+\frac{\sqrt{3}\,x(10-3x^2)}{2} \, J_1(x) \, \mathbf{H}_0(\sqrt{3}\,x) + \frac{x^2(16-x^2)}{2} \, J_1(x) \, \mathbf{H}_1(\sqrt{3}\,x) + \frac{3x^2(x^2-10)}{\pi} \, J_1(x)$$

$$\int x^4 \, I_0(x) \, \mathbf{L}_1(\sqrt{3}\,x) \, dx =$$

$$= \frac{\sqrt{3}\,x^2(x^2+10)}{2} \, I_0(x) \, \mathbf{L}_0(\sqrt{3}\,x) - \frac{x(7x^2+30)}{2} \, I_0(x) \, \mathbf{L}_1(\sqrt{3}\,x) +$$

$$+\frac{\sqrt{3}\,x(3x^2+10)}{2} \, I_1(x) \, \mathbf{L}_0(\sqrt{3}\,x) - \frac{x^2(x^2+16)}{2} \, I_1(x) \, \mathbf{L}_1(\sqrt{3}\,x) - \frac{3x^2(x^2+10)}{\pi} \, I_1(x)$$

$$\int x^4 \, K_0(x) \, \mathbf{L}_1(\sqrt{3}\,x) \, dx =$$

$$= \frac{\sqrt{3}\,(x^2+10)}{2} \, x^2 \, K_0(x) \, \mathbf{L}_0(\sqrt{3}\,x) - \frac{x(7x^2+30)}{2} \, K_0(x) \, \mathbf{L}_1(\sqrt{3}\,x) -$$

$$-\frac{\sqrt{3}\,x(3x^2+10)}{2} \, K_1(x) \, \mathbf{L}_0(\sqrt{3}\,x) + \frac{x^2(x^2+16)}{2} \, K_1(x) \, \mathbf{L}_1(\sqrt{3}\,x) + \frac{3x^2(x^2+10)}{\pi} \, K_1(x)$$

Let $\nu = \sqrt{5}\,(\sqrt{2}+i)/3 = 1.05409\,25534 + 0.74535\,59925\,i$, then holds

$$\int x^4 \, J_0(x) \, \mathbf{H}_1(\nu x) \, dx =$$

$$= -\frac{x^2[(\sqrt{10}-4\sqrt{5}\,i)x^2 + 9\sqrt{5}\,(\sqrt{2}-4i)]}{12} \, J_0(x) \, \mathbf{H}_0(\nu x) - \frac{x[(31+10\sqrt{2}\,i)x^2 - 15(7+4\sqrt{2}\,i)]}{12} \, J_0(x) \, \mathbf{H}_1(\nu x) +$$

$$+\frac{x[(11\sqrt{10}+\sqrt{5}\,i)x^2 + 33\sqrt{10} + 3\sqrt{5}\,i]}{12} \, J_1(x) \, \mathbf{H}_0(\nu x) + \frac{x^2[(2+5\sqrt{2}\,i)x^2 - 28 + 65\sqrt{2}\,i]}{12} \, J_1(x) \, \mathbf{H}_1(\nu x) +$$

$$+\frac{5(13+\sqrt{2}\,i)}{6\pi} \, J_0(x) + \frac{5x^2[(2-\sqrt{2}\,i)x^2 - 3(7+4\sqrt{2}\,i)]}{6\pi} \, J_1(x)$$

$$\int x^4 \, I_0(x) \, \mathbf{L}_1(\nu x) \, dx =$$

$$= \frac{x^2[(\sqrt{10}-4\sqrt{5}\,i)x^2 - 9\sqrt{10} + 36\sqrt{5}\,i]}{12} \, I_0(x) \, \mathbf{L}_0(\nu x) + \frac{x[(31+10\sqrt{2}\,i)x^2 - 105 - 60\sqrt{2}\,i]}{12} \, I_0(x) \, \mathbf{L}_1(\nu x) -$$

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$$-\frac{x[(11\sqrt{10}+\sqrt{5}\,i)x^2-33\sqrt{10}-3\sqrt{5}\,i]}{12}\,I_1(x)\,\mathbf{L}_0(\nu x)+\frac{x^2[(2+5\sqrt{2}\,i)x^2+28-65\sqrt{2}\,i]}{12}\,I_1(x)\,\mathbf{L}_1(\nu x)+$$

$$+\frac{(65+5\sqrt{2}\,i)x^3}{6\pi}\,I_0(x)-\frac{x^2[(10-5\sqrt{2}\,i)x^2+105+60\sqrt{2}\,i]}{6\pi}\,I_0(x)$$

$$\int x^4\,K_0(x)\,\mathbf{L}_1(\nu x)\,dx=$$

$$=\frac{x^2[(\sqrt{10}-4\sqrt{5}\,i)x^2-9\sqrt{5}\,(\sqrt{2}-4i)]}{12}\,K_0(x)\,\mathbf{L}_0(\nu x)-\frac{x[(31+10\sqrt{2}\,i)x^2-15(7+4\sqrt{2}\,i)]}{12}\,K_0(x)\,\mathbf{L}_1(\nu x)+$$

$$+\frac{x[(11\sqrt{10}+\sqrt{5}\,i)x^2-33\sqrt{10}+3\sqrt{5}\,i]}{12}\,K_1(x)\,\mathbf{L}_0(\nu x)-\frac{x^2[(2+5\sqrt{2}\,i)x^2+28-65\sqrt{2}\,i]}{12}\,K_1(x)\,\mathbf{L}_1(\nu x)+$$

$$+\frac{5(13+\sqrt{2}\,i)}{6\pi}\,x^3\,K_0(x)+\frac{5x^2[(2-\sqrt{2}\,i)x^2+3(7+4\sqrt{2}\,i)]}{6\pi}\,K_1(x)$$

With $\eta=\bar{\mu}=\sqrt{5}\,(\sqrt{2}-i)/3$ holds

$$\int x^4\,J_0(x)\,\mathbf{H}_1(\eta x)\,dx=$$

$$=-\frac{x^2[(\sqrt{10}+4\sqrt{5}\,i)x^2+9\sqrt{10}+36\sqrt{5}\,i]}{12}\,J_0(x)\,\mathbf{H}_0(\eta x)-\frac{x[(31-10\sqrt{2}\,i)x^2+105-60\sqrt{2}\,i]}{12}\,J_0(x)\,\mathbf{H}_1(\eta x)+$$

$$+\frac{x[(11\sqrt{10}-\sqrt{5}\,i)x^2+33\sqrt{10}-3\sqrt{5}\,i]}{12}\,J_1(x)\,\mathbf{H}_0(\eta x)+\frac{x^2[(2-5\sqrt{2}\,i)x^2-28-65\sqrt{2}\,i]}{12}\,J_1(x)\,\mathbf{H}_1(\eta x)+$$

$$+\frac{5(13-\sqrt{2}\,i)}{6\pi}\,J_0(x)+\frac{5x^2[(2+\sqrt{2}\,i)x^2-21+12\sqrt{2}\,i]}{6\pi}\,J_1(x)$$

$$\int x^4\,I_0(x)\,\mathbf{L}_1(\eta x)\,dx=$$

$$=\frac{x^2[(\sqrt{10}+4\sqrt{5}\,i)x^2-9\sqrt{10}-36\sqrt{5}\,i]}{12}\,I_0(x)\,\mathbf{L}_0(\eta x)+\frac{x[(31-10\sqrt{2}\,i)x^2-105+60\sqrt{2}\,i]}{12}\,I_0(x)\,\mathbf{L}_1(\eta x)-$$

$$-\frac{x[(11\sqrt{10}-\sqrt{5}\,i)x^2-33\sqrt{10}+3\sqrt{5}\,i]}{12}\,I_1(x)\,\mathbf{L}_0(\eta x)+\frac{x^2[2-5\sqrt{2}\,i)x^2+28+65\sqrt{2}\,i]}{12}\,I_1(x)\,\mathbf{L}_1(\eta x)+$$

$$+\frac{(65-5\sqrt{2}\,i)x^3}{6\pi}\,I_0(x)-\frac{x^2[(10+5\sqrt{2}\,i)x^2+105-60\sqrt{2}\,i]}{6\pi}\,I_1(x)$$

$$\int x^4\,K_0(x)\,\mathbf{L}_1(\eta x)\,dx=$$

$$=\frac{x^2[(\sqrt{10}+4\sqrt{5}\,i)x^2-9\sqrt{10}-36\sqrt{5}\,i]}{12}\,K_0(x)\,\mathbf{L}_0(\eta x)+\frac{x[(31-10\sqrt{2}\,i)x^2-105+60\sqrt{2}\,i]}{12}\,K_0(x)\,\mathbf{L}_1(\eta x)+$$

$$+\frac{x[(11\sqrt{10}-\sqrt{5}\,i)x^2-33\sqrt{10}+3\sqrt{5}\,i]}{12}\,K_1(x)\,\mathbf{L}_0(\eta x)-\frac{x^2[2-5\sqrt{2}\,i)x^2+28+65\sqrt{2}\,i]}{12}\,K_1(x)\,\mathbf{L}_1(\eta x)+$$

$$+\frac{5(13-\sqrt{2}\,i)}{6\pi}\,K_0(x)+\frac{x^2[(10+5\sqrt{2}\,i)x^2+105-60\sqrt{2}\,i]}{6\pi}\,K_1(x)$$

1.3.6. Complete Elliptic Integral $\mathbf{E}(x)$

No integrals with incomplete elliptic functions of the type

$$\int \varphi(x)\,Z_\nu(x)\,\mathbf{E}(\alpha x,k)\,dx\,,\quad \int \varphi(x)\,Z_\nu(x)\,\mathbf{E}(y,\alpha x)\,dx\quad\text{or}$$

$$\int \varphi(x)\,Z_\nu(x)\,\mathbf{F}(\alpha x,k)\,dx\,,\quad \int \varphi(x)\,Z_\nu(x)\,\mathbf{F}(y,\alpha x)\,dx\,,$$

where $Z_\nu(x)$ denotes a Bessel function and $\varphi(x)$ a simple rational function of x, $\sqrt{1-x^2}$ and $\sqrt{1-k^2x^2}$, were found.

$$\int \frac{x^3\,J_0(x)\,\mathbf{E}(x)\,dx}{1-x^2}=J_0(x)\,[\mathbf{K}(x)-\mathbf{E}(x)]-xJ_1(x)\,\mathbf{E}(x)$$

$$\int \left[1 + \frac{\alpha^2}{\alpha^2 x^2 - 1}\right] x \, \mathbf{E}(\alpha x) \, J_0(x) \, dx = \mathbf{E}(\alpha x) \left[x J_1(x) + J_0(x)\right] - \mathbf{K}(\alpha x) \, J_0(x)$$

$$\int (16x^4 + 456x^2 + 333)x \, \mathbf{E}\left(\frac{2\sqrt{3}\,ix}{3}\right) J_0(x) \, dx =$$

$$= (80x^4 + 168x^2 - 63) \, \mathbf{E}\left(\frac{2\sqrt{3}\,ix}{3}\right) J_0(x) + (16x^5 + 56x^3 - 159x) \, \mathbf{E}\left(\frac{2\sqrt{3}\,ix}{3}\right) J_1(x) -$$

$$- (16x^4 - 72x^2 - 63) \, \mathbf{K}\left(\frac{2\sqrt{3}\,ix}{3}\right) J_0(x) + (128x^3 + 96x) \, \mathbf{K}\left(\frac{2\sqrt{3}\,ix}{3}\right) J_1(x)$$

$$\int (16x^4 - 456x^2 + 333)x \, \mathbf{E}\left(\frac{2\sqrt{3}\,x}{3}\right) I_0(x) \, dx =$$

$$= -(80x^4 - 168x^2 - 63) \, \mathbf{E}\left(\frac{2\sqrt{3}\,x}{3}\right) I_0(x) + (16x^5 - 56x^3 - 159x) \, \mathbf{E}\left(\frac{2\sqrt{3}\,x}{3}\right) I_1(x) +$$

$$+ (16x^4 + 72x^2 - 63) \, \mathbf{K}\left(\frac{2\sqrt{3}\,x}{3}\right) I_0(x) - (128x^3 - 96x) \, \mathbf{K}\left(\frac{2\sqrt{3}\,x}{3}\right) I_1(x)$$

$$\int \left[\alpha^4 x^4 + (27\alpha^2 - 2)\alpha^2 x^2 + 18\alpha^4 - 3\alpha^2 + 1\right] x \, \mathbf{E}(\alpha x) \, J_0(x) \, dx =$$

$$= \left[5\alpha^4 x^4 + 6(\alpha^2 - 1)\alpha^2 x^2 + 6\alpha^2 + 1\right] \mathbf{E}(\alpha x) \, J_0(x) + \left[\alpha^4 x^4 + 2(\alpha^2 - 1)\alpha^2 x^2 + 14\alpha^2 + 1\right] x \, \mathbf{E}(\alpha x) \, J_1(x) -$$

$$- \left[\alpha^4 x^4 - 2(3\alpha^2 + 1)\alpha^2 x^2 + 6\alpha^2 + 1\right] \mathbf{K}(\alpha x) \, J_0(x) + 8\left[\alpha^2 x^2 - 1\right]\alpha^2 x \, \mathbf{K}(\alpha x) \, J_1(x)$$

$$\int (4x^4 + 120x^2 + 99) \, x \, \mathbf{E}\left(\frac{\sqrt{6}\,ix}{3}\right) J_0(x) \, dx =$$

$$= (20x^4 + 60x^2 - 27) \, \mathbf{E}\left(\frac{\sqrt{6}\,ix}{3}\right) J_0(x) + (4x^5 + 20x^3 - 75x) \, \mathbf{E}\left(\frac{\sqrt{6}\,ix}{3}\right) J_1(x) -$$

$$- (4x^4 - 12x^2 - 27) \, \mathbf{K}\left(\frac{\sqrt{6}\,ix}{3}\right) J_0(x) + 16x(2x^2 + 3) \, \mathbf{K}\left(\frac{\sqrt{6}\,ix}{3}\right) J_1(x)$$

$$\int (4x^4 - 120x^2 + 99) \, x \, \mathbf{E}\left(\frac{\sqrt{6}\,x}{3}\right) I_0(x) \, dx =$$

$$= -(20x^4 - 60x^2 - 27) \, \mathbf{E}\left(\frac{\sqrt{6}\,x}{3}\right) I_0(x) - (4x^5 - 20x^3 - 75x) \, \mathbf{E}\left(\frac{\sqrt{6}\,x}{3}\right) I_1(x) -$$

$$- (4x^4 + 12x^2 - 27) \, \mathbf{K}\left(\frac{\sqrt{6}\,x}{3}\right) I_0(x) + 16x(2x^2 - 3) \, \mathbf{K}\left(\frac{\sqrt{6}\,x}{3}\right) I_1(x)$$

$$\int \frac{(x^2 + 2)x \, \mathbf{E}(ix) \, J_0(x) \, dx}{1 + x^2} = \mathbf{E}(ix) \left[x \, J_1(x) - 2 J_0(x)\right] - \mathbf{K}(ix) \, J_0(x)$$

$$\int \frac{(x^2 + 2)x \, \mathbf{E}(ix) \, J_0(x) \, dx}{1 + x^2} = \mathbf{E}(ix) \left[x \, J_1(x) + J_0(x)\right] - \mathbf{K}(ix) \, J_0(x)$$

$$\int \frac{(x^2 - 2)x \, \mathbf{E}(x) \, I_0(x) \, dx}{x^2 - 1} = \mathbf{E}(x) \left[I_0(x) - x \, I_1(x)\right] - \mathbf{K}(x) \, I_0(x)$$

$$\int \frac{(\alpha^2 x^4 - x^2 + 1) \, \mathbf{E}(\alpha x) \, J_1(x) \, dx}{(1 - \alpha^2 x^2)x} = \mathbf{E}(\alpha x) \left[x \, J_0(x) - 2 J_1(x)\right] + \mathbf{K}(\alpha x) \, J_1(x)$$

$$\int (2x^2 + 51)x^2 \, \mathbf{E}\left(\frac{\sqrt{2}\,ix}{3}\right) J_1(x) \, dx = -(2x^4 + 21x^2 - 54) \, \mathbf{E}\left(\frac{\sqrt{2}\,ix}{3}\right) J_0(x) +$$

$$+(10x^3 + 63x)\,\mathbf{E}\left(\frac{\sqrt{2}\,ix}{3}\right)J_1(x) - (12x^2 + 54)\,\mathbf{K}\left(\frac{\sqrt{2}\,ix}{3}\right)J_0(x) - (2x^3 + 9x)\,\mathbf{K}\left(\frac{\sqrt{2}\,ix}{3}\right)J_1(x)$$

$$\int (2x^2 + 51)x^2\,\mathbf{E}\left(\frac{\sqrt{2}\,x}{3}\right)I_1(x)\,dx = -(2x^4 - 21x^2 - 54)\,\mathbf{E}\left(\frac{\sqrt{2}\,x}{3}\right)I_0(x) +$$

$$+(10x^3 - 63x)\,\mathbf{E}\left(\frac{\sqrt{2}\,x}{3}\right)I_1(x) + (12x^2 - 54)\,\mathbf{K}\left(\frac{\sqrt{2}\,x}{3}\right)I_0(x) - (2x^3 - 9x)\,\mathbf{K}\left(\frac{\sqrt{2}\,x}{3}\right)I_1(x)$$

$$\int (64x^4 + 816x^2 + 297)x\,\mathbf{K}\left(\frac{2\sqrt{6}\,ix}{3}\right)J_0(x) = (24x^2 - 63)\,\mathbf{E}\left(\frac{2\sqrt{6}\,ix}{3}\right)J_0(x) -$$

$$-144x\,\mathbf{E}\left(\frac{2\sqrt{6}\,ix}{3}\right)J_1(x) + (192x^4 + 240x^2 + 63)\,\mathbf{K}\left(\frac{2\sqrt{6}\,ix}{3}\right)J_0(x) + (64x^5 + 240x^3 + 81x)\,\mathbf{K}\left(\frac{2\sqrt{6}\,ix}{3}\right)J_1(x)$$

$$\int (64x^4 - 816x^2 + 297)x\,\mathbf{K}\left(\frac{2\sqrt{6}\,x}{3}\right)I_0(x) = (24x^2 + 63)\,\mathbf{E}\left(\frac{2\sqrt{6}\,x}{3}\right)I_0(x) -$$

$$-144x\,\mathbf{E}\left(\frac{2\sqrt{6}\,x}{3}\right)I_1(x) - (192x^4 - 240x^2 + 63)\,\mathbf{K}\left(\frac{2\sqrt{6}\,x}{3}\right)I_0(x) + (64x^5 - 240x^3 + 81x)\,\mathbf{K}\left(\frac{2\sqrt{6}\,x}{3}\right)I_1(x)$$

$$\int [\alpha^4 x^4 + 2(6\alpha^2 - 1)\alpha^2 x^2 + 3\alpha^4 - 4\alpha^2 + 1]x\,\mathbf{K}(\alpha x)\,J_0(x)\,dx = -(\alpha^2 x^2 - 3\alpha^2 - 1)\,\mathbf{E}(\alpha x)\,J_0(x) +$$

$$+6\alpha^2 x\,\mathbf{E}\,J_1(x)(\alpha x) + [3\alpha^4 x^4 + (3\alpha^2 - 2)\alpha^2 x^2 - 3\alpha^2 - 1]\,\mathbf{K}(\alpha x)\,J_0(x) +$$
$$+[\alpha^4 x^4 + (3\alpha^2 - 2)\alpha^2 x^2 - 3\alpha^2 + 1]x\,\mathbf{K}(\alpha x)\,J_1(x)$$

$$\int (x^2 + 9)x^2\,\mathbf{K}\left(\frac{\sqrt{2}\,ix}{2}\right)J_0(x) = 8\,\mathbf{E}\left(\frac{\sqrt{2}\,ix}{2}\right)J_0(x) + 2x\,\mathbf{E}\left(\frac{\sqrt{2}\,ix}{2}\right)J_1(x) -$$

$$-(x^4 + 6x^2 + 8)\,\mathbf{K}\left(\frac{\sqrt{2}\,ix}{2}\right)J_0(x) + (3x^3 + 6x)\,\mathbf{K}\left(\frac{\sqrt{2}\,ix}{2}\right)J_1(x)$$

$$\int (x^2 - 9)x^2\,\mathbf{K}\left(\frac{\sqrt{2}\,x}{2}\right)I_0(x) = -8\,\mathbf{E}\left(\frac{\sqrt{2}\,x}{2}\right)I_0(x) + 2x\,\mathbf{E}\left(\frac{\sqrt{2}\,x}{2}\right)I_1(x) +$$

$$+(x^4 - 6x^2 + 8)\,\mathbf{K}\left(\frac{\sqrt{2}\,x}{2}\right)I_0(x) - (3x^3 - 6x)\,\mathbf{K}\left(\frac{\sqrt{2}\,x}{2}\right)I_1(x)$$

1.4.7. $\int x^n\,Z_0(\sqrt{x^2+a}\,dx$ and $\int x^n\,(x+a)^{-1/2}\,Z_1(\sqrt{x^2+a}\,dx$

$$\int x\,J_0(\sqrt{x^2+a}\,)\,dx = \sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)$$

$$\int x\,I_0(\sqrt{x^2+a}\,)\,dx = \sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)$$

$$\int x\,K_0(\sqrt{x^2+a}\,)\,dx = -\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)$$

$$\int x\,\sqrt{x^2+a}\cdot J_1(\sqrt{x^2+a}\,)\,dx = -(x^2+a)\,J_0(\sqrt{x^2+a}\,) + 2\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)$$

$$\int x\,\sqrt{x^2+a}\cdot I_1(\sqrt{x^2+a}\,)\,dx = (x^2+a)\,I_0(\sqrt{x^2+a}\,) - 2\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)$$

$$\int x\,\sqrt{x^2+a}\cdot K_1(\sqrt{x^2+a}\,)\,dx = -(x^2+a)\,K_0(\sqrt{x^2+a}\,) - 2\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x\,J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = -J_0(\sqrt{x^2+a}\,)$$

$$\int \frac{x\, I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = I_0(\sqrt{x^2+a}\,)$$

$$\int \frac{x\, K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = -\,K_0(\sqrt{x^2+a}\,)$$

$$\int x^3\, J_0(\sqrt{x^2+a}\,)\,dx = 2(x^2+a)\, J_0(\sqrt{x^2+a}\,) + \sqrt{x^2+a}\,(x^2-4)\, J_1(\sqrt{x^2+a}\,)$$

$$\int x^3\, I_0(\sqrt{x^2+a}\,)\,dx = -2(x^2+a)\, I_0(\sqrt{x^2+a}\,) + \sqrt{x^2+a}\,(x^2+4)\, I_1(\sqrt{x^2+a}\,)$$

$$\int x^3\, K_0(\sqrt{x^2+a}\,)\,dx = -2(x^2+a)\, K_0(\sqrt{x^2+a}\,) - \sqrt{x^2+a}\,(x^2+4)\, K_1(\sqrt{x^2+a}\,)$$

$$\int x^3\, \sqrt{x^2+a}\cdot J_1(\sqrt{x^2+a}\,)\,dx = -[x^4-(8-a)x^2-8a]\, J_0(\sqrt{x^2+a}\,)+(4x^2-16+2a)\,\sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int x^3\, \sqrt{x^2+a}\cdot I_1(\sqrt{x^2+a}\,)\,dx = [x^4+(8+a)x^2+8a]\, I_0(\sqrt{x^2+a}\,)-(4x^2+16+2a)\,\sqrt{x^2+a}\, I_1(\sqrt{x^2+a}\,)$$

$$\int x^3\, \sqrt{x^2+a}\cdot K_1(\sqrt{x^2+a}\,)\,dx = -[x^4+(8+a)x^2+8a]\, K_0(\sqrt{x^2+a}\,)-(4x^2+16+2a)\,\sqrt{x^2+a}\, K_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^3\, J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = -x^2\, J_0(\sqrt{x^2+a}\,) + 2\sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^3\, I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = x^2\, I_0(\sqrt{x^2+a}\,) - 2\sqrt{x^2+a}\, I_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^3\, K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = -x^2\, K_0(\sqrt{x^2+a}\,) - 2\sqrt{x^2+a}\, K_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, J_0(\sqrt{x^2+a}\,)\,dx = 4\,[x^4+(a-8)x^2-8a]\, J_0(\sqrt{x^2+a}\,) + \sqrt{x^2+a}\,(x^4-16x^2-8a+64)\, J_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, I_0(\sqrt{x^2+a}\,)\,dx = -4\,[x^4+(a+8)x^2+8a]\, I_0(\sqrt{x^2+a}\,) + \sqrt{x^2+a}\,(x^4+16x^2+8a+64)\, I_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, K_0(\sqrt{x^2+a}\,)\,dx = -4\,[x^4+(a+8)x^2+8a]\, K_0(\sqrt{x^2+a}\,) - \sqrt{x^2+a}\,(x^4+16x^2+8a+64)\, K_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, \sqrt{x^2+a}\cdot J_1(\sqrt{x^2+a}\,)\,dx = -[x^6-(24-a)x^4+32(6-a)x^2+8a(24-a)]\, J_0(\sqrt{x^2+a}\,)+$$
$$+[6x^4-4(24-a)x^2+64(6-a)]\,\sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, \sqrt{x^2+a}\cdot I_1(\sqrt{x^2+a}\,)\,dx = [x^6+(24+a)x^4+32(6+a)x^2+8a(24+a)]\, I_0(\sqrt{x^2+a}\,)-$$
$$-[6x^4+4(24+a)x^2+64(6+a)]\,\sqrt{x^2+a}\, I_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, \sqrt{x^2+a}\cdot K_1(\sqrt{x^2+a}\,)\,dx = -[x^6+(24+a)x^4+32(6+a)x^2+8a(24+a)]\, K_0(\sqrt{x^2+a}\,)-$$
$$-[6x^4+4(24+a)x^2+64(6+a)]\,\sqrt{x^2+a}\, K_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^5\, J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = (-x^4+8x^2+8a)\, J_0(\sqrt{x^2+a}\,) + 4(x^2-4)\,\sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^5\, I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = (x^4+8x^2+8a)\, I_0(\sqrt{x^2+a}\,) - 4(x^2+4)\,\sqrt{x^2+a}\, I_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^5\,K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = -\,(x^4+8x^2+8a)\,K_0(\sqrt{x^2+a}\,) - 4(x^2+4)\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)$$

$$\int x^7\,J_0(\sqrt{x^2+a}\,)\,dx = 6\,[x^6+(a-24)x^4+32(6-a)x^2+8a(24-a)]\,J_0(\sqrt{x^2+a}\,)+$$

$$+\sqrt{x^2+a}\,[x^6-36x^4+24(24-a)x^2+384(a-6)]\,J_1(\sqrt{x^2+a}\,)$$

$$\int x^7\,I_0(\sqrt{x^2+a}\,)\,dx = -6\,[x^6+(a+24)x^4+32(6+a)x^2+8a(24+a)]\,I_0(\sqrt{x^2+a}\,)+$$

$$+\sqrt{x^2+a}\,[x^6+36x^4+24(24+a)x^2+384(a+6)]\,I_1(\sqrt{x^2+a}\,)$$

$$\int x^7\,K_0(\sqrt{x^2+a}\,)\,dx = -6\,[x^6+(a+24)x^4+32(6+a)x^2+8a(24+a)]\,K_0(\sqrt{x^2+a}\,)-$$

$$-\sqrt{x^2+a}\,[x^6+36x^4+24(24+a)x^2+384(a+6)]\,K_1(\sqrt{x^2+a}\,)$$

$$\int x^7\,\sqrt{x^2+a}\cdot J_1(\sqrt{x^2+a}\,)\,dx =$$

$$= -[x^8-(48-a)x^6+72(16-a)x^4-24(384-72a+a^2)x^2-576a(16-a)]\,J_0(\sqrt{x^2+a}\,)+$$

$$+[8x^6-6(48-a)x^4+288(16-a)x^2-48(384-72a+a^2)]\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)$$

$$\int x^7\,\sqrt{x^2+a}\cdot I_1(\sqrt{x^2+a}\,)\,dx =$$

$$= [x^8+(48+a)x^6+72(16+a)x^4+24(384+72a+a^2)x^2+576a(16+a)]\,I_0(\sqrt{x^2+a}\,)+$$

$$-[8x^6+6(48+a)x^4+288(16+a)x^2+48(384+72a+a^2)]\,\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)$$

$$\int x^7\,\sqrt{x^2+a}\cdot K_1(\sqrt{x^2+a}\,)\,dx =$$

$$= -[x^8+(48+a)x^6+72(16+a)x^4+24(384+72a+a^2)x^2+576a(16+a)]\,K_0(\sqrt{x^2+a}\,)+$$

$$-[8x^6+6(48+a)x^4+288(16+a)x^2+48(384+72a+a^2)]\,\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^7\,J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = (-x^6+24x^4+24(a-8)x^2-192a)\,J_0(\sqrt{x^2+a}\,)+$$

$$+6(x^4-16x^2+64-8a)\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^7\,I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = (x^6+24x^4+24(a+8)x^2+192a)\,I_0(\sqrt{x^2+a}\,)-$$

$$-6(x^4+16x^2+64+8a)\,\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^7\,K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = -\,(x^6+24x^4+24(a+8)x^2+192a)\,K_0(\sqrt{x^2+a}\,)-$$

$$-6(x^4+16x^2+64+8a)\,\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)$$

Recurrence relations:

$$\int x^{2n+1}\,J_0(\sqrt{x^2+a}\,)\,dx = 2nx^{2n-2}(x^2+a)\,J_0(\sqrt{x^2+a}\,) + x^{2n}\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)-$$

$$-4n^2\int x^{2n-1}\,J_0(\sqrt{x^2+a}\,)\,dx - 4n(n-1)a\int x^{2n-3}\,J_0(\sqrt{x^2+a}\,)\,dx$$

$$\int x^{2n+1}\,I_0(\sqrt{x^2+a}\,)\,dx = -2nx^{2n-2}(x^2+a)\,I_0(\sqrt{x^2+a}\,) + x^{2n}\,\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)+$$

$$+4n^2\int x^{2n-1}\,I_0(\sqrt{x^2+a}\,)\,dx + 4n(n-1)a\int x^{2n-3}\,I_0(\sqrt{x^2+a}\,)\,dx$$

$$\int x^{2n+1} K_0(\sqrt{x^2+a}\,)\,dx = -2nx^{2n-2}(x^2+a)\,K_0(\sqrt{x^2+a}\,) - x^{2n}\,\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)+$$

$$+4n^2 \int x^{2n-1} K_0(\sqrt{x^2+a}\,)\,dx + 4n(n-1)a \int x^{2n-3} K_0(\sqrt{x^2+a}\,)\,dx$$

$$\int x^{2n+1}\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)\,dx =$$

$$= -x^{2n}(x^2+a)\,J_0(\sqrt{x^2+a}\,) + 2x^{2n-2}\,[(n+1)x^2+na]\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)-$$

$$-4n(n+1) \int x^{2n-1}\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)\,dx - 4n(n-1)a \int x^{2n-3}\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)\,dx$$

$$\int x^{2n+1}\,\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)\,dx =$$

$$= x^{2n}(x^2+a)\,I_0(\sqrt{x^2+a}\,) - 2x^{2n-2}\,[(n+1)x^2+na]\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)+$$

$$+4n(n+1) \int x^{2n-1}\,\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)\,dx + 4n(n-1)a \int x^{2n-3}\,\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)\,dx$$

$$\int x^{2n+1}\,\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)\,dx =$$

$$= -x^{2n}(x^2+a)\,K_0(\sqrt{x^2+a}\,) - 2x^{2n-2}\,[(n+1)x^2+na]\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)+$$

$$+4n(n+1) \int x^{2n-1}\,\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)\,dx + 4n(n-1)a \int x^{2n-3}\,\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)\,dx$$

$$\int \frac{x^{2n+1} J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = -x^{2n}\,J_0(\sqrt{x^2+a}\,) + 2nx^{2n-2}\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)-$$

$$-4n(n-1) \int \frac{x^{2n-1} J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} - 4n(n-1)a \int \frac{x^{2n-3} J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}}$$

$$\int \frac{x^{2n+1} I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = x^{2n}\,I_0(\sqrt{x^2+a}\,) - 2nx^{2n-2}\,\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)+$$

$$+4n(n-1) \int \frac{x^{2n-1} I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} + 4n(n-1)a \int \frac{x^{2n-3} I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}}$$

$$\int \frac{x^{2n+1} K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = -x^{2n}\,K_0(\sqrt{x^2+a}\,) - 2nx^{2n-2}\,\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)+$$

$$+4n(n-1) \int \frac{x^{2n-1} K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} + 4n(n-1)a \int \frac{x^{2n-3} K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}}$$

1.4.8. Special cases with $Z_\nu(\sqrt{x^2+a}\,)$ and $Z_\nu(\sqrt{x^2+ax+b}\,)$

$$\int x^3\,\sqrt{x^2-9}\cdot J_0(\sqrt{x^2-9}\,)\,dx = 3(x^2-9)\sqrt{x^2-9}\,J_0(\sqrt{x^2-9}\,) + (x^4-18x^2+81)\,J_1(\sqrt{x^2-9}\,)$$

$$\int x^3\,\sqrt{x^2+9}\cdot I_0(\sqrt{x^2+9}\,)\,dx = -3(x^2+9)\sqrt{x^2+9}\,J_0(\sqrt{x^2+9}\,) + (x^4+18x^2+81)\,I_1(\sqrt{x^2+9}\,)$$

$$\int x^3\,\sqrt{x^2+9}\cdot K_0(\sqrt{x^2+9}\,)\,dx = -3(x^2+9)\sqrt{x^2+9}\,K_0(\sqrt{x^2+9}\,) - (x^4+18x^2+81)\,K_1(\sqrt{x^2+9}\,)$$

$$\int \frac{x^3 J_0(\sqrt{x^2-1}\,)\,dx}{\sqrt{x^2-1}} = \sqrt{x^2-1}\,J_0(\sqrt{x^2-1}\,) + (x^2-1)\,J_1(\sqrt{x^2-1}\,)$$

$$\int \frac{x^3 I_0(\sqrt{x^2+1})\,dx}{\sqrt{x^2+1}} = -\sqrt{x^2+1}\,I_0(\sqrt{x^2+1}) + (x^2+1)\,I_1(\sqrt{x^2+1})$$

$$\int \frac{x^3 K_0(\sqrt{x^2+1})\,dx}{\sqrt{x^2+1}} = -\sqrt{x^2+1}\,K_0(\sqrt{x^2+1}) - (x^2+1)\,K_1(\sqrt{x^2+1})$$

$$\int x^3 J_1(\sqrt{x^2-3})\,dx = -(x^2-3)\sqrt{x^2-3}\,J_0(\sqrt{x^2-3}) + 3(x^2-3)\,J_1(\sqrt{x^2-3})$$

$$\int x^3 I_1(\sqrt{x^2+3})\,dx = (x^2+3)\sqrt{x^2+3}\,I_0(\sqrt{x^2+3}) - 3(x^2-3)\,I_1(\sqrt{x^2+3})$$

$$\int x^3 K_1(\sqrt{x^2+3})\,dx = -(x^2+3)\sqrt{x^2+3}\,K_0(\sqrt{x^2+3}) - 3(x^2-3)\,K_1(\sqrt{x^2+3})$$

$$\int \frac{x^3 J_1(\sqrt{x^2+1})\,dx}{x^2+1} = -\sqrt{x^2+1}\,J_0(\sqrt{x^2+1}) + J_1(\sqrt{x^2+1})$$

$$\int \frac{x^3 I_1(\sqrt{x^2-1})\,dx}{x^2-1} = \sqrt{x^2-1}\,I_0(\sqrt{x^2-1}) - I_1(\sqrt{x^2-1})$$

$$\int \frac{x^3 K_1(\sqrt{x^2-1})\,dx}{x^2-1} = -\sqrt{x^2-1}\,K_0(\sqrt{x^2-1}) - K_1(\sqrt{x^2-1})$$

$$\int x^4 J_0\left(\sqrt{x^2+\sqrt{6}\,x+\frac{9}{4}}\right)dx = \frac{24x^3+4\sqrt{6}\,x^2-66x-45\sqrt{6}}{8}\,J_0\left(\sqrt{x^2+\sqrt{6}\,x+\frac{9}{4}}\right) +$$

$$+\frac{4x^3-2\sqrt{6}\,x^2-30x+11\sqrt{6}}{4}\sqrt{x^2+\sqrt{6}\,x+\frac{9}{4}}\,J_1\left(\sqrt{x^2+\sqrt{6}\,x+\frac{9}{4}}\right)$$

$$\int x^4\left(x^2+\sqrt{2}\,x+\frac{7}{12}\right)^{-1/2}\cdot J_1\left(\sqrt{x^2+\sqrt{2}\,x+\frac{7}{12}}\right)dx =$$

$$= -\frac{4x^3-2\sqrt{2}\,x^2-10x-7\sqrt{2}}{4}\,J_0\left(\sqrt{x^2+\sqrt{2}\,x+\frac{7}{12}}\right) + \frac{6x-5\sqrt{2}}{2}\sqrt{x^2+\sqrt{2}\,x+\frac{7}{12}}\cdot J_1\left(\sqrt{x^2+\sqrt{2}\,x+\frac{7}{12}}\right)$$

$$\int x^4\left(x^2-\sqrt{2}\,x+\frac{7}{12}\right)^{-1/2}\cdot J_1\left(\sqrt{x^2-\sqrt{2}\,x+\frac{7}{12}}\right)dx =$$

$$= -\frac{4x^3+2\sqrt{2}\,x^2-10x+7\sqrt{2}}{4}\,J_0\left(\sqrt{x^2-\sqrt{2}\,x+\frac{7}{12}}\right) + \frac{6x+5\sqrt{2}}{2}\sqrt{x^2-\sqrt{2}\,x+\frac{7}{12}}\cdot J_1\left(\sqrt{x^2-\sqrt{2}\,x+\frac{7}{12}}\right)$$

$$\int x^5\sqrt{x^2+\sqrt{30}\,x+\frac{45}{4}}\cdot J_1\left(\sqrt{x^2+\sqrt{30}\,x+\frac{45}{4}}\right)dx =$$

$$= -\frac{16x^6+8\sqrt{30}\,x^5-324x^4-198\sqrt{30}\,x^3+1722x^2+1767\sqrt{30}\,x+135}{16}\,J_0\left(\sqrt{x^2+\sqrt{30}\,x+\frac{45}{4}}\right) +$$

$$+\frac{48x^4-4\sqrt{30}\,x^3-588x^2-3\sqrt{30}\,x+1767}{8}\sqrt{x^2+\sqrt{30}\,x+\frac{45}{8}}\cdot J_1\left(\sqrt{x^2+\sqrt{30}\,x+\frac{45}{4}}\right)$$

$$\int x^5\sqrt{x^2-\sqrt{30}\,x+\frac{45}{4}}\cdot J_1\left(\sqrt{x^2-\sqrt{30}\,x+\frac{45}{4}}\right)dx =$$

$$= -\frac{16x^6-8\sqrt{30}\,x^5-324x^4+198\sqrt{30}\,x^3+1722x^2-1767\sqrt{30}\,x+135}{16}\,J_0\left(\sqrt{x^2-\sqrt{30}\,x+\frac{45}{4}}\right) +$$

$$+\frac{48x^4+4\sqrt{30}\,x^3-588x^2+3\sqrt{30}\,x+1767}{8}\sqrt{x^2-\sqrt{30}\,x+\frac{45}{8}}\cdot J_1\left(\sqrt{x^2-\sqrt{30}\,x+\frac{45}{4}}\right)$$

$$\int x^5 \left(x^2 + \sqrt{6}\,x + \frac{33}{20}\right)^{-1/2} \cdot J_1\left(\sqrt{x^2 + \sqrt{6}\,x + \frac{33}{20}}\right) dx =$$

$$= -\frac{20x^4 - 10\sqrt{6}\,x^3 - 130x^2 - 25\sqrt{6}\,x + 231}{20} J_0\left(\sqrt{x^2 + \sqrt{6}\,x + \frac{33}{20}}\right) +$$

$$+\frac{8x^2 - 7\sqrt{6}\,x - 5}{2} \sqrt{x^2 + \sqrt{6}\,x + \frac{33}{20}} \cdot J_1\left(\sqrt{x^2 + \sqrt{6}\,x + \frac{33}{20}}\right)$$

$$\int x^5 \left(x^2 - \sqrt{6}\,x + \frac{33}{20}\right)^{-1/2} \cdot J_1\left(\sqrt{x^2 - \sqrt{6}\,x + \frac{33}{20}}\right) dx =$$

$$= -\frac{20x^4 + 10\sqrt{6}\,x^3 - 130x^2 + 25\sqrt{6}\,x + 231}{20} J_0\left(\sqrt{x^2 - \sqrt{6}\,x + \frac{33}{20}}\right) +$$

$$+\frac{8x^2 + 7\sqrt{6}\,x - 5}{2} \sqrt{x^2 - \sqrt{6}\,x + \frac{33}{20}} \cdot J_1\left(\sqrt{x^2 - \sqrt{6}\,x + \frac{33}{20}}\right)$$

$$\int x^5 J_0\left(\sqrt{x^2 + 3\sqrt{2}\,x + \frac{117}{20}}\right) dx =$$

$$= \frac{160x^4 + 60\sqrt{2}\,x^3 - 1244x^2 - 1437\sqrt{2}\,x + 1989}{40} J_0\left(\sqrt{x^2 + 3\sqrt{2}\,x + \frac{117}{20}}\right) +$$

$$+\frac{20x^4 - 30\sqrt{2}\,x^3 - 230x^2 + 255\sqrt{2}\,x + 479}{20} \sqrt{x^2 + 3\sqrt{2}\,x + \frac{117}{20}}\, J_1\left(\sqrt{x^2 + 3\sqrt{2}\,x + \frac{117}{20}}\right)$$

$$\int x^5 J_0\left(\sqrt{x^2 - 3\sqrt{2}\,x + \frac{117}{20}}\right) dx =$$

$$= \frac{160x^4 - 60\sqrt{2}\,x^3 - 1244x^2 + 1437\sqrt{2}\,x + 1989}{40} J_0\left(\sqrt{x^2 - 3\sqrt{2}\,x + \frac{117}{20}}\right) +$$

$$+\frac{20x^4 + 30\sqrt{2}\,x^3 - 230x^2 - 255\sqrt{2}\,x + 479}{20} \sqrt{x^2 - 3\sqrt{2}\,x + \frac{117}{20}}\, J_1\left(\sqrt{x^2 - 3\sqrt{2}\,x + \frac{117}{20}}\right)$$

$$\int \frac{x^5 J_1(\sqrt{x^2-1}\,)\, dx}{x^2 - 1} = (2 - x^2)\sqrt{x^2 - 1}\, J_0(\sqrt{x^2-1}\,) + (3x^2 - 4)\, J_1(\sqrt{x^2-1}\,)$$

$$\int \frac{x^5 I_1(\sqrt{x^2+1}\,)\, dx}{x^2 + 1} = (2 + x^2)\sqrt{x^2 + 1}\, I_0(\sqrt{x^2+1}\,) - (3x^2 + 4)\, I_1(\sqrt{x^2+1}\,)$$

$$\int \frac{x^5 K_1(\sqrt{x^2+1}\,)\, dx}{x^2 + 1} = -(2 + x^2)\sqrt{x^2 + 1}\, K_0(\sqrt{x^2+1}\,) - (3x^2 + 4)\, K_1(\sqrt{x^2+1}\,)$$

$$\int \frac{x^5 J_1(\sqrt{x^2+3}\,)\, dx}{x^2 + 3} = -(x^2 - 6)\sqrt{x^2 + 3}\, J_0(\sqrt{x^2+3}\,) + 3x^2\, J_1(\sqrt{x^2+3}\,)$$

$$\int \frac{x^5 I_1(\sqrt{x^2-3}\,)\, dx}{x^2 + 3} = (x^2 + 6)\sqrt{x^2 - 3}\, I_0(\sqrt{x^2-3}\,) - 3x^2\, I_1(\sqrt{x^2-3}\,)$$

$$\int \frac{x^5 K_1(\sqrt{x^2-3}\,)\, dx}{x^2 + 3} = -(x^2 + 6)\sqrt{x^2 - 3}\, K_0(\sqrt{x^2-3}\,) - 3x^2\, K_1(\sqrt{x^2-3}\,)$$

$$\gamma = \sqrt{5} = 2.23606\ 79775,\ \beta = \sqrt[3]{28 + 12\gamma} = 3.79909\ 52539,\ \alpha = \frac{\beta^2 + \beta + 4}{\beta} = 5.85197\ 75147$$

$$\int \frac{x^7 \, J_1(\sqrt{x^2 + \alpha}\,)\, dx}{x^2 + \alpha} =$$

$$= \frac{1}{4} \left\{ 20x^4 + [(7 - 3\gamma)\beta^2 + 4\beta - 176]x^2 + 6(19\gamma - 45)\beta^2 + 12(\gamma - 15)\beta) \right\} \, J_1(\sqrt{x^2 - \alpha}\,) -$$

$$- \frac{1}{4} \left\{ 4x^4 - [(7 - 3\gamma)\beta^2 + 4\beta + 64]x^2 + 12(\gamma - 2)\beta^2 + 12(1 - \gamma)\beta + 192 \right\} \, \sqrt{x^2 + \alpha} \cdot J_0(\sqrt{x^2 + \alpha}\,)$$

$$\int \frac{x^7 \, I_1(\sqrt{x^2 - \alpha}\,)\, dx}{x^2 + \alpha} =$$

$$= -\frac{1}{4} \left\{ 20x^4 - [(7 - 3\gamma)\beta^2 + 4\beta - 176]x^2 + 6(19\gamma - 45)\beta^2 + 12(\gamma - 15)\beta) \right\} \, I_1(\sqrt{x^2 + \alpha}\,) +$$

$$+ \frac{1}{4} \left\{ 4x^4 + [(7 - 3\gamma)\beta^2 + 4\beta + 64]x^2 + 12(\gamma - 2)\beta^2 + 12(1 - \gamma)\beta + 192 \right\} \, \sqrt{x^2 - \alpha} \cdot I_0(\sqrt{x^2 - \alpha}\,)$$

$$\int \frac{x^7 \, K_1(\sqrt{x^2 - \alpha}\,)\, dx}{x^2 + \alpha} =$$

$$= -\frac{1}{4} \left\{ 20x^4 - [(7 - 3\gamma)\beta^2 + 4\beta - 176]x^2 + 6(19\gamma - 45)\beta^2 + 12(\gamma - 15)\beta) \right\} \, K_1(\sqrt{x^2 + \alpha}\,) -$$

$$- \frac{1}{4} \left\{ 4x^4 + [(7 - 3\gamma)\beta^2 + 4\beta + 64]x^2 + 12(\gamma - 2)\beta^2 + 12(1 - \gamma)\beta + 192 \right\} \, \sqrt{x^2 - \alpha} \cdot K_0(\sqrt{x^2 - \alpha}\,)$$

1.4.9. Integrals with $e^{\alpha x} Z_\nu(\sqrt{x + \beta}\,)$

Only special cases were found.

$$\int x\, e^{\alpha x} J_0 \left(\sqrt{x - \frac{1}{\alpha} - \frac{1}{4\alpha^2}} \right) dx =$$

$$= \frac{e^{\alpha x}}{4\alpha^3} \left[(4\alpha^2 x - 1 - 4\alpha)\, J_0 \left(\sqrt{x - \frac{1}{\alpha} - \frac{1}{4\alpha^2}} \right) + 2\alpha \sqrt{x - \frac{1}{\alpha} - \frac{1}{4\alpha^2}}\, J_1 \left(\sqrt{x - \frac{1}{\alpha} - \frac{1}{4\alpha^2}} \right) \right]$$

$$\int x\, e^{\alpha x} I_0 \left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}} \right) dx =$$

$$= \frac{e^{\alpha x}}{4\alpha^3} \left[(4\alpha^2 x + 1 - 4\alpha)\, I_0 \left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}} \right) - 2\alpha \sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}}\, I_1 \left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}} \right) \right]$$

$$\int x\, e^{\alpha x} K_0 \left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}} \right) dx =$$

$$= \frac{e^{\alpha x}}{4\alpha^3} \left[(4\alpha^2 x + 1 - 4\alpha)\, K_0 \left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}} \right) + 2\alpha \sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}}\, K_1 \left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}} \right) \right]$$

Special cases:

$$\int x\, e^{-x/2} J_0(\sqrt{x + 1}\,)\, dx = -2e^{-x/2} \left[(x + 1)\, J_0(\sqrt{x + 1}\,) - \sqrt{x + 1}\, J_1(\sqrt{x + 1}\,) \right]$$

$$\int x\, e^{x/2} I_0(\sqrt{x - 1}\,)\, dx = 2e^{x/2} \left[(x - 1)\, I_0(\sqrt{x - 1}\,) - \sqrt{x - 1}\, I_1(\sqrt{x - 1}\,) \right]$$

$$\int x\, e^{x/2} K_0(\sqrt{x - 1}\,)\, dx = 2e^{x/2} \left[(x - 1)\, K_0(\sqrt{x - 1}\,) + \sqrt{x - 1}\, K_1(\sqrt{x - 1}\,) \right]$$

$$\int x\, e^{-x/4} J_0(\sqrt{x}\,)\, dx = -4e^{-x/4} \left[x\, J_0(\sqrt{x}\,) - 2\sqrt{x}\, J_1(\sqrt{x}\,) \right]$$

$$\int x\, e^{x/4} I_0(\sqrt{x}\,)\, dx = 4e^{x/4} \left[x\, I_0(\sqrt{x}\,) - 2\sqrt{x}\, I_1(\sqrt{x}\,) \right]$$

$$\int x\, e^{x/4}\, K_0(\sqrt{x}\,)\, dx = 4e^{x/4}\, [x\, K_0(\sqrt{x}\,) + 2\sqrt{x}\, K_1(\sqrt{x}\,)]$$

$$\int x\, e^{x/2}\, J_0(\sqrt{x-3}\,)\, dx = 2e^{x/2}\, [(x-3)\, J_0(\sqrt{x-3}\,) + \sqrt{x-3}\, J_1(\sqrt{x-3}\,)]$$

$$\int x\, e^{-x/2}\, I_0(\sqrt{x+3}\,)\, dx = -2e^{-x/2}\, [(x+3)\, I_0(\sqrt{x+3}\,) + \sqrt{x+3}\, I_1(\sqrt{x+3}\,)]$$

$$\int x\, e^{-x/2}\, K_0(\sqrt{x+3}\,)\, dx = -2e^{-x/2}\, [(x+3)\, K_0(\sqrt{x+3}\,) - \sqrt{x+3}\, K_1(\sqrt{x+3}\,)]$$

$$\int x\, e^{-x/6}\, J_0(\sqrt{x-3}\,)\, dx = -6e^{-x/6}\, [(x-3)\, J_0(\sqrt{x-3}\,) - 3\sqrt{x-3}\, J_1(\sqrt{x-3}\,)]$$

$$\int x\, e^{x/6}\, I_0(\sqrt{x+3}\,)\, dx = 6e^{-x/6}\, [(x+3)\, I_0(\sqrt{x+3}\,) - 3\sqrt{x+3}\, I_1(\sqrt{x+3}\,)]$$

$$\int x\, e^{x/6}\, K_0(\sqrt{x+3}\,)\, dx = 6e^{-x/6}\, [(x+3)\, K_0(\sqrt{x+3}\,) + 3\sqrt{x+3}\, K_1(\sqrt{x+3}\,)]$$

$$\int x\, e^{x/4}\, J_0(\sqrt{x-8}\,)\, dx = 4e^{x/4}\, [(x-8)\, J_0(\sqrt{x-8}\,) + 2\sqrt{x-8}\, J_1(\sqrt{x-8}\,)]$$

$$\int x\, e^{-x/4}\, I_0(\sqrt{x+8}\,)\, dx = -4e^{-x/4}\, [(x+8)\, I_0(\sqrt{x+8}\,) + 2\sqrt{x+8}\, I_1(\sqrt{x+8}\,)]$$

$$\int x\, e^{-x/4}\, K_0(\sqrt{x+8}\,)\, dx = -4e^{-x/4}\, [(x+8)\, K_0(\sqrt{x+8}\,) - 2\sqrt{x+8}\, K_1(\sqrt{x+8}\,)]$$

$$\int x\, e^{-x/8}\, J_0(\sqrt{x-8}\,)\, dx = -8e^{-x/8}\, [(x-8)\, J_0(\sqrt{x-8}\,) - 4\sqrt{x-8}\, J_1(\sqrt{x-8}\,)]$$

$$\int x\, e^{x/8}\, I_0(\sqrt{x+8}\,)\, dx = 8e^{x/8}\, [(x+8)\, I_0(\sqrt{x+8}\,) - 4\sqrt{x+8}\, I_1(\sqrt{x+8}\,)]$$

$$\int x\, e^{x/8}\, K_0(\sqrt{x+8}\,)\, dx = 8e^{x/8}\, [(x+8)\, K_0(\sqrt{x+8}\,) + 4\sqrt{x+8}\, K_1(\sqrt{x+8}\,)]$$

Generally: Let $r \in \{\pm 1, \pm 2, \pm 3, \ldots\}$, then one gets an integral of the type

$$\int x\, e^{x/2r}\, J_0(\sqrt{x - r(r+2)}\,)\, dx \quad \text{or} \quad \int x\, e^{x/2r}\, \left\{ \begin{matrix} I_0 \\ K_0 \end{matrix} \right\} (\sqrt{x + r(r-2)}\,)\, dx \ .$$

$$\int x\, e^{x/2r}\, J_0(\sqrt{x - r(r+2)}\,)\, dx =$$

$$= \frac{2e^{x/2r}}{r^2}\, \left\{ [x - r(r+2)]\, J_0(\sqrt{x - r(r+2)}\,) + r\, \sqrt{x - r(r+2)}\, J_1(\sqrt{x - r(r+2)}\,) \right\}$$

$$\int x\, e^{x/2r}\, I_0(\sqrt{x + r(r-2)}\,)\, dx =$$

$$= \frac{2e^{x/2r}}{r^2}\, \left\{ [x + r(r-2)]\, I_0(\sqrt{x + r(r+2)}\,) - r\, \sqrt{x + r(r-2)}\, I_1(\sqrt{x + r(r-2)}\,) \right\}$$

$$\int x\, e^{x/2r}\, K_0(\sqrt{x + r(r-2)}\,)\, dx =$$

$$= \frac{2e^{x/2r}}{r^2}\, \left\{ [x + r(r-2)]\, K_0(\sqrt{x + r(r+2)}\,) + r\, \sqrt{x + r(r-2)}\, K_1(\sqrt{x + r(r-2)}\,) \right\}$$

$$\int \frac{x\, e^{\alpha x}\, J_1(\sqrt{x - 1/4\alpha^2}\,)\, dx}{\sqrt{x - 1/4\alpha^2}} = -\frac{e^{\alpha x}}{2\alpha^2}\, \left[J_0\left(\sqrt{x - \frac{1}{4\alpha^2}} \right) - 2\alpha\sqrt{x - \frac{1}{4\alpha^2}}\, J_1\left(\sqrt{x - \frac{1}{4\alpha^2}} \right) \right]$$

$$\int \frac{x\, e^{\alpha x}\, I_1(\sqrt{x + 1/4\alpha^2}\,)\, dx}{\sqrt{x + 1/4\alpha^2}} = -\frac{e^{\alpha x}}{2\alpha^2}\, \left[I_0\left(\sqrt{x + \frac{1}{4\alpha^2}} \right) - 2\alpha\sqrt{x + \frac{1}{4\alpha^2}}\, I_1\left(\sqrt{x + \frac{1}{4\alpha^2}} \right) \right]$$

$$\int \frac{x\,e^{\alpha x}\,K_1(\sqrt{x+1/4\alpha^2}\,)\,dx}{\sqrt{x+1/4\alpha^2}} = \frac{e^{\alpha x}}{2\alpha}\left[K_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right) - 2\alpha\sqrt{x+\frac{1}{4\alpha^2}}\,K_1\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)\right]$$

$(2-\sqrt{2}\,)/8 = 0.07322\,33047\,,\ (2+\sqrt{2}\,)/8 = 0.42677\,66953$

$$\int x^2\,e^{(\sqrt{2}-2)x/8}\,J_0(\sqrt{x}\,)\,dx =$$

$$= -4\,e^{(\sqrt{2}-2)x/8}\left\{[(2+\sqrt{2}\,)x - 8(4+3\sqrt{2}\,)]x\,J_0(\sqrt{x}\,) - 4[(3+2\sqrt{2}\,)x - 16 - 12\sqrt{2}\,]\,\sqrt{x}\,J_1(\sqrt{x}\,)\right\}$$

$$\int x^2\,e^{(2-\sqrt{2}\,)x/8}\,I_0(\sqrt{x}\,)\,dx =$$

$$= 4\,e^{(2-\sqrt{2}\,)x/8}\left\{[(2+\sqrt{2}\,)x + 8(4+3\sqrt{2}\,)]x\,I_0(\sqrt{x}\,) - 4[(3+2\sqrt{2}\,)x + 16 + 12\sqrt{2}\,]\,\sqrt{x}\,I_1(\sqrt{x}\,)\right\}$$

$$\int x^2\,e^{(2-\sqrt{2}\,)x/8}\,K_0(\sqrt{x}\,)\,dx =$$

$$= 4\,e^{(2-\sqrt{2}\,)x/8}\left\{[(2+\sqrt{2}\,)x + 8(4+3\sqrt{2}\,)]x\,K_0(\sqrt{x}\,) + 4[(3+2\sqrt{2}\,)x + 16 + 12\sqrt{2}\,]\,\sqrt{x}\,K_1(\sqrt{x}\,)\right\}$$

$$\int x^2\,e^{-(\sqrt{2}+2)x/8}\,J_0(\sqrt{x}\,)\,dx =$$

$$= 4\,e^{-(\sqrt{2}-2)x/8}\left\{[(\sqrt{2}-2)x + 8(4-3\sqrt{2}\,)]x\,J_0(\sqrt{x}\,) + 4[(3-2\sqrt{2}\,)x - 16 + 12\sqrt{2}\,]\,\sqrt{x}\,J_1(\sqrt{x}\,)\right\}$$

$$\int x^2\,e^{(\sqrt{2}+2)x/8}\,I_0(\sqrt{x}\,)\,dx =$$

$$= 4\,e^{(\sqrt{2}+2)x/8}\left\{[(2-\sqrt{2}\,)x + 8(4-3\sqrt{2}\,)]x\,I_0(\sqrt{x}\,) - 4[(3-2\sqrt{2}\,)x + 16 - 12\sqrt{2}\,]\,\sqrt{x}\,I_1(\sqrt{x}\,)\right\}$$

$$\int x^2\,e^{(\sqrt{2}+2)x/8}\,K_0(\sqrt{x}\,)\,dx =$$

$$= 4\,e^{(\sqrt{2}+2)x/8}\left\{[(2-\sqrt{2}\,)x + 8(4-3\sqrt{2}\,)]x\,K_0(\sqrt{x}\,) + 4[(3-2\sqrt{2}\,)x + 16 - 12\sqrt{2}\,]\,\sqrt{x}\,K_1(\sqrt{x}\,)\right\}$$

$$\int \frac{x^2\,e^{-x/4}\,J_0(\sqrt{x+2}\,)\,dx}{\sqrt{x+2}} = -4e^{-x/4}\left[x\sqrt{x+2}\,J_0\left(\sqrt{x+2}\,\right) - 2(x+2)\,J_1\left(\sqrt{x+2}\,\right)\right]$$

$$\int \frac{x^2\,e^{x/4}\,I_0(\sqrt{x-2}\,)\,dx}{\sqrt{x-2}} = 4\,e^{x/4}\left[x\sqrt{x-2}\,I_0\left(\sqrt{x-2}\,\right) - 2(x-2)\,I_1\left(\sqrt{x-2}\,\right)\right]$$

$$\int \frac{x^2\,e^{x/4}\,K_0(\sqrt{x-2}\,)\,dx}{\sqrt{x-2}} = 4\,e^{x/4}\left[x\sqrt{x-2}\,K_0\left(\sqrt{x-2}\,\right) + 2(x-2)\,K_1\left(\sqrt{x-2}\,\right)\right]$$

$$\int \frac{x^2\,e^{-\alpha x^2}\,J_1\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\right)}{\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}} = -\frac{e^{-\alpha x^2}}{4\alpha^7}\left[(2\alpha^3 x + 4\alpha - \sqrt{2}\,)\,J_0\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\right) + \right.$$

$$\left. +\alpha^2(4\alpha^3 x + 4\alpha - 2\sqrt{2}\,)\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\,J_1\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\right)\right]$$

$$\int \frac{x^2\,e^{\alpha x^2}\,I_1\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)}{\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}} = -\frac{e^{\alpha x^2}}{4\alpha^7}\left[(2\alpha^3 x - 4\alpha - \sqrt{2}\,)\,I_0\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right) - \right.$$

$$-\alpha^2(4\alpha^3 x - 4\alpha - 2\sqrt{2})\sqrt{x + \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\, I_1\left(\sqrt{x + \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\right)\Bigg]$$

$$\int \frac{x^2\, e^{\alpha x^2}\, K_1\left(\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\right)}{\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}} = \frac{e^{\alpha x^2}}{4\alpha^7}\Bigg[(2\alpha^3 x - 4\alpha + \sqrt{2})\,K_0\left(\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\right) +$$

$$+\alpha^2(4\alpha^3 x - 4\alpha + 2\sqrt{2})\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\, K_1\left(\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\right)\Bigg]$$

$$\int \frac{x^2\, e^{-\alpha x^2}\, J_1\left(\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} - \frac{1}{4\alpha^4}}\right)}{\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} - \frac{1}{4\alpha^4}}} = -\frac{e^{-\alpha x^2}}{4\alpha^7}\Bigg[(2\alpha^3 x + 4\alpha + \sqrt{2})\,J_0\left(\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} - \frac{1}{4\alpha^4}}\right) +$$

$$+\alpha^2(4\alpha^3 x + 4\alpha + 2\sqrt{2})\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} - \frac{1}{4\alpha^4}}\, J_1\left(\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} - \frac{1}{4\alpha^4}}\right)\Bigg]$$

$$\int \frac{x^2\, e^{\alpha x^2}\, I_1\left(\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\right)}{\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}} = -\frac{e^{\alpha x^2}}{4\alpha^7}\Bigg[(2\alpha^3 x - 4\alpha + \sqrt{2})\,I_0\left(\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\right) -$$

$$-\alpha^2(4\alpha^3 x - 4\alpha + 2\sqrt{2})\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\, I_1\left(\sqrt{x - \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\right)\Bigg]$$

$$\int \frac{x^2\, e^{\alpha x^2}\, K_1\left(\sqrt{x + \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\right)}{\sqrt{x + \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}} = \frac{e^{\alpha x^2}}{4\alpha^7}\Bigg[(2\alpha^3 x - 4\alpha - \sqrt{2})\,K_0\left(\sqrt{x + \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\right) +$$

$$+\alpha^2(4\alpha^3 x - 4\alpha -{}^2\sqrt{2})\sqrt{x + \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\, K_1\left(\sqrt{x + \frac{\sqrt{2}}{2\alpha^3} + \frac{1}{4\alpha^4}}\right)\Bigg]$$

Special cases:

$$\alpha = \frac{\sqrt{2}}{2}\ :\quad \int \frac{x^2\, e^{-x/2}\, J_1(\sqrt{x+1}\,)\, dx}{\sqrt{x+1}} = -2e^{-x/2}\,[(x+2)J_0(\sqrt{x+3}\,) + x\sqrt{x+1}\, J_1(\sqrt{x+1}\,)]$$

$$\alpha = -\frac{\sqrt{2}}{2}\ :\quad \int \frac{x^2\, e^{-x/2}\, J_1(\sqrt{x-3}\,)\, dx}{\sqrt{x-3}} = -2e^{-x/2}\,[(x+6)J_0(\sqrt{x-3}\,) + (x+4)\sqrt{x-3}\, J_1(\sqrt{x-3}\,)]$$

$$\alpha = \frac{\sqrt{2}}{4}\ :\quad \int x^{3/2}\, e^{-x/8}\, J_1(\sqrt{x}\,)\, dx = -8e^{x/8}\,[4x\, J_0(\sqrt{x}\,) + (x-8)\sqrt{x}\, J_1(\sqrt{x}\,)]$$

$$\alpha = -\frac{\sqrt{2}}{4}\ :\quad \int x^2\, e^{-x/8}\, J_1(\sqrt{x-32}\,)\, dx = -8e^{-x/8}\,[(4x+128)\, J_0(\sqrt{x-32}\,) + (x+24)\sqrt{x}\, J_1(\sqrt{x+32}\,)]$$

$$\alpha = \frac{\sqrt{2}}{2}\ :\quad \int \frac{x^2\, e^{x/2}\, I_1(\sqrt{x-1}\,)\, dx}{\sqrt{x-1}} = 2e^{x/2}\,[(2-x)I_0(\sqrt{x-1}\,) + x\sqrt{x-1}\, I_1(\sqrt{x-1}\,)]$$

$$\alpha = -\frac{\sqrt{2}}{2}\ :\quad \int \frac{x^2\, e^{x/2}\, I_1(\sqrt{x+3}\,)\, dx}{\sqrt{x+3}} = 2e^{x/2}\,[(6-x)I_0(\sqrt{x+3}\,) + (x-4)\sqrt{x+3}\, I_1(\sqrt{x+3}\,)]$$

$$\alpha = \frac{\sqrt{2}}{4} \;:\; \int x^{3/2}\, e^{\,x/8}\, I_1(\sqrt{x}\,)\, dx = -8e^{\,x/8}\,[4x\, I_0(\sqrt{x}\,) - (x+8)\sqrt{x}\, I_1(\sqrt{x}\,)]$$

$$\alpha = -\frac{\sqrt{2}}{4} \;:\; \int x^2\, e^{\,x/8}\, I_1(\sqrt{x+32}\,)\, dx = -8e^{\,x/8}\,[(4x-128)\, I_0(\sqrt{x+32}\,) - (x-24)\sqrt{x}\, I_1(\sqrt{x+32}\,)]$$

$$\alpha = \frac{\sqrt{2}}{2} \;:\; \int \frac{x^2\, e^{\,x/2}\, K_1(\sqrt{x+3}\,)\, dx}{\sqrt{x+3}} = 2e^{\,x/2}\,[(x-6)K_0(\sqrt{x+3}\,) + (x-4)\sqrt{x+3}\, K_1(\sqrt{x+3}\,)]$$

$$\alpha = \frac{\sqrt{2}}{2} \;:\; \int \frac{x^2\, e^{\,x/2}\, K_1(\sqrt{x-1}\,)\, dx}{\sqrt{x-1}} = 2e^{\,x/2}\,[(x-2)K_0(\sqrt{x-1}\,) + x\sqrt{x-1}\, K_1(\sqrt{x-1}\,)]$$

$$\alpha = \frac{\sqrt{2}}{4} \;:\; \int x^{3/2}\, e^{\,x/8}\, K_1(\sqrt{x}\,)\, dx = 8e^{\,x/8}\,[4x\, K_0(\sqrt{x}\,) + (x+8)\sqrt{x}\, K_1(\sqrt{x}\,)]$$

$$\alpha = -\frac{\sqrt{2}}{4} \;:\; \int x^2\, e^{\,x/8}\, K_1(\sqrt{x+32}\,)\, dx = 8e^{\,x/8}\,[(4x-128)\, K_0(\sqrt{x+32}\,) + (x-24)\sqrt{x}\, K_1(\sqrt{x+32}\,)]$$

Generally , let $v_1, v_2 \in \{-1, 1\}$ and $r \in \{\pm 1, \pm 2, \pm 3, \ldots\}$, then $\alpha = 1/\sqrt{2}\, r$ gives

$$\int \frac{x^2\, e^{\pm 1/2 r^2}\, Z_1(\sqrt{x+m}\,)\, dx}{\sqrt{x+m}} \qquad \text{with integer coefficients,} \quad m = 2v_1 r^3 + v_2 r^4 = r^3(2v_1 + v_2 r)\,.$$

In the case $v_1 = v_2 = 1$ one has $m \in \{-1, 0, 3, 27, 32, 128, 135, 375, 384, 864, 875, \ldots\}$.

$$0 < \alpha \le \frac{1}{2} \;:\; \beta_1 = \frac{1}{\alpha} - \frac{1}{4\alpha^2} + \frac{1}{2\alpha^2}\sqrt{2\alpha(1-2\alpha)}\,, \;\; \beta_2 = \frac{1}{\alpha} - \frac{1}{4\alpha^2} - \frac{1}{2\alpha^2}\sqrt{2\alpha(1-2\alpha)}\,,$$

$$\beta_3 = -\beta_2\,, \;\; \beta_4 = -\beta_1$$
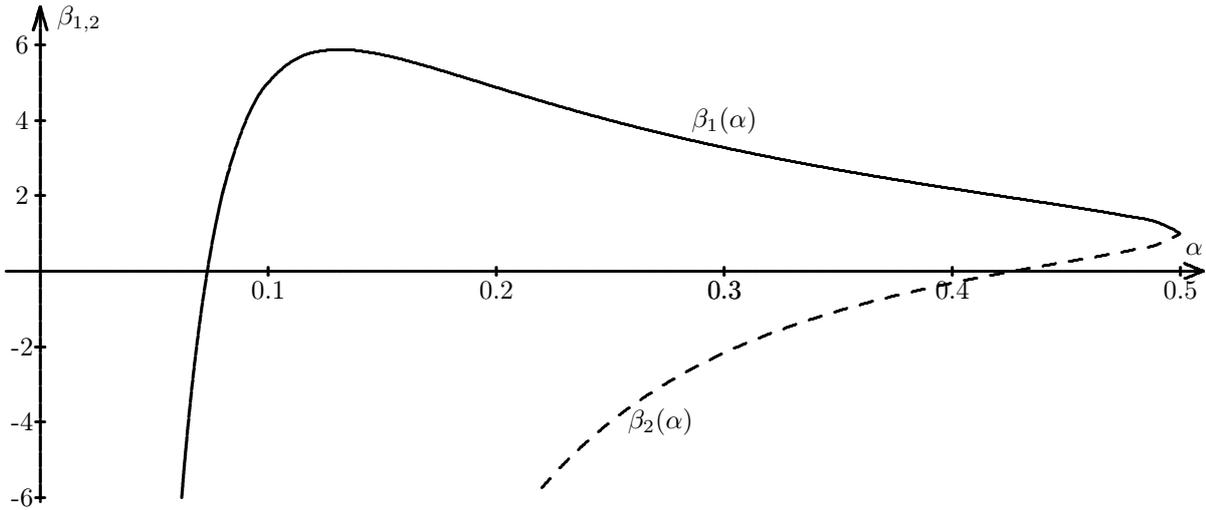
Special values: $\beta_1(0.073223) = 0$, $\beta_2(0.42678) = 0$

α	1/20	2/29	1/10	2/13	1/4	8/25	9/26	2/5	9/20	8/17	1/2
	0.05	0.0690	0.1	0.1538	0.25	0.32	0.3462	0.4	0.45	0.4708	0.5
$\beta_1(\alpha)$	-20	-29/16	5	91/16	4	775/256	221/81	35/16	140/81	391/256	1
		-1.8125		5.6875		3.0273	2.7284	2.1875	1.7284	1.5237	
$\beta_2(\alpha)$	-140	-1189/16	-35	-221/16	-4	-425/256	-91/81	-5/16	20/81	119/256	1
		-74.3125		-13.8125		-1.6602	-1.1235	-0.3125	0.2469	0.4649	

Formulas with rational coefficients may be found for $\alpha = \lambda/\mu$ with natural numbers p_s, q_s, $0 \le p_s \le q_s$, prime factors f_s and $\lambda = 2^{p_2} 3^{p_3} 5^{p_5} \ldots f_m^{p_\lambda}$, $\mu = [2^{q_2} 3^{q_3} 5^{p_5} \ldots f_\mu^{p_\mu}]^2/(2\lambda) + 2\lambda$.

$$\int x^2\, e^{-\alpha x}\, J_0(\sqrt{x+\beta_1}\,) =$$

$$= -\frac{e^{-\alpha x}}{8\alpha^5}\left[\left(8\alpha^4 x^2 + 2\alpha^2(8\alpha-1)x + \sqrt{2\alpha(1-2\alpha)} + 16\alpha^2 - 6\alpha\right) J_0(\sqrt{x+\beta_1}\,) - \right.$$

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$$- \left(4\alpha^3 x - 2\alpha\sqrt{2\alpha(1-2\alpha)} + 8\alpha^2\right)\sqrt{x+\beta_1}\,J_1(\sqrt{x+\beta_1}\,)\Big]$$

$$\int x^2\,e^{-\alpha x}\,J_0(\sqrt{x+\beta_2}\,) =$$

$$= -\frac{e^{-\alpha x}}{8\alpha^5}\Big[\left(8\alpha^4 x^2 + 2\alpha^2(8\alpha - 1)x - \sqrt{2\alpha(1-2\alpha)} + 16\alpha^2 - 6\alpha\right)J_0(\sqrt{x+\beta_2}\,) -$$

$$- \left(4\alpha^3 x + 2\alpha\sqrt{2\alpha(1-2\alpha)} + 8\alpha^2\right)\sqrt{x+\beta_2}\,J_1(\sqrt{x+\beta_2}\,)\Big]$$

$$\int x^2\,e^{\alpha x}\,I_0(\sqrt{x+\beta_3}\,) =$$

$$= \frac{e^{\alpha x}}{8\alpha^5}\Big[\left(8\alpha^4 x^2 + 2\alpha^2(1-8\alpha)x - \sqrt{2\alpha(1-2\alpha)} + 16\alpha^2 - 6\alpha\right)I_0(\sqrt{x+\beta_3}\,) -$$

$$- \left(4\alpha^3 x - 2\alpha\sqrt{2\alpha(1-2\alpha)} - 8\alpha^2\right)\sqrt{x+\beta_3}\,I_1(\sqrt{x+\beta_3}\,)\Big]$$

$$\int x^2\,e^{\alpha x}\,I_0(\sqrt{x+\beta_4}\,) =$$

$$= \frac{e^{\alpha x}}{8\alpha^5}\Big[\left(8\alpha^4 x^2 + 2\alpha^2(1-8\alpha)x + \sqrt{2\alpha(1-2\alpha)} + 16\alpha^2 - 6\alpha\right)I_0(\sqrt{x+\beta_4}\,) -$$

$$- \left(4\alpha^3 x + 2\alpha\sqrt{2\alpha(1-2\alpha)} - 8\alpha^2\right)\sqrt{x+\beta_4}\,I_1(\sqrt{x+\beta_4}\,)\Big]$$

$$\int x^2\,e^{\alpha x}\,K_0(\sqrt{x+\beta_3}\,) =$$

$$= \frac{e^{\alpha x}}{8\alpha^5}\Big[\left(8\alpha^4 x^2 + 2\alpha^2(1-8\alpha)x - \sqrt{2\alpha(1-2\alpha)} + 16\alpha^2 - 6\alpha\right)K_0(\sqrt{x+\beta_3}\,) +$$

$$+ \left(4\alpha^3 x - 2\alpha\sqrt{2\alpha(1-2\alpha)} - 8\alpha^2\right)\sqrt{x+\beta_3}\,K_1(\sqrt{x+\beta_3}\,)\Big]$$

$$\int x^2\,e^{\alpha x}\,K_0(\sqrt{x+\beta_4}\,) =$$

$$= \frac{e^{\alpha x}}{8\alpha^5}\Big[\left(8\alpha^4 x^2 + 2\alpha(1-8\alpha)x + \sqrt{2\alpha^2(1-2\alpha)} + 16\alpha^2 - 6\alpha\right)K_0(\sqrt{x+\beta_4}\,) +$$

$$+ \left(4\alpha^3 x + 2\alpha\sqrt{2\alpha(1-2\alpha)} - 8\alpha^2\right)\sqrt{x+\beta_4}\,K_1(\sqrt{x+\beta_4}\,)\Big]$$

Special parameters (for K_0 compare with the upper formulas):

$$\int x^2\,e^{-x/20}\,J_0(\sqrt{x-20}\,) =$$

$$= -20\,e^{-x/20}\left[(x^2 - 60x + 800)J_0(\sqrt{x-20}\,) - 10(x-20)\sqrt{x-20}\,J_1(\sqrt{x-20}\,)\right]$$

$$\int x^2\,e^{-x/20}\,J_0(\sqrt{x-140}\,) =$$

$$= -20\,e^{-x/20}\left[(x^2 - 60x - 11200)J_0(\sqrt{x-140}\,) - 10(x+100)\sqrt{x-140}\,J_1(\sqrt{x-140}\,)\right]$$

$$\int x^2\,e^{x/20}\,I_0(\sqrt{x+20}\,) =$$

$$= 20\,e^{x/20}\left[(x^2 + 60x + 800)I_0(\sqrt{x+20}\,) - 10(x+20)\sqrt{x+20}\,I_1(\sqrt{x+20}\,)\right]$$

$$\int x^2\,e^{x/20}\,I_0(\sqrt{x+140}\,) =$$

$$= 20\,e^{x/20}\left[(x^2 + 60x - 11200)I_0(\sqrt{x+140}\,) - 10(x-10)\sqrt{x+140}\,I_1(\sqrt{x+140}\,)\right]$$

$$\int x^2\,e^{-2x/29}\,J_0\left(\sqrt{x - \frac{29}{16}}\,\right) = -\frac{29e^{-2x/29}}{128}\left[(1856x^2 - 43732x + 73167)J_0\left(\sqrt{x - \frac{29}{16}}\,\right) -\right.$$

$$-(13456x - 97556)\sqrt{x - \frac{29}{16}}\, J_1\left(\sqrt{x - \frac{29}{16}}\right)\Bigg]$$

$$\int x^2\, e^{-2x/29}\, J_0\left(\sqrt{x - \frac{1189}{16}}\right) = -\frac{29e^{-2x/29}}{128}\left[(64x^2 - 1508x - 241367)J_0\left(\sqrt{x - \frac{1189}{16}}\right) - \right.$$

$$\left. -116(4x + 261)\sqrt{x - \frac{1189}{16}}\, J_1\left(\sqrt{x - \frac{1189}{16}}\right)\right]$$

$$\int x^2\, e^{2x/29}\, I_0\left(\sqrt{x + \frac{29}{16}}\right) = \frac{29e^{2x/29}}{128}\left[(64x^2 + 1508x + 2523)I_0\left(\sqrt{x + \frac{29}{16}}\right) - \right.$$

$$\left. -116(4x + 29)\sqrt{x + \frac{29}{16}}\, I_1\left(\sqrt{x + \frac{29}{16}}\right)\right]$$

$$\int x^2\, e^{2x/29}\, I_0\left(\sqrt{x + \frac{1189}{16}}\right) = \frac{29e^{2x/29}}{128}\left[(64x^2 + 1508x - 241367)I_0\left(\sqrt{x + \frac{1189}{16}}\right) - \right.$$

$$\left. -116(4x - 261)\sqrt{x + \frac{1189}{16}}\, I_1\left(\sqrt{x + \frac{1189}{16}}\right)\right]$$

$$\int x^2\, e^{-x/10}\, J_0(\sqrt{x+5}) = -10\, e^{-x/10}\left[(x^2 - 5x - 50)J_0(\sqrt{x+5}) - 5x\sqrt{x+5}\, J_1(\sqrt{x+5})\right]$$

$$\int x^2\, e^{-x/10}\, J_0(\sqrt{x-35}) = -10\, e^{-x/10}\left[(x^2 - 5x - 1050)J_0(\sqrt{x-35}) - 5(x+40)\sqrt{x-35}\, J_1(\sqrt{x-35})\right]$$

$$\int x^2\, e^{x/10}\, I_0(\sqrt{x-5}) =$$

$$= 10\, e^{x/10}\left[(x^2 + 5x - 50)I_0(\sqrt{x-5}) - 5x\sqrt{x-5}\, I_1(\sqrt{x-5})\right]$$

$$\int x^2\, e^{x/10}\, I_0(\sqrt{x+35}) =$$

$$= 10\, e^{x/10}\left[(x^2 + 5x - 1050)I_0(\sqrt{x+35}) - 5(x-40)\sqrt{x+35}\, I_1(\sqrt{x+35})\right]$$

$$\int x^2\, e^{-2x/13}\, J_0\left(\sqrt{x + \frac{91}{16}}\right) = -\frac{13e^{-2x/13}}{128}\left[(64x^2 + 156x - 1183)J_0\left(\sqrt{x + \frac{91}{16}}\right) - \right.$$

$$\left. -52(4x + 13)\sqrt{x + \frac{91}{16}}\, J_1\left(\sqrt{x + \frac{91}{16}}\right)\right]$$

$$\int x^2\, e^{-2x/13}\, J_0\left(\sqrt{x - \frac{221}{16}}\right) = -\frac{13e^{-2x/13}}{128}\left[(64x^2 + 156x - 14365)J_0\left(\sqrt{x - \frac{221}{16}}\right) - \right.$$

$$\left. -52(4x + 91)\sqrt{x - \frac{221}{16}}\, J_1\left(\sqrt{x - \frac{91}{16}}\right)\right]$$

$$\int x^2\, e^{2x/13}\, I_0\left(\sqrt{x - \frac{91}{16}}\right) = \frac{13e^{2x/13}}{128}\left[(64x^2 - 156x - 1183)I_0\left(\sqrt{x - \frac{91}{16}}\right) - \right.$$

$$\left. -52(4x - 13)\sqrt{x - \frac{91}{16}}\, I_1\left(\sqrt{x - \frac{91}{16}}\right)\right]$$

$$\int x^2\, e^{2x/13}\, I_0\left(\sqrt{x + \frac{221}{16}}\right) = \frac{13e^{2x/13}}{128}\left[(64x^2 - 156x - 14365)I_0\left(\sqrt{x + \frac{221}{16}}\right) - \right.$$

$$-52(4x-91)\sqrt{x+\frac{221}{16}}\,I_1\left(\sqrt{x+\frac{211}{16}}\right)\right]$$

$$\int x^2\,e^{-x/4}\,J_0(\sqrt{x+4}) = -4\,e^{-x/4}\left[(x^2+4x)J_0(\sqrt{x+4})+2(x+4)\sqrt{x+4}\,J_1(\sqrt{x+4})\right]$$

$$\int x^2\,e^{-x/4}\,J_0(\sqrt{x-4}) = -\frac{e^{-x/4}}{4}\left[(x^2+4x-32)J_0(\sqrt{x-4})+2(x+12)\sqrt{x-4}\,J_1(\sqrt{x-4})\right]$$

$$\int x^2\,e^{x/4}\,I_0(\sqrt{x+4}) = 4\,e^{x/4}\left[(x^2-4x-32)I_0(\sqrt{x+4})-2(x-12)\sqrt{x+4}\,I_1(\sqrt{x+4})\right]$$

$$\int x^2\,e^{x/4}\,I_0(\sqrt{x-4}) = \frac{e^{x/4}}{4}\left[(x^2-4x)I_0(\sqrt{x-4})-2(x-4)\sqrt{x-4}\,I_1(\sqrt{x-4})\right]$$

$$\int x^2\,e^{-8x/25}\,J_0\left(\sqrt{x+\frac{775}{256}}\right) = -\frac{25e^{-8x/25}}{65536}\left[(8192x^2+31200x+19375)J_0\left(\sqrt{x+\frac{775}{256}}\right)-\right.$$

$$\left.-400(32x+125)\sqrt{x+\frac{775}{256}}\,J_1\left(\sqrt{x+\frac{775}{256}}\right)\right]$$

$$\int x^2\,e^{-8x/25}\,J_0\left(\sqrt{x-\frac{425}{256}}\right) = -\frac{25e^{-8x/25}}{65536}\left[(8192x^2+31200x-74375)J_0\left(\sqrt{x-\frac{425}{256}}\right)-\right.$$

$$\left.-400(32x+275)\sqrt{x-\frac{425}{256}}\,J_1\left(\sqrt{x-\frac{425}{256}}\right)\right]$$

$$\int x^2\,e^{8x/25}\,I_0\left(\sqrt{x-\frac{775}{256}}\right) = \frac{25e^{8x/25}}{65536}\left[(8192x^2-31200x+19375)I_0\left(\sqrt{x-\frac{775}{256}}\right)-\right.$$

$$\left.-400(32x-125)\sqrt{x-\frac{775}{256}}\,I_1\left(\sqrt{x-\frac{775}{256}}\right)\right]$$

$$\int x^2\,e^{8x/25}\,I_0\left(\sqrt{x+\frac{425}{256}}\right) = -\frac{25e^{8x/25}}{65536}\left[(8192x^2-31200x-74375)I_0\left(\sqrt{x+\frac{425}{256}}\right)-\right.$$

$$\left.-400(32x-275)\sqrt{x+\frac{425}{256}}\,I_1\left(\sqrt{x+\frac{425}{256}}\right)\right]$$

$$\int x^2\,e^{-9x/26}\,J_0\left(\sqrt{x-\frac{91}{81}}\right) = -\frac{26e^{-9x/26}}{19683}\left[(2187x^2+8073x-11830)J_0\left(\sqrt{x-\frac{91}{81}}\right)-\right.$$

$$\left.-117(27x+208)\sqrt{x-\frac{91}{81}}\,J_1\left(\sqrt{x-\frac{91}{81}}\right)\right]$$

$$\int x^2\,e^{-9x/26}\,J_0\left(\sqrt{x+\frac{221}{81}}\right) = -\frac{26e^{-9x/26}}{19683}\left[(2187x^2+8073x+5764)J_0\left(\sqrt{x+\frac{221}{81}}\right)-\right.$$

$$\left.-117(27x+104)\sqrt{x+\frac{221}{81}}\,J_1\left(\sqrt{x+\frac{221}{81}}\right)\right]$$

$$\int x^2\,e^{9x/26}\,I_0\left(\sqrt{x+\frac{91}{81}}\right) = \frac{26e^{9x/26}}{19683}\left[(2187x^2-8073x-11830)I_0\left(\sqrt{x+\frac{91}{81}}\right)-\right.$$

$$-117(27x - 208)\sqrt{x + \frac{91}{81}}\, I_1\left(\sqrt{x + \frac{91}{81}}\right)\Bigg]$$

$$\int x^2\, e^{9x/26}\, I_0\left(\sqrt{x - \frac{221}{81}}\right) = \frac{26e^{9x/26}}{19683}\Bigg[(2187x^2 - 8073x + 5764)I_0\left(\sqrt{x - \frac{221}{81}}\right) -$$

$$-117(27x - 208)\sqrt{x - \frac{221}{81}}\, I_1\left(\sqrt{x - \frac{221}{81}}\right)\Bigg]$$

$$\int x^2\, e^{-2x/5}\, J_0\left(\sqrt{x - \frac{5}{16}}\right) = -\frac{5e^{-2x/5}}{128}\Bigg[(64x^2 + 220x - 75)\, J_0\left(\sqrt{x - \frac{5}{16}}\right) -$$

$$-20(4x + 25)\sqrt{x - \frac{5}{16}}\, J_1\left(\sqrt{x - \frac{5}{16}}\right)\Bigg]$$

$$\int x^2\, e^{-2x/5}\, J_0\left(\sqrt{x + \frac{35}{16}}\right) = -\frac{5e^{-2x/5}}{128}\Bigg[(64x^2 + 220x + 175)\, J_0\left(\sqrt{x + \frac{35}{16}}\right) -$$

$$-20(4x + 15)\sqrt{x + \frac{35}{16}}\, J_1\left(\sqrt{x + \frac{35}{16}}\right)\Bigg]$$

$$\int x^2\, e^{2x/5}\, I_0\left(\sqrt{x + \frac{5}{16}}\right) = \frac{5e^{2x/5}}{128}\Bigg[(64x^2 - 220x - 75)\, I_0\left(\sqrt{x + \frac{5}{16}}\right) -$$

$$-20(4x - 25)\sqrt{x + \frac{5}{16}}\, I_1\left(\sqrt{x + \frac{5}{16}}\right)\Bigg]$$

$$\int x^2\, e^{2x/5}\, I_0\left(\sqrt{x - \frac{35}{16}}\right) = \frac{5e^{-2x/5}}{128}\Bigg[(64x^2 - 220x + 175)\, I_0\left(\sqrt{x - \frac{35}{16}}\right) -$$

$$-20(4x - 15)\sqrt{x - \frac{35}{16}}\, I_1\left(\sqrt{x - \frac{35}{16}}\right)\Bigg]$$

$$\int x^2\, e^{-9x/20}\, J_0\left(\sqrt{x + \frac{20}{81}}\right) = -\frac{20e^{-9x/20}}{19683}\Bigg[(2187x^2 + 7020x + 1600)\, J_0\left(\sqrt{x + \frac{20}{81}}\right) -$$

$$-90(27x + 140)\sqrt{x + \frac{20}{81}}\, J_1\left(\sqrt{x + \frac{20}{81}}\right)\Bigg]$$

$$\int x^2\, e^{-9x/20}\, J_0\left(\sqrt{x + \frac{140}{81}}\right) = -\frac{20e^{-9x/20}}{19683}\Bigg[(2187x^2 + 7020x + 5600)\, J_0\left(\sqrt{x + \frac{140}{81}}\right) -$$

$$-90(27x + 100)\sqrt{x + \frac{140}{81}}\, J_1\left(\sqrt{x + \frac{140}{81}}\right)\Bigg]$$

$$\int x^2\, e^{9x/20}\, I_0\left(\sqrt{x - \frac{20}{81}}\right) = \frac{20e^{9x/20}}{19683}\Bigg[(2187x^2 - 7020x + 1600)\, I_0\left(\sqrt{x - \frac{20}{81}}\right) -$$

$$-90(27x - 140)\sqrt{x - \frac{20}{81}}\, I_1\left(\sqrt{x - \frac{20}{81}}\right)\Bigg]$$

$$\int x^2\, e^{9x/20}\, I_0\left(\sqrt{x - \frac{140}{81}}\right) = \frac{20e^{9x/20}}{19683}\Bigg[(2187x^2 - 7020x + 5600)\, I_0\left(\sqrt{x - \frac{140}{81}}\right) -$$

$$-90(27x - 100) \sqrt{x - \frac{140}{81}} \, I_1 \left(\sqrt{x - \frac{140}{81}} \right) \Bigg]$$

$$\int x^2 \, e^{-8x/17} J_0 \left(\sqrt{x + \frac{391}{256}} \right) = -\frac{17 e^{-8x/17}}{65536} \left[(8192x^2 + 25568x + 19941) J_0 \left(\sqrt{x + \frac{391}{256}} \right) - \right.$$

$$\left. -272(32x + 119) \sqrt{x + \frac{391}{256}} \, J_1 \left(\sqrt{x + \frac{391}{256}} \right) \right]$$

$$\int x^2 \, e^{-8x/17} J_0 \left(\sqrt{x + \frac{119}{256}} \right) = -\frac{17 e^{-8x/17}}{65536} \left[(8192x^2 + 25568x + 10115) J_0 \left(\sqrt{x + \frac{119}{256}} \right) - \right.$$

$$\left. -272(32x + 153) \sqrt{x + \frac{119}{256}} \, J_1 \left(\sqrt{x + \frac{119}{256}} \right) \right]$$

$$\int x^2 \, e^{8x/17} I_0 \left(\sqrt{x - \frac{391}{256}} \right) = \frac{17 e^{8x/17}}{65536} \left[(8192x^2 - 25568x + 19941) I_0 \left(\sqrt{x - \frac{391}{256}} \right) - \right.$$

$$\left. -272(32x - 119) \sqrt{x - \frac{391}{256}} \, J_1 \left(\sqrt{x - \frac{391}{256}} \right) \right]$$

$$\int x^2 \, e^{8x/17} I_0 \left(\sqrt{x - \frac{119}{256}} \right) = \frac{17 e^{8x/17}}{65536} \left[(8192x^2 - 25568x + 10115) I_0 \left(\sqrt{x - \frac{119}{256}} \right) - \right.$$

$$\left. -272(32x - 153) \sqrt{x - \frac{119}{256}} \, J_1 \left(\sqrt{x - \frac{119}{256}} \right) \right]$$

$$\int x^2 \, e^{-x/2} J_0(\sqrt{x+1}) = -2 \, e^{-x/2} \left[(x^2 + 3x + 2) J_0(\sqrt{x+1}) - (x+4) \sqrt{x+1} \, J_1(\sqrt{x+1}) \right]$$

$$\int x^2 \, e^{x/2} I_0(\sqrt{x-1}) = 2 \, e^{x/2} \left[(x^2 - 3x + 2) I_0(\sqrt{x-1}) - (x-4) \sqrt{x-1} \, I_1(\sqrt{x-1}) \right]$$

$$\int x^2 \, e^{-x/4} \sqrt{x+6} \, J_0(\sqrt{x+6}) \, dx = -4(x+6) \, e^{-x/4} \left[x \sqrt{x+6} \, J_0(\sqrt{x+6}) \right] - 2(x+6) J_1(\sqrt{x+6})$$

$$\int x^2 \, e^{x/4} \sqrt{x-6} \, I_0(\sqrt{x-6}) \, dx = 4(x-6) \, e^{x/4} \left[x \sqrt{x-6} \, I_0(\sqrt{x-6}) \right] - 2(x-6) I_1(\sqrt{x-6})$$

$$\int x^2 \, e^{x/4} \sqrt{x-6} \, K_0(\sqrt{x-6}) \, dx = 4(x-6) \, e^{x/4} \left[x \sqrt{x-6} \, K_0(\sqrt{x-6}) \right] + 2(x-6) K_1(\sqrt{x-6})$$

$$\int x^2 \, e^{-x/28} \sqrt{x-42} \, J_0(\sqrt{x-42}) \, dx =$$

$$= -28(x-42) \, e^{-x/28} \left[14(x-42) \sqrt{x-42} \, J_0(\sqrt{x-42}) \right] - (x-84) J_1(\sqrt{x-42}) \right]$$

$$\int x^2 \, e^{x/28} \sqrt{x+42} \, I_0(\sqrt{x+42}) \, dx =$$

$$= -28(x+42) \, e^{x/28} \left[14(x+42) \sqrt{x+42} \, I_0(\sqrt{x+42}) \right] - (x+84) I_1(\sqrt{x+42}) \right]$$

$$\int x^2 \, e^{x/28} \sqrt{x+42} \, K_0(\sqrt{x+42}) \, dx =$$

$$= 28(x+42) \, e^{x/28} \left[14(x+42) \sqrt{x+42} \, K_0(\sqrt{x+42}) \right] + (x+84) K_1(\sqrt{x+42}) \right]$$

$$\int \frac{x^2\, e^{-x/12}\, J_0(\sqrt{x-6}\,)\, dx}{\sqrt{x-6}} \;=\; e^{-x/12}\left[12(12-x)\,\sqrt{x-6}\,J_0\left(\sqrt{x-6}\,\right)+72(x-6)\,J_1\left(\sqrt{x-6}\,\right)\right]$$

$$\int \frac{x^2\, e^{x/12}\, I_0(\sqrt{x+6}\,)\, dx}{\sqrt{x+6}} \;=\; e^{x/12}\left[12(12+x)\,\sqrt{x+6}\,I_0\left(\sqrt{x+6}\,\right)-72(x+6)\,I_1\left(\sqrt{x+6}\,\right)\right]$$

$$\int \frac{x^2\, e^{x/12}\, K_0(\sqrt{x+6}\,)\, dx}{\sqrt{x+6}} \;=\; e^{x/12}\left[12(12+x)\,\sqrt{x+6}\,K_0\left(\sqrt{x+6}\,\right)+72(x+6)\,K_1\left(\sqrt{x+6}\,\right)\right]$$

$$\alpha = \frac{3-\sqrt{3}}{24} = 0.05283\,12164,\ \ \beta = 6(\sqrt{3}+1) = 16.39230\,48454$$

$$\int x^2\, e^{-\alpha x}\, J_1(\sqrt{x-\beta}\,)\, dx = -4e^{-\alpha x}\left\{12[(2+\sqrt{3})x - 30 - 18\sqrt{3}\,]\,\sqrt{x-\beta}\,J_0(\sqrt{x-\beta}\,)-\right.$$
$$\left.-[(3+\sqrt{3}\,)x^2 - 12(12+7\sqrt{3}\,)(x-12)]\,J_1(\sqrt{x-\beta}\,)\right\}$$

$$\int x^2\, e^{\alpha x}\, I_1(\sqrt{x+\beta}\,)\, dx = -4e^{\alpha x}\left\{12[(2+\sqrt{3})x + 30 + 18\sqrt{3}\,]\,\sqrt{x+\beta}\,I_0(\sqrt{x+\beta}\,)-\right.$$
$$\left.-[(3+\sqrt{3}\,)x^2 + 12(12+7\sqrt{3}\,)(x+12)]\,I_1(\sqrt{x+\beta}\,)\right\}$$

$$\int x^2\, e^{\alpha x}\, K_1(\sqrt{x+\beta}\,)\, dx = 4e^{\alpha x}\left\{12[(2+\sqrt{3})x + 30 + 18\sqrt{3}\,]\,\sqrt{x+\beta}\,K_0(\sqrt{x+\beta}\,)+\right.$$
$$\left.+[(3+\sqrt{3}\,)x^2 + 12(12+7\sqrt{3}\,)(x+12)]\,K_1(\sqrt{x+\beta}\,)\right\}$$

$$\gamma = \frac{3+\sqrt{3}}{24} = 0.19716\,87836,\ \ \eta = 6(\sqrt{3}-1) = 4.39230\,48454$$

$$\int x^2\, e^{-\gamma x}\, J_1(\sqrt{x+\eta}\,)\, dx = -4e^{-\gamma x}\left\{12[(2-\sqrt{3})x - 30 + 18\sqrt{3}\,]\,\sqrt{x+\eta}\,J_0(\sqrt{x+\eta}\,)+\right.$$
$$\left.+[(3-\sqrt{3}\,)x^2 - 12(12-7\sqrt{3}\,)(x-12)]\,J_1(\sqrt{x+\eta}\,)\right\}$$

$$\int x^2\, e^{\gamma x}\, I_1(\sqrt{x-\eta}\,)\, dx = -4e^{\gamma x}\left\{12[(2-\sqrt{3})x + 30 - 18\sqrt{3}\,]\,\sqrt{x-\eta}\,I_0(\sqrt{x-\eta}\,)-\right.$$
$$\left.-[(3-\sqrt{3}\,)x^2 + 12(12-7\sqrt{3}\,)(x+12)]\,I_1(\sqrt{x-\eta}\,)\right\}$$

$$\int x^2\, e^{\gamma x}\, K_1(\sqrt{x-\eta}\,)\, dx = 4e^{\gamma x}\left\{12[(2-\sqrt{3})x + 30 - 18\sqrt{3}\,]\,\sqrt{x-\eta}\,K_0(\sqrt{x-\eta}\,)+\right.$$
$$\left.+[(3-\sqrt{3}\,)x^2 + 12(12-7\sqrt{3}\,)(x+12)]\,K_1(\sqrt{x-\eta}\,)\right\}$$

1.4.10. Integrals with $\sin\alpha x\, Z_\nu(\sqrt{x+\beta}\,)$ or $\sin\alpha x\, Z_\nu(\sqrt{x+\beta}\,)$

Only special cases were found.

$$\int x\,\sin\alpha x \cdot \left(x+\frac{1}{4\alpha^2}\right)^{-1/2}\cdot J_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)dx =$$

$$= \frac{1}{2\alpha^2}\left[\sin\alpha x \cdot J_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right) - 2\alpha\,\cos\alpha x\sqrt{x+\frac{1}{4\alpha^2}}\cdot J_1\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)\right]$$

$$\int x \sin \alpha x \cdot \left(x - \frac{1}{4\alpha^2}\right)^{-1/2} \cdot I_0\left(\sqrt{x - \frac{1}{4\alpha^2}}\right) dx =$$

$$= \frac{1}{2\alpha^2}\left[\sin \alpha x \cdot I_0\left(\sqrt{x - \frac{1}{4\alpha^2}}\right) - 2\alpha \cos \alpha x \cdot \sqrt{\left(x - \frac{1}{4\alpha^2}\right)} \cdot I_1\left(\sqrt{x - \frac{1}{4\alpha^2}}\right)\right]$$

$$\int x \sin \alpha x \cdot \left(x - \frac{1}{4\alpha^2}\right)^{-1/2} \cdot K_0\left(\sqrt{x - \frac{1}{4\alpha^2}}\right) dx =$$

$$= -\frac{1}{2\alpha^2}\left[\sin \alpha x \cdot K_0\left(\sqrt{x - \frac{1}{4\alpha^2}}\right) + 2\alpha \cos \alpha x \cdot \sqrt{x - \frac{1}{4\alpha^2}} \cdot K_1\left(\sqrt{x - \frac{1}{4\alpha^2}}\right)\right]$$

$$\int x \cos \alpha x \cdot \left(x + \frac{1}{4\alpha^2}\right)^{-1/2} \cdot J_0\left(\sqrt{x + \frac{1}{4\alpha^2}}\right) dx = \quad (16)$$

$$= \frac{1}{2\alpha^2}\left[\cos \alpha x \cdot J_0\left(\sqrt{x + \frac{1}{4\alpha^2}}\right) + 2\alpha \sin \alpha x \cdot \sqrt{x + \frac{1}{4\alpha^2}} \cdot J_1\left(\sqrt{x + \frac{1}{4\alpha^2}}\right)\right] \quad (17)$$

$$\int x \cos \alpha x \cdot \left(x - \frac{1}{4\alpha^2}\right)^{-1/2} \cdot I_0\left(\sqrt{x + \frac{1}{4\alpha^2}}\right) dx =$$

$$= \frac{1}{2\alpha^2}\left[\cos \alpha x \cdot I_0\left(\sqrt{x - \frac{1}{4\alpha^2}}\right) + 2\alpha \sin \alpha x \cdot \sqrt{x - \frac{1}{4\alpha^2}} \cdot I_1\left(\sqrt{x - \frac{1}{4\alpha^2}}\right)\right]$$

$$\int x \cos \alpha x \cdot \left(x - \frac{1}{4\alpha^2}\right)^{-1/2} \cdot K_0\left(\sqrt{x + \frac{1}{4\alpha^2}}\right) dx =$$

$$= \frac{1}{2\alpha^2}\left[-\cos \alpha x \cdot K_0\left(\sqrt{x - \frac{1}{4\alpha^2}}\right) + 2\alpha \sin \alpha x \cdot \sqrt{x - \frac{1}{4\alpha^2}} \cdot K_1\left(\sqrt{x - \frac{1}{4\alpha^2}}\right)\right]$$

$$\int x^2 \sin \frac{\sqrt{2}\,x}{8} \cdot J_0(\sqrt{x + 16}\,) \, dx = 16 \left\{ \left[16(x + 16)\,J_0(\sqrt{x + 16}\,) - 4(x - 8)\,J_1(\sqrt{x + 16}\,)\right] \sin \frac{\sqrt{2}\,x}{8} - \right.$$

$$\left. - \left[\sqrt{2}\,(x^2 + 8x - 128)\,J_0(\sqrt{x + 16}\,) + 48\sqrt{2(x + 16)}\,J_1(\sqrt{x + 16}\,)\right] \cos \frac{\sqrt{2}\,x}{8} \right\}$$

$$\int x^2 \sin \frac{\sqrt{2}\,x}{8} \cdot I_0(\sqrt{x - 16}\,) \, dx = 16 \left\{ \left[16(x - 16)\,I_0(\sqrt{x - 16}\,) + 4(x + 8)\,I_1(\sqrt{x - 16}\,)\right] \sin \frac{\sqrt{2}\,x}{8} - \right.$$

$$\left. - \left[\sqrt{2}\,(x^2 - 8x - 128)\,I_0(\sqrt{x - 16}\,) - 48\sqrt{2(x + 16)}\,I_1(\sqrt{x - 16}\,)\right] \cos \frac{\sqrt{2}\,x}{8} \right\}$$

$$\int x^2 \sin \frac{\sqrt{2}\,x}{8} \cdot K_0(\sqrt{x - 16}\,) \, dx = 16 \left\{ \left[16(x - 16)\,K_0(\sqrt{x - 16}\,) - 4(x + 8)\,K_1(\sqrt{x - 16}\,)\right] \sin \frac{\sqrt{2}\,x}{8} - \right.$$

$$\left. - \left[\sqrt{2}\,(x^2 - 8x - 128)\,K_0(\sqrt{x - 16}\,) + 48\sqrt{2(x + 16)}\,K_1(\sqrt{x - 16}\,)\right] \cos \frac{\sqrt{2}\,x}{8} \right\}$$

$$\int x^2 \cos \frac{\sqrt{2}\,x}{8} \cdot J_0(\sqrt{x + 16}\,) \, dx = 16 \left\{ \left[\sqrt{2}\,(x^2 + 8x - 128)\,J_0(\sqrt{x + 16}\,) + 48\sqrt{2}\,J_1(\sqrt{x + 16}\,)\right] \sin \frac{\sqrt{2}\,x}{8} + \right.$$

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$$+ \left[16(x + 16)\, J_0(\sqrt{x + 16}\,) - 4(x - 8)\sqrt{x + 16}\; J_1(\sqrt{x + 16}\,)\right] \cos \frac{\sqrt{2}\, x}{8}\Bigg\}$$

$$\int x^2 \cos \frac{\sqrt{2}\, x}{8} \cdot I_0(\sqrt{x - 16}\,)\, dx = 16 \left\{ \left[\sqrt{2}\,(x^2 - 8x - 128)\, I_0(\sqrt{x - 16}\,) - 48\sqrt{2}\; I_1(\sqrt{x - 16}\,)\right] \sin \frac{\sqrt{2}\, x}{8} + \right.$$

$$\left. + \left[16(x - 16)\, I_0(\sqrt{x - 16}\,) + 4(x + 8)\sqrt{x - 16}\; I_1(\sqrt{x + 16}\,)\right] \cos \frac{\sqrt{2}\, x}{8}\right\}$$

$$\int x^2 \cos \frac{\sqrt{2}\, x}{8} \cdot K_0(\sqrt{x - 16}\,)\, dx = 16 \left\{ \left[\sqrt{2}\,(x^2 - 8x - 128)\, K_0(\sqrt{x - 16}\,) + 48\sqrt{2}\; K_1(\sqrt{x - 16}\,)\right] \sin \frac{\sqrt{2}\, x}{8} + \right.$$

$$\left. + \left[16(x - 16)\, K_0(\sqrt{x - 16}\,) - 4(x + 8)\sqrt{x - 16}\; K_1(\sqrt{x + 16}\,)\right] \cos \frac{\sqrt{2}\, x}{8}\right\}$$

$$\frac{2\sqrt{6} + \sqrt{23}}{4} = 2.423\ 703, \quad 236 - 20\sqrt{138} = 1.053\ 198$$

$$\int x^4 \sin \frac{(2\sqrt{6} + \sqrt{23}\,)x}{4} \cdot J_0\left(\sqrt{x + 236 - 20\sqrt{138}}\,\right) =$$

$$= 64 \left[(47 - 4\sqrt{138}\,)x^3 + (44170 - 3760\sqrt{138}\,)x^2 + (6183048 - 526336\sqrt{138}\,)x - 1975942080 + \right.$$

$$\left. + 168203360\sqrt{138}\,\right] \sin \frac{(2\sqrt{6} + \sqrt{23}\,)x}{4} \cdot J_0\left(\sqrt{x + 236 - 20\sqrt{138}}\,\right) +$$

$$+ 8\sqrt{x + 236 - 20\sqrt{138}} \left[(4\sqrt{138} - 47)x^3 + (4464 - 380\sqrt{138}\,)x^2 + (1413440 - 120320\sqrt{138}\,)x + 982229504 - \right.$$

$$\left. - 83612928\sqrt{138}\,\right] \sin \frac{(2\sqrt{6} + \sqrt{23}\,)x}{4} \cdot J_1\left(\sqrt{x + 236 - 20\sqrt{138}}\,\right) + 4\left[(\sqrt{23} - 2\sqrt{6}\,)x^4 + \right.$$

$$+ (380\sqrt{23} - 744\sqrt{6}\,)x^3 - (54336\sqrt{23} - 106384\sqrt{6}\,)x^2 - (22858240\sqrt{23} - 44753920\sqrt{6}\,)x -$$

$$\left. - 8742279680\sqrt{23} + 17116422144\sqrt{6}\,\right] \cos \frac{(2\sqrt{6} + \sqrt{23}\,)x}{4} \cdot J_0\left(\sqrt{x + 236 - 20\sqrt{138}}\,\right) +$$

$$+ 32\sqrt{x + 236 - 20\sqrt{138}} \left[(665\sqrt{23} - 1302\sqrt{6}\,)x^2 + (160152\sqrt{23} - 313560\sqrt{6}\,)x - 36783680\sqrt{23} + \right.$$

$$\left. + 72018400\sqrt{6}\,\right] \cos \frac{(2\sqrt{6} + \sqrt{23}\,)x}{4} \cdot J_1\left(\sqrt{x + 236 - 20\sqrt{138}}\,\right)$$

$$\int x^4 \cos \frac{(2\sqrt{6} + \sqrt{23}\,)x}{4} \cdot J_0\left(\sqrt{x + 236 - 20\sqrt{138}}\,\right) =$$

$$= 4 \left[(2\sqrt{6} - \sqrt{23}\,)x^4 + 4(186\sqrt{6} - 95\sqrt{23}\,)x^3 - 16(6649\sqrt{6} - 3396\sqrt{23}\,)x^2 - 2560(17482\sqrt{6} - 8929\sqrt{23}\,)x + \right.$$

$$\left. + 8742279680\sqrt{6} - 17116422144\sqrt{23}\,\right] \sin \frac{(2\sqrt{6} + \sqrt{23}\,)x}{4} \cdot J_0\left(\sqrt{x + 236 - 20\sqrt{138}}\,\right) +$$

$$+ 32\sqrt{x + 236 - 20\sqrt{138}} \left[7(186\sqrt{6} - 95\sqrt{23}\,)x^2 + 24(13065\sqrt{6} - 6673\sqrt{23}\,)x - 72018400\sqrt{6} + \right.$$

$$\left. + 36783680\sqrt{23}\,\right] \sin \frac{(2\sqrt{6} + \sqrt{23}\,)x}{4} \cdot J_1\left(\sqrt{x + 236 - 20\sqrt{138}}\,\right) + 64 \left[(47 - 4\sqrt{138}\,)x^3 + \right.$$

$$+ 10(4417 - 376\sqrt{138}\,)x^2 + 8(772881 - 65792\sqrt{138}\,)x -$$

$$\left. -1975942080 + 168203360\sqrt{138}\right] \cos \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4} \cdot J_0\left(\sqrt{x+236-20\sqrt{138}}\,\right) -$$

$$-8\sqrt{x+236-20\sqrt{138}}\,\Big[(47-4\sqrt{138}\,)x^3 - 4(1116-95\sqrt{138}\,)x^2 - 320(4417-376\sqrt{138}\,)x - 982229504 +$$

$$+83612928\sqrt{138}\,\Big]\cos \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236-20\sqrt{138}}\,\right)$$

$$\int x^4 \, \sin \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\,\right) =$$

$$= 64\,\Big[(47-4\sqrt{138}\,)x^3 + (44170-3760\sqrt{138}\,)x^2 + (6183048-526336\sqrt{138}\,)x - 1975942080 +$$

$$+168203360\sqrt{138}\,\Big]\sin \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\,\right) +$$

$$+8\sqrt{x+236-20\sqrt{138}}\,\Big[(47-4\sqrt{138}\,)x^3 + (4464-380\sqrt{138}\,)x^2 - (1413440-120320\sqrt{138}\,)x + 982229504 -$$

$$-83612928\sqrt{138}\,\Big]\sin \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236+20\sqrt{138}}\,\right) + 4\,\Big[(\sqrt{23}-2\sqrt{6}\,)x^4 -$$

$$-(380\sqrt{23}-744\sqrt{6}\,)\,)x^3 - (54336\sqrt{23}-106384\sqrt{6}\,)\,)x^2 + (22858240\sqrt{23}-44753920\sqrt{6}\,)\,)x -$$

$$-8742279680\sqrt{23}+17116422144\sqrt{6}\,\Big]\cos \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\,\right) +$$

$$+32\sqrt{x+236-20\sqrt{138}}\,\Big[(1302\sqrt{6}-665\sqrt{23}\,)x^2 + (160152\sqrt{23}-313560\sqrt{6}\,)x + 36783680\sqrt{23} -$$

$$-72018400\sqrt{6}\,\Big]\cos \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236+20\sqrt{138}}\,\right)$$

$$\int x^4 \, \cos \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\,\right) =$$

$$= 4\,\Big[(2\sqrt{6}-\sqrt{23}\,)x^4 - 4(186\sqrt{6}-95\sqrt{23}\,)x^3 + 16(6649\sqrt{6}-3396\sqrt{23}\,)x^2 + 2560(17482\sqrt{6}-8929\sqrt{23}\,)x +$$

$$+8742279680\sqrt{6}-17116422144\sqrt{23}\,\Big]\sin \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\,\right) -$$

$$-32\sqrt{x-236+20\sqrt{138}}\,\Big[7(186\sqrt{6}-95\sqrt{23}\,)x^2 - 24(13065\sqrt{6}-6673\sqrt{23}\,)x - 72018400\sqrt{6} +$$

$$+36783680\sqrt{23}\,\Big]\sin \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236+20\sqrt{138}}\,\right) + 64\,\Big[(47-4\sqrt{138}\,)\,)x^3 -$$

$$-10(4417-376\sqrt{138}\,)\,)x^2 + 8(772881-65792\sqrt{138}\,)\,)x +$$

$$+1975942080-168203360\sqrt{138}\,\Big]\cos \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\,\right) +$$

$$+8\sqrt{x-236+20\sqrt{138}}\,\Big[(47-4\sqrt{138}\,)x^3 + 4(1116-95\sqrt{138}\,)x^2 - 320(4417-376\sqrt{138}\,)x + 982229504 +$$

$$-83612928\sqrt{138}\,\Big]\cos \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236+20\sqrt{138}}\,\right)$$

$$\int x^4 \, \sin \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_0\left(\sqrt{x-236+20\sqrt{138}}\,\right) =$$

$$= 64\,\Big[(47-4\sqrt{138}\,)x^3 - (44170-3760\sqrt{138}\,)x^2 + (6183048-526336\sqrt{138}\,)x + 1975942080 -$$

$$-168203360\sqrt{138}\,\Big]\sin \frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_0\left(\sqrt{x-236+20\sqrt{138}}\,\right) +$$

$$+ 8\sqrt{x + 236 - 20\sqrt{138}} \left[(4\sqrt{138} - 47)x^3 - (4464 - 380\sqrt{138})x^2 + (1413440 - 120320\sqrt{138})x - 982229504 + \right.$$

$$\left. + 83612928\sqrt{138} \right] \sin \frac{(2\sqrt{6} + \sqrt{23})x}{4} \cdot K_1\left(\sqrt{x - 236 + 20\sqrt{138}}\right) + 4 \left[(\sqrt{23} - 2\sqrt{6})x^4 - \right.$$

$$- (380\sqrt{23} - 744\sqrt{6})x^3 - (54336\sqrt{23} - 106384\sqrt{6})x^2 + (22858240\sqrt{23} - 44753920\sqrt{6})x -$$

$$\left. - 8742279680\sqrt{23} + 17116422144\sqrt{6} \right] \cos \frac{(2\sqrt{6} + \sqrt{23})x}{4} \cdot K_0\left(\sqrt{x - 236 + 20\sqrt{138}}\right) +$$

$$+ 32\sqrt{x + 236 - 20\sqrt{138}} \left[(665\sqrt{23} - 1302\sqrt{6})x^2 - (160152\sqrt{23} - 313560\sqrt{6})x - 36783680\sqrt{23} + \right.$$

$$\left. + 72018400\sqrt{6} \right] \cos \frac{(2\sqrt{6} + \sqrt{23})x}{4} \cdot K_1\left(\sqrt{x - 236 + 20\sqrt{138}}\right)$$

$$\int x^4 \cos \frac{(2\sqrt{6} + \sqrt{23})x}{4} \cdot K_0\left(\sqrt{x - 236 + 20\sqrt{138}}\right) =$$

$$= 4 \left[(2\sqrt{6} - \sqrt{23})x^4 - 4(186\sqrt{6} - 95\sqrt{23})x^3 - 16(6649\sqrt{6} - 3396\sqrt{23})x^2 + 2560(17482\sqrt{6} - 8929\sqrt{23})x - \right.$$

$$\left. - 8742279680\sqrt{6} + 17116422144\sqrt{23} \right] \sin \frac{(2\sqrt{6} + \sqrt{23})x}{4} \cdot K_0\left(\sqrt{x - 236 + 20\sqrt{138}}\right) +$$

$$+ 32\sqrt{x - 236 + 20\sqrt{138}} \left[7(186\sqrt{6} - 95\sqrt{23})x^2 - 24(13065\sqrt{6} - 6673\sqrt{23})x - 72018400\sqrt{6} + \right.$$

$$\left. + 36783680\sqrt{23} \right] \sin \frac{(2\sqrt{6} + \sqrt{23})x}{4} \cdot K_1\left(\sqrt{x - 236 + 20\sqrt{138}}\right) + 64 \left[(47 - 4\sqrt{138})x^3 - \right.$$

$$- 10(4417 - 376\sqrt{138})x^2 + 8(772881 - 65792\sqrt{138})x +$$

$$\left. + 1975942080 - 168203360\sqrt{138} \right] \cos \frac{(2\sqrt{6} + \sqrt{23})x}{4} \cdot K_0\left(\sqrt{x - 236 + 20\sqrt{138}}\right) -$$

$$- 8\sqrt{x - 236 + 20\sqrt{138}} \left[(47 - 4\sqrt{138})x^3 - 4(1116 - 95\sqrt{138})x^2 + 320(4417 - 376\sqrt{138})x + 982229504 - \right.$$

$$\left. - 83612928\sqrt{138} \right] \cos \frac{(2\sqrt{6} + \sqrt{23})x}{4} \cdot K_1\left(\sqrt{x - 236 + 20\sqrt{138}}\right)$$

$$\frac{2\sqrt{6} - \sqrt{23}}{4} = 0.103\,148 \quad 236 + 20\sqrt{138} = 470.946\,802$$

$$\int x^4 \sin \frac{(2\sqrt{6} - \sqrt{23})x}{4} \cdot J_0\left(\sqrt{x + 236 + 20\sqrt{138}}\right) =$$

$$= -64 \left[(47 + 4\sqrt{138})x^3 + (44170 + 3760\sqrt{138})x^2 + (6183048 + 526336\sqrt{138})x - 1975942080 - \right.$$

$$\left. - 168203360\sqrt{138} \right] \sin \frac{(2\sqrt{6} - \sqrt{23})x}{4} \cdot J_0\left(\sqrt{x + 236 + 20\sqrt{138}}\right) +$$

$$+ 8\sqrt{x + 236 + 20\sqrt{138}} \left[(4\sqrt{138} + 47)x^3 - (4464 + 380\sqrt{138})x^2 - (1413440 + 120320\sqrt{138})x - 982229504 - \right.$$

$$\left. - 83612928\sqrt{138} \right] \sin \frac{(2\sqrt{6} - \sqrt{23})x}{4} \cdot J_1\left(\sqrt{x + 236 + 20\sqrt{138}}\right) - 4 \left[(\sqrt{23} + 2\sqrt{6})x^4 + \right.$$

$$+ (380\sqrt{23} + 744\sqrt{6})x^3 - (54336\sqrt{23} + 106384\sqrt{6})x^2 - (22858240\sqrt{23} + 44753920\sqrt{6})x -$$

$$\left. - 8742279680\sqrt{23} - 17116422144\sqrt{6} \right] \cos \frac{(2\sqrt{6} - \sqrt{23})x}{4} \cdot J_0\left(\sqrt{x + 236 + 20\sqrt{138}}\right) -$$

$$-32\sqrt{x+236+20\sqrt{138}}\left[(665\sqrt{23}+1302\sqrt{6}\,)x^2+(160152\sqrt{23}+313560\sqrt{6}\,)x-36783680\sqrt{23}-\right.$$

$$\left.-72018400\sqrt{6}\right]\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236+20\sqrt{138}}\right)$$

$$\int x^4\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236+20\sqrt{138}}\right)=$$

$$=4\left[(2\sqrt{6}+\sqrt{23}\,)x^4+4(186\sqrt{6}+95\sqrt{23}\,)x^3-16(6649\sqrt{6}+3396\sqrt{23}\,)x^2-2560(17482\sqrt{6}+8929\sqrt{23}\,)x-\right.$$

$$\left.-8742279680\sqrt{6}-17116422144\sqrt{23}\right]\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236+20\sqrt{138}}\right)+$$

$$+32\sqrt{x+236+20\sqrt{138}}\left[7(186\sqrt{6}+95\sqrt{23}\,)x^2+24(13065\sqrt{6}+6673\sqrt{23}\,)x-72018400\sqrt{6}-\right.$$

$$\left.-36783680\sqrt{23}\right]\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236+20\sqrt{138}}\right)+64\left[(47+4\sqrt{138}\,)\,)x^3+\right.$$

$$+10(4417+376\sqrt{138}\,)\,)x^2+8(772881+65792\sqrt{138}\,)\,)x-$$

$$\left.-1975942080-168203360\sqrt{138}\right]\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236+20\sqrt{138}}\right)-$$

$$-8\sqrt{x+236+20\sqrt{138}}\left[(47+4\sqrt{138}\,)x^3-4(1116+95\sqrt{138}\,)x^2-320(4417+376\sqrt{138}\,)x-982229504-\right.$$

$$\left.-83612928\sqrt{138}\right]\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236+20\sqrt{138}}\right)$$

$$\int x^4\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)=$$

$$=64\left[(47+4\sqrt{138}\,)x^3-(44170+3760\sqrt{138}\,)x^2+(6183048+526336\sqrt{138}\,)x+1975942080+\right.$$

$$\left.+168203360\sqrt{138}\right]\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)+$$

$$+8\sqrt{x-236-20\sqrt{138}}\left[(4\sqrt{138}+47)x^3+(4464+380\sqrt{138}\,)x^2-(1413440+120320\sqrt{138}\,)x+982229504+\right.$$

$$\left.+83612928\sqrt{138}\right]\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236-20\sqrt{138}}\right)-4\left[(\sqrt{23}+2\sqrt{6}\,)\,)x^4-\right.$$

$$-(380\sqrt{23}+744\sqrt{6}\,)\,)x^3-(54336\sqrt{23}+106384\sqrt{6}\,)\,)x^2+(22858240\sqrt{23}+44753920\sqrt{6}\,)\,)x-$$

$$\left.-8742279680\sqrt{23}-17116422144\sqrt{6}\right]\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)+$$

$$+32\sqrt{x-236-20\sqrt{138}}\left[(665\sqrt{23}+1302\sqrt{6}\,)x^2-(160152\sqrt{23}+313560\sqrt{6}\,)x-36783680\sqrt{23}-\right.$$

$$\left.-72018400\sqrt{6}\right]\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236-20\sqrt{138}}\right)$$

$$\int x^4\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)=$$

$$=4\left[(2\sqrt{6}+\sqrt{23}\,)x^4-4(186\sqrt{6}+95\sqrt{23}\,)x^3-16(6649\sqrt{6}+3396\sqrt{23}\,)x^2+2560(17482\sqrt{6}+8929\sqrt{23}\,)x-\right.$$

$$\left.-8742279680\sqrt{6}-17116422144\sqrt{23}\right]\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)-$$

$$-32\sqrt{x-236-20\sqrt{138}}\left[7(186\sqrt{6}+95\sqrt{23}\,)x^2-24(13065\sqrt{6}+6673\sqrt{23}\,)x-72018400\sqrt{6}-\right.$$

$$-36783680\sqrt{23}\Big]\sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot I_1\left(\sqrt{x-236-20\sqrt{138}}\right)+64\Big[(47+4\sqrt{138})x^3-$$

$$-10(4417+376\sqrt{138}))x^2+8(772881+65792\sqrt{138}))x+$$

$$+1975942080+168203360\sqrt{138}\Big]\cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)-$$

$$-8\sqrt{x-236-20\sqrt{138}}\Big[(47+4\sqrt{138})x^3+4(1116+95\sqrt{138})x^2-320(4417+376\sqrt{138})x+982229504+$$

$$+83612928\sqrt{138}\Big]\cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot I_1\left(\sqrt{x-236-20\sqrt{138}}\right)$$

$$\int x^4\sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)=$$

$$=64\Big[(47+4\sqrt{138})x^3-(44170+3760\sqrt{138})x^2+(6183048+526336\sqrt{138})x+1975942080+$$

$$+168203360\sqrt{138}\Big]\sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)+$$

$$-8\sqrt{x-236-20\sqrt{138}}\Big[(4\sqrt{138}+47)x^3-(4464+380\sqrt{138})x^2+(1413440+120320\sqrt{138})x+982229504+$$

$$+83612928\sqrt{138}\Big]\sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_1\left(\sqrt{x-236-20\sqrt{138}}\right)-4\Big[(\sqrt{23}+2\sqrt{6})x^4-$$

$$-(380\sqrt{23}+744\sqrt{6}))x^3-(54336\sqrt{23}+106384\sqrt{6}))x^2+(22858240\sqrt{23}+44753920\sqrt{6}))x-$$

$$-8742279680\sqrt{23}-17116422144\sqrt{6}\Big]\cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)-$$

$$-32\sqrt{x-236-20\sqrt{138}}\Big[(665\sqrt{23}+1302\sqrt{6})x^2-(160152\sqrt{23}+313560\sqrt{6})x-36783680\sqrt{23}-$$

$$-72018400\sqrt{6}\Big]\cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_1\left(\sqrt{x-236-20\sqrt{138}}\right)$$

$$\int x^4\cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)=$$

$$=4\Big[(2\sqrt{6}+\sqrt{23})x^4-4(186\sqrt{6}+95\sqrt{23})x^3-16(6649\sqrt{6}+3396\sqrt{23})x^2+2560(17482\sqrt{6}+8929\sqrt{23})x-$$

$$-8742279680\sqrt{6}-17116422144\sqrt{23}\Big]\sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)+$$

$$+32\sqrt{x-236-20\sqrt{138}}\Big[7(186\sqrt{6}+95\sqrt{23})x^2-24(13065\sqrt{6}+6673\sqrt{23})x-72018400\sqrt{6}-$$

$$-36783680\sqrt{23}\Big]\sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_1\left(\sqrt{x-236-20\sqrt{138}}\right)+64\Big[(47+4\sqrt{138})x^3-$$

$$-10(4417+376\sqrt{138}))x^2+8(772881+65792\sqrt{138}))x+$$

$$+1975942080+168203360\sqrt{138}\Big]\cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)-$$

$$-8\sqrt{x-236-20\sqrt{138}}\Big[(47+4\sqrt{138})x^3+4(1116+95\sqrt{138})x^2-320(4417+376\sqrt{138})x+982229504+$$

$$+83612928\sqrt{138}\Big]\cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_1\left(\sqrt{x-236-20\sqrt{138}}\right)$$

$$4\sqrt{42} + \frac{76}{3} = 51.256\ 296, \quad \frac{\sqrt{21} - 3\sqrt{2}}{4} = 0.084\ 984 \quad (36)$$

$$\int x^4 \sin \frac{(\sqrt{21} - 3\sqrt{2})x}{4} \cdot \left(x + \frac{76}{3} + 4\sqrt{42}\right)^{-1/2} \cdot J_1\left(\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\right) dx =$$

$$= \frac{8}{27}\left[9(13 + 2\sqrt{42})x^3 - 12(162 + 25\sqrt{42})x^2 + 768(337 + 52\sqrt{42})x - 15721344 - \right.$$

$$\left. - 2425856\sqrt{42}\right] \sin \frac{(\sqrt{21} - 3\sqrt{2})x}{4} \cdot J_0\left(\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\right) +$$

$$+ \frac{16}{81}\sqrt{x + \frac{76}{3} + 4\sqrt{42}} \cdot \left[27(13 + 2\sqrt{42})x^2 + 24(849 + 131\sqrt{42})x - 737664 - \right.$$

$$\left. - 113824\sqrt{42}\right)\right] \sin \frac{(\sqrt{21} - 3\sqrt{2})x}{4} \cdot J_1\left(\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\right) +$$

$$+ \frac{64}{27}\left[12(81\sqrt{2} + 25\sqrt{21})x^2 + 12(1429\sqrt{2} + 441\sqrt{21})x - 1036400\sqrt{2} - \right.$$

$$\left. - 319840\sqrt{21}\right] \cos \frac{(\sqrt{21} - 3\sqrt{2})x}{4} \cdot J_0\left(\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\right) -$$

$$- \frac{4}{27}\sqrt{x + \frac{76}{3} + 4\sqrt{42}} \cdot \left[3(3\sqrt{2} + \sqrt{21})x^3 - 12(13\sqrt{2} + 4\sqrt{21})x^2 - 64(81\sqrt{2} + 25\sqrt{21})x - \right.$$

$$\left. - 1076736\sqrt{2} - 332288\sqrt{21}\right] \cos \frac{(\sqrt{21} - 3\sqrt{2})x}{4} \cdot J_1\left(\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\right) =$$

$$= (69.23061705\,x^3 - 1152.065840\,x^2 - 153372.1068\,x - 9316351.912)\sin(0.084984\,x)\,J_0(\sqrt{x + 51.256296}) +$$

$$+ \sqrt{x + 51.256296}\,(138.4612341\,x^2 + 8049.668889\,x - 291422.7811)\sin(0.084984\,x)\,J_1(\sqrt{x + 51.256296}) +$$

$$+ (4887.801406\,x^2 + 114967.5253\,x - 6948460.909)\cos(0.084984\,x)\,J_0(\sqrt{x + 51.256296}) -$$

$$- \sqrt{x + 51.256296}\,(3.922318391\,x^3 - 65.27125170\,x^2 - 2172.356181\,x - $$

$$- 451180.6764)\cos(0.084984\,x)\,J_1(\sqrt{x + 51.256296})$$

$$\int x^4 \cos \frac{(\sqrt{21} - 3\sqrt{2})x}{4} \cdot \left(x + \frac{76}{3} + 4\sqrt{42}\right)^{-1/2} \cdot J_1\left(\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\right) dx =$$

$$= -\frac{64}{27}\left[9(25\sqrt{21} + 81\sqrt{2})x^2 + 12(441\sqrt{21} + 1429\sqrt{2})x - 319840\sqrt{21} - \right.$$

$$\left. - 1036400\sqrt{2}\right] \sin \frac{(\sqrt{21} - 3\sqrt{2})x}{4} \cdot J_0\left(\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\right) +$$

$$+ \frac{4}{9}\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\left[3(\sqrt{21} + 3\sqrt{2})x^3 - 12(4\sqrt{21} + 13\sqrt{2})x^2 - 64(25\sqrt{21} + 81\sqrt{2})x - 332288\sqrt{21} - \right.$$

$$\left. - 1076736\sqrt{2}\right] \sin \frac{(\sqrt{21} - 3\sqrt{2})x}{4} \cdot J_1\left(\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\right) +$$

$$+ \frac{8}{27}\left[9(13 + 2\sqrt{42})x^3 - 12(25 + 162\sqrt{42})x^2 - 768(52 + 337\sqrt{42})x - 15721344 - \right.$$

$$\left. - 2425856\sqrt{42}\right] \cos \frac{(\sqrt{21} - 3\sqrt{2})x}{4} \cdot J_0\left(\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\right) +$$

$$+ \frac{16}{27}\sqrt{x + \frac{76}{3} + 4\sqrt{42}}\left[27(13 + 2\sqrt{42})x^2 + 24(849 + 131\sqrt{42})x - 737664 - \right.$$

$$-113824\sqrt{42}\,\Big]\cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)$$

$$\int x^4\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot\left(x-\frac{76}{3}-4\sqrt{42}\right)^{-1/2}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)\,dx=$$

$$=\frac{8}{27}\Big[9(13+2\sqrt{42}\,)x^3+12(162+25\sqrt{42}\,)x^2-768(337+52\sqrt{42}\,)x+15721344+$$

$$+2425856\sqrt{42}\,\Big]\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{16}{81}\sqrt{x-\frac{76}{3}-4\sqrt{42}}\cdot\Big[27(13+2\sqrt{42}\,)x^2-24(849+131\sqrt{42}\,)x-737664-$$

$$-113824\sqrt{42}\,)\Big]\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{64}{27}\Big[12(81\sqrt{2}+25\sqrt{21}\,)x^2-12(1429\sqrt{2}+441\sqrt{21}\,)x-1036400\sqrt{2}-$$

$$-319840\sqrt{21}\,\Big]\cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)-$$

$$-\frac{4}{27}\sqrt{x-\frac{76}{3}-4\sqrt{42}}\cdot\Big[3(3\sqrt{2}+\sqrt{21}\,)x^3+12(13\sqrt{2}+4\sqrt{21}\,)x^2-64(81\sqrt{2}+25\sqrt{21}\,)x+$$

$$+1076736\sqrt{2}-+332288\sqrt{21}\,\Big]\cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)$$

$$\int x^4\cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot\left(x+\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)\,dx=$$

$$=-\frac{64}{27}\Big[9(25\sqrt{21}+81\sqrt{2}\,)x^2-12(441\sqrt{21}+1429\sqrt{2}\,)x-319840\sqrt{21}-$$

$$-1036400\sqrt{2}\,\Big]\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{9}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\,\Big[3(\sqrt{21}+3\sqrt{2}\,)x^3+12(4\sqrt{21}+13\sqrt{2}\,)x^2-64(25\sqrt{21}+81\sqrt{2}\,)x+332288\sqrt{21}+$$

$$+1076736\sqrt{2}\,\Big]\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{8}{27}\Big[9(13+2\sqrt{42}\,)x^3+12(25+162\sqrt{42}\,)x^2-768(52+337\sqrt{42}\,)x+15721344+$$

$$+2425856\sqrt{42}\,\Big]\cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{16}{27}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\,\Big[27(13+2\sqrt{42}\,)x^2-24(849+131\sqrt{42}\,)x-737664-$$

$$-113824\sqrt{42}\,\Big]\cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)$$

$$\int x^4\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot\left(x-\frac{76}{3}-4\sqrt{42}\right)^{-1/2}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)\,dx=$$

$$=-\frac{8}{27}\Big[9(13+2\sqrt{42}\,)x^3+12(162+25\sqrt{42}\,)x^2-768(337+52\sqrt{42}\,)x+15721344+$$

$$
\left. +2425856\sqrt{42}\right] \sin\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+
$$

$$
+\frac{16}{81}\sqrt{x-\frac{76}{3}-4\sqrt{42}}\cdot\left[27(13+2\sqrt{42})x^2-24(849+131\sqrt{42})x-737664-\right.
$$

$$
\left.-113824\sqrt{42})\right]\sin\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)-
$$

$$
-\frac{64}{27}\left[12(81\sqrt{2}+25\sqrt{21})x^2-12(1429\sqrt{2}+441\sqrt{21})x-1036400\sqrt{2}-\right.
$$

$$
\left.-319840\sqrt{21}\right]\cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)-
$$

$$
-\frac{4}{27}\sqrt{x-\frac{76}{3}-4\sqrt{42}}\cdot\left[3(3\sqrt{2}+\sqrt{21})x^3+12(13\sqrt{2}+4\sqrt{21})x^2-64(81\sqrt{2}+25\sqrt{21})x+\right.
$$

$$
\left.+1076736\sqrt{2}-+332288\sqrt{21}\right]\cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)
$$

$$
\int x^4\,\cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot\left(x+\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)\,dx=
$$

$$
=\frac{64}{27}\left[9(25\sqrt{21}+81\sqrt{2})x^2-12(441\sqrt{21}+1429\sqrt{2})x-319840\sqrt{21}-\right.
$$

$$
\left.-1036400\sqrt{2}\right]\sin\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot K_0\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)+
$$

$$
+\frac{4}{9}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\left[3(\sqrt{21}+3\sqrt{2})x^3+12(4\sqrt{21}+13\sqrt{2})x^2-64(25\sqrt{21}+81\sqrt{2})x+332288\sqrt{21}+\right.
$$

$$
\left.+1076736\sqrt{2}\right]\sin\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)-
$$

$$
-\frac{8}{27}\left[9(13+2\sqrt{42})x^3+12(25+162\sqrt{42})x^2-768(52+337\sqrt{42})x+15721344+\right.
$$

$$
\left.+2425856\sqrt{42}\right]\cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+
$$

$$
+\frac{16}{27}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\left[27(13+2\sqrt{42})x^2-24(849+131\sqrt{42})x-737664-\right.
$$

$$
\left.-113824\sqrt{42}\right]\cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)
$$

$$
\frac{76}{3}-4\sqrt{42}=-0.589\,629,\quad\frac{\sqrt{21}+3\sqrt{2}}{4}=2.206\,304
$$

$$
\int x^4\,\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot\left(x+\frac{76}{3}-4\sqrt{42}\right)^{-1/2}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)\,dx=
$$

$$
=-\frac{8}{27}\left[9(13+2\sqrt{42})x^3+12(162+25\sqrt{42})x^2-768(337+52\sqrt{42})x+15721344+\right.
$$

$$
\left.+2425856\sqrt{42}\right]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)+
$$

$$+\frac{16}{81}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\cdot\left[27(13+2\sqrt{42})x^2-24(849+131\sqrt{42})x-737664-\right.$$

$$\left.-113824\sqrt{42})\right]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)-$$

$$-\frac{64}{27}\left[12(81\sqrt{2}+25\sqrt{21})x^2-12(1429\sqrt{2}+441\sqrt{21})x-1036400\sqrt{2}-\right.$$

$$\left.-319840\sqrt{21}\right]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)-$$

$$-\frac{4}{27}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\cdot\left[3(3\sqrt{2}+\sqrt{21})x^3+12(13\sqrt{2}+4\sqrt{21})x^2-64(81\sqrt{2}+25\sqrt{21})x+\right.$$

$$\left.+1076736\sqrt{2}+332288\sqrt{21}\right]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)$$

$$\int x^4\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot\left(x+\frac{76}{3}-4\sqrt{42}\right)^{-1/2}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)\,dx=$$

$$=\frac{64}{27}\left[9(25\sqrt{21}-81\sqrt{2})x^2-12(441\sqrt{21}-1429\sqrt{2})x-319840\sqrt{21}+\right.$$

$$\left.+1036400\sqrt{2}\right]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{4}{9}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\left[3(\sqrt{21}-3\sqrt{2})x^3-12(4\sqrt{21}-13\sqrt{2})x^2-64(25\sqrt{21}-81\sqrt{2})x-332288\sqrt{21}+\right.$$

$$\left.+1076736\sqrt{2}\right]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{8}{27}\left[9(13-2\sqrt{42})x^3-12(25-162\sqrt{42})x^2-768(52-337\sqrt{42})x-15721344+\right.$$

$$\left.+2425856\sqrt{42}\right]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{16}{27}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\left[27(13-2\sqrt{42})x^2-24(849-131\sqrt{42})x-737664+\right.$$

$$\left.+113824\sqrt{42}\right]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)$$

$$\int x^4\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot\left(x-\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)\,dx=$$

$$=-\frac{8}{27}\left[9(13-2\sqrt{42})x^3+12(162-25\sqrt{42})x^2+768(337-52\sqrt{42})x+15721344-\right.$$

$$\left.-2425856\sqrt{42}\right]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{16}{81}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\cdot\left[27(13-2\sqrt{42})x^2-24(849-131\sqrt{42})x-737664+\right.$$

$$\left.+113824\sqrt{42})\right]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)-$$

$$-\frac{64}{27}\left[12(81\sqrt{2}-25\sqrt{21})x^2-12(1429\sqrt{2}-441\sqrt{21})x-1036400\sqrt{2}+\right.$$

$$+319840\sqrt{21}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{27}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\cdot\Big[3(3\sqrt{2}-\sqrt{21})x^3+12(13\sqrt{2}-4\sqrt{21})x^2-64(81\sqrt{2}-25\sqrt{21})x+$$

$$+1076736\sqrt{2}-332288\sqrt{21}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)$$

$$\int x^4\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot\left(x-\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)dx=$$

$$=-\frac{64}{27}\Big[9(25\sqrt{21}-81\sqrt{2})x^2-12(441\sqrt{21}-1429\sqrt{2})x-319840\sqrt{21}+$$

$$+1036400\sqrt{2}\Big]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{9}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\Big[3(\sqrt{21}-3\sqrt{2})x^3+12(4\sqrt{21}-13\sqrt{2})x^2-64(25\sqrt{21}-81\sqrt{2})x+332288\sqrt{21}-$$

$$-1076736\sqrt{2}\Big]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{8}{27}\Big[9(13-2\sqrt{42})x^3+12(25-162\sqrt{42})x^2-768(52-337\sqrt{42})x+15721344-$$

$$-2425856\sqrt{42}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{16}{27}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\Big[27(13-2\sqrt{42})x^2-24(849-131\sqrt{42})x-737664+$$

$$+113824\sqrt{42}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)$$

$$\int x^4\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot\left(x-\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)dx=$$

$$=-\frac{8}{27}\Big[9(13-2\sqrt{42})x^3+12(162-25\sqrt{42})x^2-768(337-52\sqrt{42})x+15721344-$$

$$-2425856\sqrt{42}\Big]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{16}{81}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\cdot\Big[27(13-2\sqrt{42})x^2-24(849-131\sqrt{42})x-737664+$$

$$+113824\sqrt{42})\Big]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{64}{27}\Big[12(81\sqrt{2}-25\sqrt{21})x^2-12(1429\sqrt{2}-441\sqrt{21})x-1036400\sqrt{2}+$$

$$+319840\sqrt{21}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{27}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\cdot\Big[3(3\sqrt{2}-\sqrt{21})x^3+12(13\sqrt{2}-4\sqrt{21})x^2-64(81\sqrt{2}-25\sqrt{21})x+$$

$$+1076736\sqrt{2}-332288\sqrt{21}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)$$

$$\int x^4 \, \cos \frac{(\sqrt{21}+3\sqrt{2}\,)x}{4} \cdot \left(x - \frac{76}{3} + 4\sqrt{42}\,\right)^{-1/2} \cdot K_1\left(\sqrt{x - \frac{76}{3} + 4\sqrt{42}}\,\right) \, dx =$$

$$= \frac{64}{27}\left[9(25\sqrt{21}-81\sqrt{2}\,)x^2 - 12(441\sqrt{21}-1429\sqrt{2}\,)x - 319840\sqrt{21}+\right.$$

$$\left. + 1036400\sqrt{2}\,\right]\sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\,\right)+$$

$$+\frac{4}{9}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\left[3(\sqrt{21}-3\sqrt{2}\,)x^3+12(4\sqrt{21}-13\sqrt{2}\,)x^2-64(25\sqrt{21}-81\sqrt{2}\,)x+332288\sqrt{21}-\right.$$

$$\left. -1076736\sqrt{2}\,\right]\sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\,\right)-$$

$$-\frac{8}{27}\left[9(13-2\sqrt{42}\,)x^3+12(25-162\sqrt{42}\,)x^2-768(52-337\sqrt{42}\,)x+15721344-\right.$$

$$\left. -2425856\sqrt{42}\,\right]\cos\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\,\right)+$$

$$+\frac{16}{27}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\left[27(13-2\sqrt{42}\,)x^2-24(849-131\sqrt{42}\,)x-737664+\right.$$

$$\left. +113824\sqrt{42}\,\right]\cos\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\,\right)$$

2. Products of two Bessel functions

2.1. Bessel Functions with the the the same Argument x :

See also [10], 3. .

2.1.1. Integrals of the type $\int x^{2n+1} Z_\nu^2(x)\, dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$.

$$\int \frac{J_1^2(x)}{x}\, dx = -\frac{1}{2}\left[J_0^2(x) + J_1^2(x)\right]$$

$$\int \frac{I_1^2(x)}{x}\, dx = \frac{1}{2}\left[I_0^2(x) - I_1^2(x)\right]$$

$$\int \frac{K_1^2(x)}{x}\, dx = \frac{1}{2}\left[K_0^2(x) - K_1^2(x)\right]$$

$$\int x J_0^2(x)\, dx = \frac{x^2}{2}\left[J_0^2(x) + J_1^2(x)\right]$$

$$\int x J_1^2(x)\, dx = \frac{x}{2}\left[x J_0^2(x) + x J_1^2(x) - 2 J_0(x)\cdot J_1(x)\right]$$

$$\int x I_0^2(x)\, dx = \frac{x^2}{2}\left[I_0^2(x) - I_1^2(x)\right]$$

$$\int x I_1^2(x)\, dx = \frac{x}{2}\left[x I_1^2(x) - x I_0^2(x) - 2 I_0(x)\cdot I_1(x)\right]$$

$$\int x K_0^2(x)\, dx = \frac{x^2}{2}\left[K_0^2(x) - K_1^2(x)\right]$$

$$\int x K_1^2(x)\, dx = \frac{x}{2}\left[x K_1^2(x) - x K_0^2(x) + 2 K_0(x)\cdot K_1(x)\right]$$

$$\int x^3 J_0^2(x)\, dx = \frac{x^4}{6} J_0^2(x) + \frac{x^3}{3} J_0(x)J_1(x) + \left(\frac{x^4}{6} - \frac{x^2}{3}\right) J_1^2(x)$$

$$\int x^3 J_1^2(x)\, dx = \frac{x^4}{6} J_0^2(x) - \frac{2x^3}{3} J_0(x)J_1(x) + \left(\frac{x^4}{6} + \frac{2x^2}{3}\right) J_1^2(x)$$

$$\int x^3 I_0^2(x)\, dx = \frac{x^4}{6} I_0^2(x) + \frac{x^3}{3} I_0(x)I_1(x) - \left(\frac{x^4}{6} + \frac{x^2}{3}\right) I_1^2(x)$$

$$\int x^3 I_1^2(x)\, dx = -\frac{x^4}{6} I_0^2(x) + \frac{2x^3}{3} I_0(x)I_1(x) + \left(\frac{x^4}{6} - \frac{2x^2}{3}\right) I_1^2(x)$$

$$\int x^3 K_0^2(x)\, dx = \frac{x^4}{6} K_0^2(x) - \frac{x^3}{3} K_0(x)K_1(x) - \left(\frac{x^4}{6} + \frac{x^2}{3}\right) K_1^2(x)$$

$$\int x^3 K_1^2(x)\, dx = -\frac{x^4}{6} K_0^2(x) - \frac{2x^3}{3} K_0(x)K_1(x) + \left(\frac{x^4}{6} - \frac{2x^2}{3}\right) K_1^2(x)$$

$$\int x^5 J_0^2(x)\, dx = \left(\frac{x^6}{10} + \frac{4x^4}{15}\right) J_0^2(x) + \left(\frac{2x^5}{5} - \frac{16x^3}{15}\right) J_0(x)J_1(x) + \left(\frac{x^6}{10} - \frac{8x^4}{15} + \frac{16x^2}{15}\right) J_1^2(x)$$

$$\int x^5 J_1^2(x)\, dx = \left(\frac{x^6}{10} - \frac{2x^4}{5}\right) J_0^2(x) + \left(-\frac{3x^5}{5} + \frac{8x^3}{5}\right) J_0(x)J_1(x) + \left(\frac{x^6}{10} + \frac{4x^4}{5} - \frac{8x^2}{5}\right) J_1^2(x)$$

$$\int x^5 I_0^2(x)\, dx = \left(\frac{x^6}{10} - \frac{4x^4}{15}\right) I_0^2(x) + \left(\frac{2x^5}{5} + \frac{16x^3}{15}\right) I_0(x)I_1(x) - \left(\frac{x^6}{10} + \frac{8x^4}{15} + \frac{16x^2}{15}\right) I_1^2(x)$$

$$\int x^5 I_1^2(x)\, dx = -\left(\frac{x^6}{10} + \frac{2x^4}{5}\right) I_0^2(x) + \left(\frac{3x^5}{5} + \frac{8x^3}{5}\right) I_0(x)I_1(x) + \left(\frac{x^6}{10} - \frac{4x^4}{5} - \frac{8x^2}{5}\right) I_1^2(x)$$

$$\int x^5 K_0^2(x)\,dx \;=\; \left(\frac{x^6}{10}-\frac{4x^4}{15}\right)K_0^2(x)-\left(\frac{2x^5}{5}+\frac{16x^3}{15}\right)K_0(x)K_1(x)-\left(\frac{x^6}{10}+\frac{8x^4}{15}+\frac{16x^2}{15}\right)K_1^2(x)$$

$$\int x^5 K_1^2(x)\,dx \;=\; -\left(\frac{x^6}{10}+\frac{2x^4}{5}\right)K_0^2(x)-\left(\frac{3x^5}{5}+\frac{8x^3}{5}\right)K_0(x)K_1(x)+\left(\frac{x^6}{10}-\frac{4x^4}{5}-\frac{8x^2}{5}\right)K_1^2(x)$$

$$\int x^7 J_0^2(x)\,dx \;=\; \left(\frac{x^8}{14}+\frac{18x^6}{35}-\frac{72x^4}{35}\right)J_0^2(x)+\left(\frac{3x^7}{7}-\frac{108x^5}{35}+\frac{288x^3}{35}\right)J_0(x)J_1(x)+$$
$$+\left(\frac{x^8}{14}-\frac{27x^6}{35}+\frac{144x^4}{35}-\frac{288x^2}{35}\right)J_1^2(x)$$

$$\int x^7 J_1^2(x)\,dx \;=\; \left(\frac{x^8}{14}-\frac{24x^6}{35}+\frac{96x^4}{35}\right)J_0^2(x)+\left(-\frac{4x^7}{7}+\frac{144x^5}{35}-\frac{384x^3}{35}\right)J_0(x)J_1(x)+$$
$$+\left(\frac{x^8}{14}+\frac{36x^6}{35}-\frac{192x^4}{35}+\frac{384x^2}{35}\right)J_1^2(x)$$

$$\int x^7 I_0^2(x)\,dx \;=\; \left(\frac{x^8}{14}-\frac{18x^6}{35}-\frac{72x^4}{35}\right)I_0^2(x)+\left(\frac{3x^7}{7}+\frac{108x^5}{35}+\frac{288x^3}{35}\right)I_0(x)I_1(x)-$$
$$-\left(\frac{x^8}{14}+\frac{27x^6}{35}+\frac{144x^4}{35}+\frac{288x^2}{35}\right)I_1^2(x)$$

$$\int x^7 I_1^2(x)\,dx \;=\; -\left(\frac{x^8}{14}+\frac{24x^6}{35}+\frac{96x^4}{35}\right)I_0^2(x)+\left(\frac{4x^7}{7}+\frac{144x^5}{35}+\frac{384x^3}{35}\right)I_0(x)I_1(x)+$$
$$+\left(\frac{x^8}{14}-\frac{36x^6}{35}-\frac{192x^4}{35}-\frac{384x^2}{35}\right)I_1^2(x)$$

$$\int x^7 K_0^2(x)\,dx \;=\; \left(\frac{x^8}{14}-\frac{18x^6}{35}-\frac{72x^4}{35}\right)K_0^2(x)-\left(\frac{3x^7}{7}+\frac{108x^5}{35}+\frac{288x^3}{35}\right)K_0(x)K_1(x)-$$
$$-\left(\frac{x^8}{14}+\frac{27x^6}{35}+\frac{144x^4}{35}+\frac{288x^2}{35}\right)K_1^2(x)$$

$$\int x^7 K_1^2(x)\,dx \;=\; -\left(\frac{x^8}{14}+\frac{24x^6}{35}+\frac{96x^4}{35}\right)K_0^2(x)-\left(\frac{4x^7}{7}+\frac{144x^5}{35}+\frac{384x^3}{35}\right)K_0(x)K_1(x)+$$
$$+\left(\frac{x^8}{14}-\frac{36x^6}{35}-\frac{192x^4}{35}-\frac{384x^2}{35}\right)K_1^2(x)$$

$$\int x^9 J_0^2(x)\,dx \;=\; \left(\frac{x^{10}}{18}+\frac{16x^8}{21}-\frac{256x^6}{35}+\frac{1024x^4}{35}\right)J_0^2(x)+$$
$$+\left(\frac{4x^9}{9}-\frac{128x^7}{21}+\frac{1536x^5}{35}-\frac{4096x^3}{35}\right)J_0(x)J_1(x)+$$
$$+\left(\frac{x^{10}}{18}-\frac{64x^8}{63}+\frac{384x^6}{35}-\frac{2048x^4}{35}+\frac{4096x^2}{35}\right)J_1^2(x)$$

$$\int x^9 J_1^2(x)\,dx \;=\; \left(\frac{x^{10}}{18}-\frac{20x^8}{21}+\frac{64x^6}{7}-\frac{256x^4}{7}\right)J_0^2(x)+$$
$$+\left(-\frac{5x^9}{9}+\frac{160x^7}{21}-\frac{384x^5}{7}+\frac{1024x^3}{7}\right)J_0(x)J_1(x)+$$
$$+\left(\frac{x^{10}}{18}+\frac{80x^8}{63}-\frac{96x^6}{7}+\frac{512x^4}{7}-\frac{1024x^2}{7}\right)J_1^2(x)$$

$$\int x^9 I_0^2(x)\,dx \;=\; \left(\frac{x^{10}}{18}-\frac{16x^8}{21}-\frac{256x^6}{35}-\frac{1024x^4}{35}\right)I_0^2(x)+$$
$$+\left(\frac{4x^9}{9}+\frac{128x^7}{21}+\frac{1536x^5}{35}+\frac{4096x^3}{35}\right)I_0(x)I_1(x)-$$

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$$-\left(\frac{x^{10}}{18} + \frac{64x^8}{63} + \frac{384x^6}{35} + \frac{2048x^4}{35} + \frac{4096x^2}{35}\right) I_1^2(x)$$

$$\int x^9 I_1^2(x)\,dx \;=\; -\left(\frac{x^{10}}{18} + \frac{20x^8}{21} + \frac{64x^6}{7} + \frac{256x^4}{7}\right) I_0^2(x)+$$

$$+\left(\frac{5x^9}{9} + \frac{160x^7}{21} + \frac{384x^5}{7} + \frac{1024x^3}{7}\right) I_0(x)I_1(x)+$$

$$+\left(\frac{x^{10}}{18} - \frac{80x^8}{63} - \frac{96x^6}{7} - \frac{512x^4}{7} - \frac{1024x^2}{7}\right) I_1^2(x)$$

$$\int x^9 K_0^2(x)\,dx \;=\; \left(\frac{x^{10}}{18} - \frac{16x^8}{21} - \frac{256x^6}{35} - \frac{1024x^4}{35}\right) K_0^2(x)-$$

$$-\left(\frac{4x^9}{9} + \frac{128x^7}{21} + \frac{1536x^5}{35} + \frac{4096x^3}{35}\right) K_0(x)K_1(x)-$$

$$-\left(\frac{x^{10}}{18} + \frac{64x^8}{63} + \frac{384x^6}{35} + \frac{2048x^4}{35} + \frac{4096x^2}{35}\right) K_1^2(x)$$

$$\int x^9 K_1^2(x)\,dx \;=\; -\left(\frac{x^{10}}{18} + \frac{20x^8}{21} + \frac{64x^6}{7} + \frac{256x^4}{7}\right) K_0^2(x)-$$

$$-\left(\frac{5x^9}{9} + \frac{160x^7}{21} + \frac{384x^5}{7} + \frac{1024x^3}{7}\right) K_0(x)K_1(x)+$$

$$+\left(\frac{x^{10}}{18} - \frac{80x^8}{63} - \frac{96x^6}{7} - \frac{512x^4}{7} - \frac{1024x^2}{7}\right) K_1^2(x)$$

Let

$$\int x^m \cdot J_0^2(x)\,dx = A_m(x)\cdot J_0^2(x) + B_m(x)\cdot J_0(x)\cdot J_1(x) + C_m(x)\cdot J_1^2(x)\,,$$

$$\int x^m \cdot J_1^2(x)\,dx = D_m(x)\cdot J_0^2(x) + E_m(x)\cdot J_0(x)\cdot J_1(x) + F_m(x)\cdot J_1^2(x)\,,$$

$$\int x^m \cdot I_0^2(x)\,dx = A_m^*(x)\cdot I_0^2(x) + B_m^*(x)\cdot I_0(x)\cdot I_1(x) + C_m^*(x)\cdot I_1^2(x)\,,$$

$$\int x^m \cdot I_1^2(x)\,dx = D_m^*(x)\cdot I_0^2(x) + E_m^*(x)\cdot I_0(x)\cdot I_1(x) + F_m^*(x)\cdot I_1^2(x)$$

and

$$\int x^m \cdot K_0^2(x)\,dx = A_m^*(x)\cdot K_0^2(x) - B_m^*(x)\cdot K_0(x)\cdot K_1(x) + C_m^*(x)\cdot K_1^2(x)\,,$$

$$\int x^m \cdot K_1^2(x)\,dx = D_m^*(x)\cdot K_0^2(x) - E_m^*(x)\cdot K_0(x)\cdot K_1(x) + F_m^*(x)\cdot K_1^2(x)\,,$$

then holds

$$A_{11} = \frac{1}{22}\,x^{12} + \frac{100}{99}\,x^{10} - \frac{4000}{231}\,x^8 + \frac{12800}{77}\,x^6 - \frac{51200}{77}\,x^4$$

$$B_{11} = \frac{5}{11}\,x^{11} - \frac{1000}{99}\,x^9 + \frac{32000}{231}\,x^7 - \frac{76800}{77}\,x^5 + \frac{204800}{77}\,x^3$$

$$C_{11} = \frac{1}{22}\,x^{12} - \frac{125}{99}\,x^{10} + \frac{16000}{693}\,x^8 - \frac{19200}{77}\,x^6 + \frac{102400}{77}\,x^4 - \frac{204800}{77}\,x^2$$

$$D_{11} = \frac{1}{22}\,x^{12} - \frac{40}{33}\,x^{10} + \frac{1600}{77}\,x^8 - \frac{15360}{77}\,x^6 + \frac{61440}{77}\,x^4$$

$$E_{11} = -\frac{6}{11}\,x^{11} + \frac{400}{33}\,x^9 - \frac{12800}{77}\,x^7 + \frac{92160}{77}\,x^5 - \frac{245760}{77}\,x^3$$

$$F_{11} = \frac{1}{22}\,x^{12} + \frac{50}{33}\,x^{10} - \frac{6400}{231}\,x^8 + \frac{23040}{77}\,x^6 - \frac{122880}{77}\,x^4 + \frac{245760}{77}\,x^2$$

$$A_{11}^* = \frac{1}{22}\,x^{12} - \frac{100}{99}\,x^{10} - \frac{4000}{231}\,x^8 - \frac{12800}{77}\,x^6 - \frac{51200}{77}\,x^4$$

$$B_{11}^* = \frac{5}{11}\,x^{11} + \frac{1000}{99}\,x^9 + \frac{32000}{231}\,x^7 + \frac{76800}{77}\,x^5 + \frac{204800}{77}\,x^3$$

$$C_{11}^* = -\frac{1}{22}\,x^{12} - \frac{125}{99}\,x^{10} - \frac{16000}{693}\,x^8 - \frac{19200}{77}\,x^6 - \frac{102400}{77}\,x^4 - \frac{204800}{77}\,x^2$$

$$D_{11}^* = -\frac{1}{22}\,x^{12} - \frac{40}{33}\,x^{10} - \frac{1600}{77}\,x^8 - \frac{15360}{77}\,x^6 - \frac{61440}{77}\,x^4$$

$$E_{11}^* = \frac{6}{11}\,x^{11} + \frac{400}{33}\,x^9 + \frac{12800}{77}\,x^7 + \frac{92160}{77}\,x^5 + \frac{245760}{77}\,x^3$$

$$F_{11}^* = \frac{1}{22}\,x^{12} - \frac{50}{33}\,x^{10} - \frac{6400}{231}\,x^8 - \frac{23040}{77}\,x^6 - \frac{122880}{77}\,x^4 - \frac{245760}{77}\,x^2$$

$$A_{13} = \frac{1}{26}\,x^{14} + \frac{180}{143}\,x^{12} - \frac{4800}{143}\,x^{10} + \frac{576000}{1001}\,x^8 - \frac{5529600}{1001}\,x^6 + \frac{22118400}{1001}\,x^4$$

$$B_{13} = \frac{6}{13}\,x^{13} - \frac{2160}{143}\,x^{11} + \frac{48000}{143}\,x^9 - \frac{4608000}{1001}\,x^7 + \frac{33177600}{1001}\,x^5 - \frac{88473600}{1001}\,x^3$$

$$C_{13} = \frac{1}{26}\,x^{14} - \frac{216}{143}\,x^{12} + \frac{6000}{143}\,x^{10} - \frac{768000}{1001}\,x^8 + \frac{8294400}{1001}\,x^6 - \frac{44236800}{1001}\,x^4 + \frac{88473600}{1001}\,x^2$$

$$D_{13} = \frac{1}{26}\,x^{14} - \frac{210}{143}\,x^{12} + \frac{5600}{143}\,x^{10} - \frac{96000}{143}\,x^8 + \frac{921600}{143}\,x^6 - \frac{3686400}{143}\,x^4$$

$$E_{13} = -\frac{7}{13}\,x^{13} + \frac{2520}{143}\,x^{11} - \frac{56000}{143}\,x^9 + \frac{768000}{143}\,x^7 - \frac{5529600}{143}\,x^5 + \frac{14745600}{143}\,x^3$$

$$F_{13} = \frac{1}{26}\,x^{14} + \frac{252}{143}\,x^{12} - \frac{7000}{143}\,x^{10} + \frac{128000}{143}\,x^8 - \frac{1382400}{143}\,x^6 + \frac{7372800}{143}\,x^4 - \frac{14745600}{143}\,x^2$$

$$A_{13}^* = \frac{1}{26}\,x^{14} - \frac{180}{143}\,x^{12} - \frac{4800}{143}\,x^{10} - \frac{576000}{1001}\,x^8 - \frac{5529600}{1001}\,x^6 - \frac{22118400}{1001}\,x^4$$

$$B_{13}^* = \frac{6}{13}\,x^{13} + \frac{2160}{143}\,x^{11} + \frac{48000}{143}\,x^9 + \frac{4608000}{1001}\,x^7 + \frac{33177600}{1001}\,x^5 + \frac{88473600}{1001}\,x^3$$

$$C_{13}^* = -\frac{1}{26}\,x^{14} - \frac{216}{143}\,x^{12} - \frac{6000}{143}\,x^{10} - \frac{768000}{1001}\,x^8 - \frac{8294400}{1001}\,x^6 - \frac{44236800}{1001}\,x^4 - \frac{88473600}{1001}\,x^2$$

$$D_{13}^* = -\frac{1}{26}\,x^{14} - \frac{210}{143}\,x^{12} - \frac{5600}{143}\,x^{10} - \frac{96000}{143}\,x^8 - \frac{921600}{143}\,x^6 - \frac{3686400}{143}\,x^4$$

$$E_{13}^* = \frac{7}{13}\,x^{13} + \frac{2520}{143}\,x^{11} + \frac{56000}{143}\,x^9 + \frac{768000}{143}\,x^7 + \frac{5529600}{143}\,x^5 + \frac{14745600}{143}\,x^3$$

$$F_{13}^* = \frac{1}{26}\,x^{14} - \frac{252}{143}\,x^{12} - \frac{7000}{143}\,x^{10} - \frac{128000}{143}\,x^8 - \frac{1382400}{143}\,x^6 - \frac{7372800}{143}\,x^4 - \frac{14745600}{143}\,x^2$$

$$A_{15} = \frac{1}{30}\,x^{16} + \frac{98}{65}\,x^{14} - \frac{8232}{143}\,x^{12} + \frac{219520}{143}\,x^{10} - \frac{3763200}{143}\,x^8 + \frac{36126720}{143}\,x^6 - \frac{144506880}{143}\,x^4$$

$$B_{15} = \frac{7}{15}\,x^{15} - \frac{1372}{65}\,x^{13} + \frac{98784}{143}\,x^{11} - \frac{2195200}{143}\,x^9 + \frac{30105600}{143}\,x^7 - \frac{216760320}{143}\,x^5 + \frac{578027520}{143}\,x^3$$

$$C_{15} = \frac{x^{16}}{30} - \frac{343}{195}\,x^{14} + \frac{49392}{715}\,x^{12} - \frac{274400}{143}\,x^{10} + \frac{5017600}{143}\,x^8 - \frac{54190080}{143}\,x^6 + \frac{289013760}{143}\,x^4 - \frac{578027520}{143}\,x^2$$

$$D_{15} = \frac{x^{16}}{30} - \frac{112}{65}\,x^{14} + \frac{9408}{143}\,x^{12} - \frac{250880}{143}\,x^{10} + \frac{4300800}{143}\,x^8 - \frac{41287680}{143}\,x^6 + \frac{165150720}{143}\,x^4$$

$$E_{15} = -\frac{8}{15}\,x^{15} + \frac{1568}{65}\,x^{13} - \frac{112896}{143}\,x^{11} + \frac{2508800}{143}\,x^9 - \frac{34406400}{143}\,x^7 + \frac{247726080}{143}\,x^5 - \frac{660602880}{143}\,x^3$$

$$F_{15} = \frac{x^{16}}{30} + \frac{392}{195}\,x^{14} - \frac{56448}{715}\,x^{12} + \frac{313600}{143}\,x^{10} - \frac{5734400}{143}\,x^8 + \frac{61931520}{143}\,x^6 - \frac{330301440}{143}\,x^4 + \frac{660602880}{143}\,x^2$$

$$A_{15}^* = \frac{x^{16}}{30} - \frac{98}{65}\,x^{14} - \frac{8232}{143}\,x^{12} - \frac{219520}{143}\,x^{10} - \frac{3763200}{143}\,x^8 - \frac{36126720}{143}\,x^6 - \frac{144506880}{143}\,x^4$$

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$$B_{15}^* = \frac{7}{15}\,x^{15} + \frac{1372}{65}\,x^{13} + \frac{98784}{143}\,x^{11} + \frac{2195200}{143}\,x^{9} + \frac{30105600}{143}\,x^{7} + \frac{216760320}{143}\,x^{5} + \frac{578027520}{143}\,x^{3}$$

$$C_{15}^* = -\frac{x^{16}}{30} - \frac{343}{195}\,x^{14} - \frac{49392}{715}\,x^{12} - \frac{274400}{143}\,x^{10} - \frac{5017600}{143}\,x^{8} - \frac{54190080}{143}\,x^{6} - \frac{289013760}{143}\,x^{4} - \frac{578027520}{143}\,x^{2}$$

$$D_{15}^* = -\frac{x^{16}}{30} - \frac{112}{65}\,x^{14} - \frac{9408}{143}\,x^{12} - \frac{250880}{143}\,x^{10} - \frac{4300800}{143}\,x^{8} - \frac{41287680}{143}\,x^{6} - \frac{165150720}{143}\,x^{4}$$

$$E_{15}^* = \frac{8}{15}\,x^{15} + \frac{1568}{65}\,x^{13} + \frac{112896}{143}\,x^{11} + \frac{2508800}{143}\,x^{9} + \frac{34406400}{143}\,x^{7} + \frac{247726080}{143}\,x^{5} + \frac{660602880}{143}\,x^{3}$$

$$F_{15}^* = \frac{x^{16}}{30} - \frac{392}{195}\,x^{14} - \frac{56448}{715}\,x^{12} - \frac{313600}{143}\,x^{10} - \frac{5734400}{143}\,x^{8} - \frac{61931520}{143}\,x^{6} - \frac{330301440}{143}\,x^{4} - \frac{660602880}{143}\,x^{2}$$

Recurrence Formulas:

$$\int x^{2n+1} J_0^2(x)\,dx = \frac{x^{2n}}{4n+2}\left\{(x^2+2n^2)J_0^2(x) + x^2 J_1^2(x) + 2nx\,J_0(x)J_1(x)\right\} - \frac{2n^3}{2n+1}\int x^{2n-1} J_0^2(x)\,dx$$

$$\int x^{2n+1} J_1^2(x)\,dx =$$

$$= \frac{x^{2n}}{4n+2}\left\{x^2\,J_0^2(x) + \left[x^2 + 2n(n+1)\right]J_1^2(x) - 2(n+1)xJ_0(x)J_1(x)\right\} - \frac{2n(n^2-1)}{2n+1}\int x^{2n-1} J_1^2(x)\,dx$$

$$\int x^{2n+1} I_0^2(x)\,dx = \frac{x^{2n}}{4n+2}\left\{(x^2-2n^2)I_0^2(x) - x^2 I_1^2(x) + 2nx\,I_0(x)I_1(x)\right\} + \frac{2n^3}{2n+1}\int x^{2n-1} I_0^2(x)\,dx$$

$$\int x^{2n+1} I_1^2(x)\,dx =$$

$$= \frac{x^{2n}}{4n+2}\left\{-x^2\,I_0^2(x) + \left[x^2 - 2n(n+1)\right]I_1^2(x) + 2(n+1)xI_0(x)I_1(x)\right\} + \frac{2n(n^2-1)}{2n+1}\int x^{2n-1} I_1^2(x)\,dx$$

$$\int x^{2n+1} K_0^2(x)\,dx = \frac{x^{2n}}{4n+2}\left\{(x^2-2n^2)K_0^2(x) - x^2 K_1^2(x) - 2nx\,K_0(x)K_1(x)\right\} + \frac{2n^3}{2n+1}\int x^{2n-1} K_0^2(x)\,dx$$

$$\int x^{2n+1} K_1^2(x)\,dx =$$

$$= \frac{x^{2n}}{4n+2}\left\{-x^2\,K_0^2(x) + \left[x^2 - 2n(n+1)\right]K_1^2(x) - 2(n+1)xK_0(x)K_1(x)\right\} + \frac{2n(n^2-1)}{2n+1}\int x^{2n-1} K_1^2(x)\,dx$$

2.1.2. Integrals of the type $\int x^{-2n} Z_\nu^2(x)\, dx$

See also [4], 1.8.3.

Concerning the case $x^{+2n} Z_\nu^2(x)$ see 2.1.3., p. 278 .

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$.

$$\int \frac{J_0^2(x)}{x^2}\, dx = -\left(2x + \frac{1}{x}\right) J_0^2(x) + 2 J_0(x) J_1(x) - 2x J_1^2(x)$$

$$\int \frac{I_0^2(x)}{x^2}\, dx = \left(2x - \frac{1}{x}\right) I_0^2(x) - 2 I_0(x) I_1(x) - 2x I_1^2(x)$$

$$\int \frac{K_0^2(x)}{x^2}\, dx = \left(2x - \frac{1}{x}\right) K_0^2(x) + 2 K_0(x) K_1(x) - 2x K_1^2(x)$$

$$\int \frac{J_1^2(x)}{x^2}\, dx = \frac{2x}{3} J_0^2(x) - \frac{2}{3} J_0(x) J_1(x) + \left(\frac{2x}{3} - \frac{1}{3x}\right) J_1^2(x)$$

$$\int \frac{I_1^2(x)}{x^2}\, dx = \frac{2x}{3} I_0^2(x) - \frac{2}{3} I_0(x) I_1(x) - \left(\frac{2x}{3} + \frac{1}{3x}\right) I_1^2(x)$$

$$\int \frac{K_1^2(x)}{x^2}\, dx = \frac{2x}{3} K_0^2(x) + \frac{2}{3} K_0(x) K_1(x) - \left(\frac{2x}{3} + \frac{1}{3x}\right) K_1^2(x)$$

$$\int \frac{J_0^2(x)}{x^4}\, dx = \frac{1}{27x^3}(16x^4 + 6x^2 - 9) J_0^2(x) + \frac{1}{27x^2}(-16x^2 + 6) J_0(x) J_1(x) + \frac{1}{27x}(16x^2 - 2) J_1^2(x)$$

$$\int \frac{I_0^2(x)}{x^4}\, dx = \frac{1}{27x^3}(16x^4 - 6x^2 - 9) I_0^2(x) - \frac{1}{27x^2}(16x^2 + 6) I_0(x) I_1(x) - \frac{1}{27x}(16x^2 + 2) I_1^2(x)$$

$$\int \frac{K_0^2(x)}{x^4}\, dx = \frac{1}{27x^3}(16x^4 - 6x^2 - 9) K_0^2(x) + \frac{1}{27x^2}(16x^2 + 6) K_0(x) K_1(x) - \frac{1}{27x}(16x^2 + 2) K_1^2(x)$$

$$\int \frac{J_1^2(x)}{x^4}\, dx = \frac{-16x^2 - 6}{45x} J_0^2(x) + \frac{16x^2 - 6}{45x^2} J_0(x) J_1(x) + \frac{-16x^4 + 2x^2 - 9}{45x^3} J_1^2(x)$$

$$\int \frac{I_1^2(x)}{x^4}\, dx = \frac{16x^2 - 6}{45x} I_0^2(x) - \frac{16x^2 + 6}{45x^2} I_0(x) I_1(x) - \frac{16x^4 + 2x^2 + 9}{45x^3} I_1^2(x)$$

$$\int \frac{K_1^2(x)}{x^4}\, dx = \frac{16x^2 - 6}{45x} K_0^2(x) + \frac{16x^2 + 6}{45x^2} K_0(x) K_1(x) - \frac{16x^4 + 2x^2 + 9}{45x^3} K_1^2(x)$$

$$\int \frac{J_0^2(x)}{x^6}\, dx = \frac{-256x^6 - 96x^4 + 90x^2 - 675}{3375x^5} J_0^2(x) + \frac{256x^4 - 96x^2 + 270}{3375x^4} J_0(x) J_1(x) +$$
$$+ \frac{-256x^4 + 32x^2 - 54}{3375x^3} J_1^2(x)$$

$$\int \frac{I_0^2(x)}{x^6}\, dx = \frac{256x^6 - 96x^4 - 90x^2 - 675}{3375x^5} I_0^2(x) - \frac{256x^4 + 96x^2 + 270}{3375x^4} I_0(x) I_1(x) -$$
$$- \frac{256x^4 + 32x^2 + 54}{3375x^3} I_1^2(x)$$

$$\int \frac{K_0^2(x)}{x^6}\, dx = \frac{256x^6 - 96x^4 - 90x^2 - 675}{3375x^5} K_0^2(x) + \frac{256x^4 + 96x^2 + 270}{3375x^4} K_0(x) K_1(x) -$$
$$- \frac{256x^4 + 32x^2 + 54}{3375x^3} K_1^2(x)$$

$$\int \frac{J_1^2(x)}{x^6}\, dx = \frac{1}{4725x^5}\left[x^2(256x^4 + 96x^2 - 90) J_0^2(x) + x(-256x^4 + 96x^2 - 270) J_0(x) J_1(x) + \right.$$
$$\left. + (256x^6 - 32x^4 + 54x^2 - 675) J_1^2(x) \right]$$

$$\int \frac{I_1^2(x)}{x^6}\, dx = \frac{1}{4725x^5}\left[x^2(256x^4 - 96x^2 - 90) I_0^2(x) - x(256x^4 + 96x^2 + 270) I_0(x) I_1(x) - \right.$$

$$-(256\,x^6 + 32\,x^4 + 54\,x^2 + 675)\,I_1^2(x)\big]$$

$$\int \frac{K_1^2(x)}{x^6}\,dx = \frac{1}{4725x^5}\,\big[\,x^2(256\,x^4 - 96\,x^2 - 90)\,K_0^2(x) + x(256\,x^4 + 96\,x^2 + 270)\,K_0(x)\,K_1(x) -$$

$$-(256\,x^6 + 32\,x^4 + 54\,x^2 + 675)\,K_1^2(x)\big]$$

$$\int \frac{J_0^2(x)}{x^8}\,dx = \frac{1}{385875x^7}\,\big[\,(2048\,x^8 + 768\,x^6 - 720\,x^4 + 3150\,x^2 - 55125)\,J_0^2(x) +$$

$$+x(-2048\,x^6 + 768\,x^4 - 2160\,x^2 + 15750)\,J_0(x)\,J_1(x) + x^2\,(2048\,x^6 - 256\,x^4 + 432\,x^2 - 2250)\,J_1^2(x)\big]$$

$$\int \frac{I_0^2(x)}{x^8}\,dx = \frac{1}{385875x^7}\,\big[\,(2048\,x^8 - 768\,x^6 - 720\,x^4 - 3150\,x^2 - 55125)\,I_0^2(x) -$$

$$-x(2048\,x^6 + 768\,x^4 + 2160\,x^2 + 15750)\,I_0(x)\,I_1(x) - x^2\,(2048\,x^6 + 256\,x^4 + 432\,x^2 + 2250)\,I_1^2(x)\big]$$

$$\int \frac{K_0^2(x)}{x^8}\,dx = \frac{1}{385875x^7}\,\big[\,(2048\,x^8 - 768\,x^6 - 720\,x^4 - 3150\,x^2 - 55125)\,K_0^2(x) +$$

$$+x(2048\,x^6 + 768\,x^4 + 2160\,x^2 + 15750)\,K_0(x)\,K_1(x) - x^2\,(2048\,x^6 + 256\,x^4 + 432\,x^2 + 2250)\,K_1^2(x)\big]$$

$$\int \frac{J_1^2(x)}{x^8}\,dx = \frac{1}{496125x^7}\,\big[\,x^2(-2048\,x^6 - 768\,x^4 + 720\,x^2 - 3150)\,J_0^2(x) +$$

$$+x(2048\,x^6 - 768\,x^4 + 2160\,x^2 - 15750)\,J_0(x)\,J_1(x) + (-2048\,x^8 + 256\,x^6 - 432\,x^4 + 2250\,x^2 - 55125)\,J_1^2(x)\big]$$

$$\int \frac{I_1^2(x)}{x^8}\,dx = \frac{1}{496125x^7}\,\big[\,x^2(2048\,x^6 - 768\,x^4 - 720\,x^2 - 3150)\,I_0^2(x) -$$

$$-x(2048\,x^6 + 768\,x^4 + 2160\,x^2 + 15750)\,I_0(x)\,I_1(x) - (2048\,x^8 + 256\,x^6 + 432\,x^4 + 2250\,x^2 + 55125)\,I_1^2(x)\big]$$

$$\int \frac{K_1^2(x)}{x^8}\,dx = \frac{1}{496125x^7}\,\big[\,x^2(2048\,x^6 - 768\,x^4 - 720\,x^2 - 3150)\,K_0^2(x) +$$

$$+x(2048\,x^6 + 768\,x^4 + 2160\,x^2 + 15750)\,K_0(x)\,K_1(x) - (2048\,x^8 + 256\,x^6 + 432\,x^4 + 2250\,x^2 + 55125)\,K_1^2(x)\big]$$

$$\int \frac{J_0^2(x)}{x^{10}}\,dx = \frac{1}{281302875x^9}\,\big[\,(-65536\,x^{10} - 24576\,x^8 + 23040\,x^6 - 100800\,x^4 + 992250\,x^2 - 31255875)\,J_0^2(x) +$$

$$+x(65536\,x^8 - 24576\,x^6 + 69120\,x^4 - 504000\,x^2 + 6945750)\,J_0(x)\,J_1(x) +$$

$$+x^2\,(-65536\,x^8 + 8192\,x^6 - 13824\,x^4 + 72000\,x^2 - 771750)\,J_1^2(x)\big]$$

$$\int \frac{I_0^2(x)}{x^{10}}\,dx = \frac{1}{281302875x^9}\,\big[\,(65536\,x^{10} - 24576\,x^8 - 23040\,x^6 - 100800\,x^4 - 992250\,x^2 - 31255875)\,I_0^2(x) -$$

$$-x(65536\,x^8 + 24576\,x^6 + 69120\,x^4 + 504000\,x^2 + 6945750)\,I_0(x)\,I_1(x) -$$

$$-x^2\,(65536\,x^8 + 8192\,x^6 + 13824\,x^4 + 72000\,x^2 + 771750)\,I_1^2(x)\big]$$

$$\int \frac{K_0^2(x)}{x^{10}}\,dx = \frac{1}{281302875x^9}\,\big[\,(65536\,x^{10} - 24576\,x^8 - 23040\,x^6 - 100800\,x^4 - 992250\,x^2 - 31255875)\,K_0^2(x) +$$

$$+x(65536\,x^8 + 24576\,x^6 + 69120\,x^4 + 504000\,x^2 + 6945750)\,K_0(x)\,K_1(x) -$$

$$-x^2\,(65536\,x^8 + 8192\,x^6 + 13824\,x^4 + 72000\,x^2 + 771750)\,K_1^2(x)\big]$$

$$\int \frac{J_1^2(x)}{x^{10}}\,dx = \frac{1}{343814625x^9}\,\big[\,x^2(65536\,x^8 + 24576\,x^6 - 23040\,x^4 + 100800\,x^2 - 992250)\,J_0^2(x) +$$

$$+x(-65536\,x^8 + 24576\,x^6 - 69120\,x^4 + 504000\,x^2 - 6945750)\,J_0(x)\,J_1(x) +$$

$$+(65536\,x^{10} - 8192\,x^8 + 13824\,x^6 - 72000\,x^4 + 771750\,x^2 - 31255875)\,J_1^2(x)\big]$$

$$\int \frac{I_1^2(x)}{x^{10}}\,dx = \frac{1}{343814625x^9}\,\big[\,x^2(65536\,x^8 - 24576\,x^6 - 23040\,x^4 - 100800\,x^2 - 992250)\,I_0^2(x) -$$

$$-x(65536\,x^8 + 24576\,x^6 + 69120\,x^4 + 504000\,x^2 + 6945750)\,I_0(x)\,I_1(x) -$$

$$-(65536\,x^{10} + 8192\,x^8 + 13824\,x^6 + 72000\,x^4 + 771750\,x^2 - 31255875)\,I_1^2(x)\big]$$

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$$\int \frac{K_1^2(x)}{x^{10}}\,dx \;=\; \frac{1}{343814625x^9}\left[x^2(65536\,x^8 - 24576\,x^6 - 23040\,x^4 - 100800\,x^2 - 992250)\,K_0^2(x)+\right.$$

$$+x(65536\,x^8 + 24576\,x^6 + 69120\,x^4 + 504000\,x^2 + 6945750)\,K_0(x)\,K_1(x)-$$

$$\left.-(65536\,x^{10} + 8192\,x^8 + 13824\,x^6 + 72000\,x^4 + 771750\,x^2 - 31255875)\,K_1^2(x)\right]$$

<u>Recurrence formulas:</u>

$$\int \frac{J_0^2(x)}{x^{2n+2}}\,dx = (2n+1)^{-3}\cdot$$

$$\cdot\left[\frac{-[2x^2 + (2n+1)^2]\,J_0^2(x) + (4n+2)xJ_0(x)J_1(x) - 2x^2\,J_1^2(x)}{x^{2n+1}} - 8n\int \frac{J_0^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{I_0^2(x)}{x^{2n+2}}\,dx = (2n+1)^{-3}\cdot$$

$$\cdot\left[\frac{[2x^2 - (2n+1)^2]\,I_0^2(x) - (4n+2)xI_0(x)I_1(x) - 2x^2\,I_1^2(x)}{x^{2n+1}} + 8n\int \frac{I_0^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{K_0^2(x)}{x^{2n+2}}\,dx = (2n+1)^{-3}\cdot$$

$$\cdot\left[\frac{[2x^2 - (2n+1)^2]\,K_0^2(x) + (4n+2)xK_0(x)K_1(x) - 2x^2\,K_1^2(x)}{x^{2n+1}} + 8n\int \frac{K_0^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{J_1^2(x)}{x^{2n+2}}\,dx = -\,\frac{1}{(2n+3)(2n+1)(2n-1)}\cdot$$

$$\cdot\left[\frac{2x^2\,J_0^2(x) + (4n-2)xJ_0(x)J_1(x) + (4n^2 - 1 + 2x^2)\,J_1^2(x)}{x^{2n+1}} + 8n\int \frac{J_1^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{I_1^2(x)}{x^{2n+2}}\,dx = -\,\frac{1}{(2n+3)(2n+1)(2n-1)}\cdot$$

$$\cdot\left[\frac{2x^2\,I_0^2(x) + (4n-2)xI_0(x)I_1(x) + (4n^2 - 1 - 2x^2)\,I_1^2(x)}{x^{2n+1}} - 8n\int \frac{I_1^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{K_1^2(x)}{x^{2n+2}}\,dx = -\,\frac{1}{(2n+3)(2n+1)(2n-1)}\cdot$$

$$\cdot\left[\frac{2x^2\,K_0^2(x) - (4n-2)xK_0(x)K_1(x) + (4n^2 - 1 - 2x^2)\,K_1^2(x)}{x^{2n+1}} - 8n\int \frac{K_1^2(x)}{x^{2n}}\,dx\right]$$

2.1.3. Integrals of the type $\int x^{2n} Z_\nu^2(x)\, dx$

a) The functions $\Theta(x)$ and $\Omega(x)$:

From HANKEL's asymptotic expansion of $J_\nu(x)$ and $Y_\nu(x)$ (see [1], 9.2, or [5], XIII. A. 4) and such of $\mathbf{H}_\nu(x)$ follows, that no finite representations of the integrals $\int Z_\nu^2(x)\, dx$ by functions of the type

$$A(x)\, J_0^2(x) + B(x)\, J_0(x)\, J_1(x) + C(x)\, J_1^2(x) + [D(x)\, J_0(x) + E(x)\, J_1(x)]\, \Phi(x) + F(x)\, \Phi^2(x)$$

with

$$A(x) \quad = \quad \sum_{i=-m}^{n} a_i\, x^i\ ,\ \dots\ ,$$

can be expected. Indeed, one has

$$\lim_{x \to +\infty} \frac{1}{\ln x} \int_0^x J_\nu^2(t)\, dt \quad = \quad \frac{1}{\pi}$$

and this contradicts the upper statement.
At least should be given some other representations or approximations.

$$\Theta(x) = \int_0^x J_0^2(t)\, dt = 2 \sum_{k=0}^{\infty} \frac{(-1)^k \cdot (2k)!}{(2k+1) \cdot (k!)^4} \cdot \left(\frac{x}{2}\right)^{2k+1}\ ,$$

$$\Omega(x) = \int_0^x I_0^2(t)\, dt = 2 \sum_{k=0}^{\infty} \frac{(2k)!}{(2k+1) \cdot (k!)^4} \cdot \left(\frac{x}{2}\right)^{2k+1}\ .$$
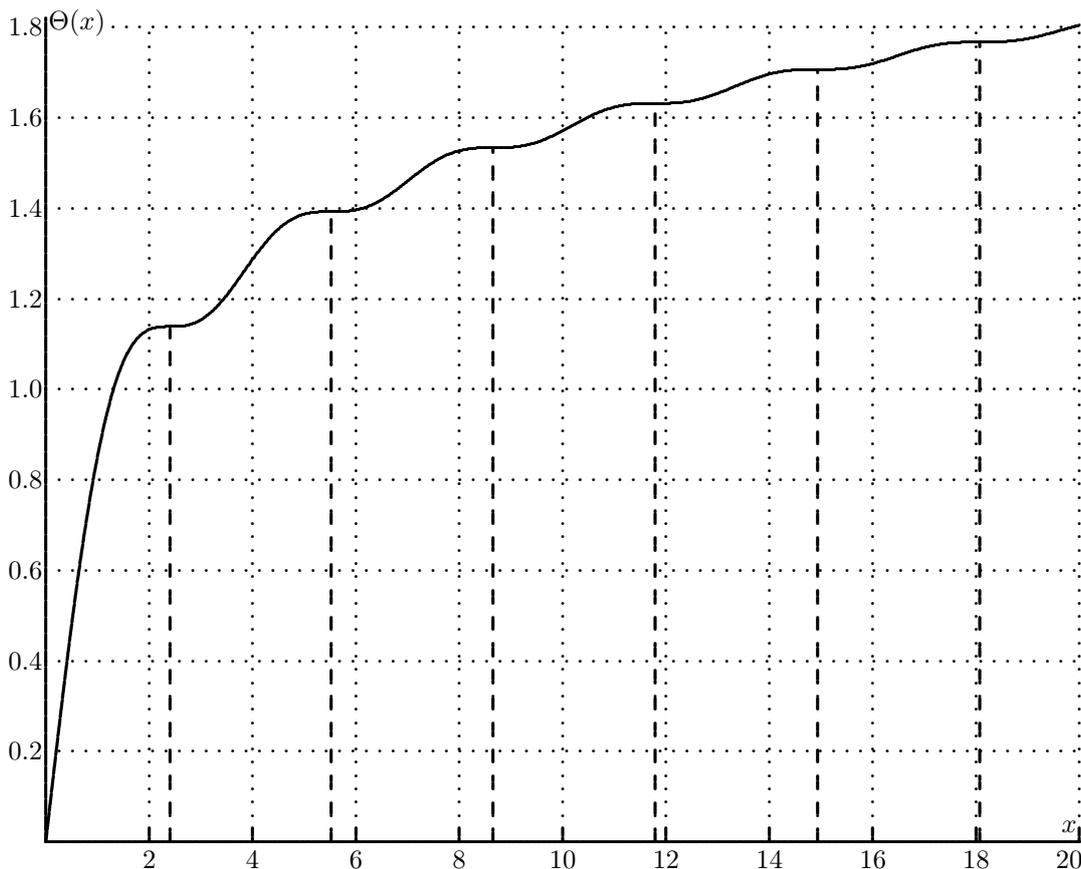
FIGURE 6 :   *Function* $\Theta(x)$

The dashed lines are located in the zeros of $J_0(x)$.

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If $\Theta(x)$ is computed by its series expansion with floating point numbers with n decimal digits, then the rounding error is (roughly spoken) about $10^{-n} \cdot \Omega(x)$. The computation of $\Omega(x)$ does not cause problems.

x	$\Theta(x)$	$\Omega(x)$	x	$\Theta(x)$	$\Omega(x)$
1	0.850 894 480	1.186 711 080	11	1.623 448 675	27 934 437.937
2	1.132 017 958	4.122 544 686	12	1.631 897 146	187 937 123.616
3	1.153 502 059	16.143 998 37	13	1.653 795 366	1 274 682 776.62
4	1.286 956 020	77.509 947 74	14	1.696 509 451	8 704 524 383.83
5	1.386 983 380	425.031 292 0	15	1.706 616 878	59 786 647 515.3
6	1.396 339 284	2 509.864 255	16	1.719 735 792	412 698 941 831.
7	1.460 064 224	15 483.965 76	17	1.755 251 443	2 861 234 688 170
8	1.527 171 173	98 307.748 55	18	1.767 226 854	19 912 983 676 244
9	1.534 810 723	637 083.688 6	19	1.774 861 457	139 056 981 172 080
10	1.571 266 461	4 193 041.1057	20	1.804 335 251	974 012 122 207 867

Differential equations:

$$2x\Theta''' \cdot \Theta' - 2\Theta'' \cdot \Theta' - x\Theta''^2 + 4x\Theta'^2 \;=\; 0$$

$$2x\Omega''' \cdot \Omega' - 2\Omega'' \cdot \Omega' - x\Omega''^2 - 4x\Omega'^2 \;=\; 0$$

Approximation by Chebyshev polynomials:

From [2], table 9.1., follows, that in the case $-8 \le x \le 8$ holds

$$\Theta(x) \quad \approx \quad \sum_{k=0}^{23} c_k \, T_{2k+1}\left(\frac{x}{8}\right) \; . \qquad (*)$$

k	c_k	k	c_k
0	1.80296 30053 02073 59034	12	0.00000 59437 33032 29013
1	-0.41322 52443 66465 65056	13	-0.00000 05921 36076 87261
2	0.21926 25129 41565 79685	14	0.00000 00501 50462 51669
3	-0.12660 62713 07010 86382	15	-0.00000 00036 59712 13919
4	0.08920 60707 21441 83736	16	0.00000 00002 32718 06621
5	-0.08107 02495 61597 55273	17	-0.00000 00000 13019 50604
6	0.05544 00433 79678 61623	18	0.00000 00000 00646 15991
7	-0.02523 73073 64048 13366	19	-0.00000 00000 00028 65553
8	0.00802 84592 74213 97781	20	0.00000 00000 00001 14279
9	-0.00188 87924 36267 70784	21	-0.00000 00000 00000 04122
10	0.00034 34505 49931 43439	22	0.00000 00000 00000 00135
11	-0.00004 99025 73931 36611	23	-0.00000 00000 00000 00004

This approximation differs from $\Theta(x)$ as shown in the following figure:
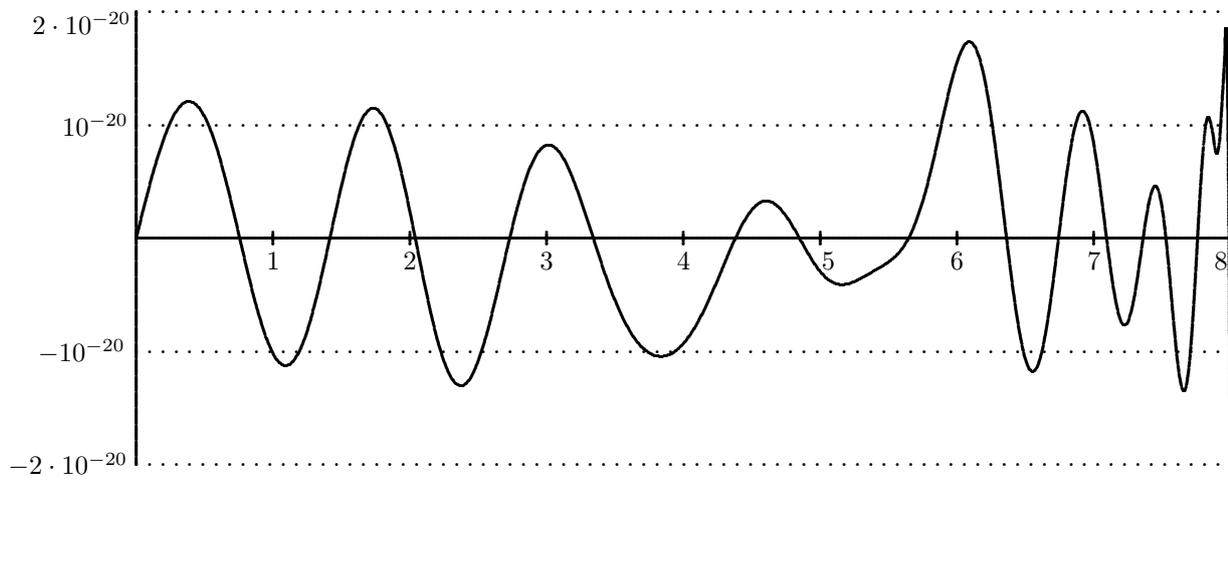
Asymptotic series of $\Theta(x)$ for $x \to +\infty$:

$$\Theta(x) \quad \sim \quad \frac{1}{\pi} \left[\ln 8x + \mathbf{C} + \mathcal{A}(x)\cos 2x + \mathcal{B}(x)\sin 2x + \mathcal{C}(x)\right]$$

with Euler's constant $\mathbf{C} = 0.577\,215\,664\,901\,533\,\ldots$ and

$$\mathcal{A}(x) = -\frac{1}{2x} + \frac{29}{64x^3} - \frac{6747}{4096x^5} + \frac{1796265}{131072x^7} - \frac{3447866835}{16777216x^9} + \frac{2611501938675}{536870912x^{11}} - \frac{5739627264576975}{34359738368x^{13}} +$$

$$+\frac{8634220069330080225}{1099511627776x^{15}} - \frac{13632639239279010838 3875}{281474976710656x^{17}} + \frac{34175257161344197762100 7375}{9007199254740992x^{19}} - \cdots ,$$

$$\mathcal{B}(x) = -\frac{3}{8x^2} + \frac{195}{256x^4} - \frac{71505}{16384x^6} + \frac{26103735}{524288x^8} - \frac{63761381145}{67108864x^{10}} + \frac{58671892003725}{2147483648x^{12}} - \frac{151798966421827725}{137438953472x^{14}} +$$

$$+\frac{262762002151603329375}{4398046511104x^{16}} - \frac{4692430263630584633783625}{1125899906842624x^{18}} + \frac{13126880101429581600348860625}{36028797018963968x^{20}} - \cdots ,$$

$$\mathcal{C}(x) = \frac{1}{16x^2} - \frac{27}{512x^4} + \frac{375}{2048x^6} - \frac{385875}{262144x^8} + \frac{11252115}{524288x^{10}} - \frac{8320313925}{16777216x^{12}} -$$

$$+\frac{1119167124075}{67108864x^{14}} - \frac{26440323306271875}{34359738368x^{16}} + \frac{1603719856835971875}{34359738368x^{18}} - \frac{3959969219293655192625}{1099511627776x^{20}} + \cdots .$$

The asymptotic series

$$\mathcal{A}(x) = \sum_{k=1}^{\infty} \frac{a_k}{x^{2k-1}} , \qquad \mathcal{B}(x) = \sum_{k=1}^{\infty} \frac{b_k}{x^{2k}} , \qquad \mathcal{C}(x) = \sum_{k=0}^{\infty} \frac{c_k}{x^{2k}}$$

begin with

k	a_k	b_k	c_k
1	-0.500 000 000 000 000	-0.375 000 000 000 000	0.062 500 000 000 000
2	0.453 125 000 000 000	0.761 718 750 000 000	-0.052 734 375 000 000
3	-1.647 216 796 875 000	-4.364 318 847 656 250	0.183 105 468 750 000
4	13.704 414 367 675 78	49.788 923 263 549 80	-1.471 996 307 373 047
5	-205.508 877 933 025 4	-950.118 618 384 003 6	21.461 706 161 499 02
6	4 864.301 418 280 229	27 321.228 759 235 24	-495.929 355 919 361 1
7	-167 045.138 793 094 5	-1 104 482.845 562 072	16 676.889 718 696 48
8	7 852 777.406 997 193	59 745 162.196 032 50	-769 514.686 726 961 4
9	-484 328 639.035 390 4	-4 167 715 296.104 454	46 674 390.813 451 37
10	37 942 157 373.010 10	364 344 113 252.523 3	-3 601 571 024.131 458

Let x_k denote the k-th positive zero of $J_0(x)$, then holds

$$\Theta(x_k) \sim \frac{1}{\pi}\left[\ln x_k + \frac{5}{2^4 \cdot x_k^2} - \frac{331}{2^9 \cdot x_k^4} + \frac{7\,987}{2^{11} \cdot x_k^6} - \frac{753\,375}{2^{14} \cdot x_k^8} + \frac{246\,293\,295}{2^{18} \cdot x_k^{10}} - \cdots\right] =$$

$$= \frac{1}{\pi}\left[\ln x_k + \frac{0.312\,500}{x_k^2} - \frac{0.646\,484}{x_k^4} + \frac{3.899\,902}{x_k^6} - \frac{45.982\,36}{x_k^8} + \frac{939.5344}{x_k^{10}} - \cdots\right] .$$

Simple approximation: $\Theta(x) \approx (\ln 8x + \mathbf{C})/\pi$:

$$\frac{\ln 8x + \mathbf{C}}{\pi} - \Theta(x)$$
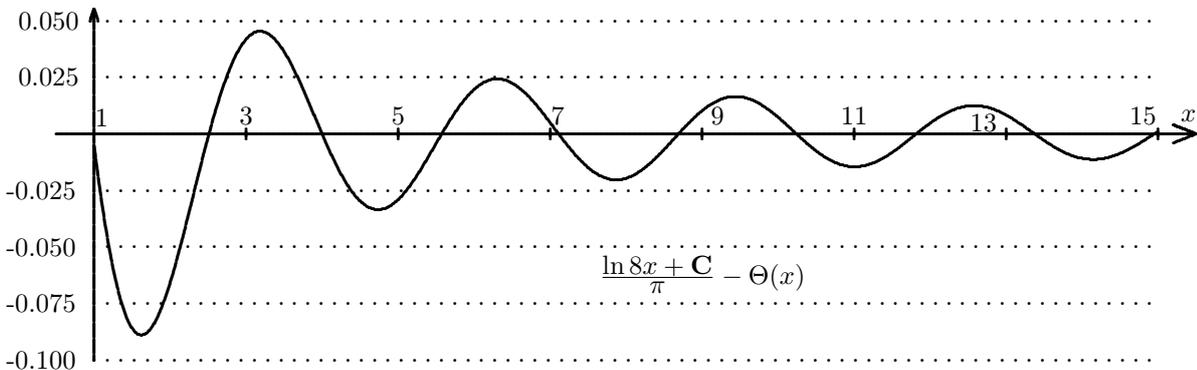
Figure 7

Let

$$\Delta_n(x) = \frac{1}{\pi}\left[\ln 8x + \mathbf{C} + \sum_{k=1}^{n} \frac{a_k x \cos 2x + b_k \sin 2x + c_k}{x^{2k}}\right] - \Theta(x)$$

with $\Delta_0(x) = (\ln 8x + \mathbf{C})/\pi$.

In the following table are given some first consecutive maxima and minima $\Delta_{n,k}^*$ of the differences $\Delta_n(x)$

$$\Delta_{n,k}^* = \Delta_n(x_{n,k}) .$$

Values	$x_{n,1},\ \Delta_{n,1}^*$	$x_{n,2},\ \Delta_{n,2}^*$	$x_{n,3},\ \Delta_{n,3}^*$	$x_{n,4},\ \Delta_{n,4}^*$	$x_{n,5},\ \Delta_{n,5}^*$	$x_{n,6},\ \Delta_{n,6}^*$	$x_{n,7},\ \Delta_{n,7}^*$
x	1.6216	3.1847	4.7356	6.3043	7.8688	9.4386	11.0064
$\Delta_0(x)$	-8.886E-02	4.536E-02	-3.347E-02	2.432E-02	-2.031E-02	1.651E-02	-1.454E-02
x	1.5839	3.1735	4.7253	6.2993	7.8633	9.4353	11.0027
$\Delta_1(x)$	2.104E-02	-3.483E-03	1.237E-03	-5.267E-04	2.863E-04	-1.638E-04	1.066E-04
x	1.5694	3.1664	4.7195	6.2969	7.8602	9.4338	11.0007
$\Delta_2(x)$	-1.894E-02	1.007E-03	-1.756E-04	4.430E-05	-1.583E-05	6.405E-06	-3.092E-06
x	1.5650	3.1612	4.7160	6.2952	7.8583	9.4330	10.9995
$\Delta_3(x)$	3.978E-02	-6.484E-04	5.541E-05	-8.322E-06	1.965E-06	-5.644E-07	2.029E-07
x	1.5642	3.1574	4.7138	6.2939	7.8569	9.4324	10.9986
$\Delta_4(x)$	-1.578E-01	7.486E-04	-3.105E-05	2.773E-06	-4.335E-07	8.855E-08	-2.376E-08
x	1.5644	3.1545	4.7124	6.2928	7.8559	9.4319	10.9979
$\Delta_5(x)$	1.041E+00	-1.376E-03	2.730E-05	-1.444E-06	1.494E-07	-2.172E-08	4.352E-09
x	1.5649	3.1523	4.7115	6.2918	7.8551	9.4314	10.9973
$\Delta_6(x)$	-1.045E+01	3.722E-03	-3.486E-05	1.085E-06	-7.420E-08	7.673E-09	-1.149E-09
x	1.5655	3.1507	4.7110	6.2909	7.8545	9.4310	10.9968
$\Delta_7(x)$	1.493E+02	-1.403E-02	6.123E-05	-1.115E-06	5.025E-08	-3.692E-09	4.132E-10
x	1.5660	3.1494	4.7107	6.2902	7.8540	9.4305	10.9964
$\Delta_8(x)$	-2.891E+03	7.055E-02	-1.421E-04	1.506E-06	-4.457E-08	2.324E-09	-1.942E-10
x	1.5664	3.1483	4.7105	6.2895	7.8537	9.4302	10.9961
$\Delta_9(x)$	7.303E+04	-4.582E-01	4.224E-04	-2.590E-06	5.022E-08	-1.854E-09	1.156E-10

If $x > 8$, then $|\Delta_n(x)|$ is restricted by $|\Delta_n(x)| \le |\Delta_{n,5}^*|$. More accurate:

n	0	1	2	3	4
$|\Delta_n(x)| <$	1.9625E-02	2.7573E-04	1.5193E-05	1.8792E-06	4.1271E-07

n	5	6	7	8	9
$|\Delta_n(x)| <$	1.4150E-07	6.9856E-08	4.6990E-08	4.1368E-08	4.6224E-08

Therefore the formula (*) from page 279 may be continued to $x > 8$ by an asymptotic formula ($n = 7, 8$ or 9) with a uniformly bounded absolute error less then 0.5E-07.

The following figure shows the values of $-\lg|\Delta_{n,k}^*|$ from the preceding table, connected by a polygonal line. It gives the intervals where some special asymptotic formula is preferable.
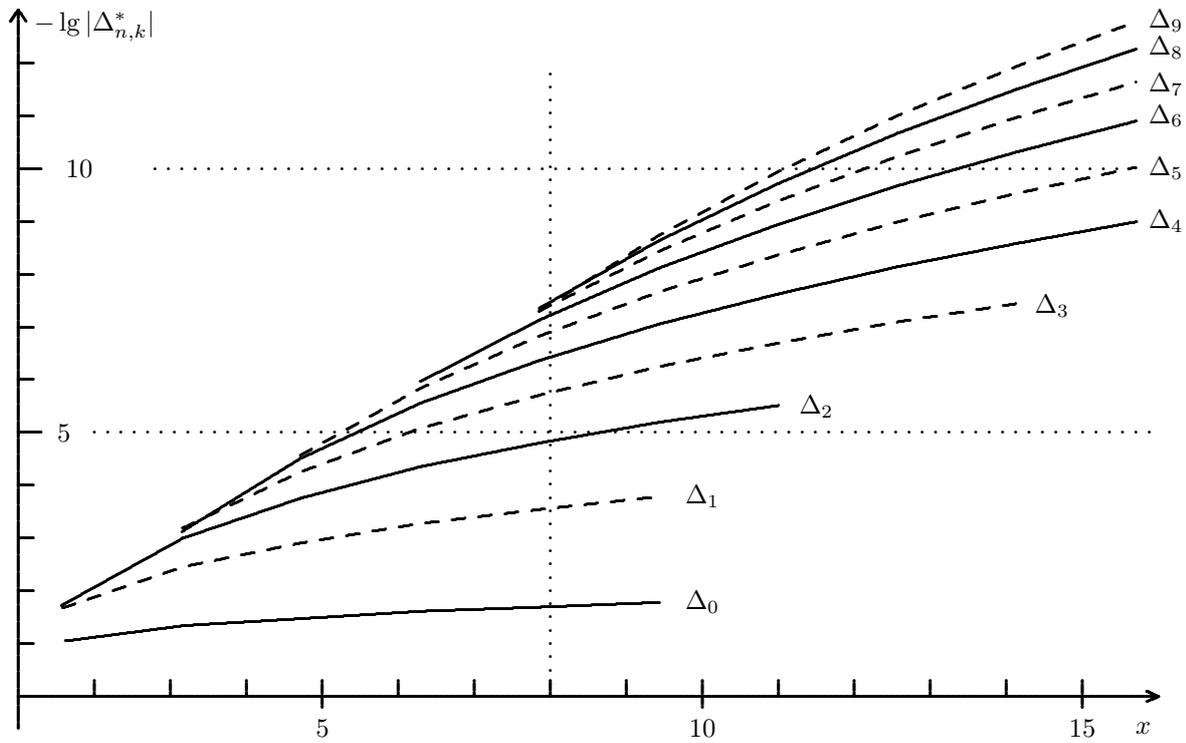
The differences $\Delta_n(x)$ are shown in the following figures:
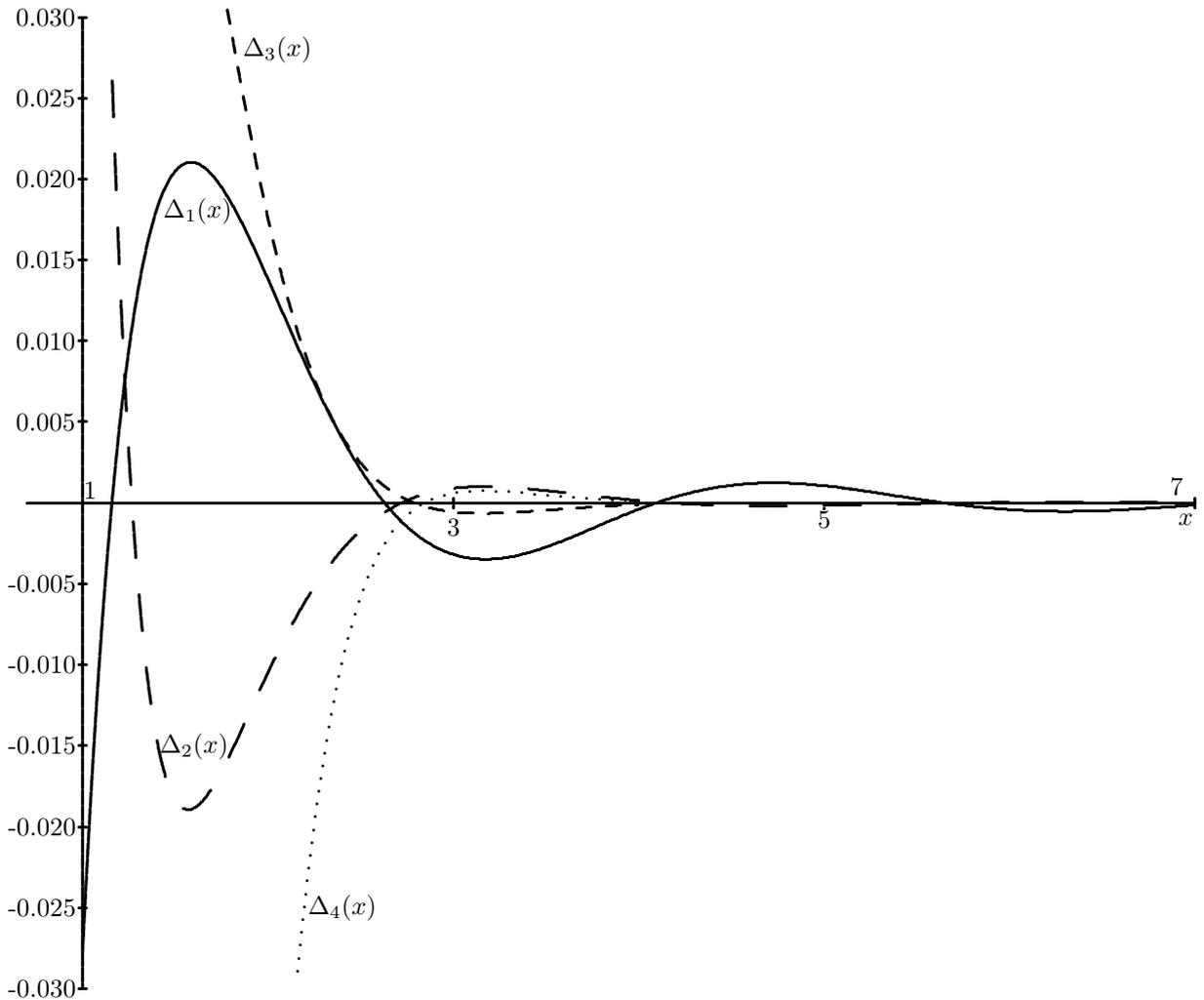
FIGURE 8 :   *Differences* $\Delta_{1...4}(x)$, $1 \le x \le 7$

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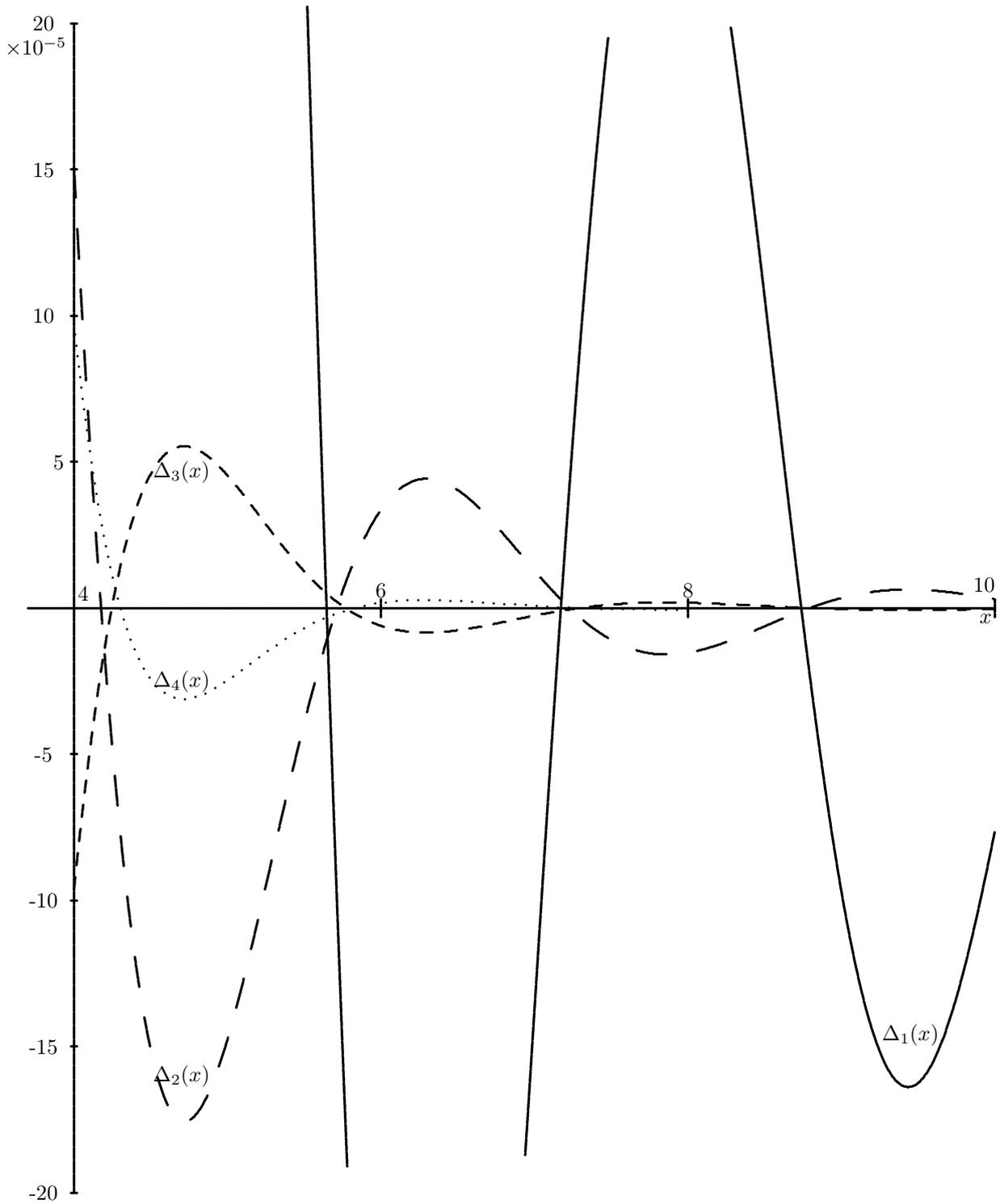
FIGURE 9 :   *Differences $\Delta_{1\ldots4}(x)$, $4 \le x \le 10$*

Asymptotic behaviour of $\Omega(x)$ for $x \to \infty$:

$$\Omega(x) \sim$$

$$\sim \frac{e^{2x}}{4\pi x}\left[1 + \frac{3}{4x} + \frac{29}{32x^2} + \frac{195}{128x^3} + \frac{6\,747}{2\,048x^4} + \frac{71\,505}{8\,192x^5} + \frac{1\,796\,265}{65\,536x^6} + \frac{26\,103\,735}{262\,144x^7} + \frac{430\,983\,354}{1\,048\,576x^8} + \ldots\right]$$

$$= \frac{e^{2x}}{4\pi x}\left[1 + \frac{0.75}{x} + \frac{0.90625}{x^2} + \frac{1.5234}{x^3} + \frac{3.2944}{x^4} + \frac{8.7286}{x^5} + \frac{27.409}{x^6} + \frac{99.578}{x^7} + \frac{411.02}{x^8} + \ldots\right]$$

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b) Integrals:

Holds

$$\int J_0^2(x)\,dx = x[J_0^2(x) + J_1^2(x)] + \int J_1^2(x)\,dx \ ,$$

$$\int I_0^2(x)\,dx = x[I_0^2(x) - I_1^2(x)] - \int I_1^2(x)\,dx \ .$$

These formulas express every integral by the other one. Therefore the next integrals are given with $\int J_0^2(x)\,dx$ or $\int I_0^2(x)\,dx$. The last integrals are represented by the functions $\Theta(x)$ respectively $\Omega(x)$, see page 278 .

$$\int x^2\,J_0^2(x)\,dx = \frac{1}{8}\left[(2\,x^3 + x)\,J_0^2(x) + 2x^2\,J_0(x)J_1(x) + 2x^3\,J_1^2(x)\right] - \frac{1}{8}\int J_0^2(x)\,dx$$

$$\int x^2\,I_0^2(x)\,dx = \frac{1}{8}\left[(2\,x^3 - x)\,I_0^2(x) + 2x^2\,I_0(x)I_1(x) - 2x^3\,I_1^2(x)\right] + \frac{1}{8}\int I_0^2(x)\,dx$$

$$\int x^2\,J_1^2(x)\,dx = \frac{1}{8}\left[(2\,x^3 - 3x)\,J_0^2(x) - 6x^2\,J_0(x)J_1(x) + 2x^3\,J_1^2(x)\right] + \frac{3}{8}\int J_0^2(x)\,dx$$

$$\int x^2\,I_1^2(x)\,dx = \frac{1}{8}\left[(-2\,x^3 - 3x)\,I_0^2(x) + 6x^2\,I_0(x)I_1(x) + 2x^3\,I_1^2(x)\right] + \frac{3}{8}\int I_0^2(x)\,dx$$

$$\int x^4\,J_0^2(x)\,dx =$$

$$= \frac{1}{128}\left[(16\,x^5 + 18\,x^3 - 27\,x)\,J_0^2(x) + (48\,x^4 - 54\,x^2)J_0(x)J_1(x) + (16\,x^5 - 54\,x^3)\,J_1^2(x)\right] + \frac{27}{128}\int J_0^2(x)\,dx$$

$$\int x^4\,I_0^2(x)\,dx =$$

$$= \frac{1}{128}\left[(16\,x^5 - 18\,x^3 - 27\,x)\,I_0^2(x) + (48\,x^4 + 54\,x^2)I_0(x)I_1(x) - (16\,x^5 + 54\,x^3)\,I_1^2(x)\right] + \frac{27}{128}\int I_0^2(x)\,dx$$

$$\int x^4\,J_1^2(x)\,dx =$$

$$= \frac{1}{128}\left[(16\,x^5 - 30\,x^3 + 45\,x)\,J_0^2(x) + (-80\,x^4 + 90\,x^2)J_0(x)J_1(x) + (16\,x^5 + 90\,x^3)\,J_1^2(x)\right] -$$

$$- \frac{45}{128}\int J_0^2(x)\,dx$$

$$\int x^4\,I_1^2(x)\,dx =$$

$$= \frac{1}{128}\left[(-16\,x^5 - 30\,x^3 - 45\,x)\,I_0^2(x) + (80\,x^4 + 90\,x^2)I_0(x)I_1(x) + (16\,x^5 - 90\,x^3)\,I_1^2(x)\right] +$$

$$+ \frac{45}{128}\int I_0^2(x)\,dx$$

With

$$\int x^{2n}\,J_0^2(x)\,dx = \frac{1}{\beta_n}\left[A_n(x)\,J_0^2(x) + B_n(x)J_0(x)J_1(x) + C_n(x)\,J_1^2(x) + \gamma_n\int J_0^2(x)\,dx\right]$$

$$\int x^{2n}\,I_0^2(x)\,dx = \frac{1}{\beta_n^*}\left[A_n^*(x)\,I_0^2(x) + B_n^*(x)I_0(x)I_1(x) + C_n^*(x)\,I_1^2(x) + \gamma_n^*\int I_0^2(x)\,dx\right]$$

$$\int x^{2n}\,J_1^2(x)\,dx = \frac{1}{\xi_n}\left[P_n(x)\,J_0^2(x) + Q_n(x)J_0(x)J_1(x) + R_n(x)\,J_1^2(x) + \varrho_n\int J_0^2(x)\,dx\right]$$

$$\int x^{2n}\,I_1^2(x)\,dx = \frac{1}{\xi_n^*}\left[P_n^*(x)\,I_0^2(x) + Q_n^*(x)I_0(x)I_1(x) + R_n^*(x)\,I_1^2(x) + \varrho_n^*\int I_0^2(x)\,dx\right]$$

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holds

$$\beta_3 = 3072 \ , \quad \gamma_3 = -3375$$

$$A_3(x) = 256\,x^7 + 1200\,x^5 - 2250\,x^3 + 3375\,x$$

$$B_3(x) = 1280\,x^6 - 6000\,x^4 + 6750\,x^2 \ , \quad C_3(x) = 256\,x^7 - 2000\,x^5 + 6750\,x^3$$

$$\beta_3^* = 3072 \ , \quad \gamma_3^* = 3375$$

$$A_3^*(x) = 256\,x^7 - 1200\,x^5 - 2250\,x^3 - 3375\,x$$

$$B_3^*(x) = 1280\,x^6 + 6000\,x^4 + 6750\,x^2 \ , \quad C_3^*(x) = -256\,x^7 - 2000\,x^5 - 6750\,x^3$$

$$\xi_3 = 3072 \ , \quad \varrho_3 = 4725$$

$$P_3(x) = 256\,x^7 - 1680\,x^5 + 3150\,x^3 - 4725\,x$$

$$Q_3(x) = -1792\,x^6 + 8400\,x^4 - 9450\,x^2 \ , \quad R_3(x) = 256\,x^7 + 2800\,x^5 - 9450\,x^3$$

$$\xi_3^* = 3072 \ , \quad \varrho_3^* = 4725$$

$$P_3^*(x) = -256\,x^7 - 1680\,x^5 - 3150\,x^3 - 4725\,x$$

$$Q_3^*(x) = 1792\,x^6 + 8400\,x^4 + 9450\,x^2 \ , \quad R_3^*(x) = 256\,x^7 - 2800\,x^5 - 9450\,x^3$$

$$\beta_4 = 98304 \ , \quad \gamma_4 = 1157625$$

$$A_4(x) = 6144\,x^9 + 62720\,x^7 - 411600\,x^5 + 771750\,x^3 - 1157625\,x$$

$$B_4(x) = 43008\,x^8 - 439040\,x^6 + 2058000\,x^4 - 2315250\,x^2$$

$$C_4(x) = 6144\,x^9 - 87808\,x^7 + 686000\,x^5 - 2315250\,x^3$$

$$\beta_4^* = 98304 \ , \quad \gamma_4^* = 1157625$$

$$A_4^*(x) = 6144\,x^9 - 62720\,x^7 - 411600\,x^5 - 771750\,x^3 - 1157625\,x$$

$$B_4^*(x) = 43008\,x^8 + 439040\,x^6 + 2058000\,x^4 + 2315250\,x^2$$

$$C_4^*(x) = -6144\,x^9 - 87808\,x^7 - 686000\,x^5 - 2315250\,x^3$$

$$\xi_4 = 32768 \ , \quad \varrho_4 = -496125$$

$$P_4(x) = 2048\,x^9 - 26880\,x^7 + 176400\,x^5 - 330750\,x^3 + 496125\,x$$

$$Q_4(x) = -18432\,x^8 + 188160\,x^6 - 882000\,x^4 + 992250\,x^2$$

$$R_4(x) = 2048\,x^9 + 37632\,x^7 - 294000\,x^5 + 992250\,x^3$$

$$\xi_4^* = 32768 \ , \quad \varrho_4^* = 496125$$

$$P_4^*(x) = -2048\,x^9 - 26880\,x^7 - 176400\,x^5 - 330750\,x^3 - 496125\,x$$

$$Q_4^*(x) = 18432\,x^8 + 188160\,x^6 + 882000\,x^4 + 992250\,x^2$$

$$R_4^*(x) = 2048\,x^9 - 37632\,x^7 - 294000\,x^5 - 992250\,x^3$$

$$\beta_5 = 1310720 \ , \quad \gamma_5 = -281302875$$

$$A_5(x) = 65536\,x^{11} + 1161216\,x^9 - 15240960\,x^7 + 100018800\,x^5 - 187535250\,x^3 + 281302875\,x$$

$$B_5(x) = 589824\,x^{10} - 10450944\,x^8 + 106686720\,x^6 - 500094000\,x^4 + 562605750\,x^2$$

$$C_5(x) = 65536\,x^{11} - 1492992\,x^9 + 21337344\,x^7 - 166698000\,x^5 + 562605750\,x^3$$

$$\beta_5^* = 1310720 \ , \quad \gamma_5^* = 281302875^*$$

$$A_5(x)^* = 65536\,x^{11} - 1161216\,x^9 - 15240960\,x^7 - 100018800\,x^5 - 187535250\,x^3 - 281302875\,x$$

$$B_5(x)^* = 589824\,x^{10} + 10450944\,x^8 + 106686720\,x^6 + 500094000\,x^4 + 562605750\,x^2$$

$$C_5(x)^* = -65536\,x^{11} - 1492992\,x^9 - 21337344\,x^7 - 166698000\,x^5 - 562605750\,x^3$$

$$\xi_5 = 1310720 \,, \quad \varrho_5 = 343814625$$

$$P_5(x) = 65536\,x^{11} - 1419264\,x^9 + 18627840\,x^7 - 122245200\,x^5 + 229209750\,x^3 - 343814625\,x$$

$$Q_5(x) = -720896\,x^{10} + 12773376\,x^8 - 130394880\,x^6 + 611226000\,x^4 - 687629250\,x^2$$

$$R_5(x) = 65536\,x^{11} + 1824768\,x^9 - 26078976\,x^7 + 203742000\,x^5 - 687629250\,x^3$$

$$\xi_5^* = 1310720 \,, \quad \varrho_5^* = 343814625$$

$$P_5^*(x) = -65536\,x^{11} - 1419264\,x^9 - 18627840\,x^7 - 122245200\,x^5 - 229209750\,x^3 - 343814625\,x$$

$$Q_5^*(x) = 720896\,x^{10} + 12773376\,x^8 + 130394880\,x^6 + 611226000\,x^4 + 687629250\,x^2$$

$$R_5^*(x) = 65536\,x^{11} - 1824768\,x^9 - 26078976\,x^7 - 203742000\,x^5 - 687629250\,x^3$$

Recurrence relations:

$$\int x^{2n+2}\,J_0^2(x)\,dx =$$

$$= \frac{x^{2n+1}}{8(n+1)}\left\{[x^2 + (2n+1)^2]\,J_0^2(x) + 2x^2\,J_1^2(x) + 2(2n+1)x\,J_0(x)\,J_1(x)\right\} - \frac{(2n+1)^3}{8(n+1)}\int x^{2n}\,J_0^2(x)\,dx$$

$$\int x^{2n+2}\,I_0^2(x)\,dx =$$

$$= \frac{x^{2n+1}}{8(n+1)}\left\{[x^2 - (2n+1)^2]\,I_0^2(x) - 2x^2\,I_1^2(x) + 2(2n+1)x\,I_0(x)\,I_1(x)\right\} + \frac{(2n+1)^3}{8(n+1)}\int x^{2n}\,I_0^2(x)\,dx$$

$$\int x^{2n+2}\,J_1^2(x)\,dx = \frac{x^{2n+1}}{8(n+1)}\left\{[x^2 + (2n+1)(2n+3)]\,J_1^2(x) + 2x^2\,J_0^2(x) - 2(2n+3)x\,J_0(x)\,J_1(x)\right\} -$$

$$- \frac{(2n-1)(2n+1)(2n+3)}{8(n+1)}\int x^{2n}\,J_0^2(x)\,dx$$

$$\int x^{2n+2}\,I_1^2(x)\,dx = \frac{x^{2n+1}}{8(n+1)}\left\{[x^2 - (2n+1)(2n+3)]\,I_1^2(x) - 2x^2\,I_0^2(x) + 2(2n+3)x\,I_0(x)\,I_1(x)\right\} +$$

$$+ \frac{(2n-1)(2n+1)(2n+3)}{8(n+1)}\int x^{2n}\,I_0^2(x)\,dx$$

2.1.4. Integrals of the type $\int x^{2n} Z_0(x) Z_1(x)\, dx$

In the following formulas both $J_0(x)$ and $J_1(x)$ together may be substituted by $Y_0(x)$ and $Y_1(x)$ respectively or $H_0^{(p)}(x)$, $H_1^{(p)}(x)$, $p = 1, 2$.

$$\int J_0(x) J_1(x)\, dx = -\frac{1}{2} J_0^2(x)$$

$$\int I_0(x) I_1(x)\, dx = \frac{1}{2} I_0^2(x)$$

$$\int K_0(x) K_1(x)\, dx = -\frac{1}{2} K_0^2(x)$$

$$\int x^2 \cdot J_0(x) J_1(x)\, dx = \frac{x^2}{2} J_1^2(x)$$

$$\int x^2 \cdot I_0(x) I_1(x)\, dx = \frac{x^2}{2} I_1^2(x)$$

$$\int x^2 \cdot K_0(x) I_1(x)\, dx = -\frac{x^2}{2} K_1^2(x)$$

$$\int x^4 \cdot J_0(x) J_1(x)\, dx = -\frac{x^4}{6} J_0^2(x) + \frac{2x^3}{3} J_0(x) J_1(x) + \left(\frac{x^4}{3} - \frac{2x^2}{3} \right) J_1^2(x)$$

$$\int x^4 \cdot I_0(x) I_1(x)\, dx = \frac{x^4}{6} I_0^2(x) - \frac{2x^3}{3} I_0(x) I_1(x) + \left(\frac{x^4}{3} + \frac{2x^2}{3} \right) I_1^2(x)$$

$$\int x^4 \cdot K_0(x) K_1(x)\, dx = -\frac{x^4}{6} K_0^2(x) - \frac{2x^3}{3} K_0(x) K_1(x) - \left(\frac{x^4}{3} + \frac{2x^2}{3} \right) K_1^2(x)$$

$$\int x^6 \cdot J_0(x) J_1(x)\, dx = \left(-\frac{x^6}{5} + \frac{4x^4}{5} \right) J_0^2(x) + \left(\frac{6x^5}{5} - \frac{16x^3}{5} \right) J_0(x) J_1(x) + \left(\frac{3x^6}{10} - \frac{8x^4}{5} + \frac{16x^2}{5} \right) J_1^2(x)$$

$$\int x^6 \cdot I_0(x) I_1(x)\, dx = \left(\frac{x^6}{5} + \frac{4x^4}{5} \right) I_0^2(x) - \left(\frac{6x^5}{5} + \frac{16x^3}{5} \right) I_0(x) I_1(x) + \left(\frac{3x^6}{10} + \frac{8x^4}{5} + \frac{16x^2}{5} \right) I_1^2(x)$$

$$\int x^6 \cdot K_0(x) K_1(x)\, dx =$$

$$= - \left(\frac{x^6}{5} + \frac{4x^4}{5} \right) K_0^2(x) - \left(\frac{6x^5}{5} + \frac{16x^3}{5} \right) K_0(x) K_1(x) - \left(\frac{3x^6}{10} + \frac{8x^4}{5} + \frac{16x^2}{5} \right) K_1^2(x)$$

$$\int x^8 \cdot J_0(x) J_1(x)\, dx = \left(-\frac{3x^8}{14} + \frac{72x^6}{35} - \frac{288x^4}{35} \right) J_0^2(x) + \left(\frac{12x^7}{7} - \frac{432x^5}{35} + \frac{1152x^3}{35} \right) J_0(x) J_1(x) +$$

$$+ \left(\frac{2x^8}{7} - \frac{108x^6}{35} + \frac{576x^4}{35} - \frac{1152x^2}{35} \right) J_1^2(x)$$

$$\int x^8 \cdot I_0(x) I_1(x)\, dx = \left(\frac{3x^8}{14} + \frac{72x^6}{35} + \frac{288x^4}{35} \right) I_0^2(x) - \left(\frac{12x^7}{7} + \frac{432x^5}{35} + \frac{1152x^3}{35} \right) I_0(x) I_1(x) +$$

$$+ \left(\frac{2x^8}{7} + \frac{108x^6}{35} + \frac{576x^4}{35} + \frac{1152x^2}{35} \right) I_1^2(x)$$

$$\int x^8 \cdot K_0(x) K_1(x)\, dx = - \left(\frac{3x^8}{14} + \frac{72x^6}{35} + \frac{288x^4}{35} \right) K_0^2(x) - \left(\frac{12x^7}{7} + \frac{432x^5}{35} + \frac{1152x^3}{35} \right) K_0(x) K_1(x) -$$

$$- \left(\frac{2x^8}{7} + \frac{108x^6}{35} + \frac{576x^4}{35} + \frac{1152x^2}{35} \right) K_1^2(x)$$

$$\int x^{10} \cdot J_0(x) J_1(x)\, dx = \left(-\frac{2x^{10}}{9} + \frac{80x^8}{21} - \frac{256x^6}{7} + \frac{1024x^4}{7} \right) J_0^2(x) +$$

$$+ \left(\frac{20x^9}{9} - \frac{640x^7}{21} + \frac{1536x^5}{7} - \frac{4096x^3}{7} \right) J_0(x) J_1(x) +$$

$$+\left(\frac{5x^{10}}{18}-\frac{320x^8}{63}+\frac{384x^6}{7}-\frac{2048x^4}{7}+\frac{4096x^2}{7}\right)J_1^2(x)$$

$$\int x^{10}\cdot I_0(x)I_1(x)\,dx=\left(\frac{2x^{10}}{9}+\frac{80x^8}{21}+\frac{256x^6}{7}+\frac{1024x^4}{7}\right)I_0^2(x)-$$

$$-\left(\frac{20x^9}{9}+\frac{640x^7}{21}+\frac{1536x^5}{7}+\frac{4096x^3}{7}\right)I_0(x)I_1(x)+$$

$$+\left(\frac{5x^{10}}{18}+\frac{320x^8}{63}+\frac{384x^6}{7}+\frac{2048x^4}{7}+\frac{4096x^2}{7}\right)I_1^2(x)$$

$$\int x^{10}\cdot K_0(x)K_1(x)\,dx=-\left(\frac{2x^{10}}{9}+\frac{80x^8}{21}+\frac{256x^6}{7}+\frac{1024x^4}{7}\right)K_0^2(x)-$$

$$-\left(\frac{20x^9}{9}+\frac{640x^7}{21}+\frac{1536x^5}{7}+\frac{4096x^3}{7}\right)K_0(x)K_1(x)-$$

$$-\left(\frac{5x^{10}}{18}+\frac{320x^8}{63}+\frac{384x^6}{7}+\frac{2048x^4}{7}+\frac{4096x^2}{7}\right)K_1^2(x)$$

Let

$$\int x^m J_0(x)J_1(x)\,dx=P_m(x)J_0^2(x)+Q_m(x)J_0(x)J_1(x)+R_m(x)J_1^2(x)\,,$$

$$\int x^m I_0(x)I_1(x)\,dx=P_m^*(x)I_0^2(x)+Q_m^*(x)I_0(x)I_1(x)+R_m^*(x)I_1^2(x)\,,$$

$$\int x^m K_0(x)K_1(x)\,dx=-P_m^*(x)K_0^2(x)+Q_m^*(x)K_0(x)K_1(x)-R_m^*(x)K_1^2(x)\,,$$

then holds

$$P_{12}=-\frac{5}{22}\,x^{12}+\frac{200}{33}\,x^{10}-\frac{8000}{77}\,x^8+\frac{76800}{77}\,x^6-\frac{307200}{77}\,x^4$$

$$Q_{12}=\frac{30}{11}\,x^{11}-\frac{2000}{33}\,x^9+\frac{64000}{77}\,x^7-\frac{460800}{77}\,x^5+\frac{1228800}{77}\,x^3$$

$$R_{12}=\frac{3}{11}\,x^{12}-\frac{250}{33}\,x^{10}+\frac{32000}{231}\,x^8-\frac{115200}{77}\,x^6+\frac{614400}{77}\,x^4-\frac{1228800}{77}\,x^2$$

$$P_{12}^*=\frac{5}{22}\,x^{12}+\frac{200}{33}\,x^{10}+\frac{8000}{77}\,x^8+\frac{76800}{77}\,x^6+\frac{307200}{77}\,x^4$$

$$Q_{12}^*=-\frac{30}{11}\,x^{11}-\frac{2000}{33}\,x^9-\frac{64000}{77}\,x^7-\frac{460800}{77}\,x^5-\frac{1228800}{77}\,x^3$$

$$R_{12}^*=\frac{3}{11}\,x^{12}+\frac{250}{33}\,x^{10}+\frac{32000}{231}\,x^8+\frac{115200}{77}\,x^6+\frac{614400}{77}\,x^4+\frac{1228800}{77}\,x^2$$

$$P_{14}=-\frac{3}{13}\,x^{14}+\frac{1260}{143}\,x^{12}-\frac{33600}{143}\,x^{10}+\frac{576000}{143}\,x^8-\frac{5529600}{143}\,x^6+\frac{22118400}{143}\,x^4$$

$$Q_{14}=\frac{42}{13}\,x^{13}-\frac{15120}{143}\,x^{11}+\frac{336000}{143}\,x^9-\frac{4608000}{143}\,x^7+\frac{33177600}{143}\,x^5-\frac{88473600}{143}\,x^3$$

$$R_{14}=\frac{7}{26}\,x^{14}-\frac{1512}{143}\,x^{12}+\frac{42000}{143}\,x^{10}-\frac{768000}{143}\,x^8+\frac{8294400}{143}\,x^6-\frac{44236800}{143}\,x^4+\frac{88473600}{143}\,x^2$$

$$P_{14}^*=\frac{3}{13}\,x^{14}+\frac{1260}{143}\,x^{12}+\frac{33600}{143}\,x^{10}+\frac{576000}{143}\,x^8+\frac{5529600}{143}\,x^6+\frac{22118400}{143}\,x^4$$

$$Q_{14}^*=-\frac{42}{13}\,x^{13}-\frac{15120}{143}\,x^{11}-\frac{336000}{143}\,x^9-\frac{4608000}{143}\,x^7-\frac{33177600}{143}\,x^5-\frac{88473600}{143}\,x^3$$

$$R_{14}^*=\frac{7}{26}\,x^{14}+\frac{1512}{143}\,x^{12}+\frac{42000}{143}\,x^{10}+\frac{768000}{143}\,x^8+\frac{8294400}{143}\,x^6+\frac{44236800}{143}\,x^4+\frac{88473600}{143}\,x^2$$

$$P_{16}=-\frac{7}{30}\,x^{16}+\frac{784}{65}\,x^{14}-\frac{65856}{143}\,x^{12}+\frac{1756160}{143}\,x^{10}-\frac{30105600}{143}\,x^8+\frac{289013760}{143}\,x^6-\frac{1156055040}{143}\,x^4$$

$$Q_{16} = \frac{56}{15}x^{15} - \frac{10976}{65}x^{13} + \frac{790272}{143}x^{11} - \frac{17561600}{143}x^9 + \frac{240844800}{143}x^7 - \frac{1734082560}{143}x^5 + \frac{4624220160}{143}x^3$$

$$R_{16} =$$

$$= \frac{4}{15}x^{16} - \frac{2744}{195}x^{14} + \frac{395136}{715}x^{12} - \frac{2195200}{143}x^{10} + \frac{40140800}{143}x^8 - \frac{433520640}{143}x^6 + \frac{2312110080}{143}x^4 - \frac{4624220160}{143}x^2$$

$$P_{16}^* = \frac{7}{30}x^{16} + \frac{784}{65}x^{14} + \frac{65856}{143}x^{12} + \frac{1756160}{143}x^{10} + \frac{30105600}{143}x^8 + \frac{289013760}{143}x^6 + \frac{1156055040}{143}x^4$$

$$Q_{16}^* = -\frac{56}{15}x^{15} - \frac{10976}{65}x^{13} - \frac{790272}{143}x^{11} - \frac{17561600}{143}x^9 - \frac{240844800}{143}x^7 - \frac{1734082560}{143}x^5 - \frac{4624220160}{143}x^3$$

$$R_{16}^* =$$

$$= \frac{4}{15}x^{16} + \frac{2744}{195}x^{14} + \frac{395136}{715}x^{12} + \frac{2195200}{143}x^{10} + \frac{40140800}{143}x^8 + \frac{433520640}{143}x^6 + \frac{2312110080}{143}x^4 + \frac{4624220160}{143}x^2$$

Recurrence Formulas:

$$\int x^{2n+2} J_0(x) J_1(x)\, dx =$$

$$= \frac{x^{2n+1}}{4n+2}\left[-nxJ_0^2(x) + 2n(n+1)J_0(x)J_1(x) + (n+1)xJ_1^2(x)\right] - \frac{2n^2(n+1)}{2n+1}\int x^{2n} J_0(x) J_1(x)\, dx$$

$$\int x^{2n+2} I_0(x) I_1(x)\, dx =$$

$$= \frac{x^{2n+1}}{4n+2}\left[nxI_0^2(x) - 2n(n+1)I_0(x)I_1(x) + (n+1)xI_1^2(x)\right] + \frac{2n^2(n+1)}{2n+1}\int x^{2n} I_0(x) I_1(x)\, dx$$

$$\int x^{2n+2} K_0(x) K_1(x)\, dx =$$

$$= -\frac{x^{2n+1}}{4n+2}\left[nxK_0^2(x) + 2n(n+1)K_0(x)K_1(x) + (n+1)xK_1^2(x)\right] + \frac{2n^2(n+1)}{2n+1}\int x^{2n} K_0(x) K_1(x)\, dx$$

2.1.5. Integrals of the type $\int x^{2n+1} Z_0(x) Z_1(x)\, dx$

The integrals $\int J_0^2(x)\, dx$ and $\int I_0^2(x)\, dx$ may be defined as the functions $\Theta(x)$ and $\Omega(x)$ in 2.1.3., page 279
.

$$\int x\, J_0(x) J_1(x)\, dx = -\frac{x}{2}\, J_0^2(x) + \frac{1}{2} \int J_0^2(x)\, dx$$

$$\int x\, I_0(x) I_1(x)\, dx = \frac{x}{2}\, I_0^2(x) - \frac{1}{2} \int I_0^2(x)\, dx$$

$$\int x^3\, J_0(x) J_1(x)\, dx = \frac{1}{16}\left[(-2\,x^3 + 3\,x)\, J_0^2(x) + 6\,x^2 J_0(x) J_1(x) + 6\,x^3\, J_1^2(x)\right] - \frac{3}{16} \int J_0^2(x)\, dx$$

$$\int x^3\, I_0(x) I_1(x)\, dx = \frac{1}{16}\left[(2\,x^3 + 3\,x)\, I_0^2(x) - 6x^2 I_0(x) I_1(x) + 6x^3\, I_1^2(x)\right] - \frac{3}{16} \int I_0^2(x)\, dx$$

With

$$\int x^{2n+1}\, J_0(x)\cdot J_1(x)\, dx = \frac{1}{\beta_n}\left[A_n(x)\, J_0^2(x) + B_n(x) J_0(x) J_1(x) + C_n(x)\, J_1^2(x) + \gamma_n \int J_0^2(x)\, dx\right]$$

$$\int x^{2n+1}\, I_0(x)\cdot I_1(x)\, dx = \frac{1}{\beta_n^*}\left[A_n^*(x)\, I_0^2(x) + I_n^*(x) I_0(x) I_1(x) + C_n^*(x)\, I_1^2(x) + \gamma_n^* \int I_0^2(x)\, dx\right]$$

holds

$$\beta_2 = 256\,, \qquad \gamma_2 = 135$$
$$A_2(x) = -48\,x^5 + 90\,x^3 - 135\,x\,, \quad B_2(x) = 240\,x^4 - 270\,x^2\,, \quad C_2(x) = 80\,x^5 - 270\,x^3$$

$$\beta_2^* = 256\,, \qquad \gamma_2^* = -135$$
$$A_2^*(x) = 48\,x^5 + 90\,x^3 + 135\,x\,, \quad B_2^*(x) = -240\,x^4 - 270\,x^2\,, \quad C_2^*(x) = 80\,x^5 + 270\,x^3$$

$$\beta_3 = 6144\,, \qquad \gamma_3 = -23625$$
$$A_3(x) = -1280\,x^7 + 8400\,x^5 - 15750\,x^3 + 23625\,x$$
$$B_3(x) = 8960\,x^6 - 42000\,x^4 + 47250\,x^2\,, \quad C_3(x) = 1792\,x^7 - 14000\,x^5 + 47250\,x^3$$

$$\beta_3^* = 6144\,, \qquad \gamma_3^* = -23625$$
$$A_3^*(x) = 1280\,x^7 + 8400\,x^5 + 15750\,x^3 + 23625\,x$$
$$B_3^*(x) = -8960\,x^6 - 42000\,x^4 - 47250\,x^2\,, \quad C_3^*(x) = 1792\,x^7 + 14000\,x^5 + 47250\,x^3$$

$$\beta_4 = 65536\,, \qquad \gamma_4 = 3472875$$
$$A_4(x) = -14336\,x^9 + 188160\,x^7 - 1234800\,x^5 + 2315250\,x^3 - 3472875\,x$$
$$B_4(x) = 129024\,x^8 - 1317120\,x^6 + 6174000\,x^4 - 6945750\,x^2$$
$$C_4(x) = 18432\,x^9 - 263424\,x^7 + 2058000\,x^5 - 6945750\,x^3$$

$$\beta_4^* = 65536\,, \qquad \gamma_4^* = -3472875$$
$$A_4^*(x) = 14336\,x^9 + 188160\,x^7 + 1234800\,x^5 + 2315250\,x^3 + 3472875\,x$$
$$B_4^*(x) = -129024\,x^8 - 1317120\,x^6 - 6174000\,x^4 - 6945750\,x^2$$
$$C_4^*(x) = 18432\,x^9 + 263424\,x^7 + 2058000\,x^5 + 6945750\,x^3$$

$$\beta_5 = 2621440\,, \qquad \gamma_5 = -3094331625$$
$$A_5(x) = -589824\,x^{11} + 12773376\,x^9 - 167650560\,x^7 + 1100206800\,x^5 - 2062887750\,x^3 + 3094331625\,x$$

$$B_5(x) = 6488064\,x^{10} - 114960384\,x^8 + 1173553920\,x^6 - 5501034000\,x^4 + 6188663250\,x^2$$
$$C_5(x) = 720896\,x^{11} - 16422912\,x^9 + 234710784\,x^7 - 1833678000\,x^5 + 6188663250\,x^3$$

$$\beta_5^* = 2621440\,, \qquad \gamma_5^* = -3094331625$$
$$A_5^*(x) = 589824\,x^{11} + 12773376\,x^9 + 167650560\,x^7 + 1100206800\,x^5 + 2062887750\,x^3 + 3094331625\,x$$
$$B_5^*(x) = -6488064\,x^{10} - 114960384\,x^8 - 1173553920\,x^6 - 5501034000\,x^4 - 6188663250\,x^2$$
$$C_5^*(x) = 720896\,x^{11} + 16422912\,x^9 + 234710784\,x^7 + 1833678000\,x^5 + 6188663250\,x^3$$

$$\beta_6 = 125829120\,, \qquad \gamma_6 = 4867383646125$$
$$A_6(x) = -28835840\,x^{13} + 927793152\,x^{11} - 20092520448\,x^9 + 263714330880\,x^7 - 1730625296400\,x^5 +$$
$$+3244922430750\,x^3 - 4867383646125\,x$$
$$B_6(x) = 374865920\,x^{12} - 10205724672\,x^{10} + 180832684032\,x^8 - 1846000316160\,x^6 + 8653126482000\,x^4 -$$
$$-9734767292250\,x^2$$
$$C_6(x) = 34078720\,x^{13} - 1133969408\,x^{11} + 25833240576\,x^9 - 369200063232\,x^7 + 2884375494000\,x^5 -$$
$$-9734767292250\,x^3$$

$$\beta_6^* = 125829120\,, \qquad \gamma_6^* = 4867383646125$$
$$A_6^*(x) = 28835840\,x^{13} + 927793152\,x^{11} + 20092520448\,x^9 + 263714330880\,x^7 + 1730625296400\,x^5 +$$
$$+3244922430750\,x^3 + 4867383646125\,x$$
$$B_6^*(x) = -374865920\,x^{12} - 10205724672\,x^{10} - 180832684032\,x^8 - 1846000316160\,x^6 - 8653126482000\,x^4 -$$
$$-9734767292250\,x^2$$
$$C_6^*(x) = 34078720\,x^{13} + 1133969408\,x^{11} + 25833240576\,x^9 + 369200063232\,x^7 + 2884375494000\,x^5 +$$
$$+9734767292250\,x^3$$

Recurrence formulas:

$$\int x^{2n+1}\, J_0(x)\, J_1(x)\, dx =$$
$$= -\frac{x^{2n}}{8n}\left[(2n-1)xJ_0^2(x) - (2n+1)xJ_1^2(x) - (4n^2-1)\,J_0(x)J_1(x)\right] - (2n-1)^2(2n+1)\int x^{2n-1}\,J_0(x)\,J_1(x)\,dx$$

$$\int x^{2n+1}\, I_0(x)\, I_1(x)\, dx =$$
$$= \frac{x^{2n}}{8n}\left[(2n-1)xI_0^2(x) + (2n+1)xI_1^2(x) - (4n^2-1)\,I_0(x)I_1(x)\right] + (2n-1)^2(2n+1)\int x^{2n-1}\,I_0(x)\,I_1(x)\,dx$$

2.1.6. Integrals of the type $\int x^{-(2n+1)} Z_0(x) Z_1(x)\, dx$

See also [4], 1.8.3..

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x} = x[J_0^2(x) + J_1^2(x)] - J_0(x)\, J_1(x)$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x} = x[I_0^2(x) - I_1^2(x)] - I_0(x)\, I_1(x)$$

$$\int \frac{K_0(x)\, K_1(x)\, dx}{x} = x[-K_0^2(x) + K_1^2(x)] - K_0(x)\, K_1(x)$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x} = -x[K_0(x)\, I_0(x) + K_1(x)\, I_1(x)] - I_0(x)\, K_1(x)$$

$$\int \frac{K_0(x)\, I_1(x)\, dx}{x} = x[K_0(x)\, I_0(x) + K_1(x)\, I_1(x)] - K_0(x)\, I_1(x)$$

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x^3} = \frac{1}{9x^2}\left[x(-8\,x^2 - 3)\, J_0^2(x) + (8\,x^2 - 3)\, J_0(x)\, J_1(x) + x(-8\,x^2 + 1)\, J_1^2(x)\right]$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x^3} = \frac{1}{9x^2}\left[x(8\,x^2 - 3)\, I_0^2(x) - (8\,x^2 + 3)\, I_0(x)\, J_1(x) - x(8\,x^2 + 1)\, I_1^2(x)\right]$$

$$\int \frac{K_0(x)\, K_1(x)\, dx}{x^3} = \frac{1}{9x^2}\left[x(-8\,x^2 + 3)\, K_0^2(x) + (-8\,x^2 - 3)\, K_0(x)\, K_1(x) + x(8\,x^2 + 1)\, K_1^2(x)\right]$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x^3} =$$

$$= \frac{1}{9x^2}\left[(-8\,x^3 + 3\,x)\, I_0(x)K_0(x) + (-4\,x^2 - 3)\, I_0(x)K_1(x) + 4\,x^2\, I_1(x)K_0(x) - (8\,x^3 + x)\, I_1(x)K_1(x)\right]$$

$$\int \frac{I_1(x)\, K_0(x)\, dx}{x^3} =$$

$$= \frac{1}{9x^2}\left[(8\,x^3 - 3\,x)\, I_0(x)K_0(x) + 4\,x^2\, I_0(x)K_1(x) - (4\,x^2 + 3)\, I_1(x)K_0(x) + (8\,x^3 + x)\, I_1(x)K_1(x)\right]$$

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x^5} = \frac{1}{675x^4}\left[x(128\,x^4 + 48\,x^2 - 45)\, J_0^2(x) + (-128\,x^4 + 48\,x^2 - 135)\, J_0(x)\, J_1(x) + \right.$$
$$\left. + x(128\,x^4 - 16\,x^2 + 27)\, J_1^2(x)\right]$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x^5} = \frac{1}{675x^4}\left[x(128\,x^4 - 48\,x^2 - 45)\, I_0^2(x) - (128\,x^4 + 48\,x^2 + 135)\, I_0(x)\, I_1(x) - \right.$$
$$\left. - x(128\,x^4 + 16\,x^2 + 27)\, I_1^2(x)\right]$$

$$\int \frac{K_0(x)\, K_1(x)\, dx}{x^5} = \frac{1}{675x^4}\left[x(-128\,x^4 + 48\,x^2 + 45)\, K_0^2(x) + (-128\,x^4 - 48\,x^2 - 135)\, K_0(x)\, K_1(x) + \right.$$
$$\left. + x(128\,x^4 + 16\,x^2 + 27)\, K_1^2(x)\right]$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x^5} = \frac{1}{675x^4}\left[(-128\,x^5 + 48\,x^3 + 45\,x)\, I_0(x)K_0(x) + (-64\,x^4 - 24\,x^2 - 135)\, I_0(x)K_1(x) + \right.$$
$$\left. + (64\,x^4 + 24\,x^2)\, I_1(x)K_0(x) + (-128\,x^5 - 16\,x^3 - 27\,x)\, I_1(x)K_1(x)\right]$$

$$\int \frac{I_1(x)\, K_0(x)\, dx}{x^5} = \frac{1}{675x^4}\left[(128\,x^5 - 48\,x^3 - 45\,x)\, I_0(x)K_0(x) + (64\,x^4 + 24\,x^2)\, I_0(x)K_1(x) + \right.$$
$$\left. + (-64\,x^4 - 24\,x^2 - 135)\, I_1(x)K_0(x) + (128\,x^5 + 16\,x^3 + 27\,x)\, I_1(x)K_1(x)\right]$$

The integrals for $K_0(x)K_1(x)$ may be found in a simple way from such for $I_0(x)I_1(x)$. The same holds for $I_1(x)K_0(x)$, which is similar to $I_0(x)K_1(x)$.

$$\int \frac{J_0(x)\,J_1(x)\,dx}{x^7} = \frac{1}{55\,125x^6}\left[x(-1\,024\,x^6 - 384\,x^4 + 360\,x^2 - 1\,575)\,J_0^2(x)+\right.$$

$$+(1\,024\,x^6 - 384\,x^4 + 1\,080\,x^2 - 7\,875)\,J_0(x)\,J_1(x) + x\left(-1\,024\,x^6 + 128\,x^4 - 216\,x^2 + 1\,125\right)J_1^2(x)\Big]$$

$$\int \frac{I_0(x)\,I_1(x)\,dx}{x^7} = \frac{1}{55\,125x^6}\left[x(1\,024\,x^6 - 384\,x^4 - 360\,x^2 - 1\,575)\,I_0^2(x)-\right.$$

$$-(1\,024\,x^6 + 384\,x^4 + 1\,080\,x^2 + 7\,875)\,I_0(x)\,I_1(x) - x\left(1\,024\,x^6 + 128\,x^4 + 216\,x^2 + 1\,125\right)I_1^2(x)\Big]$$

$$\int \frac{I_0(x)\,K_1(x)\,dx}{x^7} = \frac{1}{55125x^6}\left[(-1024\,x^7 + 384\,x^5 + 360\,x^3 + 1575\,x)\,I_0(x)K_0(x)-\right.$$

$$-(512\,x^6 + 192\,x^4 + 540\,x^2 + 7875)\,I_0(x)K_1(x) + (512\,x^6 + 192\,x^4 + 540\,x^2)\,I_1(x)K_0(x)-$$

$$-(1024\,x^7 + 128\,x^5 + 216\,x^3 + 1125\,x)\,I_1(x)K_1(x)\Big]$$

$$\int \frac{J_0(x)\,J_1(x)\,dx}{x^9} = \frac{1}{31\,255\,875x^8}\left[x\,(32\,768\,x^8 + 12\,288\,x^6 - 11\,520\,x^4 + 50\,400\,x^2 - 496\,125)\,J_0^2(x)+\right.$$

$$+(-32\,768\,x^8 + 12\,288\,x^6 - 34\,560\,x^4 + 252\,000\,x^2 - 3\,472\,875)\,J_0(x)\,J_1(x)+$$

$$+x\,(32\,768\,x^8 - 4\,096\,x^6 + 6\,912\,x^4 - 36\,000\,x^2 + 385\,875)\,J_1^2(x)\Big]$$

$$\int \frac{I_0(x)\,I_1(x)\,dx}{x^9} = \frac{1}{31\,255\,875x^8}\left[x\,(32\,768\,x^8 - 12\,288\,x^6 - 11\,520\,x^4 - 50\,400\,x^2 - 496\,125)\,I_0^2(x)-\right.$$

$$-(32\,768\,x^8 + 12\,288\,x^6 + 34\,560\,x^4 + 252\,000\,x^2 + 3\,472\,875)\,I_0(x)\,I_1(x)-$$

$$-x\,(32\,768\,x^8 + 4\,096\,x^6 + 6\,912\,x^4 + 36\,000\,x^2 + 385\,875)\,I_1^2(x)\Big]$$

$$\int \frac{I_0(x)\,K_1(x)\,dx}{x^9} = \frac{1}{31255875x^8}\left[(-32768\,x^9 + 12288\,x^7 + 11520\,x^5 + 50400\,x^3 + 496125\,x)\,I_0(x)K_0(x)-\right.$$

$$-(16384\,x^8 + 6144\,x^6 + 17280\,x^4 + 126000\,x^2 + 3472875)\,I_0(x)K_1(x)+$$

$$+(16384\,x^8 + 6144\,x^6 + 17280\,x^4 + 126000\,x^2)\,I_1(x)K_0(x)-$$

$$-(32768\,x^9 + 4096\,x^7 + 6912\,x^5 + 36000\,x^3 + 385875\,x)\,I_1(x)K_1(x)\Big]$$

$$\int \frac{J_0(x)\,J_1(x)\,dx}{x^{11}} = \frac{1}{3\,403\,7647\,875x^{10}}\left[x\,(-1\,310\,720\,x^{10}-\right.$$

$$-491\,520\,x^8 + 460\,800\,x^6 - 2\,016\,000\,x^4 + 19\,845\,000\,x^2 - 343\,814\,625)\,J_0^2(x)+$$

$$+(1\,310\,720\,x^{10} - 491\,520\,x^8 + 1\,382\,400\,x^6 - 10\,080\,000\,x^4 + 138\,915\,000\,x^2 - 3\,094\,331\,625)\,J_0(x)\,J_1(x)+$$

$$+x\,(-1\,310\,720\,x^{10} + 163\,840\,x^8 - 276\,480\,x^6 + 1\,440\,000\,x^4 - 15\,435\,000\,x^2 + 281\,302\,875)\,J_1^2(x)\Big]$$

$$\int \frac{I_0(x)\,I_1(x)\,dx}{x^{11}} = \frac{1}{34\,037\,647\,875x^{10}}\left[x\,(1\,310\,720\,x^{10}-\right.$$

$$-491\,520\,x^8 - 460\,800\,x^6 - 2\,016\,000\,x^4 - 19\,845\,000\,x^2 - 343\,814\,625)\,I_0^2(x)-$$

$$-(1\,310\,720\,x^{10} + 491\,520\,x^8 + 1\,382\,400\,x^6 + 10\,080\,000\,x^4 + 138\,915\,000\,x^2 + 3\,094\,331\,625)\,I_0(x)\,I_1(x)-$$

$$-x\,(1\,310\,720\,x^{10} + 163\,840\,x^8 + 276\,480\,x^6 + 1\,440\,000\,x^4 + 15\,435\,000\,x^2 + 281\,302\,875)\,I_1^2(x)\Big]$$

$$\int \frac{I_0(x)\,K_1(x)\,dx}{x^{11}} = \frac{1}{6807529575x^{10}}\cdot$$

$$\cdot\left[(-262144\,x^{11} + 98304\,x^9 + 92160\,x^7 + 403200\,x^5 + 3969000\,x^3 + 68762925\,x)\,I_0(x)K_0(x)-\right.$$

$$-(131072\,x^{10} + 49152\,x^8 + 138240\,x^6 + 1008000\,x^4 + 13891500\,x^2 + 618866325)\,I_0(x)K_1(x)+$$

$$+\left(131072\,x^{10}+49152\,x^{8}+138240\,x^{6}+1008000\,x^{4}+13891500\,x^{2}\right)I_{1}(x)K_{0}(x)-$$
$$-\left(262144\,x^{11}+32768\,x^{9}+55296\,x^{7}+288000\,x^{5}+3087000\,x^{3}+56260575\,x\right)I_{1}(x)K_{1}(x)\big]$$

$$\int \frac{J_{0}(x)\,J_{1}(x)\,dx}{x^{13}} \;=\; \frac{1}{4\,218\,399\,159\,975\,x^{12}}\,\big[x\left(4\,194\,304\,x^{12}+1\,572\,864\,x^{10}-1\,474\,560\,x^{8}+\right.$$
$$+6\,451\,200\,x^{6}-63\,504\,000\,x^{4}+1\,100\,206\,800\,x^{2}-29\,499\,294\,825\big)\,J_{0}^{2}(x)+$$
$$+\left(-4\,194\,304\,x^{12}+1\,572\,864\,x^{10}-4\,423\,680\,x^{8}+32\,256\,000\,x^{6}-\right.$$
$$-444\,528\,000\,x^{4}+9\,901\,861\,200\,x^{2}-324\,492\,243\,075\big)\,J_{0}(x)\,J_{1}(x)+$$
$$+x\left(4\,194\,304\,x^{12}-524\,288\,x^{10}+884\,736\,x^{8}-4\,608\,000\,x^{6}+\right.$$
$$+49\,392\,000\,x^{4}-900\,169\,200\,x^{2}+24\,960\,941\,775\big)\,J_{1}^{2}(x)\big]$$

$$\int \frac{I_{0}(x)\,I_{1}(x)\,dx}{x^{13}} \;=\; \frac{1}{4\,218\,399\,159\,975\,x^{12}}\,\big[x\left(4\,194\,304\,x^{12}-1\,572\,864\,x^{10}-1\,474\,560\,x^{8}-\right.$$
$$-6\,451\,200\,x^{6}-63\,504\,000\,x^{4}-1\,100\,206\,800\,x^{2}-29\,499\,294\,825\big)\,I_{0}^{2}(x)-$$
$$-\left(4\,194\,304\,x^{12}+1\,572\,864\,x^{10}+4\,423\,680\,x^{8}+32\,256\,000\,x^{6}+\right.$$
$$+444\,528\,000\,x^{4}+9\,901\,861\,200\,x^{2}+324\,492\,243\,075\big)\,I_{0}(x)\,I_{1}(x)-$$
$$-x\left(4\,194\,304\,x^{12}+524\,288\,x^{10}+884\,736\,x^{8}+4\,608\,000\,x^{6}+\right.$$
$$+49\,392\,000\,x^{4}+900\,169\,200\,x^{2}+24\,960\,941\,775\big)\,I_{1}^{2}(x)\big]$$

$$\int \frac{I_{0}(x)\,K_{1}(x)\,dx}{x^{13}} \;=\; \frac{1}{4218399159975\,x^{12}}\,\cdot$$
$$\cdot\,\big[\left(-4194304\,x^{13}+1572864\,x^{11}+1474560\,x^{9}+6451200\,x^{7}+63504000\,x^{5}+1100206800\,x^{3}+\right.$$
$$+29499294825\,x\big)\,I_{0}(x)K_{0}(x)-\left(2097152\,x^{12}+786432\,x^{10}+2211840\,x^{8}+16128000\,x^{6}+222264000\,x^{4}+\right.$$
$$*4950930600\,x^{2}+324492243075\big)\,I_{0}(x)K_{1}(x)+\left(2097152\,x^{12}+786432\,x^{10}+2211840\,x^{8}+16128000\,x^{6}+\right.$$
$$+222264000\,x^{4}+4950930600\,x^{2}\big)\,I_{1}(x)K_{0}(x)-\left(4194304\,x^{13}+524288\,x^{11}+884736\,x^{9}+4608000\,x^{7}+\right.$$
$$+49392000\,x^{5}+900169200\,x^{3}+24960941775\,x\big)\,I_{1}(x)K_{1}(x)\big]$$

Recurrence Relations:

$$\int \frac{J_{0}(x)\,J_{1}(x)\,dx}{x^{2n+1}} =$$
$$=\frac{1}{(2n+1)x^{2n}}\left[-\frac{x\,J_{0}^{2}(x)}{2n-1}-J_{0}(x)\,J_{1}(x)+\frac{x\,J_{1}^{2}(x)}{2n+1}\right]-\frac{8n}{(2n+1)^{2}(2n-1)}\int \frac{J_{0}(x)\,J_{1}(x)\,dx}{x^{2n-1}}$$

$$\int \frac{I_{0}(x)\,I_{1}(x)\,dx}{x^{2n+1}} =$$
$$=-\frac{1}{(2n+1)x^{2n}}\left[\frac{x\,I_{0}^{2}(x)}{2n-1}+I_{0}(x)\,I_{1}(x)+\frac{x\,I_{1}^{2}(x)}{2n+1}\right]+\frac{8n}{(2n+1)^{2}(2n-1)}\int \frac{I_{0}(x)\,I_{1}(x)\,dx}{x^{2n-1}}$$

$$\int \frac{K_{0}(x)\,K_{1}(x)\,dx}{x^{2n+1}} =$$
$$=\frac{1}{(2n+1)x^{2n}}\left[\frac{x\,K_{0}^{2}(x)}{2n-1}-K_{0}(x)\,K_{1}(x)+\frac{x\,K_{1}^{2}(x)}{2n+1}\right]+\frac{8n}{(2n+1)^{2}(2n-1)}\int \frac{K_{0}(x)\,K_{1}(x)\,dx}{x^{2n-1}}$$

$$\int \frac{I_{0}(x)\,K_{1}(x)\,dx}{x^{2n+1}} =\frac{1}{(2n+1)x^{2n}}\left[\frac{x\,I_{0}(x)\,K_{0}(x)}{2n-1}-I_{0}(x)\,K_{1}(x)-\frac{x\,I_{1}(x)\,K_{1}(x)}{2n+1}\right]+$$
$$+\frac{4n}{(2n+1)^{2}(2n-1)}\int \frac{I_{0}(x)\,K_{1}(x)-I_{1}(x)\,K_{0}(x)}{x^{2n-1}}\,dx$$

$$\int \frac{I_{1}(x)\,K_{0}(x)\,dx}{x^{2n+1}} =\frac{1}{(2n+1)x^{2n}}\left[-\frac{x\,I_{0}(x)\,K_{0}(x)}{2n-1}-I_{1}(x)\,K_{0}(x)+\frac{x\,I_{1}(x)\,K_{1}(x)}{2n+1}\right]+$$
$$+\frac{4n}{(2n+1)^{2}(2n-1)}\int \frac{I_{1}(x)\,K_{0}(x)-I_{0}(x)\,K_{1}(x)}{x^{2n-1}}\,dx$$

2.1.7. Integrals of the type $\int x^{2n+1} \cdot J_\nu(x) \cdot \left\{ \begin{array}{c} I_\nu(x) \\ K_\nu(x) \end{array} \right\} dx$

a) $\nu = 0$:

$$\int x \cdot J_0(x) \cdot I_0(x)\, dx = \frac{x}{2} \left[J_0(x) \cdot I_1(x) + J_1(x) \cdot I_0(x) \right]$$

$$\int x \cdot J_0(x) \cdot K_0(x)\, dx = \frac{x}{2} \left[-J_0(x) \cdot K_1(x) + J_1(x) \cdot K_0(x) \right]$$

$$\int x^3 \cdot J_0(x) \cdot I_0(x)\, dx = \frac{1}{2} \left[x^3 J_0(x) \cdot I_1(x) + x^3 J_1(x) \cdot I_0(x) - 2x^2 J_1(x) \cdot I_1(x) \right]$$

$$\int x^3 \cdot J_0(x) \cdot K_0(x)\, dx = \frac{1}{2} \left[-x^3 J_0(x) \cdot K_1(x) + x^3 J_1(x) \cdot K_0(x) + 2x^2 J_1(x) \cdot K_1(x) \right]$$

$$\int x^5 \cdot J_0(x) \cdot I_0(x)\, dx =$$

$$= \frac{1}{2} \left[8\,x^2 J_0(x) \cdot I_0(x) + (x^5 - 4\,x^3 - 8\,x)J_0(x) \cdot I_1(x) + (x^5 + 4\,x^3 - 8\,x)J_1(x) \cdot I_0(x) - 4\,x^4 J_1(x) \cdot I_1(x) \right]$$

$$\int x^5 \cdot J_0(x) \cdot K_0(x)\, dx =$$

$$= \frac{1}{2} \left[8\,x^2 J_0(x) \cdot K_0(x) - (x^5 - 4\,x^3 - 8\,x)J_0(x) \cdot K_1(x) + (x^5 + 4\,x^3 - 8\,x)J_1(x) \cdot K_0(x) + 4\,x^4 J_1(x) \cdot I_K(x) \right]$$

$$\int x^7 \cdot J_0(x) \cdot I_0(x)\, dx = \frac{1}{2} \left[48\,x^4 J_0(x) \cdot I_0(x) + (x^7 - 12\,x^5 - 96\,x^3)J_0(x) \cdot I_1(x) + \right.$$

$$\left. + (x^7 + 12\,x^5 - 96\,x^3)J_1(x) \cdot I_0(x) + (-6\,x^6 + 192\,x^2)J_1(x) \cdot I_1(x) \right]$$

$$\int x^7 \cdot J_0(x) \cdot K_0(x)\, dx = \frac{1}{2} \left[48\,x^4 J_0(x) \cdot K_0(x) - (x^7 - 12\,x^5 - 96\,x^3)J_0(x) \cdot K_1(x) + \right.$$

$$\left. + (x^7 + 12\,x^5 - 96\,x^3)J_1(x) \cdot K_0(x) + (6\,x^6 - 192\,x^2)J_1(x) \cdot K_1(x) \right]$$

$$\int x^9 \cdot J_0(x) \cdot I_0(x)\, dx =$$

$$= \frac{1}{2} \left[(144\,x^6 - 3456\,x^2)J_0(x) \cdot I_0(x) + (x^9 - 24\,x^7 - 432\,x^5 + 1728\,x^3 + 3456\,x)J_0(x) \cdot I_1(x) + \right.$$

$$\left. + (x^9 + 24\,x^7 - 432\,x^5 - 1728\,x^3 + 3456\,x)J_1(x) \cdot I_0(x) + (-8\,x^8 + 1728\,x^4)J_1(x) \cdot I_1(x) \right]$$

$$\int x^9 \cdot J_0(x) \cdot K_0(x)\, dx =$$

$$= \frac{1}{2} \left[(144\,x^6 - 3456\,x^2)J_0(x) \cdot K_0(x) - (x^9 - 24\,x^7 - 432\,x^5 + 1728\,x^3 + 3456\,x)J_0(x) \cdot K_1(x) + \right.$$

$$\left. + (x^9 + 24\,x^7 - 432\,x^5 - 1728\,x^3 + 3456\,x)J_1(x) \cdot K_0(x) + (8\,x^8 - 1728\,x^4)J_1(x) \cdot K_1(x) \right]$$

With

$$\int x^n \cdot J_0(x) \cdot I_0(x)\, dx = \frac{1}{2} \left[P_n(x)J_0(x) \cdot I_0(x) + Q_n(x)J_0(x) \cdot I_1(x) + R_n(x)J_1(x) \cdot I_0(x) + S_n(x)J_1(x) \cdot I_1(x) \right]$$

holds

$$\int x^n \cdot J_0(x) \cdot K_0(x)\, dx =$$

$$= \frac{1}{2} \left[P_n(x)J_0(x) \cdot K_0(x) - Q_n(x)J_0(x) \cdot K_1(x) + R_n(x)J_1(x) \cdot K_0(x) - S_n(x)J_1(x) \cdot K_1(x) \right] \ .$$

$$P_{11}(x) = 320\,x^8 - 61440\,x^4$$

$$Q_{11}(x) = x^{11} - 40\,x^9 - 1280\,x^7 + 15360\,x^5 + 122880\,x^3$$

$$R_{11}(x) = x^{11} + 40\,x^9 - 1280\,x^7 - 15360\,x^5 + 122880\,x^3$$
$$S_{11}(x) = -10\,x^{10} + 7680\,x^6 - 245760\,x^2$$

$$P_{13}(x) = 600\,x^{10} - 432000\,x^6 + 10368000\,x^2$$
$$Q_{13}(x) = x^{13} - 60\,x^{11} - 3000\,x^9 + 72000\,x^7 + 1296000\,x^5 - 5184000\,x^3 - 10368000\,x$$
$$R_{13}(x) = x^{13} + 60\,x^{11} - 3000\,x^9 - 72000\,x^7 + 1296000\,x^5 + 5184000\,x^3 - 10368000\,x$$
$$S_{13}(x) = -12\,x^{12} + 24000\,x^8 - 5184000\,x^4$$

$$P_{15}(x) = 1008\,x^{12} - 1935360\,x^8 + 371589120\,x^4$$
$$Q_{15}(x) = x^{15} - 84\,x^{13} - 6048\,x^{11} + 241920\,x^9 + 7741440\,x^7 - 92897280\,x^5 - 743178240\,x^3$$
$$R_{15}(x) = x^{15} + 84\,x^{13} - 6048\,x^{11} - 241920\,x^9 + 7741440\,x^7 + 92897280\,x^5 - 743178240\,x^3$$
$$S_{15}(x) = -14\,x^{14} + 60480\,x^{10} - 46448640\,x^6 + 1486356480\,x^2$$

b) $\nu = 1$:

$$\int x \cdot J_1(x) \cdot I_1(x)\,dx = \frac{x}{2}\,[-J_0(x) \cdot I_1(x) + J_1(x) \cdot I_0(x)]$$

$$\int x \cdot J_1(x) \cdot K_1(x)\,dx = -\frac{x}{2}\,[J_0(x) \cdot K_1(x) + J_1(x) \cdot K_0(x)]$$

$$\int x^3 \cdot J_1(x) \cdot I_1(x)\,dx = \frac{1}{2}\,\left[2\,x^2 J_0(x) \cdot I_0(x) - (x^3 + 2\,x)J_0(x) \cdot I_1(x) + (x^3 - 2\,x)J_1(x) \cdot I_0(x)\right]$$

$$\int x^3 \cdot J_1(x) \cdot K_1(x)\,dx = \frac{1}{2}\,\left[-2\,x^2 J_0(x) \cdot K_0(x) - (x^3 + 2\,x)J_0(x) \cdot K_1(x) - (x^3 - 2\,x)J_1(x) \cdot K_0(x)\right]$$

With

$$\int x^n \cdot J_1(x) \cdot I_1(x)\,dx = \frac{1}{2}\,[P_n(x)J_0(x) \cdot I_0(x) + Q_n(x)J_0(x) \cdot I_1(x) + R_n(x)J_1(x) \cdot I_0(x) + S_n(x)J_1(x) \cdot I_1(x)]$$

holds

$$\int x^n \cdot J_1(x) \cdot K_1(x)\,dx =$$

$$= \frac{1}{2}\,[-P_n(x)J_0(x) \cdot K_0(x) + Q_n(x)J_0(x) \cdot K_1(x) - R_n(x)J_1(x) \cdot K_0(x) + S_n(x)J_1(x) \cdot K_1(x)]\ .$$

$$P_5(x) = 4\,x^4\ , \quad Q_5(x) = -x^5 - 8\,x^3\ , \quad R_5(x) = x^5 - 8\,x^3\ , \quad S_5(x) = 16\,x^2$$

$$P_7(x) = 6\,x^6 - 144\,x^2\ , \quad Q_7(x) = -x^7 - 18\,x^5 + 72\,x^3 + 144\,x$$
$$R_7(x) = x^7 - 18\,x^5 - 72\,x^3 + 144\,x\ , \quad S_7(x) = 72\,x^4$$

$$P_9(x) = 8\,x^8 - 1536\,x^4\ , \quad Q_9(x) = -x^9 - 32\,x^7 + 384\,x^5 + 3072\,x^3$$
$$R_9(x) = x^9 - 32\,x^7 - 384\,x^5 + 3072\,x^3\ , \quad S_9(x) = 192\,x^6 - 6144\,x^2$$

$$P_{11}(x) = 10\,x^{10} - 7200\,x^6 + 172800\,x^2$$
$$Q_{11}(x) = -x^{11} - 50\,x^9 + 1200\,x^7 + 21600\,x^5 - 86400\,x^3 - 172800\,x$$
$$R_{11}(x) = x^{11} - 50\,x^9 - 1200\,x^7 + 21600\,x^5 + 86400\,x^3 - 172800\,x$$
$$S_{11}(x) = 400\,x^8 - 86400\,x^4$$

$$P_{13}(x) = 12\,x^{12} - 23040\,x^8 + 4423680\,x^4$$

$$Q_{13}(x) = -x^{13} - 72\,x^{11} + 2880\,x^9 + 92160\,x^7 - 1105920\,x^5 - 8847360\,x^3$$

$$R_{13}(x) = x^{13} - 72\,x^{11} - 2880\,x^9 + 92160\,x^7 + 1105920\,x^5 - 8847360\,x^3$$

$$S_{13}(x) = 720\,x^{10} - 552960\,x^6 + 17694720\,x^2$$

$$P_{15}(x) = 14\,x^{14} - 58800\,x^{10} + 42336000\,x^6 - 1016064000\,x^2$$

$$Q_{15}(x) = -x^{15} - 98\,x^{13} + 5880\,x^{11} + 294000\,x^9 - 7056000\,x^7 - 127008000\,x^5 + 508032000\,x^3 + 1016064000\,x$$

$$R_{15}(x) = x^{15} - 98\,x^{13} - 5880\,x^{11} + 294000\,x^9 + 7056000\,x^7 - 127008000\,x^5 - 508032000\,x^3 + 1016064000\,x$$

$$S_{15}(x) = 1176\,x^{12} - 2352000\,x^8 + 508032000\,x^4$$

c) Recurrence relations:

$$\int x^{2n+1}\,J_0(x)\,I_0(x)\,dx =$$

$$= \frac{x^{2n}}{2}\left[x\,J_0(x)\,I_1(x) + x\,J_1(x)\,I_0(x) - 2n\,J_1(x)\,I_1(x)\right] + 2n(n-1)\int x^{2n-1}\,J_1(x)\,I_1(x)\,dx$$

$$\int x^{2n+1}\,J_1(x)\,I_1(x)\,dx =$$

$$= \frac{x^{2n}}{2}\left[-x\,J_0(x)\,I_1(x) + x\,J_1(x)\,I_0(x) + 2n\,J_0(x)\,I_0(x)\right] - 2n^2\int x^{2n-1}\,J_0(x)\,I_0(x)\,dx$$

$$\int x^{2n+1}\,K_0(x)\,I_0(x)\,dx =$$

$$= \frac{x^{2n}}{2}\left[-x\,J_0(x)\,K_1(x) + x\,J_1(x)\,K_0(x) + 2n\,J_1(x)\,K_1(x)\right] - 2n(n-1)\int x^{2n-1}\,J_1(x)\,K_1(x)\,dx$$

$$\int x^{2n+1}\,J_1(x)\,K_1(x)\,dx =$$

$$= -\frac{x^{2n}}{2}\left[x\,J_0(x)\,K_1(x) + x\,J_1(x)\,K_0(x) + 2n\,J_0(x)\,K_0(x)\right] + 2n^2\int x^{2n-1}\,J_0(x)\,K_0(x)\,dx$$

2.1.8. Integrals of the type $\int x^{2n} \cdot J_\nu(x) \cdot \left\{ \begin{array}{c} I_{1-\nu}(x) \\ K_{1-\nu}(x) \end{array} \right\} dx$

a) $\nu = 0$:

$$\int x^2 \cdot J_0(x) \cdot I_1(x)\, dx = \frac{1}{2} \left[x^2 J_0(x) \cdot I_0(x) - x J_0(x) \cdot I_1(x) - x J_1(x) \cdot I_0(x) + x^2 J_1(x) \cdot I_1(x) \right]$$

$$\int x^2 \cdot J_0(x) \cdot K_1(x)\, dx = \frac{1}{2} \left[-x^2 J_0(x) \cdot K_0(x) - x J_0(x) \cdot K_1(x) + x J_1(x) \cdot K_0(x) + x^2 J_1(x) \cdot K_1(x) \right]$$

$$\int x^4 \cdot J_0(x) \cdot I_1(x)\, dx =$$

$$= \frac{1}{2} \left[(x^4 - 2\,x^2) J_0(x) \cdot I_0(x) + (-x^3 + 2\,x) J_0(x) \cdot I_1(x) + (-3\,x^3 + 2\,x) J_1(x) \cdot I_0(x) + (x^4 + 4\,x^2) J_1(x) \cdot I_1(x) \right]$$

$$\int x^4 \cdot J_0(x) \cdot K_1(x)\, dx =$$

$$= \frac{1}{2} \left[-(x^4 - 2\,x^2) J_0(x) \cdot K_0(x) - (x^3 - 2\,x) J_0(x) \cdot K_1(x) + (3\,x^3 - 2\,x) J_1(x) \cdot K_0(x) + (x^4 + 4\,x^2) J_1(x) \cdot K_1(x) \right]$$

With

$$\int x^n \cdot J_0(x) \cdot I_1(x)\, dx = \frac{1}{2} \left[P_n(x) J_0(x) \cdot I_0(x) + Q_n(x) J_0(x) \cdot I_1(x) + R_n(x) J_1(x) \cdot I_0(x) + S_n(x) J_1(x) \cdot I_1(x) \right]$$

holds

$$\int x^n \cdot J_0(x) \cdot K_1(x)\, dx =$$

$$= \frac{1}{2} \left[-P_n(x) J_0(x) \cdot K_0(x) + Q_n(x) J_0(x) \cdot K_1(x) - R_n(x) J_1(x) \cdot K_0(x) + S_n(x) J_1(x) \cdot K_1(x) \right] \; .$$

$$P_6(x) = x^6 - 8\,x^4 - 24\,x^2 \; , \quad Q_6(x) = -x^5 + 28\,x^3 + 24\,x$$

$$R_6(x) = -5\,x^5 + 4\,x^3 + 24\,x \; , \quad S_6(x) = x^6 + 12\,x^4 - 32\,x^2$$

$$P_8(x) = x^8 - 18\,x^6 - 192\,x^4 + 432\,x^2 \; , \quad Q_8(x) = -x^7 + 102\,x^5 + 168\,x^3 - 432\,x$$

$$R_8(x) = -7\,x^7 + 6\,x^5 + 600\,x^3 - 432\,x \; , \quad S_8(x) = x^8 + 24\,x^6 - 216\,x^4 - 768\,x^2$$

$$P_{10}(x) = x^{10} - 32\,x^8 - 720\,x^6 + 6144\,x^4 + 17280\,x^2$$

$$Q_{10}(x) = -x^9 + 248\,x^7 + 624\,x^5 - 20928\,x^3 - 17280\,x$$

$$R_{10}(x) = -9\,x^9 + 8\,x^7 + 3696\,x^5 - 3648\,x^3 - 17280\,x$$

$$S_{10}(x) = x^{10} + 40\,x^8 - 768\,x^6 - 8640\,x^4 + 24576\,x^2$$

$$P_{12}(x) = x^{12} - 50\,x^{10} - 1920\,x^8 + 36000\,x^6 + 368640\,x^4 - 864000\,x^2$$

$$Q_{12}(x) = -x^{11} + 490\,x^9 + 1680\,x^7 - 200160\,x^5 - 305280\,x^3 + 864000\,x$$

$$R_{12}(x) = -11\,x^{11} + 10\,x^9 + 13680\,x^7 - 15840\,x^5 - 1169280\,x^3 + 864000\,x$$

$$S_{12}(x) = x^{12} + 60\,x^{10} - 2000\,x^8 - 46080\,x^6 + 432000\,x^4 + 1474560\,x^2$$

$$P_{14}(x) = x^{14} - 72\,x^{12} - 4200\,x^{10} + 138240\,x^8 + 3024000\,x^6 - 26542080\,x^4 - 72576000\,x^2$$

$$Q_{14}(x) = -x^{13} + 852\,x^{11} + 3720\,x^9 - 1056960\,x^7 - 2436480\,x^5 + 89372160\,x^3 + 72576000\,x$$

$$R_{14}(x) = -13\,x^{13} + 12\,x^{11} + 38280\,x^9 - 48960\,x^7 - 15707520\,x^5 + 16796160\,x^3 + 72576000\,x$$

$$S_{14}(x) = x^{14} + 84\,x^{12} - 4320\,x^{10} - 168000\,x^8 + 3317760\,x^6 + 36288000\,x^4 - 106168320\,x^2$$

$$P_{16}(x) =$$
$$= x^{16} - 98\,x^{14} - 8064\,x^{12} + 411600\,x^{10} + 15482880\,x^8 - 296352000\,x^6 - 2972712960\,x^4 + 7112448000\,x^2$$

$$Q_{16}(x) =$$
$$= -x^{15} + 1358\,x^{13} + 7224\,x^{11} - 3993360\,x^9 - 12539520\,x^7 + 1632234240\,x^5 + 2389201920\,x^3 - 7112448000\,x$$

$$R_{16}(x) =$$
$$= -15\,x^{15} + 14\,x^{13} + 89544\,x^{11} - 122640\,x^9 - 111323520\,x^7 + 145877760\,x^5 + 9501649920\,x^3 - 7112448000\,x$$

$$S_{16}(x) =$$
$$= x^{16} + 112\,x^{14} - 8232\,x^{12} - 483840\,x^{10} + 16464000\,x^8 + 371589120\,x^6 - 3556224000\,x^4 - 11890851840\,x^2$$

b) $\nu = 1$:

$$\int x^2 \cdot J_1(x) \cdot I_0(x)\,dx = \frac{1}{2}\left[-x^2 J_0(x) \cdot I_0(x) + x J_0(x) \cdot I_1(x) + x J_1(x) \cdot I_0(x) + x^2 J_1(x) \cdot I_1(x)\right]$$

$$\int x^2 \cdot J_1(x) \cdot K_0(x)\,dx = \frac{1}{2}\left[-x^2 J_0(x) \cdot K_0(x) - x J_0(x) \cdot K_1(x) + x J_1(x) \cdot K_0(x) - x^2 J_1(x) \cdot K_1(x)\right]$$

$$\int x^4 \cdot J_1(x) \cdot I_0(x)\,dx =$$

$$= \frac{1}{2}\left[(-x^4 - 2\,x^2)J_0(x) \cdot I_0(x) + (3\,x^3 + 2\,x)J_0(x) \cdot I_1(x) + (x^3 + 2\,x)J_1(x) \cdot I_0(x) + (x^4 - 4\,x^2)J_1(x) \cdot I_1(x)\right]$$

$$\int x^4 \cdot J_1(x) \cdot K_0(x)\,dx =$$

$$= \frac{1}{2}\left[(-x^4 - 2\,x^2)J_0(x) \cdot K_0(x) - (3\,x^3 + 2\,x)J_0(x) \cdot K_1(x) + (x^3 + 2\,x)J_1(x) \cdot K_0(x) - (x^4 - 4\,x^2)J_1(x) \cdot K_1(x)\right]$$

With

$$\int x^n \cdot J_1(x) \cdot I_0(x)\,dx = \frac{1}{2}\left[P_n(x)J_0(x) \cdot I_0(x) + Q_n(x)J_0(x) \cdot I_1(x) + R_n(x)J_1(x) \cdot I_0(x) + S_n(x)J_1(x) \cdot I_1(x)\right]$$

holds

$$\int x^n \cdot J_1(x) \cdot K_0(x)\,dx = \frac{1}{2}\left[P_n(x)J_0(x) \cdot K_0(x) - Q_n(x)J_0(x) \cdot K_1(x) + R_n(x)J_1(x) \cdot K_0(x) - S_n(x)J_1(x) \cdot K_1(x)\right] \ .$$

$$P_6(x) = -x^6 - 8\,x^4 + 24\,x^2 \ , \quad Q_6(x) = 5\,x^5 + 4\,x^3 - 24\,x$$
$$R_6(x) = x^5 + 28\,x^3 - 24\,x \ , \quad S_6(x) = x^6 - 12\,x^4 - 32\,x^2$$

$$P_8(x) = -x^8 - 18\,x^6 + 192\,x^4 + 432\,x^2 \ , \quad Q_8(x) = 7\,x^7 + 6\,x^5 - 600\,x^3 - 432\,x$$
$$R_8(x) = x^7 + 102\,x^5 - 168\,x^3 - 432\,x \ , \quad S_8(x) = x^8 - 24\,x^6 - 216\,x^4 + 768\,x^2$$

$$P_{10}(x) = -x^{10} - 32\,x^8 + 720\,x^6 + 6144\,x^4 - 17280\,x^2$$
$$Q_{10}(x) = 9\,x^9 + 8\,x^7 - 3696\,x^5 - 3648\,x^3 + 17280\,x$$
$$R_{10}(x) = x^9 + 248\,x^7 - 624\,x^5 - 20928\,x^3 + 17280\,x$$
$$S_{10}(x) = x^{10} - 40\,x^8 - 768\,x^6 + 8640\,x^4 + 24576\,x^2$$

$$P_{12}(x) = -x^{12} - 50\,x^{10} + 1920\,x^8 + 36000\,x^6 - 368640\,x^4 - 864000\,x^2$$
$$Q_{12}(x) = 11\,x^{11} + 10\,x^9 - 13680\,x^7 - 15840\,x^5 + 1169280\,x^3 + 864000\,x$$
$$R_{12}(x) = x^{11} + 490\,x^9 - 1680\,x^7 - 200160\,x^5 + 305280\,x^3 + 864000\,x$$

$$S_{12}(x) = x^{12} - 60\,x^{10} - 2000\,x^8 + 46080\,x^6 + 432000\,x^4 - 1474560\,x^2$$

$$P_{14}(x) = -x^{14} - 72\,x^{12} + 4200\,x^{10} + 138240\,x^8 - 3024000\,x^6 - 26542080\,x^4 + 72576000\,x^2$$

$$Q_{14}(x) = 13\,x^{13} + 12\,x^{11} - 38280\,x^9 - 48960\,x^7 + 15707520\,x^5 + 16796160\,x^3 - 72576000\,x$$

$$R_{14}(x) = x^{13} + 852\,x^{11} - 3720\,x^9 - 1056960\,x^7 + 2436480\,x^5 + 89372160\,x^3 - 72576000\,x$$

$$S_{14}(x) = x^{14} - 84\,x^{12} - 4320\,x^{10} + 168000\,x^8 + 3317760\,x^6 - 36288000\,x^4 - 106168320\,x^2$$

$$P_{16}(x) =$$
$$= -x^{16} - 98\,x^{14} + 8064\,x^{12} + 411600\,x^{10} - 15482880\,x^8 - 296352000\,x^6 + 2972712960\,x^4 + 7112448000\,x^2$$

$$Q_{16}(x) =$$
$$= 15\,x^{15} + 14\,x^{13} - 89544\,x^{11} - 122640\,x^9 + 111323520\,x^7 + 145877760\,x^5 - 9501649920\,x^3 - 7112448000\,x$$

$$R_{16}(x) =$$
$$= x^{15} + 1358\,x^{13} - 7224\,x^{11} - 3993360\,x^9 + 12539520\,x^7 + 1632234240\,x^5 - 2389201920\,x^3 - 7112448000\,x$$

$$S_{16}(x) =$$
$$= x^{16} - 112\,x^{14} - 8232\,x^{12} + 483840\,x^{10} + 16464000\,x^8 - 371589120\,x^6 - 3556224000\,x^4 + 11890851840\,x^2$$

c) Recurrence Relations:

$$\int x^{2n+2}\, J_0(x)\, I_1(x)\, dx = \frac{x^{2n+1}}{2}\,[x\,J_0(x)\,I_0(x) - J_0(x)\,I_1(x) - (2n+1)\,J_1(x)\,I_0(x) + x\,J_1(x)\,I_1(x)] +$$
$$+ n\int x^{2n}\, J_0(x)\, I_1(x)\, dx + n(2n+1)\int x^{2n}\, J_1(x)\, I_0(x)\, dx$$

$$\int x^{2n+2}\, J_1(x)\, I_0(x)\, dx = \frac{x^{2n+1}}{2}\,[-x\,J_0(x)\,I_0(x) + (2n+1)\,J_0(x)\,I_1(x) + J_1(x)\,I_0(x) + x\,J_1(x)\,I_1(x)] -$$
$$- n(2n+1)\int x^{2n}\, J_0(x)\, I_1(x)\, dx - n\int x^{2n}\, J_1(x)\, I_0(x)\, dx$$

$$\int x^{2n+2}\, J_0(x)\, K_1(x)\, dx = \frac{x^{2n+1}}{2}\,[-x\,J_0(x)\,K_0(x) - J_0(x)\,K_1(x) + (2n+1)\,J_1(x)\,K_0(x) + x\,J_1(x)\,K_1(x)] +$$
$$+ n\int x^{2n}\, J_0(x)\, K_1(x)\, dx - n(2n+1)\int x^{2n}\, J_1(x)\, K_0(x)\, dx$$

$$\int x^{2n+2}\, J_1(x)\, K_0(x)\, dx = \frac{x^{2n+1}}{2}\,[-x\,J_0(x)\,K_0(x) - (2n+1)\,J_0(x)\,K_1(x) + J_1(x)\,K_0(x) - x\,J_1(x)\,K_1(x)] +$$
$$+ n(2n+1)\int x^{2n}\, J_0(x)\, K_1(x)\, dx - n\int x^{2n}\, J_1(x)\, K_0(x)\, dx$$

2.1.9. Integrals of the type $\int x^{2n[+1]} J_\mu(x)\, Y_\nu\, dx$, :

a) $\int x^{2n+1} J_0(x) Y_0(x)\, dx$:

$$\int x\, J_0(x) Y_0(x)\, dx = \frac{x^2}{2}\left[J_0(x)Y_0(x) + J_1(x)Y_1(x)\right]$$

$$\int x^3\, J_0(x) Y_0(x)\, dx =$$

$$= \frac{x^4}{6} J_0(x)Y_0(x) + \frac{x^3}{6}\left[J_0(x)Y_1(x) + J_1(x)Y_0(x)\right] + \left(\frac{x^4}{6} - \frac{x^2}{3}\right) J_1(x)Y_1(x)$$

$$\int x^5\, J_0(x) Y_0(x)\, dx = \left(\frac{x^6}{10} + \frac{4}{15}\, x^4\right) J_0(x)Y_0(x) +$$

$$+ \left(\frac{x^5}{5} - \frac{8}{15}\, x^3\right)\left[J_0(x)Y_1(x) + J_1(x)Y_0(x)\right] + \left(\frac{x^6}{10} - \frac{8}{15}\, x^4 + \frac{16}{15}\, x^2\right) J_1(x)Y_1(x)$$

$$\int x^7\, J_0(x) Y_0(x)\, dx = \left(\frac{x^8}{14} + \frac{18}{35}\, x^6 - \frac{72}{35}\, x^4\right) J_0(x)Y_0(x) +$$

$$+ \left(\frac{3}{14}\, x^7 - \frac{54}{35}\, x^5 + \frac{144}{35}\, x^3\right)\left[J_0(x)Y_1(x) + J_1(x)Y_0(x)\right] +$$

$$+ \left(\frac{x^8}{14} - \frac{27}{35}\, x^6 + \frac{144}{35}\, x^4 - \frac{288}{35}\, x^2\right) J_1(x)Y_1(x)$$

$$\int x^9\, J_0(x) Y_0(x)\, dx = \left(\frac{x^{10}}{18} + \frac{16}{21}\, x^8 - \frac{256}{35}\, x^6 + \frac{1024}{35}\, x^4\right) J_0(x)Y_0(x) +$$

$$+ \left(\frac{2}{9}\, x^9 - \frac{64}{21}\, x^7 + \frac{768}{35}\, x^5 - \frac{2048}{35}\, x^3\right)\left[J_0(x)Y_1(x) + J_1(x)Y_0(x)\right] +$$

$$+ \left(\frac{x^{10}}{18} - \frac{64}{63}\, x^8 + \frac{384}{35}\, x^6 - \frac{2048}{35}\, x^4 + \frac{4096}{35}\, x^2\right) J_1(x)Y_1(x)$$

$$\int x^{11}\, J_0(x) Y_0(x)\, dx = \left(\frac{x^{12}}{22} + \frac{100}{99}\, x^{10} - \frac{4000}{231}\, x^8 + \frac{12800}{77}\, x^6 - \frac{51200}{77}\, x^4\right) J_0(x)Y_0(x) +$$

$$+ \left(\frac{5}{22}\, x^{11} - \frac{500}{99}\, x^9 + \frac{16000}{231}\, x^7 - \frac{38400}{77}\, x^5 + \frac{102400}{77}\, x^3\right)\left[J_0(x)Y_1(x) + J_1(x)Y_0(x)\right] +$$

$$+ \left(\frac{x^{12}}{22} - \frac{125}{99}\, x^{10} + \frac{16000}{693}\, x^8 - \frac{19200}{77}\, x^6 + \frac{102400}{77}\, x^4 - \frac{204800}{77}\, x^2\right) J_1(x)Y_1(x)$$

$$\int x^{13}\, J_0(x) Y_0(x)\, dx =$$

$$= \left(\frac{x^{14}}{26} + \frac{180}{143}\, x^{12} - \frac{4800}{143}\, x^{10} + \frac{576000}{1001}\, x^8 - \frac{5529600}{1001}\, x^6 + \frac{22118400}{1001}\, x^4\right) J_0(x)Y_0(x) +$$

$$+ \left(\frac{3}{13}\, x^{13} - \frac{1080}{143}\, x^{11} + \frac{24000}{143}\, x^9 - \frac{2304000}{1001}\, x^7 + \frac{16588800}{1001}\, x^5 - \frac{44236800}{1001}\, x^3\right)\left[J_0(x)Y_1(x) + J_1(x)Y_0(x)\right] +$$

$$+ \left(\frac{x^{14}}{26} - \frac{216}{143}\, x^{12} + \frac{6000}{143}\, x^{10} - \frac{768000}{1001}\, x^8 + \frac{8294400}{1001}\, x^6 - \frac{44236800}{1001}\, x^4 + \frac{88473600}{1001}\, x^2\right) J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+1} J_0(x) Y_0(x)\, dx =$$

$$= \frac{x^{2n}}{4n+2}\left\{(2n^2 + x^2) J_0(x)Y_0(x) + nx[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + x^2 J_1(x)Y_1(x)\right\} -$$

$$- \frac{2n^3}{2n+1} \int x^{2n-1} J_0(x) Y_0(x)\, dx$$

b) $\int x^{-2n} J_0(x)Y_0(x)\,dx$:

$$\int \frac{J_0(x)Y_0(x)}{x^2}\,dx = -\frac{2x^2+1}{x}\,J_0(x)Y_0(x) + J_0(x)Y_1(x) + J_1(x)Y_0(x) - 2x\,J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_0(x)}{x^4}\,dx = \frac{1}{27\,x^3}\,\left\{(16\,x^4 + 6\,x^2 - 9)\,J_0(x)Y_0(x) + (-8\,x^3 + 3\,x)\,[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \right.$$

$$\left. + (16\,x^4 - 2\,x^2)\,J_1(x)Y_1(x)\right\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^6}\,dx = \frac{1}{3375\,x^5}\,\left\{(-256\,x^6 - 96\,x^4 + 90\,x^2 - 675)\,J_0(x)Y_0(x) + \right.$$

$$+ (128\,x^5 - 48\,x^3 + 135\,x)\,[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$\left. + (-256\,x^6 + 32\,x^4 - 54\,x^2)\,J_1(x)Y_1(x)\right\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^8}\,dx = \frac{1}{385875\,x^7}\,\left\{(2048\,x^8 + 768\,x^6 - 720\,x^4 + 3150\,x^2 - 55125)\,J_0(x)Y_0(x) + \right.$$

$$+ (-1024\,x^7 + 384\,x^5 - 1080\,x^3 + 7875\,x)\,[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$\left. + (2048\,x^8 - 256\,x^6 + 432\,x^4 - 2250\,x^2)\,J_1(x)Y_1(x)\right\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^{10}}\,dx = \frac{1}{281302875\,x^9}\,\cdot$$

$$\cdot\left\{(-65536\,x^{10} - 24576\,x^8 + 23040\,x^6 - 100800\,x^4 + 992250\,x^2 - 31255875)\,J_0(x)Y_0(x) + \right.$$

$$+ (32768\,x^9 - 12288\,x^7 + 34560\,x^5 - 252000\,x^3 + 3472875\,x)\,[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$\left. + (-65536\,x^{10} + 8192\,x^8 - 13824\,x^6 + 72000\,x^4 - 771750\,x^2)\,J_1(x)Y_1(x)\right\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^{12}}\,dx = \frac{1}{74882825325\,x^{11}}\,\cdot$$

$$\cdot\left\{(524288\,x^{12} + 196608\,x^{10} - 184320\,x^8 + 806400\,x^6 - 7938000\,x^4 + 137525850\,x^2 - 6807529575)\,J_0(x)Y_0(x) + \right.$$

$$+ (-262144\,x^{11} + 98304\,x^9 - 276480\,x^7 + 2016000\,x^5 - 27783000\,x^3 + 618866325\,x)\,[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$\left. + (524288\,x^{12} - 65536\,x^{10} + 110592\,x^8 - 576000\,x^6 + 6174000\,x^4 - 112521150\,x^2)\,J_1(x)Y_1(x)\right\}$$

Recurrence Formula:

$$\int \frac{J_0(x)Y_0(x)}{x^{2n+2}}\,dx = \frac{1}{(2n+1)^3}\,\cdot$$

$$\cdot\left\{\frac{-(4n^2 + 4n + 1 + 2x^2)J_0(x)Y_0(x) + (2n+1)x\,[J_0(x)Y_1(x) + J_1(x)Y_0(x)] - 2x^2 J_1(x)Y_1(x)}{x^{2n+1}} - \right.$$

$$\left. - 8n \int \frac{J_0(x)Y_0(x)}{x^{2n}}\,dx\right\}$$

c) $\int x^{2n} J_0(x)Y_1(x)\,dx$:

$$\int x^2 J_0(x)Y_1(x)\,dx = \frac{x^3}{4}J_0(x)Y_1(x) - \frac{x^3}{4}J_1(x)Y_0(x) + \frac{x^2}{2}J_1(x)Y_1(x)$$

$$\int x^4 J_0(x)Y_1(x)\,dx =$$

$$= -\frac{x^4}{6}J_0(x)Y_0(x) + \frac{3\,x^5 + 8\,x^3}{24}J_0(x)Y_1(x) - \frac{3\,x^5 - 8\,x^3}{24}J_1(x)Y_0(x) + \frac{x^4 - 2\,x^2}{3}J_1(x)Y_1(x)$$

$$\int x^6 J_0(x)Y_1(x)\,dx = -\frac{x^6 - 4\,x^4}{5}J_0(x)Y_0(x) + \frac{5\,x^7 + 36\,x^5 - 96\,x^3}{60}J_0(x)Y_1(x)-$$

$$-\frac{5\,x^7 - 36\,x^5 + 96\,x^3}{60}J_1(x)Y_0(x) + \frac{3\,x^6 - 16\,x^4 + 32\,x^2}{10}J_1(x)Y_1(x)$$

$$\int x^8 J_0(x)Y_1(x)\,dx = -\frac{15\,x^8 - 144\,x^6 + 576\,x^4}{70}J_0(x)Y_0(x)+$$

$$+\frac{35\,x^9 + 480\,x^7 - 3456\,x^5 + 9216\,x^3}{560}J_0(x)Y_1(x)-$$

$$-\frac{35\,x^9 - 480\,x^7 + 3456\,x^5 - 9216\,x^3}{560}J_1(x)Y_0(x)+$$

$$+\frac{10\,x^8 - 108\,x^6 + 576\,x^4 - 1152\,x^2}{35}J_1(x)Y_1(x)$$

$$\int x^{10} J_0(x)Y_1(x)\,dx = -\frac{14\,x^{10} - 240\,x^8 + 2304\,x^6 - 9216\,x^4}{63}J_0(x)Y_0(x)+$$

$$+\frac{63\,x^{11} + 1400\,x^9 - 19200\,x^7 + 138240\,x^5 - 368640\,x^3}{1260}J_0(x)Y_1(x)-$$

$$-\frac{63\,x^{11} - 1400\,x^9 + 19200\,x^7 - 138240\,x^5 + 368640\,x^3}{1260}J_1(x)Y_0(x)+$$

$$+\frac{35\,x^{10} - 640\,x^8 + 6912\,x^6 - 36864\,x^4 + 73728\,x^2}{126}J_1(x)Y_1(x)$$

$$\int x^{12} J_0(x)Y_1(x)\,dx = -\frac{105\,x^{12} - 2800\,x^{10} + 48000\,x^8 - 460800\,x^6 + 1843200\,x^4}{462}J_0(x)Y_0(x)+$$

$$+\frac{77\,x^{13} + 2520\,x^{11} - 56000\,x^9 + 768000\,x^7 - 5529600\,x^5 + 14745600\,x^3}{1848}J_0(x)Y_1(x)-$$

$$-\frac{77\,x^{13} - 2520\,x^{11} + 56000\,x^9 - 768000\,x^7 + 5529600\,x^5 - 14745600\,x^3}{1848}J_1(x)Y_0(x)+$$

$$+\frac{63\,x^{12} - 1750\,x^{10} + 32000\,x^8 - 345600\,x^6 + 1843200\,x^4 - 3686400\,x^2}{231}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+2} J_0(x)Y_1(x)\,dx = x^{2n}\left[-\frac{nx^2}{2(2n+1)}J_0(x)Y_0(x) + \left(\frac{n(n+1)}{2n+1} + \frac{x^2}{4(n+1)}\right)xJ_0(x)Y_1(x)+\right.$$

$$\left. -\frac{x^3}{4(n+1)}J_1(x)Y_0(x) + \frac{(n+1)x^2}{2(2n+1)}J_1(x)Y_1(x)\right] - \frac{2n^2(n+1)}{2n+1}\int x^{2n} J_0(x)Y_1(x)\,dx$$

d) $\int x^{-2n-1} J_0(x)Y_1(x)\,dx$:

$$\int \frac{J_0(x)Y_1(x)\,dx}{x} = xJ_0(x)Y_0(x) - J_0(x)Y_1(x) + xJ_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^3} =$$

$$= -\frac{8\,x^2+3}{9x}J_0(x)Y_0(x) + \frac{4\,x^2-3}{9x^2}J_0(x)Y_1(x) + \frac{4}{9}J_1(x)Y_0(x) - \frac{8\,x^2-1}{9x}J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^5} = \frac{128\,x^4+48\,x^2-45}{675\,x^3}J_0(x)Y_0(x) - \frac{64\,x^4-24\,x^2+135}{675\,x^4}J_0(x)Y_1(x) -$$

$$-\frac{64\,x^2-24}{675\,x^2}J_1(x)Y_0(x) + \frac{128\,x^4-16\,x^2+27}{675\,x^3}J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^7} = -\frac{1024\,x^6+384\,x^4-360\,x^2+1575}{55125\,x^5}J_0(x)Y_0(x) +$$

$$+\frac{512\,x^6-192\,x^4+540\,x^2-7875}{55125\,x^6}J_0(x)Y_1(x) + \frac{512\,x^4-192\,x^2+540}{55125\,x^4}J_1(x)Y_0(x) -$$

$$-\frac{1024\,x^6-128\,x^4+216\,x^2-1125}{55125\,x^5}J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^9} = \frac{32768\,x^8+12288\,x^6-11520\,x^4+50400\,x^2-496125}{31255875\,x^7}J_0(x)Y_0(x) -$$

$$-\frac{16384\,x^8-6144\,x^6+17280\,x^4-126000\,x^2+3472875}{31255875\,x^8}J_0(x)Y_1(x) -$$

$$-\frac{16384\,x^6-6144\,x^4+17280\,x^2-126000}{31255875\,x^6}J_1(x)Y_0(x) +$$

$$+\frac{32768\,x^8-4096\,x^6+6912\,x^4-36000\,x^2+385875}{31255875\,x^7}J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^{11}} =$$

$$-\frac{262144\,x^{10}+98304\,x^8-92160\,x^6+403200\,x^4-3969000\,x^2+68762925}{6807529575\,x^9}J_0(x)Y_0(x) +$$

$$+\frac{131072\,x^{10}-49152\,x^8+138240\,x^6-1008000\,x^4+13891500\,x^2-618866325}{6807529575\,x^{10}}J_0(x)Y_1(x) +$$

$$+\frac{131072\,x^8-49152\,x^6+138240\,x^4-1008000\,x^2+13891500}{6807529575\,x^8}J_1(x)Y_0(x) -$$

$$-\frac{262144\,x^{10}-32768\,x^8+55296\,x^6-288000\,x^4+3087000\,x^2-56260575}{6807529575\,x^9}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^{2n+1}} = \frac{1}{x^{2n+1}} \left\{ -\frac{x^2}{4n^2-1}J_0(x)Y_0(x) - \left[\frac{x}{2n+1} + \frac{4nx^3}{(4n^2-1)^2}\right]J_0(x)Y_1(x) + \right.$$

$$\left. +\frac{4nx^3}{(4n^2-1)^2}J_1(x)Y_0(x) + \frac{x^2}{(2n+1)^2}J_1(x)Y_1(x) \right\} - \frac{8n}{(2n+1)^2(2n-1)}\int \frac{J_0(x)Y_1(x)\,dx}{x^{2n-1}}$$

e) $\int x^{2n} J_1(x)Y_0(x)\,dx$:

$$\int x^2 J_1(x)Y_0(x)\,dx = -\frac{x^3}{4}J_0(x)Y_1(x) + \frac{x^3}{4}J_1(x)Y_0(x) + \frac{x^2}{2}J_1(x)Y_1(x)$$

$$\int x^4 J_1(x)Y_0(x)\,dx =$$

$$= -\frac{x^4}{6}J_0(x)Y_0(x) - \frac{3\,x^5 - 8\,x^3}{24}J_0(x)Y_1(x) + \frac{3\,x^5 + 8\,x^3}{24}J_1(x)Y_0(x) + \frac{x^4 - 2\,x^2}{3}J_1(x)Y_1(x)$$

$$\int x^6 J_1(x)Y_0(x)\,dx = -\frac{x^6 - 4\,x^4}{5}J_0(x)Y_0(x) - \frac{5\,x^7 - 36\,x^5 + 96\,x^3}{60}J_0(x)Y_1(x) +$$

$$+ \frac{5\,x^7 + 36\,x^5 - 96\,x^3}{60}J_1(x)Y_0(x) + \frac{3\,x^6 - 16\,x^4 + 32\,x^2}{10}J_1(x)Y_1(x)$$

$$\int x^8 J_1(x)Y_0(x)\,dx = -\frac{15\,x^8 - 144\,x^6 + 576\,x^4}{70}J_0(x)Y_0(x) -$$

$$- \frac{35\,x^9 - 480\,x^7 + 3456\,x^5 - 9216\,x^3}{560}J_0(x)Y_1(x) +$$

$$+ \frac{35\,x^9 + 480\,x^7 - 3456\,x^5 + 9216\,x^3}{560}J_1(x)Y_0(x) +$$

$$+ \frac{10\,x^8 - 108\,x^6 + 576\,x^4 - 1152\,x^2}{35}J_1(x)Y_1(x)$$

$$\int x^{10} J_1(x)Y_0(x)\,dx = -\frac{14\,x^{10} - 240\,x^8 + 2304\,x^6 - 9216\,x^4}{63}J_0(x)Y_0(x) -$$

$$- \frac{63\,x^{11} - 1400\,x^9 + 19200\,x^7 - 138240\,x^5 + 368640\,x^3}{1260}J_0(x)Y_1(x) +$$

$$+ \frac{63\,x^{11} + 1400\,x^9 - 19200\,x^7 + 138240\,x^5 - 368640\,x^3}{1260}J_1(x)Y_0(x) +$$

$$+ \frac{35\,x^{10} - 640\,x^8 + 6912\,x^6 - 36864\,x^4 + 73728\,x^2}{126}J_1(x)Y_1(x)$$

$$\int x^{12} J_1(x)Y_0(x)\,dx = -\frac{105\,x^{12} - 2800\,x^{10} + 48000\,x^8 - 460800\,x^6 + 1843200\,x^4}{462}J_0(x)Y_0(x) -$$

$$- \frac{77\,x^{13} - 2520\,x^{11} + 56000\,x^9 - 768000\,x^7 + 5529600\,x^5 - 14745600\,x^3}{1848}J_0(x)Y_1(x) +$$

$$+ \frac{77\,x^{13} + 2520\,x^{11} - 56000\,x^9 + 768000\,x^7 - 5529600\,x^5 + 14745600\,x^3}{1848}J_1(x)Y_0(x) +$$

$$+ \frac{63\,x^{12} - 1750\,x^{10} + 32000\,x^8 - 345600\,x^6 + 1843200\,x^4 - 3686400\,x^2}{231}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+2} J_1(x)Y_0(x)\,dx = x^{2n}\left[-\frac{nx^2}{2(2n+1)}J_0(x)Y_0(x) - \frac{x^3}{4(n+1)}J_0(x)Y_1(x) - \right.$$

$$\left. + \left(\frac{n(n+1)}{2n+1} + \frac{x^2}{4(n+1)}\right)xJ_1(x)Y_0(x) + \frac{(n+1)x^2}{2(2n+1)}J_1(x)Y_1(x)\right] - \frac{2n^2(n+1)}{2n+1}\int x^{2n} J_1(x)Y_0(x)\,dx$$

f) $\int x^{-2n-1} J_1(x)Y_0(x)\,dx$:

$$\int \frac{J_1(x)Y_0(x)\,dx}{x} = xJ_0(x)Y_0(x) - J_1(x)Y_0(x) + xJ_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^3} =$$

$$= -\frac{8\,x^2+3}{9x}J_0(x)Y_0(x) + \frac{4}{9}J_0(x)Y_1(x) + \frac{4\,x^2-3}{9x^2}J_1(x)Y_0(x) - \frac{8\,x^2-1}{9x}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^5} = \frac{128\,x^4+48\,x^2-45}{675\,x^3}J_0(x)Y_0(x) - \frac{64\,x^2-24}{675\,x^2}J_0(x)Y_1(x)-$$

$$-\frac{64\,x^4-24\,x^2+135}{675\,x^4}J_1(x)Y_0(x) + \frac{128\,x^4-16\,x^2+27}{675\,x^3}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^7} = -\frac{1024\,x^6+384\,x^4-360\,x^2+1575}{55125\,x^5}J_0(x)Y_0(x)+$$

$$+\frac{512\,x^4-192\,x^2+540}{55125\,x^4}J_0(x)Y_1(x) + \frac{512\,x^6-192\,x^4+540\,x^2-7875}{55125\,x^6}J_1(x)Y_0(x)-$$

$$-\frac{1024\,x^6-128\,x^4+216\,x^2-1125}{55125\,x^5}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^9} = \frac{32768\,x^8+12288\,x^6-11520\,x^4+50400\,x^2-496125}{31255875\,x^7}J_0(x)Y_0(x)-$$

$$-\frac{16384\,x^6-6144\,x^4+17280\,x^2-126000}{31255875\,x^6}J_0(x)Y_1(x)-$$

$$-\frac{16384\,x^8-6144\,x^6+17280\,x^4-126000\,x^2+3472875}{31255875\,x^8}J_1(x)Y_0(x)+$$

$$+\frac{32768\,x^8-4096\,x^6+6912\,x^4-36000\,x^2+385875}{31255875\,x^7}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^{11}} =$$

$$= -\frac{262144\,x^{10}+98304\,x^8-92160\,x^6+403200\,x^4-3969000\,x^2+68762925}{6807529575\,x^9}J_0(x)Y_0(x)+$$

$$+\frac{131072\,x^8-49152\,x^6+138240\,x^4-1008000\,x^2+13891500}{6807529575\,x^8}J_0(x)Y_1(x)+$$

$$+\frac{131072\,x^{10}-49152\,x^8+138240\,x^6-1008000\,x^4+13891500\,x^2-618866325}{6807529575\,x^{10}}J_1(x)Y_0(x)-$$

$$-\frac{262144\,x^{10}-32768\,x^8+55296\,x^6-288000\,x^4+3087000\,x^2-56260575}{6807529575\,x^9}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^{2n+1}} = \frac{1}{x^{2n+1}}\left\{-\frac{x^2}{4n^2-1}J_0(x)Y_0(x) + \frac{4nx^3}{(4n^2-1)^2}J_0(x)Y_1(x)-\right.$$

$$\left.-\left[\frac{x}{2n+1}+\frac{4nx^3}{(4n^2-1)^2}\right]J_1(x)Y_0(x) + \frac{x^2}{(2n+1)^2}J_1(x)Y_1(x)\right\} - \frac{8n}{(2n-1)(2n+1)^2}\int \frac{J_1(x)Y_0(x)\,dx}{x^{2n-1}}$$

g) $\int x^{2n+1} J_1(x)Y_1(x)\,dx$:

$$\int x\, J_1(x)Y_1(x)\,dx = \frac{x^2}{2}J_0(x)Y_0(x) - xJ_0(x)Y_1(x) + \frac{x^2}{2}J_1(x)Y_1(x)$$

$$\int x^3\, J_1(x)Y_1(x)\,dx =$$

$$= \frac{x^4}{6}J_0(x)Y_0(x) - \frac{x^3}{3}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{x^4 + 4\,x^2}{6}J_1(x)Y_1(x)$$

$$\int x^5\, J_1(x)Y_1(x)\,dx = \frac{x^6 - 4\,x^4}{10}J_0(x)Y_0(x)-$$

$$-\frac{3\,x^5 - 8\,x^3}{10}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{x^6 + 8\,x^4 - 16\,x^2}{10}J_1(x)Y_1(x)$$

$$\int x^7\, J_1(x)Y_1(x)\,dx = \frac{5\,x^8 - 48\,x^6 + 192\,x^4}{70}J_0(x)Y_0(x)-$$

$$-\frac{10\,x^7 - 72\,x^5 + 192\,x^3}{35}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{5\,x^8 + 72\,x^6 - 384\,x^4 + 768\,x^2}{70}J_1(x)Y_1(x)$$

$$\int x^9\, J_1(x)Y_1(x)\,dx = \frac{7\,x^{10} - 120\,x^8 + 1152\,x^6 - 4608\,x^4}{126}J_0(x)Y_0(x)-$$

$$-\frac{35\,x^9 - 480\,x^7 + 3456\,x^5 - 9216\,x^3}{126}[J_0(x)Y_1(x) + J_1(x)Y_0(x)]+$$

$$+\frac{7\,x^{10} + 160\,x^8 - 1728\,x^6 + 9216\,x^4 - 18432\,x^2}{126}J_1(x)Y_1(x)$$

$$\int x^{11}\, J_1(x)Y_1(x)\,dx = \frac{21\,x^{12} - 560\,x^{10} + 9600\,x^8 - 92160\,x^6 + 368640\,x^4}{462}J_0(x)Y_0(x)-$$

$$-\frac{63\,x^{11} - 1400\,x^9 + 19200\,x^7 - 138240\,x^5 + 368640\,x^3}{231}[J_0(x)Y_1(x) + J_1(x)Y_0(x)]+$$

$$+\frac{21\,x^{12} + 700\,x^{10} - 12800\,x^8 + 138240\,x^6 - 737280\,x^4 + 1474560\,x^2}{462}J_1(x)Y_1(x)$$

$$\int x^{13}\, J_1(x)Y_1(x)\,dx =$$

$$= \frac{11\,x^{14} - 420\,x^{12} + 11200\,x^{10} - 192000\,x^8 + 1843200\,x^6 - 7372800\,x^4}{286}J_0(x)Y_0(x)-$$

$$-\frac{77\,x^{13} - 2520\,x^{11} + 56000\,x^9 - 768000\,x^7 + 5529600\,x^5 - 14745600\,x^3}{286}[J_0(x)Y_1(x) + J_1(x)Y_0(x)]+$$

$$+\frac{11\,x^{14} + 504\,x^{12} - 14000\,x^{10} + 256000\,x^8 - 2764800\,x^6 + 14745600\,x^4 - 29491200\,x^2}{286}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+1} J_1(x)Y_1(x)\,dx =$$

$$= \frac{x^{2n}}{4n+2}\left\{x^2 J_0(x)Y_0(x) - (n+1)x\,[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + [2n(n+1) + x^2]J_1(x)Y_1(x)\right\} -$$

$$-\frac{2n(n^2 - 1)}{2n+1}\int x^{2n-1} J_1(x)Y_1(x)\,dx$$

h) $\int x^{-2n} J_1(x)Y_1(x)\,dx$:

Exception:

$$\int \frac{J_1(x)Y_1(x)\,dx}{x} = -\frac{J_0(x)Y_0(x) + J_1(x)Y_1(x)}{2}$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^2} = \frac{2x}{3}J_0(x)Y_0(x) - \frac{1}{3}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{2x^2-1}{3x}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^4} =$$

$$= -\frac{16\,x^2 + 6}{45x}J_0(x)Y_0(x) + \frac{8\,x^2 - 3}{45x^2}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] - \frac{16\,x^4 - 2\,x^2 + 9}{45x^3}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^6} = \frac{256\,x^4 + 96\,x^2 - 90}{4725\,x^3}J_0(x)Y_0(x)-$$

$$-\frac{128\,x^4 - 48\,x^2 + 135}{4725\,x^4}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{256\,x^6 - 32\,x^4 + 54\,x^2 - 675}{4725\,x^5}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^8} = -\frac{2048\,x^6 + 768\,x^4 - 720\,x^2 + 3150}{496125\,x^5}J_0(x)Y_0(x)+$$

$$+\frac{1024\,x^6 - 384\,x^4 + 1080\,x^2 - 7875}{496125\,x^6}[J_0(x)Y_1(x) + J_1(x)Y_0(x)]-$$

$$-\frac{2048\,x^8 - 256\,x^6 + 432\,x^4 - 2250\,x^2 + 55125}{496125\,x^7}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^{10}} = \frac{65536\,x^8 + 24576\,x^6 - 23040\,x^4 + 100800\,x^2 - 992250}{343814625\,x^7}J_0(x)Y_0(x)-$$

$$-\frac{32768\,x^8 - 12288\,x^6 + 34560\,x^4 - 252000\,x^2 + 3472875}{343814625\,x^8}[J_0(x)Y_1(x) + J_1(x)Y_0(x)]+$$

$$+\frac{65536\,x^{10} - 8192\,x^8 + 13824\,x^6 - 72000\,x^4 + 771750\,x^2 - 31255875}{343814625\,x^9}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^{12}} =$$

$$= -\frac{524288\,x^{10} + 196608\,x^8 - 184320\,x^6 + 806400\,x^4 - 7938000\,x^2 + 137525850}{88497884475\,x^9}J_0(x)Y_0(x)+$$

$$+\frac{262144\,x^{10} - 98304\,x^8 + 276480\,x^6 - 2016000\,x^4 + 27783000\,x^2 - 618866325}{88497884475\,x^{10}}[J_0(x)Y_1(x) + J_1(x)Y_0(x)]-$$

$$-\frac{524288\,x^{12} - 65536\,x^{10} + 110592\,x^8 - 576000\,x^6 + 6174000\,x^4 - 112521150\,x^2 + 6807529575}{88497884475\,x^{11}}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^{2n+2}} =$$

$$= -\frac{2x^2 J_0(x)Y_0(x) + (2n-1)x[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + (4n^2 - 1 + 2x^2)J_1(x)Y_1(x)}{(2n+3)(4n^2-1)\,x^{2n+1}} -$$

$$- \frac{8n}{(2n+3)(4n^2-1)}\int \frac{J_1(x)Y_1(x)\,dx}{x^{2n}}$$

2.2. Bessel Functions with different Arguments αx and βx :

See also [10], 4. - 6. .

One-step recurrence formulas:

Let

$$\alpha^2 + \beta^2 = \sigma \quad \text{and} \quad \alpha^2 - \beta^2 = \Delta$$

and

$$Z_{\mu\nu,UW}^{(m)} = \int x^m U_\mu(\alpha x) W_\nu(\beta x)\, dx \ ,$$

then the following systems hold:

$$Z_{00,JJ}^{(2n+3)} = x^{2n+2} \cdot Z_{00,JJ}^{(1)} - \frac{2(n+1)}{\Delta} \left[\alpha Z_{10,JJ}^{(2n+2)} - \beta Z_{01,JJ}^{(2n+2)} \right]$$

$$Z_{11,JJ}^{(2n+3)} = x^{2n+2} \cdot Z_{11,JJ}^{(1)} - \frac{2(n+1)}{\Delta} \left[\beta Z_{10,JJ}^{(2n+2)} - \alpha Z_{01,JJ}^{(2n+2)} \right]$$

$$Z_{01,JJ}^{(2n+2)} = x^{2n} \cdot Z_{01,JJ}^{(2)} - \frac{4n\beta}{\Delta^2} \left[\beta Z_{01,JJ}^{(2n)} - \alpha Z_{10,JJ}^{(2n)} \right] - \frac{2n}{\Delta} \left[\beta Z_{00,JJ}^{(2n+1)} + \alpha Z_{11,JJ}^{(2n+1)} \right]$$

$$Z_{10,JJ}^{(2n+2)} = x^{2n} \cdot Z_{10,JJ}^{(2)} + \frac{4n\alpha}{\Delta^2} \left[\beta Z_{01,JJ}^{(2n)} - \alpha Z_{10,JJ}^{(2n)} \right] + \frac{2n}{\Delta} \left[\alpha Z_{00,JJ}^{(2n+1)} + \beta Z_{11,JJ}^{(2n+1)} \right]$$

$$Z_{00,II}^{(2n+3)} = x^{2n+2} \cdot Z_{00,II}^{(1)} - \frac{2(n+1)}{\Delta} \left[\alpha Z_{10,II}^{(2n+2)} - \beta Z_{01,II}^{(2n+2)} \right]$$

$$Z_{11,II}^{(2n+3)} = x^{2n+2} \cdot Z_{11,II}^{(1)} + \frac{2(n+1)}{\Delta} \left[\beta Z_{10,II}^{(2n+2)} - \alpha Z_{01,II}^{(2n+2)} \right]$$

$$Z_{01,II}^{(2n+2)} = x^{2n} \cdot Z_{01,II}^{(2)} + \frac{4n\beta}{\Delta^2} \left[\beta Z_{01,II}^{(2n)} - \alpha Z_{10,II}^{(2n)} \right] + \frac{2n}{\Delta} \left[\beta Z_{00,II}^{(2n+1)} - \alpha Z_{11,II}^{(2n+1)} \right]$$

$$Z_{10,II}^{(2n+2)} = x^{2n} \cdot Z_{10,II}^{(2)} - \frac{4n\alpha}{\Delta^2} \left[\beta Z_{01,II}^{(2n)} - \alpha Z_{10,II}^{(2n)} \right] - \frac{2n}{\Delta} \left[\alpha Z_{00,II}^{(2n+1)} - \beta Z_{11,II}^{(2n+1)} \right]$$

$$Z_{00,KK}^{(2n+3)} = x^{2n+2} \cdot Z_{00,KK}^{(1)} + \frac{2(n+1)}{\Delta} \left[\alpha Z_{10,KK}^{(2n+2)} - \beta Z_{01,KK}^{(2n+2)} \right]$$

$$Z_{11,KK}^{(2n+3)} = x^{2n+2} \cdot Z_{11,KK}^{(1)} - \frac{2(n+1)}{\Delta} \left[\beta Z_{10,KK}^{(2n+2)} - \alpha Z_{01,KK}^{(2n+2)} \right]$$

$$Z_{01,KK}^{(2n+2)} = x^{2n} \cdot Z_{01,KK}^{(2)} + \frac{4n\beta}{\Delta^2} \left[\beta Z_{01,KK}^{(2n)} - \alpha Z_{10,KK}^{(2n)} \right] - \frac{2n}{\Delta} \left[\beta Z_{00,KK}^{(2n+1)} - \alpha Z_{11,KK}^{(2n+1)} \right]$$

$$Z_{10,KK}^{(2n+2)} = x^{2n} \cdot Z_{10,KK}^{(2)} - \frac{4n\alpha}{\Delta^2} \left[\beta Z_{01,KK}^{(2n)} - \alpha Z_{10,KK}^{(2n)} \right] + \frac{2n}{\Delta} \left[\alpha Z_{00,KK}^{(2n+1)} - \beta Z_{11,KK}^{(2n+1)} \right]$$

$$Z_{00,JI}^{(2n+3)} = x^{2n+2} \cdot Z_{00,JI}^{(1)} - \frac{2(n+1)}{\sigma} \left[\alpha Z_{10,JI}^{(2n+2)} + \beta Z_{01,JI}^{(2n+2)} \right]$$

$$Z_{11,JI}^{(2n+3)} = x^{2n+2} \cdot Z_{11,JI}^{(1)} - \frac{2(n+1)}{\sigma} \left[\beta Z_{10,JI}^{(2n+2)} - \alpha Z_{01,JI}^{(2n+2)} \right]$$

$$Z_{01,JI}^{(2n+2)} = x^{2n} \cdot Z_{01,JI}^{(2)} + \frac{4n\beta}{\sigma^2} \left[\beta Z_{01,JI}^{(2n)} + \alpha Z_{10,JI}^{(2n)} \right] - \frac{2n}{\sigma} \left[\beta Z_{00,JI}^{(2n+1)} + \alpha Z_{11,JI}^{(2n+1)} \right]$$

$$Z_{10,JI}^{(2n+2)} = x^{2n} \cdot Z_{10,JI}^{(2)} - \frac{4n\alpha}{\sigma^2} \left[\beta Z_{01,JI}^{(2n)} + \alpha Z_{10,JI}^{(2n)} \right] + \frac{2n}{\sigma} \left[\alpha Z_{00,JI}^{(2n+1)} - \beta Z_{11,JI}^{(2n+1)} \right]$$

$$Z_{00,JK}^{(2n+3)} = x^{2n+2} \cdot Z_{00,JK}^{(1)} - \frac{2(n+1)}{\sigma} \left[\alpha Z_{10,JK}^{(2n+2)} - \beta Z_{01,JK}^{(2n+2)} \right]$$

$$Z_{11,JK}^{(2n+3)} = x^{2n+2} \cdot Z_{11,JK}^{(1)} + \frac{2(n+1)}{\sigma} \left[\beta Z_{10,JK}^{(2n+2)} + \alpha Z_{01,JK}^{(2n+2)} \right]$$

$$Z_{01,JK}^{(2n+2)} = x^{2n} \cdot Z_{01,JK}^{(2)} + \frac{4n\beta}{\sigma^2} \left[\beta Z_{01,JK}^{(2n)} - \alpha Z_{10,JK}^{(2n)} \right] + \frac{2n}{\sigma} \left[\beta Z_{00,JK}^{(2n+1)} - \alpha Z_{11,JK}^{(2n+1)} \right]$$

$$Z_{10,JK}^{(2n+2)} = x^{2n} \cdot Z_{10,JK}^{(2)} + \frac{4n\alpha}{\sigma^2} \left[\beta Z_{01,JK}^{(2n)} - \alpha Z_{10,JK}^{(2n)} \right] + \frac{2n}{\sigma} \left[\alpha Z_{00,JK}^{(2n+1)} + \beta Z_{11,JK}^{(2n+1)} \right]$$

$$Z_{00,IK}^{(2n+3)} = x^{2n+2} \cdot Z_{00,IK}^{(1)} - \frac{2(n+1)}{\Delta} \left[\alpha Z_{10,IK}^{(2n+2)} + \beta Z_{01,IK}^{(2n+2)} \right]$$

$$Z_{11,IK}^{(2n+3)} = x^{2n+2} \cdot Z_{11,IK}^{(1)} - \frac{2(n+1)}{\Delta} \left[\beta Z_{10,IK}^{(2n+2)} + \alpha Z_{01,IK}^{(2n+2)} \right]$$

$$Z_{01,IK}^{(2n+2)} = x^{2n} \cdot Z_{01,IK}^{(2)} + \frac{4n\beta}{\Delta^2} \left[\beta Z_{01,IK}^{(2n)} + \alpha Z_{10,IK}^{(2n)} \right] - \frac{2n}{\Delta} \left[\beta Z_{00,IK}^{(2n+1)} + \alpha Z_{11,IK}^{(2n+1)} \right]$$

$$Z_{10,IK}^{(2n+2)} = x^{2n} \cdot Z_{10,IK}^{(2)} + \frac{4n\alpha}{\Delta^2} \left[\beta Z_{01,IK}^{(2n)} + \alpha Z_{10,IK}^{(2n)} \right] - \frac{2n}{\Delta} \left[\alpha Z_{00,IK}^{(2n+1)} + \beta Z_{11,IK}^{(2n+1)} \right]$$

2.2.1. Integrals of the type $\int x^{2n} \cdot J_\nu(\alpha x) \cdot J_\nu(\beta x)\, dx$ and $\int x^{2n} \cdot I_\nu(\alpha x) \cdot I_\nu(\beta x)\, dx$

a) Basic Integrals:

In the case $\alpha = \beta$ it was necessary to define the new functions $\Theta(x)$ and $\Omega(x)$ (see page 278). The more there is no solution with already defined functions expected in the described class of functions if $\alpha \neq \beta$. Let $0 < \beta < \alpha$ and $\gamma = \beta/\alpha < 1$. The integrals may be reduced to the single parameter γ by

$$\int x^{2n} \cdot Z_\nu(\alpha x) \cdot Z_\nu(\beta x)\, dx = \alpha^{-2n-1} \int t^{2n} \cdot Z_\nu(t) \cdot Z_\nu(\gamma t)\, dt , \quad t = \alpha x .$$

The functions $\Theta(x)$ and $\Omega(x)$ from page 278 are generalized to

$$\Theta_0(x;\gamma) = \int_0^x J_0(s) \cdot J_0(\gamma s)\, ds \qquad \text{and} \qquad \Omega_0(x;\gamma) = \int_0^x I_0(s) \cdot I_0(\gamma s)\, ds .$$

From (for instance)

$$I_0(s) \cdot I_0(\gamma s) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\sum_{l=0}^{k} \binom{k}{l}^2 \gamma^2 \right) \left(\frac{s}{2} \right)^{2k}$$

one gets with [1], 22.3.1 the power series

$$\Theta_0(x;\gamma) = \sum_{k=0}^{\infty} \frac{(\gamma^2 - 1)^k}{(k!)^2 \cdot 4^k \cdot (2k+1)} \cdot P_k \left(\frac{1+\gamma^2}{1-\gamma^2} \right) x^{2k+1}$$

and

$$\Omega_0(x;\gamma) = \sum_{k=0}^{\infty} \frac{(1 - \gamma^2)^k}{(k!)^2 \cdot 4^k \cdot (2k+1)} \cdot P_k \left(\frac{1+\gamma^2}{1-\gamma^2} \right) x^{2k+1} ,$$

where

$$P_n(x) = \frac{(2n)!}{2^n \cdot (n!)^2} x^n + \ldots$$

denotes the Legendre polynomials. Their values may be found by the recurrence relation

$$P_{n+1} \left(\frac{1+\gamma^2}{1-\gamma^2} \right) = \frac{2n+1}{n+1} \cdot \frac{1+\gamma^2}{1-\gamma^2} \cdot P_n \left(\frac{1+\gamma^2}{1-\gamma^2} \right) - \frac{n}{n+1} P_{n-1} \left(\frac{1+\gamma^2}{1-\gamma^2} \right)$$

with

$$P_0 \left(\frac{1+\gamma^2}{1-\gamma^2} \right) = 1 \qquad \text{and} \qquad P_1 \left(\frac{1+\gamma^2}{1-\gamma^2} \right) = \frac{1+\gamma^2}{1-\gamma^2} .$$

Some first terms of the power series:

$$\Theta_0(x;\gamma) = x - \frac{\gamma^2 + 1}{12} x^3 + \frac{\gamma^4 + 4\gamma^2 + 1}{320} x^5 - \frac{\gamma^6 + 9\gamma^4 + 9\gamma^2 + 1}{16128} x^7 + \frac{\gamma^8 + 16\gamma^6 + 36\gamma^4 + 16\gamma^2 + 1}{1327104} x^9 -$$

$$-\frac{\gamma^{10} + 25\,\gamma^8 + 100\,\gamma^6 + 100\,\gamma^4 + 25\,\gamma^2 + 1}{162201600}\,x^{11} + \frac{\gamma^{12} + 36\,\gamma^{10} + 225\,\gamma^8 + 400\,\gamma^6 + 225\,\gamma^4 + 36\,\gamma^2 + 1}{27603763200}\,x^{13} -$$

$$-\frac{\gamma^{14} + 49\,\gamma^{12} + 441\,\gamma^{10} + 1225\,\gamma^8 + 1225\,\gamma^6 + 441\,\gamma^4 + 49\,\gamma^2 + 1}{6242697216000}\,x^{15} +$$

$$+\frac{\gamma^{16} + 64\,\gamma^{14} + 784\,\gamma^{12} + 3136\,\gamma^{10} + 4900\,\gamma^8 + 3136\,\gamma^6 + 784\,\gamma^4 + 64\,\gamma^2 + 1}{1811214552268800}\,x^{17} - \ldots$$

If $x > \gamma x >> 1$ one has

$$\Omega_0(x;\gamma) \quad \approx \quad \frac{e^{(1+\gamma)x}}{2\pi\sqrt{\gamma}(1+\gamma)x} \; .$$

Let $\Theta_0(x;\gamma)$ be computed with n decimal signs, then in the case $x > \gamma x >> 1$ the loss of significant digits can be expected. Only about

$$n - \lg \frac{e^{(1+\gamma)x}}{2\pi\sqrt{\gamma}(1+\gamma)x}$$

significant digits are left.

The upper integral with primary parameters:

$$\int_0^x J_0(\alpha s) \cdot J_0(\beta s)\,ds = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 \cdot 4^k \cdot (2k+1)} \cdot (\alpha^2 - \beta^2)^k \cdot P_k\left(\frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2}\right) x^{2k+1}$$

Asymptotic series for $x > \gamma x >> 1$:

$$\Theta_0(x;\gamma) \sim \frac{2}{\pi}\,\mathbf{K}(\gamma) + \frac{1}{\pi\sqrt{\gamma}\,x}\left[\frac{\sin(1-\gamma)x}{1-\gamma} - \frac{\cos(\gamma+1)x}{\gamma+1}\right] +$$

$$+\frac{1}{8\pi\gamma^{3/2}\,x^2}\left[\frac{\gamma^2 - 10\,\gamma + 1}{(1-\gamma)^2}\cos(1-\gamma)x - \frac{\gamma^2 + 10\,\gamma + 1}{(\gamma+1)^2}\sin(\gamma+1)x\right] +$$

$$+\frac{1}{128\pi\gamma^{5/2}\,x^3}\left[\frac{9\,\gamma^4 + 52\,\gamma^3 + 342\,\gamma^2 + 52\,\gamma + 9}{(\gamma+1)^3}\cos(\gamma+1)x - \frac{9\,\gamma^4 - 52\,\gamma^3 + 342\,\gamma^2 - 52\,\gamma + 9}{(1-\gamma)^3}\sin(1-\gamma)x\right] +$$

$$+\frac{3}{1024\pi\gamma^{7/2}\,x^4}\left[\frac{25\,\gamma^6 + 150\,\gamma^5 + 503\,\gamma^4 + 2804\,\gamma^3 + 503\,\gamma^2 + 150\,\gamma + 25}{(\gamma+1)^4}\sin(\gamma+1)x - \right.$$

$$\left. -\frac{25\,\gamma^6 - 150\,\gamma^5 + 503\,\gamma^4 - 2804\,\gamma^3 + 503\,\gamma^2 - 150\,\gamma + 25}{(1-\gamma)^4}\cos(1-\gamma)x\right] + \ldots \; ,$$

where $\mathbf{K}$ denotes the complete elliptic integral of the first kind, see [1] or [5]. Particulary follows

$$\lim_{x \to \infty} \Theta_0(x;\gamma) = \frac{2}{\pi}\,\mathbf{K}(\gamma) \qquad ([4], 2.12.31.1.) \; .$$

Some values of this limit:

γ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	1.0000	1.0000	1.0001	1.0002	1.0004	1.0006	1.0009	1.0012	1.0016	1.0020
0.1	1.0025	1.0030	1.0036	1.0043	1.0050	1.0057	1.0065	1.0073	1.0083	1.0092
0.2	1.0102	1.0113	1.0124	1.0136	1.0149	1.0162	1.0176	1.0190	1.0205	1.0221
0.3	1.0237	1.0254	1.0272	1.0290	1.0309	1.0329	1.0350	1.0371	1.0394	1.0417
0.4	1.0441	1.0465	1.0491	1.0518	1.0545	1.0574	1.0603	1.0634	1.0665	1.0698
0.5	1.0732	1.0767	1.0803	1.0841	1.0880	1.0920	1.0962	1.1006	1.1051	1.1097
0.6	1.1146	1.1196	1.1248	1.1302	1.1359	1.1417	1.1479	1.1542	1.1609	1.1678
0.7	1.1750	1.1826	1.1905	1.1988	1.2074	1.2166	1.2262	1.2363	1.2470	1.2583
0.8	1.2702	1.2830	1.2965	1.3110	1.3265	1.3432	1.3613	1.3809	1.4023	1.4258
0.9	1.4518	1.4810	1.5139	1.5517	1.5959	1.6489	1.7145	1.8004	1.9232	2.1369

This picture shows $\Theta_0(x; 1/2)$ (solid line), the asymptotic approximation with x^{-1} (long dashes) and with x^{-2} (short dashes):

$\gamma = 1/4$

$\gamma = 1/3$

$\gamma = 2/3$

$\gamma = 3/4$
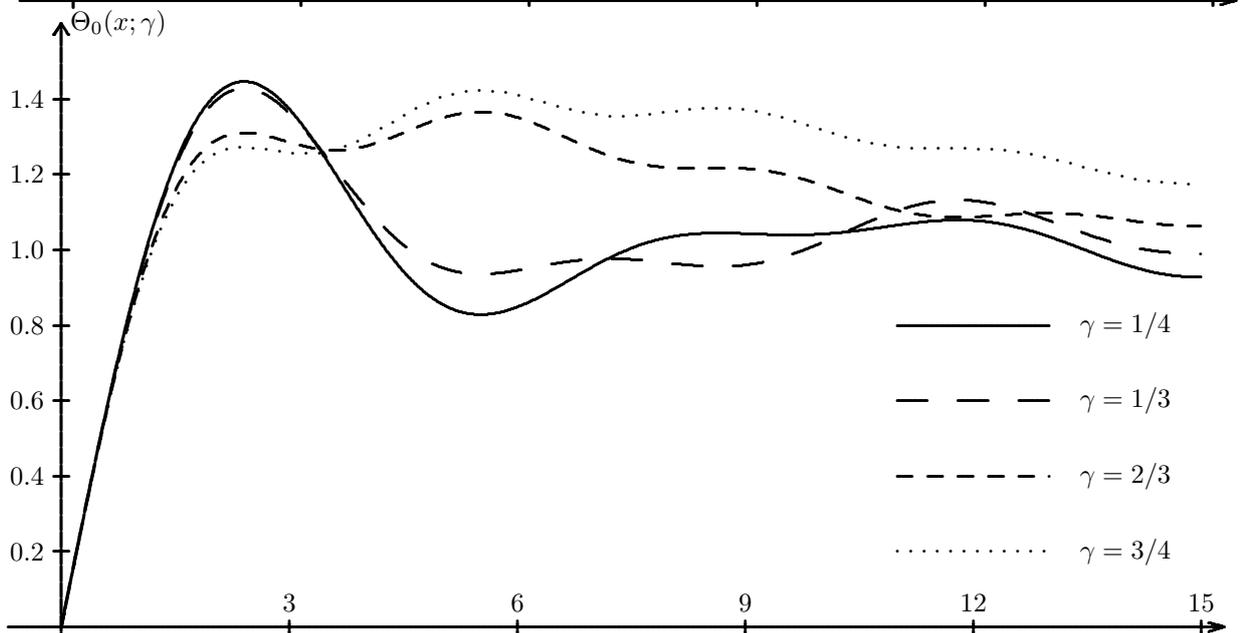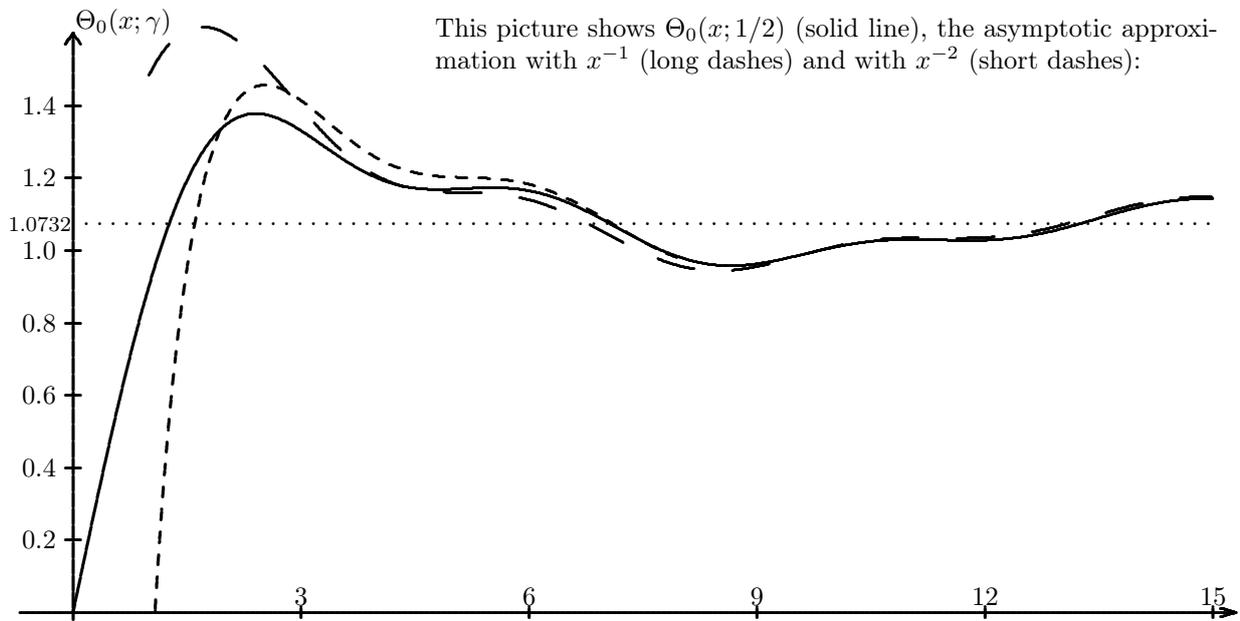
Let

$$\Theta_1(x; \gamma) = \int_0^x J_1(s) \cdot J_1(\gamma s)\, ds \qquad \text{and} \qquad \Omega_1(x; \gamma) = \int_0^x I_1(s) \cdot I_1(\gamma s)\, ds .$$

Power series: From (for instance)

$$I_1(x)\, I_1(\gamma x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\sum_{l=1}^{k} \binom{k}{l} \binom{k}{l-1} \gamma^{2l-1} \right) \left(\frac{x}{2} \right)^{2k}$$

and

$$\sum_{l=1}^{k} \binom{k}{l} \binom{k}{l-1} \gamma^{2l-1} = k\, (1 - \gamma^2)^k\, L_k^{-1} \left(\frac{1 + \gamma^2}{1 + \gamma^2} \right)$$

follows

$$\Theta_1(x; \gamma) = - \sum_{k=1}^{\infty} \frac{2k\, (\gamma^2 - 1)^k}{(k!)^2 \cdot (2k+1)}\, P_{k+1}^{-1} \left(\frac{1 + \gamma^2}{1 - \gamma^2} \right) \left(\frac{x}{2} \right)^{2k+1}$$

and

$$\Omega_1(x; \gamma) = \sum_{k=1}^{\infty} \frac{2k\, (1 - \gamma^2)^k}{(k!)^2 \cdot (2k+1)}\, P_{k+1}^{-1} \left(\frac{1 + \gamma^2}{1 - \gamma^2} \right) \left(\frac{x}{2} \right)^{2k+1} ,$$

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where $P_n^{-1}(x)$ denotes the associated Legendre functions of the first kind. Their values may be found by the recurrence relation, starting with $n = 2$:

$$P_{n+1}^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{2n+1}{n+2} \cdot \frac{1+\gamma^2}{1-\gamma^2} \cdot P_n^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) - \frac{n-1}{n+2} P_{n-1}^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right)$$

from

$$P_1^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{\gamma}{1-\gamma^2} \qquad \text{and} \qquad P_2^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{\gamma^3+\gamma}{(1-\gamma^2)^2} .$$

Some first terms of the power series:

$$\Theta_1(x;\gamma) = \frac{\gamma}{12} x^3 - \frac{\gamma^3+\gamma}{160} x^5 + \frac{\gamma^5+3\gamma^3+\gamma}{5376} x^7 - \frac{\gamma^7+6\gamma^5+6\gamma^3+\gamma}{331776} x^9 + \frac{\gamma^9+10\gamma^7+20\gamma^5+10\gamma^3+\gamma}{32440320} x^{11} -$$

$$- \frac{\gamma^{11}+15\gamma^9+50\gamma^7+50\gamma^5+15\gamma^3+\gamma}{4600627200} x^{13} + \frac{\gamma^{13}+21\gamma^{11}+105\gamma^9+175\gamma^7+105\gamma^5+21\gamma^3+\gamma}{891813888000} x^{15} -$$

$$- \frac{\gamma^{15}+28\gamma^{13}+196\gamma^{11}+490\gamma^9+490\gamma^7+196\gamma^5+28\gamma^3+\gamma}{226401819033600} x^{17} + \dots$$

In the case $x > \gamma x \gg 1$ one has once again $\Omega_1(x;\gamma) \approx e^{(1+\gamma)x}/[2\pi\sqrt{\gamma}(1+\gamma)x]$.
Asymptotic series for $x > \gamma x \gg 1$ ([4], 2.12.31.1. and [5], IX, Definitions):

$$\Theta_1(x;\gamma) \sim \frac{2}{\pi\gamma}\left[\mathbf{K}(\gamma) - \mathbf{E}(\gamma)\right] + \frac{1}{\pi\sqrt{\gamma}}\left[\frac{1}{x}\left(\frac{\cos(1+\gamma)x}{1+\gamma} - \frac{\sin(1-\gamma)x}{1-\gamma}\right) - \right.$$

$$- \frac{1}{8g\,x^2}\left(\frac{3\gamma^2-2\gamma+3}{(1+\gamma)^2}\sin(1+\gamma)x + \frac{3\gamma^2+2\gamma+3}{(1-\gamma)^2}\cos(1-\gamma)x\right) +$$

$$+ \frac{1}{128g^3\,x^3}\left(\frac{15\gamma^4+108\gamma^3-70\gamma^2+108\gamma+15}{(1+\gamma)^3}\cos(1+\gamma)x - \frac{15\gamma^4-108\gamma^3-70\gamma^2-108\gamma+15}{(1-\gamma)^3}\sin(1-\gamma)x\right) -$$

$$- \frac{3}{1024\gamma^3\,x^4}\left(\frac{35\gamma^6+210\gamma^5+909\gamma^4-580\gamma^3+909\gamma^2+210\gamma+35}{(1+\gamma)^4}\sin(1+\gamma)x + \right.$$

$$\left.\left. + \frac{35\gamma^6-210\gamma^5+909\gamma^4+580\gamma^3+909\gamma^2-210\gamma+35}{(1-\gamma)^4}\cos(1-\gamma)x\right) + \dots\right]$$

with the complete elliptic integrals of the first and second kind.
Particulary follows

$$\lim_{x\to\infty}\Theta_1(x;\gamma) = \frac{2}{\pi\gamma}\left[\mathbf{K}(\gamma) - \mathbf{E}(\gamma)\right] \qquad ([4], 2.12.31.1.) .$$

Some values of this limit:

γ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0050	0.0100	0.0150	0.0200	0.0250	0.0300	0.0351	0.0401	0.0451
0.1	0.0502	0.0553	0.0603	0.0654	0.0705	0.0756	0.0808	0.0859	0.0911	0.0963
0.2	0.1015	0.1068	0.1121	0.1174	0.1227	0.1280	0.1334	0.1389	0.1443	0.1498
0.3	0.1554	0.1609	0.1666	0.1722	0.1780	0.1837	0.1895	0.1954	0.2013	0.2073
0.4	0.2134	0.2195	0.2257	0.2319	0.2382	0.2446	0.2511	0.2577	0.2643	0.2711
0.5	0.2779	0.2849	0.2919	0.2991	0.3064	0.3138	0.3214	0.3290	0.3369	0.3448
0.6	0.3530	0.3613	0.3698	0.3785	0.3873	0.3964	0.4058	0.4153	0.4252	0.4353
0.7	0.4457	0.4564	0.4674	0.4789	0.4907	0.5029	0.5157	0.5289	0.5427	0.5571
0.8	0.5721	0.5879	0.6046	0.6221	0.6407	0.6605	0.6816	0.7042	0.7287	0.7552
0.9	0.7844	0.8165	0.8525	0.8934	0.9407	0.9967	1.0654	1.1543	1.2803	1.4971

Some Functions $\Theta_1\,(x;\gamma)$

——— $\gamma = 1/4$ — — $\gamma = 1/3$ — — — $\gamma = 1/2$ ········· $\gamma = 3/4$ — — — — $\gamma = 2/3$
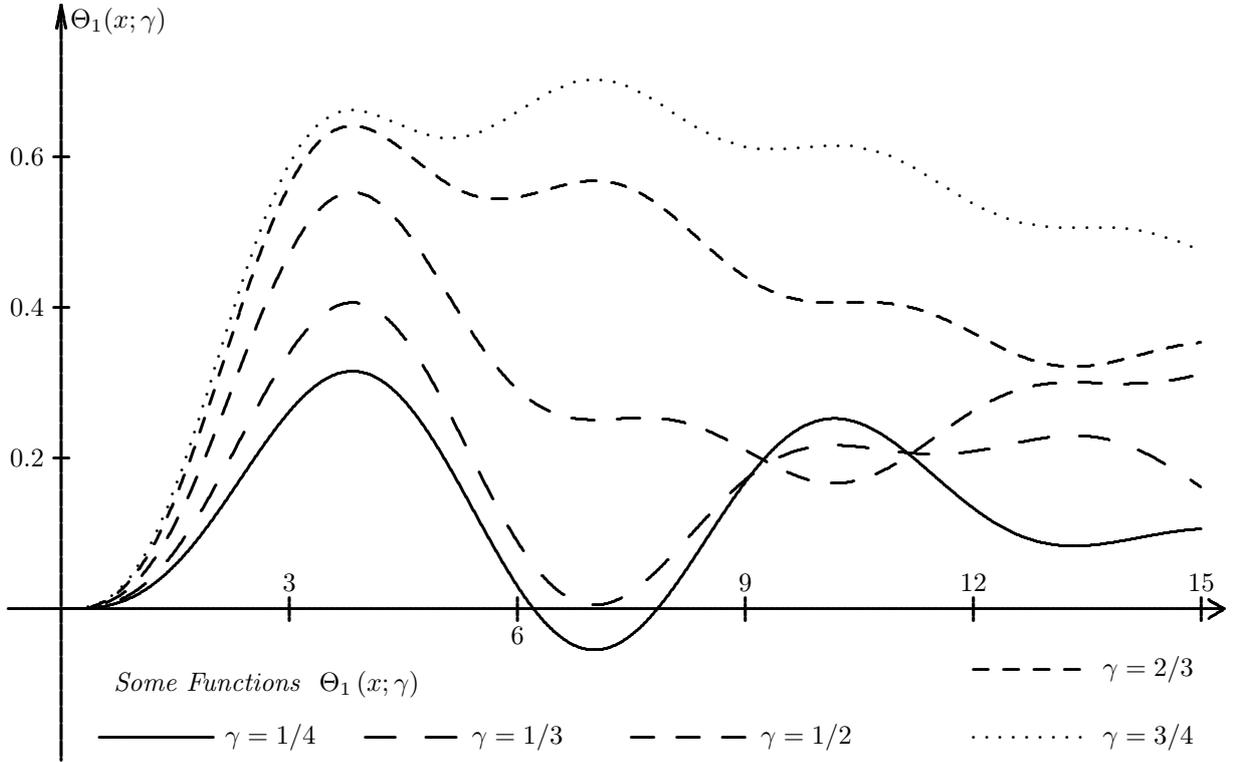
The value of x may be too large to use the power series for $\Theta_0(x;\gamma)$ and γx may be too small to apply the asymptotic formula. In this case

$$\Theta_0(x;\gamma) \sim \frac{2}{\pi}\,\mathbf{K}(\gamma) +$$

$$+\frac{A_0(x;\gamma)\,\cos x\,J_0(\gamma x) + A_1(x;\gamma)\,\cos x\,J_1(\gamma x) + B_0(x;\gamma)\,\sin x\,J_0(\gamma x) + B_1(x;\gamma)\,\sin x\,J_1(\gamma x)}{\sqrt{\pi x}}$$

is applicable. Let

$$A_\mu(x;\gamma) = \sum_{k=0}^{\infty} \frac{a_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k} \qquad \text{and} \qquad B_\mu(x;\gamma) = \sum_{k=0}^{\infty} \frac{b_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k}\ ,$$

then holds

$$a_0^{(0)}(x;\gamma) = -1\ , \quad a_1^{(0)}(x;\gamma) = -\frac{11\gamma^2 + 5}{8}\ , \quad a_2^{(0)}(x;\gamma) = -\frac{31\,\gamma^4 - 926\,\gamma^2 - 129}{128}\ ,$$

$$a_3^{(0)}(x;\gamma) = \frac{3\,\left(\gamma^2 + 15\right)\left(59\,\gamma^4 + 906\,\gamma^2 + 59\right)}{1024}\ ,$$

$$a_4^{(0)}(x;\gamma) = \frac{7125\,\gamma^8 + 15468\,\gamma^6 - 4088898\,\gamma^4 - 8215572\,\gamma^2 - 301035}{32768}\ ,$$

$$a_5^{(0)}(x;\gamma) = -\frac{102165\,\gamma^{10} - 208569\,\gamma^8 + 25390098\,\gamma^6 + 501398862\,\gamma^4 + 469053609\,\gamma^2 + 10896795}{262144}\ ,$$

$$a_6^{(0)}(x;\gamma) =$$

$$-\frac{45\,[84231\,\gamma^{12} - 348490\,\gamma^{10} + 2847497\,\gamma^8 - 451498956\,\gamma^6 - 2481377623\,\gamma^4 - 1343311306\,\gamma^2 - 21362649]}{4194304}$$

$$a_0^{(1)}(x;\gamma) = -\gamma\ , \quad a_1^{(1)}(x;\gamma) = -\frac{\gamma\,\left(\gamma^2 - 17\right)}{8}\ , \quad a_2^{(1)}(x;\gamma) = \frac{\gamma\,\left(9\,\gamma^4 + 206\,\gamma^2 + 809\right)}{128}\ ,$$

$$a_3^{(1)}(x;\gamma) = \frac{\gamma\,\left(75\,\gamma^6 + 143\,\gamma^4 - 24063\,\gamma^2 - 25307\right)}{1024}\ ,$$

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$$a_4^{(1)}(x;\gamma) = -\frac{3\gamma\left(1225\,\gamma^8 - 1892\,\gamma^6 + 201078\,\gamma^4 + 2678812\,\gamma^2 + 1315081\right)}{32768}\,,$$

$$a_5^{(1)}(x;\gamma) = -\frac{3\gamma\left(19845\,\gamma^{10} - 67625\,\gamma^8 + 467314\,\gamma^6 - 58112658\,\gamma^4 - 216355367\,\gamma^2 - 61495829\right)}{262144}\,,$$

$$a_6^{(1)}(x;\gamma) =$$

$$\frac{3\gamma\left(800415\,\gamma^{12} - 3869530\,\gamma^{10} + 12921201\,\gamma^8 + 680167252\,\gamma^6 + 20763422609\,\gamma^4 + 36255061542\,\gamma^2 + 6716005951\right)}{4194304}$$

$$b_0^{(0)}(x;\gamma) = 1\,,\quad b_1^{(0)}(x;\gamma) = -\frac{11\gamma^2 + 5}{8} = a_1^{(0)}\,,\quad b_2^{(0)}(x;\gamma) = \frac{31\,\gamma^4 - 926\,\gamma^2 - 129}{128} = -a_2^{(0)}\,,$$

$$b_3^{(0)}(x;\gamma) = \frac{3\left(\gamma^2 + 15\right)\left(59\,\gamma^4 + 906\,\gamma^2 + 59\right)}{1024} = a_3^{(0)}\,,$$

$$b_4^{(0)}(x;\gamma) = -\frac{7125\,\gamma^8 + 15468\,\gamma^6 - 4088898\,\gamma^4 - 8215572\,\gamma^2 - 301035}{32768} = -a_4^{(0)}\,,$$

$$b_5^{(0)}(x;\gamma) = -\frac{102165\,\gamma^{10} - 208569\,\gamma^8 + 25390098\,\gamma^6 + 501398862\,\gamma^4 + 469053609\,\gamma^2 + 10896795}{262144} = a_5^{(0)}\,,$$

$$b_6^{(0)}(x;\gamma) = -a_6^{(0)} =$$

$$\frac{45\left[84231\,\gamma^{12} - 348490\,\gamma^{10} + 2847497\,\gamma^8 - 451498956\,\gamma^6 - 2481377623\,\gamma^4 - 1343311306\,\gamma^2 - 21362649\right]}{4194304}$$

$$b_0^{(1)}(x;\gamma) = -\gamma\,,\quad b_1^{(1)}(x;\gamma) = \frac{\gamma\left(\gamma^2 - 17\right)}{8} = a_1^{(1)}\,,\quad b_2^{(1)}(x;\gamma) = \frac{\gamma\left(9\,\gamma^4 + 206\,\gamma^2 + 809\right)}{128} = a_2^{(1)}\,,$$

$$b_3^{(1)}(x;\gamma) = -\frac{\gamma\left(75\,\gamma^6 + 143\,\gamma^4 - 24063\,\gamma^2 - 25307\right)}{1024} = -a_3^{(0)}\,,$$

$$b_4^{(1)}(x;\gamma) = -\frac{3\gamma\left(1225\,\gamma^8 - 1892\,\gamma^6 + 201078\,\gamma^4 + 2678812\,\gamma^2 + 1315081\right)}{32768} = a_4^{(1)}\,,$$

$$b_5^{(1)}(x;\gamma) = \frac{3\gamma\left(19845\,\gamma^{10} - 67625\,\gamma^8 + 467314\,\gamma^6 - 58112658\,\gamma^4 - 216355367\,\gamma^2 - 61495829\right)}{262144} = -a_5^{(1)}\,,$$

$$b_6^{(1)}(x;\gamma) = a_6^{(1)} =$$

$$\frac{3\gamma\left(800415\,\gamma^{12} - 3869530\,\gamma^{10} + 12921201\,\gamma^8 + 680167252\,\gamma^6 + 20763422609\,\gamma^4 + 36255061542\,\gamma^2 + 6716005951\right)}{4194304}$$

When $\gamma << 1$ one has approximately

$$A_0(x;\gamma) \approx A_0(x;0) = -1 - \frac{0.625}{x} + \frac{1.0078}{x^2} + \frac{2.5928}{x^3} - \frac{9.1869}{x^4} - \frac{41.568}{x^5} + \frac{229.20}{x^6} + \ldots$$

$$B_0(x;\gamma) \approx B_0(x;0) = 1 - \frac{0.625}{x} - \frac{1.0078}{x^2} + \frac{2.5928}{x^3} + \frac{9.1869}{x^4} - \frac{41.568}{x^5} - \frac{229.20}{x^6} + \ldots$$

$$A_1(x;\gamma) \approx \gamma\,\frac{\partial A_1}{\partial \gamma}(x;0) = \gamma\left[-1 + \frac{2.125}{x} + \frac{6.3203}{x^2} - \frac{24.714}{x^3} - \frac{120.40}{x^4} + \frac{703.76}{x^5} + \frac{4803.7}{x^6} + \ldots\right]$$

$$B_1(x;\gamma) \approx \gamma\,\frac{\partial B_1}{\partial \gamma}(x;0) = \gamma\left[-1 - \frac{2.125}{x} + \frac{6.3203}{x^2} + \frac{24.714}{x^3} - \frac{120.40}{x^4} - \frac{703.76}{x^5} + \frac{4803.7}{x^6} + \ldots\right]$$

$$\left|\frac{a_3^{(0)}(x;0)}{a_2^{(0)}(x;0)}\right| = \left|\frac{b_3^{(0)}(x;0)}{b_2^{(0)}(x;0)}\right| = 2.57\,,\quad \left|\frac{a_4^{(0)}(x;0)}{a_3^{(0)}(x;0)}\right| = 3.54\,,\quad \left|\frac{a_5^{(0)}(x;0)}{a_4^{(0)}(x;0)}\right| = 4.52\,,\quad \left|\frac{a_6^{(0)}(x;0)}{a_5^{(0)}(x;0)}\right| = 5.51$$

The summand $a_k^{(0)}(x;\gamma)/[(1-\gamma^2)^{k+1}\,x^k]$ can be used if $|x| > |a_k^{(0)}(x;\gamma)/a_{k-1}^{(0)}(x;\gamma)|$.
The same holds for $b_k^{(0)}(x;\gamma)$.

Let

$$\Delta_n(x;\gamma) = -\Theta_0(x;\gamma)+$$

$$+\frac{1}{\sqrt{\pi x}}\left[A_0^{(n)}(x;\gamma)\,\cos x\,J_0(\gamma x) + A_1^{(n)}(x;\gamma)\,\cos x\,J_1(\gamma x) + B_0^{(n)}(x;\gamma)\,\sin x\,J_0(\gamma x) + B_1^{(n)}(x;\gamma)\,\sin x\,J_1(\gamma x)\right]$$

with

$$A_\mu^{(n)}(x;\gamma) = \sum_{k=0}^{n}\frac{a_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k}\qquad\text{and}\qquad B_\mu^{(n)}(x;\gamma) = \sum_{k=0}^{n}\frac{b_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k}\ .$$

For the case $\gamma = 0.1$ some of these differences are shown:
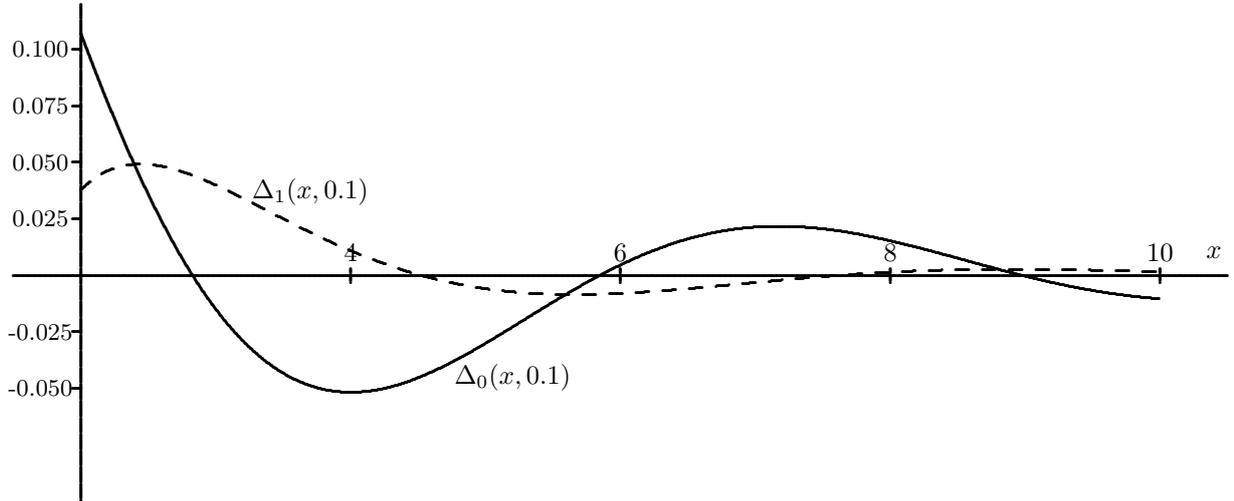
FIGURE 10 :   *Differences $\Delta_0\,(x;\gamma)$ and $\Delta_1\,(x;\gamma)$ with $\gamma = 0.1$*

FIGURE 11 :   *Differences $\Delta_1\,(x;\gamma)$ and $\Delta_2\,(x;\gamma)$ with $\gamma = 0.1$*
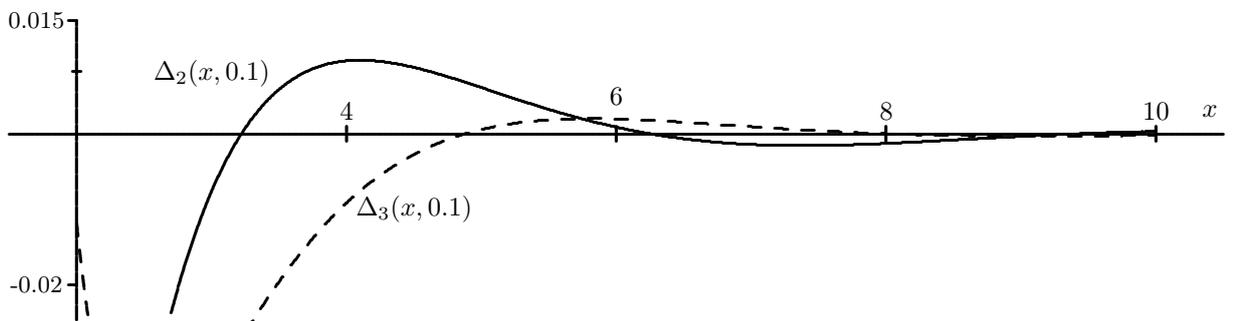
FIGURE 12 :   *Differences $\Delta_2\,(x;\gamma)$ and $\Delta_3\,(x;\gamma)$ with $\gamma = 0.1$*

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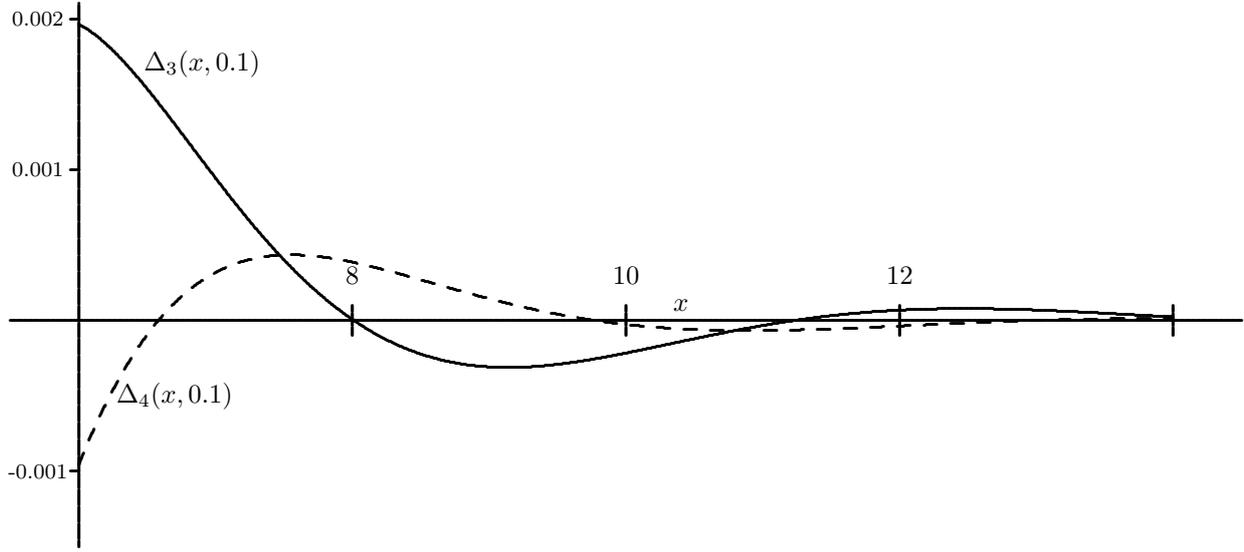
FIGURE 13 :   *Differences $\Delta_3(x;\gamma)$ and $\Delta_4(x;\gamma)$ with $\gamma = 0.1$*

The same way the asymptotic expansion

$$\Theta_1(x;\gamma) \sim \frac{2}{\pi\gamma}\left[\mathbf{K}(\gamma) - \mathbf{E}(\gamma)\right]+$$

$$+\frac{A_0^*(x;\gamma)\ \cos x\, J_0(\gamma x) + A_1^*(x;\gamma)\ \cos x\, J_1(\gamma x) + B_0^*(x;\gamma)\ \sin x\, J_0(\gamma x) + B_1^*(x;\gamma)\ \sin x\, J_1(\gamma x)}{\sqrt{\pi x}}$$

is applicable in the case $x >> 1$ and $\gamma x \approx 1$. Let

$$A_\mu^*(x;\gamma) = \sum_{k=0}^{\infty} \frac{a_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k} \qquad \text{and} \qquad B_\mu^*(x;\gamma) = \sum_{k=0}^{\infty} \frac{b_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k}\ ,$$

then holds

$$a_0^{(0,*)}(x;\gamma) = -\gamma\ , \quad a_1^{(0,*)}(x;\gamma) = -\frac{\gamma\left(3\gamma^2 + 13\right)}{8}\ , \quad a_2^{(0,*)}(x;\gamma) = -\frac{\gamma\left(15\gamma^4 - 382\gamma^2 - 657\right)}{128}\ ,$$

$$a_3^{(0,*)}(x;\gamma) = \frac{3\gamma\left(35\gamma^6 + 327\gamma^4 + 8457\gamma^2 + 7565\right)}{1024}$$

$$a_4^{(0,*)}(x;\gamma) = \frac{3\gamma\left(1575\gamma^8 - 860\gamma^6 - 455382\gamma^4 - 2435292\gamma^2 - 1304345\right)}{32768}$$

$$a_5^{(0,*)}(x;\gamma) = -\frac{3\gamma\left(24255\gamma^{10} - 74795\gamma^8 + 1750326\gamma^6 + 76252650\gamma^4 + 190548859\gamma^2 + 67043025\right)}{262144}$$

$$a_6^{(0,*)}(x;\gamma) =$$

$$\frac{45\gamma\left(63063\gamma^{12} - 295722\gamma^{10} + 1295545\gamma^8 - 137114124\gamma^6 - 1469273511\gamma^4 - 2157119402\gamma^2 - 532523145\right)}{4194304}$$

$$a_0^{(1,*)}(x;\gamma) = -1\ , \quad a_1^{(1,*)}(x;\gamma) = \frac{7\gamma^2 + 9}{8}\ , \quad a_2^{(1,*)}(x;\gamma) = \frac{57\gamma^4 + 622\gamma^2 + 345}{128}\ ,$$

$$a_3^{(1,*)}(x;\gamma) = \frac{195\gamma^6 - 8921\gamma^4 - 30871\gamma^2 - 9555}{1024}$$

$$a_4^{(1,*)}(x;\gamma) = -\frac{7035\gamma^8 + 100692\gamma^6 + 4097826\gamma^4 + 7006164\gamma^2 + 1371195}{32768}$$

$$a_5^{(1,*)}(x;\gamma) = -\frac{97335\gamma^{10} - 38595\gamma^8 - 54339354\gamma^6 - 442588230\gamma^4 - 449504301\gamma^2 - 60259815}{262144}$$

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$$a_6^{(1,*)}(x;\gamma) =$$

$$\frac{3565485\,\gamma^{12} - 12841710\,\gamma^{10} + 423532419\,\gamma^8 + 25838749116\,\gamma^6 + 96291507171\,\gamma^4 + 64464832914\,\gamma^2 + 6264182925}{4194304}$$

$$b_0^{(0,*)}(x;\gamma) = \gamma\,, \quad b_1^{(0,*)}(x;\gamma) = -\frac{\gamma\,(3\,\gamma^2 + 13)}{8} = a_1^{(0,*)}\,, \quad b_2^{(0,*)}(x;\gamma) = \frac{\gamma\,(15\,\gamma^4 - 382\,\gamma^2 - 657)}{128} = -a_2^{(0,*)}\,,$$

$$b_3^{(0,*)}(x;\gamma) = \frac{3\,\gamma\,(35\,\gamma^6 + 327\,\gamma^4 + 8457\,\gamma^2 + 7565)}{1024} = a_3^{(0,*)}$$

$$b_4^{(0,*)}(x;\gamma) = -\frac{3\,\gamma\,(1575\,\gamma^8 - 860\,\gamma^6 - 455382\,\gamma^4 - 2435292\,\gamma^2 - 1304345)}{32768} = -a_4^{(0,*)}$$

$$b_5^{(0,*)}(x;\gamma) = -\frac{3\,\gamma\,(24255\,\gamma^{10} - 74795\,\gamma^8 + 1750326\,\gamma^6 + 76252650\,\gamma^4 + 190548859\,\gamma^2 + 67043025)}{262144} = a_5^{(0,*)}$$

$$b_6^{(0,*)}(x;\gamma) = -a_6^{(0,*)} =$$

$$\frac{45\,\gamma\,(63063\,\gamma^{12} - 295722\,\gamma^{10} + 1295545\,\gamma^8 - 137114124\,\gamma^6 - 1469273511\,\gamma^4 - 2157119402\,\gamma^2 - 532523145)}{4194304}$$

$$b_0^{(1,*)}(x;\gamma) = -1\,, \quad b_1^{(1,*)}(x;\gamma) = -\frac{7\,\gamma^2 + 9}{8} = -a_1^{(1,*)}\,, \quad b_2^{(1,*)}(x;\gamma) = \frac{57\,\gamma^4 + 622\,\gamma^2 + 345}{128} = a_2^{(1,*)}\,,$$

$$b_3^{(1,*)}(x;\gamma) = \frac{195\,\gamma^6 - 8921\,\gamma^4 - 30871\,\gamma^2 - 9555}{1024} = -a_3^{(1,*)}$$

$$b_4^{(1,*)}(x;\gamma) = -\frac{7035\,\gamma^8 + 100692\,\gamma^6 + 4097826\,\gamma^4 + 7006164\,\gamma^2 + 1371195}{32768} = a_4^{(1,*)}$$

$$b_5^{(1,*)}(x;\gamma) = \frac{97335\,\gamma^{10} - 38595\,\gamma^8 - 54339354\,\gamma^6 - 442588230\,\gamma^4 - 449504301\,\gamma^2 - 60259815}{262144} = -a_5^{(1,*)}$$

$$b_6^{(1,*)}(x;\gamma) = a_6^{(1,*)} =$$

$$\frac{3565485\,\gamma^{12} - 12841710\,\gamma^{10} + 423532419\,\gamma^8 + 25838749116\,\gamma^6 + 96291507171\,\gamma^4 + 64464832914\,\gamma^2 + 6264182925}{4194304}$$

When $\gamma << 1$ one has approximately

$$A_0^*(x;\gamma) \approx \gamma\,\frac{\partial A_0^*}{\partial \gamma}(x;0) = \gamma\left[-1 - \frac{1.625}{x} + \frac{5.1328}{x^2} + \frac{22.163}{x^3} - \frac{119.42}{x^4} - \frac{767.25}{x^5} + \frac{5713.4}{x^6} + \ldots\right]$$

$$B_0^*(x;\gamma) \approx \gamma\,\frac{\partial B_0^*}{\partial \gamma}(x;0) = \gamma\left[1 - \frac{1.625}{x} - \frac{5.1328}{x^2} + \frac{22.163}{x^3} + \frac{119.42}{x^4} - \frac{767.25}{x^5} - \frac{5713.4}{x^6} + \ldots\right]$$

$$A_1^*(x;\gamma) \approx A_1^*(x;0) = -1 + \frac{1.125}{x} + \frac{2.6953}{x^2} - \frac{9.3311}{x^3} - \frac{41.846}{x^4} + \frac{229.87}{x^5} + \frac{1493.5}{x^6} + \ldots$$

$$B_1^*(x;\gamma) \approx B_1^*(x;0) = -1 - \frac{1.125}{x} + \frac{2.6953}{x^2} + \frac{9.3311}{x^3} - \frac{41.846}{x^4} - \frac{229.87}{x^5} + \frac{1493.5}{x^6} + \ldots$$

$$\left|\frac{a_3^{(1,*)}(x;0)}{a_2^{(1,*)}(x;0)}\right| = \left|\frac{b_3^{(1,*)}(x;0)}{b_2^{(1,*)}(x;0)}\right| = 3.46\,, \quad \left|\frac{a_4^{(1,*)}(x;0)}{a_3^{(1,*)}(x;0)}\right| = 4.48\,, \quad \left|\frac{a_5^{(1,*)}(x;0)}{a_4^{(1,*)}(x;0)}\right| = 5.49\,, \quad \left|\frac{a_6^{(1,*)}(x;0)}{a_5^{(1,*)}(x;0)}\right| = 6.50$$

The summand $a_k^{(0,*)}(x;\gamma)/[(1-\gamma^2)^{k+1}\,x^k]$ can be used if $|x| > |a_k^{(0,*)}(x;\gamma)/a_{k-1}^{(0)}(x;\gamma)|$.
The same holds for $b_k^{(0,*)}(x;\gamma)$.
Let

$$\Delta_n^*(x;\gamma) = -\Theta_1(x;\gamma)+$$

$$+\frac{1}{\sqrt{\pi x}}\left[A_0^{(n,*)}(x;\gamma)\,\cos x\,J_0(\gamma x) + A_1^{(n,*)}(x;\gamma)\,\cos x\,J_1(\gamma x) + B_0^{(n,*)}(x;\gamma)\,\sin x\,J_0(\gamma x) + B_1^{(n,*)}(x;\gamma)\,\sin x\,J_1(\gamma x)\right]$$

with

$$A_\mu^{(n,*)}(x;\gamma) = \sum_{k=0}^{n} \frac{a_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k} \qquad \text{and} \qquad B_\mu^{(n,*)}(x;\gamma) = \sum_{k=0}^{n} \frac{b_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k} \ .$$

For the case $\gamma = 0.1$ some of these differences are shown:
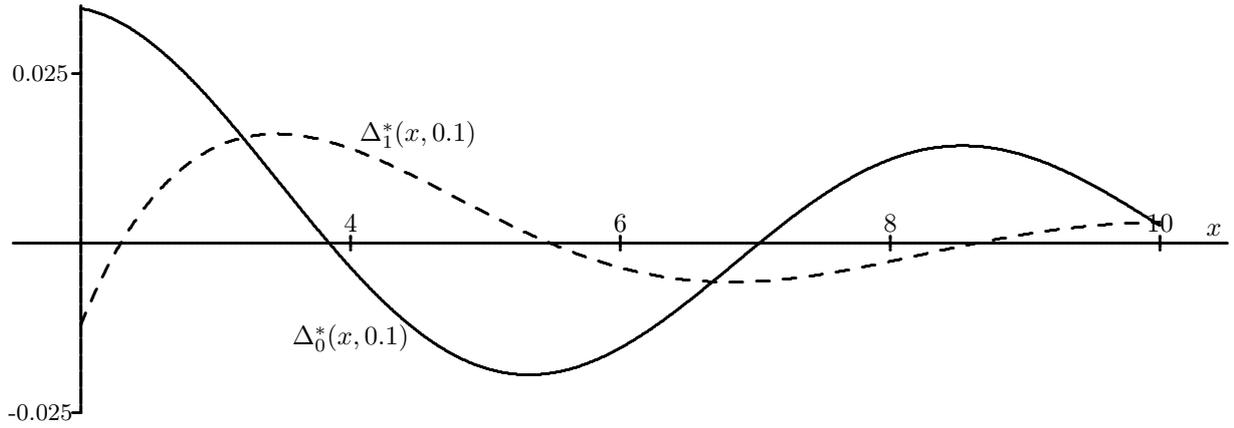
FIGURE 14 :   *Differences $\Delta_0^*(x;\gamma)$ and $\Delta_1^*(x;\gamma)$ with $\gamma = 0.1$*
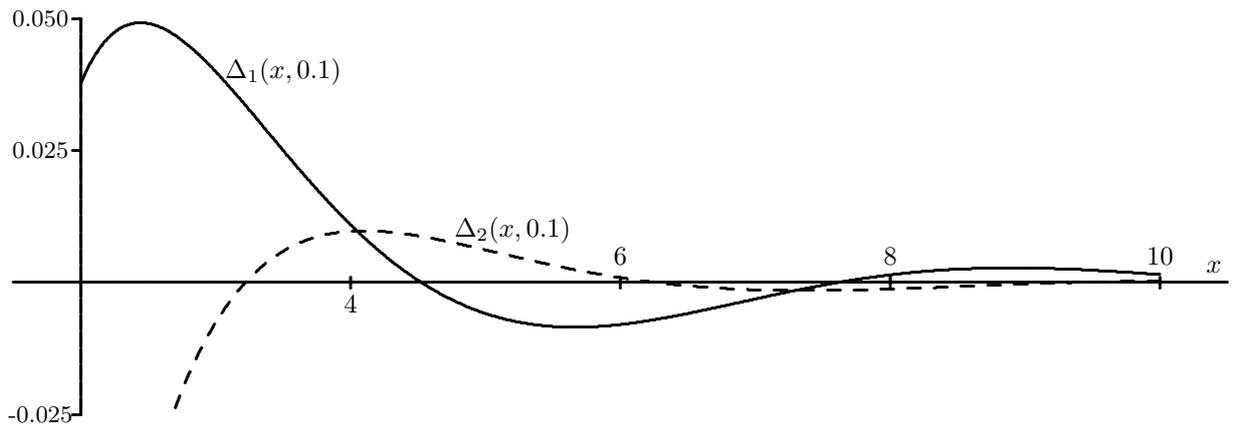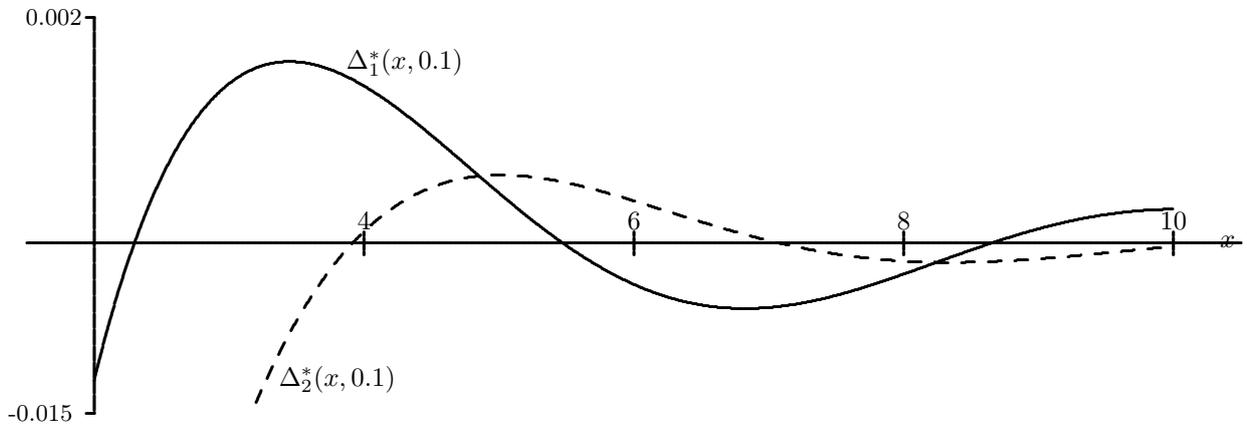

FIGURE 15 :   *Differences $\Delta_1^*(x;\gamma)$ and $\Delta_2^*(x;\gamma)$ with $\gamma = 0.1$*
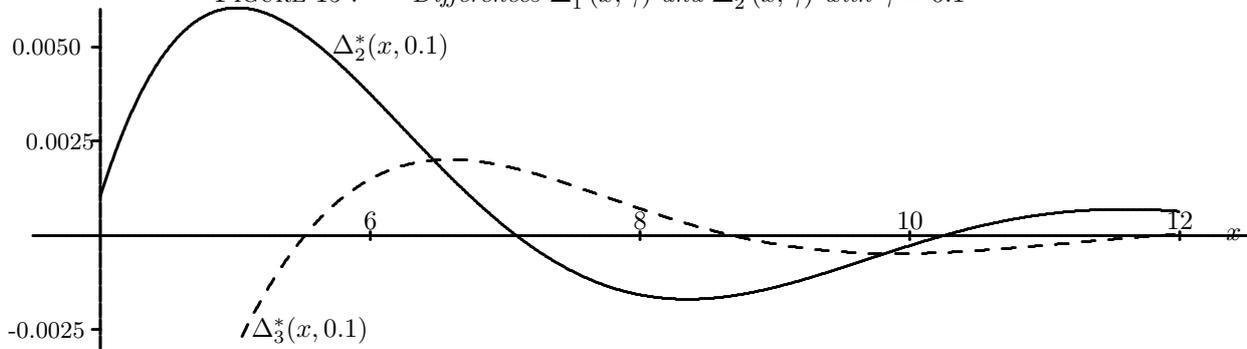
FIGURE 16 :   *Differences $\Delta_2^*(x;\gamma)$ and $\Delta_3^*(x;\gamma)$ with $\gamma = 0.1$*

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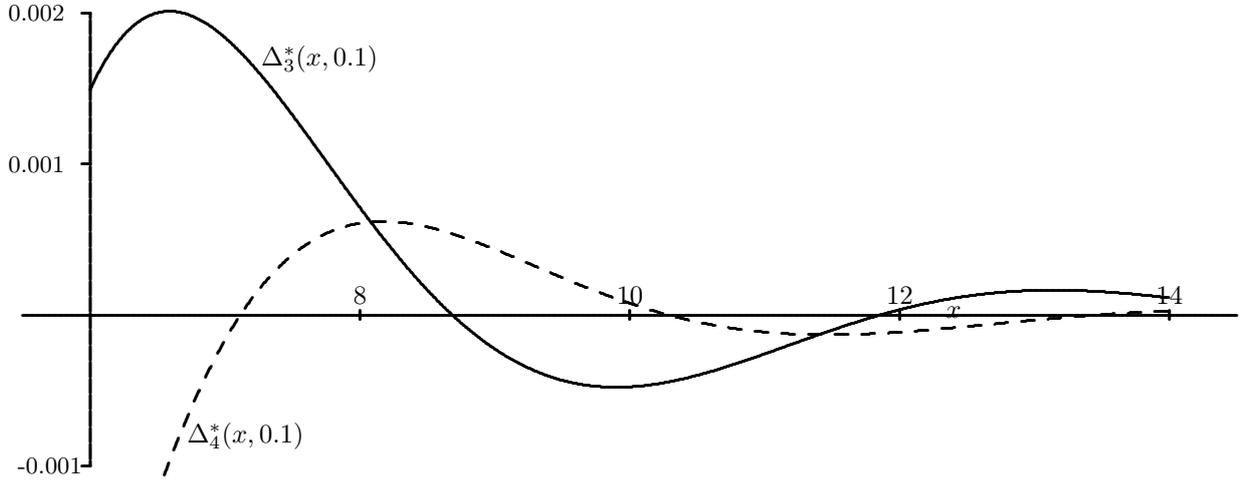
FIGURE 17 :   *Differences* $\Delta_3^*\,(x;\gamma)$ *and* $\Delta_4^*\,(x;\gamma)$ *with* $\gamma = 0.1$

b) Integrals:

Holds (with $0 < \beta < \alpha$ and $\beta/\alpha = \gamma < 1$)

$$\int J_0(\alpha x)\, J_0(\beta x)\, dx = \frac{1}{\alpha}\, \Theta_0\left(ax; \frac{b}{a}\right) , \qquad \int J_1(\alpha x)\, J_1(\beta x)\, dx = \frac{1}{\alpha}\, \Theta_1\left(ax; \frac{b}{a}\right)$$

Let
$$\boxed{\alpha^2 + \beta^2 = \sigma \quad \text{and} \quad \alpha^2 - \beta^2 = \Delta .}$$

$$\int I_0(\alpha x)\, I_0(\beta x)\, dx = \frac{1}{\alpha}\, \Omega_0\left(ax; \frac{b}{a}\right) , \qquad \int I_1(\alpha x)\, I_1(\beta x)\, dx = \frac{1}{\alpha}\, \Omega_1\left(ax; \frac{b}{a}\right)$$

(Θ_ν and Ω_ν as definded on pages 310 and 312.)

$$\int x^2 \cdot J_0(\alpha x)\, J_0(\beta x)\, dx = \frac{\sigma x}{\Delta^2}\, J_0(\alpha x)\, J_0(\beta x) - \frac{\beta x^2}{\Delta}\, J_0(\alpha x)\, J_1(\beta x) + \frac{\alpha x^2}{\Delta}\, J_1(\alpha x)\, J_0(\beta x) +$$

$$+ \frac{2\,\alpha\,\beta x}{\Delta^2}\, J_1(\alpha x)\, J_1(\beta x) - \frac{\sigma}{\Delta^2}\int J_0(\alpha x)\, J_0(\beta x)\, dx + \frac{2\,\alpha\,\beta}{\Delta^2}\int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^2 \cdot J_1(\alpha x)\, J_1(\beta x)\, dx = \frac{2\,\alpha\,\beta x}{\Delta^2}\, J_0(\alpha x)\, J_0(\beta x) - \frac{\alpha x^2}{\Delta}\, J_0(\alpha x)\, J_1(\beta x) + \frac{\beta x^2}{\Delta}\, J_1(\alpha x)\, J_0(\beta x) +$$

$$+ \frac{\sigma x}{\Delta^2}\, J_1(\alpha x)\, J_1(\beta x) + \frac{2\,\alpha\,\beta}{\Delta^2}\int J_0(\alpha x)\, J_0(\beta x)\, dx - \frac{\sigma}{\Delta^2}\int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^2 \cdot I_0(\alpha x)\, I_0(\beta x)\, dx = -\frac{\sigma x}{\Delta^2}\, I_0(\alpha x)\, I_0(\beta x) - \frac{\beta x^2}{\Delta}\, I_0(\alpha x)\, I_1(\beta x) + \frac{\alpha x^2}{\Delta}\, I_1(\alpha x)\, I_0(\beta x) +$$

$$+ \frac{2\,\alpha\,\beta x}{\Delta^2}\, I_1(\alpha x)\, I_1(\beta x) + \frac{\sigma}{\Delta^2}\int I_0(\alpha x)\, I_0(\beta x)\, dx + \frac{2\,\alpha\,\beta}{\Delta^2}\int I_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x^2 \cdot I_1(\alpha x)\, I_1(\beta x)\, dx = \frac{2\alpha\,\beta x}{\Delta^2}\, I_0(\alpha x)\, I_0(\beta x) + \frac{\alpha x^2}{\Delta}\, I_0(\alpha x)\, I_1(\beta x) - \frac{\beta x^2}{\Delta}\, I_1(\alpha x)\, I_0(\beta x) -$$

$$- \frac{\sigma x}{\Delta^2}\, I_1(\alpha x)\, I_1(\beta x) - \frac{2\,\alpha\,\beta}{\Delta^2}\int I_0(\alpha x)\, I_0(\beta x)\, dx - \frac{\sigma}{\Delta^2}\int I_1(\alpha x)\, I_1(\beta x)\, dx$$

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$$\int x^4 \cdot J_0(\alpha x)\, J_0(\beta x)\, dx = \frac{3\left[\sigma\,\Delta^2\,x^2 - 3\,\sigma^2 - 4\,\alpha^2\beta^2\right]x}{\Delta^4}\, J_0(\alpha x)\, J_0(\beta x) -$$

$$-\frac{(\Delta^2\,x^2 - 15\,\alpha^2 - 9\,\beta^2)\beta\,x^2}{\Delta^3}\, J_0(\alpha x)\, J_1(\beta x) + \frac{(\Delta^2\,x^2 - 9\,\alpha^2 - 15\,\beta^2)\,\alpha\,x^2}{\Delta^3}\, J_1(\alpha x)\, J_0(\beta x) +$$

$$+\frac{6\,(\Delta^2\,x^2 - 4\,\sigma)\alpha\,\beta\,x}{\Delta^4}\, J_1(\alpha x)\, J_1(\beta x) + \frac{9\,\sigma^2 + 12\,\alpha^2\beta^2}{\Delta^4}\int J_0(\alpha x)\, J_0(\beta x)\, dx - \frac{24\,\alpha\,\beta\,\sigma}{\Delta^4}\int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^4 \cdot J_1(\alpha x)\, J_1(\beta x)\, dx = \frac{6\,(\Delta^2\,x^2\alpha^4 - 4\,\sigma)\,\beta\,\alpha\,x}{\Delta^4}\, J_0(\alpha x)\, J_0(\beta x) -$$

$$-\frac{(\Delta^2\,x^2 - 3\,\alpha^2 - 21\,\beta^2)\alpha\,x^2}{\Delta^3}\, J_0(\alpha x)\, J_1(\beta x) + \frac{(\Delta^2\,x^2 - 21\,\alpha^2 - 3\,\beta^2)\,\beta\,x^2}{\Delta}\, J_1(\alpha x)\, J_0(\beta x) +$$

$$+\frac{3\left[\sigma\,\Delta^2\,x^2 - \sigma^2 - 12\,\alpha^2\beta^2\right]x}{\Delta^4}\, J_1(\alpha x)\, J_1(\beta x) -$$

$$+\frac{24\,\alpha\,\beta\,\sigma}{\Delta^4}\int J_0(\alpha x)\, J_0(\beta x)\, dx - \frac{3\,\sigma^2 + 36\,\alpha^2\beta^2}{\Delta^4}\int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^4 \cdot I_0(\alpha x)\, I_0(\beta x)\, dx = -\frac{3\left[\sigma\,\Delta^2\,x^2 + 3\,\sigma^2 + 4\,\alpha^2\beta^2\right]x}{\Delta^4}\, I_0(\alpha x)\, I_0(\beta x) -$$

$$-\frac{(\Delta^2\,x^2 + 15\,\alpha^2 + 9\,\beta^2)\,\beta\,x^2}{\Delta^3}\, I_0(\alpha x)\, I_1(\beta x) + \frac{(\Delta^2\,x^2 + 9\,\alpha^2 + 15\,\beta^2)\,\alpha\,x^2}{\Delta^3}\, I_1(\alpha x)\, I_0(\beta x) +$$

$$+\frac{6\,(\Delta^2\,x^2 + 4\,\sigma)\,\alpha\,\beta\,x}{\Delta^4}\, I_1(\alpha x)\, I_1(\beta x) -$$

$$+\frac{3\,(3\,\sigma^2 + 4*\alpha^2\beta^2)}{\Delta^4}\int I_0(\alpha x)\, I_0(\beta x)\, dx + \frac{24\,\alpha\,\beta\,\sigma}{\Delta^4}\int I_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x^4 \cdot I_1(\alpha x)\, I_1(\beta x)\, dx = \frac{6\,(\Delta^2\,x^2 + 4\,\sigma)\,\alpha\,\beta\,x}{\Delta^4}\, I_0(\alpha x)\, I_0(\beta x) +$$

$$+\frac{(\Delta^2\,x^2 + 3\,\alpha^2 + 21\,\beta^2)\alpha\,x^2}{\Delta^3}\, I_0(\alpha x)\, I_1(\beta x) - \frac{(\Delta^2\,x^2 + 21\,\alpha^2 + 3\,\beta^2)\,\beta\,x^2}{\Delta^3}\, I_1(\alpha x)\, I_0(\beta x) -$$

$$-\frac{3\,(\sigma\,\Delta^2\,x^2 + \sigma^2 + 12\,\alpha^2\beta^2)\,x}{\Delta^4}\, I_1(\alpha x)\, I_1(\beta x) -$$

$$-\frac{24\,\alpha\,\beta\,\sigma}{\Delta^4}\int I_0(\alpha x)\, I_0(\beta x)\, dx - \frac{3\,\sigma^2 + 36\,\alpha^2\beta^2}{\Delta^4}\int I_1(\alpha x)\, I_1(\beta x)\, dx$$

Let

$$\int x^n \cdot J_\nu(\alpha x)\, J_\nu(\beta x)\, dx = \frac{P_n^{(\nu)}(x)}{\Delta^n}\, J_0(\alpha x)\, J_0(\beta x) + \frac{Q_n^{(\nu)}(x)}{\Delta^{n-1}}\, J_0(\alpha x)\, J_1(\beta x) +$$

$$+\frac{R_n^{(\nu)}(x)}{\Delta^{n-1}}\, J_1(\alpha x)\, J_0(\beta x) + \frac{S_n^{(\nu)}(x)}{\Delta^n}\, J_1(\alpha x)\, J_1(\beta x) +$$

$$+\frac{U_n^{(\nu)}(x)}{\Delta^n}\int J_0(\alpha x)\, J_0(\beta x)\, dx + \frac{V_n^{(\nu)}(x)}{\Delta^n}\int J_1(\alpha x)\, J_1(\beta x)\, dx$$

and

$$\int x^n \cdot I_\nu(\alpha x)\, I_\nu(\beta x)\, dx = \frac{\bar{P}_n^{(\nu)}(x)}{\Delta^n}\, I_0(\alpha x)\, I_0(\beta x) + \frac{\bar{Q}_n^{(\nu)}(x)}{\Delta^{n-1}}\, I_0(\alpha x)\, I_1(\beta x) +$$

$$+\frac{\bar{R}_n^{(\nu)}(x)}{\Delta^{n-1}}\, I_1(\alpha x)\, I_0(\beta x) + \frac{\bar{S}_n^{(\nu)}(x)}{\Delta^n}\, I_1(\alpha x)\, I_1(\beta x) +$$

$$+\frac{\bar{U}_n^{(\nu)}}{\Delta^n}\int I_0(\alpha x)\, I_0(\beta x)\, dx + \frac{\bar{V}_n^{(\nu)}}{\Delta^n}\int I_1(\alpha x)\, I_1(\beta x)\, dx\ .$$

$$P_6^{(0)} = 5\,x[\sigma\,\Delta^4\,x^4 - 3\,(5\,\sigma^2 + 12\,\alpha^2\,\beta^2)\Delta^2 x^2 + 3\,\sigma\,(15\,\sigma^2 + 68\,\alpha^2\,\beta^2)]$$

$$Q_6^{(0)} = -\beta\, x^2[\Delta^4 x^4 - 5\,(11\,\alpha^2 + 5\,\beta^2)\Delta^2 x^2 + 465\,\alpha^4 + 1230\,\alpha^2\,\beta^2 + 225\,\beta^4]$$

$$R_6^{(0)} = \alpha\, x^2[\Delta^4 x^4 - 5\,(5\,\alpha^2 + 11\,\beta^2)\Delta^2 x^2 + 225\,\alpha^4 + 1230\,\alpha^2\,\beta^2 + 465\,\beta^4]$$

$$S_6^{(0)} = 10\,\alpha\,\beta\, x[\Delta^4 x^4 - 24\,\sigma\Delta^2 x^2 + 69\,\sigma^2 + 108\alpha^2\,\beta^2]$$

$$U_6^{(0)} = -15\,\sigma(15\,\sigma^2 + 68\alpha^2\,\beta^2)\,, \quad V_6^{(0)} = 30\,\alpha\,\beta\,(23\,\sigma^2 + 36\,\alpha^2\,\beta^2)$$

$$P_6^{(1)} = 10\,\alpha\,\beta\,[\Delta^4 x^4 - 24\,\sigma\Delta^2 x^2 + 81\,\sigma^2 + 60\,\alpha^2\beta^2]\,x$$

$$Q_6^{(1)} = -\alpha\,[\Delta^4 x^4 - 5\,(3\,\alpha^2 + 13\,\beta^2)\,\Delta^2\,x^2 + 45\,\alpha^4 + 1110\,\alpha^2\,\beta^2 + 765\,\beta^4]\,x^2$$

$$R_6^{(1)} = \beta\,[\Delta^4\,x^4 - 5\,(13\,\alpha^2 + 3\,\beta^2)\,\Delta^2\,x^2 + 765\,\alpha^4 + 1110\,\alpha^2\,\beta^2 + 45\,\beta^4]\,x^2$$

$$S_6^{(1)} = 5\,[\sigma\,\Delta^4\,x^4 - 3\,(3\,\sigma^2 + 20\,\alpha^2\,\beta^2)\Delta^2\,x^2 + 3\,\sigma\,(3\,\sigma^2 + 116\,\alpha^2\,\beta^2)]\,x$$

$$U_6^{(1)} = -30\,(27\,\sigma^2 + 10\,\alpha^2\,\beta^2)\alpha\,\beta\,, \quad V_6^{(1)} = 15\,\sigma(3\,\sigma^2 + 116\,\alpha^2\,\beta^2)$$

$$\bar{P}_6^{(0)} = -5\,[\sigma\Delta^4\,x^4 + 3\,(5\,\sigma^2 + 12\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 + 3\,\sigma(15\,\sigma^2 + 68\,\alpha^2\,\beta^2)]\,x$$

$$\bar{Q}_6^{(0)} = -\beta\,[\Delta^4\,x^4 + 5\,(11\,\alpha^2 + 5\,\beta^2)\,\Delta^2\,x^2 + 465\,\alpha^4 + 1230\,\alpha^2\,\beta^2 + 225\,\beta^4]\,x^2$$

$$\bar{R}_6^{(0)} = \alpha\,[\Delta^4\,x^4 + 5\,(5\,\alpha^2 + 11\,\beta^2)\,\Delta^2\,x^2 + 225\,\alpha^4 + 1230\,\alpha^2\,\beta^2 + 465\,\beta^4]\,x^2$$

$$\bar{S}_6^{(0)} = 10\,\alpha\,\beta\,[\Delta^4\,x^4 + 24\,\sigma\,\Delta^2\,x^2 + 69\,\sigma^2 + 108\,\alpha^2\,\beta^2]\,x$$

$$\bar{U}_6^{(0)} = 15\,\sigma(15\,\sigma^2 + 68\,\alpha^2\,\beta^2)\,, \quad \bar{V}_6^{(0)} = 30\,\alpha\,\beta\,(23\,\sigma^2 + 36\,\alpha^2\,\beta^2)$$

$$\bar{P}_6^{(1)} = 10\,\alpha\,\beta\,[\Delta^4 x^4 + 24\,\sigma\Delta^2 x^2 + 81\,\sigma^2 + 60\,\alpha^2\,\beta^2]\,x$$

$$\bar{Q}_6^{(1)} = \alpha\,[\Delta^4\,x^4 + 5\,(3\,\alpha^2 + 13\,\beta^2)\,\Delta^2\,x^2 + 45\,\alpha^4 + 1110\,\alpha^2\,\beta^2 + 765\,\beta^4]\,x^2$$

$$\bar{R}_6^{(1)} = -\beta\,[\Delta^4\,x^4 + 5\,(13\,\alpha^2 + 3\,\beta^2)\,\Delta^2\,x^2 + 765\,\alpha^4 + 1110\,\alpha^2\,\beta^2 + 45\,\beta^4]\,x^2$$

$$\bar{S}_6^{(1)} = -5\,x[\sigma\,\Delta^4\,x^4 + 3\,(3\,\sigma^2 + 20\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 + 3\,\sigma(3\,\sigma^2 + 116\,\alpha^2\,\beta^2)]$$

$$\bar{U}_6^{(1)} = -30\,(27\,\sigma^2 + 20\,\alpha^2\,\beta^2)\alpha\,\beta\,, \quad \bar{V}_6^{(1)} = -15\,\sigma(3\,\sigma^2 + 116\,\alpha^2\,\beta^2)$$

$$P_8^{(0)} = 7\,[\sigma\Delta^6\,x^6 - 5\,(7\,\sigma^2 + 20\,\alpha^2\,\beta^2)\,\Delta^4\,x^4 + 15\,\sigma\,(35\,\sigma^2 + 244\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 -$$
$$-\,1575\,\sigma^4 - 15240\,\alpha^2\,\beta^2\,\sigma^2 - 6000\,\alpha^4\,\beta^4]\,x$$

$$Q_8^{(0)} = -\beta\,[\Delta^6 x^6 - 7\,(17\,\alpha^2 + 7\,\beta^2)\Delta^4 x^4 + 35\,(107\,\alpha^4 + 242\,\alpha^2\,\beta^2 + 35\,\beta^4)\Delta^2 x^2 -$$
$$-\,25935\,\alpha^6 - 160755\,\alpha^4\beta^2 - 124845\,\alpha^2\beta^4 - 11025\,\beta^6]\,x^2$$

$$R_8^{(0)} = \alpha\,[\Delta^6 x^6 - 7\,(7\,\alpha^2 + 17\,\beta^2)\,\Delta^4\,x^4 + 35\,(35\,\alpha^4 + 242\,\alpha^2\,\beta^2 + 107\,\beta^4)\Delta^2 x^2 -$$
$$-\,11025\alpha^6 - 124845\,\alpha^4\beta^2 - 160755\,\alpha^2\beta^4 - 25935\,\beta^6]\,x^2$$

$$S_8^{(0)} = 14\,\alpha\,\beta\,[\Delta^6 x^6 - 60\,\sigma\Delta^4 x^4 + 15\,(71\,\sigma^2 + 100\,\alpha^2\,\beta^2)\Delta^2\,x^2 - 240\,\sigma(11\,\sigma^2 + 52\,\alpha^2\,\beta^2)]\,x$$

$$U_8^{(0)} = 11025\,\sigma^4 + 106680\,\alpha^2\beta^2\sigma^2 + 42000\,\alpha^4\beta^4$$

$$V_8^{(0)} = -3360\,\alpha\,\beta\,\sigma(11\,\sigma^2 + 52\,\alpha^2\,\beta^2)$$

$$P_8^{(1)} = 14\,\alpha\,\beta\,[\Delta^6 x^6 - 60\,\sigma\Delta^4 x^4 + 45\,(25\,\sigma^2 + 28\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 - 720\,\sigma\,(5\,\sigma^2 + 12\,\alpha^2\,\beta^2)]\,x$$

$$Q_8^{(1)} = -\alpha\,[\Delta^6 x^6 - 7\,(5\,\alpha^2 + 19\,\beta^2)\Delta^4 x^4 +$$
$$+\,105\,(5\,\alpha^2 + 3\,\beta^2)(\alpha^2 + 15\,\beta^2)\,\Delta^2\,x^2 - 1575\,\alpha^6 - 85995\,\alpha^4\beta^2 - 186165\,\alpha^2\beta^4 - 48825\,\beta^6]\,x^2$$

$$R_8^{(1)} = \beta\,[\Delta^6 x^6 - 7\,(19\,\alpha^2 + 5\,\beta^2)\,\Delta^4\,x^4 +$$

$$+105\,(3\,\alpha^2 + 5\,\beta^2)(15\,\alpha^2 + \beta^2)\,\Delta^2\,x^2 - 48825\,\alpha^6 - 186165\,\alpha^4\beta^2 - 85995\,\alpha^2\beta^4 - 1575\,\beta^6]\,x^2$$

$$S_8^{(1)} = 7\,[\sigma\,\Delta^6\,x^6 - 5\,(5\,\sigma^2 + 28\,\alpha^2\,\beta^2)\,\Delta^2\,x^4 + 45\,\sigma\,(5\,\sigma^2 + 108\,\alpha^2\beta^2)\,\Delta^2\,x^2 -$$
$$-225\,\sigma^4 - 18360\,\alpha^2\beta^2\sigma^2 - 15120\,\alpha^4\beta^4]\,x$$

$$U_8^{(1)} = 10080\,\alpha\,\beta\,\sigma(5\,\sigma^2 + 12\,\alpha^2\,\beta^2)$$

$$V_8^{(1)} = -(1575\,\sigma^4 + 128520\,\alpha^2\beta^2\sigma^2 + 105840\,\alpha^4\beta^4)$$

$$\bar{P}_8^{(0)} = -7\,[\sigma\Delta^6 x^6 + 5\,(7\,\sigma^2 + 20\,\alpha^2\,\beta^2)\,\Delta^4\,x^4 + 15\,\sigma\,(35\,\sigma^2 + 244\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 +$$
$$+1575\,\sigma^4 + 15240\,\alpha^2\beta^2\sigma^2 + 6000\,\alpha^4\,\beta^4]\,x$$

$$\bar{Q}_8^{(0)} = -\beta\,[\Delta^6 x^6 + 7\,(17\,\alpha^2 + 7\,\beta^2)\,\Delta^4\,x^4 + 35\,(107\,\alpha^4 + 242\,\alpha^2\,\beta^2 + 35\,\beta^4)\,\Delta^2\,x^2 +$$
$$+25935\,\alpha^6 + 160755\,\alpha^4\beta^2 + 124845\,\alpha^2\beta^4 + 11025\,\beta^6]\,x^2$$

$$\bar{R}_8^{(0)} = \alpha\,[\Delta^6 x^6 + 7\,(7\,\alpha^2 + 17\,\beta^2)\,\Delta^4\,x^4 + 35\,(35\,\alpha^4 + 242\,\alpha^2\,\beta^2 + 107\,\beta^4)\,\Delta^2\,x^2 +$$
$$+11025\,\alpha^6 + 124845\,\alpha^4\beta^2 + 160755\,\alpha^2\beta^4 + 25935\,\beta^6]\,x^2$$

$$\bar{S}_8^{(0)} = 14\,\alpha\,\beta\,[\Delta^6 x^6 + 60\,\sigma\,\Delta^4\,x^4 + 15\,(71\,\sigma^2 + 100\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 + 240\,\sigma(11\,\sigma^2 + 52\,\alpha^2\beta^2)]\,x$$

$$\bar{U}_8^{(0)} = 11025\,\sigma^4 + 106680\,\alpha^2\beta^2\sigma^2 + 42000\,\alpha^4\beta^4$$

$$\bar{V}_8^{(0)} = 3360\,\alpha\,\beta\,\sigma\,(11\,\sigma^2 + 52\,\alpha^2\beta^2)$$

$$\bar{P}_8^{(1)} = 14\,\alpha\,\beta\,[\Delta^6 x^6 + 60\,\sigma\Delta^4 x^4 + 45\,(25\,\sigma^2 + 28\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 + 720\,\sigma\,(5\,\sigma^2 + 12\,\alpha^2\beta^2)]\,x$$

$$\bar{Q}_8^{(1)} = \alpha\,[\Delta^6\,x^6 + 7\,(5\,\alpha^2 + 19\,\beta^2)\Delta^4 x^4 +$$
$$+105\,(5\,\alpha^2 + 3\,\beta^2)(\alpha^2 + 15\,\beta^2)\,\Delta^2\,x^2 + 1575\,\alpha^6 + 85995\,\alpha^4\beta^2 + 186165\,\alpha^2\beta^4 + 48825\,\beta^6]\,x^2$$

$$\bar{R}_8^{(1)} = -\beta\,[\Delta^6 x^6 + 7\,(19\,\alpha^2 + 5\,\beta^2)\,\Delta^4\,x^4 +$$
$$+105\,(3\,\alpha^2 + 5\,\beta^2)(15\,\alpha^2 + \beta^2)\,\Delta^2\,x^2 + 48825\,\alpha^6 + 186165\,\alpha^4\beta^2 + 85995\,\alpha^2\beta^4 + 1575\,\beta^6]\,x^2$$

$$\bar{S}_8^{(1)} = -7\,[\sigma\,\Delta^6\,x^6 + 5\,(5\,\sigma^2 + 28\,\alpha^2\,\beta^2)\,\Delta^4\,x^4 + 45\,\sigma\,(5\,\sigma^2 + 108\,\alpha^2\beta^2)\,\Delta^2\,x^2 +$$
$$+225\,\sigma^4 + 18360\,\alpha^2\beta^2\sigma^2 + 15120\,\alpha^4\beta^4]\,x$$

$$\bar{U}_8^{(1)} = -10080\,\alpha\,\beta\,\sigma\,(5\,\sigma^2 + 12\,\alpha^2\,\beta^2)$$

$$\bar{V}_8^{(1)} = -(1575\,\alpha^8 + 134820\,\alpha^6\beta^2 + 372330\,\alpha^4\beta^4 + 134820\,\alpha^2\beta^6 + 1575\,\beta^8)$$

Recurrence relations: (see also page 309)

$$\int x^{2n+2}\,J_0(\alpha x)\,J_0(\beta x)\,dx = \frac{(2n+1)\,\sigma\,x^{2n+1}}{\Delta^2}\,J_0(\alpha x)\,J_0(\beta x) - \frac{\beta\,x^{2n+2}}{\Delta}\,J_0(\alpha x)\,J_1(\beta x) +$$
$$+\frac{\alpha\,x^{2n+2}}{\Delta}\,J_1(\alpha x)\,J_0(\beta x) + \frac{(4n+2)\,\alpha\,\beta\,x^{2n+1}}{\Delta^2}\,J_1(\alpha x)\,J_1(\beta x) -$$
$$-\frac{(2n+1)^2\,\sigma}{\Delta^2}\int x^{2n}\,J_0(\alpha x)\,J_0(\beta x)\,dx - \frac{(8n^2-2)\,\alpha\,\beta}{\Delta^2}\int x^{2n}\,J_1(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^{2n+2}\,J_1(\alpha x)\,J_1(\beta x)\,dx = \frac{(4n+2)\,\alpha\,\beta\,x^{2n+1}}{\Delta^2}\,J_0(\alpha x)\,J_0(\beta x) - \frac{\alpha\,x^{2n+2}}{\Delta}\,J_0(\alpha x)\,J_1(\beta x) +$$
$$+\frac{\beta\,x^{2n+2}}{\Delta}\,J_1(\alpha x)\,J_0(\beta x) + \frac{(2n+1)\,\sigma\,x^{2n+1}}{\Delta^2}\,J_1(\alpha x)\,J_1(\beta x) -$$
$$-\frac{2\,(2n+1)^2\,\alpha\,\beta}{\Delta^2}\int x^{2n}\,J_0(\alpha x)\,J_0(\beta x)\,dx - \frac{(4n^2-1)\,\sigma}{\Delta^2}\int x^{2n}\,J_1(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^{2n+2} I_0(\alpha x) I_0(\beta x)\, dx = -\frac{(2n+1)\,\sigma\, x^{2n+1}}{\Delta^2} I_0(\alpha x) I_0(\beta x) - \frac{\beta\, x^{2n+2}}{\Delta} I_0(\alpha x) I_1(\beta x) +$$

$$+ \frac{\alpha\, x^{2n+2}}{\Delta} I_1(\alpha x) I_0(\beta x) + \frac{(4n+2)\,\alpha\,\beta\, x^{2n+1}}{\Delta^2} I_1(\alpha x) I_1(\beta x) +$$

$$+ \frac{(2n+1)^2\,\sigma}{\Delta^2} \int x^{2n} I_0(\alpha x) I_0(\beta x)\, dx - \frac{(8n^2-2)\,\alpha\,\beta}{\Delta^2} \int x^{2n} I_1(\alpha x) I_1(\beta x)\, dx$$

$$\int x^{2n+2} I_1(\alpha x) I_1(\beta x)\, dx = \frac{(4n+2)\,\alpha\,\beta\, x^{2n+1}}{\Delta^2} I_0(\alpha x) I_0(\beta x) + \frac{\alpha\, x^{2n+2}}{\Delta} I_0(\alpha x) I_1(\beta x) -$$

$$- \frac{\beta\, x^{2n+2}}{\Delta} I_1(\alpha x) I_0(\beta x) - \frac{(2n+1)\,\sigma\, x^{2n+1}}{\Delta^2} I_1(\alpha x) I_1(\beta x) -$$

$$- \frac{2\,(2n+1)^2\,\alpha\,\beta}{\Delta^2} \int x^{2n} I_0(\alpha x) I_0(\beta x)\, dx + \frac{(4n^2-1)\,\sigma}{\Delta^2} \int x^{2n} I_1(\alpha x) I_1(\beta x)\, dx$$

2.2.2. Integrals of the type $\int x^{2n} Z_0(\alpha x) Z_1(\beta x)\, dx$ and $\int x^{2n} V_0(\alpha x)\, W_1(\beta x)\, dx$, $\alpha^2 \neq \beta^2$

See the remark in 2.2.3, page 332.

Let

$$\boxed{\quad \alpha^2 + \beta^2 = \sigma \qquad \text{and} \qquad \alpha^2 - \beta^2 = \Delta \ .\quad}$$

$$\int x^2 \cdot J_0(\alpha x) J_1(\beta x)\, dx \;=$$

$$= \; \frac{2\beta x}{\Delta^2} \left[\beta J_0(\alpha x) J_1(\beta x) - \alpha J_1(\alpha x) J_0(\beta x) \right] + \frac{x^2}{\Delta} \left[\beta J_0(\alpha x) J_0(\beta x) + \alpha J_1(\alpha x) J_1(\beta x) \right]$$

$$\int x^2 \cdot I_0(\alpha x) I_1(\beta x)\, dx \;=$$

$$= \; -\frac{2\beta x}{\Delta^2} \left[\beta I_0(\alpha x) I_1(\beta x) - \alpha I_1(\alpha x) I_0(\beta x) \right] - \frac{x^2}{\Delta} \left[\beta I_0(\alpha x) I_0(\beta x) - \alpha I_1(\alpha x) I_1(\beta x) \right]$$

$$\int x^2 \cdot K_0(\alpha x) K_1(\beta x)\, dx \;=$$

$$= \; -\frac{2\beta x}{\Delta^2} \left[\beta K_0(\alpha x) K_1(\beta x) - \alpha K_1(\alpha x) K_0(\beta x) \right] + \frac{x^2}{\Delta} \left[\beta K_0(\alpha x) K_0(\beta x) - \alpha K_1(\alpha x) K_1(\beta x) \right]$$

$$\int x^2 \cdot J_0(\alpha x) I_1(\beta x)\, dx \;=$$

$$= \; -\frac{2\beta x}{\sigma^2} \left[\beta J_0(\alpha x) I_1(\beta x) + \alpha J_1(\alpha x) I_0(\beta x) \right] + \frac{x^2}{\sigma} \left[\beta J_0(\alpha x) I_0(\beta x) + \alpha J_1(\alpha x) I_1(\beta x) \right]$$

$$\int x^2 \cdot J_1(\alpha x) I_0(\beta x)\, dx \;=$$

$$= \; \frac{2\alpha x}{\sigma^2} \left[\beta J_0(\alpha x) I_1(\beta x) + \alpha J_1(\alpha x) I_0(\beta x) \right] - \frac{x^2}{\sigma} \left[\alpha J_0(\alpha x) I_0(\beta x) - \beta J_1(\alpha x) I_1(\beta x) \right]$$

$$\int x^2 \cdot J_0(\alpha x) K_1(\beta x)\, dx \;=$$

$$= \; -\frac{2\beta x}{\sigma^2} \left[\beta J_0(\alpha x) K_1(\beta x) - \alpha J_1(\alpha x) K_0(\beta x) \right] - \frac{x^2}{\sigma} \left[\beta J_0(\alpha x) K_0(\beta x) - \alpha J_1(\alpha x) K_1(\beta x) \right]$$

$$\int x^2 \cdot J_1(\alpha x) K_0(\beta x)\, dx \;=$$

$$= \; -\frac{2\alpha x}{\sigma^2} \left[\beta J_0(\alpha x) K_1(\beta x) - \alpha J_1(\alpha x) K_0(\beta x) \right] - \frac{x^2}{\sigma} \left[\alpha J_0(\alpha x) K_0(\beta x) + \beta J_1(\alpha x) K_1(\beta x) \right]$$

$$\int x^2 \cdot I_0(\alpha x) K_1(\beta x)\, dx \;=$$

$$= \; -\frac{2\beta x}{\Delta^2} \left[\beta I_0(\alpha x) K_1(\beta x) + \alpha I_1(\alpha x) K_0(\beta x) \right] + \frac{x^2}{\Delta} \left[\beta I_0(\alpha x) K_0(\beta x) + \alpha I_1(\alpha x) K_1(\beta x) \right]$$

$$\int x^2 \cdot I_1(\alpha x) K_0(\beta x)\, dx \;=$$

$$= \; -\frac{2\alpha x}{\Delta^2} \left[\beta I_0(\alpha x) K_1(\beta x) + \alpha I_1(\alpha x) K_0(\beta x) \right] + \frac{x^2}{\Delta} \left[\alpha I_0(\alpha x) K_0(\beta x) + \beta I_1(\alpha x) K_1(\beta x) \right]$$

Let

$$\int x^m \, U_0(\alpha x) W_1(\beta x)\, dx =$$

$$= P_m^{[UW]}(x) U_0(\alpha x) W_0(\beta x) + Q_m^{[UW]}(x) U_0(\alpha x) W_1(\beta x) + R_m^{[UW]}(x) U_1(\alpha x) W_0(\beta x) + S_m^{[UW]}(x) U_1(\alpha x) W_1(\beta x)\ .$$

One has

$$P_m^{[JJ]} = P_m^{[YY]} = P_m^{[H^{(1)} H^{(1)}]} = P_m^{[H^{(2)} H^{(2)}]} = P_m^{[JY]} = P_m^{[JH^{(1)}]} = P_m^{[JH^{(2)}]} = P_m^{[YH^{(1)}]} = P_m^{[YH^{(2)}]} = P_m^{[H^{(1)} H^{(2)}]}\ ,$$

$$P_m^{[JI]} = P_m^{[YI]} = P_m^{[H^{(1)}I]} = P_m^{[H^{(2)}I]} \quad \text{and} \quad P_m^{[JK]} = P_m^{[YK]} = P_m^{[H^{(1)}K]} = P_m^{[H^{(2)}K]} \ .$$

The same holds for the polynomials $Q_m(x)$, $R_m(x)$ and $S_m(x)$.

$$P_4^{[JJ]}(x) = \frac{\beta x^4}{\Delta} - \frac{8\beta\,(2\alpha^2 + \beta^2)\,x^2}{\Delta^3} \quad , \qquad Q_4^{[JJ]}(x) = \frac{2\,(\alpha^2 + 2\,\beta^2)\,x^3}{\Delta^2} - \frac{16\beta^2(2\alpha^2 + \beta^2)\,x}{\Delta^4}$$

$$R_4^{[JJ]}(x) = -\frac{6\,\alpha\,\beta\,x^3}{\Delta^2} + \frac{16\,\alpha\,\beta\,(2\alpha^2 + \beta^2)\,x}{\Delta^4} \quad , \qquad S_4^{[JJ]}(x) = \frac{\alpha x^4}{\Delta} - \frac{4\,\alpha\,(\alpha^2 + 5\,\beta^2)\,x^2}{\Delta^3}$$

$$P_4^{[II]}(x) = -P_4^{[KK]}(x) = -\frac{\beta\,x^4}{\Delta} - \frac{8\,\beta\,(2\alpha^2 + \beta^2)\,x^2}{\Delta^3}$$

$$Q_4^{[II]}(x) = Q_4^{[KK]}(x) = -\frac{2\,(\alpha^2 + 2\,\beta^2)\,x^3}{\Delta^2} - \frac{16\,\beta^2\,(2\,\alpha^2 + \beta^2)\,x}{\Delta^4}$$

$$R_4^{[II]}(x) = R_4^{[KK]}(x) = \frac{6\,\alpha\,\beta\,x^3}{\Delta^2} + \frac{16\,\alpha\,\beta\,(2\alpha^2 + \beta^2)\,x}{\Delta^4}$$

$$S_4^{[II]}(x) = -S_4^{[KK]}(x) = \frac{\alpha\,x^4}{\Delta} + \frac{4\,\alpha\,(\alpha^2 + 5\,\beta^2)\,x^2}{\Delta^3}$$

$$P_4^{[JI]}(x) = -P_4^{[JK]}(x) = \frac{\beta\,x^4}{\sigma} - \frac{8\,\beta\,(2\,\alpha^2 - \beta^2)\,x^2}{\sigma^3}$$

$$Q_4^{[JI]}(x) = Q_4^{[JK]}(x) = \frac{2\,(\alpha^2 - 2\,\beta^2)\,x^3}{\sigma^2} + \frac{16\,\beta^2(2\alpha^2 - \beta^2)\,x}{\sigma^4}$$

$$R_4^{[JI]}(x) = -R_4^{[JK]}(x) = -\frac{6\,\alpha\,\beta\,x^3}{\sigma^2} + \frac{16\,\alpha\,\beta\,(2\,\alpha^2 - \beta^2)\,x}{\sigma^4}$$

$$S_4^{[JI]}(x) = S_4^{[JK]}(x) = \frac{\alpha\,x^4}{\sigma} - \frac{4\,\alpha\,(\alpha^2 - 5\,\beta^2)\,x^2}{\sigma^3}$$

$$P_4^{[IJ]}(x) = P_4^{[KJ]}(x) = -\frac{\beta\,x^4}{\sigma} - \frac{8\,\beta\,(2\,\alpha^2 - \beta^2)\,x^2}{\sigma^3}$$

$$Q_4^{[IJ]}(x) = Q_4^{[KJ]}(x) = -\frac{2\,(\alpha^2 - 2\,\beta^2)\,x^3}{\sigma^2} + \frac{16\,\beta^2(2\,\alpha^2 - \beta^2)\,x}{\sigma^4}$$

$$R_4^{[IJ]}(x) = -R_4^{[KJ]}(x) = \frac{6\,\alpha\,\beta\,x^3}{\sigma^2} + \frac{16\,\alpha\,\beta\,(2\,\alpha^2 - \beta^2)\,x}{\sigma^4}$$

$$S_4^{[IJ]}(x) = -S_4^{[KJ]}(x) = \frac{\alpha\,x^4}{\sigma} + \frac{4\,\alpha\,(\alpha^2 - 5\,\beta^2)\,x^2}{\sigma^3}$$

$$P_4^{[IK]}(x) = -P_4^{[KI]}(x) = \frac{\beta\,x^4}{\Delta} + \frac{8\,\beta\,(2\alpha^2 + \beta^2)\,x^2}{\Delta^3}$$

$$Q_4^{[IK]}(x) = Q_4^{[KI]}(x) = -\frac{2\,(\alpha^2 + 2\,\beta^2)\,x^3}{\Delta^2} - \frac{16\,\beta^2(2\alpha^2 + \beta^2)\,x}{\Delta^4}$$

$$R_4^{[IK]}(x) = R_4^{[KI]}(x) = -\frac{6\,\alpha\,\beta\,x^3}{\Delta^2} - \frac{16\,\alpha\,\beta\,(2\,\alpha^2 + \beta^2)\,x}{\Delta^4}$$

$$S_4^{[IK]}(x) = -S_4^{[KI]}(x) = \frac{\alpha\,x^4}{\Delta} + \frac{4\,\alpha\,(\alpha^2 + 5\,\beta^2)\,x^2}{\Delta^3}$$

$$P_6^{[JJ]}(x) = \frac{\beta\,x^6}{\Delta} - \frac{8\,\beta\,(7\,\alpha^2 + 3\,\beta^2)\,x^4}{\Delta^3} + \frac{192\,\beta\,(3\,\sigma^2 - 2\beta^4)\,x^2}{\Delta^5}$$

$$Q_6^{[JJ]}(x) = \frac{2\,(2\,\alpha^2 + 3\,\beta^2)\,x^5}{\Delta^2} - \frac{32\,(\alpha^4 + 11\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\Delta^4} + \frac{384\,\beta^2(3\,\sigma^2 - 2\beta^4)\,x}{\Delta^6}$$

$$R_6^{[JJ]}(x) = -\frac{10\,\alpha\,\beta\,x^5}{\Delta^2} + \frac{32\,\alpha\,\beta\,(8\,\alpha^2 + 7\,\beta^2)\,x^3}{\Delta^4} - \frac{384\,\alpha\,\beta\,(3\,\sigma^2 - 2\beta^4)\,x}{\Delta^6}$$

$$S_6^{[JJ]}(x) = \frac{\alpha\,x^6}{\Delta} - \frac{16\,\alpha\,(\alpha^2 + 4\,\beta^2)\,x^4}{\Delta^3} + \frac{64\,\alpha\,(\alpha^4 + 19\,\alpha^2\beta^2 + 10\,\beta^4)\,x^2}{\Delta^5}$$

$$P_6^{[II]}(x) = -P_6^{[KK]}(x) = -\frac{\beta\,x^6}{\Delta} - \frac{8\,\beta\,(7\,\alpha^2 + 3\,\beta^2)\,x^4}{\Delta^3} - \frac{192\,\beta\,(3\,\sigma^2 - 2\,\beta^4)\,x^2}{\Delta^5}$$

$$Q_6^{[II]}(x) = Q_6^{[KK]}(x) = -\frac{2\,(2\,\alpha^2 + 3\,\beta^2)\,x^5}{\Delta^2} - \frac{32\,(\alpha^4 + 11\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\Delta^4} - \frac{384\,\beta^2(3\,\sigma^2 - 2\,\beta^4)\,x}{\Delta^6}$$

$$R_6^{[II]}(x) = R_6^{[KK]}(x) = \frac{10\,\alpha\,\beta\,x^5}{\Delta^2} + \frac{32\,\alpha\,\beta\,(8\,\alpha^2 + 7\,\beta^2)\,x^3}{\Delta^4} + \frac{384\,\alpha\,\beta\,(3\,\sigma^2 - 2\,\beta^4)\,x}{\Delta^6}$$

$$S_6^{[II]}(x) = -S_6^{[KK]}(x) = \frac{\alpha\,x^6}{\Delta} + \frac{16\,\alpha\,(\alpha^2 + 4\,\beta^2)\,x^4}{\Delta^3} + \frac{64\,\alpha\,(\alpha^4 + 19\,\alpha^2\beta^2 + 10\,\beta^4)\,x^2}{\Delta^5}$$

$$P_6^{[JI]}(x) = -P_6^{[JK]}(x) = \frac{\beta\,x^6}{\sigma} - \frac{8\,\beta\,(7\,\alpha^2 - 3\,\beta^2)\,x^4}{\sigma^3} + \frac{192\,\beta\,(3\,\Delta^2 - 2\,\beta^4)\,x^2}{\sigma^5}$$

$$Q_6^{[JI]}(x) = Q_6^{[JK]}(x) = \frac{2\,(2\,\alpha^2 - 3\,\beta^2)\,x^5}{\sigma^2} - \frac{32\,(\alpha^4 - 11\,\beta^2\alpha^2 + 3\,\beta^4)\,x^3}{\sigma^4} - \frac{384\,\beta^2(3\,\Delta^2 - 2\beta^4)\,x}{\sigma^6}$$

$$R_6^{[JI]}(x) = -R_6^{[JK]}(x) = -\frac{10\,\alpha\,\beta\,x^5}{\sigma^2} + \frac{32\,\alpha\,\beta\,(8\,\alpha^2 - 7\,\beta^2)\,x^3}{\sigma^4} - \frac{384\,\alpha\,\beta\,(3\,\Delta^2 - 2\,\beta^4)\,x}{\sigma^6}$$

$$S_6^{[JI]}(x) = S_6^{[JK]}(x) = \frac{\alpha\,x^6}{\sigma} - \frac{16\,\alpha\,(\alpha^2 - 4\,\beta^2)\,x^4}{\sigma^3} + \frac{64\,\alpha\,(\alpha^4 - 19\,\beta^2\alpha^2 + 10\,\beta^4)\,x^2}{\sigma^5}$$

$$P_6^{[IJ]}(x) = P_6^{[KJ]}(x) = -\frac{\beta\,x^6}{\sigma} - \frac{8\,\beta\,(7\,\alpha^2 - 3\,\beta^2)\,x^4}{\sigma^3} - \frac{192\,\beta\,(3\,\Delta^2 - 2\,\beta^4)\,x^2}{\sigma^5}$$

$$Q_6^{[IJ]}(x) = Q_6^{[KJ]}(x) = -\frac{2\,(2\,\alpha^2 - 3\,\beta^2)\,x^5}{\sigma^2} - \frac{32\,(\alpha^4 - 11\,\alpha^2\,\beta^2 + 3\,\beta^4)\,x^3}{\sigma^4} + \frac{384\,\beta^2(3\,\Delta^2 - 2\beta^4)\,x}{\sigma^6}$$

$$R_6^{[IJ]}(x) = -R_6^{[KJ]}(x) = \frac{10\,\alpha\,\beta\,x^5}{\sigma^2} + \frac{32\,\alpha\,\beta\,(8\,\alpha^2 - 7\,\beta^2)\,x^3}{\sigma^4} + \frac{384\,\alpha\,\beta\,(3\,\Delta^2 - 2\,\beta^4)\,x}{\sigma^6}$$

$$S_6^{[IJ]}(x) = -S_6^{[KJ]}(x) = \frac{\alpha\,x^6}{\sigma} + \frac{16\,\alpha\,(\alpha^2 - 4\,\beta^2)\,x^4}{\sigma^3} + \frac{64\,\alpha\,(\alpha^4 - 19\,\alpha^2\,\beta^2 + 10\,\beta^4)\,x^2}{\sigma^5}$$

$$P_6^{[IK]}(x) = -P_6^{[KI]}(x) = \frac{\beta\,x^6}{\Delta} + \frac{8\,\beta\,(7\,\alpha^2 + 3\,\beta^2)\,x^4}{\Delta^3} + \frac{192\,\beta\,(3\,\sigma^2 - 2\beta^2)\,x^2}{\Delta^5}$$

$$Q_6^{[IK]}(x) = Q_6^{[KI]}(x) = -\frac{2\,(2\,\alpha^2 + 3\,\beta^2)\,x^5}{\Delta^2} - \frac{32\,(\alpha^4 + 11\,\beta^2\alpha^2 + 3\,\beta^4)\,x^3}{\Delta^4} - \frac{384\,\beta^2(3\,\sigma^2 - 2\beta^2)\,x}{\Delta^6}$$

$$R_6^{[IK]}(x) = R_6^{[KI]}(x) = -\frac{10\,\alpha\,\beta\,x^5}{\Delta^2} - \frac{32\,\alpha\,\beta\,(8\,\alpha^2 + 7\,\beta^2)\,x^3}{\Delta^4} - \frac{384\,\alpha\,\beta\,(3\,\sigma^2 - 2\beta^2)\,x}{\Delta^6}$$

$$S_6^{[IK]}(x) = -S_6^{[KI]}(x) = \frac{\alpha\,x^6}{\Delta} + \frac{16\,\alpha\,(\alpha^2 + 4\,\beta^2)\,x^4}{\Delta^3} + \frac{64\,\alpha\,(\alpha^4 + 10\,\beta^4 + 19\,\beta^2\alpha^2)\,x^2}{\Delta^5}$$

$$P_8^{[JJ]}(x) = \frac{\beta}{\Delta}\,x^8 - \frac{24\,\beta\,(5\,\alpha^2 + 2\,\beta^2)\,x^6}{\Delta^3} + \frac{192\,\beta\,(21\,\alpha^4 + 43\,\alpha^2\beta^2 + 6\,\beta^4)\,x^4}{\Delta^5} -$$
$$- \frac{9216\,\beta\,(4\,\alpha^6 + 18\,\alpha^4\beta^2 + 12\,\alpha^2\beta^4 + \beta^6)\,x^2}{\Delta^7}$$

$$Q_8^{[JJ]}(x) = \frac{2\,(3\,\alpha^2 + 4\,\beta^2)\,x^7}{\Delta^2} - \frac{48\,(3\,\alpha^4 + 26\,\alpha^2\beta^2 + 6\,\beta^4)\,x^5}{\Delta^4} + \frac{384\,(3\,\alpha^6 + 86\,\alpha^4\beta^2 + 109\,\alpha^2\beta^4 + 12\,\beta^6)\,x^3}{\Delta^6} -$$
$$- \frac{18432\,\beta^2(4\,\alpha^6 + 18\,\alpha^4\beta^2 + 12\,\alpha^2\beta^4 + \beta^6)\,x}{\Delta^8}$$

$$R_8^{[JJ]}(x) = -\frac{14\,\alpha\,\beta\,x^7}{\Delta^2} + \frac{48\,\alpha\,\beta\,(18\,\alpha^2 + 17\,\beta^2)\,x^5}{\Delta^4} - \frac{1920\,\alpha\,\beta\,(9\,\alpha^4 + 26\,\alpha^2\beta^2 + 7\,\beta^4)\,x^3}{\Delta^6} +$$
$$+ \frac{18432\,\alpha\,\beta\,(4\,\alpha^6 + 18\,\alpha^4\beta^2 + 12\,\alpha^2\beta^4 + \beta^6)\,x}{\Delta^8}$$

$$S_8^{[JJ]}(x) = \frac{\alpha\, x^8}{\Delta} - \frac{12\,\alpha\,(3\,\alpha^2 + 11\,\beta^2)\,x^6}{\Delta^3} + \frac{192\,\alpha\,(3\,\alpha^4 + 44\,\alpha^2\beta^2 + 23\,\beta^4)\,x^4}{\Delta^5} -$$
$$- \frac{768\,\alpha\,(3\,\alpha^6 + 131\,\alpha^4\beta^2 + 239\,\alpha^2\beta^4 + 47\,\beta^6)\,x^2}{\Delta^7}$$

$$P_8^{[II]}(x) = -P_8^{[KK]}(x) = -\frac{\beta\, x^8}{\Delta} - \frac{24\,\beta\,(5\,\alpha^2 + 2\,\beta^2)\,x^6}{\Delta^3} - \frac{192\,\beta\,(21\,\alpha^4 + 43\,\alpha^2\beta^2 + 6\,\beta^4)\,x^4}{\Delta^5} -$$
$$- \frac{9216\,\beta\,(4\,\alpha^6 + 18\,\alpha^4\beta^2 + 12\,\alpha^2\beta^4 + \beta^6)\,x^2}{\Delta^7}$$

$$Q_8^{[II]}(x) = Q_8^{[KK]}(x) = -\frac{2\,(3\,\alpha^2 + 4\,\beta^2)\,x^7}{\Delta^2} - \frac{48\,(3\,\alpha^4 + 26\,\alpha^2\beta^2 + 6\,\beta^4)\,x^5}{\Delta^4} -$$
$$- \frac{384\,(3\,\alpha^6 + 86\,\alpha^4\beta^2 + 109\,\alpha^2\beta^4 + 12\,\beta^6)\,x^3}{\Delta^6} - \frac{18432\,\beta^2(4\,\alpha^6 + 18\,\alpha^4\beta^2 + 12\,\alpha^2\beta^4 + \beta^6)\,x}{\Delta^8}$$

$$R_8^{[II]}(x) = R_8^{[KK]}(x) = \frac{14\,\alpha\,\beta\, x^7}{\Delta^2} + \frac{48\,\alpha\,\beta\,(18\,\alpha^2 + 17\,\beta^2)\,x^5}{\Delta^4} + \frac{1920\,\alpha\,\beta\,(9\,\alpha^4 + 26\,\alpha^2\beta^2 + 7\,\beta^4)\,x^3}{\Delta^6} +$$
$$+ \frac{18432\,\alpha\,\beta\,(4\,\alpha^6 + 18\,\alpha^4\beta^2 + 12\,\alpha^2\beta^4 + \beta^6)\,x}{\Delta^8}$$

$$S_8^{[II]}(x) = -S_8^{[KK]}(x) = \frac{\alpha\, x^8}{\Delta} + \frac{12\,\alpha\,(3\,\alpha^2 + 11\,\beta^2)\,x^6}{\Delta^3} + \frac{192\,\alpha\,(3\,\alpha^4 + 44\,\alpha^2\beta^2 + 23\,\beta^4)\,x^4}{\Delta^5} +$$
$$+ \frac{768\,\alpha\,(3\,\alpha^6 + 131\,\alpha^4\beta^2 + 239\,\alpha^2\beta^4 + 47\,\beta^6)\,x^2}{\Delta^7}$$

$$P_8^{[JI]}(x) = -P_8^{[JK]}(x) = \frac{\beta\, x^8}{\sigma} - \frac{24\,\beta\,(5\,\alpha^2 - 2\,\beta^2)\,x^6}{\sigma^3} + \frac{192\,\beta\,(21\,\alpha^4 - 43\,\alpha^2\,\beta^2 + 6\,\beta^4)\,x^4}{\sigma^5} -$$
$$- \frac{9216\,\beta\,(4\,\alpha^6 - 18\,\alpha^4\,\beta^2 + 12\,\alpha^2\beta^4 - \beta^6)\,x^2}{\sigma^7}$$

$$Q_8^{[JI]}(x) = Q_8^{[JK]}(x) = \frac{2\,(3\,\alpha^2 - 4\,\beta^2)\,x^7}{\sigma^2} - \frac{48\,(3\,\alpha^4 - 26\,\alpha^2\,\beta^2 + 6\,\beta^4)\,x^5}{\sigma^4} +$$
$$+ \frac{384\,(3\,\alpha^6 - 86\,\alpha^4\,\beta^2 + 109\,\alpha^2\beta^4 - 12\,\beta^6)\,x^3}{(\sigma)^6} + \frac{18432\,\beta^2(4\,\alpha^6 - 18\,\alpha^4\,\beta^2 + 12\,\alpha^2\beta^4 - \beta^6)\,x}{\sigma^8}$$

$$R_8^{[JI]}(x) = -R_8^{[JK]}(x) = -\frac{14\,\alpha\,\beta\, x^7}{\sigma^2} + \frac{48\,\alpha\,\beta\,(18\,\alpha^2 - 17\,\beta^2)\,x^5}{\sigma^4} - \frac{1920\,\alpha\,\beta\,(9\,\alpha^4 - 26\,\alpha^2\,\beta^2 + 7\,\beta^4)\,x^3}{\sigma^6} +$$
$$+ \frac{18432\,\alpha\,\beta\,(4\,\alpha^6 - 18\,\alpha^4\,\beta^2 + 12\,\alpha^2\,\beta^4 - \beta^6)\,x}{\sigma^8}$$

$$S_8^{[JI]}(x) = S_8^{[JK]}(x) = \frac{\alpha\, x^8}{\sigma} - \frac{12\,\alpha\,(3\,\alpha^2 - 11\,\beta^2)\,x^6}{\sigma^3} + \frac{192\,\alpha\,(3\,\alpha^4 - 44\,\alpha^2\,\beta^2 + 23\,\beta^4)\,x^4}{\sigma^5} -$$
$$- \frac{768\,\alpha\,(3\,\alpha^6 - 131\,\alpha^4\,\beta^2 + 239\,\alpha^2\,\beta^4 - 47\,\beta^6)\,x^2}{\sigma^7}$$

$$P_8^{[IJ]}(x) = P_8^{[KJ]}(x) = -\frac{\beta\, x^8}{\sigma} - \frac{24\,\beta\,(5\,\alpha^2 - 2\,\beta^2)\,x^6}{\sigma^3} - \frac{192\,\beta\,(21\,\alpha^4 - 43\,\alpha^2\,\beta^2 + 6\,\beta^4)\,x^4}{\sigma^5} -$$
$$- \frac{9216\,\beta\,(4\,\alpha^6 - 18\alpha^4\,\beta^2 + 12\,\alpha^2\,\beta^4 - \beta^6)\,x^2}{\sigma^7}$$

$$Q_8^{[IJ]}(x) = Q_8^{[KJ]}(x) = -\frac{2\,(3\,\alpha^2 - 4\,\beta^2)\,x^7}{\sigma^2} - \frac{48\,(3\,\alpha^4 - 26\,\alpha^2\,\beta^2 + 6\,\beta^4)\,x^5}{\sigma^4} -$$
$$- \frac{384\,(3\,\alpha^6 - 86\alpha^4\,\beta^2 + 109\,\alpha^2\,\beta^4 - 12\,\beta^6)\,x^3}{\sigma^6} + \frac{18432\,\beta^2(4\,\alpha^6 - 18\alpha^4\,\beta^2 + 12\,\alpha^2\,\beta^4 - \beta^6)\,x}{\sigma^8}$$

$$R_8^{[IJ]}(x) = -R_8^{[KJ]}(x) = \frac{14\,\alpha\,\beta\,x^7}{\sigma^2} + \frac{48\,\alpha\,\beta\,(18\,\alpha^2 - 17\,\beta^2)\,x^5}{\sigma^4} + \frac{1920\,\alpha\,\beta\,(9\,\alpha^4 - 26\,\alpha^2\,\beta^2 + 7\,\beta^4)\,x^3}{\sigma^6} +$$

$$+ \frac{18432\,\alpha\,\beta\,(4\,\alpha^6 - 18\alpha^4\,\beta^2 + 12\,\alpha^2\,\beta^4 - \beta^6)\,x}{\sigma^8}$$

$$S_8^{[IJ]}(x) = -S_8^{[KJ]}(x) = \frac{\alpha\,x^8}{\sigma} + \frac{12\,\alpha\,(3\,\alpha^2 - 11\,\beta^2)\,x^6}{\sigma^3} + \frac{192\,\alpha\,(3\,\alpha^4 - 44\,\alpha^2\,\beta^2 + 23\,\beta^4)\,x^4}{\sigma^5} +$$

$$+ \frac{768\,\alpha\,(3\,\alpha^6 - 131\alpha^4\,\alpha^2 + 239\,\alpha^2\,\beta^4 - 47\,\beta^6)\,x^2}{\sigma^7}$$

$$P_8^{[IK]}(x) = -P_8^{[KI]}(x) = \frac{\beta\,x^8}{\Delta} + \frac{24\,\beta\,(5\,\alpha^2 + 2\,\beta^2)\,x^6}{\Delta^3} + \frac{192\,\beta\,(21\,\alpha^4 + 43\,\alpha^2\,\beta^2 + 6\,\beta^4)\,x^4}{\Delta^5} +$$

$$+ \frac{9216\,\beta\,(4\,\alpha^6 + 18\,\alpha^4\,\beta^2 + 12\,\alpha^2\beta^4 + \beta^6)\,x^2}{\Delta^7}$$

$$Q_8^{[IK]}(x) = Q_8^{[KI]}(x) = -\frac{2\,(3\,\alpha^2 + 4\,\beta^2)\,x^7}{\Delta^2} - \frac{48\,(3\,\alpha^4 + 26\,\alpha^2\,\beta^2 + 6\,\beta^4)\,x^5}{\Delta^4} -$$

$$- \frac{384\,(3\,\alpha^6 + 86\,\alpha^4\,\beta^2 + 109\,\alpha^2\beta^4 + 12\,\beta^6)\,x^3}{\Delta^6} - \frac{18432\,\beta^2(4\,\alpha^6 + 18\,\alpha^4\,\beta^2 + 12\,\alpha^2\beta^4 + \beta^6)\,x}{\Delta^8}$$

$$R_8^{[IK]}(x) = R_8^{[KI]}(x) = -\frac{14\,\alpha\,\beta\,x^7}{\Delta^2} - \frac{48\,\alpha\,\beta\,(18\,\alpha^2 + 17\,\beta^2)\,x^5}{\Delta^4} -$$

$$- \frac{1920\,\alpha\,\beta\,(9\,\alpha^4 + 26\,\alpha^2\,\beta^2 + 7\,\beta^4)\,x^3}{\Delta^6} - \frac{18432\,\alpha\,\beta\,(4\,\alpha^6 + 18\,\alpha^4\,\beta^2 + 12\,\alpha^2\beta^4 + \beta^6)\,x}{\Delta^8}$$

$$S_8^{[IK]}(x) = -S_8^{[KI]}(x) = \frac{\alpha\,x^8}{\Delta} + \frac{12\,\alpha\,(3\,\alpha^2 + 11\,\beta^2)\,x^6}{\Delta^3} + \frac{192\,\alpha\,(3\,\alpha^4 + 44\,\alpha^2\,\beta^2 + 23\,\beta^4)\,x^4}{\Delta^5} +$$

$$+ \frac{768\,\alpha\,(3\,\alpha^6 + 131\,\alpha^4\,\beta^2 + 239\,\alpha^2\beta^4 + 47\,\beta^6)\,x^2}{\Delta^7}$$

Recurrence relations: See also page 309.

$$\int x^{2n+2}\,J_0(\alpha x)\,J_1(\beta x)\,dx = x^{2n-1}\left\{\left[\frac{\beta\,x^3}{\Delta} - \frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\Delta^2}\right]J_0(\alpha x)\,J_0(\beta x) +\right.$$

$$+ \left[\frac{2\,(n\,\sigma + \beta^2)\,x^2}{\Delta^2} + \frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\Delta^2}\right]J_0(\alpha x)\,J_1(\beta x) - \frac{2\,(2n+1)\alpha\beta\,x^2}{\Delta^2}\,J_1(\alpha x)\,J_0(\beta x) +$$

$$+ \left.\left[\frac{\alpha\,x^3}{\Delta} + \frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\Delta^2}\right]J_1(\alpha x)\,J_1(\beta x)\right\} -$$

$$- \frac{8(2n^2\,\sigma - \beta^2)}{(2n-1)\,\Delta^2}\int x^{2n}\,J_0(\alpha x)\,J_1(\beta x)\,dx - \frac{16\,(n-1)^2 n^2(2n+1)}{(2n-1)\,\Delta^2}\int x^{2n-2}\,J_0(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^{2n+2}\,J_0(\alpha x)\,I_1(\beta x)\,dx = x^{2n-1}\left\{\left[\frac{\beta\,x^3}{\sigma} - \frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\sigma^2}\right]J_0(\alpha x)\,I_0(\beta x) +\right.$$

$$+ \left[\frac{2\,(n\,\sigma - \beta^2)\,x^2}{\sigma^2} + \frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\sigma^2}\right]J_0(\alpha x)\,I_1(\beta x) - \frac{2\,(2n+1)\alpha\beta\,x^2}{\sigma^2}\,J_1(\alpha x)\,I_0(\beta x) +$$

$$+ \left.\left[\frac{\alpha\,x^3}{\sigma} + \frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\sigma^2}\right]J_1(\alpha x)\,I_1(\beta x)\right\} -$$

$$- \frac{8(2n^2\,\sigma + \beta^2)}{(2n-1)\,\sigma^2}\int x^{2n}\,J_0(\alpha x)\,I_1(\beta x)\,dx + \frac{16\,(n-1)^2 n^2(2n+1)}{(2n-1)\,\sigma^2}\int x^{2n-2}\,J_0(\alpha x)\,I_1(\beta x)\,dx$$

$$\int x^{2n+2}\,J_0(\alpha x)\,K_1(\beta x)\,dx = x^{2n-1}\left\{\left[\frac{\beta\,x^3}{\sigma} - \frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\sigma^2}\right]J_0(\alpha x)\,K_0(\beta x) +\right.$$

$$+ \left[\frac{2\,(n\,\sigma - \beta^2)\,x^2}{\sigma^2} + \frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\sigma^2} \right] J_0(\alpha x)\,K_1(\beta x) - \frac{2\,(2n+1)\alpha\beta\,x^2}{\sigma^2}\,J_1(\alpha x)\,K_0(\beta x) +$$

$$+ \left[\frac{\alpha\,x^3}{\sigma} + \frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\sigma^2} \right] J_1(\alpha x)\,K_1(\beta x) \Bigg\} -$$

$$- \frac{8(2n^2\,\sigma + \beta^2)}{(2n-1)\,\sigma^2} \int x^{2n}\,J_0(\alpha x)\,K_1(\beta x)\,dx + \frac{16\,(n-1)^2 n^2 (2n+1)}{(2n-1)\,\sigma^2} \int x^{2n-2}\,J_0(\alpha x)\,K_1(\beta x)\,dx$$

$$\int x^{2n+2}\,I_0(\alpha x)\,J_1(\beta x)\,dx = x^{2n-1} \left\{ - \left[\frac{\beta\,x^3}{\sigma} + \frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\sigma^2} \right] I_0(\alpha x)\,J_0(\beta x) - \right.$$

$$- \left[\frac{2\,(n\,\Delta - \beta^2)\,x^2}{\sigma^2} - \frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\sigma^2} \right] I_0(\alpha x)\,J_1(\beta x) + \frac{2\,(2n+1)\alpha\beta\,x^2}{\sigma^2}\,I_1(\alpha x)\,J_0(\beta x) +$$

$$+ \left[\frac{\alpha\,x^3}{\sigma} - \frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\sigma^2} \right] I_1(\alpha x)\,J_1(\beta x) \Bigg\} +$$

$$+ \frac{8(2n^2\,\Delta + \beta^2)}{(2n-1)\,\sigma^2} \int x^{2n}\,I_0(\alpha x)\,J_1(\beta x)\,dx - \frac{16\,(n-1)^2 n^2 (2n+1)}{(2n-1)\,\sigma^2} \int x^{2n-2}\,I_0(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^{2n+2}\,I_0(\alpha x)\,I_1(\beta x)\,dx = x^{2n-1} \left\{ - \left[\frac{\beta\,x^3}{\Delta} + \frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\Delta^2} \right] I_0(\alpha x)\,I_0(\beta x) - \right.$$

$$- \left[\frac{2\,(n\,\sigma + \beta^2)\,x^2}{\Delta^2} - \frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\Delta^2} \right] I_0(\alpha x)\,I_1(\beta x) + \frac{2\,(2n+1)\alpha\beta\,x^2}{\Delta^2}\,I_1(\alpha x)\,I_0(\beta x) +$$

$$+ \left[\frac{\alpha\,x^3}{\Delta} - \frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\Delta^2} \right] I_1(\alpha x)\,I_1(\beta x) \Bigg\} +$$

$$+ \frac{8(2n^2\,\sigma - \beta^2)}{(2n-1)\,\Delta^2} \int x^{2n}\,I_0(\alpha x)\,I_1(\beta x)\,dx - \frac{16\,(n-1)^2 n^2 (2n+1)}{(2n-1)\,\Delta^2} \int x^{2n-2}\,I_0(\alpha x)\,I_1(\beta x)\,dx$$

$$\int x^{2n+2}\,K_0(\alpha x)\,J_1(\beta x)\,dx = x^{2n-1} \left\{ - \left[\frac{\beta\,x^3}{\sigma} + \frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\sigma^2} \right] K_0(\alpha x)\,J_0(\beta x) - \right.$$

$$- \left[\frac{2\,(n\,\sigma - \beta^2)\,x^2}{\sigma^2} - \frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\sigma^2} \right] K_0(\alpha x)\,J_1(\beta x) - \frac{2\,(2n+1)\alpha\beta\,x^2}{\sigma^2}\,K_1(\alpha x)\,J_0(\beta x) -$$

$$- \left[\frac{\alpha\,x^3}{\sigma} - \frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\sigma^2} \right] K_1(\alpha x)\,J_1(\beta x) \Bigg\} +$$

$$+ \frac{8(2n^2\,\Delta + \beta^2)}{(2n-1)\,\sigma^2} \int x^{2n}\,K_0(\alpha x)\,J_1(\beta x)\,dx - \frac{16\,(n-1)^2 n^2 (2n+1)}{(2n-1)\,\sigma^2} \int x^{2n-2}\,K_0(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^{2n+2}\,K_0(\alpha x)\,I_1(\beta x)\,dx = x^{2n-1} \left\{ - \left[\frac{\beta\,x^3}{\Delta} + \frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\Delta^2} \right] K_0(\alpha x)\,I_0(\beta x) - \right.$$

$$- \left[\frac{2\,(n\,\sigma + \beta^2)\,x^2}{\Delta^2} + \frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\Delta^2} \right] K_0(\alpha x)\,I_1(\beta x) - \frac{2\,(2n+1)\alpha\beta\,x^2}{\Delta^2}\,K_1(\alpha x)\,I_0(\beta x) -$$

$$- \left[\frac{\alpha\,x^3}{\Delta} - \frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\Delta^2} \right] K_1(\alpha x)\,I_1(\beta x) \Bigg\} -$$

$$+ \frac{8(2n^2\,\sigma - \beta^2)}{(2n-1)\,\Delta^2} \int x^{2n}\,K_0(\alpha x)\,I_1(\beta x)\,dx - \frac{16\,(n-1)^2 n^2 (2n+1)}{(2n-1)\,\Delta^2} \int x^{2n-2}\,K_0(\alpha x)\,I_1(\beta x)\,dx$$

$$\int x^{2n+2}\,K_0(\alpha x)\,K_1(\beta x)\,dx = x^{2n-1} \left\{ \left[\frac{\beta\,x^3}{\Delta} + \frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\Delta^2} \right] K_0(\alpha x)\,K_0(\beta x) - \right.$$

$$- \left[\frac{2\,(n\,\sigma + \beta^2)\,x^2}{\Delta^2} - \frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\Delta^2} \right] K_0(\alpha x)\,K_1(\beta x) + \frac{2\,(2n+1)\alpha\beta\,x^2}{\Delta^2}\,K_1(\alpha x)\,K_0(\beta x) -$$

$$- \left[\frac{\alpha\,x^3}{\Delta} - \frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\Delta^2} \right] K_1(\alpha x)\,K_1(\beta x) \Bigg\} +$$

$$+ \frac{8(2n^2\,\sigma - \beta^2)}{(2n-1)\,\Delta^2} \int x^{2n}\,K_0(\alpha x)\,K_1(\beta x)\,dx - \frac{16\,(n-1)^2 n^2(2n+1)}{(2n-1)\,\Delta^2} \int x^{2n-2}\,K_0(\alpha x)\,K_1(\beta x)\,dx$$

2.2.3. Integrals of the type $\int x^{2n+1} Z_\nu(\alpha x) Z_\nu(\beta x)\, dx$ and $x^{2n+1} V_\nu(\alpha x) W_\nu(\beta x)$, $\alpha^2 \neq \beta^2$

Let

$$\int x^{2n+1} J_\nu(\alpha x)\, J_\nu(\beta x)\, dx =$$

$$= A_\nu(x)\, J_0(\alpha x)\, J_0(\beta x) + B_\nu(x)\, J_0(\alpha x)\, J_1(\beta x) + C_\nu(x)\, J_1(\alpha x)\, J_0(\beta x) + D_\nu(x)\, J_1(\alpha x)\, J_1(\beta x)\,,$$

then in this formula $J_\mu(\alpha x)$ or $J_\mu(\beta x)$ may be substituted by $Y_\mu(\alpha x)$, $H_\mu^{(p)}(\alpha x)$, $p = 1, 2$, or $Y_\mu(\beta x)$, $H_\mu^{(p)}(\beta x)$, $p = 1, 2$, respectively. The functions $A_\mu(x)$, $B_\mu(x)$, $C_\mu(x)$ and $D_\mu(x)$ are always the same in all cases. Therefore the integrals are given with $J_\nu(\alpha x)\, J_\nu(\beta x)$ only.
The same way in

$$\int x^{2n+1} J_\nu(\alpha x)\, I_\nu(\beta x)\, dx =$$

$$= P_\nu(x)\, J_0(\alpha x)\, I_0(\beta x) + Q_\nu(x)\, J_0(\alpha x)\, I_1(\beta x) + R_\nu(x)\, J_1(\alpha x)\, I_0(\beta x) + S_\nu(x)\, J_1(\alpha x)\, I_1(\beta x)\,,$$

$J_\mu(\alpha x)$ may be substituted by $Y_\mu(\alpha x)$ or $H_\mu^{(p)}(\alpha x)$ without changing the coefficients $P_\nu(x)$ (and so on). The same holds for the integrals $\int x^{2n+1} J_\nu(\alpha x)\, K_\nu(\beta x)\, dx$. In both cases the integrals are given with $J_\nu(\alpha x)$ only.

Let

$$\boxed{\Delta = \alpha^2 - \beta^2 \qquad \text{and} \qquad \sigma = \alpha^2 + \beta^2\,.}$$

a) $\nu = 0$:

$$\int x \cdot J_0(\alpha x) J_0(\beta x)\, dx = \frac{\alpha x J_1(\alpha x) J_0(\beta x) - \beta x J_0(\alpha x) J_1(\beta x)}{\Delta}$$

$$\int x \cdot I_0(\alpha x) I_0(\beta x)\, dx = \frac{\alpha x I_1(\alpha x) I_0(\beta x) - \beta x I_0(\alpha x) I_1(\beta x)}{\Delta}$$

$$\int x \cdot K_0(\alpha x) K_0(\beta x)\, dx = -\frac{\alpha x K_1(\alpha x) K_0(\beta x) - \beta x K_0(\alpha x) K_1(\beta x)}{\Delta}$$

$$\int x \cdot J_0(\alpha x) I_0(\beta x)\, dx = \frac{\alpha x J_1(\alpha x) I_0(\beta x) + \beta x J_0(\alpha x) I_1(\beta x)}{\sigma}$$

$$\int x \cdot J_0(\alpha x) K_0(\beta x)\, dx = \frac{\alpha x J_1(\alpha x) K_0(\beta x) - \beta x J_0(\alpha x) K_1(\beta x)}{\sigma}$$

$$\int x \cdot I_0(\alpha x) K_0(\beta x)\, dx = \frac{\alpha x I_1(\alpha x) K_0(\beta x) + \beta x I_0(\alpha x) k_1(\beta x)}{\Delta}$$

$$\int x^3 \cdot J_0(\alpha x) J_0(\beta x)\, dx =$$

$$= \frac{2x^2}{\Delta^2}\left[\sigma J_0(\alpha x) J_0(\beta x) + 2\alpha\beta J_1(\alpha x) J_1(\beta x)\right] + \left[\frac{4\sigma x}{\Delta^3} - \frac{x^3}{\Delta}\right] \cdot \left[\beta J_0(\alpha x) J_1(\beta x) - \alpha J_1(\alpha x) J_0(\beta x)\right]$$

$$\int x^3 \cdot I_0(\alpha x) I_0(\beta x)\, dx =$$

$$= \frac{-2x^2}{\Delta^2}\left[\sigma I_0(\alpha x) I_0(\beta x) - 2\alpha\beta I_1(\alpha x) I_1(\beta x)\right] - \left[\frac{4\sigma x}{\Delta^3} + \frac{x^3}{\Delta}\right] \cdot \left[\beta I_0(\alpha x) I_1(\beta x) - \alpha I_1(\alpha x) I_0(\beta x)\right]$$

$$\int x^3 \cdot K_0(\alpha x) K_0(\beta x)\, dx =$$

$$= \frac{-2x^2}{\Delta^2}\left[\sigma K_0(\alpha x) K_0(\beta x) - 2\alpha\beta K_1(\alpha x) K_1(\beta x)\right] + \left[\frac{4\sigma x}{\Delta^3} + \frac{x^3}{\Delta}\right] \cdot \left[\beta K_0(\alpha x) K_1(\beta x) - \alpha K_1(\alpha x) K_0(\beta x)\right]$$

$$\int x^3 \cdot J_0(\alpha x) I_0(\beta x)\, dx =$$

$$= \frac{2x^2}{\sigma^2} \left[\Delta\, J_0(\alpha x)I_0(\beta x) - 2\alpha\beta J_1(\alpha x)I_1(\beta x)\right] - \left[\frac{4\Delta\, x}{\sigma^3} - \frac{x^3}{\sigma}\right] \cdot \left[\beta J_0(\alpha x)I_1(\beta x) + \alpha J_1(\alpha x)I_0(\beta x)\right]$$

$$\int x^3 \cdot J_0(\alpha x)K_0(\beta x)\, dx \; =$$

$$= \frac{2x^2}{\sigma^2} \left[\Delta\, J_0(\alpha x)K_0(\beta x) + 2\alpha\beta J_1(\alpha x)K_1(\beta x)\right] + \left[\frac{4\Delta\, x}{\sigma^3} - \frac{x^3}{\sigma}\right] \cdot \left[\beta J_0(\alpha x)K_1(\beta x) - \alpha J_1(\alpha x)K_0(\beta x)\right]$$

$$\int x^3 \cdot I_0(\alpha x)K_0(\beta x)\, dx \; =$$

$$= -\frac{2x^2}{\Delta^2} \left[\sigma I_0(\alpha x)K_0(\beta x) + 2\alpha\beta I_1(\alpha x)K_1(\beta x)\right] + \left[\frac{4\sigma x}{\Delta^3} + \frac{x^3}{\Delta}\right] \cdot \left[\beta I_0(\alpha x)K_1(\beta x) + \alpha I_1(\alpha x)K_0(\beta x)\right]$$

Let

$$\int x^m\, F_0(\alpha x)G_0(\beta x)\, dx =$$

$$= P_m^{[FG]}(x)F_0(\alpha x)G_0(\beta x) + Q_m^{[FG]}(x)F_0(\alpha x)G_1(\beta x) + R_m^{[FG]}(x)F_1(\alpha x)G_0(\beta x) + S_m^{[FG]}(x)F_1(\alpha x)G_1(\beta x) \;.$$

One has

$$P_m^{[JJ]} = P_m^{[YY]} = P_m^{[H^{(1)}H^{(1)}]} = P_m^{[H^{(2)}H^{(2)}]} = P_m^{[JY]} = P_m^{[JH^{(1)}]} = P_m^{[JH^{(2)}]} = P_m^{[YH^{(1)}]} = P_m^{[YH^{(2)}]} = P_m^{[H^{(1)}H^{(2)}]} \;,$$

$$P_m^{[JI]} = P_m^{[YI]} = P_m^{[H^{(1)}I]} = P_m^{[H^{(2)}I]} \quad \text{and} \quad P_m^{[JK]} = P_m^{[YK]} = P_m^{[H^{(1)}K]} = P_m^{[H^{(2)}K]} \;.$$

The same holds analogous for the polynomials $Q_m(x)$, $R_m(x)$ and $S_m(x)$.

$$P_5^{[JJ]}(x) = \frac{4\sigma\, x^4}{\Delta^2} - \frac{32\,(\sigma^2 + 2\,\alpha^2\beta^2)\, x^2}{\Delta^4}$$

$$Q_5^{[JJ]}(x) = -\frac{\beta\, x^5}{\Delta} + \frac{16\,\beta\,(2\,\alpha^2 + \beta^2)\, x^3}{\Delta^3} - \frac{64\,\beta\,(\sigma^2 + 2\,\alpha^2\beta^2)\, x}{\Delta^5}$$

$$R_5^{[JJ]}(x) = \frac{\alpha\, x^5}{\Delta} - \frac{16\,\alpha\,(\alpha^2 + 2\,\beta^2)\, x^3}{\Delta^3} + \frac{64\,\alpha\,(\sigma^2 + 2\,\alpha^2\beta^2)\, x}{\Delta^5}$$

$$S_5^{[JJ]}(x) = \frac{8\,\alpha\,\beta\, x^4}{\Delta^2} - \frac{96\,\alpha\,\beta\,\sigma\, x^2}{\Delta^4}$$

$$P_5^{[II]}(x) = P_5^{[KK]}(x) = -\frac{4\,\sigma\, x^4}{\Delta^2} - \frac{32\,(\sigma^2 + \alpha^2\beta^2)\, x^2}{\Delta^4}$$

$$Q_5^{[II]}(x) = -Q_5^{[KK]}(x) = -\frac{\beta\, x^5}{\Delta} - \frac{16\,\beta\,(2\alpha^2 + \beta^2)\, x^3}{\Delta^3} - \frac{64\,\beta\,(\sigma^2 + 2\alpha^2\beta^2)}{\Delta^5}$$

$$R_5^{[II]}(x) = -R_5^{[KK]}(x) = \frac{\alpha\, x^5}{\Delta} + \frac{16\,\alpha\,(\alpha^2 + 2\,\beta^2)\, x^3}{\Delta^3} + \frac{64\,\alpha\,(\sigma^2 + 2\alpha^2\beta^2)\, x}{\Delta^5}$$

$$S_5^{[II]}(x) = S_5^{[KK]}(x) = \frac{8\,\alpha\,\beta\, x^4}{\Delta^2} + \frac{96\,\alpha\,\beta\,\sigma\, x^2}{\Delta^4}$$

$$P_5^{[JI]}(x) = \frac{4\,\Delta\, x^4}{\sigma^2} - \frac{32\,(\Delta^2 - 2\alpha^2\beta^2)\, x^2}{\sigma^4}$$

$$Q_5^{[JI]}(x) = \frac{\beta\, x^5}{\sigma} - \frac{16\,\beta\,(2\alpha^2 + \beta^2)\, x^3}{\sigma^3} + \frac{64\,\beta\,(\Delta^2 - 2\alpha^2\beta^2)\, x}{\sigma^5}$$

$$R_5^{[JI]}(x) = \frac{\alpha\, x^5}{\sigma} - \frac{16\,\alpha\,(\alpha^2 + 2\beta^2)\, x^3}{\sigma^3} + \frac{64\,\alpha\,(\Delta^2 - 2\alpha^2\beta^2)\, x}{\sigma^5}$$

$$S_5^{[JI]}(x) = -\frac{8\,\alpha\,\beta\, x^4}{\sigma^2} + \frac{96\,\alpha\,\beta\,\Delta\, x^2}{\sigma^4}$$

$$P_5^{[JK]}(x) = \frac{4\,\Delta\,x^4}{\sigma^2} - \frac{32\,(\Delta^2 - 2\alpha^2\beta^2)x^2}{\sigma^4}$$

$$Q_5^{[JK]}(x) = -\frac{\beta\,x^5}{\sigma} + \frac{16\,\beta\,(2\,\alpha^2 - \beta^2)\,x^3}{\sigma^3} - \frac{64\,\beta\,(\Delta^2 - 2\alpha^2\beta^2)\,x}{\sigma^5}$$

$$R_5^{[JK]}(x) = \frac{\alpha\,x^5}{\sigma} - \frac{16\,\alpha\,(\alpha^2 - 2\beta^2)\,x^3}{\sigma^3} + \frac{64\,\alpha\,(\Delta^2 - 2\alpha^2\beta^2)\,x}{\sigma^5}$$

$$S_5^{[JK]}(x) = \frac{8\,\alpha\,\beta\,x^4}{\sigma^2} - \frac{96\,\alpha\,\beta\,\Delta\,x^2}{\sigma^4}$$

$$P_5^{[IK]}(x) = -\frac{4\,\sigma\,x^4}{\Delta^2} - \frac{32\,(\sigma^2 + 2\alpha^2\beta^2)\,x^2}{\Delta^4}$$

$$Q_5^{[IK]}(x) = \frac{\beta\,x^5}{\Delta} + \frac{16\,\beta\,(2\sigma^2 - \beta^2)\,x^3}{\Delta^3} + \frac{64\,\beta\,(\sigma^2 + 2\alpha^2\beta^2)\,x}{\Delta^5}$$

$$R_5^{[IK]}(x) = \frac{\alpha\,x^5}{\Delta} + \frac{16\,\alpha\,(2\sigma - \alpha^2)\,x^3}{\Delta^3} + \frac{64\,\alpha\,(\sigma^2 + 2\alpha^2\beta^2)\,x}{\Delta^5}$$

$$S_5^{[IK]}(x) = -\frac{8\,\alpha\,\beta\,x^4}{\Delta^2} - \frac{96\,\alpha\,\beta\,\sigma\,x^2}{\Delta^4}$$

$$P_7^{[JJ]}(x) = \frac{6\,\sigma\,x^6}{\Delta^2} - \frac{48\,(3\,\sigma^2 + 8\,\alpha^2\beta^2)\,x^4}{\Delta^4} + \frac{1152\,\sigma(\sigma^2 + 6\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$Q_7^{[JJ]}(x) = -\frac{\beta\,x^7}{\Delta} + \frac{12\,\beta\,(7\,\alpha^2 + 3\,\beta^2)\,x^5}{\Delta^3} -$$

$$-\frac{192\,\beta\,(8\,\alpha^4 + 19\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\Delta^5} + \frac{2304\,\beta\,\sigma(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$R_7^{[JJ]}(x) = \frac{\alpha\,x^7}{\Delta} - \frac{12\,\alpha\,(3\,\alpha^2 + 7\,\beta^2)\,x^5}{\Delta^3} +$$

$$+\frac{192\,\alpha\,(3\,\alpha^4 + 19\,\alpha^2\beta^2 + 8\,\beta^4)\,x^3}{\Delta^5} - \frac{2304\,\alpha\,\sigma(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$S_7^{[JJ]}(x) = \frac{12\,\alpha\,\beta\,x^6}{\Delta^2} - \frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4} + \frac{384\,\alpha\,\beta\,(11\,\sigma^2 + 16\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$P_7^{[II]}(x) = P_7^{[KK]}(x) = -\frac{6\,\sigma\,x^6}{\Delta^2} - \frac{48\,(3\,\sigma^2 + 8\,\alpha^2\beta^2)\,x^4}{\Delta^4} - \frac{1152\,\sigma\,(\sigma^2 + 6\,\alpha^2)\,x^2}{\Delta^6}$$

$$Q_7^{[II]}(x) = -Q_7^{[KK]}(x) = -\frac{\beta\,x^7}{\Delta} - \frac{12\,\beta\,(7\,\alpha^2 + 3\,\beta^2)\,x^5}{\Delta^3} -$$

$$-\frac{192\,\beta\,(8\,\alpha^4 + 19\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\Delta^5} - \frac{2304\,\beta\,\sigma(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$R_7^{[II]}(x) = -R_7^{[KK]}(x) = \frac{\alpha\,x^7}{\Delta} + \frac{12\,\alpha\,(3\,\alpha^2 + 7\,\beta^2)\,x^5}{\Delta^3} +$$

$$+\frac{192\,\alpha\,(3\,\alpha^4 + 19\,\alpha^2\beta^2 + 8\,\beta^4)\,x^3}{\Delta^5} + \frac{2304\,\alpha\,\sigma\,(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$S_7^{[II]}(x) = S_7^{[KK]}(x) = \frac{12\,\alpha\,\beta\,x^6}{\Delta^2} + \frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4} + \frac{384\,\alpha\,\beta\,(11\,\sigma^2 + 16\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$P_7^{[JI]}(x) = \frac{6\,\Delta\,x^6}{\sigma^2} - \frac{48\,(3\,\Delta^2 - 8\,\alpha^2\beta^2)\,x^4}{\sigma^4} + \frac{1152\,\Delta\,(\Delta^2 - 6\alpha^2\,\beta^2)\,x^2}{\sigma^6}$$

$$Q_7^{[JI]}(x) = \frac{\beta\,x^7}{\sigma} - \frac{12\,\beta\,(7\,\alpha^2 - 3\,\beta^2)\,x^5}{\sigma^3} + \frac{192\,\beta\,(8\,\alpha^4 - 19\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\sigma^5} - \frac{2304\,\beta\,\Delta\,(\Delta^2 - 6\,\alpha^2\beta^2)\,x}{\sigma^7}$$

$$R_7^{[JI]}(x) = \frac{\alpha\,x^7}{\sigma} - \frac{12\,\alpha\,(3\,\alpha^2 - 7\,\beta^2)\,x^5}{\sigma^3} + \frac{192\,\alpha\,(3\,\alpha^4 - 19\,\alpha^2\beta^2 + 8\,\beta^4)\,x^3}{\sigma^5} - \frac{2304\,\alpha\,\Delta\,(\Delta^2 - 6\,\alpha^2\beta^2)\,x}{\sigma^7}$$

$$S_7^{[JI]}(x) = -\frac{12\,\alpha\,\beta\,x^6}{\sigma^2} + \frac{480\,\alpha\,\beta\,\Delta\,x^4}{\sigma^4} - \frac{384\,\alpha\,\beta\,(11\,\Delta^2 - 16\alpha^2\,\beta^2)\,x^2}{\sigma^6}$$

$$P_7^{[JK]}(x) = \frac{6\,\Delta x^6}{\sigma^2} - \frac{48\,(3\,\Delta^2 - 8\,\alpha^2\beta^2)\,x^4}{\sigma^4} + \frac{1152\,\Delta\,(\Delta^2 - 6\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$Q_7^{[JK]}(x) = -\frac{\beta\,x^7}{\sigma} + \frac{12\,\beta\,(7\,\alpha^2 - 3\,\beta^2)\,x^5}{\sigma^3} - \frac{192\,\beta\,(8\,\alpha^4 - 19\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\sigma^5} + \frac{2304\,\beta\,\Delta\,(\Delta^2 - 6\,\alpha^2\beta^2)\,x}{\sigma^7}$$

$$R_7^{[JK]}(x) = \frac{\alpha\,x^7}{\sigma} - \frac{12\,\alpha\,(3\,\alpha^2 - 7\,\beta^2)\,x^5}{\sigma^3} + \frac{192\,\alpha\,(3\,\alpha^4 - 19\,\alpha^2\beta^2 + 8\,\beta^4)\,x^3}{\sigma^5} - \frac{2304\,\alpha\,\Delta\,(\Delta^2 - 6\,\alpha^2\beta^2)\,x}{\sigma^7}$$

$$S_7^{[JK]}(x) = \frac{12\,\alpha\,\beta\,x^6}{\sigma^2} - \frac{480\,\alpha\,\beta\,\Delta\,x^4}{\sigma^4} + \frac{384\,\alpha\,\beta\,(11\,\Delta^2 - 16\,\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$P_7^{[IK]}(x) = -\frac{6\,\sigma\,x^6}{\Delta^2} - \frac{48\,(3\,\sigma^2 + 8\,\alpha^2\beta^2)\,x^4}{\Delta^4} - \frac{1152\,\sigma\,(\sigma^2 + 6\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$Q_7^{[IK]}(x) = \frac{\beta\,x^7}{\Delta} + \frac{12\,\beta\,(7\,\alpha^2 + 3\,\beta^2)\,x^5}{\Delta^3} + \frac{192\,\beta\,(8\,\alpha^4 + 19\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\Delta^5} + \frac{2304\,\beta\,\sigma\,(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$R_7^{[IK]}(x) = \frac{\alpha\,x^7}{\Delta} + \frac{12\,\alpha\,(3\,\alpha^2 + 7\,\beta^2)x^5}{\Delta^3} + \frac{12\,\alpha\,(3\,\alpha^4 + 19\,\alpha^2\beta^2 + 8\,\beta^4)\,x^3}{\Delta^5} + \frac{2304\,\alpha\,\sigma\,(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$S_7^{[IK]}(x) = -\frac{12\,\alpha\,\beta\,x^6}{\Delta^2} - \frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4} - \frac{384\,\alpha\,\beta\,(11\,\sigma^2 + 16\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

Recurrence formulas: See also page 309.

$$\int x^{2n+1}\,J_0(\alpha x)\,J_0(\beta x)\,dx = x^{2n-2}\,\left\{ \frac{\left[2\,n\sigma\,x^2 + 8\,(n-1)^2\right]\,J_0(\alpha x)\,J_0(\beta x)}{\Delta^2} - \right.$$

$$-\left[\frac{\beta\,x^3}{\Delta}-\frac{4\,n(n-1)\,\beta\,x}{\Delta^2}\right]J_0(\alpha x)\,J_1(\beta x)+\left[\frac{\alpha\,x^3}{\Delta}+\frac{4\,n(n-1)\,\alpha\,x}{\Delta^2}\right]J_1(\alpha x)\,J_0(\beta x)+\frac{4\,\alpha\beta\,n\,x^2\,J_1(\alpha x)\,J_1(\beta x)}{\Delta^2}\Bigg\}-$$

$$-\frac{4\,(2n-1)n\,\sigma}{\Delta^2}\int x^{2n-1}\,J_0(\alpha x)\,J_0(\beta x)\,dx-\frac{16\,(n-1)^3 n}{\Delta^2}\int x^{2n-3}\,J_0(\alpha x)\,J_0(\beta x)\,dx$$

$$\int x^{2n+1}\,I_0(\alpha x)\,I_0(\beta x)\,dx=x^{2n-2}\left\{-\frac{\left[2\,n\sigma\,x^2-8\,(n-1)^2\right]\,I_0(\alpha x)\,I_0(\beta x)}{\Delta^2}-\right.$$

$$-\left[\frac{\beta\,x^3}{\Delta}+\frac{4\,n(n-1)\,\beta\,x}{\Delta^2}\right]I_0(\alpha x)\,I_1(\beta x)+\left[\frac{\alpha\,x^3}{\Delta}-\frac{4\,n(n-1)\,\alpha\,x}{\Delta^2}\right]I_1(\alpha x)\,I_0(\beta x)+\frac{4\,\alpha\beta\,n\,x^2\,I_1(\alpha x)\,I_1(\beta x)}{\Delta^2}\Bigg\}+$$

$$+\frac{4\,(2n-1)n\,\sigma}{\Delta^2}\int x^{2n-1}\,I_0(\alpha x)\,I_0(\beta x)\,dx-\frac{16\,(n-1)^3 n}{\Delta^2}\int x^{2n-3}\,I_0(\alpha x)\,I_0(\beta x)\,dx$$

$$\int x^{2n+1}\,K_0(\alpha x)\,K_0(\beta x)\,dx=x^{2n-2}\left\{-\frac{\left[2\,n\sigma\,x^2-8\,(n-1)^2\right]\,K_0(\alpha x)\,K_0(\beta x)}{\Delta^2}+\right.$$

$$+\left[\frac{\beta\,x^3}{\Delta}+\frac{4\,n(n-1)\,\beta\,x}{\Delta^2}\right]K_0(\alpha x)\,K_1(\beta x)-\left[\frac{\alpha\,x^3}{\Delta}-\frac{4\,n(n-1)\,\alpha\,x}{\Delta^2}\right]K_1(\alpha x)\,K_0(\beta x)+\frac{4\,\alpha\beta\,n\,x^2\,K_1(\alpha x)\,K_1(\beta x)}{\Delta^2}\Bigg\}+$$

$$+\frac{4\,(2n-1)n\,\sigma}{\Delta^2}\int x^{2n-1}\,K_0(\alpha x)\,K_0(\beta x)\,dx-\frac{16\,(n-1)^3 n}{\Delta^2}\int x^{2n-3}\,K_0(\alpha x)\,K_0(\beta x)\,dx$$

$$\int x^{2n+1}\,J_0(\alpha x)\,I_0(\beta x)\,dx=x^{2n-2}\left\{\frac{\left[2\,n\Delta\,x^2+8\,(n-1)^2\right]\,J_0(\alpha x)\,I_0(\beta x)}{\sigma^2}+\right.$$

$$+\left[\frac{\beta\,x^3}{\sigma}-\frac{4\,n(n-1)\,\beta\,x}{\sigma^2}\right]J_0(\alpha x)\,I_1(\beta x)+\left[\frac{\alpha\,x^3}{\sigma}+\frac{4\,n(n-1)\,\alpha\,x}{\sigma^2}\right]J_1(\alpha x)\,I_0(\beta x)-\frac{4\,\alpha\beta\,n\,x^2\,J_1(\alpha x)\,I_1(\beta x)}{\sigma^2}\Bigg\}-$$

$$-\frac{4\,(2n-1)n\,\Delta}{\sigma^2}\int x^{2n-1}\,J_0(\alpha x)\,I_0(\beta x)\,dx-\frac{16\,(n-1)^3 n}{\sigma^2}\int x^{2n-3}\,J_0(\alpha x)\,I_0(\beta x)\,dx$$

$$\int x^{2n+1}\,J_0(\alpha x)\,K_0(\beta x)\,dx=x^{2n-2}\left\{\frac{\left[2\,n\Delta\,x^2+8\,(n-1)^2\right]\,J_0(\alpha x)\,K_0(\beta x)}{\sigma^2}-\right.$$

$$-\left[\frac{\beta\,x^3}{\sigma}-\frac{4\,n(n-1)\,\beta\,x}{\sigma^2}\right]J_0(\alpha x)\,K_1(\beta x)+\left[\frac{\alpha\,x^3}{\sigma}+\frac{4\,n(n-1)\,\alpha\,x}{\sigma^2}\right]J_1(\alpha x)\,K_0(\beta x)+\frac{4\,\alpha\beta\,n\,x^2\,J_1(\alpha x)\,I_K(\beta x)}{\sigma^2}\Bigg\}-$$

$$-\frac{4\,(2n-1)n\,\Delta}{\sigma^2}\int x^{2n-1}\,J_0(\alpha x)\,K_0(\beta x)\,dx-\frac{16\,(n-1)^3 n}{\sigma^2}\int x^{2n-3}\,J_0(\alpha x)\,K_0(\beta x)\,dx$$

$$\int x^{2n+1}\,I_0(\alpha x)\,K_0(\beta x)\,dx=x^{2n-2}\left\{-\frac{\left[2\,n\sigma\,x^2-8\,(n-1)^2\right]\,I_0(\alpha x)\,K_0(\beta x)}{\Delta^2}+\right.$$

$$+\left[\frac{\beta\,x^3}{\Delta}+\frac{4\,n(n-1)\,\beta\,x}{\Delta^2}\right]I_0(\alpha x)\,K_1(\beta x)+\left[\frac{\alpha\,x^3}{\Delta}-\frac{4\,n(n-1)\,\alpha\,x}{\Delta^2}\right]I_1(\alpha x)\,K_0(\beta x)-\frac{4\,\alpha\beta\,n\,x^2\,I_1(\alpha x)\,K_1(\beta x)}{\Delta^2}\Bigg\}+$$

$$+\frac{4\,(2n-1)n\,\sigma}{\Delta^2}\int x^{2n-1}\,I_0(\alpha x)\,K_0(\beta x)\,dx-\frac{16\,(n-1)^3 n}{\Delta^2}\int x^{2n-3}\,I_0(\alpha x)\,K_0(\beta x)\,dx$$

b) $\nu=1$:

Let again

$$\boxed{\Delta=\alpha^2-\beta^2\qquad\text{and}\qquad\sigma=\alpha^2+\beta^2\,.}$$

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$$\int x \cdot J_1(\alpha x) J_1(\beta x)\, dx \;=\; \frac{x}{\Delta}\left[\beta J_1(\alpha x) J_0(\beta x) - \alpha J_0(\alpha x) J_1(\beta x)\right]$$

$$\int x \cdot I_1(\alpha x) I_1(\beta x)\, dx \;=\; \frac{x}{\Delta}\left[\alpha I_0(\alpha x) I_1(\beta x) - \beta I_1(\alpha x) I_0(\beta x)\right]$$

$$\int x \cdot K_1(\alpha x) K_1(\beta x)\, dx \;=\; \frac{x}{\Delta}\left[\beta K_1(\alpha x) K_0(\beta x) - \alpha K_0(\alpha x) K_1(\beta x)\right]$$

$$\int x \cdot J_1(\alpha x) I_1(\beta x)\, dx \;=\; \frac{x}{\sigma}\left[\beta J_1(\alpha x) I_0(\beta x) - \alpha J_0(\alpha x) I_1(\beta x)\right]$$

$$\int x \cdot J_1(\alpha x) K_1(\beta x)\, dx \;=\; -\frac{x}{\sigma}\left[\beta J_1(\alpha x) K_0(\beta x) + \alpha J_0(\alpha x) K_1(\beta x)\right]$$

$$\int x \cdot I_1(\alpha x) K_1(\beta x)\, dx \;=\; \frac{x}{\Delta}\left[\beta I_1(\alpha x) K_0(\beta x) + \alpha I_0(\alpha x) K_1(\beta x)\right]$$

$$\int x^3 \cdot J_1(\alpha x) J_1(\beta x)\, dx \;=\; \frac{2x^2}{\Delta^2}\left[2\alpha\beta J_0(\alpha x) J_0(\beta x) + \sigma\, J_1(\alpha x) J_1(\beta x)\right] +$$

$$+\frac{8\alpha\beta x}{\Delta^3}\cdot\left[\beta J_0(\alpha x) J_1(\beta x) - \alpha J_1(\alpha x) J_0(\beta x)\right] - \frac{x^3}{\Delta}\cdot\left[\alpha J_0(\alpha x) J_1(\beta x) - \beta J_1(\alpha x) J_0(\beta x)\right]$$

$$\int x^3 \cdot I_1(\alpha x) I_1(\beta x)\, dx \;=\; \frac{2x^2}{\Delta^2}\left[2\alpha\beta I_0(\alpha x) I_0(\beta x) - \sigma\, I_1(\alpha x) I_1(\beta x)\right] +$$

$$+\frac{8\alpha\beta x}{\Delta^3}\cdot\left[\beta I_0(\alpha x) I_1(\beta x) - \alpha I_1(\alpha x) I_0(\beta x)\right] + \frac{x^3}{\Delta}\cdot\left[\alpha I_0(\alpha x) I_1(\beta x) - \beta I_1(\alpha x) I_0(\beta x)\right]$$

$$\int x^3 \cdot K_1(\alpha x) K_1(\beta x)\, dx \;=\; \frac{2x^2}{\Delta^2}\left[2\alpha\beta\, K_0(\alpha x) K_0(\beta x) - \sigma\, K_1(\alpha x) K_1(\beta x)\right] -$$

$$-\frac{8\alpha\beta x}{\Delta^3}\cdot\left[\beta K_0(\alpha x) K_1(\beta x) - \alpha K_1(\alpha x) K_0(\beta x)\right] - \frac{x^3}{\Delta}\cdot\left[\alpha K_0(\alpha x) K_1(\beta x) - \beta K_1(\alpha x) K_0(\beta x)\right]$$

$$\int x^3 \cdot J_1(\alpha x) I_1(\beta x)\, dx \;=\; \frac{2x^2}{\sigma^2}\left[2\alpha\beta J_0(\alpha x) I_0(\beta x) + \Delta\, J_1(\alpha x) I_1(\beta x)\right] -$$

$$-\frac{8\alpha\beta x}{\sigma^3}\cdot\left[\beta J_0(\alpha x) I_1(\beta x) + \alpha J_1(\alpha x) I_0(\beta x)\right] - \frac{x^3}{\sigma}\cdot\left[\alpha J_0(\alpha x) I_1(\beta x) - \beta J_1(\alpha x) I_0(\beta x)\right]$$

$$\int x^3 \cdot J_1(\alpha x) K_1(\beta x)\, dx \;=\; \frac{2x^2}{\sigma^2}\left[-2\alpha\beta J_0(\alpha x) K_0(\beta x) + \Delta\, J_1(\alpha x) K_1(\beta x)\right] -$$

$$-\frac{8\alpha\beta x}{\sigma^3}\cdot\left[\beta J_0(\alpha x) K_1(\beta x) - \alpha J_1(\alpha x) K_0(\beta x)\right] - \frac{x^3}{\sigma}\cdot\left[\alpha J_0(\alpha x) K_1(\beta x) + \beta J_1(\alpha x) K_0(\beta x)\right]$$

$$\int x^3 \cdot I_1(\alpha x) K_1(\beta x)\, dx \;=\; -\frac{2x^2}{\Delta^2}\left[2\alpha\beta I_0(\alpha x) K_0(\beta x) + \sigma\, I_1(\alpha x) K_1(\beta x)\right] +$$

$$+\frac{8\alpha\beta x}{\Delta^3}\cdot\left[\beta I_0(\alpha x) K_1(\beta x) + \alpha I_1(\alpha x) K_0(\beta x)\right] + \frac{x^3}{\Delta}\cdot\left[\alpha I_0(\alpha x) K_1(\beta x) + \beta I_1(\alpha x) K_0(\beta x)\right]$$

Let
$$\int x^m\, F_1(\alpha x) G_1(\beta x)\, dx =$$

$$= T_m^{[FG]}(x) F_0(\alpha x) G_0(\beta x) + U_m^{[FG]}(x) F_0(\alpha x) G_1(\beta x) + V_m^{[FG]}(x) F_1(\alpha x) G_0(\beta x) + W_m^{[FG]}(x) F_1(\alpha x) G_1(\beta x)\ .$$

One has

$$T_m^{[JJ]} = T_m^{[YY]} = T_m^{[H^{(1)}H^{(1)}]} = T_m^{[H^{(2)}H^{(2)}]} = T_m^{[JY]} = T_m^{[JH^{(1)}]} = T_m^{[JH^{(2)}]} = T_m^{[YH^{(1)}]} = T_m^{[YH^{(2)}]} = T_m^{[H^{(1)}H^{(2)}]}\ ,$$

$$T_m^{[JI]} = T_m^{[YI]} = T_m^{[H^{(1)}I]} = T_m^{[H^{(2)}I]} \quad \text{and} \quad T_m^{[JK]} = T_m^{[YK]} = T_m^{[H^{(1)}K]} = T_m^{[H^{(2)}K]} \;.$$

The same holds analogous for the polynomials $U_m(x)$, $V_m(x)$ and $W_m(x)$.

$$T_5^{[JJ]}(x) = \frac{8\,\alpha\,\beta\,x^4}{\Delta^2} - \frac{96\,\alpha\,\beta\,\sigma\,x^2}{\Delta^4} \;, \quad U_5^{[JJ]}(x) = -\frac{\alpha\,x^5}{\Delta} + \frac{8\,\alpha\,(\alpha^2 + 5\,\beta^2)\,x^3}{\Delta^3} - \frac{192\,\alpha\,\beta^2\sigma\,x}{\Delta^5}$$

$$V_5^{[JJ]}(x) = \frac{\beta\,x^5}{\Delta} - \frac{8\,\beta\,(5\,\alpha^2 + \beta^2)\,x^3}{\Delta^3} + \frac{192\,\alpha^2\beta\,\sigma\,x}{\Delta^5} \;, \quad W_5^{[JJ]}(x) = \frac{4\,\sigma\,x^4}{\Delta^2} - \frac{16\,(\sigma^2 + 8\,\alpha^2\beta^2)\,x^2}{\Delta^4}$$

$$T_5^{[II]}(x) = T_5^{[KK]}(x) = \frac{8\,\alpha\,\beta\,x^4}{\Delta^2} + \frac{96\,\alpha\,\beta\,\sigma\,x^2}{\Delta^4} \;, \quad U_5^{[II]}(x) = -U_5^{[KK]}(x) = \frac{\alpha\,x^5}{\Delta} + \frac{8\,\alpha\,(\alpha^2 + 5\,\beta^2)\,x^3}{\Delta^3} + \frac{192\,\alpha\,\beta^2\sigma\,x}{\Delta^5}$$

$$V_5^{[II]}(x) = -V_5^{[KK]}(x) = -\frac{\beta\,x^5}{\Delta} - \frac{8\,\beta\,(5\,\alpha^2 + \beta^2)\,x^3}{\Delta^3} - \frac{192\,\alpha^2\beta\,\sigma\,x}{\Delta^5}$$

$$W_5^{[II]}(x) = W_5^{[KK]}(x) = -\frac{4\,\sigma\,x^4}{\Delta^2} - \frac{16\,(\sigma^2 + 8\,\alpha^2\beta^2)\,x^2}{\Delta^4}$$

$$T_5^{[JI]}(x) = \frac{8\,\alpha\beta\,x^4}{\sigma^2} - \frac{96\,\alpha\beta\,\Delta\,x^2}{\sigma^4} \;, \quad U_5^{[JI]}(x) = -\frac{\alpha\,x^5}{\sigma} + \frac{\alpha\,(\alpha^2 - 5\,\beta^2)\,x^3}{\sigma^3} + \frac{192\,\alpha\,\beta^2\Delta\,x}{\sigma^5}$$

$$V_5^{[JI]}(x) = \frac{\beta\,x^5}{\sigma} - \frac{8\,\beta\,(5\,\alpha^2 - \beta^2)\,x^3}{\sigma^3} + \frac{192\,\alpha^2\beta\,\Delta\,x}{\sigma^5} \;, \quad W_5^{[JI]}(x) = \frac{4\,\Delta\,x^4}{\sigma^2} - \frac{16\,(\Delta^2 - 8\,\alpha^2\beta^2)\,x^2}{\sigma^4}$$

$$T_5^{[JK]}(x) = -\frac{8\,\alpha\,\beta\,x^4}{\sigma^2} + \frac{96\,\alpha\,\beta\,\Delta\,x^2}{\sigma^4} \;, \quad U_5^{[JK]}(x) = -\frac{\alpha\,x^5}{\sigma} + \frac{8\,\alpha\,(\alpha^2 - 5\,\beta^2)\,x^3}{\sigma^3} + \frac{192\,\alpha\,\beta^2\Delta\,x}{\sigma^5}$$

$$V_5^{[JK]}(x) = -\frac{\beta\,x^5}{\sigma} + \frac{8\,\beta\,(5\,\alpha^2 - \beta^2)\,x^3}{\sigma^3} - \frac{192\,\alpha^2\beta\,\Delta\,x}{\sigma^5} \;, \quad W_5^{[JK]}(x) = \frac{4\,\Delta\,x^4}{\sigma^2} - \frac{16\,(\Delta^2 - 8\,\alpha^2\beta^2)\,x^2}{\sigma^4}$$

$$T_5^{[IK]}(x) = -\frac{8\,\alpha\,\beta\,x^4}{\Delta^2} - \frac{96\,\alpha\,\beta\,\sigma\,x^2}{\Delta^4} \;, \quad U_5^{[IK]}(x) = \frac{\alpha\,x^5}{\Delta} + \frac{8\,\alpha\,(\alpha^2 + 5\,\beta^2)\,x^3}{\Delta^3} + \frac{192\,\alpha\,\beta^2\sigma\,x}{\Delta^5}$$

$$V_5^{[IK]}(x) = \frac{\beta\,x^5}{\Delta} + \frac{8\,\beta\,(5\,\alpha^2 + \beta^2)\,x^3}{\Delta^3} + \frac{192\,\alpha^2\beta\,\sigma\,x}{\Delta^5} \;, \quad W_5^{[IK]}(x) = -\frac{4\,\sigma\,x^4}{\Delta^2} - \frac{16\,(\sigma^2 + 8\,\alpha^2\beta^2)\,x^2}{\Delta^4}$$

$$T_7^{[JJ]}(x) = \frac{12\,\alpha\,\beta\,x^6}{\Delta^2} - \frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4} + \frac{4608\,\alpha\,\beta\,(\sigma^2 + \alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$U_7^{[JJ]}(x) = -\frac{\alpha\,x^7}{\Delta} + \frac{24\,\alpha\,(\alpha^2 + 4\,\beta^2)\,x^5}{\Delta^3} -$$

$$-\frac{192\,\alpha\,(\alpha^4 + 18\,\alpha^2\beta^2 + 11\,\beta^4)\,x^3}{\Delta^5} + \frac{9216\,\alpha\,\beta^2(\sigma^2 + \alpha^2\beta^2)\,x}{\Delta^7}$$

$$V_7^{[JJ]}(x) = \frac{\beta\,x^7}{\Delta} - \frac{24\,\beta\,(4\alpha^2 + \beta^2)\,x^5}{\Delta^3} +$$

$$+\frac{192\,\beta\,(11\,\alpha^4 + 18\,\alpha^2\beta^2 + \beta^4)\,x^3}{\Delta^5} - \frac{9216\,\alpha^2\beta\,(\sigma^2 + \alpha^2\beta^2)\,x}{\Delta^7}$$

$$W_7^{[JJ]}(x) = \frac{6\,\sigma\,x^6}{\Delta^2} - \frac{96\,(\sigma^2 + 6\,\alpha^2\beta^2)\,x^4}{\Delta^4} + \frac{384\,\sigma\,(\sigma^2 + 26\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$T_7^{[II]}(x) = T_7^{[KK]}(x) = \frac{12\,\alpha\,\beta\,x^6}{\Delta^2} + \frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4} + \frac{4608\,\alpha\,\beta\,(\sigma^2 + \alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$U_7^{[II]}(x) = -U_7^{[KK]}(x) = \frac{\alpha\,x^7}{\Delta} + \frac{24\,\alpha\,(\alpha^2 + 4\,\beta^2)\,x^5}{\Delta^3} +$$

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$$+\frac{192\,\alpha\,(\sigma^2+16\,\alpha^2\beta^2)\,x^3}{\Delta^5}+\frac{9216\,\alpha\,\beta^2\,(\sigma^2+3\alpha^2\beta^2)\,x}{\Delta^7}$$

$$V_7^{[II]}(x)=-V_7^{[KK]}(x)=-\frac{\beta\,x^7}{\Delta}-\frac{24\,\beta\,(4\,\alpha^2+\beta^2)\,x^5}{\Delta^3}-$$

$$-\frac{192\,\beta\,(11\,\alpha^4+18\,\alpha^2\beta^2+\beta^4)\,x^3}{\Delta^5}-\frac{9216\,\alpha^2\beta\,(\sigma^2+\alpha^2\beta^2)\,x}{\Delta^7}$$

$$W_7^{[II]}(x)=W_7^{[KK]}(x)=-\frac{6\,\sigma\,x^6}{\Delta^2}-\frac{96\,(\sigma^2+6\,\alpha^2\beta^2)\,x^4}{\Delta^4}-\frac{384\,\sigma\,(\sigma^2+26\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$T_7^{[JI]}(x)=\frac{12\,\alpha\,\beta\,x^6}{\sigma^2}-\frac{480\,\alpha\,\beta\,\Delta\,x^4}{\sigma^4}+\frac{4608\,\alpha\,\beta\,(\Delta^2-\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$U_7^{[JI]}(x)=-\frac{\alpha\,x^7}{\sigma}+\frac{24\,\alpha\,(\alpha^2-4\,\beta^2)\,x^5}{\sigma^3}-\frac{192\,\alpha\,(\alpha^4-18\,\alpha^2\beta^2+11\,\beta^4)\,x^3}{\sigma^5}-$$

$$-\frac{9216\,\alpha\,\beta^2(\Delta^2-5\,\alpha^2\beta^2)\,x}{\sigma^7}$$

$$V_7^{[JI]}(x)=\frac{\beta\,x^7}{\sigma}-\frac{24\,\beta\,(4\,\alpha^2-\beta^2)\,x^5}{\sigma^3}+\frac{192\,\beta\,(11\,\alpha^4-18\,\alpha^2\beta^2+\beta^4)\,x^3}{\sigma^5}-$$

$$-\frac{9216\,\alpha^2\beta(\Delta^2-\alpha^2\beta^2)\,x}{\sigma^7}$$

$$W_7^{[JI]}(x)=\frac{6\,\Delta\,x^6}{\sigma^2}-\frac{96\,(\Delta^2-6\,\alpha^2\beta^2)\,x^4}{\sigma^4}+\frac{384\,\Delta\,(\Delta^2-26\,\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$T_7^{[JK]}(x)=-\frac{12\,\alpha\,\beta\,x^6}{\sigma^2}+\frac{480\,\alpha\,\beta\,\Delta\,x^4}{\sigma^4}-\frac{4608\,\alpha\,\beta\,(\Delta^2-\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$U_7^{[JK]}(x)=-\frac{\alpha\,x^7}{\sigma}+\frac{24\,\alpha\,(\alpha^2-4\,\beta^2)\,x^5}{\sigma^3}-\frac{192\,\alpha\,(\alpha^4-18\,\alpha^2\beta^2+11\,\beta^4)\,x^3}{\sigma^5}-$$

$$-\frac{9216\,\alpha\,\beta^2\,(\Delta^2-\alpha^2\beta^2)\,x}{\sigma^7}$$

$$V_7^{[JK]}(x)=-\frac{\beta\,x^7}{\sigma}+\frac{24\,\beta\,(4\,\alpha^2-\beta^2)\,x^5}{\sigma^3}-\frac{192\,\beta\,(11\,\alpha^4-18\,\alpha^2\beta^2+\beta^4)\,x^3}{\sigma^5}+$$

$$+\frac{9216\,\alpha^2\beta\,(\Delta^2-\alpha^2\beta^2)\,x}{\sigma^7}$$

$$W_7^{[JK]}(x)=\frac{6\,\Delta\,x^6}{\sigma^2}-\frac{96\,(\Delta^2-6\,\alpha^2\beta^2)\,x^4}{\sigma^4}+\frac{384\,\Delta\,(\Delta^2-26\,\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$T_7^{[IK]}(x)=-\frac{12\,\alpha\,\beta\,x^6}{\Delta^2}-\frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4}-\frac{4608\,\alpha\,\beta\,(\sigma^2+\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$U_7^{[IK]}(x)=\frac{\alpha\,x^7}{\Delta}+\frac{24\,\alpha\,(\alpha^2+4\,\beta^2)\,x^5}{\Delta^3}+\frac{192\,\alpha\,(\alpha^4+18\,\alpha^2\beta^2+11\,\beta^4)\,x^3}{\Delta^5}+$$

$$+\frac{9216\,\alpha\,\beta^2\,(\sigma^2+\alpha^2\beta^2)\,x}{\Delta^7}$$

$$V_7^{[IK]}(x)=\frac{\beta\,x^7}{\Delta}+\frac{24\,\beta\,(4\,\alpha^2+\beta^2)\,x^5}{\Delta^3}+\frac{192\,\beta\,(11\,\alpha^4+18\,\alpha^2\beta^2+\beta^4)\,x^3}{\Delta^5}+$$

$$+\frac{9216\,\alpha^2\beta\,(\sigma^2+\alpha^2\beta^2)\,x}{\Delta^7}$$

$$W_7^{[IK]}(x)=-\frac{6\,\sigma\,x^6}{\Delta^2}-\frac{96\,(\sigma^2+6\,\alpha^2\beta^2)\,x^4}{\Delta^4}-\frac{384\,\sigma\,(\sigma^3+26\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$\int x^{2n+1} J_1(\alpha x)\, J_1(\beta x)\, dx = x^{2n-2}\left\{ \frac{4\,n\alpha\beta\,x^2\,J_0(\alpha x)\,J_0(\beta x)}{\Delta^2} - \left(\frac{\alpha\,x^3}{\Delta} + \frac{4\,n^2\alpha\,x}{\Delta^2}\right) J_0(\alpha x)\,J_1(\beta x)+ \right.$$

$$\left. + \left(\frac{\beta\,x^3}{\Delta} - \frac{4\,n^2\beta\,x}{\Delta^2}\right) J_1(\alpha x)\,J_0(\beta x) + \frac{[2\,n\sigma\,x^2 + 8\,n^2(n-1)]\,J_1(\alpha x)\,J_1(\beta x)}{\Delta^2} \right\} -$$

$$-\frac{4\,(2n-1)n\,\sigma}{\Delta^2}\int x^{2n-1} J_1(\alpha x)\,J_1(\beta x)\, dx - \frac{16\,n(n-1)(n-2)}{\Delta^2}\int x^{2n-3} J_1(\alpha x)\,J_1(\beta x)\, dx$$

$$\int x^{2n+1} I_1(\alpha x)\, I_1(\beta x)\, dx = x^{2n-2}\left\{ \frac{4\,n\alpha\beta\,x^2\,I_0(\alpha x)\,I_0(\beta x)}{\Delta^2} + \left(\frac{\alpha\,x^3}{\Delta} - \frac{4\,n^2\alpha\,x}{\Delta^2}\right) I_0(\alpha x)\,I_1(\beta x)- \right.$$

$$\left. - \left(\frac{\beta\,x^3}{\Delta} + \frac{4\,n^2\beta\,x}{\Delta^2}\right) I_1(\alpha x)\,I_0(\beta x) - \frac{[2\,n\sigma\,x^2 - 8\,n^2(n-1)]\,I_1(\alpha x)\,I_1(\beta x)}{\Delta^2} \right\} +$$

$$+\frac{4\,(2n-1)n\,\sigma}{\Delta^2}\int x^{2n-1} I_1(\alpha x)\,I_1(\beta x)\, dx - \frac{16\,n(n-1)(n-2)}{\Delta^2}\int x^{2n-3} I_1(\alpha x)\,I_1(\beta x)\, dx$$

$$\int x^{2n+1} K_1(\alpha x)\, K_1(\beta x)\, dx = x^{2n-2}\left\{ \frac{4\,n\alpha\beta\,x^2\,K_0(\alpha x)\,K_0(\beta x)}{\Delta^2} - \left(\frac{\alpha\,x^3}{\Delta} - \frac{4\,n^2\alpha\,x}{\Delta^2}\right) K_0(\alpha x)\,K_1(\beta x)+ \right.$$

$$\left. + \left(\frac{\beta\,x^3}{\Delta} + \frac{4\,n^2\beta\,x}{\Delta^2}\right) K_1(\alpha x)\,K_0(\beta x) - \frac{[2\,n\sigma\,x^2 - 8\,n^2(n-1)]\,K_1(\alpha x)\,K_1(\beta x)}{\Delta^2} \right\} +$$

$$+\frac{4\,(2n-1)n\,\sigma}{\Delta^2}\int x^{2n-1} K_1(\alpha x)\,K_1(\beta x)\, dx - \frac{16\,n(n-1)(n-2)}{\Delta^2}\int x^{2n-3} K_1(\alpha x)\,K_1(\beta x)\, dx$$

$$\int x^{2n+1} J_1(\alpha x)\, I_1(\beta x)\, dx = x^{2n-2}\left\{ \frac{4\,n\alpha\beta\,x^2\,J_0(\alpha x)\,I_0(\beta x)}{\sigma^2} - \left(\frac{\alpha\,x^3}{\sigma} + \frac{4\,n^2\alpha\,x}{\sigma^2}\right) J_0(\alpha x)\,I_1(\beta x)+ \right.$$

$$\left. + \left(\frac{\beta\,x^3}{\sigma} - \frac{4\,n^2\beta\,x}{\sigma^2}\right) J_1(\alpha x)\,I_0(\beta x) + \frac{[2\,n\Delta\,x^2 + 8\,n^2(n-1)]\,J_1(\alpha x)\,I_1(\beta x)}{\sigma^2} \right\} -$$

$$-\frac{4\,(2n-1)n\,\Delta}{\sigma^2}\int x^{2n-1} J_1(\alpha x)\,I_1(\beta x)\, dx - \frac{16\,n(n-1)(n-2)}{\sigma^2}\int x^{2n-3} J_1(\alpha x)\,I_1(\beta x)\, dx$$

$$\int x^{2n+1} J_1(\alpha x)\, K_1(\beta x)\, dx = x^{2n-2}\left\{ -\frac{4\,n\alpha\beta\,x^2\,J_0(\alpha x)\,K_0(\beta x)}{\sigma^2} - \left(\frac{\alpha\,x^3}{\sigma} + \frac{4\,n^2\alpha\,x}{\sigma^2}\right) J_0(\alpha x)\,K_1(\beta x)- \right.$$

$$\left. - \left(\frac{\beta\,x^3}{\sigma} - \frac{4\,n^2\beta\,x}{\sigma^2}\right) J_1(\alpha x)\,K_0(\beta x) + \frac{[2\,n\Delta\,x^2 + 8\,n^2(n-1)]\,J_1(\alpha x)\,I_1(\beta x)}{\sigma^2} \right\} -$$

$$-\frac{4\,(2n-1)n\,\Delta}{\sigma^2}\int x^{2n-1} J_1(\alpha x)\,K_1(\beta x)\, dx - \frac{16\,n(n-1)(n-2)}{\sigma^2}\int x^{2n-3} J_1(\alpha x)\,K_1(\beta x)\, dx$$

$$\int x^{2n+1} I_1(\alpha x)\, K_1(\beta x)\, dx = x^{2n-2}\left\{ -\frac{4\,n\alpha\beta\,x^2\,I_0(\alpha x)\,K_0(\beta x)}{\Delta^2} + \left(\frac{\alpha\,x^3}{\Delta} - \frac{4\,n^2\alpha\,x}{\Delta^2}\right) I_0(\alpha x)\,K_1(\beta x)+ \right.$$

$$\left. + \left(\frac{\beta\,x^3}{\Delta} + \frac{4\,n^2\beta\,x}{\Delta^2}\right) I_1(\alpha x)\,K_0(\beta x) - \frac{[2\,n\sigma\,x^2 - 8\,n^2(n-1)]\,I_1(\alpha x)\,K_1(\beta x)}{\Delta^2} \right\} +$$

$$+\frac{4\,(2n-1)n\,\sigma}{\Delta^2}\int x^{2n-1} I_1(\alpha x)\,K_1(\beta x)\, dx - \frac{16\,n(n-1)(n-2)}{\Delta^2}\int x^{2n-3} I_1(\alpha x)\,K_1(\beta x)\, dx$$

2.2.4. Integrals of the type $\int x^{2n+1} \cdot J_\nu(\alpha x) \, Y_\nu(\beta x) \, dx$

Compare with 2.2.3.b) .

Let

$$\alpha^2 + \beta^2 = \sigma \qquad \text{and} \qquad \alpha^2 - \beta^2 = \Delta \ .$$

$$\int x \, J_0(\alpha x) \, Y_0(\beta x) \, dx = \frac{x}{\Delta} \left[\alpha \, J_1(\alpha x) \, Y_0(\beta x) - \beta J_0(\alpha x) \, Y_1(\beta x) \right]$$

$$\int x \, J_1(\alpha x) \, Y_1(\beta x) \, dx = \frac{x}{\Delta} \left[\beta \, J_1(\alpha x) \, Y_0(\beta x) - \alpha J_0(\alpha x) \, Y_1(\beta x) \right]$$

$$\int x^3 \, J_0(\alpha x) \, Y_0(\beta x) \, dx = \frac{2 \, \sigma \, x^2}{\Delta^2} \, J_0(\alpha x) \, Y_0(\beta x) + \frac{4 \, \alpha \, \beta x^2}{\Delta^2} \, J_1(\alpha x) \, Y_1(\beta x) +$$

$$+ \frac{\Delta^2 x^3 - 4 \, \sigma x}{\Delta^3} \left[\alpha \, J_1(\alpha x) \, Y_0(\beta x) - \beta \, J_0(\alpha x) \, Y_1(\beta x) \right]$$

$$\int x^3 \, J_1(\alpha x) \, Y_1(\beta x) \, dx = \frac{4 \, \alpha \, \beta \, x^2}{\Delta^2} \, J_0(\alpha x) \, Y_0(\beta x) + \frac{2 \, \sigma \, x^2}{\Delta^2} \, J_1(\alpha x) \, Y_1(\beta x) -$$

$$- \frac{\alpha[\Delta^2 \, x^3 - 8\beta^2 \, x]}{\Delta^3} \, J_0(\alpha x) \, Y_1(\beta x) + \frac{\beta[\Delta^2 \, x^3 - 8\alpha^2 \, x]}{\Delta^3} \, J_1(\alpha x) \, Y_0(\beta x)$$

$$\int x^5 \, J_0(\alpha x) \, Y_0(\beta x) \, dx = \frac{4 \, \sigma \Delta^2 x^4 - 32(\, \Delta^2 + 2 \, \alpha^2 \beta^2) x^2}{\Delta^4} \, J_0(\alpha x) \, Y_0(\beta x) -$$

$$- \frac{\beta \, [\Delta^4 x^5 - 16 \, (2 \, \alpha^2 + \beta^2) \Delta^2 x^3 + 64 \, (\Delta^2 + 2 \, \alpha^2 \beta^2) x]}{\Delta^5} \, J_0(\alpha x) \, Y_1(\beta x) +$$

$$+ \frac{\alpha \, [\Delta^4 x^5 - 16 \, (\alpha^2 + 2 \, \beta^2) \Delta^2 x^3 + 64 \, (\Delta^2 + 2 \, \alpha^2 \beta^2) \, x]}{\Delta^5} \, J_1(\alpha x) \, Y_0(\beta x) +$$

$$+ \frac{8 \, \alpha \, \beta \, \Delta^2 x^4 - 96 \, \alpha \, \beta \, \sigma \, x^2}{\Delta^4} \, J_1(\alpha x) \, Y_1(\beta x)$$

$$\int x^5 \, J_1(\alpha x) \, Y_1(\beta x) \, dx = \frac{8 \, \alpha \, \beta \, \Delta^2 \, x^4 - 96 \, \alpha \, \beta \, \sigma \, x^2}{\Delta^4} \, J_0(\alpha x) \, Y_0(\beta x) -$$

$$- \frac{\alpha[\Delta^4 \, x^5 - 8 \, (\alpha^2 + 5 \, \beta^2) \Delta^2 \, x^3 + 192 \, \beta^2 \, \sigma \, x]}{\Delta^5} \, J_0(\alpha x) \, Y_1(\beta x)] +$$

$$+ \frac{\beta[\Delta^4 \, x^5 - 8 \, (5 \, \alpha^2 + \beta^2) \Delta^2 \, x^3 + 192 \, \alpha^2 \, \sigma \, x]}{\Delta^5} \, J_1(\alpha x) \, Y_0(\beta x) +$$

$$+ \frac{4 \, (\alpha^2 + \beta^2) \Delta^2 \, x^4 - 16 \, (\Delta^2 + 8 \, \alpha^2 \beta^2) \, x^2}{\Delta^4} \, J_1(\alpha x) \, Y_1(\beta x)$$

2.2.5. Integrals of the type $\int x^{2n+1} \cdot J_0(\alpha x) \cdot J_1(\beta x)\,dx$ **and** $\int x^{2n+1} \cdot I_0(\alpha x) \cdot I_1(\beta x)\,dx$

Holds (for $\beta \neq \alpha$)

$$\int J_0(\alpha x)\, J_0(\beta x)\, dx = \frac{1}{\alpha}\, \Theta_0\left(\alpha x; \frac{\beta}{\alpha}\right) = \frac{1}{\beta}\, \Theta_0\left(\beta x; \frac{\alpha}{\beta}\right)\,,$$

$$\int J_1(\alpha x)\, J_1(\beta x)\, dx = \frac{1}{\alpha}\, \Theta_1\left(\alpha x; \frac{\beta}{\alpha}\right) = \frac{1}{\beta}\, \Theta_1\left(\beta x; \frac{\alpha}{\beta}\right)\,,$$

$$\int I_0(\alpha x)\, I_0(\beta x)\, dx = \frac{1}{\alpha}\, \Omega_0\left(\alpha x; \frac{\beta}{\alpha}\right) = \frac{1}{\beta}\, \Omega_0\left(\beta x; \frac{\alpha}{\beta}\right)\,,$$

$$\int I_1(\alpha x)\, I_1(\beta x)\, dx = \frac{1}{\alpha}\, \Omega_1\left(\alpha x; \frac{\beta}{\alpha}\right) = \frac{1}{\beta}\, \Omega_1\left(\beta x; \frac{\alpha}{\beta}\right)\,.$$

Θ_ν and Ω_ν as definded on pages 310 and 312. In these integrals both Bessel functions are of the same order, so one can suppose $\beta < \alpha$. This relation is not presumed for the product $Z_0(\alpha x) \cdot Z_1(\beta x)$, that means for the following integrals. They were expressed by Θ_ν and Ω_ν, so one can use the according right side of the previous equations.

Let $$\boxed{\alpha^2 + \beta^2 = \sigma \qquad \text{and} \qquad \alpha^2 - \beta^2 = \Delta\,.}$$

$$\int \frac{J_0(\alpha x)\, J_1(\beta x)\, dx}{x} = -J_0(\alpha x)\, J_1(\beta x) + \beta \int J_0(\alpha x)\, J_0(\beta x)\, dx - \alpha \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int \frac{I_0(\alpha x)\, I_1(\beta x)\, dx}{x} = -I_0(\alpha x)\, I_1(\beta x) + \beta \int I_0(\alpha x)\, I_1(\beta x)\, dx + \alpha \int I_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x\, J_0(\alpha x)\, J_1(\beta x)\, dx =$$

$$= \frac{x}{\Delta}\, [\beta\, J_0(\alpha x)\, J_0(\beta x) + \alpha\, J_1(\alpha x)\, J_1(\beta x)] - \frac{\beta}{\Delta} \int J_0(\alpha x)\, J_0(\beta x)\, dx + \frac{\alpha}{\Delta} \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x\, I_0(\alpha x)\, I_1(\beta x)\, dx =$$

$$= \frac{x}{\Delta}\, [\alpha\, I_1(\alpha x)\, I_1(\beta x) - \beta\, I_0(\alpha x)\, I_0(\beta x)] + \frac{\beta}{\Delta} \int I_0(\alpha x)\, I_0(\beta x)\, dx + \frac{\alpha}{\Delta} \int I_1(\alpha x)\, I_1(\beta x)\, dx$$

Let $n = 2m + 1$ and

$$\int x^n \cdot J_0(\alpha x)\, J_1(\beta x)\, dx = \frac{P_n(x)}{\Delta^n}\, J_0(\alpha x)\, J_0(\beta x) + \frac{Q_n(x)}{\Delta^{n-1}}\, J_0(\alpha x)\, J_1(\beta x) + \frac{R_n(x)}{\Delta^{n-1}}\, J_1(\alpha x)\, J_0(\beta x) +$$

$$+ \frac{S_n(x)}{\Delta^n}\, J_1(\alpha x)\, J_1(\beta x) + \frac{U_n}{\Delta^n} \int J_0(\alpha x)\, J_0(\beta x)\, dx + \frac{V_n}{\Delta^n} \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

and

$$\int x^n \cdot I_0(\alpha x)\, I_1(\beta x)\, dx = \frac{\bar{P}_n(x)}{\Delta^n}\, I_0(\alpha x)\, I_0(\beta x) + \frac{\bar{Q}_n(x)}{\Delta^{n-1}}\, I_0(\alpha x)\, I_1(\beta x) + + \frac{\bar{R}_n(x)}{\Delta^{n-1}}\, I_1(\alpha x)\, I_0(\beta x) +$$

$$+ \frac{\bar{S}_n(x)}{\Delta^n}\, I_1(\alpha x)\, I_1(\beta x) + \frac{\bar{U}_n}{\Delta^n} \int I_0(\alpha x)\, I_0(\beta x)\, dx + \frac{\bar{V}_n}{\Delta^n} \int I_1(\alpha x)\, I_1(\beta x)\, dx\,,$$

then holds

$$P_3 = \beta\, x(\Delta^2 x^2 - 5\,\alpha^2 - 3\,\beta^2)\,, \quad Q_3 = x^2(\alpha^2 + 3\,\beta^2)\,, \quad R_3 = -4\,\alpha\,\beta\, x^2\,,$$

$$S_3 = \alpha\, x\, (\Delta^2 x^2 - \alpha^2 - 7\,\beta^2)\,, \quad U_3 = \beta(5\,\alpha^2 + 3\,\beta^2)\,, \quad V_3 = -\alpha\,(\alpha^2 + 7\,\beta^2)$$

$$\bar{P}_3 = -\beta\, x\,(\Delta^2\, x^2 + 5\,\alpha^2 + 3\,\beta^2)\,, \quad \bar{Q}_3 = -(\alpha^2 + 3\,\beta^2)\, x^2\,, \quad \bar{R}_3 = 4\,\alpha\,\beta\, x^2$$

$$\bar{S}_3 = \alpha\, x\,(\Delta^2\, x^2 + \alpha^2 + 7\,\beta^2)\,, \quad \bar{U}_3 = \beta(5\,\alpha^2 + 3\,\beta^2)\,, \quad \bar{V}_3 = \alpha\,(\alpha^2 + 7\,\beta^2)$$

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$$P_5 = \beta\, x\, [\Delta^4\, x^4 - 3\,(11\,\alpha^2 + 5\,\beta^2)\Delta^2\, x^2 + 117\,\alpha^4 + 222\,\alpha^2\beta^2 + 45\,\beta^4]$$

$$Q_5 = x^2[(3\,\alpha^2 + 5\,\beta^2)\Delta^2\, x^2 - 9\,\alpha^4 - 138\,\alpha^2\beta^2 - 45\,\beta^4]$$

$$R_5 = -4\,\alpha\,\beta\,x^2[2\,\Delta^2\, x^2 - 27\,\alpha^2 - 21\,\beta^2]$$

$$S_5 = \alpha\, x\, [\Delta^4\, x^4 - 3\,(3\,\alpha^2 + 13\,\beta^2)\Delta^2\, x^2 + 9\,\alpha^4 + 246\,\alpha^2\beta^2 + 129\,\beta^4]$$

$$U_5 = 3\beta\,(39\,\alpha^4 + 74\,\beta^2\alpha^2 + 15\,\beta^4)\,, \quad V_5 = -3\,\alpha\,(3\,\alpha^4 + 82\,\alpha^2\beta^2 + 43\,\beta^4)$$

$$\bar{P}_5 = -\beta\, x\,(\Delta^4 x^4 + 3\,(11\,\alpha^2 + 5\,\beta^2)\Delta^2\, x^2 + 117\,\alpha^4 + 222\,\alpha^2\beta^2 + 455\,\beta^4)$$

$$\bar{Q}_5 = -x^2\,[(3\,\alpha^2 + 5\,\beta^2)\Delta^2\, x^2 + 9\,\alpha^4 + 138\,\beta^2\alpha^2 + 45\,\beta^4)]$$

$$\bar{R}_5 = 4\,\alpha\,\beta\,x^2\,[2\,\Delta^2\, x^2 + 27\,\alpha^2 + 21\,\beta^2]$$

$$\bar{S}_5 = \alpha\, x\,[\Delta^4\, x^4 + 3\,(3\,\alpha^2 + 13\,\beta^2)\Delta^2\, x^2 + 9\,\alpha^4 + 246\,\alpha^2\beta^2 + 129\,\beta^4]$$

$$\bar{U}_5 = 3\,\beta\,(39\,\alpha^4 + 74\,\alpha^2\beta^2 + 15\,\beta^4)\,, \quad \bar{V}_5 = 3\,\alpha\,(3\,\alpha^4 + 82\,\alpha^2\beta^2 + 43\,\beta^4)$$

$$P_7 = \beta\, x\,[\Delta^6\, x^6 - 5\,(17\,\alpha^2 + 7\,\beta^2)\Delta^4\, x^4 +$$
$$+15\,(115\,\alpha^4 + 234\,\alpha^2\beta^2 + 35\,\beta^4)\,\Delta^2\, x^2 - 5625\,\alpha^6 - 22965\,\alpha^4\beta^2 - 15915\,\alpha^2\beta^4 - 1575\,\beta^6]$$

$$Q_7 = x^2\,[(5\,\alpha^2 + 7\,\beta^2)\,\Delta^4\, x^4 - 5\,(15\,\alpha^4 + 142\,\alpha^2\beta^2 + 35\,\beta^4)\,\Delta^2\, x^2 +$$
$$225\,\alpha^6 + 8805\,\alpha^4\beta^2 + 12435\,\alpha^2\beta^4 + 1575\,\beta^6]$$

$$R_7 = -4\,\alpha\,\beta\,x^2\,[3\,\Delta^4\, x^4 - 5\,(25\,\alpha^2 + 23\,\beta^2)\Delta^2\, x^2 + 1350\,\alpha^4 + 3540\,\alpha^2\beta^2 + 870\,\beta^4]$$

$$S_7 = \alpha\, x\,[\Delta^6\, x^6 - 5\,(5\,\alpha^2 + 19\,\beta^2)\Delta^4\, x^4 +$$
$$+15\,(15\,\alpha^4 + 242\,\alpha^2\beta^2 + 127\,\beta^4)\Delta^2\, x^2 - 225\,\alpha^6 - 14205\,\alpha^4\beta^2 - 26595\,\alpha^2\beta^4 - 5055\,\beta^6]$$

$$U_7 = 15\,\beta\,(375\,\alpha^6 + 1531\,\alpha^4\beta^2 + 1061\,\alpha^2\beta^4 + 105\,\beta^6)$$

$$V_7 = -15\,\alpha\,(15\,\alpha^6 + 947\,\alpha^4\beta^2 + 1773\,\alpha^2\beta^4 + 337\,\beta^6)$$

$$\bar{P}_7 = -\beta\, x\,[\Delta^6\, x^6 + 5\,(17\,\alpha^2 + 7\,\beta^2)\Delta^4\, x^4 +$$
$$+15\,(115\,\alpha^4 + 234\,\alpha^2\beta^2 + 35\,\beta^4)\,\Delta^2\, x^2 + 5625\,\alpha^6 + 22965\,\alpha^4\beta^2 + 15915\,\alpha^2\beta^4 + 1575\,\beta^6]$$

$$\bar{Q}_7 = -x^2\,[(5\,\alpha^2 + 7\,\beta^2)\,\Delta^4\, x^4 + 5\,(15\,\alpha^4 + 142\,\alpha^2\beta^2 + 35\,\beta^4)\,\Delta^2\, x^2 +$$
$$+225\,\alpha^6 + 8805\,\alpha^4\beta^2 + 12435\,\alpha^2\beta^4 + 1575\,\beta^6]$$

$$\bar{R}_7 = 4\,\alpha\,\beta\,x^2\,[3\,\Delta^4\, x^4 + 5\,(25\,\alpha^2 + 23\,\beta^2)\,\Delta^2\, x^2 + 1350\,\alpha^4 + 3540\,\alpha^2\beta^2 + 870\,\beta^4]$$

$$\bar{S}_7 = \alpha\, x\,[\Delta^6\, x^6 + 5\,(5\,\alpha^2 + 19\,\beta^2)\,\Delta^4\, x^4 +$$
$$+15\,(15\,\alpha^4 + 242\,\alpha^2\beta^2 + 127\,\beta^4)\,\Delta^2\, x^2 + 225\,\alpha^6 + 14205\,\alpha^4\beta^2 + 26595\,\alpha^2\beta^4 + 5055\,\beta^6]$$

$$\bar{U}_7 = 15\,\beta\,(375\,\alpha^6 + 1531\,\alpha^4\beta^2 + 1061\,\alpha^2\beta^4 + 105\,\beta^6)$$

$$\bar{V}_7 = 15\,\alpha\,(15\,\alpha^6 + 947\,\alpha^4\beta^2 + 1773\,\alpha^2\beta^4 + 337\,\beta^6)$$

Recurrence relations:

$$\int x^{2n+1}\, J_0(\alpha x)\, J_1(\beta x)\, dx = \frac{x^{2n-2}}{(n-1)\,\Delta^2}\,\Big\{ bx\,[(n-1)\,\Delta\, x^2 - (2n-1)(2n-3)n]\, J_0(\alpha x)\, J_0(\beta x) +$$
$$+[(2\,\sigma\, n - \Delta)\,(n-1)\, x^2 + (2n-1)^2(2n-3)n]\, J_0(\alpha x)\, J_1(\beta x) -$$
$$- 4\,\alpha\beta\,(n-1)n\, x^2\, J_1(\alpha x)\, J_0(\beta x) + ax\,[(n-1)\,\Delta\, x^2 + (2n-1)^2 n]\, J_1(\alpha x)\, J_1(\beta x) \Big\} -$$
$$-\frac{(2n-1)\,[4\,n(n-1)\,\sigma + \Delta]}{(n-1)\,\Delta^2}\, \int x^{2n-1}\, J_0(\alpha x)\, J_1(\beta x)\, dx - \frac{(2n-1)^2\,(2n-3)^2\, n}{(n-1)\,\Delta^2}\, \int x^{2n-3}\, J_0(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, dx = \frac{x^{2n-2}}{(n-1)\,\Delta^2}\,\Big\{ - bx\,[(n-1)\,\Delta\, x^2 + (2n-1)(2n-3)n]\, I_0(\alpha x)\, I_0(\beta x) -$$
$$-[(2\,\sigma\, n - \Delta)\,(n-1)\, x^2 - (2n-1)^2(2n-3)n]\, I_0(\alpha x)\, I_1(\beta x) +$$
$$+ 4\,\alpha\beta\,(n-1)n\, x^2\, I_1(\alpha x)\, I_0(\beta x) + \alpha\, x\,[(n-1)\,\Delta\, x^2 - (2n-1)^2\, n]\, I_1(\alpha x)\, I_1(\beta x) \Big\} +$$
$$+\frac{(2n-1)\,[4\,n(n-1)\,\sigma + \Delta]}{(n-1)\,\Delta^2}\, \int x^{2n-1}\, I_0(\alpha x)\, I_1(\beta x)\, dx - \frac{(2n-1)^2\,(2n-3)^2\, n}{(n-1)\,\Delta^2}\, \int x^{2n-3}\, I_0(\alpha x)\, I_1(\beta x)\, dx$$

2.3. Bessel Functions with different Arguments x and $x + \alpha$:

2.3.1. Integrals of the type $\int x^{-1} Z_\nu(x+\alpha)Z_1(x)\,dx$ and $\int [x(x+\alpha)]^{-1} Z_1(x+\alpha)Z_1(x)\,dx$

$$\int \frac{J_1(x)J_0(x+\alpha)}{x}\,dx = \frac{x+\alpha}{\alpha}\left(J_0(x)J_1(x+\alpha) - J_1(x)J_0(x+\alpha)\right)$$

$$\int \frac{I_1(x)I_0(x+\alpha)}{x}\,dx = \frac{x+\alpha}{\alpha}\left(I_0(x)I_1(x+\alpha) - I_1(x)I_0(x+\alpha)\right)$$

$$\int \frac{K_1(x)I_0(x+\alpha)}{x}\,dx = \frac{x+\alpha}{\alpha}\left(K_0(x)K_1(x+\alpha) - K_1(x)K_0(x+\alpha)\right)$$

$$\int \frac{J_1(x)J_1(x+\alpha)}{x}\,dx =$$

$$= -\frac{x+\alpha}{\alpha}J_0(x)J_0(x+\alpha) - \frac{x}{\alpha^2}J_1(x)J_0(x+\alpha) + \frac{x+\alpha}{\alpha^2}J_0(x)J_1(x+\alpha) - \frac{x+\alpha}{\alpha}J_1(x)J_1(x+\alpha)$$

$$\int \frac{I_1(x)I_1(x+\alpha)}{x}\,dx =$$

$$= \frac{x+\alpha}{\alpha}I_0(x)I_0(x+\alpha) + \frac{x}{\alpha^2}I_1(x)I_0(x+\alpha) - \frac{x+\alpha}{\alpha^2}I_0(x)I_1(x+\alpha) - \frac{x+\alpha}{\alpha}I_1(x)I_1(x+\alpha)$$

$$\int \frac{K_1(x)K_1(x+\alpha)}{x}\,dx =$$

$$= \frac{x+\alpha}{\alpha}K_0(x)K_0(x+\alpha) - \frac{x}{\alpha^2}K_1(x)K_0(x+\alpha) + \frac{x+\alpha}{\alpha^2}K_0(x)K_1(x+\alpha) - \frac{x+\alpha}{\alpha}K_1(x)K_1(x+\alpha)$$

From this

$$\int \frac{J_1(x)\,J_1(x+\alpha)\,dx}{x(x+\alpha)} =$$

$$= \frac{1}{\alpha^3}\left\{2(x+\alpha)J_1(x)\,J_0(x+\alpha) - 2x\,J_0(x)\,J_1(x+\alpha) - \alpha(2x+\alpha)\left[\,J_0(x)\,J_0(x+\alpha) + J_1(x)\,J_1(x+\alpha)\right]\right\}$$

$$\int \frac{I_1(x)\,I_1(x+\alpha)\,dx}{x(x+\alpha)} =$$

$$= \frac{1}{\alpha^3}\left\{-2(x+\alpha)I_1(x)\,I_0(x+\alpha) + 2x\,I_0(x)\,I_1(x+\alpha) + \alpha(2x+\alpha)\left[\,I_0(x)\,I_0(x+\alpha) - I_1(x)\,I_1(x+\alpha)\right]\right\}$$

$$\int \frac{K_1(x)\,K_1(x+\alpha)\,dx}{x(x+\alpha)} =$$

$$= \frac{1}{\alpha^3}\left\{2(x+\alpha)K_1(x)\,K_0(x+\alpha) - 2x\,K_0(x)\,K_1(x+\alpha) + \alpha(2x+\alpha)\left[\,K_0(x)\,K_0(x+\alpha) - K_1(x)\,K_1(x+\alpha)\right]\right\}$$

2.4. Elementary Function and two Bessel Functions:

2.4.1. Integrals of the type $\int x^{2n+1} \ln x \, Z_\nu^2(x) \, dx$ and $\int x^{2n} \ln x \, Z_0(x) \, Z_1(x) \, dx$

In the following integrals $J_\nu(x)$ may be substituted by $Y_\nu(x)$, $H_\nu^{(1)}(x)$ or $H_\nu^{(2)}(x)$.

$$\int x \ln x \, J_0^2(x) \, dx = \frac{x^2(\ln x - 1)}{2} \, [J_0^2(x) + J_1^2(x)] + \frac{x}{2} \, J_0(x) \, J_1(x)$$

$$\int x \ln x \, I_0^2(x) \, dx = \frac{x^2(\ln x - 1)}{2} \, [I_0^2(x) - I_1^2(x)] + \frac{x}{2} \, I_0(x) \, I_1(x)$$

$$\int x \ln x \, K_0^2(x) \, dx = \frac{x^2(\ln x - 1)}{2} \, [K_0^2(x) - K_1^2(x)] - \frac{x}{2} \, K_0(x) \, K_1(x)$$

$$\int x \ln x \, J_1^2(x) \, dx = \frac{x^2(\ln x - 1) - 1}{2} \, J_0^2(x) + \frac{x(1 - 2\ln x)}{2} \, J_0(x) \, J_1(x) + \frac{x^2(\ln x - 1)}{2} \, J_1^2(x)$$

$$\int x \ln x \, I_1^2(x) \, dx = \frac{x^2(1 - \ln x) - 1}{2} \, I_0^2(x) + \frac{x(2\ln x - 1)}{2} \, I_0(x) \, I_1(x) + \frac{x^2(\ln x - 1)}{2} \, I_1^2(x)$$

$$\int x \ln x \, K_1^2(x) \, dx = \frac{x^2(\ln x - 1) - 1}{2} \, K_0^2(x) - \frac{x(2\ln x - 1)}{2} \, K_0(x) \, K_1(x) + \frac{x^2(\ln x - 1)}{2} \, K_1^2(x)$$

$$\int x^2 \ln x \, J_0(x) \, J_1(x) \, dx = -\frac{x^2}{4} \, J_0^2(x) + \frac{x}{2} \, J_0(x) \, J_1(x) + \frac{x^2(2\ln x - 1)}{4} \, J_1^2(x)$$

$$\int x^2 \ln x \, I_0(x) \, I_1(x) \, dx = \frac{x^2}{4} \, I_0^2(x) - \frac{x}{2} \, I_0(x) \, I_1(x) + \frac{x^2(2\ln x - 1)}{4} \, I_1^2(x)$$

$$\int x^2 \ln x \, K_0(x) \, K_1(x) \, dx = -\frac{x^2}{4} \, K_0^2(x) - \frac{x}{2} \, K_0(x) \, K_1(x) + \frac{x^2(1 - 2\ln x)}{4} \, K_1^2(x)$$

$$\int x^3 \ln x \, J_0^2(x) \, dx =$$

$$= \frac{3x^2 - x^4 + 3x^4 \ln x}{18} \, J_0^2(x) + \frac{x^3 - 6x + 6x^3 \ln x}{18} \, J_0(x) \, J_1(x) - \frac{x^2 + x^4 + (6x^2 - 3x^4)\ln x}{18} \, J_1^2(x)$$

$$\int x^3 \ln x \, I_0^2(x) \, dx =$$

$$= \frac{-3x^2 - x^4 + 3x^4 \ln x}{18} \, I_0^2(x) + \frac{x^3 + 6x + 6x^3 \ln x}{18} \, I_0(x) \, I_1(x) - \frac{x^2 - x^4 + (6x^2 + 3x^4)\ln x}{18} \, I_1^2(x)$$

$$\int x^3 \ln x \, K_0^2(x) \, dx =$$

$$= \frac{-3x^2 - x^4 + 3x^4 \ln x}{18} \, K_0^2(x) - \frac{x^3 + 6x + 6x^3 \ln x}{18} \, K_0(x) \, K_1(x) - \frac{x^2 - x^4 + (6x^2 + 3x^4)\ln x}{18} \, K_1^2(x)$$

$$\int x^3 \ln x \, J_1^2(x) \, dx =$$

$$= -\frac{6x^2 + x^4 - 3x^4 \ln x}{18} \, J_0^2(x) + \frac{12x + x^3 - 12x^3 \ln x}{18} \, J_0(x) \, J_1(x) - \frac{x^2 + x^4 - (12x^2 + 3x^4)\ln x}{18} \, J_1^2(x)$$

$$\int x^3 \ln x \, I_1^2(x) \, dx =$$

$$= \frac{-6x^2 + x^4 - 3x^4 \ln x}{18} \, I_0^2(x) + \frac{12x - x^3 + 12x^3 \ln x}{18} \, I_0(x) \, I_1(x) + \frac{x^2 - x^4 - (12x^2 - 3x^4)\ln x}{18} \, I_1^2(x)$$

$$\int x^3 \ln x \, K_1^2(x) \, dx =$$

$$= \frac{-6\,x^2 + x^4 - 3\,x^4 \ln x}{18}\, K_0^2(x) - \frac{12\,x - x^3 + 12\,x^3 \ln x}{18}\, K_0(x)\,K_1(x) + \frac{x^2 - x^4 - (12x^2 - 3\,x^4)\ln x}{18}\, K_1^2(x)$$

$$\int x^4\, \ln x\, J_0(x)\, J_1(x)\, dx = \frac{12\,x^2 - x^4 - 6\,x^4 \ln x}{36}\, J_0^2(x) +$$

$$+ \frac{5\,x^3 - 12\,x + 12\,x^3 \ln x}{18}\, J_0(x)\, J_1(x) - \frac{10\,x^2 + x^4 + (24x^2 - 12\,x^4)\ln x}{36}\, J_1^2(x)$$

$$\int x^4\, \ln x\, I_0(x)\, I_1(x)\, dx = \frac{12\,x^2 + x^4 + 6\,x^4 \ln x}{36}\, I_0^2(x) -$$

$$- \frac{5\,x^3 + 12\,x + 12\,x^3 \ln x}{18}\, I_0(x)\, I_1(x) + \frac{10\,x^2 - x^4 + (24x^2 + 12\,x^4)\ln x}{36}\, I_1^2(x)$$

$$\int x^4\, \ln x\, K_0(x)\, K_1(x)\, dx = -\frac{12\,x^2 + x^4 + 6\,x^4 \ln x}{36}\, K_0^2(x) -$$

$$- \frac{5\,x^3 + 12\,x + 12\,x^3 \ln x}{18}\, K_0(x)\, K_1(x) - \frac{10\,x^2 - x^4 + (24x^2 + 12\,x^4)\ln x}{36}\, K_1^2(x)$$

Let

$$\int x^{2n+1}\, \ln x\, J_0^2(x)\, dx =$$

$$= \frac{A_n(x) + B_n(x)\,\ln x}{N_n^{(20,20)}}\, J_0^2(x) + \frac{C_n(x) + D_n(x)\,\ln x}{N_n^{(20,11)}}\, J_0(x)\,J_1(x) + \frac{E_n(x) + F_n(x)\,\ln x}{N_n^{(20,02)}}\, J_1^2(x)\,,$$

$$\int x^{2n}\, \ln x\, J_0(x)\, J_1(x)\, dx =$$

$$= \frac{G_n(x) + H_n(x)\,\ln x}{N_n^{(11,20)}}\, J_0^2(x) + \frac{I_n(x) + K_n(x)\,\ln x}{N_n^{(11,11)}}\, J_0(x)\,J_1(x) + \frac{L_n(x) + M_n(x)\,\ln x}{N_n^{(11,02)}}\, J_1^2(x)\,,$$

$$\int x^{2n+1}\, \ln x\, J_1^2(x)\, dx =$$

$$= \frac{P_n(x) + Q_n(x)\,\ln x}{N_n^{(02,20)}}\, J_0^2(x) + \frac{R_n(x) + S_n(x)\,\ln x}{N_n^{(02,11)}}\, J_0(x)\,J_1(x) + \frac{T_n(x) + U_n(x)\,\ln x}{N_n^{(02,02)}}\, J_1^2(x)\,,$$

and let the integrals with $I_\nu(x)$ be described with the polynomials $A_n^*(x), \ldots$ and such with $K_\nu(x)$ written with $A_n^{**}(x) \ldots$ (the denominators $N_n^{(\mu,\nu,\lambda,\kappa)}$ are the same), then holds

$$N_2^{(20,20)} = 450\,, \quad A_2(x) = -9\,x^6 + 56\,x^4 - 240\,x^2\,, \quad B_2(x) = 45\,x^6 + 120\,x^4\,,$$

$$N_2^{(20,11)} = 450\,, \quad C_2(x) = 9\,x^5 - 344\,x^3 + 480\,x\,, \quad D_2(x) = 180\,x^5 - 480\,x^3\,,$$

$$N_2^{(20,02)} = 450\,, \quad E_2(x) = -9\,x^6 - 52\,x^4 + 344\,x^2\,, \quad F_2(x) = 45\,x^6 - 240\,x^4 + 480\,x^2\,,$$

$$A_2^*(x) = -9\,x^6 - 56\,x^4 - 240\,x^2\,, \quad B_2^*(x) = 45\,x^6 - 120\,x^4\,,$$

$$C_2^*(x) = 9\,x^5 + 344\,x^3 + 480\,x\,, \quad D_2^*(x) = 180\,x^5 + 480\,x^3\,,$$

$$E_2^*(x) = 9\,x^6 - 52\,x^4 - 344\,x^2\,, \quad F_2^*(x) = -45\,x^6 - 240\,x^4 - 480\,x^2\,,$$

$$A_2^{**}(x) = -9\,x^6 - 56\,x^4 - 240\,x^2\,, \quad B_2^{**}(x) = 45\,x^6 - 120\,x^4\,,$$

$$C_2^{**}(x) = -9\,x^5 - 344\,x^3 - 480\,x\,, \quad D_2^{**}(x) = -180\,x^5 - 480\,x^3\,,$$

$$E_2^{**}(x) = 9\,x^6 - 52\,x^4 - 344\,x^2\,, \quad F_2^{**}(x) = -45\,x^6 - 240\,x^4 - 480\,x^2\,,$$

$\int x^4\, \ln x\, Z_0(x)\, Z_1(x)\, dx$ see before.

$$N_2^{(02,02)} = 150\,, \quad P_2(x) = -3x^6 - 23\,x^4 + 120\,x^2\,, \quad Q_2(x) = 15\,x^6 - 60\,x^4\,,$$

$$N_2^{(02,11)} = 150\,, \quad R_2(x) = 3\,x^5 + 152\,x^3 - 240\,x\,, \quad S_2(x) = -90\,x^5 + 240\,x^3\,,$$

$$N_2^{(02,02)} = 150\,, \quad T_2(x) = -3\,x^6 + 16\,x^4 - 152\,x^2\,, \quad U_2(x) = 15\,x^6 + 120\,x^4 - 240\,x^2\,,$$

$$P_2^*(x) = 3\,x^6 - 23\,x^4 - 120\,x^2\,, \quad Q_2^*(x) = -15\,x^6 - 60\,x^4\,,$$
$$R_2^*(x) = -3\,x^5 + 152\,x^3 + 240\,x\,, \quad S_2^*(x) = 90\,x^5 + 240\,x^3\,,$$
$$T_2^*(x) = -3\,x^6 - 16\,x^4 - 152\,x^2\,, \quad U_2^*(x) = 15\,x^6 - 120\,x^4 - 240\,x^2\,,$$

$$P_2^{**}(x) = 3\,x^6 - 23\,x^4 - 120\,x^2\,, \quad Q_2^{**}(x) = -15\,x^6 - 60\,x^4\,,$$
$$R_2^{**}(x) = 3\,x^5 - 152\,x^3 - 240\,x\,, \quad S_2^{**}(x) = -90\,x^5 - 240\,x^3\,,$$
$$T_2^{**}(x) = -3\,x^6 - 16\,x^4 - 152\,x^2\,, \quad U_2^{**}(x) = 15\,x^6 - 120\,x^4 - 240\,x^2\,,$$

$$N_3^{(20,20)} = 2450\,, \quad A_3(x) = -25\,x^8 + 303\,x^6 - 4152\,x^4 + 10080\,x^2\,, \quad B_3(x) = 175\,x^8 + 1260\,x^6 - 5040\,x^4\,,$$

$$N_3^{(20,11)} = 2450\,, \quad C_3(x) = 25\,x^7 - 3078\,x^5 + 21648\,x^3 - 20160\,x\,, \quad D_3(x) = 1050\,x^7 - 7560\,x^5 + 20160\,x^3\,,$$

$$N_3^{(20,02)} = 2450\,, \quad E_3(x) = -25\,x^8 - 297\,x^6 + 5784\,x^4 - 21648\,x^2\,, \quad F_3(x) = 175\,x^8 - 1890\,x^6 + 10080\,x^4 - 20160\,x^2\,,$$

$$A_3^*(x) = -25\,x^8 - 303\,x^6 - 4152\,x^4 - 10080\,x^2\,, \quad B_3^*(x) = 175\,x^8 - 1260\,x^6 - 5040\,x^4\,,$$
$$C_3^*(x) = 25\,x^7 + 3078\,x^5 + 21648\,x^3 + 20160\,x\,, \quad D_3^*(x) = 1050\,x^7 + 7560\,x^5 + 20160\,x^3\,,$$
$$E_3^*(x) = 25\,x^8 - 297\,x^6 - 5784\,x^4 - 21648\,x^2\,, \quad F_3^*(x) = -175\,x^8 - 1890\,x^6 - 10080\,x^4 - 20160\,x^2\,,$$

$$A_3^{**}(x) = -25\,x^8 - 303\,x^6 - 4152\,x^4 - 10080\,x^2\,, \quad B_3^{**}(x) = 175\,x^8 - 1260\,x^6 - 5040\,x^4\,,$$
$$C_3^{**}(x) = -25\,x^7 - 3078\,x^5 - 21648\,x^3 - 20160\,x\,, \quad D_3^{**}(x) = -1050\,x^7 - 7560\,x^5 - 20160\,x^3\,,$$
$$E_3^{**}(x) = 25\,x^8 - 297\,x^6 - 5784\,x^4 - 21648\,x^2\,, \quad F_3^{**}(x) = -175\,x^8 - 1890\,x^6 - 10080\,x^4 - 20160\,x^2\,,$$

$$N_3^{(11,20)} = 300\,, \quad G_3(x) = -3\,x^6 + 152\,x^4 - 480\,x^2\,, \quad H_3(x) = -60\,x^6 + 240\,x^4\,,$$
$$N_3^{(11,11)} = 150\,, \quad I_3(x) = 39\,x^5 - 424\,x^3 + 480\,x\,, \quad K_3(x) = 180\,x^5 - 480\,x^3\,,$$
$$N_3^{(11,02)} = 300\,, \quad L_3(x) = -3\,x^6 - 184\,x^4 + 848\,x^2\,, \quad M_3(x) = 90\,x^6 - 480\,x^4 + 960\,x^2\,,$$

$$G_3^*(x) = 3\,x^6 + 152\,x^4 + 480\,x^2\,, \quad H_3^*(x) = 60\,x^6 + 240\,x^4\,,$$
$$I_3^*(x) = -39\,x^5 - 424\,x^3 - 480\,x\,, \quad K_3^*(x) = -180\,x^5 - 480\,x^3\,,$$
$$L_3^*(x) = -3\,x^6 + 184\,x^4 + 848\,x^2\,, \quad M_3^*(x) = 90\,x^6 + 480\,x^4 + 960\,x^2\,,$$

$$G_3^{**}(x) = -3\,x^6 - 152\,x^4 - 480\,x^2\,, \quad H_3^{**}(x) = -60\,x^6 - 240\,x^4\,,$$
$$I_3^{**}(x) = -39\,x^5 - 424\,x^3 - 480\,x\,, \quad K_3^{**}(x) = -180\,x^5 - 480\,x^3\,,$$
$$L_3^{**}(x) = 3\,x^6 - 184\,x^4 - 848\,x^2\,, \quad M_3^{**}(x) = -90\,x^6 - 480\,x^4 - 960\,x^2\,,$$

$$N_3^{(02,20)} = 2450\,, \quad P_3(x) = -25\,x^8 - 334\,x^6 + 5256\,x^4 - 13440\,x^2\,, \quad Q_3(x) = 175\,x^8 - 1680\,x^6 + 6720\,x^4\,,$$

$$N_3^{(02,11)} = 2450\,, \quad R_3(x) = 25\,x^7 + 3684\,x^5 - 27744\,x^3 + 26880\,x\,, \quad S_3(x) = -1400\,x^7 + 10080\,x^5 - 26880\,x^3\,,$$

$$N_3^{(02,02)} = 2450\,, \quad T_3(x) = -25\,x^8 + 291\,x^6 - 7152\,x^4 + 27744\,x^2\,, \quad U_3(x) = 175\,x^8 + 2520\,x^6 - 13440\,x^4 + 26880\,x^2\,,$$

$$P_3^*(x) = 25\,x^8 - 334\,x^6 - 5256\,x^4 - 13440\,x^2\,, \quad Q_3^*(x) = -175\,x^8 - 1680\,x^6 - 6720\,x^4\,,$$
$$R_3^*(x) = -25\,x^7 + 3684\,x^5 + 27744\,x^3 + 26880\,x\,, \quad S_3^*(x) = 1400\,x^7 + 10080\,x^5 + 26880\,x^3\,,$$

$$T_3^*(x) = -25\,x^8 - 291\,x^6 - 7152\,x^4 - 27744\,x^2\,, \quad U_3^*(x) = 175\,x^8 - 2520\,x^6 - 13440\,x^4 - 26880\,x^2\,,$$

$$P_3^{**}(x) = 25\,x^8 - 334\,x^6 - 5256\,x^4 - 13440\,x^2\,, \quad Q_3^{**}(x) = -175\,x^8 - 1680\,x^6 - 6720\,x^4\,,$$

$$R_3^{**}(x) = 25\,x^7 - 3684\,x^5 - 27744\,x^3 - 26880\,x\,, \quad S_3^{**}(x) = -1400\,x^7 - 10080\,x^5 - 26880\,x^3\,,$$

$$T_3^{**}(x) = -25\,x^8 - 291\,x^6 - 7152\,x^4 - 27744\,x^2\,, \quad U_3^{**}(x) = 175\,x^8 - 2520\,x^6 - 13440\,x^4 - 26880\,x^2\,,$$

$$N_4^{(20,20)} = 198450\,, \quad A_4(x) = -1225\,x^{10} + 24600\,x^8 - 732096\,x^6 + 6315264\,x^4 - 11612160\,x^2\,,$$

$$B_4(x) = 11025\,x^{10} + 151200\,x^8 - 1451520\,x^6 + 5806080\,x^4\,,$$

$$N_4^{(20,11)} = 198450\,, \quad C_4(x) = 1225\,x^9 - 348000\,x^7 + 5844096\,x^5 - 31067136\,x^3 + 23224320\,x\,,$$

$$D_4(x) = 88200\,x^9 - 1209600\,x^7 + 8709120\,x^5 - 23224320\,x^3\,,$$

$$N_4^{(20,02)} = 198450\,, \quad E_4(x) = -1225\,x^{10} - 24400\,x^8 + 916704\,x^6 - 9727488\,x^4 + 31067136\,x^2\,,$$

$$F_4(x) = 11025\,x^{10} - 201600\,x^8 + 2177280\,x^6 - 11612160\,x^4 + 23224320\,x^2\,,$$

$$A_4^*(x) = -1225\,x^{10} - 24600\,x^8 - 732096\,x^6 - 6315264\,x^4 - 11612160\,x^2\,,$$

$$B_4^*(x) = 11025\,x^{10} - 151200\,x^8 - 1451520\,x^6 - 5806080\,x^4\,,$$

$$C_4^*(x) = 1225\,x^9 + 348000\,x^7 + 5844096\,x^5 + 31067136\,x^3 + 23224320\,x\,,$$

$$D_4^*(x) = 88200\,x^9 + 1209600\,x^7 + 8709120\,x^5 + 23224320\,x^3\,,$$

$$E_4^*(x) = 1225\,x^{10} - 24400\,x^8 - 916704\,x^6 - 9727488\,x^4 - 31067136\,x^2\,,$$

$$F_4^*(x) = -11025\,x^{10} - 201600\,x^8 - 2177280\,x^6 - 11612160\,x^4 - 23224320\,x^2\,,$$

$$A_4^{**}(x) = -1225\,x^{10} - 24600\,x^8 - 732096\,x^6 - 6315264\,x^4 - 11612160\,x^2\,,$$

$$B_4^{**}(x) = 11025\,x^{10} - 151200\,x^8 - 1451520\,x^6 - 5806080\,x^4\,,$$

$$C_4^{**}(x) = -1225\,x^9 - 348000\,x^7 - 5844096\,x^5 - 31067136\,x^3 - 23224320\,x\,,$$

$$D_4^{**}(x) = -88200\,x^9 - 1209600\,x^7 - 8709120\,x^5 - 23224320\,x^3\,,$$

$$E_4^{**}(x) = 1225\,x^{10} - 24400\,x^8 - 916704\,x^6 - 9727488\,x^4 - 31067136\,x^2\,,$$

$$F_4^{**}(x) = -11025\,x^{10} - 201600\,x^8 - 2177280\,x^6 - 11612160\,x^4 - 23224320\,x^2\,,$$

$$N_4^{(11,20)} = 4900\,, \quad G_4(x) = -25\,x^8 + 3684\,x^6 - 38256\,x^4 + 80640\,x^2\,, \quad H_4(x) = -1050\,x^8 + 10080\,x^6 - 40320\,x^4\,,$$

$$N_4^{(11,11)} = 2450\,, \quad I_4(x) = 625\,x^7 - 16092\,x^5 + 96672\,x^3 - 80640\,x\,, \quad K_4(x) = 4200\,x^7 - 30240\,x^5 + 80640\,x^3\,,$$

$$N_4^{(11,02)} = 4900\,, \quad L_4(x) = -25\,x^8 - 4266\,x^6 + 56352\,x^4 - 193344\,x^2\,,$$

$$M_4(x) = 1400\,x^8 - 15120\,x^6 + 80640\,x^4 - 161280\,x^2\,,$$

$$G_4^*(x) = 25\,x^8 + 3684\,x^6 + 38256\,x^4 + 80640\,x^2\,, \quad H_4^*(x) = 1050\,x^8 + 10080\,x^6 + 40320\,x^4\,,$$

$$I_4^*(x) = -625\,x^7 - 16092\,x^5 - 96672\,x^3 - 80640\,x\,, \quad K_4^*(x) = -4200\,x^7 - 30240\,x^5 - 80640\,x^3\,,$$

$$L_4^*(x) = -25\,x^8 + 4266\,x^6 + 56352\,x^4 + 193344\,x^2\,, \quad M_4^*(x) = 1400\,x^8 + 15120\,x^6 + 80640\,x^4 + 161280\,x^2\,,$$

$$G_4^{**}(x) = -25\,x^8 - 3684\,x^6 - 38256\,x^4 - 80640\,x^2\,, \quad H_4^{**}(x) = -1050\,x^8 - 10080\,x^6 - 40320\,x^4\,,$$

$$I_4^{**}(x) = -625\,x^7 - 16092\,x^5 - 96672\,x^3 - 80640\,x\,, \quad K_4^{**}(x) = -4200\,x^7 - 30240\,x^5 - 80640\,x^3\,,$$

$$L_4^{**}(x) = 25\,x^8 - 4266\,x^6 - 56352\,x^4 - 193344\,x^2\,, \quad M_4^{**}(x) = -1400\,x^8 - 15120\,x^6 - 80640\,x^4 - 161280\,x^2\,,$$

$$N_4^{(02,20)} = 39690\,, \quad P_4(x) = -245\,x^{10} - 5205\,x^8 + 173952\,x^6 - 1542528\,x^4 + 2903040\,x^2\,,$$
$$Q_4(x) = 2205\,x^{10} - 37800\,x^8 + 362880\,x^6 - 1451520\,x^4\,,$$
$$N_4^{(02,11)} = 39690\,, \quad R_4(x) = 245\,x^9 + 79440\,x^7 - 1406592\,x^5 + 7621632\,x^3 - 5806080\,x\,,$$
$$S_4(x) = -22050\,x^9 + 302400\,x^7 - 2177280\,x^5 + 5806080\,x^3\,,$$
$$N_4^{(02,02)} = 39690\,, \quad T_4(x) = -245\,x^{10} + 4840\,x^8 - 215568\,x^6 + 2359296\,x^4 - 7621632\,x^2\,,$$
$$U_4(x) = 2205\,x^{10} + 50400\,x^8 - 544320\,x^6 + 2903040\,x^4 - 5806080\,x^2\,,$$

$$P_4^*(x) = 245\,x^{10} - 5205\,x^8 - 173952\,x^6 - 1542528\,x^4 - 2903040\,x^2\,,$$
$$Q_4^*(x) = -2205\,x^{10} - 37800\,x^8 - 362880\,x^6 - 1451520\,x^4\,,$$
$$R_4^*(x) = -245\,x^9 + 79440\,x^7 + 1406592\,x^5 + 7621632\,x^3 + 5806080\,x\,,$$
$$S_4^*(x) = 22050\,x^9 + 302400\,x^7 + 2177280\,x^5 + 5806080\,x^3\,,$$
$$T_4^*(x) = -245\,x^{10} - 4840\,x^8 - 215568\,x^6 - 2359296\,x^4 - 7621632\,x^2\,,$$
$$U_4^*(x) = 2205\,x^{10} - 50400\,x^8 - 544320\,x^6 - 2903040\,x^4 - 5806080\,x^2\,,$$

$$P_4^{**}(x) = 245\,x^{10} - 5205\,x^8 - 173952\,x^6 - 1542528\,x^4 - 2903040\,x^2\,,$$
$$Q_4^{**}(x) = -2205\,x^{10} - 37800\,x^8 - 362880\,x^6 - 1451520\,x^4\,,$$
$$R_4^{**}(x) = 245\,x^9 - 79440\,x^7 - 1406592\,x^5 - 7621632\,x^3 - 5806080\,x\,,$$
$$S_4^{**}(x) = -22050\,x^9 - 302400\,x^7 - 2177280\,x^5 - 5806080\,x^3\,,$$
$$T_4^{**}(x) = -245\,x^{10} - 4840\,x^8 - 215568\,x^6 - 2359296\,x^4 - 7621632\,x^2\,,$$
$$U_4^{**}(x) = 2205\,x^{10} - 50400\,x^8 - 544320\,x^6 - 2903040\,x^4 - 5806080\,x^2\,,$$

$$N_5^{(20,20)} = 960498\,, \quad A_5(x) = -3969\,x^{12} + 119315\,x^{10} - 6183600\,x^8 + 113915520\,x^6 - 828218880\,x^4 + 1277337600\,x^2\,,$$
$$B_5(x) = 43659\,x^{12} + 970200\,x^{10} - 16632000\,x^8 + 159667200\,x^6 - 638668800\,x^4\,,$$
$$N_5^{(20,11)} = 960498\,, \quad C_5(x) = 3969\,x^{11} - 2163350\,x^9 + 66100800\,x^7 - 843160320\,x^5 + 3951544320\,x^3 - 2554675200\,x\,,$$
$$D_5(x) = 436590\,x^{11} - 9702000\,x^9 + 133056000\,x^7 - 958003200\,x^5 + 2554675200\,x^3\,,$$
$$N_5^{(20,02)} = 960498\,, \quad E_5(x) = -3969\,x^{12} - 118825\,x^{10} + 7320800\,x^8 - 150914880\,x^6 + 1337103360\,x^4 - 3951544320\,x^2\,,$$
$$F_5(x) = 43659\,x^{12} - 1212750\,x^{10} + 22176000\,x^8 - 239500800\,x^6 + 1277337600\,x^4 - 2554675200\,x^2\,,$$

$$A_5^*(x) = -3969\,x^{12} - 119315\,x^{10} - 6183600\,x^8 - 113915520\,x^6 - 828218880\,x^4 - 1277337600\,x^2\,,$$
$$B_5^*(x) = 43659\,x^{12} - 970200\,x^{10} - 16632000\,x^8 - 159667200\,x^6 - 638668800\,x^4\,,$$
$$C_5^*(x) = 3969\,x^{11} + 2163350\,x^9 + 66100800\,x^7 + 843160320\,x^5 + 3951544320\,x^3 + 2554675200\,x\,,$$
$$D_5^*(x) = 436590\,x^{11} + 9702000\,x^9 + 133056000\,x^7 + 958003200\,x^5 + 2554675200\,x^3\,,$$
$$E_5^*(x) = 3969\,x^{12} - 118825\,x^{10} - 7320800\,x^8 - 150914880\,x^6 - 1337103360\,x^4 - 3951544320\,x^2\,,$$
$$F_5^*(x) = -43659\,x^{12} - 1212750\,x^{10} - 22176000\,x^8 - 239500800\,x^6 - 1277337600\,x^4 - 2554675200\,x^2\,,$$

$$A_5^{**}(x) = -3969\,x^{12} - 119315\,x^{10} - 6183600\,x^8 - 113915520\,x^6 - 828218880\,x^4 - 1277337600\,x^2\,,$$
$$B_5^{**}(x) = 43659\,x^{12} - 970200\,x^{10} - 16632000\,x^8 - 159667200\,x^6 - 638668800\,x^4\,,$$

$$C_5^{**}(x) = -3969\,x^{11} - 2163350\,x^9 - 66100800\,x^7 - 843160320\,x^5 - 3951544320\,x^3 - 2554675200\,x\,,$$

$$D_5^{**}(x) = -436590\,x^{11} - 9702000\,x^9 - 133056000\,x^7 - 958003200\,x^5 - 2554675200\,x^3\,,$$

$$E_5^{**}(x) = 3969\,x^{12} - 118825\,x^{10} - 7320800\,x^8 - 150914880\,x^6 - 1337103360\,x^4 - 3951544320\,x^2\,,$$

$$F_5^{**}(x) = -43659\,x^{12} - 1212750\,x^{10} - 22176000\,x^8 - 239500800\,x^6 - 1277337600\,x^4 - 2554675200\,x^2\,,$$

$$N_5^{(11,20)} = 79380\,, \quad G_5(x) = -245\,x^{10} + 79440\,x^8 - 1754496\,x^6 + 13791744\,x^4 - 23224320\,x^2\,,$$

$$H_5(x) = -17640\,x^{10} + 302400\,x^8 - 2903040\,x^6 + 11612160\,x^4\,,$$

$$N_5^{(11,11)} = 39690\,, \quad I_5(x) = 10045\,x^9 - 468960\,x^7 + 6715008\,x^5 - 33389568\,x^3 + 23224320\,x\,,$$

$$K_5(x) = 88200\,x^9 - 1209600\,x^7 + 8709120\,x^5 - 23224320\,x^3\,,$$

$$N_5^{(11,02)} = 79380\,, \quad L_5(x) = -245\,x^{10} - 89120\,x^8 + 2268864\,x^6 - 21777408\,x^4 + 66779136\,x^2\,,$$

$$M_5(x) = 22050\,x^{10} - 403200\,x^8 + 4354560\,x^6 - 23224320\,x^4 + 46448640\,x^2\,,$$

$$G_5^*(x) = 245\,x^{10} + 79440\,x^8 + 1754496\,x^6 + 13791744\,x^4 + 23224320\,x^2\,,$$

$$H_5^*(x) = 17640\,x^{10} + 302400\,x^8 + 2903040\,x^6 + 11612160\,x^4\,,$$

$$I_5^*(x) = -10045\,x^9 - 468960\,x^7 - 6715008\,x^5 - 33389568\,x^3 - 23224320\,x\,,$$

$$K_5^*(x) = -88200\,x^9 - 1209600\,x^7 - 8709120\,x^5 - 23224320\,x^3\,,$$

$$L_5^*(x) = -245\,x^{10} + 89120\,x^8 + 2268864\,x^6 + 21777408\,x^4 + 66779136\,x^2\,,$$

$$M_5^*(x) = 22050\,x^{10} + 403200\,x^8 + 4354560\,x^6 + 23224320\,x^4 + 46448640\,x^2\,,$$

$$G_5^{**}(x) = -245\,x^{10} - 79440\,x^8 - 1754496\,x^6 - 13791744\,x^4 - 23224320\,x^2\,,$$

$$H_5^{**}(x) = -17640\,x^{10} - 302400\,x^8 - 2903040\,x^6 - 11612160\,x^4\,,$$

$$I_5^{**}(x) = -10045\,x^9 - 468960\,x^7 - 6715008\,x^5 - 33389568\,x^3 - 23224320\,x\,,$$

$$K_5^{**}(x) = -88200\,x^9 - 1209600\,x^7 - 8709120\,x^5 - 23224320\,x^3\,,$$

$$L_5^{**}(x) = 245\,x^{10} - 89120\,x^8 - 2268864\,x^6 - 21777408\,x^4 - 66779136\,x^2\,,$$

$$M_5^{**}(x) = -22050\,x^{10} - 403200\,x^8 - 4354560\,x^6 - 23224320\,x^4 - 46448640\,x^2\,,$$

$$N_5^{(02,20)} = 320166\,, \quad P_5(x) = -1323\,x^{12} - 41258\,x^{10} + 2362560\,x^8 - 44501760\,x^6 + 327029760\,x^4 - 510935040\,x^2\,,$$

$$Q_5(x) = 14553\,x^{12} - 388080\,x^{10} + 6652800\,x^8 - 63866880\,x^6 + 255467520\,x^4\,,$$

$$N_5^{(02,11)} = 320166\,, \quad R_5(x) = 1323\,x^{11} + 800660\,x^9 - 25553280\,x^7 + 330877440\,x^5 - 1563586560\,x^3 + 1021870080\,x\,,$$

$$S_5(x) = -174636\,x^{11} + 3880800\,x^9 - 53222400\,x^7 + 383201280\,x^5 - 1021870080\,x^3\,,$$

$$N_5^{(02,02)} = 320166\,, \quad T_5(x) = -1323\,x^{12} + 39445\,x^{10} - 2780480\,x^8 + 58769280\,x^6 - 526325760\,x^4 + 1563586560\,x^2\,,$$

$$U_5(x) = 14553\,x^{12} + 485100\,x^{10} - 8870400\,x^8 + 95800320\,x^6 - 510935040\,x^4 + 1021870080\,x^2\,,$$

$$P_5^*(x) = 1323\,x^{12} - 41258\,x^{10} - 2362560\,x^8 - 44501760\,x^6 - 327029760\,x^4 - 510935040\,x^2\,,$$

$$Q_5^*(x) = -14553\,x^{12} - 388080\,x^{10} - 6652800\,x^8 - 63866880\,x^6 - 255467520\,x^4\,,$$

$$R_5^*(x) = -1323\,x^{11} + 800660\,x^9 + 25553280\,x^7 + 330877440\,x^5 + 1563586560\,x^3 + 1021870080\,x\,,$$

$$S_5^*(x) = 174636\,x^{11} + 3880800\,x^9 + 53222400\,x^7 + 383201280\,x^5 + 1021870080\,x^3\,,$$

$$T_5^*(x) = -1323\,x^{12} - 39445\,x^{10} - 2780480\,x^8 - 58769280\,x^6 - 526325760\,x^4 - 1563586560\,x^2\,,$$

$$U_5^*(x) = 14553\,x^{12} - 485100\,x^{10} - 8870400\,x^8 - 95800320\,x^6 - 510935040\,x^4 - 1021870080\,x^2\,,$$

$$P_5^{**}(x) = 1323\,x^{12} - 41258\,x^{10} - 2362560\,x^8 - 44501760\,x^6 - 327029760\,x^4 - 510935040\,x^2\,,$$

$$Q_5^{**}(x) = -14553\,x^{12} - 388080\,x^{10} - 6652800\,x^8 - 63866880\,x^6 - 255467520\,x^4\,,$$

$$R_5^{**}(x) = 1323\,x^{11} - 800660\,x^9 - 25553280\,x^7 - 330877440\,x^5 - 1563586560\,x^3 - 1021870080\,x\,,$$

$$S_5^{**}(x) = -174636\,x^{11} - 3880800\,x^9 - 53222400\,x^7 - 383201280\,x^5 - 1021870080\,x^3\,,$$

$$T_5^{**}(x) = -1323\,x^{12} - 39445\,x^{10} - 2780480\,x^8 - 58769280\,x^6 - 526325760\,x^4 - 1563586560\,x^2\,,$$

$$U_5^{**}(x) = 14553\,x^{12} - 485100\,x^{10} - 8870400\,x^8 - 95800320\,x^6 - 510935040\,x^4 - 1021870080\,x^2\,,$$

Recurrence relations:

$$\int x^{2n+1}\,\ln x\, J_0^2(x)\,dx =$$

$$= -\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 + n\left(4n^2 + n - 2\right) - \left[(2n+1)x^2 + 2n^2\left(4n^2 + n - 2\right)\right]\ln x\right\}J_0^2(x) -$$

$$-\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 + (4n+3)n^2 - \left[(2n+1)x^2 + 2(4n+3)(n-1)n^2\right]\ln x\right\}J_1^2(x) +$$

$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1 + 2n(2n+1)\,\ln x\right]J_0(x)\,J_1(x) -$$

$$-\frac{2\left(4n^2 + n - 2\right)n^3}{(2n+1)^2}\int x^{2n-1}\,\ln x\, J_0^2(x)\,dx - \frac{2\left(4n+3\right)(n-1)^2 n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\, J_1^2(x)\,dx$$

$$\int x^{2n}\,\ln x\, J_0(x)\,J_1(x)\,dx = \frac{x^{2n}}{4}\left\{(1 - 2n\ln x)J_0^2(x) + \left[1 - 2(n-1)\ln x\right]J_1^2(x)\right\} +$$

$$+n^2\int x^{2n-1}\,\ln x\, J_0^2(x)\,dx + (n-1)^2\int x^{2n-1}\,\ln x\, J_1^2(x)\,dx$$

$$\int x^{2n+1}\,\ln x\, J_1^2(x)\,dx =$$

$$= \frac{x^{2n}}{2(2n+1)^2}\left\{-x^2 + 4n^3 + 3n^2 - 1 + \left[(2n+1)x^2 - 2n\left(4n^3 + 3n^2 - 1\right)\right]\ln x\right\}J_0^2(x) +$$

$$+\frac{x^{2n}}{2(2n+1)^2}\left\{-x^2 + \left(4n^2 + 5n + 2\right)n + \left[(2n+1)x^2 - 2(n-1)\left(4n^2 + 5n + 2\right)n\right]\ln x\right\}J_1^2(x) +$$

$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1 - 2(n+1)(2n+1)\,\ln x\right]J_0(x)\,J_1(x) +$$

$$+\frac{2\left(4n^3 + 3n^2 - 1\right)n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\, J_0^2(x)\,dx + \frac{2\left(4n^2 + 5n + 2\right)(n-1)^2 n}{(2n+1)^2}\int x^{2n-1}\,\ln x\, J_1^2(x)\,dx$$

$$\int x^{2n+1}\,\ln x\, I_0^2(x)\,dx =$$

$$= -\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 - n\left(4n^2 + n - 2\right) + \left[-(2n+1)x^2 + 2n^2\left(4n^2 + n - 2\right)\right]\ln x\right\}I_0^2(x) +$$

$$+\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 - (4n+3)n^2 - \left[(2n+1)x^2 - 2(4n+3)(n-1)n^2\right]\ln x\right\}I_1^2(x) +$$

$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1 + 2n(2n+1)\,\ln x\right]I_0(x)\,I_1(x) +$$

$$+\frac{2\left(4\,n^2+n-2\right)n^3}{(2n+1)^2}\int x^{2n-1}\,\ln x\, I_0^2(x)\,dx-\frac{2\left(4\,n+3\right)(n-1)^2\,n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\, I_1^2(x)\,dx$$

$$\int x^{2n}\,\ln x\, I_0(x)\, I_1(x)\,dx=-\frac{x^{2n}}{4}\left\{(1-2n\ln x)\, I_0^2(x)-[1-2(n-1)\ln x]\, I_1^2(x)\right\}+$$

$$-n^2\int x^{2n-1}\,\ln x\, I_0^2(x)\,dx+(n-1)^2\int x^{2n-1}\,\ln x\, I_1^2(x)\,dx$$

$$\int x^{2n+1}\,\ln x\, I_1^2(x)\,dx=$$

$$=\frac{x^{2n}}{2(2n+1)^2}\left\{x^2+4\,n^3+3\,n^2-1-[(2n+1)x^2+2\,n\left(4\,n^3+3\,n^2-1\right)]\ln x\right\}I_0^2(x)-$$

$$-\frac{x^{2n}}{2(2n+1)^2}\left\{x^2+\left(4\,n^2+5\,n+2\right)n-[(2n+1)x^2+2\,(n-1)\left(4\,n^2+5\,n+2\right)n]\ln x\right\}I_1^2(x)+$$

$$-\frac{x^{2n+1}}{2(2n+1)^2}\left[1-2\,(n+1)\,(2\,n+1)\,\ln x\right]I_0(x)\, I_1(x)+$$

$$+\frac{2\left(4\,n^3+3\,n^2-1\right)n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\, I_0^2(x)\,dx-\frac{2\left(4\,n^2+5\,n+2\right)(n-1)^2\,n}{(2n+1)^2}\int x^{2n-1}\,\ln x\, I_1^2(x)\,dx$$

$$\int x^{2n+1}\,\ln x\, K_0^2(x)\,dx=$$

$$=-\frac{x^{2n}}{2(2n+1)^2}\left\{x^2-n\left(4\,n^2+n-2\right)-[(2n+1)x^2-2\,n^2\left(4\,n^2+n-2\right)]\ln x\right\}K_0^2(x)+$$

$$+\frac{x^{2n}}{2(2n+1)^2}\left\{x^2-\left(4\,n+3\right)n^2-[(2n+1)x^2-2\,(4\,n+3)\,(n-1)\,n^2]\ln x\right\}K_1^2(x)+$$

$$-\frac{x^{2n+1}}{2(2n+1)^2}\left[1+2n(2n+1)\,\ln x\right]K_0(x)\, K_1(x)+$$

$$+\frac{2\left(4\,n^2+n-2\right)n^3}{(2n+1)^2}\int x^{2n-1}\,\ln x\, K_0^2(x)\,dx-\frac{2\left(4\,n+3\right)(n-1)^2\,n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\, K_1^2(x)\,dx$$

$$\int x^{2n}\,\ln x\, K_0(x)\, K_1(x)\,dx=\frac{x^{2n}}{4}\left\{(1-2n\ln x)\, K_0^2(x)-[1-2(n-1)\ln x]\, K_1^2(x)\right\}+$$

$$+n^2\int x^{2n-1}\,\ln x\, K_0^2(x)\,dx-(n-1)^2\int x^{2n-1}\,\ln x\, K_1^2(x)\,dx$$

$$\int x^{2n+1}\,\ln x\, K_1^2(x)\,dx=$$

$$=\frac{x^{2n}}{2(2n+1)^2}\left\{x^2+4\,n^3+3\,n^2-1-[(2n+1)x^2+2\,n\left(4\,n^3+3\,n^2-1\right)]\ln x\right\}K_0^2(x)-$$

$$-\frac{x^{2n}}{2(2n+1)^2}\left\{x^2+\left(4\,n^2+5\,n+2\right)n-[(2n+1)x^2+2\,(n-1)\left(4\,n^2+5\,n+2\right)n]\ln x\right\}K_1^2(x)+$$

$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1-2\,(n+1)\,(2\,n+1)\,\ln x\right]K_0(x)\, K_1(x)+$$

$$+\frac{2\left(4\,n^3+3\,n^2-1\right)n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\, K_0^2(x)\,dx-\frac{2\left(4\,n^2+5\,n+2\right)(n-1)^2\,n}{(2n+1)^2}\int x^{2n-1}\,\ln x\, K_1^2(x)\,dx$$

2.4.2. Integrals of the Type $\int x^n \ln x \, Z_\mu(x) \, Z_\nu^*(x) \, dx$:

Integrals were found in the following cases:

n	$J_0 I_0$	$J_0 K_0$	$J_1 I_1$	$J_1 K_1$	$I_0 K_0$	$I_0 K_1$	$I_1 K_0$	$I_1 K_1$
1			*	*	*			*
2						*	*	
3	*	*			*			*
4						*	*	
5			*	*	*			*
6						*	*	
7	*	*			*			*
8						*	*	
9			*	*	*			*
10						*	*	

Therefore one can expect, that integrals of the types

$$\int x^{4n+3-2\nu} \ln x \, J_\nu(x) \, I_\nu(x) \, dx \quad \text{and} \quad \int x^{4n+3-2\nu} \ln x \, J_\nu(x) \, K_\nu(x) \, dx$$

may be expressed.

For the integrals $\int x^{2n+1} \ln x \, I_\nu(x) \, K_\nu(x) \, dx$ and $\int x^{2n} \ln x \, I_\nu(x) \, K_{1-\nu}(x) \, dx$ recurrence relations were found. Holds $x[I_0(x)K_1(x) + I_1(x)K_0(x)] = 1$ ([5], XIII B. 2.), so any multiple of this expression may be added to these antiderivatives.

a) Integrals with $J_0(x) \, Z_0(x)$:

$$\int x^3 \, \ln x \, J_0(x) \, I_0(x) \, dx = \frac{x^2 \ln x}{2} \, [x \, J_0(x)I_1(x) + x \, J_1(x)I_0(x) - 2 \, J_1(x)I_1(x)] -$$

$$- \frac{x}{2} \, [J_0(x)I_1(x) - J_1(x)I_0(x) + x \, J_1(x)I_1(x)]$$

$$\int x^3 \, \ln x \, J_0(x) \, K_0(x) \, dx = \frac{x^2 \ln x}{2} \, [-x \, J_0(x)K_1(x) + x \, J_1(x)K_0(x) + 2 \, J_1(x)K_1(x)] +$$

$$+ \frac{x}{2} \, [J_0(x)K_1(x) + J_1(x)K_0(x) + x \, J_1(x)K_1(x)]$$

$$\int x^7 \, \ln x \, J_0(x) \, I_0(x) \, dx = \frac{x^2 \ln x}{2} \, \Big[48x^2 \, J_0(x) \, I_0(x) + x \, \big(x^4 - 12 \, x^2 - 96\big) \, J_0(x) \, I_1(x) +$$

$$+ x \, \big(x^4 + 12 \, x^2 - 96\big) \, J_1(x) \, I_0(x) + (192 - 6 \, x^4) \, J_1(x) \, I_1(x)\Big] + \frac{x}{2} \, \Big[32x^3 \, J_0(x) \, I_0(x) +$$

$$+ (96 - 88 \, x^2 - 5 \, x^4) \, J_0(x) \, I_1(x) + (5 \, x^4 - 96 - 88 \, x^2) \, J_1(x) \, I_0(x) + x \, \big(272 - x^4\big) \, J_1(x) \, I_1(x)\Big]$$

$$\int x^7 \, \ln x \, J_0(x) \, K_0(x) \, dx = \frac{x^2 \ln x}{2} \, \Big[48x^2 \, J_0(x) \, K_0(x) - x \, \big(x^4 - 12 \, x^2 - 96\big) \, J_0(x) \, K_1(x) +$$

$$+ x \, \big(x^4 + 12 \, x^2 - 96\big) \, J_1(x) \, K_0(x) + (6 \, x^4 - 192) \, J_1(x) \, K_1(x)\Big] + \frac{x}{2} \, \Big[32x^3 \, J_0(x) \, K_0(x) +$$

$$+ (88 \, x^2 + 5 \, x^4 - 96) \, J_0(x) \, K_1(x) + (5 \, x^4 - 96 - 88 \, x^2) \, J_1(x) \, K_0(x) + x \, \big(x^4 - 272\big) \, J_1(x) \, K_1(x)\Big]$$

b) Integrals with $J_1(x) \, Z_1(x)$:

$$\int x \, \ln x \, J_1(x) \, I_1(x) \, dx = \frac{x \ln x}{2} \, [J_1(x)I_0(x) - J_0(x)I_1(x)] + \frac{J_0(x)I_0(x)}{2}$$

$$\int x \ln x\, J_1(x)\, K_1(x)\, dx = -\frac{x \ln x}{2}\left[J_1(x)K_0(x) + J_0(x)K_1(x)\right] - \frac{J_0(x)K_0(x)}{2}$$

$$\int x^5 \ln x\, J_1(x)\, I_1(x)\, dx =$$

$$= \frac{x^2 \ln x}{2}\left[4x^2\, J_0(x)\, I_0(x) - x(x^2+8)\, J_0(x)\, I_1(x) + x(x^2-8)\, J_1(x)\, I_1(x) + 16\, J_1(x)\, I_1(x)\right] +$$

$$+ \frac{x}{2}\left[x^3\, J_0(x)\, I_0(x) + (8-4x^2)\, J_0(x)\, I_1(x) - (4x^2+8)\, J_1(x)\, I_0(x) + 16x\, J_1(x)\, I_1(x)\right]$$

$$\int x^5 \ln x\, J_1(x)\, K_1(x)\, dx =$$

$$= \frac{x^2 \ln x}{2}\left[-4x^2\, J_0(x)\, K_0(x) - x(x^2+8)\, J_0(x)\, K_1(x) - x(x^2-8)\, J_1(x)\, K_0(x) + 16\, J_1(x)\, K_1(x)\right] +$$

$$+ \frac{x}{2}\left[-x^3\, J_0(x)\, K_0(x) + (8-4x^2)\, J_0(x)\, K_1(x) + (8+4x^2)\, J_1(x)\, K_0(x) + 16x\, J_1(x)\, K_1(x)\right]$$

$$\int x^9 \ln x\, J_1(x)\, I_1(x)\, dx = \frac{x^2 \ln x}{2}\left[8\,x^2\left(x^4-192\right)\, J_0(x)\, I_0(x) + \right.$$

$$+ x\left(3072 + 384\,x^2 - 32\,x^4 - x^6\right)\, J_0(x)\, I_1(x) + x\left(3072 - 384\,x^2 - 32\,x^4 + x^6\right)\, J_1(x)\, I_0(x) -$$

$$- (6144 - 192\,x^4)\, J_1(x)\, I_1(x)\big] + \frac{x}{2}\left[x^3\left(x^4-1408\right)\, J_0(x)\, I_0(x) + \right.$$

$$+ (-3072 + 3584\,x^2 + 256\,x^4 - 8\,x^6)\, J_0(x)\, I_1(x) + (3072 + 3584\,x^2 - 256\,x^4 - 8\,x^6)\, J_1(x)\, I_0(x) +$$

$$+ 80\,x\left(x^4-128\right)\, J_1(x)\, I_1(x)\big]$$

$$\int x^9 \ln x\, J_1(x)\, K_1(x)\, dx = \frac{x^2 \ln x}{2}\left[8\,x^2\left(192-x^4\right)\, J_0(x)\, K_0(x) + \right.$$

$$+ x\left(3072 + 384\,x^2 - 32\,x^4 - x^6\right)\, J_0(x)\, K_1(x) - x\left(3072 - 384\,x^2 - 32\,x^4 + x^6\right)\, J_1(x)\, K_0(x) -$$

$$- (6144 - 192\,x^4)\, J_1(x)\, K_1(x)\big] + \frac{x}{2}\left[x^3\left(1408 - x^4\right)\, J_0(x)\, K_0(x) + \right.$$

$$+ (-3072 + 3584\,x^2 + 256\,x^4 - 8\,x^6)\, J_0(x)\, K_1(x) + (-3072 - 3584\,x^2 + 256\,x^4 + 8\,x^6)\, J_1(x)\, K_0(x) +$$

$$+ 80\,x\left(x^4-128\right)\, J_1(x)\, K_1(x)\big]$$

c) Integrals with $I_\nu(x)\, K_\nu(x)$:

$$\int x \ln x\, I_0(x)\, K_0(x)\, dx =$$

$$= \frac{x^2 \ln x}{2}\left[I_0(x)K_0(x) + I_1(x)K_1(x)\right] - \frac{x}{2}\left[x\, I_0(x)K_0(x) + I_0(x)K_1(x) + x\, I_1(x)K_1(x)\right]$$

$$\int x \ln x\, I_1(x)\, K_1(x)\, dx = \frac{x \ln x}{2}\left[x\, I_0(x)K_0(x) + I_0(x)K_1(x) - I_1(x)K_0(x) + x\, I_1(x)K_1(x)\right] +$$

$$+ \frac{1}{2}\left[(1-x^2)\, I_0(x)K_0(x) - x\, I_0(x)K_1(x) - x^2\, I_1(x)K_1(x)\right]$$

$$\int x^3 \ln x\, I_0(x)\, K_0(x)\, dx =$$

$$= \frac{x^2 \ln x}{6}\left[x^2\, I_0(x)K_0(x) - x\, I_0(x)K_1(x) + x\, I_1(x)K_0(x) + (x^2+2)\, I_1(x)K_1(x)\right] -$$

$$- \frac{x}{36}\left[2x(x^2+3)\, I_0(x)K_0(x) + (x^2-4)\, I_0(x)K_1(x) - (x^2+16)\, I_1(x)K_0(x) + 2x(x^2-1)\, I_1(x)K_1(x)\right]$$

$$\int x^3 \ln x \, I_1(x) \, K_1(x) \, dx =$$

$$= \frac{x^2 \ln x}{6} \left[x^2 \, I_0(x)K_0(x) + 2x \, I_0(x)K_1(x) - 2x \, I_1(x)K_0(x) + (x^2 - 4) \, I_1(x)K_1(x) \right] -$$

$$- \frac{x}{36} \left[2x(x^2 - 6) \, I_0(x)K_0(x) + (x^2 - 4) \, I_0(x)K_1(x) - (x^2 - 20) \, I_1(x)K_0(x) + 2x(x^2 - 1) \, I_1(x)K_1(x) \right]$$

$$\int x^5 \ln x \, I_0(x) \, K_0(x) \, dx =$$

$$= \frac{x^2 \ln x}{30} \left[x^2 \left(3x^2 - 8 \right) I_0(x) K_0(x) - 2x \left(8 + 3x^2 \right) I_0(x) K_1(x) + \right.$$

$$\left. + 2x \left(8 + 3x^2 \right) I_1(x) K_0(x) + (32 + 16x^2 + 3x^4) I_1(x) K_1(x) \right] +$$

$$+ \frac{x}{900} \left[-2x \left(240 + 56x^2 + 9x^4 \right) I_0(x) K_0(x) + (-9x^4 - 344x^2 + 1376) I_1(x) K_0(x) + \right.$$

$$\left. + (9x^4 + 344x^2 + 2336) I_0(x) K_1(x) - 2x \left(9x^4 - 52x^2 - 344 \right) I_1(x) K_1(x) \right]$$

$$\int x^5 \ln x \, I_1(x) \, K_1(x) \, dx =$$

$$= \frac{x^2 \ln x}{10} \left[x^2(x^2 + 4) I_0(x) K_0(x) + x(3x^2 + 8) I_0(x) K_1(x) - x(3x^2 + 8) I_1(x) K_0(x) + \right.$$

$$\left. + (x^4 - 8x^2 - 16) I_1(x) K_1(x) \right] + \frac{x}{300} \left[2x \left(120 + 23x^2 - 3x^4 \right) I_0(x) K_0(x) - \right.$$

$$- (3x^4 + 608 - 152x^2) I_0(x) K_1(x) + (3x^4 - 152x^2 - 1088) I_1(x) K_0(x) - 2x \left(152 + 16x^2 + 3x^4 \right) I_1(x) K_1(x) \right]$$

$$\int x^7 \ln x \, I_0(x) \, K_0(x) \, dx = \frac{x^2 \ln x}{70} \left[x^2 \left(5x^4 - 144 - 36x^2 \right) I_0(x) K_0(x) - \right.$$

$$- 3x \left(96 + 36x^2 + 5x^4 \right) I_0(x) K_1(x) + 3x \left(96 + 36x^2 + 5x^4 \right) I_1(x) K_0(x) +$$

$$+ (576 + 288x^2 + 54x^4 + 5x^6) I_1(x) K_1(x) \left] + \frac{x}{4900} \left[-2x \left(10080 + 4152x^2 + 303x^4 + 25x^6 \right) I_0(x) K_0(x) - \right.$$

$$- (86592 + 21648x^2 + 3078x^4 + 25x^6) I_0(x) K_1(x) + \left(126912 + 21648x^2 + 3078x + 25x^6 \right) I_1(x) K_0(x) +$$

$$+ 2x \left(21648 + 5784x^2 + 297x^4 - 25x^6 \right) I_1(x) K_1(x) \right]$$

$$\int x^7 \ln x \, I_1(x) \, K_1(x) \, dx = \frac{x^2 \ln x}{70} \left[x^2 \left(192 + 48x^2 + 5x^4 \right) I_0(x) K_0(x) + \right.$$

$$+ 4x \left(96 + 36x^2 + 5x^4 \right) I_0(x) K_1(x) - 4x \left(96 + 36x^2 + 5x^4 \right) I_1(x) K_0(x) +$$

$$+ (5x^6 - 768 - 384x^2 - 72x^4) I_1(x) K_1(x) \left] + \frac{x}{4900} \left[2x \left(13440 + 5256x^2 + 334x^4 - 25x^6 \right) I_0(x) K_0(x) + \right.$$

$$+ (27744x^2 + 3684x^4 - 25x^6 - 110976) I_0(x) K_1(x) + \left(25x^6 - 3684x^4 - 27744x^2 - 164736 \right) I_1(x) K_0(x) -$$

$$- 2x \left(27744 + 7152x^2 + 291x^4 + 25x^6 \right) I_1(x) K_1(x) \right]$$

$$\int x^9 \ln x \, I_0(x) \, K_0(x) \, dx = \frac{x^2 \ln x}{630} \left[x^2 \left(35x^6 - 18432 - 4608x^2 - 480x^4 \right) I_0(x) K_0(x) - \right.$$

$$- 4x \left(9216 + 3456x^2 + 480x^4 + 35x^6 \right) I_0(x) K_1(x) + 4x \left(9216 + 3456x^2 + 480x^4 + 35x^6 \right) I_1(x) K_0(x) +$$

$$+ (73728 + 36864x^2 + 6912x^4 + 640x^6 + 35x^8) I_1(x) K_1(x) \right] +$$

$$+ \frac{x}{396900} \left[-2x \left(11612160 + 6315264x^2 + 732096x^4 + 24600x^6 + 1225x^8 \right) I_0(x) K_0(x) - \right.$$

$$- (31067136x^2 + 5844096x^4 + 348000x^6 + 1225x^8 - 124268544) I_0(x) K_1(x) +$$

$$+ \left(170717184 + 31067136x^2 + 5844096\,x^4 + 348000\,x^6 + 1225\,x^8\right)\,I_1(x)\,K_0(x)+$$

$$+2\,x\left(31067136 + 9727488\,x^2 + 916704\,x^4 + 24400\,x^6 - 1225\,x^8\right)\,I_1(x)\,K_1(x)\Big]$$

$$\int x^9\,\ln x\,I_1(x)\,K_1(x)\,dx = \frac{x^2\,\ln x}{126}\,\Big[x^2\left(4608 + 1152\,x^2 + 120\,x^4 + 7\,x^6\right)\,I_0(x)\,K_0(x)+$$

$$+x\left(9216 + 3456\,x^2 + 480\,x^4 + 35\,x^6\right)\,I_0(x)\,K_1(x) - x\left(9216 + 3456\,x^2 + 480\,x^4 + 35\,x^6\right)\,I_1(x)\,K_0(x)+$$

$$+(7\,x^8 - 18432 - 9216\,x^2 - 1728\,x^4 - 160\,x^6)\,I_1(x)\,K_1(x)\Big] +$$

$$+\frac{x}{79380}\,\Big[2\,x\left(2903040 + 1542528\,x^2 + 173952\,x^4 + 5205\,x^6 - 245\,x^8\right)\,I_0(x)\,K_0(x)+$$

$$+(7621632\,x^2 + 1406592\,x^4 + 79440\,x^6 - 245\,x^8 - 30486528)\,I_0(x)\,K_1(x)+$$

$$+\left(245\,x^8 - 7621632x^2 - 1406592\,x^4 - 79440\,x^6 - 42098688\right)\,I_1(x)\,K_0(x)-$$

$$-2\,x\left(7621632 + 2359296\,x^2 + 215568\,x^4 + 4840\,x^6 + 245\,x^8\right)\,I_1(x)\,K_1(x)\Big]$$

About recurrence relations see page 358.

d) Integrals with $I_\nu(x)\,K_{1-\nu}(x)$:

$$\int x^2\,\ln x\,I_0(x)\,K_1(x)\,dx = \frac{x^2\,\ln x}{4}\,[x\,I_0(x)K_1(x) + x\,I_1(x)K_0(x) + 2\,I_1(x)K_1(x)] -$$

$$-\frac{x}{8}\,\Big[2x\,I_0(x)K_0(x) + (x^2 + 4)\,I_0(x)K_1(x) + x^2\,I_1(x)K_0(x) + 2x\,I_1(x)K_1(x)\Big]$$

$$\int x^2\,\ln x\,I_1(x)\,K_0(x)\,dx = \frac{x^2\,\ln x}{4}\,[x\,I_1(x)K_0(x) + x\,I_0(x)K_1(x) - 2\,I_1(x)K_1(x)] +$$

$$+\frac{x}{8}\,\Big[2x\,I_0(x)K_0(x) - (x^2 + 4)\,I_1(x)K_0(x) - x^2\,I_0(x)K_1(x) + 2x\,I_1(x)K_1(x)\Big]$$

$$\int x^4\,\ln x\,I_0(x)\,K_1(x)\,dx =$$

$$= \frac{x^2\,\ln x}{24}\,\Big[-4x^2\,I_0(x)K_0(x) + x(3x^2 - 8)\,I_0(x)K_1(x) + x(3x^2 + 8)\,I_1(x)K_0(x) + 8(x^2 + 2)\,I_1(x)K_1(x)\Big] -$$

$$-\frac{x}{288}\,\Big[8x(x^2 + 12)\,I_0(x)K_0(x) + (9x^4 + 40x^2 - 160)\,I_0(x)K_1(x)+$$

$$+(9x^4 - 40x^2 - 352)\,I_1(x)K_0(x) + 8x(x^2 - 10)\,I_1(x)K_1(x)\Big]$$

$$\int x^4\,\ln x\,I_1(x)\,K_0(x)\,dx =$$

$$= \frac{x^2\,\ln x}{24}\,\Big[4x^2\,I_0(x)K_0(x) + x(3x^2 + 8)\,I_0(x)K_1(x) + x(3x^2 - 8)\,I_1(x)K_0(x) - 8(x^2 + 2)\,I_1(x)K_1(x)\Big] +$$

$$+\frac{x}{288}\,\Big[8x(x^2 + 12)\,I_0(x)K_0(x) - (160 - 40x^2 + 9x^4)\,I_0(x)K_1(x)-$$

$$-(9x^4 + 40x^2 + 352)\,I_1(x)K_0(x) + 8x(x^2 - 10)\,I_1(x)K_1(x)\Big]$$

$$\int x^6\,\ln x\,I_0(x)\,K_1(x)\,dx =$$

$$= \frac{x^2\,\ln x}{60}\,\Big[-12\,x^2\left(4 + x^2\right)\,I_0(x)\,K_0(x) + x\left(-96 - 36\,x^2 + 5\,x^4\right)\,I_0(x)\,K_1(x)+$$

$$+x\left(96 + 36\,x^2 + 5\,x^4\right)\,I_1(x)\,K_0(x) + (192 + 96\,x^2 + 18\,x^4)\,I_1(x)\,K_1(x)\Big] +$$

$$+\frac{x}{1800}\,\Big[-6\,x\left(480 + 152\,x^2 + 3\,x^4\right)\,I_0(x)\,K_0(x) - (25\,x^6 + 234\,x^4 + 2544\,x^2 - 10176)\,I_1(x)\,K_0(x)-$$

$$-(25\,x^6 - 234\,x^4 - 2544\,x^2 - 15936)\,I_0(x)\,K_1(x) + 6\,x\left(848 + 184\,x^2 - 3\,x^4\right)\,I_1(x)\,K_1(x)\Big]$$

$$\int x^6 \ln x \, I_1(x) \, K_0(x) \, dx =$$

$$= \frac{x^2 \ln x}{60} \left[12 \, x^2 \left(4 + x^2 \right) I_0(x) \, K_0(x) + x \left(5 \, x^4 + 36 \, x^2 + 96 \right) I_0(x) \, K_1(x) + \right.$$

$$\left. + x \left(5 \, x^4 - 36 \, x^2 - 96 \right) I_1(x) \, K_0(x) - \left(192 + 96 \, x^2 + 18 \, x^4 \right) I_1(x) \, K_1(x) \right] +$$

$$+ \frac{x}{1800} \left[6x \left(480 + 152 \, x^2 + 3 \, x^4 \right) I_0(x) \, K_0(x) + \left(-10176 + 2544 \, x^2 + 234 \, x^4 - 25 \, x^6 \right) I_0(x) \, K_1(x) - \right.$$

$$\left. - \left(15936 + 2544 \, x^2 + 234 \, x^4 + 25 \, x^6 \right) I_1(x) \, K_0(x) + 6 \, x \left(3 \, x^4 - 848 - 184 \, x^2 \right) I_1(x) \, K_1(x) \right]$$

$$\int x^8 \ln x \, I_0(x) \, K_1(x) \, dx = \frac{x^2 \ln x}{560} \left[-24 \, x^2 \left(192 + 48 \, x^2 + 5 \, x^4 \right) I_0(x) \, K_0(x) + \right.$$

$$+ x \left(35 \, x^6 - 9216 - 3456 \, x^2 - 480 \, x^4 \right) I_0(x) \, K_1(x) + x \left(9216 + 3456 \, x^2 + 480 \, x^4 + 35 \, x^6 \right) I_1(x) \, K_0(x) +$$

$$\left. + \left(18432 + 9216 \, x^2 + 1728 \, x^4 + 160 \, x^6 \right) I_1(x) \, K_1(x) \right] +$$

$$+ \frac{x}{156800} \left[-32 \, x \left(80640 + 38256 \, x^2 + 3684 \, x^4 + 25 \, x^6 \right) I_0(x) \, K_0(x) - \right.$$

$$- \left(3093504 \, x^2 + 514944 \, x^4 + 20000 \, x^6 + 1225 \, x^8 - 12374016 \right) I_0(x) \, K_1(x) +$$

$$+ \left(17534976 + 3093504 x^2 + 514944 \, x^4 + 20000 \, x^6 - 1225 \, x^6 \right) I_1(x) \, K_0(x) +$$

$$\left. + 32 \, x \left(193344 + 56352 \, x^2 + 4266 \, x^4 - 25 \, x^6 \right) I_1(x) \, K_1(x) \right]$$

$$\int x^8 \ln x \, I_1(x) \, K_0(x) \, dx = \frac{x^2 \ln x}{560} \left[24 \, x^2 \left(192 + 48 \, x^2 + 5 \, x^4 \right) I_0(x) \, K_0(x) + \right.$$

$$+ x \left(9216 + 3456 \, x^2 + 480 \, x^4 + 35 \, x^6 \right) I_0(x) \, K_1(x) + x \left(35 \, x^6 - 9216 - 3456 \, x^2 - 480 \, x^4 \right) I_1(x) \, K_0(x) -$$

$$\left. - \left(18432 + 9216 \, x^2 + 1728 \, x^4 + 160 \, x^6 \right) I_1(x) \, K_1(x) \right] +$$

$$+ \frac{x}{156800} \left[32 \, x \left(80640 + 38256 \, x^2 + 3684 \, x^4 + 25 \, x^6 \right) I_0(x) \, K_0(x) + \right.$$

$$+ \left(3093504 \, x^2 + 514944 \, x^4 + 20000 \, x^6 - 1225 \, x^8 - 12374016 \right) I_0(x) \, K_1(x) -$$

$$- \left(17534976 + 3093504 x^2 + 514944 \, x^4 + 20000 \, x^6 + 1225 \, x^8 \right) I_1(x) \, K_0(x) +$$

$$\left. + +32 \, x \left(-193344 - 56352 \, x^2 - 4266 \, x^4 + 25 \, x^6 \right) I_1(x) \, K_1(x) \right]$$

$$\int x^{10} \ln x \, I_0(x) \, K_1(x) \, dx = \frac{x^2 \ln x}{1260} \left[-40 \, x^2 \left(4608 + 1152 \, x^2 + 120 \, x^4 + 7 \, x^6 \right) I_0(x) \, K_0(x) + \right.$$

$$+ x \left(63 \, x^8 - 368640 - 138240 \, x^2 - 19200 \, x^4 - 1400 \, x^6 \right) I_0(x) \, K_1(x) +$$

$$+ x \left(368640 + 138240 \, x^2 + 19200 \, x^4 + 1400 \, x^6 + 63 \, x^8 \right) I_1(x) \, K_0(x) +$$

$$\left. + \left(737280 + 368640 \, x^2 + 69120 \, x^4 + 6400 \, x^6 + 350 \, x^8 \right) I_1(x) \, K_1(x) \right] +$$

$$+ \frac{x}{793800} \left[-10 \, x \left(23224320 + 13791744 \, x^2 + 1754496 \, x^4 + 79440 \, x^6 + 245 \, x^8 \right) I_0(x) \, K_0(x) - \right.$$

$$- \left(333895680 \, x^2 + 67150080 \, x^4 + 4689600 \, x^6 + 100450 \, x^8 + 3969 \, x^{10} - 1335582720 \right) I_0(x) \, K_1(x) -$$

$$- \left(3969 \, x^{10} - 1800069120 - 333895680 x^2 - 67150080 \, x^4 - 4689600 \, x^6 - 100450 \, x^8 \right) I_1(x) \, K_0(x) -$$

$$\left. - 10 \, x \left(245 \, x^8 - 66779136 - 21777408 \, x^2 - 2268864 \, x^4 - 89120 \, x^6 \right) I_1(x) \, K_1(x) \right]$$

$$\int x^{10} \ln x \, I_1(x) \, K_0(x) \, dx = \frac{x^2 \ln x}{1260} \left[40 \, x^2 \left(4608 + 1152 \, x^2 + 120 \, x^4 + 7 \, x^6 \right) I_0(x) \, K_0(x) + \right.$$

$$+ x \left(368640 + 138240 \, x^2 + 19200 \, x^4 + 1400 \, x^6 + 63 \, x^8 \right) I_0(x) \, K_1(x) +$$

$$+ x \left(63 \, x^8 - 368640 - 138240 \, x^2 - 19200 \, x^4 - 1400 \, x^6 \right) I_1(x) \, K_0(x) -$$

$$-(737280 + 368640\,x^2 + 69120\,x^4 + 6400\,x^6 + 350\,x^8)\,I_1(x)\,K_1(x)\Big] +$$

$$+\frac{x}{793800}\Big[10\,x\left(23224320 + 13791744\,x^2 + 1754496\,x^4 + 79440\,x^6 + 245\,x^8\right)\,I_0(x)\,K_0(x)+$$

$$+(333895680\,x^2 + 67150080\,x^4 + 4689600\,x^6 + 100450\,x^8 - 3969\,x^{10} - 1335582720)\,I_0(x)\,K_1(x)-$$

$$-\left(1800069120 + 333895680x^2 + 67150080\,x^4 + 4689600\,x^6 + 100450\,x^8 + 3969\,x^{10}\right)\,I_1(x)\,K_0(x)+$$

$$+10\,x\left(245\,x^8 - 66779136 - 21777408\,x^2 - 2268864\,x^4 - 89120\,x^6\right)\,I_1(x)\,K_1(x)\Big]$$

Recurrence Relations:

$$\int x^{2n+1}\,\ln x\,I_0(x)\,K_0(x)\,dx = \frac{x^{2n-1}}{2(2n+1)^2}\left\{\Big[((2n+1)x^3 + 4n^2(n+1)x)\,I_0(x)\,K_0(x) - 2n^2x^2\,I_0(x)\,K_1(x) +\right.$$

$$+2n(n+1)x^2\,I_1(x)\,K_0(x) + (2n+1)x^3\,I_1(x)\,K_1(x)\Big]\,\ln x - x\,[x^2+2n(n+1)]\,I_0(x)\,K_0(x) - x^2\,I_0(x)\,K_1(x)-$$

$$\left.- x^3\,I_1(x)\,K_1(x)\right\} - \frac{4n^3(n+1)}{(2n-1)^2}\int x^{2n+1}\,\ln x\,I_0(x)\,K_0(x)\,dx + \frac{2n^2}{2n+1}\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx-$$

$$-\frac{4n^2(n+1)}{(2n+1)^2}\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+1}\,\ln x\,I_1(x)\,K_1(x)\,dx = \frac{x^{2n-1}}{2(2n+1)^2}\left\{\Big[((2n+1)x^3 - 2n(2n^2 + 2n + 1)x)\,I_0(x)\,K_0(x)+\right.$$

$$+(2n^2 + 4n + 1)x^2\,I_0(x)\,K_1(x) - (2n^2 + 2n + 1)x^2\,I_1(x)\,K_0(x) + (2n+1)x^3\,I_1(x)\,K_1(x)\Big]\,\ln x-$$

$$\left.-[x^3 - (2n^2 + 2n + 1)x]\,I_0(x)\,K_0(x) - x^2\,I_0(x)\,K_1(x) - x^3\,I_1(x)\,K_1(x)\right\} +$$

$$+\frac{2n^2(2n^2 + 2n + 1)}{(2n+1)^2}\int x^{2n-1}\,\ln x\,I_0(x)\,K_0(x)\,dx - \frac{2n(n+1)}{2n+1}\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx+$$

$$+\frac{2n(2n^2 + 2n + 1)}{(2n+1)^2}\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+2}\,\ln x\,I_0(x)\,K_1(x)\,dx = \frac{x^{2n-1}}{2(2n+1)^2}\left\{\Big[(2n^2(4n^2 + 5n + 2)x - n(2n+1)x^3)\,I_0(x)\,K_0(x)+\right.$$

$$+\left(\frac{(2n+1)^2x^2}{2(n+1)} - n^2\right)\,x^2\,I_0(x)\,K_1(x) + \left(\frac{(2n+1)^2x^2}{2(n+1)} + n(4n^2 + 5n + 2)\right)\,x^2\,I_1(x)\,K_0(x)+$$

$$+ (2n+1)(n+1)x^3\,I_1(x)\,K_1(x)\Big]\,\ln x - \frac{x^3 + 2n(4n^2 + 5n + 2)x}{2}\,I_0(x)\,K_0(x)-$$

$$-\left(\frac{(2n+1)^2\,x^2}{2(n+1)^2} + 2n^2 + 2n + 1\right)\,x^2\,I_0(x)\,K_1(x) - \frac{(2n+1)^2x^4}{4(n+1)^2}\,I_1(x)\,K_0(x) - \frac{x^3}{2}\,I_1(x)\,K_1(x)\right\} -$$

$$-\frac{2n^3(4n^2 + 5n + 2)}{(2n+1)^2}\int x^{2n-1}\,\ln x\,I_0(x)\,K_0(x)\,dx+$$

$$+\frac{2n^2(n+1)}{2n+1}\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx - \frac{2n^2(4n^2 + 5n + 2)}{(2n+1)^2}\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+2}\,\ln x\,I_1(x)\,K_0(x)\,dx = \frac{x^{2n-1}}{2(2n+1)^2}\left\{\Big[((2n+1)nx^3 - 2n^2(4n^2 + 5n + 2)x)\,I_0(x)\,K_0(x)+\right.$$

$$+\left(\frac{(2n+1)^2}{2(n+1)}\,x^2 + n^2\right)\,x^2\,I_0(x)\,K_1(x) + \left(\frac{(2n+1)^2}{2(n+1)}\,x^2 - n(4n^2 + 5n + 2)\right)\,x^2\,I_1(x)\,K_0(x)-$$

$$\left.- (2n+1)(n+1)x^3\,I_1(x)\,K_1(x)\Big]\,\ln x + \left(\frac{x^3}{2} + n(4n^2 + 5n + 2)x\right)\,I_0(x)\,K_0(x)+\right.$$

$$\left(2n^2 + 2n + 1 - \frac{(2n+1)^2\, x^2}{4(n+1)^2}\right) x^2\, I_0(x)\, K_1(x) - \frac{(2n+1)^2 x^4}{(2n+2)^2}\, I_1(x)\, K_0(x) +$$

$$+ \left. \frac{x^3}{2}\, I_1(x)\, K_1(x)\right\} + \frac{2n^3(4n^2 + 5n + 2)}{(2n+1)^2} \int x^{2n-1}\, \ln x\, I_0(x)\, K_0(x)\, dx -$$

$$-\frac{2n^2(n+1)}{2n+1} \int x^{2n}\, \ln x\, I_0(x)\, K_1(x)\, dx + \frac{2n^2(4n^2 + 5n + 2)}{(2n+1)^2} \int x^{2n}\, \ln x\, I_1(x)\, K_0(x)\, dx$$

2.4.3. Some Cases of $\int x^n \ln x \, Z_\mu(x) \, Z_\nu^*(\alpha x) \, dx$:

Some integrals of the type $\int x^n Z_0(\alpha x) \, Z_1^*(\beta x) \, dx$ are already found. Then the integral

$$\int x^n \ln x \, Z_\mu\left(\frac{x}{\alpha}\right) Z_\nu^*(x) \, dx \quad \text{may be expressed by} \quad \int x^n \ln x \, Z_\mu(x) \, Z_\nu^*(\alpha x) \, dx \ ,$$

but nevertheless both integrals are given in the following.

Numerical values of some α :

$(\sqrt{3}+1)/\sqrt{2} = \sqrt{2+\sqrt{3}} = 1.93185\ 16526, \ (\sqrt{3}-1)/\sqrt{2} = \sqrt{2-\sqrt{3}} = 0.51763\ 80902$

$\sqrt{3+\sqrt{6}} = 2.33441\ 42183, \ \sqrt{3-\sqrt{6}} = 0.74196\ 37843$

n = 4

$$\int x^4 \ln x \, J_1(x) \, I_0\left(\frac{x}{\sqrt{2}}\right) dx =$$

$$= \frac{x^2 \ln x}{3}\left[-2x^2 J_0(x) I_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\, x\, J_0(x) I_1\left(\frac{x}{\sqrt{2}}\right) + 4x\, J_1(x) I_0\left(\frac{x}{\sqrt{2}}\right) + \sqrt{2}\,(x^2-8) J_1(x) I_1\left(\frac{x}{\sqrt{2}}\right)\right] -$$

$$- \frac{x}{81}\left[24x\, J_0(x) I_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,(32-9x^2) J_0(x) I_1\left(\frac{x}{\sqrt{2}}\right) - (18x^2+176) J_1(x) I_0\left(\frac{x}{\sqrt{2}}\right) +\right.$$

$$\left. + 168\sqrt{2}\, x\, J_1(x) I_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4 \ln x \, J_1(x) \, K_0\left(\frac{x}{\sqrt{2}}\right) dx =$$

$$= - \frac{x^2 \ln x}{3}\left[2x^2 J_0(x) K_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\, x\, J_0(x) K_1\left(\frac{x}{\sqrt{2}}\right) - 4x\, J_1(x) K_0\left(\frac{x}{\sqrt{2}}\right) +\right.$$

$$\left. + \sqrt{2}\,(x^2-8) J_1(x) K_1\left(\frac{x}{\sqrt{2}}\right)\right] - \frac{x}{81}\left[24x\, J_0(x) K_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,(9x^2-32) J_0(x) K_1\left(\frac{x}{\sqrt{2}}\right) -\right.$$

$$\left. - (18x^2+176) J_1(x) K_0\left(\frac{x}{\sqrt{2}}\right) - 168\sqrt{2}\, x\, J_1(x) K_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4 \ln x \, J_0\left(\frac{x}{\sqrt{2}}\right) I_1(x) \, dx =$$

$$= \frac{x^2 \ln x}{3}\left[2x^2 I_0(x) J_0\left(\frac{x}{\sqrt{2}}\right) - 4\sqrt{2}\, x\, I_1(x) J_0\left(\frac{x}{\sqrt{2}}\right) - 4x\, I_0(x) J_1\left(\frac{x}{\sqrt{2}}\right) +\right.$$

$$\left. + \sqrt{2}\,(x^2+8) I_1(x) J_1\left(\frac{x}{\sqrt{2}}\right)\right] - \frac{x}{81}\left[24x\, I_0(x) J_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,(32+9x^2) I_1(x) J_0\left(\frac{x}{\sqrt{2}}\right) +\right.$$

$$\left. + (18x^2-176) I_0(x) J_1\left(\frac{x}{\sqrt{2}}\right) - 168\sqrt{2}\, x\, I_1(x) J_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4 \ln x \, K_1(x) \, J_0\left(\frac{x}{\sqrt{2}}\right) dx = \frac{x^2 \ln x}{3}\cdot$$

$$\cdot \left[-2x^2 K_0(x) J_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\, x\, K_0(x) J_1\left(\frac{x}{\sqrt{2}}\right) - 4x\, K_1(x) J_0\left(\frac{x}{\sqrt{2}}\right) + \sqrt{2}\,(x^2+8) K_1(x) J_1\left(\frac{x}{\sqrt{2}}\right)\right] +$$

$$+ \frac{x}{81}\left[24x\, K_0(x) J_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\,(9x^2+32) K_0(x) J_1\left(\frac{x}{\sqrt{2}}\right) + (176-18x^2) K_1(x) J_0\left(\frac{x}{\sqrt{2}}\right) +\right.$$

$$\left. + 168\sqrt{2}\, x\, K_1(x) J_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4 \ln x \, J_0(x) \, I_1\left(\sqrt{2}\, x\right) dx =$$

$$= \frac{x^2 \ln x}{3}\left[\sqrt{2}\, x^2 J_0(x) I_0\left(\sqrt{2}\, x\right) - 2x\, J_0(x) I_1\left(\sqrt{2}\, x\right) - 2\sqrt{2}\, x\, J_1(x) I_0\left(\sqrt{2}\, x\right) + (x^2+4) J_1(x) I_1\left(\sqrt{2}\, x\right)\right] -$$

$$-\frac{81}{x}\left[6\sqrt{2}\,x\,J_0(x)\,I_0\left(\sqrt{2}\,x\right)+(9x^2-44)\,J_0(x)\,I_1\left(\sqrt{2}\,x\right)+\right.$$

$$\left.+2\sqrt{2}\,(9x^2+16)\,J_1(x)\,I_0\left(\sqrt{2}\,x\right)-84x\,J_1(x)\,I_1\left(\sqrt{2}\,x\right)\right]$$

$$\int x^4\,\ln x\,J_0(x)\,K_1\left(\sqrt{2}\,x\right)\,dx=\frac{x^2\,\ln x}{3}\left[-\sqrt{2}\,x^2\,J_0(x)\,K_0\left(\sqrt{2}\,x\right)-2x\,J_0(x)\,K_1\left(\sqrt{2}\,x\right)+\right.$$

$$\left.+2\sqrt{2}\,x\,J_1(x)\,K_0\left(\sqrt{2}\,x\right)+(x^2+4)\,J_1(x)\,K_1\left(\sqrt{2}\,x\right)\right]+\frac{x}{81}\left[6\sqrt{2}\,x\,J_0(x)\,K_0\left(\sqrt{2}\,x\right)+\right.$$

$$\left.+(44-9x^2)\,J_0(x)\,K_1\left(\sqrt{2}\,x\right)+2\sqrt{2}\,(9x^2+16)\,J_1(x)\,K_0\left(\sqrt{2}\,x\right)+84x\,J_1(x)\,K_1\left(\sqrt{2}\,x\right)\right]$$

$$\int x^4\,\ln x\,I_0(x)\,J_1\left(\sqrt{2}\,x\right)\,dx=\frac{x^2\,\ln x}{3}\left[-2\sqrt{2}\,x\,I_0(x)\,J_0\left(\sqrt{2}\,x\right)+\right.$$

$$+2x\,I_0(x)\,J_0\left(\sqrt{2}\,x\right)+2\sqrt{2}\,x\,I_1(x)\,J_0\left(\sqrt{2}\,x\right)+(x^2-4)\,I_1(x)\,J_1\left(\sqrt{2}\,x\right)\bigg]+\frac{x}{81}\left[-6\sqrt{2}\,x\,I_0(x)\,J_0\left(\sqrt{2}\,x\right)+\right.$$

$$\left.+(9x^2+44)\,I_0(x)\,J_1\left(\sqrt{2}\,x\right)+2\sqrt{2}\,(9x^2-16)\,I_1(x)\,J_0\left(\sqrt{2}\,x\right)-84x\,I_1(x)\,J_1\left(\sqrt{2}\,x\right)\right]$$

$$\int x^4\,\ln x\,K_0(x)\,J_1\left(\sqrt{2}\,x\right)\,dx=\frac{x^2\,\ln x}{3}\left[-\sqrt{2}\,x^2\,K_0(x)\,J_0\left(\sqrt{2}\,x\right)+2x\,K_0(x)\,J_1\left(\sqrt{2}\,x\right)-\right.$$

$$-2\sqrt{2}\,x\,K_1(x)\,J_0\left(\sqrt{2}\,x\right)-(x^2-4)\,K_1(x)\,J_1\left(\sqrt{2}\,x\right)\bigg]+\frac{x}{81}\left[-6\sqrt{2}\,x\,K_0(x)\,J_0\left(\sqrt{2}\,x\right)+\right.$$

$$\left.+(9x^2+44)\,K_0(x)\,J_1\left(\sqrt{2}\,x\right)-2\sqrt{2}\,(9x^2-16)\,K_1(x)\,J_0\left(\sqrt{2}\,x\right)+84x\,K_1(x)\,J_1\left(\sqrt{2}\,x\right)\right]$$

m = 5

$$\int x^5\,\ln x\,J_0(x)\cdot I_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\,dx=\frac{x^2\,\ln x}{18}\left[-12\,x^2\left(\sqrt{3}-1\right)\,J_0(x)\cdot I_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+\right.$$

$$+3\,x\sqrt{2}\left(-8\,\sqrt{3}+16+\sqrt{3}x^2\right)\,J_0(x)\cdot I_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$-3\,x\left(8-8\,\sqrt{3}-3\,x^2+\sqrt{3}x^2\right)\,J_1(x)\cdot I_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$-12\,\sqrt{2}\left(-4\,\sqrt{3}+8+\sqrt{3}x^2-x^2\right)\,J_1(x)\cdot I_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\bigg]+$$

$$+\frac{x}{594}\left[33\left(\sqrt{3}-1\right)x\left(4\,\sqrt{3}-4-3\,x^2\right)\,J_0(x)\cdot I_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\right.$$

$$-2\,\sqrt{2}\left(8\,\sqrt{3}-15\right)\left(4\,\sqrt{3}-64+33\,x^2\right)\,J_0(x)\cdot I_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$-2\left(-3+5\,\sqrt{3}\right)\left(-116+56\,\sqrt{3}-33\,x^2\right)\,J_1(x)\cdot I_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$-33\,\sqrt{2}\left(\sqrt{3}-1\right)x\left(26\,\sqrt{3}-26+3\,x^2\right)\,J_1(x)\cdot I_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\bigg]$$

$$\int x^5\,\ln x\,J_0(x)\cdot K_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\,dx=\frac{x^2\,\ln x}{18}\left[-12\,x^2\left(\sqrt{3}-1\right)\,J_0(x)\cdot K_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+\right.$$

$$-\sqrt{2}\sqrt{3}x\left(16\,\sqrt{3}-24+3\,x^2\right)\,J_0(x)\cdot K_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+\left(3-\sqrt{3}\right)x\left(8\,\sqrt{3}+3\,x^2\right)\,J_1(x)\cdot K_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+$$

$$\left. +12\sqrt{2}\left(\sqrt{3}-1\right)\left(x^2+2\sqrt{3}-2\right)J_1(x)\cdot K_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}x\right)\right]\ln x+$$

$$+\frac{x}{594}\left[\left(33\left(\sqrt{3}-1\right)x\left(4\sqrt{3}-4-3x^2\right)\right)J_0(x)\cdot K_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}x\right)+\right.$$

$$+2\sqrt{2}\left(-15+8\sqrt{3}\right)\left(4\sqrt{3}-64+33x^2\right)J_0(x)\cdot K_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}x\right)-$$

$$-2\left(-3+5\sqrt{3}\right)\left(-116+56\sqrt{3}-33x^2\right)J_1(x)\cdot K_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}x\right)+$$

$$\left. +33\sqrt{2}\left(\sqrt{3}-1\right)x\left(-26+26\sqrt{3}+3x^2\right)J_1(x)\cdot K_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}x\right)\right]$$

$$\int x^5\ \ln x\, J_0(x)\cdot I_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)dx=\frac{x^2\ln x}{18}\left[12x^2\left(1+\sqrt{3}\right)J_0(x)\cdot I_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+\right.$$

$$+3x\sqrt{2}\left(-8\sqrt{3}-16+\sqrt{3}x^2\right)J_0(x)\cdot I_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+$$

$$+3x\left(-8-8\sqrt{3}+3x^2+\sqrt{3}x^2\right)J_1(x)\cdot I_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)-$$

$$\left. -12\sqrt{2}\left(-4\sqrt{3}-8+\sqrt{3}x^2+x^2\right)J_1(x)\cdot I_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)\right]+$$

$$+\frac{x}{594}\left[33\left(1+\sqrt{3}\right)\left(4\sqrt{3}+4+3x^2\right)x\,J_0(x)\cdot I_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+\right.$$

$$+2\sqrt{2}\left(15+8\sqrt{3}\right)\left(4\sqrt{3}+64-33x^2\right)J_0(x)\cdot I_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)-$$

$$-2\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}+33x^2\right)J_1(x)\cdot I_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+$$

$$\left. +33\sqrt{2}\left(1+\sqrt{3}\right)\left(26+26\sqrt{3}-3x^2\right)x\,J_1(x)\cdot I_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)\right]$$

$$\int x^5\ \ln x\, J_0(x)\cdot K_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)dx=\frac{x^2\ln x}{18}\left[12x^2\left(1+\sqrt{3}\right)J_0(x)\cdot K_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)-\right.$$

$$-3x\sqrt{2}\left(-8\sqrt{3}-16+\sqrt{3}x^2\right)J_0(x)\cdot K_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+$$

$$+3x\left(-8-8\sqrt{3}+3x^2+\sqrt{3}x^2\right)J_1(x)\cdot K_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+$$

$$\left. +12\sqrt{2}\left(-4\sqrt{3}-8+\sqrt{3}x^2+x^2\right)J_1(x)\cdot K_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)\right]\ln x+$$

$$+\frac{x}{594}\left[33\left(1+\sqrt{3}\right)\left(4\sqrt{3}+4+3x^2\right)x\,J_0(x)\cdot K_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)-\right.$$

$$-2\sqrt{2}\left(15+8\sqrt{3}\right)\left(4\sqrt{3}+64-33x^2\right)J_0(x)\cdot K_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)-$$

$$-2\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}+33\,x^2\right)J_1(x)\cdot K_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$\left.-33\sqrt{2}\left(1+\sqrt{3}\right)x\left(26+26\sqrt{3}-3\,x^2\right)J_1(x)\cdot K_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\right]$$

$$\int x^5\ \ln x\,I_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)dx=\frac{x^2\,\ln x}{6}\left[4\left(\sqrt{3}-1\right)x^2\,I_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+\right.$$

$$+\sqrt{2}\left(8\sqrt{3}-16+\sqrt{3}x^2\right)x\,I_0(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\left(-8+8\sqrt{3}-3\,x^2+\sqrt{3}x^2\right)x\,I_1(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$\left.-4\sqrt{2}\left(4\sqrt{3}-8+\sqrt{3}x^2-x^2\right)I_1(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\right]+$$

$$+\frac{x}{594}\left[33\left(\sqrt{3}-1\right)x\left(4\sqrt{3}-4+3\,x^2\right)I_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\right.$$

$$-2\sqrt{2}\left(8\sqrt{3}-15\right)\left(4\sqrt{3}-64-33\,x^2\right)I_0(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$-2\left(-3+5\sqrt{3}\right)\left(-116+56\sqrt{3}+33\,x^2\right)I_1(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)+$$

$$\left.+33\sqrt{2}\left(\sqrt{3}-1\right)\left(-26+26\sqrt{3}-3\,x^2\right)x\,I_1(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\right]$$

$$\int x^5\ \ln x\,I_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)dx=\frac{x^2\,\ln x}{18}\left[-12\,x^2\left(1+\sqrt{3}\right)I_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+\right.$$

$$+\sqrt{6}\,x\left(24+16\sqrt{3}+3\,x^2\right)I_0(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+\left(3+\sqrt{3}\right)\left(8\sqrt{3}+3\,x^2\right)x\,I_1(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$\left.-12\sqrt{2}\left(1+\sqrt{3}\right)\left(2+2\sqrt{3}+x^2\right)I_1(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\right]+$$

$$+\frac{x}{594}\left[33\left(1+\sqrt{3}\right)x\left(4\sqrt{3}+4-3\,x^2\right)I_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)+\right.$$

$$+2\sqrt{2}\left(8\sqrt{3}+15\right)\left(4\sqrt{3}+64+33\,x^2\right)I_0(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$-2\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}-33\,x^2\right)I_1(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)-$$

$$\left.-33\sqrt{2}\left(1+\sqrt{3}\right)\left(26+26\sqrt{3}+3\,x^2\right)x\,I_1(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}\,x\right)\right]$$

$$\int x^5\ \ln x\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)dx=\frac{x^2\,\ln x}{18}\left[12\left(\sqrt{3}-1\right)x^2\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\right.$$

$$-\sqrt{6}\left(-24+16\sqrt{3}-3\,x^2\right)x\,K_0(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-\left(-3+\sqrt{3}\right)\left(8\sqrt{3}-3\,x^2\right)x\,K_1(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)-$$

$$\left.-12\sqrt{2}\left(\sqrt{3}-1\right)\left(-2+2\sqrt{3}-x^2\right)K_1(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}\,x\right)\right]+$$

$$+\frac{x}{594}\left[33\left(\sqrt{3}-1\right)\left(4\sqrt{3}-4+3x^2\right)x\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}x\right)-\right.$$

$$-2\sqrt{2}\left(-15+8\sqrt{3}\right)\left(4\sqrt{3}-64-33x^2\right)K_0(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}x\right)+$$

$$+2\left(-3+5\sqrt{3}\right)\left(-116+56\sqrt{3}+33x^2\right)K_1(x)\cdot J_0\left(\frac{\sqrt{3}+1}{\sqrt{2}}x\right)-$$

$$\left.-33\sqrt{2}\left(\sqrt{3}-1\right)\left(-26+26\sqrt{3}-3x^2\right)x\,K_1(x)\cdot J_1\left(\frac{\sqrt{3}+1}{\sqrt{2}}x\right)\right]$$

$$\int x^5\,\ln x\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)dx=\frac{x^2\ln x}{18}\left[-12x^2\left(1+\sqrt{3}\right)K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+\right.$$

$$+\sqrt{6}\left(24+16\sqrt{3}+3x^2\right)x\,K_0(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)-\left(3+\sqrt{3}\right)x\left(8\sqrt{3}+3x^2\right)K_1(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+$$

$$\left.+12\sqrt{2}\left(1+\sqrt{3}\right)\left(2+2\sqrt{3}+x^2\right)K_1(x)\cdot J_1\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)\right]+$$

$$+\frac{x}{594}\left[33\left(1+\sqrt{3}\right)\left(4\sqrt{3}+4-3x^2\right)x\,K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+\right.$$

$$+2\sqrt{2}\left(15+8\sqrt{3}\right)\left(4\sqrt{3}+64+33x^2\right)K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+$$

$$+2\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}-33x^2\right)K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)+$$

$$\left.+33\sqrt{2}\left(1+\sqrt{3}\right)x\left(26\sqrt{3}+26+3x^2\right)K_0(x)\cdot J_0\left(\frac{\sqrt{3}-1}{\sqrt{2}}x\right)\right]$$

n = 6

$$\int x^6\,\ln x\,J_0(x)\cdot I_1\left(\sqrt{3+\sqrt{6}}\,x\right)dx=$$

$$=\frac{x^2\ln x}{70+30\sqrt{6}}\left[5\sqrt{3+\sqrt{6}}\left(2+\sqrt{6}\right)\left(-8+x^2+4\sqrt{6}\right)x^2\,J_0(x)\cdot I_0\left(\sqrt{3+\sqrt{6}}\,x\right)+\right.$$

$$+10\left(5+2\sqrt{6}\right)x\left(-40-x^2+16\sqrt{6}\right)J_0(x)\cdot I_1\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-2\sqrt{3+\sqrt{6}}\left(1+\sqrt{6}\right)x\left(-8+8\sqrt{6}+5x^2\right)J_1(x)\cdot I_0\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$\left.+5\left(2+\sqrt{6}\right)\left(-32+8x^2+x^4+16\sqrt{6}\right)J_1(x)\cdot I_1\left(\sqrt{3+\sqrt{6}}\,x\right)\right]+$$

$$+\frac{x}{625\left(4+\sqrt{6}\right)^3\left(2+\sqrt{6}\right)\left(11+4\sqrt{6}\right)}\cdot$$

$$\cdot\left[-\frac{100}{73}\sqrt{3+\sqrt{6}}\left(331+134\sqrt{6}\right)x\left(-1825x^2+436\sqrt{6}+176\right)J_0(x)\cdot I_0\left(\sqrt{3+\sqrt{6}}\,x\right)+\right.$$

$$+2\left(664+271\sqrt{6}\right)\left(-18552\sqrt{6}+47168+11850x^2\sqrt{6}-30400x^2-625x^4\right)J_0(x)\cdot I_1\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-4\sqrt{3+\sqrt{6}}\left(149+61\sqrt{6}\right)\left(5472\sqrt{6}-11448-2850x^2+3650x^2\sqrt{6}+625x^4\right)J_1(x)\cdot I_0\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$+\frac{100}{67}\left(1672+683\sqrt{6}\right)x\left(1675\,x^2-21804+10706\sqrt{6}\right)J_1(x)\cdot I_1\left(\sqrt{3+\sqrt{6}}\,x\right)\Bigg]$$

$$\int x^6\,\ln x\,J_0(x)\cdot I_1\left(\sqrt{3-\sqrt{6}}\,x\right)dx=$$

$$=\frac{x^2\ln x}{70-30\sqrt{6}}\left[5\sqrt{3-\sqrt{6}}\left(2-\sqrt{6}\right)\left(-8+x^2+4\sqrt{6}\right)x^2\,J_0(x)\cdot I_0\left(\sqrt{3-\sqrt{6}}\,x\right)-\right.$$

$$-10\left(5-2\sqrt{6}\right)x\left(40+x^2+16\sqrt{6}\right)J_0(x)\cdot I_1\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+2\sqrt{3-\sqrt{6}}\left(1-\sqrt{6}\right)x\left(8+8\sqrt{6}-5\,x^2\right)J_1(x)\cdot I_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$\left.+5\left(2-\sqrt{6}\right)\left(-32+8\,x^2+x^4-16\sqrt{6}\right)J_1(x)\cdot I_1\left(\sqrt{3-\sqrt{6}}\,x\right)\right]+$$

$$+\frac{x}{625\left(4-\sqrt{6}\right)^4\left(-2+\sqrt{6}\right)}\cdot$$

$$\cdot\left[\frac{100}{73}\sqrt{3-\sqrt{6}}\left(-104+41\sqrt{6}\right)x\left(1825\,x^2-176+436\sqrt{6}\right)J_0(x)\cdot I_0\left(\sqrt{3-\sqrt{6}}\,x\right)-\right.$$

$$-4\left(-103+42\sqrt{6}\right)\left(-18552\sqrt{6}-47168+11850\,x^2\sqrt{6}+30400\,x^2+625\,x^4\right)J_0(x)\cdot I_1\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+4\sqrt{3-\sqrt{6}}\left(-46+19\sqrt{6}\right)\left(11448+5472\sqrt{6}+3650\,x^2\sqrt{6}+2850\,x^2-625\,x^4\right)J_1(x)\cdot I_0\left(\sqrt{3-\sqrt{6}}\,x\right)-$$

$$\left.-\frac{200}{67}\left(-259+106\sqrt{6}\right)x\left(-1675\,x^2+10706\sqrt{6}+21804\right)J_1(x)\cdot I_1\left(\sqrt{3-\sqrt{6}}\,x\right)\right.$$

$$\int x^6\,\ln x\,J_0(x)\cdot K_1\left(\sqrt{3+\sqrt{6}}\,x\right)dx=$$

$$=\frac{x^2\ln x}{70+30\sqrt{6}}\left[-5\sqrt{3+\sqrt{6}}\left(2+\sqrt{6}\right)\left(-8+x^2+4\sqrt{6}\right)x^2\,J_0(x)\cdot K_0\left(\sqrt{3+\sqrt{6}}\,x\right)+\right.$$

$$+10\left(5+2\sqrt{6}\right)x\left(-40-x^2+16\sqrt{6}\right)J_0(x)\cdot K_1\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$+2\sqrt{3+\sqrt{6}}\left(\sqrt{6}+1\right)x\left(8\sqrt{6}-8+5\,x^2\right)J_1(x)\cdot K_0\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$\left.+5\left(2+\sqrt{6}\right)\left(-32+8\,x^2+x^4+16\sqrt{6}\right)J_1(x)\cdot K_1\left(\sqrt{3+\sqrt{6}}\,x\right)\right]+$$

$$+\frac{x}{152843750}\cdot\left[670\sqrt{3+\sqrt{6}}\left(-337+143\sqrt{6}\right)x\left(176+436\sqrt{6}-1825\,x^2\right)J_0(x)\cdot K_0\left(\sqrt{3+\sqrt{6}}\,x\right)+\right.$$

$$+4891\left(3\sqrt{6}-2\right)\left(47168-18552\sqrt{6}+11850\,x^2\sqrt{6}-30400\,x^2-625\,x^4\right)J_0(x)\cdot K_1\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-9782\sqrt{3+\sqrt{6}}\left(4\sqrt{6}-11\right)\left(-11448+5472\sqrt{6}-2850\,x^2+3650\,x^2\sqrt{6}+625\,x^4\right)J_1(x)\cdot K_0\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$\left.-730\left(37\sqrt{6}-158\right)x\left(-21804+10706\sqrt{6}+1675\,x^2\right)J_1(x)\cdot K_1\left(\sqrt{3+\sqrt{6}}\,x\right)\right]$$

$$\int x^6\,\ln x\,J_0(x)\cdot K_1\left(\sqrt{3-\sqrt{6}}\,x\right)dx=$$

$$=\frac{x^2\ln x}{70-30\sqrt{6}}\left[-5\sqrt{3-\sqrt{6}}\left(-2+\sqrt{6}\right)\left(8-x^2+4\sqrt{6}\right)x^2\,J_0(x)\cdot K_0\left(\sqrt{3-\sqrt{6}}\,x\right)+\right.$$

$$+10\left(-5+2\sqrt{6}\right)\left(40+x^2+16\sqrt{6}\right)x\,J_0(x)\cdot K_1\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+2\sqrt{3-\sqrt{6}}\left(-1+\sqrt{6}\right)\left(8-5x^2+8\sqrt{6}\right)x\,J_1(x)\cdot K_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+5\left(-2+\sqrt{6}\right)\left(32-8x^2-x^4+16\sqrt{6}\right)J_1(x)\cdot K_1\left(\sqrt{3-\sqrt{6}}\,x\right)\Bigg]+\frac{x}{61180296250-24977725625\sqrt{6}}\cdot$$

$$\cdot\Bigg[1675\sqrt{3-\sqrt{6}}\left(867\sqrt{6}-2128\right)\left(436\sqrt{6}-176+1825x^2\right)x\,J_0(x)\cdot K_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+4891\left(874\sqrt{6}-2141\right)\left(-47168-18552\sqrt{6}+30400x^2+11850x^2\sqrt{6}+625x^4\right)J_0(x)\cdot K_1\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+4891\sqrt{3-\sqrt{6}}\left(393\sqrt{6}-962\right)\left(11448+5472\sqrt{6}+3650x^2\sqrt{6}+2850x^2-625x^4\right)J_1(x)\cdot K_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+3650\left(2202\sqrt{6}-5393\right)\left(10706\sqrt{6}+21804-1675x^2\right)x\,J_1(x)\cdot K_1\left(\sqrt{3-\sqrt{6}}\,x\right)\Bigg]$$

$$\int x^6\,\ln x\,K_0(x)\cdot J_1\left(\sqrt{3+\sqrt{6}}\,x\right)\,dx=$$

$$=\frac{x^2\,\ln x}{70+30\sqrt{6}}\Bigg[5\sqrt{3+\sqrt{6}}\left(2+\sqrt{6}\right)\left(-8-x^2+4\sqrt{6}\right)x^2\,K_0(x)\cdot J_0\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-10\left(5+2\sqrt{6}\right)\left(x-4+2\sqrt{6}\right)\left(-x-4+2\sqrt{6}\right)x\,K_0(x)\cdot J_1\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-2\sqrt{3+\sqrt{6}}\left(1+\sqrt{6}\right)\left(8\sqrt{6}-8-5x^2\right)x\,K_1(x)\cdot J_0\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-5\left(2+\sqrt{6}\right)\left(-32-8x^2+x^4+16\sqrt{6}\right)K_1(x)\cdot J_1\left(\sqrt{3+\sqrt{6}}\,x\right)\Bigg]+$$

$$+\frac{x}{152843750}\Bigg[670\sqrt{3+\sqrt{6}}\left(-337+143\sqrt{6}\right)\left(176+436\sqrt{6}+1825x^2\right)x\,K_0(x)\cdot J_0\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$+4891\left(3\sqrt{6}-2\right)\left(-47168+18552\sqrt{6}-30400x^2+11850x^2\sqrt{6}+625x^4\right)K_0(x)\cdot J_1\left(\sqrt{3+\sqrt{6}}\,x\right)-$$

$$-9782\sqrt{3+\sqrt{6}}\left(-11+4\sqrt{6}\right)\left(-5472\sqrt{6}+11448-2850x^2+3650x^2\sqrt{6}-625x^4\right)K_1(x)\cdot J_0\left(\sqrt{3+\sqrt{6}}\,x\right)+$$

$$+730\left(37\sqrt{6}-158\right)\left(-21804+10706\sqrt{6}-1675x^2\right)x\,K_1(x)\cdot J_1\left(\sqrt{3+\sqrt{6}}\,x\right)\Bigg]$$

$$\int x^6\,\ln x\,K_0(x)\cdot J_1\left(\sqrt{3-\sqrt{6}}\,x\right)\,dx=$$

$$=\frac{x^2\,\ln x}{70-30\sqrt{6}}\Bigg[5\sqrt{3-\sqrt{6}}\left(-2+\sqrt{6}\right)\left(8+x^2+4\sqrt{6}\right)x^2\,K_0(x)\cdot J_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+10\left(-5+2\sqrt{6}\right)\left(-x+4+2\sqrt{6}\right)x\left(x+4+2\sqrt{6}\right)K_0(x)\cdot J_1\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+2\sqrt{3-\sqrt{6}}\left(-1+\sqrt{6}\right)x\left(8+5x^2+8\sqrt{6}\right)K_1(x)\cdot J_0\left(\sqrt{3-\sqrt{6}}\,x\right)-$$

$$-5\left(-2+\sqrt{6}\right)\left(32+8x^2-x^4+16\sqrt{6}\right)K_1(x)\cdot J_1\left(\sqrt{3-\sqrt{6}}\,x\right)\Bigg]+$$

$$+\frac{x}{152843750}\Bigg[670\sqrt{3-\sqrt{6}}\left(337+143\sqrt{6}\right)\left(436\sqrt{6}-176-1825x^2\right)x\,K_0(x)\cdot J_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+4891 \left(3\sqrt{6}+2\right)\left(47168+18552\sqrt{6}+30400\,x^2+11850\,x^2\sqrt{6}-625\,x^4\right)K_0(x)\cdot J_1\left(\sqrt{3-\sqrt{6}}\,x\right)-$$

$$-9782\sqrt{3-\sqrt{6}}\left(11+4\sqrt{6}\right)\left(-11448-5472\sqrt{6}+3650x^2\sqrt{6}+2850x^2+625\,x^4\right)K_1(x)\cdot J_0\left(\sqrt{3-\sqrt{6}}\,x\right)+$$

$$+730\left(158+37\sqrt{6}\right)x\left(10706\sqrt{6}+21804+1675\,x^2\right)K_1(x)\cdot J_1\left(\sqrt{3-\sqrt{6}}\,x\right)\Big]$$

2.4.4. Integrals of the type $\int x^{-1} \cdot \exp/\sin/\cos(2x)\, Z_\nu(x)\, Z_1(x)\, dx$

$$\int \frac{e^{2x}\, I_0(x)\, I_1(x)\, dx}{x} = e^{2x}\left[(1-x)\, I_0^2(x) + (2x-1)I_0(x)I_1(x) - x\, I_1^2(x)\right]$$

$$\int \frac{e^{2x}\, K_0(x)\, K_1(x)\, dx}{x} = e^{2x}\left[(x-1)\, K_0^2(x) + (2x-1)K_0(x)K_1(x) + x\, K_1^2(x)\right]$$

$$\int \frac{e^{-2x}\, I_0(x)\, I_1(x)\, dx}{x} = -e^{-2x}\left[(1+x)\, I_0^2(x) + (2x+1)I_0(x)I_1(x) + x\, I_1^2(x)\right]$$

$$\int \frac{e^{-2x}\, K_0(x)\, K_1(x)\, dx}{x} = e^{-2x}\left[(1+x)\, K_0^2(x) - (2x+1)K_0(x)K_1(x) + x\, K_1^2(x)\right]$$

$$\int \frac{e^{2x}\, I_1^2(x)\, dx}{x} = \frac{e^{2x}}{2}\left[(1-2x)\, I_0^2(x) + 4xI_0(x)I_1(x) - (2x+1)\, I_1^2(x)\right]$$

$$\int \frac{e^{2x}\, K_1^2(x)\, dx}{x} = \frac{e^{2x}}{2}\left[(1-2x)\, K_0^2(x) - 4xK_0(x)K_1(x) - (2x+1)\, K_1^2(x)\right]$$

$$\int \frac{e^{-2x}\, I_1^2(x)\, dx}{x} = \frac{e^{-2x}}{2}\left[(1+2x)\, I_0^2(x) + 4xI_0(x)I_1(x) + (2x-1)\, I_1^2(x)\right]$$

$$\int \frac{e^{-2x}\, K_1^2(x)\, dx}{x} = \frac{e^{-2x}}{2}\left[(1+2x)\, K_0^2(x) - 4xK_0(x)K_1(x) + (2x-1)\, K_1^2(x)\right]$$

$$\int \frac{\sin 2x\, J_0(x)\, J_1(x)\, dx}{x} = \sin 2x\left[-x\, J_0^2(x) - J_0(x)J_1(x) + x\, J_1^2(x)\right] + \cos 2x\left[2x\, J_0(x)J_1(x) - J_0^2(x)\right]$$

$$\int \frac{\cos 2x\, J_0(x)\, J_1(x)\, dx}{x} = \sin 2x\left[J_0^2(x) - 2x\, J_0(x)J_1(x)\right] - \cos 2x\left[x\, J_0^2(x) + J_0(x)J_1(x) - x\, J_1^2(x)\right]$$

$$\int \frac{\sin 2x\, J_1^2(x)\, dx}{x} = \frac{\sin 2x}{2}\left[-J_0^2(x) + 4x\, J_0(x)J_1(x) - J_1^2(x)\right] + x\, \cos 2x\left[J_0^2(x) - J_1^2(x)\right]$$

$$\int \frac{\cos 2x\, J_1^2(x)\, dx}{x} = \frac{\cos 2x}{2}\left[-J_0^2(x) + 4x\, J_0(x)J_1(x) - J_1^2(x)\right] - x\, \sin 2x\left[J_0^2(x) - J_1^2(x)\right]$$

From this:

$$\int \frac{\sin^2 x\, J_0(x)\, J_1(x)\, dx}{x} =$$

$$= \frac{1}{2}\left[xJ_0^2(x) - J_0(x)J_1(x) + xJ_1^2(x)\right] + \frac{\sin 2x}{2}\left[J_0^2(x) - 2x\, J_0(x)J_1(x)\right] +$$

$$+ \frac{\cos 2x}{2}\left[x\, J_0^2(x) + J_0(x)J_1(x) - x\, J_1^2(x)\right]$$

$$\int \frac{\cos^2 x\, J_0(x)\, J_1(x)\, dx}{x} =$$

$$= \frac{1}{2}\left[xJ_0^2(x) - J_0(x)J_1(x) + xJ_1^2(x)\right] + \frac{\sin 2x}{2}\left[J_0^2(x) - 2x\, J_0(x)J_1(x)\right] -$$

$$- \frac{\cos 2x}{2}\left[x\, J_0^2(x) + J_0(x)J_1(x) - x\, J_0^2(x)\right]$$

$$\int \frac{\sin^2 x\, J_1^2(x)\, dx}{x} =$$

$$= -\frac{J_0^2(x) + J_1^2(x)}{4} + \frac{\cos 2x}{4}\left[J_0^2(x) - 4x\, J_0(x)J_1(x) + J_1^2(x)\right] + \frac{x\, \sin 2x}{2}\left[J_0^2(x) - J_1^2(x)\right]$$

$$\int \frac{\cos^2 x\, J_1^2(x)\, dx}{x} =$$

$$= -\frac{J_0^2(x) + J_1^2(x)}{4} - \frac{\cos 2x}{4}\left[J_0^2(x) - 4x\, J_0(x)J_1(x) + J_1^2(x)\right] - \frac{x\, \sin 2x}{2}\left[J_0^2(x) - J_1^2(x)\right]$$

2.3.5. Some Cases of $\int x^n \cdot \exp\left(\alpha x\right) \cdot Z_\mu(x)\, Z_\nu(\beta x)\, dx$

<u>**n = 1 :**</u>

$$\int x\, e^{4x}\, I_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[-4x\, I_0(x)\, I_0(3x) + (4x+3)\, I_0(x)\, I_1(3x) + (4x-1)\, I_0(x)\, I_1(3x) - 4x\, I_1(x)\, I_1(3x)\right]$$

$$\int x\, e^{-4x}\, I_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[4x\, I_0(x)\, I_0(3x) + (4x-3)\, I_0(x)\, I_1(3x) + (4x+1)\, I_0(x)\, I_1(3x) + 4x\, I_1(x)\, I_1(3x)\right]$$

$$\int x\, e^{4x}\, K_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[4x\, K_0(x)\, K_0(3x) + (4x+3)\, K_0(x)\, K_1(3x) + (4x-1)\, K_0(x)\, K_1(3x) + 4x\, K_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{-4x}\, K_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[-4x\, K_0(x)\, K_0(3x) + (4x-3)\, K_0(x)\, K_1(3x) + (4x+1)\, K_0(x)\, K_1(3x) - 4x\, K_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{4x}\, I_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[4x\, I_0(x)\, K_0(3x) + (4x+3)\, I_0(x)\, K_1(3x) - (4x-1)\, I_0(x)\, K_1(3x) - 4x\, I_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{-4x}\, I_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[-4x\, I_0(x)\, K_0(3x) + (4x-3)\, I_0(x)\, K_1(3x) - (4x+1)\, I_0(x)\, K_1(3x) + 4x\, I_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{4x}\, K_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[-4x\, K_0(x)\, I_0(3x) + (4x+3)\, K_0(x)\, I_1(3x) - (4x-1)\, K_0(x)\, I_1(3x) + 4x\, K_1(x)\, I_1(3x)\right]$$

$$\int x\, e^{-4x}\, K_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[4x\, K_0(x)\, I_0(3x) + (4x-3)\, K_0(x)\, I_1(3x) - (4x+1)\, K_0(x)\, I_1(3x) - 4x\, K_1(x)\, I_1(3x)\right]$$

<u>**n = 2 :**</u>

$$\int x^2 \exp\left(\frac{8x}{3}\right) \cdot I_0(x)\, I_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512}\exp\left(\frac{5x}{3}\right)\left[(-64\,x^2 + 24\,x)\, I_0(x)\, I_0\left(\frac{5x}{3}\right) + \right.$$

$$\left. +(64\,x^2 + 120\,x - 45)\, I_0(x)\, I_1\left(\frac{5x}{3}\right) + (64\,x^2 - 72\,x + 27)\, I_1(x)\, I_0\left(\frac{5x}{3}\right) + (-64\,x^2 + 24\,x)\, I_1(x)\, I_1\left(\frac{5x}{3}\right)\right]$$

$$\int x^2 \exp\left(-\frac{8x}{3}\right) \cdot I_0(x)\, I_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512}\exp\left(-\frac{8x}{3}\right)\left[(64\,x^2 + 24\,x)\, I_0(x)\, I_0\left(\frac{5x}{3}\right) + \right.$$

$$\left. +(64\,x^2 - 120\,x - 45)\, I_0(x)\, I_1\left(\frac{5x}{3}\right) + (64\,x^2 + 72\,x + 27)\, I_1(x)\, I_0\left(\frac{5x}{3}\right) + (64\,x^2 + 24\,x)\, I_1(x)\, I_1\left(\frac{5x}{3}\right)\right]$$

$$\int x^2 \exp\left(\frac{8x}{3}\right) \cdot I_0(x)\, K_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512}\exp\left(\frac{8x}{3}\right)\left[(64\,x^2 - 24\,x)\, I_0(x)\, K_0\left(\frac{5x}{3}\right) + \right.$$

$$+(64\,x^2 + 120\,x - 45)\,I_0(x)\,K_1\left(\frac{5x}{3}\right) - (64\,x^2 - 72\,x + 27)\,I_1(x)\,K_0\left(\frac{5x}{3}\right) + (-64\,x^2 + 24\,x)\,I_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2 \exp\left(-\frac{8x}{3}\right)\cdot I_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx = \frac{x}{512}\exp\left(-\frac{8x}{3}\right)\Bigg[-(64\,x^2 + 24\,x)\,I_0(x)\,K_0\left(\frac{5x}{3}\right) +$$

$$+(64\,x^2 - 120\,x - 45)\,I_0(x)\,K_1\left(\frac{5x}{3}\right) - (64\,x^2 + 72\,x + 27)\,I_1(x)\,K_0\left(\frac{5x}{3}\right) + (64\,x^2 + 24\,x)\,I_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2 \exp\left(\frac{8x}{3}\right)\cdot K_0(x)\,I_1\left(\frac{5x}{3}\right)\,dx = \frac{x}{512}\exp\left(\frac{8x}{3}\right)\Bigg[(-64\,x^2 + 24\,x)\,K_0(x)\,I_0\left(\frac{5x}{3}\right) +$$

$$+(64\,x^2 + 120\,x - 45)\,K_0(x)\,I_1\left(\frac{5x}{3}\right) - (64\,x^2 - 72\,x + 27)\,K_1(x)\,I_0\left(\frac{5x}{3}\right) + (64\,x^2 - 24\,x)\,K_1(x)\,I_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2 \exp\left(-\frac{8x}{3}\right)\cdot K_0(x)\,I_1\left(\frac{5x}{3}\right)\,dx = \frac{x}{512}\exp\left(-\frac{8x}{3}\right)\Bigg[(64\,x^2 + 24\,x)\,K_0(x)\,I_0\left(\frac{5x}{3}\right) +$$

$$+(64\,x^2 - 120\,x - 45)\,K_0(x)\,I_1\left(\frac{5x}{3}\right) - (64\,x^2 + 72\,x + 27)\,K_1(x)\,I_0\left(\frac{5x}{3}\right) - (64\,x^2 + 24\,x)\,K_1(x)\,I_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2 \exp\left(\frac{8x}{3}\right)\cdot K_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx = \frac{x}{512}\exp\left(\frac{8x}{3}\right)\Bigg[(64\,x^2 - 24\,x)\,K_0(x)\,K_0\left(\frac{5x}{3}\right) +$$

$$+(64\,x^2 + 120\,x - 45)\,K_0(x)\,K_1\left(\frac{5x}{3}\right) + (64\,x^2 - 72\,x + 27)\,K_1(x)\,K_0\left(\frac{5x}{3}\right) +$$

$$+(64\,x^2 - 24\,x)\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2 \exp\left(-\frac{8x}{3}\right)\cdot K_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx = \frac{x}{512}\exp\left(-\frac{8x}{3}\right)\Bigg[-(64\,x^2 + 24\,x)\,K_0(x)\,K_0\left(\frac{5x}{3}\right) +$$

$$+(64\,x^2 - 120\,x - 45)\,K_0(x)\,K_1\left(\frac{5x}{3}\right) + (64\,x^2 + 72\,x + 27)\,K_1(x)\,K_0\left(\frac{5x}{3}\right) -$$

$$-(64\,x^2 + 24\,x)\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

$\underline{n=3:}$ $\qquad 8/\sqrt{51} = 1.12022\,40672,\ \sqrt{35/51} = 0.82841\,68696$

$$\int x^3 \exp\left(\frac{8x}{\sqrt{51}}\right)\cdot J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx =$$

$$= \frac{x}{20480}\exp\left(\frac{8x}{\sqrt{51}}\right)\Bigg[\sqrt{35}\,(-64\,\sqrt{51}x^2 + 408\,x)\,J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right) +$$

$$+(1600\,\sqrt{51}x^2 - 14280\,x + 1785\,\sqrt{51})\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right) +$$

$$+\sqrt{35}\,(-1088\,x^2 + 408\,\sqrt{51}x - 2601\,)\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right) + (5440\,x^2 - 680\,\sqrt{51}x)\,J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\Bigg]$$

$\underline{n=4:}$ $\qquad 2\sqrt{3}/5 = 0.69282\,03230,\ \sqrt{13}/5 = 0.72111\,02551$

$$\int x^4 \exp\left(\frac{2\sqrt{3}\,x}{5}\right)\cdot J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)\,dx =$$

$$= \frac{x}{96}\exp\left(\frac{2\sqrt{3}\,x}{5}\right)\Bigg[(40\,\sqrt{3}x^3 - 180\,x^2 + 150\,\sqrt{3}x)\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right) +$$

$$+\sqrt{13}\,(20\,\sqrt{3}x^2 - 150\,x + 125\,\sqrt{3})\,J_0(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right) +$$

$$+(48\,x^3 - 140\,\sqrt{3}x^2 + 750\,x - 625\,\sqrt{3})\,J_1(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right) +$$

$$+\sqrt{13}\,(8\,\sqrt{3}x^3 - 60\,x^2 + 50\,\sqrt{3}x)\,J_1(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)\Bigg]$$

$4/\sqrt{5} = 1.78885\ 43820,\ \sqrt{7/15} = 0.68313\ 00511$

$$\int x^4 \exp\left(\frac{4x}{\sqrt{5}}\right) \cdot J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\,dx =$$

$$= \frac{x}{7168}\exp\left(\frac{4x}{\sqrt{5}}\right)\left[\sqrt{21}\,(-64\,\sqrt{5}x^3 + 240\,x^2 - 60\,\sqrt{5}x)\,J_0(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right) +\right.$$

$$+(1344\,\sqrt{5}x^3 - 3360\,x^2 + 1260\,\sqrt{5}x - 1575)\,J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right) +$$

$$+\sqrt{21}\,(-192\,x^3 + 288\,\sqrt{5}x^2 - 900\,x + 225\,\sqrt{5})\,J_1(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right) +$$

$$+(1344\,x^3 - 1008\,\sqrt{5}x^2 + 1260\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]$$

2.4.6. Some Cases of $\int x^n \cdot \left\{ \begin{array}{c} \sin\,/\,\cos \\ \sinh\,/\,\cosh \end{array} \right\} \alpha x \cdot Z_\mu(x)\, Z_\nu(\beta x)\, dx$

Some integrals, where α and β are roots of cubic equations, are left out.

With the integral $\int w(\alpha x)\, Z_\nu(x)\, Z_\nu(\beta x)\, dx$ the integral

$$\int w\left(\frac{\alpha}{\beta}x\right) Z_\nu(x)\, Z_\nu\left(\frac{x}{\beta}\right) dx$$

may be found. So in the following tables only one of both integrals is given.

Numerical values of the coefficients:

α		β		α/β	$1/\beta$
$8/\sqrt{51}$	1.12022 40672	$\sqrt{35/51}$	0.82841 68696	1.35224 68076	1.20712 17242
$2\sqrt{3/13}$	0.96076 89228	$5/\sqrt{13}$	1.38675 04906	0.69282 03230	0.72111 02551
$4/\sqrt{5}$	1.78885 43820	$\sqrt{7/15}$	0.68313 00511	2.61861 46828	1.46385 01094
$2/\sqrt{7}$	0.75592 89460	$\sqrt{3/7}$	0.65465 36707	1.15470 05384	1.52752 52317
$2\sqrt{3/5}$	0.69282 03230	$\sqrt{13/5}$	0.72111 02551	.96076 89228	1.38675 04906
$4/\sqrt{11}$	1.20604 53783	$\sqrt{3/11}$	0.52223 29679	2.30940 10768	1.91485 42155

2.4.6 a) $\int x^n \cdot \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} \alpha x \cdot Z_\mu(x)\, Z_\nu(\beta x)\, dx$:

n = 1 :

$$\int x \sin 4x \cdot J_0(x)\, J_1(3x)\, dx = \frac{x^2 \sin 4x}{4}\left[J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x)\right] +$$

$$+\frac{x \cos 4x}{16}\left[4x\, J_0(x)\, J_0(3x) - 3\, J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x) - 4x\, J_1(x)\, J_1(3x)\right]$$

$$\int x \cos 4x \cdot J_0(x)\, J_1(3x)\, dx = \frac{x^2 \cos 4x}{4}\left[J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x)\right] +$$

$$-\frac{x \sin 4x}{16}\left[4x\, J_0(x)\, J_0(3x) - 3\, J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x) - 4x\, J_1(x)\, J_1(3x)\right]$$

n = 2 :

$$\int x^2 \sin\frac{8x}{3}\cdot J_0(x)\, J_1\left(\frac{5x}{3}\right) dx = \frac{x}{512}\cdot \sin\frac{8x}{3}\cdot$$

$$\cdot\left[-24x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) + (64x^2+45)\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + (64x^2-27)\, J_1(x)\, J_0\left(\frac{5x}{3}\right) + 24x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right] +$$

$$+\frac{x^2}{64}\cdot\cos\frac{8x}{3}\left[8x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) - 15\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + 9\, J_1(x)\, J_0\left(\frac{5x}{3}\right) - 8x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right]$$

$$\int x^2 \cos\frac{8x}{3}\cdot J_0(x)\, J_1\left(\frac{5x}{3}\right) dx = \frac{x}{512}\cdot \cos\frac{8x}{3}\cdot$$

$$\cdot\left[-24x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) + (64x^2+45)\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + (64x^2-27)\, J_1(x)\, J_0\left(\frac{5x}{3}\right) + 24x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right] -$$

$$-\frac{x^2}{64}\cdot\sin\frac{8x}{3}\left[8x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) - 15\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + 9\, J_1(x)\, J_0\left(\frac{5x}{3}\right) - 8x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right]$$

n = 3 :

$$\int x^3 \sin\left(\frac{8x}{\sqrt{51}}\right)\cdot I_0(x)\, I_1\left(\sqrt{\frac{35}{51}}\,x\right) dx = \frac{x^2}{2560}\left[8\sqrt{1785}\,x\, I_0(x)\, I_0\left(\sqrt{\frac{35}{51}}\,x\right) + 1785\, I_0(x)\, I_1\left(\sqrt{\frac{35}{51}}\,x\right) - \right.$$

$$\left. -51\sqrt{1785}\, I_1(x)\, I_0\left(\sqrt{\frac{35}{51}}\,x\right) + 680x\, I_1(x)\, I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right) +$$

$$+\frac{x}{20480}\left[408\sqrt{35}\,x\,I_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\sqrt{51}\,(1785-1600\,x^2)\,I_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\sqrt{51}\,x\,I_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\cos\left(\frac{8x}{\sqrt{51}}\right)\cdot I_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x}{20480}\left[-408\sqrt{35}x\,I_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+\sqrt{51}\,(1600\,x^2-1785)\,I_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)+\sqrt{35}\,(2601-1088\,x^2)\,I_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)-$$

$$\left.-680\sqrt{51}\,x\,I_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)+\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,I_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+1785\,I_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)-51\sqrt{1785}\,I_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\,x\,I_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\sin\left(\frac{8x}{\sqrt{51}}\right)\cdot K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=-\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-\right.$$

$$-1785\,K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,K_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680x\,K_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)-$$

$$-\frac{x}{20480}\left[408\sqrt{35}\,x\,K_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-\sqrt{51}\,(1785-1600\,x^2)K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.-\sqrt{35}\,(1088\,x^2-2601)K_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\sqrt{51}\,x\,K_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\cos\left(\frac{8x}{\sqrt{51}}\right)\cdot K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x}{20480}\left[408\sqrt{35}x\,K_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+\sqrt{51}\,(1600\,x^2-1785)\,K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+\sqrt{35}\,(2601-1088\,x^2)\,K_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+$$

$$\left.+680\sqrt{51}\,x\,K_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)-\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-\right.$$

$$-1785\,K_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,K_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\,x\,K_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\sin\left(\frac{8x}{\sqrt{51}}\right)\cdot I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x^2}{2560}\left[-8\sqrt{1785}\,x\,I_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+1785\,I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,I_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680x\,I_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)-$$

$$-\frac{x}{20480}\left[408\sqrt{35}\,x\,I_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-\sqrt{51}\,(1785-1600\,x^2)I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-680\sqrt{51}\,x\,I_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\cos\left(\frac{8x}{\sqrt{51}}\right)\cdot I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x^2}{2560}\left[-8\sqrt{1785}\,x\,I_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+1785\,I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,I_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680x\,I_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)+$$

$$+\frac{x}{20480}\left[408\,\sqrt{35}\,x\,I_0(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-\sqrt{51}\,(1785-1600\,x^2)I_0(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)-680\,\sqrt{51}\,x\,I_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\sin\left(\frac{8x}{\sqrt{51}}\right)\cdot K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+1785\,K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,K_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)-680x\,K_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)+$$

$$+\frac{x}{20480}\left[408\,\sqrt{35}\,x\,K_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\sqrt{51}\,(1785-1600\,x^2)K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)-\right.$$

$$\left.-\sqrt{35}\,(1088\,x^2-2601)\,K_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)-680\,\sqrt{51}\,x\,K_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)$$

$$\int x^3\cos\left(\frac{8x}{\sqrt{51}}\right)\cdot K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$+1785\,K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)+51\sqrt{1785}\,K_1(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)-680x\,K_1(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cos\left(\frac{8x}{\sqrt{51}}\right)+$$

$$+\frac{x}{20480}\left[-408\,\sqrt{35}\,x\,K_0(x)\,I_0\left(\sqrt{\frac{35}{51}}\,x\right)-\sqrt{51}\,(1785-1600\,x^2)\,K_0(x)\,I_1\left(\sqrt{\frac{35}{51}}\,x\right)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,K_0\left(\sqrt{\frac{35}{51}}\,x\right)+680\,\sqrt{51}\,x\,I_1(x)\,K_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sin\left(\frac{8x}{\sqrt{51}}\right)$$

n = 4 :

$$\int x^4\sin\left(2\sqrt{\frac{3}{13}}\,x\right)I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)\,dx=\frac{x^2}{80}\left[78x\,I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)+\sqrt{13}(8\,x^2-65)\,I_0(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)+\right.$$

$$+169\,I_1(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)-26\,\sqrt{13}x\,I_1(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)\right]\sin\left(2\sqrt{\frac{3}{13}}\,x\right)+\frac{x}{480}\left[\sqrt{39}\,x(78-40\,x^2)\,I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)+\right.$$

$$+\sqrt{3}\,(364\,x^2-845)\,I_0(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)+\sqrt{39}(169-52\,x^2)\,I_1(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)+$$

$$+\sqrt{3}\,x(104\,x^2-338)\,I_1(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)\right]\cos\left(2\sqrt{\frac{3}{13}}\,x\right)$$

$$\int x^4\cos\left(2\sqrt{\frac{3}{13}}\,x\right)I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)\,dx=\frac{x^2}{80}\left[78x\,I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)+\sqrt{13}(8\,x^2-65)\,I_0(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)+\right.$$

$$+169\,I_1(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)-26\,\sqrt{13}x\,I_1(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)\right]\cos\left(2\sqrt{\frac{3}{13}}\,x\right)+\frac{x}{480}\left[\sqrt{39}\,x(40\,x^2-78)\,I_0(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)-\right.$$

$$-\sqrt{3}\,(364\,x^2-845)\,I_0(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)-\sqrt{39}(169-52\,x^2)\,I_1(x)\,I_0\left(\frac{5\,x}{\sqrt{13}}\right)-$$

$$\left.-\sqrt{3}\,x(104\,x^2-338)\,I_1(x)\,I_1\left(\frac{5\,x}{\sqrt{13}}\right)\right]\sin\left(2\sqrt{\frac{3}{13}}\,x\right)$$

$$\int x^4\,\sin\left(2\sqrt{\frac{3}{13}}\,x\right)K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)dx=\frac{x^2}{80}\left[78x\,K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)+\right.$$

$$\left.+\sqrt{13}\,(65-8x^2)\,K_0(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)-169\,K_1(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)-26\,\sqrt{13}\,x\,K_1(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)\right]\sin\left(2\sqrt{\frac{3}{13}}\right)+$$

$$+\frac{x}{480}\left[\sqrt{39}\,x(78-40x^2)\,K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)+\sqrt{3}\,(845-364x^2)\,K_0(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)+\right.$$

$$\left.-\sqrt{39}\,(169-52x^2)\,K_1(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)-\sqrt{3}\,x(338-104x^2)\,K_1(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)\right]\cos\left(2\sqrt{\frac{3}{13}}\right)$$

$$\int x^4\,\cos\left(2\sqrt{\frac{3}{13}}\,x\right)K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)dx=\frac{x^2}{80}\left[78x\,K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)+\right.$$

$$\left.+\sqrt{13}\,(65-8x^2)\,K_0(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)-169\,K_1(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)-26\,\sqrt{13}\,x\,K_1(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)\right]\cos\left(2\sqrt{\frac{3}{13}}\right)-$$

$$-\frac{x}{480}\left[\sqrt{39}\,x(78-40x^2)\,K_0(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)-\sqrt{3}\,(845-364x^2)\,K_0(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)-\right.$$

$$\left.-\sqrt{39}\,(169-52x^2)\,K_1(x)\,K_0\left(\frac{5x}{\sqrt{13}}\,x\right)-\sqrt{3}\,x(338-104x^2)\,K_1(x)\,K_1\left(\frac{5x}{\sqrt{13}}\,x\right)\right]\sin\left(2\sqrt{\frac{3}{13}}\right)$$

$$\int x^4\,\sin\frac{4x}{\sqrt{5}}\cdot I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)dx=\frac{x}{7168}\left[\sqrt{105}\,(64\,x^3-60\,x)\,I_0(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+\right.$$

$$+(3360\,x^2-1575)\,I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{105}\,(225-288\,x^2)\,I_1(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$\left.+(1344\,x^3-1260\,x)\,I_1(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\sin\frac{4x}{\sqrt{5}}+\frac{x^2}{1792}\left[60\,\sqrt{21}\,x\,I_0(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+\right.$$

$$+\sqrt{5}\,(315-336\,x^2)\,I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{21}\,(48\,x^2-225)\,I_1(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$\left.+252\,\sqrt{5}\,x\,I_1(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\cos\frac{4x}{\sqrt{5}}$$

$$\int x^4\,\cos\frac{4x}{\sqrt{5}}\cdot I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)dx=\frac{x^2}{1792}\left[-60\,\sqrt{21}\,x\,I_0(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+\right.$$

$$+\sqrt{5}\,(336\,x^2-315)\,I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{21}\,(225-48\,x^2)\,I_1(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)-$$

$$\left.-252\,\sqrt{5}\,x\,I_1(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\sin\frac{4x}{\sqrt{5}}+\frac{x}{7168}\left[\sqrt{105}\,x(64x^2-60)\,I_0(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+\right.$$

$$+(3360\,x^2-1575)\,I_0(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)+\sqrt{105}\,(225-288x^2)\,I_1(x)\,I_0\left(\sqrt{\frac{7}{15}}\,x\right)+$$

$$\left.+(1344\,x^3-1260\,x)\,I_1(x)\,I_1\left(\sqrt{\frac{7}{15}}\,x\right)\right]\cos\frac{4x}{\sqrt{5}}$$

$$\int x^4 \sin \frac{4x}{\sqrt{5}} \cdot K_0(x)\, K_1\left(\sqrt{\frac{7}{15}}\, x\right) dx = \frac{x}{7168}\left[-\sqrt{105}\,(64\,x^3 - 60\,x)\, K_0(x)\, K_0\left(\sqrt{\frac{7}{15}}\, x\right) + \right.$$

$$+ (3360\,x^2 - 1575)\, K_0(x)\, K_1\left(\sqrt{\frac{7}{15}}\, x\right) + \sqrt{105}\,(225 - 288\,x^2)\, K_1(x)\, K_0\left(\sqrt{\frac{7}{15}}\, x\right) +$$

$$\left. -(1344\,x^3 - 1260\,x)\, K_1(x)\, K_1\left(\sqrt{\frac{7}{15}}\, x\right) \right] \sin \frac{4x}{\sqrt{5}} - \frac{x^2}{1792}\left[60\,\sqrt{21}\, x\, K_0(x)\, K_0\left(\sqrt{\frac{7}{15}}\, x\right) + \right.$$

$$-\sqrt{5}\,(315 - 336\,x^2)\, K_0(x)\, K_1\left(\sqrt{\frac{7}{15}}\, x\right) - \sqrt{21}\,(48\,x^2 - 225)\, K_1(x)\, K_0\left(\sqrt{\frac{7}{15}}\, x\right) +$$

$$\left. + 252\,\sqrt{5}\, x\, K_1(x)\, K_1\left(\sqrt{\frac{7}{15}}\, x\right) \right] \cos \frac{4x}{\sqrt{5}}$$

$$\int x^4 \cos \frac{4x}{\sqrt{5}} \cdot K_0(x)\, K_1\left(\sqrt{\frac{7}{15}}\, x\right) dx = \frac{x^2}{1792}\left[60\,\sqrt{21}\, x\, K_0(x)\, K_0\left(\sqrt{\frac{7}{15}}\, x\right) + \right.$$

$$+ \sqrt{5}\,(336\,x^2 - 315)\, K_0(x)\, K_1\left(\sqrt{\frac{7}{15}}\, x\right) + \sqrt{21}\,(225 - 48\,x^2)\, K_1(x)\, K_0\left(\sqrt{\frac{7}{15}}\, x\right) +$$

$$\left. + 252\,\sqrt{5}\, x\, K_1(x)\, K_1\left(\sqrt{\frac{7}{15}}\, x\right) \right] \sin \frac{4x}{\sqrt{5}} + \frac{x}{7168}\left[-\sqrt{105}\, x(64x^2 - 60)\, K_0(x)\, K_0\left(\sqrt{\frac{7}{15}}\, x\right) + \right.$$

$$+ (3360\,x^2 - 1575)\, K_0(x)\, K_1\left(\sqrt{\frac{7}{15}}\, x\right) + \sqrt{105}\,(225 - 288x^2)\, K_1(x)\, K_0\left(\sqrt{\frac{7}{15}}\, x\right) -$$

$$\left. -(1344\,x^3 - 1260\,x)\, K_1(x)\, K_1\left(\sqrt{\frac{7}{15}}\, x\right) \right] \cos \frac{4x}{\sqrt{5}}$$

$$\int x^4 \sin \frac{2x}{\sqrt{7}} \cdot I_1(x)\, I_1\left(\sqrt{\frac{3}{7}}\, x\right) dx = \frac{x^2}{16}\left[-6\,\sqrt{21}\, x\, I_0(x)\, I_0\left(\sqrt{\frac{3}{7}}\, x\right) + (8\,x^2 - 63)\, I_0(x)\, I_1\left(\sqrt{\frac{3}{7}}\, x\right) + \right.$$

$$\left. + 21\,\sqrt{21}\, I_1(x)\, I_0\left(\sqrt{\frac{3}{7}}\, x\right) + 14\, x\, I_1(x)\, I_1\left(\sqrt{\frac{3}{7}}\, x\right) \right] \sin \frac{2x}{\sqrt{7}} + \frac{x}{32}\left[\sqrt{3}\,(8\, x^3 - 42x)\, I_0(x)\, I_0\left(\sqrt{\frac{3}{7}}\, x\right) + \right.$$

$$+ \sqrt{7}\,(20\, x^2 - 63)\, I_0(x)\, I_1\left(\sqrt{\frac{3}{7}}\, x\right) + \sqrt{3}\,(147 - 28\, x^2)\, I_1(x)\, I_0\left(\sqrt{\frac{3}{7}}\, x\right) +$$

$$\left. + \sqrt{7}\,(14x - 8\, x^3)\, I_1(x)\, I_1\left(\sqrt{\frac{3}{7}}\, x\right) \right] \cos \frac{2x}{\sqrt{7}}$$

$$\int x^4 \cos \frac{2x}{\sqrt{7}} \cdot I_1(x)\, I_1\left(\sqrt{\frac{3}{7}}\, x\right) dx = \frac{x}{32}\left[\sqrt{3}\,(42x - 8x^3)\, I_0(x)\, I_0\left(\sqrt{\frac{3}{7}}\, x\right) + \right.$$

$$+ \sqrt{7}\,(63 - 20x^2)\, I_0(x)\, I_1\left(\sqrt{\frac{3}{7}}\, x\right) + \sqrt{3}\,(28x^2 - 147)\, I_1(x)\, I_0\left(\sqrt{\frac{3}{7}}\, x\right) +$$

$$\left. + \sqrt{7}\,(8x^3 - 14x)\, I_1(x)\, I_1\left(\sqrt{\frac{3}{7}}\, x\right) \right] \sin \frac{2x}{\sqrt{7}} + \frac{x^2}{16}\left[-6\sqrt{21}\, x\, I_0(x)\, I_0\left(\sqrt{\frac{3}{7}}\, x\right) + \right.$$

$$\left. + (8x^2 - 63)\, I_0(x)\, I_1\left(\sqrt{\frac{3}{7}}\, x\right) + 21\sqrt{21}\, I_1(x)\, I_0\left(\sqrt{\frac{3}{7}}\, x\right) + 14x\, I_1(x)\, I_1\left(\sqrt{\frac{3}{7}}\, x\right) \right] \cos \frac{2x}{\sqrt{7}}$$

$$\int x^4 \sin \frac{2x}{\sqrt{7}} \cdot K_1(x)\, K_1\left(\sqrt{\frac{3}{7}}\, x\right) dx = \frac{x^2}{16}\left[-6\,\sqrt{21}\, x\, K_0(x)\, K_0\left(\sqrt{\frac{3}{7}}\, x\right) - (8\, x^2 - 63)\, K_0(x)\, K_1\left(\sqrt{\frac{3}{7}}\, x\right) - \right.$$

$$-21\sqrt{21}\,K_1(x)\,K_0\left(\sqrt{\tfrac{3}{7}}\,x\right)+14\,x\,K_1(x)\,K_1\left(\sqrt{\tfrac{3}{7}}\,x\right)\Bigg]\sin\frac{2x}{\sqrt{7}}+\frac{x}{32}\Bigg[\sqrt{3}\,(8\,x^3-42x)\,K_0(x)\,K_0\left(\sqrt{\tfrac{3}{7}}\,x\right)-$$

$$-\sqrt{7}\,(20\,x^2-63)\,K_0(x)\,K_1\left(\sqrt{\tfrac{3}{7}}\,x\right)-\sqrt{3}\,(147-28\,x^2)\,K_1(x)\,K_0\left(\sqrt{\tfrac{3}{7}}\,x\right)+$$

$$+\sqrt{7}\,(14x-8\,x^3)\,K_1(x)\,K_1\left(\sqrt{\tfrac{3}{7}}\,x\right)\Bigg]\cos\frac{2x}{\sqrt{7}}$$

$$\int x^4\,\cos\frac{2x}{\sqrt{7}}\cdot K_1(x)\,K_1\left(\sqrt{\tfrac{3}{7}}\,x\right)dx=\frac{x}{32}\Bigg[\sqrt{3}\,(42x-8x^3)\,K_0(x)\,K_0\left(\sqrt{\tfrac{3}{7}}\,x\right)+$$

$$-\sqrt{7}\,(63-20x^2)\,K_0(x)\,K_1\left(\sqrt{\tfrac{3}{7}}\,x\right)-\sqrt{3}\,(28x^2-147)\,K_1(x)\,K_0\left(\sqrt{\tfrac{3}{7}}\,x\right)+$$

$$+\sqrt{7}\,(8x^3-14x)\,K_1(x)\,K_1\left(\sqrt{\tfrac{3}{7}}\,x\right)\Bigg]\sin\frac{2x}{\sqrt{7}}-\frac{x^2}{16}\Bigg[6\sqrt{21}\,x\,K_0(x)\,K_0\left(\sqrt{\tfrac{3}{7}}\,x\right)+$$

$$+(8x^2-63)\,K_0(x)\,K_1\left(\sqrt{\tfrac{3}{7}}\,x\right)+21\sqrt{21}\,K_1(x)\,K_0\left(\sqrt{\tfrac{3}{7}}\,x\right)-14x\,K_1(x)\,K_1\left(\sqrt{\tfrac{3}{7}}\,x\right)\Bigg]\cos\frac{2x}{\sqrt{7}}$$

n = 5 :

$$\int x^5\,\sin\frac{4x}{\sqrt{11}}\cdot I_1(x)\,I_1\left(\sqrt{\tfrac{3}{11}}\,x\right)dx=\frac{x}{24576}\Bigg[\sqrt{33}\,x(7260-4224\,x^2)\,I_0(x)\,I_0\left(\sqrt{\tfrac{3}{11}}\,x\right)+$$

$$+(8448\,x^4-52272\,x^2+59895)\,I_0(x)\,I_1\left(\sqrt{\tfrac{3}{11}}\,x\right)+\sqrt{33}\,(256\,x^4+11088x^2-19965x)\,I_1(x)\,I_0\left(\sqrt{\tfrac{3}{11}}\,x\right)+$$

$$+(12672\,x^3-4356\,x)\,I_1(x)\,I_1\left(\sqrt{\tfrac{3}{11}}\,x\right)\Bigg]\sin\frac{4x}{\sqrt{11}}+\frac{x^2}{6144}\Bigg[\sqrt{3}\,(704\,x^3-7260x)\,I_0(x)\,I_0\left(\sqrt{\tfrac{3}{11}}\,x\right)+$$

$$+\sqrt{11}\,(2112\,x^2-5445)\,I_0(x)\,I_1\left(\sqrt{\tfrac{3}{11}}\,x\right)+\sqrt{3}\,(19965-1408\,x^2)\,I_1(x)\,I_0\left(\sqrt{\tfrac{3}{11}}\,x\right)+$$

$$+\sqrt{11}\,(396x-960\,x^3)\,I_1(x)\,I_1\left(\sqrt{\tfrac{3}{11}}\,x\right)\Bigg]\cos\frac{4x}{\sqrt{11}}$$

$$\int x^5\,\cos\frac{4x}{\sqrt{11}}\cdot I_1(x)\,I_1\left(\sqrt{\tfrac{3}{11}}\,x\right)dx=\frac{x^2}{6144}\Bigg[\sqrt{3}\,(7260x-704\,x^3)\,I_0(x)\,I_0\left(\sqrt{\tfrac{3}{11}}\,x\right)+$$

$$+\sqrt{11}\,(5445-2112\,x^2)\,I_0(x)\,I_1\left(\sqrt{\tfrac{3}{11}}\,x\right)+\sqrt{3}\,(1408\,x^2-19965)\,I_1(x)\,I_0\left(\sqrt{\tfrac{3}{11}}\,x\right)+$$

$$+\sqrt{11}\,(960\,x^3-396x)\,I_1(x)\,I_1\left(\sqrt{\tfrac{3}{11}}\,x\right)\Bigg]\sin\frac{4x}{\sqrt{11}}+\frac{x}{24576}\Bigg[\sqrt{33}\,(7260x-4224\,x^3)\,I_0(x)\,I_0\left(\sqrt{\tfrac{3}{11}}\,x\right)+$$

$$+(8448\,x^4-52272\,x^2+59895)\,I_0(x)\,I_1\left(\sqrt{\tfrac{3}{11}}\,x\right)+\sqrt{33}\,(256x^4+11088x^2-19965)\,I_1(x)\,I_0\left(\sqrt{\tfrac{3}{11}}\,x\right)+$$

$$+(12672\,x^3-4356\,x)\,I_1(x)\,I_1\left(\sqrt{\tfrac{3}{11}}\,x\right)\Bigg]\cos\frac{4x}{\sqrt{11}}$$

$$\int x^5\,\sin\frac{4x}{\sqrt{11}}\cdot K_1(x)\,K_1\left(\sqrt{\tfrac{3}{11}}\,x\right)dx=\frac{x}{24576}\Bigg[-\sqrt{33}\,x(4224\,x^2-7260)\,K_0(x)\,K_0\left(\sqrt{\tfrac{3}{11}}\,x\right)-$$

$$-(8448\,x^4-52272\,x^2+59895)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)-\sqrt{33}\,(256\,x^4+11088x^2-19965x)\,K_1(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+(12672\,x^3-4356\,x)\,K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)\Bigg]\sin\frac{4x}{\sqrt{11}}+\frac{x^2}{6144}\Bigg[\sqrt{3}\,(704\,x^3-7260x)\,K_0(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)-$$

$$-\sqrt{11}\,(2112\,x^2-5445)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)-\sqrt{3}\,(19965-1408\,x^2)\,K_1(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+\sqrt{11}\,(396x-960\,x^3)\,K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)\Bigg]\cos\frac{4x}{\sqrt{11}}$$

$$\int x^5\,\cos\frac{4x}{\sqrt{11}}\cdot K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)dx=\frac{x}{24576}\Bigg[\sqrt{33}\,x(7260-4224\,x^2)\,K_0(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)-$$

$$-(8448\,x^4-52272\,x^2+59895)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)-\sqrt{33}\,(256\,x^4+11088x^2-19965x)\,K_1(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+(12672\,x^3-4356\,x)\,K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)\Bigg]\cos\frac{4x}{\sqrt{11}}-\frac{x^2}{6144}\Bigg[\sqrt{3}\,(704\,x^3-7260x)\,K_0(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)-$$

$$-\sqrt{11}\,(2112\,x^2-5445)\,K_0(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)-\sqrt{3}\,(19965-1408\,x^2)\,K_1(x)\,K_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+\sqrt{11}\,(396x-960\,x^3)\,K_1(x)\,K_1\left(\sqrt{\frac{3}{11}}\,x\right)\Bigg]\sin\frac{4x}{\sqrt{11}}$$

2.4.6 b) $\int x^n\cdot\left\{\begin{array}{c}\sinh\\\cosh\end{array}\right\}\alpha x\cdot Z_\mu(x)\,Z_\nu(\beta x)\,dx$:

$\underline{\mathbf{n=1:}}$

$$\int x\,\sinh 4x\cdot I_0(x)\,I_1(3x)\,dx=\frac{x^2}{4}\,[I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)]\,\sinh 4x-$$

$$-\frac{x}{16}\,[4x\,I_0(x)\,I_0(3x)-3\,I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)+4x\,I_1(x)\,I_1(3x)]\,\cosh 4x$$

$$\int x\,\cosh 4x\cdot I_0(x)\,I_1(3x)\,dx=\frac{x^2}{4}\,[I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)]\,\cosh 4x-$$

$$-\frac{x}{16}\,[4x\,I_0(x)\,I_0(3x)-3\,I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)+4x\,I_1(x)\,I_1(3x)]\,\sinh 4x$$

$$\int x\,\sinh 4x\cdot K_0(x)\,K_1(3x)\,dx=\frac{x^2}{4}\,[K_0(x)\,K_1(3x)+K_1(x)\,K_0(3x)]\,\sinh 4x+$$

$$+\frac{x}{16}\,[4x\,K_0(x)\,K_0(3x)+3\,K_0(x)\,K_1(3x)-K_1(x)\,K_0(3x)+4x\,K_1(x)\,K_1(3x)]\,\cosh 4x$$

$$\int x\,\cosh 4x\cdot K_0(x)\,K_1(3x)\,dx=\frac{x^2}{4}\,[K_0(x)\,K_1(3x)+K_1(x)\,K_0(3x)]\,\cosh 4x+$$

$$+\frac{x}{16}\,[4x\,K_0(x)\,K_0(3x)+3\,K_0(x)\,K_1(3x)-K_1(x)\,K_0(3x)+4x\,K_1(x)\,K_1(3x)]\,\sinh 4x$$

$\underline{\mathbf{n=2:}}$

$$\int x^2\,\sinh\frac{8x}{3}\cdot I_0(x)\,I_1\left(\frac{5x}{3}\right)dx=\frac{x}{512}\Bigg[24x\,I_0(x)\,I_0\left(\frac{5x}{3}\right)+(64\,x^2-45)\,I_0(x)\,I_1\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2+27)\,I_1(x)\,I_0\left(\frac{5x}{3}\right)+24x\,I_1(x)\,I_1\left(\frac{5x}{3}\right)\Bigg]\sinh\frac{8x}{3}-$$

$$-\frac{x^2}{64}\left[8x\,I_0(x)\,I_0\left(\frac{5x}{3}\right) - 15\,I_0(x)\,I_1\left(\frac{5x}{3}\right) + 9\,I_1(x)\,I_0\left(\frac{5x}{3}\right) + 8x\,I_1(x)\,I_1\left(\frac{5x}{3}\right)\right]\cosh\frac{8x}{3}$$

$$\int x^2\cosh\frac{8x}{3}\cdot I_0(x)\,I_1\left(\frac{5x}{3}\right)dx = \frac{x}{512}\left[24x\,I_0(x)\,I_0\left(\frac{5x}{3}\right) + (64\,x^2 - 45)\,J_0(x)\,J_1\left(\frac{5x}{3}\right) + \right.$$

$$\left. +(64\,x^2 + 27)\,I_1(x)\,I_0\left(\frac{5x}{3}\right) + 24x\,I_1(x)\,I_1\left(\frac{5x}{3}\right)\right]\cosh\frac{8x}{3} -$$

$$-\frac{x^2}{64}\left[8x\,I_0(x)\,I_0\left(\frac{5x}{3}\right) - 15\,I_0(x)\,I_1\left(\frac{5x}{3}\right) + 9\,I_1(x)\,I_0\left(\frac{5x}{3}\right) + 8x\,I_1(x)\,I_1\left(\frac{5x}{3}\right)\right]\sinh\frac{8x}{3}$$

$$\int x^2\sinh\frac{8x}{3}\cdot K_0(x)\,K_1\left(\frac{5x}{3}\right)dx = \frac{x}{512}\left[-24x\,K_0(x)\,K_0\left(\frac{5x}{3}\right) + (64\,x^2 - 45)\,K_0(x)\,K_1\left(\frac{5x}{3}\right) + \right.$$

$$\left. +(64\,x^2 + 27)\,K_1(x)\,K_0\left(\frac{5x}{3}\right) - 24x\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\right]\sinh\frac{8x}{3} +$$

$$+\frac{x^2}{64}\left[8x\,K_0(x)\,K_0\left(\frac{5x}{3}\right) + 15\,K_0(x)\,K_1\left(\frac{5x}{3}\right) - 9\,K_1(x)\,K_0\left(\frac{5x}{3}\right) + 8x\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\right]\cosh\frac{8x}{3}$$

$$\int x^2\cosh\frac{8x}{3}\cdot K_0(x)\,K_1\left(\frac{5x}{3}\right)dx = \frac{x}{512}\left[-24x\,K_0(x)\,K_0\left(\frac{5x}{3}\right) + (64\,x^2 - 45)\,K_0(x)\,K_1\left(\frac{5x}{3}\right) + \right.$$

$$\left. +(64\,x^2 + 27)\,K_1(x)\,K_0\left(\frac{5x}{3}\right) - 24x\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\right]\cosh\frac{8x}{3} +$$

$$+\frac{x^2}{64}\left[8x\,K_0(x)\,K_0\left(\frac{5x}{3}\right) + 15\,K_0(x)\,K_1\left(\frac{5x}{3}\right) - 9\,K_1(x)\,K_0\left(\frac{5x}{3}\right) + 8x\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\right]\sinh\frac{8x}{3}$$

n = 3 :

$$\int x^3\sinh\frac{8x}{\sqrt{51}}\cdot J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)dx = \frac{x^2}{2560}\left[-8\sqrt{1785}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right) - 1785\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right) + \right.$$

$$\left. +51\sqrt{1785}\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right) + 680x\,J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sinh\frac{8x}{\sqrt{51}} +$$

$$+\frac{x}{20480}\left[408\sqrt{35}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right) + \sqrt{51}\,(1600x^2 + 1785)\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right) - \right.$$

$$\left. -\sqrt{35}\,(1088x^2 + 2601)\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right) - 680\sqrt{51}\,x\,J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cosh\frac{8x}{\sqrt{51}}$$

$$\int x^3\cosh\frac{8x}{\sqrt{51}}\cdot J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)dx = \frac{x}{20480}\left[408\sqrt{35}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right) + \right.$$

$$+\sqrt{51}\,(1600\,x^2 + 1785)\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right) - \sqrt{35}\,(1088\,x^2 + 2601)\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right) -$$

$$-680\sqrt{51}\,x\,J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\sinh\frac{8x}{\sqrt{51}} + \frac{x^2}{2560}\left[-8\sqrt{1785}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right) - 1785\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right) + \right.$$

$$\left. +51\sqrt{1785}\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right) + 680\,x\,J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\right]\cosh\frac{8x}{\sqrt{51}}$$

n = 4 :

$$\int x^4\sinh\frac{2\sqrt{3}\,x}{5}\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)dx = \left[-\frac{15x^3}{8}\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right) - \frac{25\sqrt{13}\,x^2}{16}\,J_0(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right) + \right.$$

$$+\frac{8x^4 + 125x^2}{16}\, J_1(x)\, J_0\left(\frac{\sqrt{13}\,x}{5}\right) - \frac{5\sqrt{13}\,x^3}{8}\, J_1(x)\, J_1\left(\frac{\sqrt{13}\,x}{5}\right)\Bigg]\, \sinh\frac{2\sqrt{3}\,x}{5} +$$

$$+\left[\frac{5\sqrt{3}\,(4x^4+15x^2)}{48}\, J_0(x)\, J_0\left(\frac{\sqrt{13}\,x}{5}\right) + \frac{5\sqrt{39}\,(4x^3+25x)}{96}\, J_0(x)\, J_1\left(\frac{\sqrt{13}\,x}{5}\right) -\right.$$

$$-\frac{5\sqrt{3}\,(28x^3+125x)}{96}\, J_1(x)\, J_0\left(\frac{\sqrt{13}\,x}{5}\right) + \frac{\sqrt{39}\,(4x^4+25x^2)}{48}\, J_1(x)\, J_1\left(\frac{\sqrt{13}\,x}{5}\right)\Bigg]\, \cosh\frac{2\sqrt{3}\,x}{5}$$

$$\int x^4 \cosh\frac{2\sqrt{3}\,x}{5}\, J_0(x)\, J_0\left(\frac{\sqrt{13}\,x}{5}\right)\, dx = \frac{\sqrt{3}\,x}{96}\left[(40x^3+150x)\, J_0(x)\, J_0\left(\frac{\sqrt{13}\,x}{5}\right) +\right.$$

$$+\sqrt{13}\,(20x^2+125)\, J_0(x)\, J_1\left(\frac{\sqrt{13}\,x}{5}\right) - (140x^2+625)\, J_1(x)\, J_0\left(\frac{\sqrt{13}\,x}{5}\right) +$$

$$+\sqrt{13}\,(8x^3+50x)\, J_1(x)\, J_1\left(\frac{\sqrt{13}\,x}{5}\right)\Bigg]\, \sinh\frac{2\sqrt{3}\,x}{5} - \frac{x^2}{16}\left[30x\, J_0(x)\, J_0\left(\frac{\sqrt{13}\,x}{5}\right) + 25\sqrt{13}\, J_0(x)\, J_1\left(\frac{\sqrt{13}\,x}{5}\right) -\right.$$

$$-(8x^2+125)\, J_1(x)\, J_0\left(\frac{\sqrt{13}\,x}{5}\right) + 10\sqrt{13}x\, J_1(x)\, J_1\left(\frac{\sqrt{13}\,x}{5}\right)\Bigg]\, \cosh\frac{2\sqrt{3}\,x}{5}$$

$$\int x^4 \sinh\frac{4x}{\sqrt{5}}\cdot J_0(x)\, J_1\left(\sqrt{\frac{7}{15}}\,x\right)\, dx = \frac{x}{7168}\left[-\sqrt{105}\,(64x^3+60x)\, J_0(x)\, J_0\left(\sqrt{\frac{7}{15}}\,x\right) -\right.$$

$$-(3360x^2+1575)\, J_0(x)\, J_1\left(\sqrt{\frac{7}{15}}\,x\right) + \sqrt{105}\,(288x^2+225)\, J_1(x)\, J_0\left(\sqrt{\frac{7}{15}}\,x\right) +$$

$$+(1344x^3+1260x)\, J_1(x)\, J_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]\, \sinh\frac{4x}{\sqrt{5}} + \frac{x^2}{1792}\left[60\sqrt{21}x\, J_0(x)\, J_0\left(\sqrt{\frac{7}{15}}\,x\right) +\right.$$

$$+\sqrt{5}\,(336x^2+315)\, J_0(x)\, J_1\left(\sqrt{\frac{7}{15}}\,x\right) - \sqrt{21}\,(48x^2+225)\, J_1(x)\, J_0\left(\sqrt{\frac{7}{15}}\,x\right) -$$

$$-252\sqrt{5}x\, J_1(x)\, J_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]\, \cosh\frac{4x}{\sqrt{5}}$$

$$\int x^4 \cosh\frac{4x}{\sqrt{5}}\cdot J_0(x)\, J_1\left(\sqrt{\frac{7}{15}}\,x\right)\, dx = \frac{x^2}{1792}\left[60\sqrt{21}\,x\, J_0(x)\, J_0\left(\sqrt{\frac{7}{15}}\,x\right) +\right.$$

$$+\sqrt{5}\,(336x^2+315)\, J_0(x)\, J_1\left(\sqrt{\frac{7}{15}}\,x\right) - \sqrt{21}\,(48x^2+225)\, J_1(x)\, J_0\left(\sqrt{\frac{7}{15}}\,x\right) -$$

$$-252\sqrt{5}\,x\, J_1(x)\, J_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]\, \sinh\frac{4x}{\sqrt{5}} + \frac{x}{7168}\left[-\sqrt{105}\,(64x^3+60x)\, J_0(x)\, J_0\left(\sqrt{\frac{7}{15}}\,x\right) -\right.$$

$$-(3360x^2+1575)\, J_0(x)\, J_1\left(\sqrt{\frac{7}{15}}\,x\right) + \sqrt{105}\,(288x^2+225)\, J_1(x)\, J_0\left(\sqrt{\frac{7}{15}}\,x\right) +$$

$$+(1344x^3+1260x)\, J_1(x)\, J_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]\, \cosh\frac{4x}{\sqrt{5}}$$

$$\int x^4 \sinh\frac{2x}{\sqrt{7}}\cdot J_1(x)\, J_1\left(\sqrt{\frac{3}{7}}\,x\right)\, dx = \frac{x^2}{16}\left[-6\sqrt{21}\,x\, J_0(x)\, J_0\left(\sqrt{\frac{3}{7}}\,x\right) -\right.$$

$$-(8x^2+63)\, J_0(x)\, J_1\left(\sqrt{\frac{3}{7}}\,x\right) + 21\sqrt{21}\, J_1(x)\, J_0\left(\sqrt{\frac{3}{7}}\,x\right) - 14x\, J_1(x)\, J_1\left(\sqrt{\frac{3}{7}}\,x\right)\Bigg]\, \sinh\frac{2x}{\sqrt{7}} +$$

$$+\frac{x}{32}\left[\sqrt{3}\left(8\,x^3+42\,x\right)J_0(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)+\sqrt{7}\left(20x^2+63\right)J_0(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)-\right.$$

$$\left.-\sqrt{3}\left(28\,x^2+147\right)J_1(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)+\sqrt{7}\left(8\,x^3+14\,x\right)J_1(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\cosh\frac{2x}{\sqrt{7}}$$

$$\int x^4\cosh\frac{2x}{\sqrt{7}}\cdot J_1(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)dx=\frac{x^2}{16}\left[-6\,\sqrt{21}\,x\,J_0(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)-\right.$$

$$\left.-\left(8\,x^2+63\right)J_0(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)+21\,\sqrt{21}\,J_1(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)-14\,x\,J_1(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\cosh\frac{2x}{\sqrt{7}}+$$

$$+\frac{x}{32}\left[\sqrt{3}\left(8\,x^3+42\,x\right)J_0(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)+\sqrt{7}\left(20\,x^2+63\right)J_0(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)-\right.$$

$$\left.-\sqrt{3}\left(28x^2+147\right)J_1(x)\,J_0\left(\sqrt{\frac{3}{7}}\,x\right)+\sqrt{7}\left(8x^3+14\,x\right)J_1(x)\,J_1\left(\sqrt{\frac{3}{7}}\,x\right)\right]\sinh\frac{2x}{\sqrt{7}}$$

n = 5 :

$$\int x^5\sinh\frac{4x}{\sqrt{11}}\cdot J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)dx=\frac{x}{24\,576}\left[-\sqrt{33}\left(4224\,x^3+7260x\right)J_0(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)-\right.$$

$$-\left(8448\,x^4+52272\,x^2+59895\right)J_0(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$+\sqrt{33}\left(-256\,x^4+11088\,x^2+19965\right)J_1(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)-$$

$$\left.-\left(12672\,x^3+4356\,x\right)J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)\right]\sinh\frac{4x}{\sqrt{11}}+$$

$$+\frac{x^2}{6144}\left[\sqrt{3}\left(704\,x^3+7260\,x\right)J_0(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+\sqrt{11}\left(2112\,x^2+5445\right)J_0(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)-\right.$$

$$\left.-\sqrt{3}\left(1408\,x^2+19965\right)J_1(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+\sqrt{11}\left(960\,x^3+396\,x\right)J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)\right]\cosh\frac{4x}{\sqrt{11}}$$

$$\int x^5\cosh\frac{4x}{\sqrt{11}}\cdot J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)dx=\frac{x^2}{6144}\left[\sqrt{3}\left(704\,x^3+7260\,x\right)J_0(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+\right.$$

$$+\sqrt{11}\left(2112\,x^2+5445\right)J_0(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)-\sqrt{3}\left(1408\,x^2+19965\right)J_1(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+$$

$$\left.+\sqrt{11}\left(960\,x^3+396\,x\right)J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)\right]\sinh\frac{4x}{\sqrt{11}}-$$

$$-\frac{x}{24576}\left[\sqrt{33}\left(4224\,x^3+7260\,x\right)J_0(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+\left(8448\,x^4+52272\,x^2+59895\right)J_0(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)-\right.$$

$$\left.-\sqrt{33}\left(-256\,x^4+11088\,x^2+19965\right)J_1(x)\,J_0\left(\sqrt{\frac{3}{11}}\,x\right)+\left(12672\,x^3+4356\,x\right)J_1(x)\,J_1\left(\sqrt{\frac{3}{11}}\,x\right)\right]\cosh\frac{4x}{\sqrt{11}}$$

3. Products of three Bessel Functions

3.1. Integrals of the type $\int x^n Z_0^m(x) Z_1^{3-m}(x)\, dx$

These integrals are expressed by three basic integrals with $m = 0, 1, 3$. One has obviously

$$\int J_0^2(x) J_1(x)\, dx = -\frac{1}{3} J_0^3(x) \quad , \qquad \int I_0^2(x) I_1(x)\, dx = \frac{1}{3} I_0^3(x) .$$

a) Basic integral $\int Z_0^3(x)\, dx$:
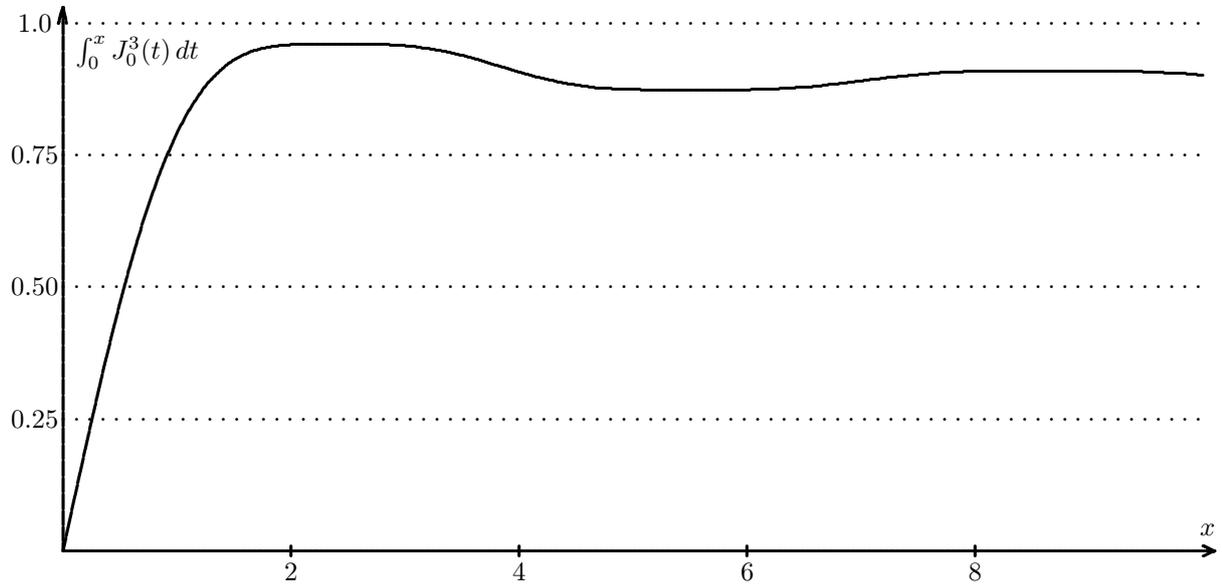
$$\int_0^\infty J_0^3(x)\, dx \;=\; \frac{\Gamma\left(\dfrac{1}{6}\right)}{3\,\Gamma\left(\dfrac{5}{6}\right)\cdot\Gamma^2\left(\dfrac{2}{3}\right)} \;=\; \frac{2\pi}{3\,\Gamma^2\left(\dfrac{5}{6}\right)\cdot\Gamma^2\left(\dfrac{2}{3}\right)} \;=\; 0.89644\;07887\;76762\;86423\;\ldots$$

Formula 2.12.42.4 from [4] gives $2\sqrt{3}/3\pi = 0.36755\ldots$. This does not fit to the result of computations. The formula 2.12.42.18 offers $2\,\Gamma(1/6)/[3\,\Gamma(5/6)\cdot\Gamma^2(2/3)] = 2\cdot 0.896\ldots$.

With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x J_0^3(t)\, dt = \sum_{k=0}^\infty c_k^{(30)}\, T_{2k+1}\left(\frac{x}{8}\right) \ .$$

The first coefficients are

k	$c_k^{(30)}$	k	$c_k^{(30)}$
0	1.14145 15823 43066 65430	15	-0.00001 19647 92899 04163
1	-0.37698 69057 03625 95863	16	0.00000 19134 34477 76782
2	0.24193 82520 26895 89401	17	-0.00000 02652 90698 65709
3	-0.15672 12348 70401 19757	18	0.00000 00322 34936 59711
4	0.09593 25955 24494 24624	19	-0.00000 00034 64486 68661
5	-0.06183 39745 85568 16834	20	0.00000 00003 31986 30964
6	0.04061 90786 50027 57290	21	-0.00000 00000 28561 98591
7	-0.02860 28881 71908 58991	22	0.00000 00000 02219 71843
8	0.02158 63885 61883 43325	23	-0.00000 00000 00156 67592
9	-0.01457 50343 38917 46000	24	0.00000 00000 00010 09261
10	0.00781 65287 61390 63536	25	-0.00000 00000 00000 59593
11	-0.00326 40051 49421 93383	26	0.00000 00000 00000 03238
12	0.00107 89280 58820 94395	27	-0.00000 00000 00000 00162
13	-0.00028 89222 57879 12574	28	0.00000 00000 00000 00008
14	0.00006 40382 21736 47959	-	-

The given approximation differs from the true function as shown in the following figure:

$10^{20}\,\bar{\Delta}^{(30)}(x)$

$$\bar{\Delta}^{(30)}(x) = \sum_{k=0}^{28} c_k^{(30)} T_{2k+1}(x/8) - \int_0^x J_0^3(t)\,dt$$
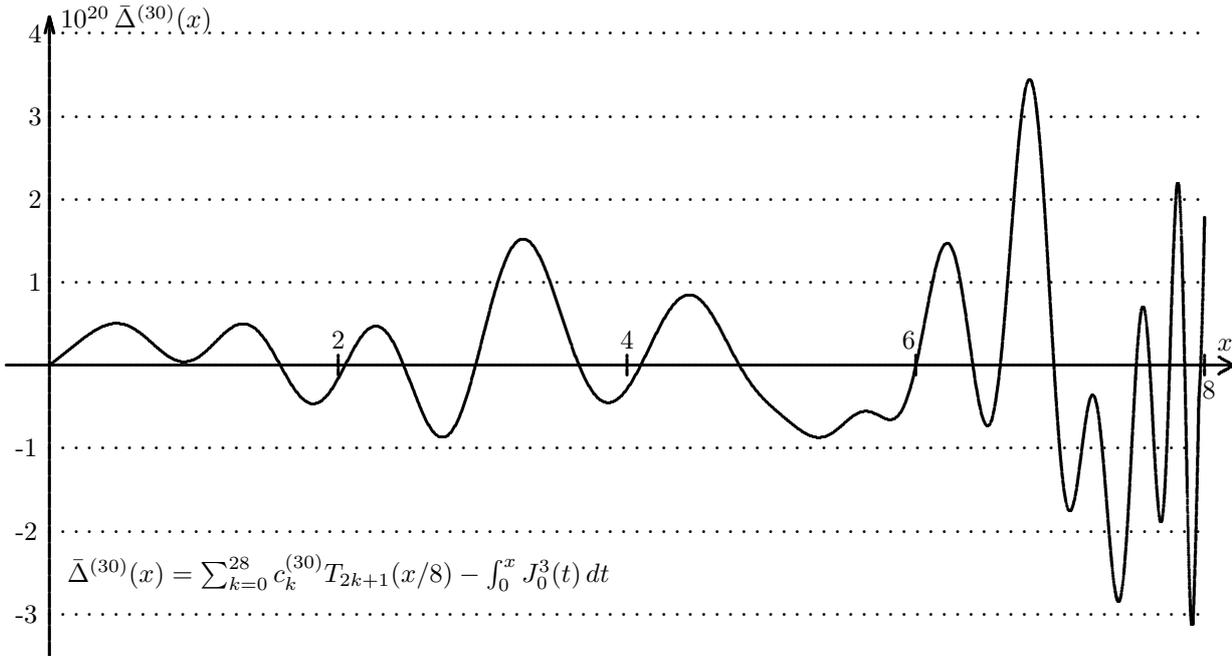
Asymptotic formula:

$$\int_0^x J_0^3(t)\,dt \sim 0.89644\ 07887\ 76762\ 86423\ \ldots +$$

$$+\sqrt{\frac{2}{\pi^3 x}} \sum_{k=1}^{\infty} \frac{1}{x^k} \left[a_k^{(30)} \sin\left(3x + \frac{7-2k}{4}\,\pi\right) + b_k^{(30)} \sin\left(x + \frac{1-2k}{4}\,\pi\right) \right]$$

with the first values

k	$a_k^{(30)}$	$a_k^{(30)}$
1	1/6	0.16666 66666 66666 66667
2	7/48	0.14583 33333 33333 33333
3	379/2304	0.16449 65277 77777 77778
4	13141/55296	0.23764 82928 24074 07407
5	250513/589824	0.42472 50027 12673 61111
6	12913841/14155776	0.91226 65546 55852 14120
7	1565082415/679477248	2.30336 25034 63839 30724
8	36535718855/5435817984	6.72129 18023 63631 16606
9	23344474269635/1043677052928	22.36778 53260 66262 13087
10	2103860629922855/25048249270272	83.99232 24662 16381 92796

k	$b_k^{(30)}$	$b_k^{(30)}$
1	3/2	1.50000 00000 00000 00000
2	39/16	2.43750 00000 00000 00000
3	1635/256	6.38671 87500 00000 00000
4	46053/2048	22.48681 64062 50000 00000
5	6664257/65536	101.68849 18212 89062 5000
6	293433849/524288	559.68065 07110 595703 125
7	30538511055/8388608	3640.47420 68052 2918 70
8	1832502818925/67108864	27306.41989 29816 48445 1
9	996997642437465/4294967296	2 32131.60281 01055 41721
10	75773171001327165/34359738368	22 05289.52199 17166 1050

The first consecutive maxima and minima of

$$\Delta_n^{(30)}(x) = 0.896\ldots + \sqrt{\frac{2}{\pi^3 x}} \sum_{k=1}^{n} \frac{1}{x^k} \left[a_k^{(30)} \sin\left(3x + \frac{7-2k}{4}\pi\right) + b_k^{(30)} \sin\left(x + \frac{1-2k}{4}\pi\right) \right] - \int_0^x J_0^3(t)\,dt \; :$$

i	x_i	$\Delta_1^{(30)}(x_i)$	x_i	$\Delta_2^{(30)}(x_i)$	x_i	$\Delta_3^{(30)}(x_i)$	x_i	$\Delta_4^{(30)}(x_i)$	x_i	$\Delta_5^{(30)}(x_i)$
1	3.953	-1.551E-02	2.356	3.014E-02	3.933	5.971E-03	2.356	-4.857E-02	3.928	-6.275E-03
2	7.084	4.314E-03	5.498	-2.889E-03	7.072	-6.255E-04	5.498	1.167E-03	7.069	2.440E-04
3	10.221	-1.832E-03	8.639	7.036E-04	10.213	1.380E-04	8.639	-1.302E-04	10.211	-2.818E-05
4	13.360	9.631E-04	11.781	-2.547E-04	13.354	-4.417E-05	11.781	2.690E-05	13.352	5.533E-06
5	16.500	-5.762E-04	14.923	1.153E-04	16.495	1.771E-05	14.923	-7.843E-06	16.494	-1.496E-06
6	19.641	3.758E-04	18.064	-6.022E-05	19.636	-8.262E-06	18.064	2.852E-06	19.635	5.014E-07
7	22.781	-2.607E-04	21.206	3.478E-05	22.778	4.297E-06	21.206	-1.211E-06	22.777	-1.962E-07
8	25.922	1.894E-04	24.347	-2.161E-05	25.919	-2.426E-06	24.347	5.757E-07	25.918	8.626E-08
9	29.064	-1.427E-04	27.489	1.421E-05	29.061	1.460E-06	27.489	-2.988E-07	29.060	-4.154E-08
10	32.205	1.106E-04	30.631	-9.769E-06	32.202	-9.244E-07	30.631	1.662E-07	32.202	2.152E-08

i	x_i	$\Delta_6^{(30)}(x_i)$	x_i	$\Delta_7^{(30)}(x_i)$	x_i	$\Delta_8^{(30)}(x_i)$	x_i	$\Delta_9^{(30)}(x_i)$	x_i	$\Delta_{10}^{(30)}(x_i)$
1	5.498	-1.017E-03	3.927	1.332E-02	5.498	1.579E-03	3.927	-4.858E-02	5.498	-3.900E-03
2	8.639	5.162E-05	7.069	-1.853E-04	8.639	-3.588E-05	7.069	2.333E-04	8.639	3.891E-05
3	11.781	-6.116E-06	10.210	1.115E-05	11.781	2.431E-06	10.210	-7.235E-06	11.781	-1.498E-06
4	14.923	1.154E-06	13.352	-1.345E-06	14.923	-2.973E-07	13.352	5.351E-07	14.923	1.186E-07
5	18.064	-2.934E-07	16.493	2.458E-07	18.064	5.293E-08	16.493	-6.616E-08	18.064	-1.479E-08
6	21.206	9.182E-08	19.635	-5.934E-08	21.206	-1.224E-08	19.635	1.152E-08	21.206	2.528E-09
7	24.347	-3.349E-08	22.777	1.750E-08	24.347	3.429E-09	22.777	-2.565E-09	24.347	-5.448E-10
8	27.489	1.374E-08	25.918	-6.004E-09	27.489	-1.114E-09	25.918	6.874E-10	27.489	1.403E-10
9	30.631	-6.192E-09	29.060	2.317E-09	30.631	4.071E-10	29.060	-2.129E-10	30.631	-4.162E-11
10	33.772	3.009E-09	32.201	-9.833E-10	33.772	-1.636E-10	32.201	7.404E-11	33.772	1.384E-11

In the case $x \geq 8$ one has $g_n^{(30)} \leq \Delta_n^{(30)}(x) \leq G_n^{(30)}$ with such values:

n	$g_n^{(30)}$	$G_n^{(30)}$	n	$g_n^{(30)}$	$G_n^{(30)}$
1	-1.832E-03	2.384E-03	6	5.162E-05	-6.116E-06
2	-2.547E-04	7.036E-04	7	-9.887E-05	1.115E-05
3	-3.729E-04	1.380E-04	8	-3.588E-05	2.431E-06
4	-1.302E-04	2.690E-05	9	1.129E-04	-7.235E-06
5	1.404E-04	-2.818E-05	10	3.891E-05	-1.498E-06

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{30}(x) = 0.896\ldots + \sum_{k=1}^{10} \frac{1}{x^{k+1/2}} \left[\tilde{a}_k^{(30)} \sin\left(3x + \frac{7-2k}{4}\pi\right) + \tilde{b}_k^{(30)} \sin\left(x + \frac{1-2k}{4}\pi\right) \right] \quad .$$

The values of the coefficients are

k	$\tilde{a}_k^{(30)}$	$\tilde{b}_k^{(30)}$
1	0.042328861175	0.380959694283
2	0.037041314549	0.619046345961
3	0.041506694202	1.619134748282
4	0.063280735888	5.687020985444
5	0.009657085591	24.391624641402
6	0.955932046063	129.687141434225
7	-12.477953789465	581.699380578201
8	57.157092072696	3774.819577903598
9	-530.170189213242	9530.689591084210
10	649.179657827779	65374.322605554143

With $8 \leq x \leq 30$ holds

$$-1.8 \cdot 10^{-9} \;\leq\; F_{30}(x) - \int_0^x J_0^3(t)\,dt \;\leq\; 1.1 \cdot 10^{-9} \quad .$$

Power series for the modified Bessel function:

$$\int_0^x I_0^3(t)\,dt \;=\; \sum_{k=0}^{\infty} d_k^{(30)}\,x^{2k+1} = x + \frac{1}{4}\,x^3 + \frac{3}{64}\,x^5 + \frac{31}{5376}\,x^7 + \frac{71}{147456}\,x^9 + \frac{47}{1638400}\,x^{11} + \frac{11723}{9201254400}\,x^{13} + \ldots$$

With $n \geq 1$ the following recurrence relation holds:

$$d_{n+1}^{(30)} = \frac{4\,\sigma_1^{(30)}(n,d) \;+\; 9\,\sigma_2^{(30)}(n,d)}{12(2n+3)(n+1)^2}$$

with

$$\sigma_1^{(30)}(n,d) = \sum_{k=1}^{n}(2k+1)k(2n-2k+3)(2n-5k+2)\cdot d_k^{(30)}\cdot d_{n-k+1}^{(30)}$$

and

$$\sigma_2^{(30)}(n,d) = \sum_{k=0}^{n}(2k+1)(2n-2k+1)\cdot d_k^{(30)}\cdot d_{n-k}^{(30)}\;.$$

Asymptotic formula for the modified Bessel function:

$$\int_0^x I_0^3(t)\,dt \sim \frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3\,x}}\sum_{k=1}^{\infty}\frac{c_k^{(30)}}{x^k}$$

with the first values

k	$c_k^{(30)}$	$c_k^{(30)}$
1	1/12	0.08333 33333 33333 3
2	7/96	0.07291 66666 66666 7
3	379/4608	0.08224 82638 88888 9
4	13141/110592	0.11882 41464 12037
5	250513/1179648	0.21236 25013 56337
6	12913841/28311552	0.45613 32773 27926
7	1565082415/1358954496	1.15168 12517 3192
8	36535718855/10871635968	3.36064 59011 8182
9	23344744269635/2087354105856	11.18389 26630 331
10	2103860629922855/50096498540544	41.99616 12331 082

Let

$$\delta_n^{(30)}(x) = \left[\frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3\,x}}\sum_{k=1}^{n}\frac{c_k^{(30)}}{x^k}\right]\cdot\left[\int_0^x I_0^3(t)\,dt\right]^{-1} - 1$$

be the relative error, then one has the following values of $\delta_n^{(30)}(x)$:

n	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
1	-1.897E-01	-9.019E-02	-5.944E-02	-4.435E-02	-3.538E-02
2	-4.794E-02	-1.058E-02	-4.579E-03	-2.545E-03	-1.618E-03
3	-1.595E-02	-1.597E-03	-4.529E-04	-1.874E-04	-9.492E-05
4	-6.704E-03	-3.000E-04	-5.555E-05	-1.710E-05	-6.896E-06
5	-3.400E-03	-6.813E-05	-8.209E-06	-1.877E-06	-6.026E-07
6	-1.981E-03	-1.833E-05	-1.430E-06	-2.427E-07	-6.198E-08
7	-1.264E-03	-5.756E-06	-2.887E-07	-3.632E-08	-7.376E-09
8	-8.462E-04	-2.087E-06	-6.672E-08	-6.211E-09	-1.002E-09
9	-5.678E-04	-8.660E-07	-1.747E-08	-1.201E-09	-1.539E-10
10	-3.588E-04	-4.075E-07	-5.140E-09	-2.603E-10	-2.647E-11

b) Basic integral $\int Z_0(x)\, Z_1^2(x)\, dx$:
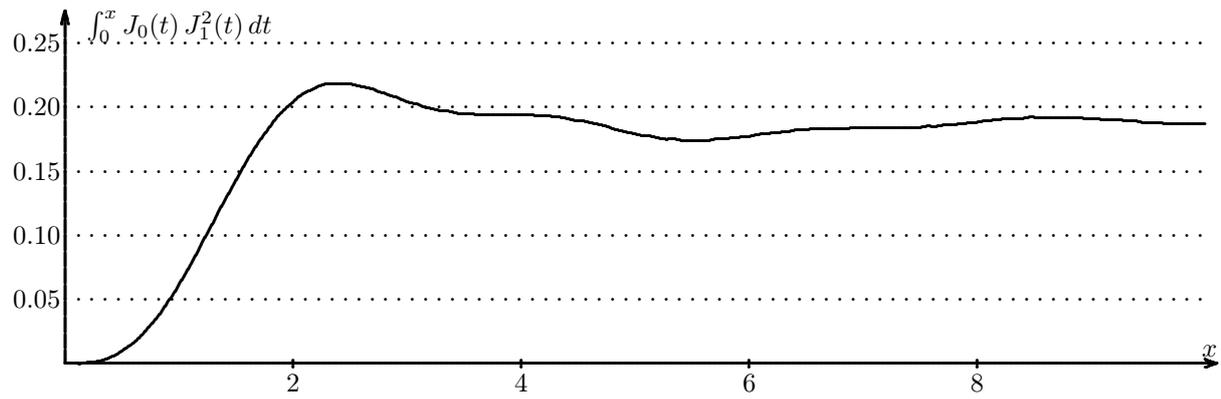
$$\int_0^\infty J_0(x)\, J_1^2(x)\, dx \;=\; 0.18578\ 75214\ 63065\ 82253\ \ldots$$

It differs from formula 2.12.42.4 in [4].

With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x J_0(t)\, J_1^2(t)\, dt = \sum_{k=0}^\infty c_k^{(12)}\, T_{2k+1}\!\left(\frac{x}{8}\right)\ .$$

The first coefficients are

k	$c_k^{(12)}$	k	$c_k^{(12)}$
0	0.23402 35970 33006 35445	15	0.00001 04741 31410 16206
1	-0.07444 03661 88704 59538	16	-0.00000 16973 09596 00078
2	0.04383 45295 19423 05343	17	0.00000 02378 37605 87389
3	-0.02088 20217 52543 47233	18	-0.00000 00291 52869 89193
4	0.00418 98619 72133 53415	19	0.00000 00031 56305 17549
5	0.00544 25004 57533 63387	20	-0.00000 00003 04352 15694
6	-0.00771 94350 97286 08113	21	0.00000 00000 26326 39770
7	0.00909 15741 08442 81722	22	-0.00000 00000 02055 66180
8	-0.01063 96487 30041 15062	23	0.00000 00000 00145 70197
9	0.00922 86127 36398 55548	24	-0.00000 00000 00009 42056
10	-0.00564 70826 05794 70935	25	0.00000 00000 00000 55809
11	0.00254 11445 11845 57857	26	-0.00000 00000 00000 03042
12	-0.00087 99927 23272 67696	27	0.00000 00000 00000 00153
13	0.00024 31559 72294 70777	28	-0.00000 00000 00000 00007
14	-0.00005 51197 58270 08008	-	-

The given approximation differs from the true function as shown in the following figure:

$$\bar{\Delta}^{(12)}(x) = \sum_{k=0}^{28} c_k^{(12)} T_{2k+1}(x/8) - \int_0^x J_0(t)\, J_1^2(t)\, dt$$

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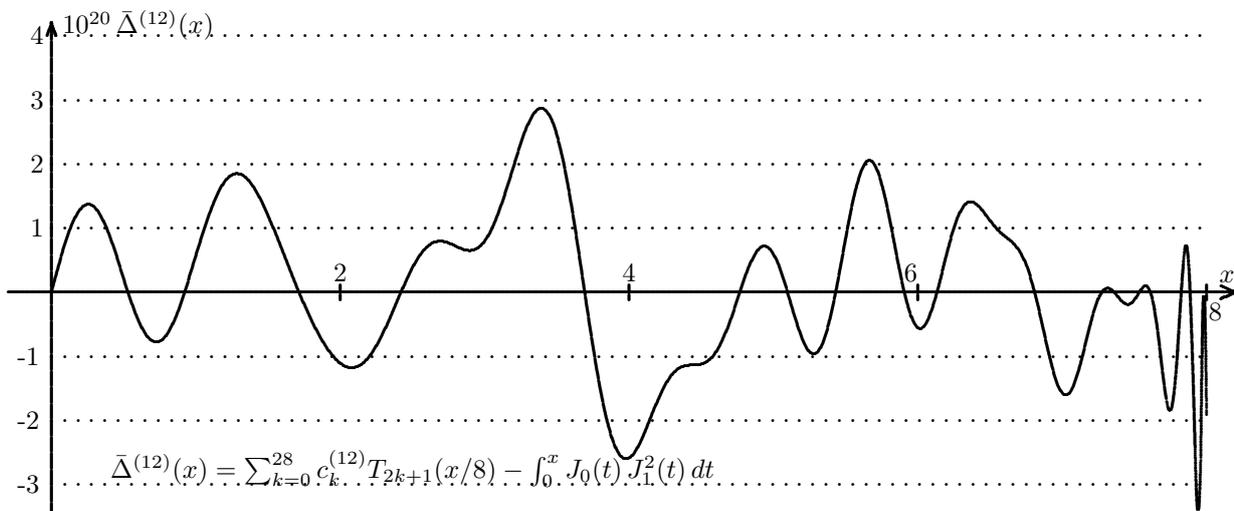
Asymptotic formula:

$$\int_0^x J_0(t)\, J_1^2(t)\, dt \sim 0.18578\ 75214\ 63065\ 82253\ \ldots + \sqrt{\frac{2}{\pi^3 x}} \left\{ \frac{\left[a_0^{(12)} \sin\left(3x + \frac{\pi}{4}\right) + b_0^{(12)} \sin\left(x + \frac{7\pi}{4}\right) \right]}{x} + \right.$$

$$\left. + \sum_{k=2}^{\infty} \frac{1}{x^k} \left[a_k^{(12)} \sin\left(3x + \frac{7-2k}{4}\pi\right) + b_k^{(12)} \sin\left(x + \frac{1-2k}{4}\pi\right) \right] \right\}$$

with the first values

k	$a_k^{(12)}$	$a_k^{(12)}$
1	1/6	0.16666 66666 66666 66667
2	1/48	0.02083 33333 33333 33333
3	85/2304	0.03689 23611 11111 11111
4	3379/55296	0.06110 74942 12962 96296
5	69967/589824	0.11862 35215 92881 94444
6	3833063/14155776	0.27077 73137 97562 21065
7	487468417/679477248	0.71741 68354 19925 64260
8	11835711665/5435817984	2.17735 61402 97136 1890
9	7811427811325/1043677052928	7.48452 57825 78345 0049
10	722951995177505/25048249270272	28.86237 62633 79242 299

k	$b_k^{(12)}$	$b_k^{(12)}$
1	1/2	0.50000 00000 00000 00000
2	21/16	1.31250 00000 00000 00000
3	753/256	2.94140 62500 00000 00000
4	21375/2048	10.43701 17187 50000 0000
5	3061323/65536	46.71208 19091 79687 5000
6	134966187/524288	257.42757 22503 66210 938
7	14029169013/8388608	1672.40727 10275 65002 44
8	842027324535/67108864	12547.18489 25203 08494 6
9	458031686444595/4294967296	1 06643.81236 87624 46567
10	34812460139616855/34359738368	10 13175.93768 52090 6451

The first consecutive maxima and minima of

$$\Delta_n^{(12)}(x) = 0.185\ldots + \sqrt{\frac{2}{\pi^3 x}} \sum_{k=1}^{n} \frac{1}{x^k} \left[a_k^{(12)} \sin\left(3x + \ldots\right) + b_k^{(12)} \sin\left(x + \ldots\right) \right] - \int_0^x J_0(t)\, J_1^2(t)\, dt \quad :$$

i	x_i	$\Delta_1^{(12)}(x_i)$	x_i	$\Delta_2^{(12)}(x_i)$	x_i	$\Delta_3^{(12)}(x_i)$	x_i	$\Delta_4^{(12)}(x_i)$	x_i	$\Delta_5^{(12)}(x_i)$
1	3.880	-8.324E-03	2.365	1.432E-02	3.923	2.771E-03	2.358	-2.242E-02	3.927	-2.887E-03
2	7.042	2.261E-03	5.503	-1.356E-03	7.066	-2.895E-04	5.499	5.375E-04	7.068	1.122E-04
3	10.192	-9.530E-04	8.643	3.293E-04	10.209	6.382E-05	8.640	-5.993E-05	10.210	-1.296E-05
4	13.338	4.995E-04	11.784	-1.191E-04	13.351	-2.042E-05	11.782	1.238E-05	13.352	2.544E-06
5	16.482	-2.984E-04	14.925	5.386E-05	16.492	8.184E-06	14.923	-3.609E-06	16.493	-6.876E-07
6	19.625	1.944E-04	18.066	-2.813E-05	19.634	-3.818E-06	18.065	1.313E-06	19.635	2.305E-07
7	22.768	-1.348E-04	21.207	1.624E-05	22.776	1.986E-06	21.206	-5.571E-07	22.776	-9.018E-08
8	25.911	9.790E-05	24.349	-1.009E-05	25.918	-1.121E-06	24.348	2.649E-07	25.918	3.965E-08
9	29.053	-7.371E-05	27.490	6.636E-06	29.059	6.745E-07	27.489	-1.375E-07	29.060	-1.910E-08
10	32.196	5.712E-05	30.632	-4.561E-06	32.201	-4.271E-07	30.631	7.645E-08	32.201	9.893E-09

i	x_i	$\Delta_6^{(12)}(x_i)$	x_i	$\Delta_7^{(12)}(x_i)$	x_i	$\Delta_8^{(12)}(x_i)$	x_i	$\Delta_9^{(12)}(x_i)$	x_i	$\Delta_{10}^{(12)}(x_i)$
1	2.356	8.374E-02	3.927	6.119E-03	2.356	-6.023E-01	3.927	-2.232E-02	2.356	7.281E+00
2	5.498	-4.673E-04	7.069	-8.515E-05	5.498	7.255E-04	7.069	1.072E-04	5.498	-1.792E-03
3	8.640	2.372E-05	10.210	5.122E-06	8.639	-1.649E-05	10.210	-3.324E-06	8.639	1.788E-05
4	11.781	-2.810E-06	13.352	-6.178E-07	11.781	1.117E-06	13.352	2.458E-07	11.781	-6.882E-07
5	14.923	5.304E-07	16.493	1.129E-07	14.923	-1.366E-07	16.493	-3.040E-08	14.923	5.448E-08
6	18.064	-1.348E-07	19.635	-2.726E-08	18.064	2.432E-08	19.635	5.293E-09	18.064	-6.794E-09
7	21.206	4.219E-08	22.777	8.042E-09	21.206	-5.621E-09	22.777	-1.179E-09	21.206	1.161E-09
8	24.347	-1.539E-08	25.918	-2.759E-09	24.347	1.575E-09	25.918	3.158E-10	24.347	-2.503E-10
9	27.489	6.316E-09	29.060	1.065E-09	27.489	-5.119E-10	29.060	-9.780E-11	27.489	6.446E-11
10	30.631	-2.845E-09	32.201	-4.518E-10	30.631	1.871E-10	32.201	3.402E-11	30.631	-1.912E-11

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In the case $x \geq 8$ one has $g_n^{(12)} \leq \Delta_n^{(12)}(x) \leq G_n^{(12)}$ with such values:

n	$g_n^{(12)}$	$G_n^{(12)}$	n	$g_n^{(12)}$	$G_n^{(12)}$
1	-9.530E-04	1.314E-03	6	-2.810E-06	2.372E-05
2	-1.191E-04	3.293E-04	7	-4.543E-05	5.122E-06
3	-1.735E-04	6.382E-05	8	-1.649E-05	1.117E-06
4	-5.993E-05	1.238E-05	9	-3.324E-06	5.189E-05
5	6.457E-05	1.238E-05	10	1.788E-05	-6.882E-07

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{12}(x) = 0.18578\ 75214\ 63 + \sum_{k=1}^{10} \frac{1}{x^{k+1/2}} \left[\tilde{a}_k^{(12)} \sin\left(3x + \frac{7 - 2k + 4\delta_{1k}}{4}\pi \right) + \tilde{b}_k^{(12)} \sin\left(x + \frac{1 - 2k}{4}\pi \right) \right] \quad .$$

Here δ_{kl} denotes the Kronecker symbol.

The values of the coefficients are

k	$\tilde{a}_k^{(12)}$	$\tilde{b}_k^{(12)}$
1	0.042329196006	0.126986297337
2	0.005292679875	0.333333960255
3	0.009245148401	0.745696621275
4	0.016862928651	2.639682546435
5	-0.014976122776	11.204590967086
6	0.401440149083	59.656792209975
7	-5.815950738680	267.223468914520
8	26.023951772804	1734.742283706764
9	-244.011181624276	4378.452764670006
10	296.167068374575	30038.714778003193

With $8 \leq x \leq 30$ holds

$$-8.1 \cdot 10^{-10} \ \leq \ F_{12}(x) - \int_0^x J_0(t)\, J_1^2(t)\, dx \ \leq \ 5.5 \cdot 10^{-10} \quad .$$

Power series for the modified Bessel functions:

$$\int_0^x I_0(t)\, I_1^2(t)\, dt \ = \ \sum_{k=1}^{\infty} d_k^{(12)}\, x^{2k+1} = \frac{x^3}{12} - \frac{x^5}{40} + \frac{5}{1344}x^7 - \frac{19}{55296}x^9 + \frac{707}{32440320}x^{11} - \frac{581}{575078400}x^{13} + \ldots$$

With $n \geq 1$ the coefficients $d_k^{(12)}$ are represented bei $d_k^{(03)}$ (see page 392):

$$d_k^{(12)} \quad = \quad \frac{4(k+1)(k+2)}{6k+3}\, d_{k+1}^{(03)} \quad .$$

Asymptotic formula for the modified Bessel function:

$$\int_0^x I_0(t)\, I_1^2(t)\, dt \ \sim \ \frac{\sqrt{2}\, e^{3x}}{\sqrt{\pi^3\, x}} \sum_{k=1}^{\infty} \frac{c_k^{(12)}}{x^k}$$

with the first values

k	$c_k^{(12)}$	$c_k^{(12)}$
1	1/12	0.08333 33333 33333 3
2	-1/96	-0.010416666666667
3	-85/4608	-0.018446180555556
4	-3379/110592	-0.030553747106481
5	-69967/1179648	-0.059311760796441
6	-3833063/28311552	-0.135388656898781
7	-487468417/1358954496	-0.358708417709963
8	-11835711665/10871635968	-1.088678070148568
9	-7811427811325/2087354105856	-3.742262891289173
10	-722951995177505/50096498540544	-14.431188131689621

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Let

$$\delta_n^{(12)}(x) = \left[\frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3\,x}} \sum_{k=1}^{n} \frac{c_k^{(12)}}{x^k}\right] \cdot \left[\int_0^x I_0(t)\,I_1^2(t)\,dt\right]^{-1} - 1$$

be the relative error, then one has the following values of $\delta_n^{(12)}(x)$:

n	$x = 5$	$x = 10$	$x = 15$	$x = 20$	$x = 25$
1	4.085E-02	1.541E-02	9.532E-03	6.902E-03	5.409E-03
2	1.483E-02	2.715E-03	1.120E-03	6.084E-04	3.817E-04
3	5.617E-03	4.676E-04	1.265E-04	5.122E-05	2.561E-05
4	2.564E-03	9.526E-05	1.684E-05	5.070E-06	2.019E-06
5	1.379E-03	2.299E-05	2.643E-06	5.914E-07	1.875E-07
6	8.376E-04	6.492E-06	4.829E-07	8.020E-08	2.023E-08
7	5.508E-04	2.121E-06	1.014E-07	1.248E-08	2.503E-09
8	3.768E-04	7.943E-07	2.419E-08	2.204E-09	3.513E-10
9	2.571E-04	3.383E-07	6.500E-09	4.374E-10	5.537E-11
10	1.648E-04	1.624E-07	1.952E-09	9.679E-11	9.724E-12

c) Basic integral $\int Z_1^3(x)\,dx$:
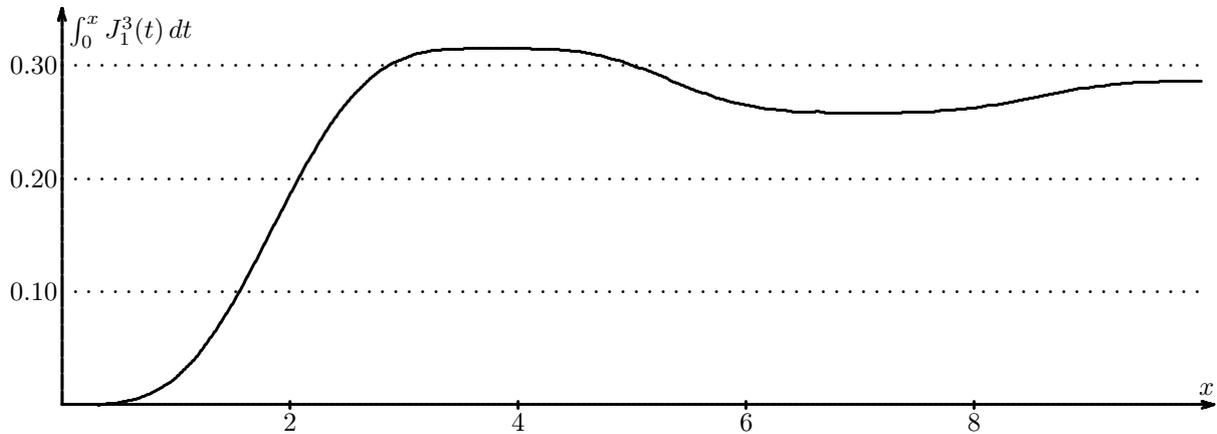
$$\int_0^\infty J_1^3(x)\,dx \;=\; \frac{\sqrt{3}}{2\pi} \;=\; 0.27566\ 44477\ 10896\ 02476\ \ldots$$

Formula 2.12.42.18 from [4] gives $2 \cdot 0.27566 \ldots$.

With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x J_1^3(t)\,dt = \sum_{k=0}^{\infty} c_k^{(03)}\, T_{2k+1}\left(\frac{x}{8}\right)\ .$$

Using 30 items of the series, the given approximation differs from the true function as shown in the following figure:

$$\bar{\Delta}^{(03)}(x) = \sum_{k=0}^{29} c_k^{(03)} T_{2k}(x/8) - \int_0^x J_1^3(t)\,dt$$
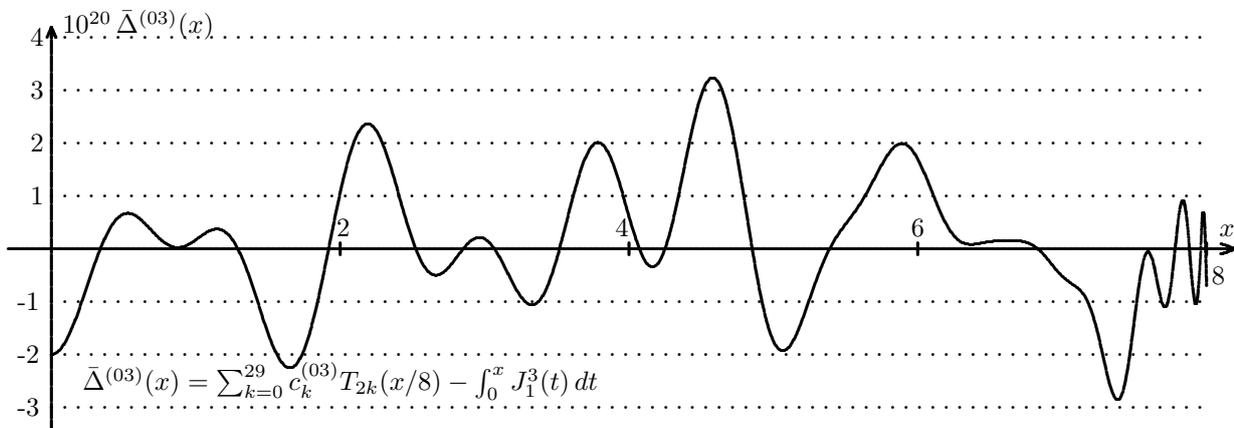
The first coefficients of the series are

k	$c_k^{(03)}$	k	$c_k^{(03)}$
0	0.23821 24114 31419 64234	15	-0.00002 28518 02498 15054
1	0.05565 99710 46458 43436	16	0.00000 40330 04736 70912
2	-0.07298 97702 38741 38530	17	-0.00000 06104 34965 75114
3	0.06987 40694 66707 70829	18	0.00000 00802 87294 38422
4	-0.04171 62455 49522 66601	19	-0.00000 00092 76147 24624
5	0.01717 16060 47354 59698	20	0.00000 00009 50125 56893
6	-0.00484 82605 99791 99458	21	-0.00000 00000 86955 13371
7	-0.00029 03839 52557 94438	22	0.00000 00000 07159 24024
8	0.00492 11828 87493 16152	23	-0.00000 00000 00533 44393
9	-0.00739 18550 17718 86749	24	0.00000 00000 00036 16224
10	0.00611 65128 72946 06394	25	-0.00000 00000 00002 24086
11	-0.00338 52693 70801 23544	26	0.00000 00000 00000 12747
12	0.00137 35573 01229 79344	27	-0.00000 00000 00000 00668
13	-0.00043 18988 45664 76700	28	0.00000 00000 00000 00032
14	0.00010 92654 18291 14430	29	-0.00000 00000 00000 00001

Asymptotic formula:

$$\int_0^x J_1^3(t)\,dt \sim \frac{\sqrt{3}}{2\pi} + \sqrt{\frac{2}{\pi^3 x}}\left\{ \frac{a_1^{(03)}\sin(3x-\frac{\pi}{4}) + b_1^{(03)}\sin(x+\frac{5\pi}{4})}{x} + \right.$$

$$\left. + \sum_{k=2}^{\infty}\frac{1}{x^k}\left[a_k^{(03)}\sin\left(3x+\frac{5-2k}{4}\pi\right) + b_k^{(03)}\sin\left(x+\frac{7-2k}{4}\pi\right)\right]\right\}$$

with $\sqrt{3}/2\pi = 0.27566\ 44477\ 10896\ 02476\ \ldots$ (see [8], 13.46) and the first values

k	$a_k^{(03)}$	$a_k^{(03)}$
1	1/6	0.16666 66666 66666 66667
2	5/48	0.10416 66666 66666 66667
3	173/2304	0.07508 68055 55555 55556
4	5735/55296	0.10371 45543 98148 14815
5	112415/589824	0.19059 07524 95659 72222
6	6113875/14155776	0.43189 96712 01352 71991
7	790059305/679477248	1.16274 57833 58444 40225
8	19738125085/5435817984	3.63112 32537 76703 35181
9	13496143234525/13496143234525	12.93134 03956 34871 2675
10	1298437733131525/25048249270272	51.83746 45318 04589 8562

k	$b_k^{(03)}$	$b_k^{(03)}$
1	3/2	1.50000 00000 00000 00000
2	27/16	1.68750 00000 00000 00000
3	891/256	3.48046 87500 00000 00000
4	25065/2048	12.23876 95312 50000 0000
5	3564945/65536	54.39674 37744 14062 5000
6	156773205/156773205	299.02115 82183 83789 063
7	16277745015/8388608	1940.45841 87269 21081 54
8	976536193185/67108864	14551.52322 62760 40077 2
9	531096920069625/4294967296	1 23655.63774 23985 39558
10	40362305845577625/34359738368	11 74697.70617 25228 8503

The first consecutive maxima and minima of

$$\Delta_n^{(03)}(x) = \frac{\sqrt{3}}{2\pi} + \sqrt{\frac{2}{\pi^3 x}} \left\{ \frac{a_1^{(03)} \sin(3x - \frac{\pi}{4}) + b_1^{(03)} \sin(x + \frac{5\pi}{4})}{x} + \right.$$

$$\left. + \sum_{k=1}^{n} \frac{1}{x^k} \left[a_k^{(03)} \sin\left(3x + \frac{7-2k}{4}\pi \right) + b_k^{(03)} \sin\left(x + \frac{1-2k}{4}\pi \right) \right] \right\} - \int_0^x J_0^3(t)\, dt \quad :$$

i	x_i	$\Delta_1^{(03)}(x_i)$	x_i	$\Delta_2^{(03)}(x_i)$	x_i	$\Delta_3^{(03)}(x_i)$	x_i	$\Delta_4^{(03)}(x_i)$	x_i	$\Delta_5^{(03)}(x_i)$
1	2.023	2.977E-02	3.933	4.556E-03	2.338	-1.901E-02	3.927	-2.969E-03	2.354	4.614E-02
2	5.336	-4.810E-03	7.072	-7.718E-04	5.487	9.013E-04	7.069	1.899E-04	5.496	-5.376E-04
3	8.533	1.690E-03	10.213	2.352E-04	8.632	-1.477E-04	10.210	-3.029E-05	8.638	4.048E-05
4	11.702	-8.051E-04	13.354	-9.624E-05	11.775	4.028E-05	13.352	7.586E-06	11.780	-6.330E-06
5	14.860	4.533E-04	16.495	4.706E-05	14.918	-1.461E-05	16.493	-2.496E-06	14.922	1.484E-06
6	18.013	-2.838E-04	19.636	-2.593E-05	18.060	6.359E-06	19.635	9.863E-07	18.063	-4.510E-07
7	21.162	1.911E-04	22.778	1.557E-05	21.203	-3.145E-06	22.777	-4.448E-07	21.205	1.643E-07
8	24.309	-1.358E-04	25.919	-9.966E-06	24.345	1.709E-06	25.918	2.215E-07	24.347	-6.841E-08
9	27.455	1.005E-04	29.061	6.707E-06	27.486	-9.979E-07	29.060	-1.192E-07	27.488	3.156E-08
10	30.600	-7.684E-05	32.202	-4.698E-06	30.628	6.168E-07	32.201	6.828E-08	30.630	-1.579E-08

i	x_i	$\Delta_6^{(03)}(x_i)$	x_i	$\Delta_7^{(03)}(x_i)$	x_i	$\Delta_8^{(03)}(x_i)$	x_i	$\Delta_9^{(03)}(x_i)$	x_i	$\Delta_{10}^{(03)}(x_i)$
1	3.927	4.522E-03	2.356	-2.426E-01	3.927	-1.276E-02	2.356	2.291E+00	3.927	5.861E-02
2	7.069	-1.056E-04	5.497	6.337E-04	7.069	1.047E-04	5.498	-1.254E-03	7.069	-1.634E-04
3	10.210	8.813E-06	8.639	-2.157E-05	10.210	-4.529E-06	8.639	1.893E-05	10.210	3.623E-06
4	13.352	-1.356E-06	11.781	1.933E-06	13.352	4.279E-07	11.781	-9.676E-07	13.352	-2.095E-07
5	16.493	3.014E-07	14.922	-2.937E-07	16.493	-6.433E-08	14.923	9.523E-08	16.493	2.130E-08
6	19.635	-8.572E-08	18.064	6.248E-08	19.635	1.319E-08	18.064	-1.419E-08	19.635	-3.150E-09
7	22.777	2.911E-08	21.206	-1.680E-08	22.777	-3.380E-09	21.206	2.820E-09	22.777	6.096E-10
8	25.918	-1.130E-08	24.347	5.371E-09	25.918	1.025E-09	24.347	-6.931E-10	25.918	-1.444E-10
9	29.060	4.873E-09	27.489	-1.961E-09	29.060	-3.542E-10	27.489	2.005E-10	29.060	4.007E-11
10	32.201	-2.284E-09	30.630	7.955E-10	32.201	1.361E-10	30.631	-6.600E-11	32.201	-1.262E-11

In the case $x \geq 8$ one has $g_n^{(03)} \leq \Delta_n^{(03)}(x) \leq G_n^{(03)}$ with such values:

n	g_n	G_n	n	g_n	G_n
1	-8.051E-04	1.690E-03	6	-5.870E-05	8.813E-06
2	-4.559E-04	2.352E-04	7	-2.157E-05	1.933E-06
3	-1.477E-04	4.028E-05	8	-4.529E-06	5.333E-05
4	-3.029E-05	1.120E-04	9	-9.676E-07	1.893E-05
5	-6.330E-06	4.048E-05	10	-7.485E-05	3.623E-06

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{03}(x) = \frac{\sqrt{3}}{2\pi} + \sum_{k=1}^{10} \frac{1}{x^{k+1/2}} \left[\tilde{a}_k^{(03)} \sin\left(3x + \frac{5-2k+4\delta_{1k}}{4}\pi \right) + \tilde{b}_k^{(03)} \sin\left(x + \frac{7-2k}{4}\pi \right) \right] \quad .$$

Here δ_{kl} denotes the Kronecker symbol.
The values of the coefficients are

k	$\tilde{a}_k^{(12)}$	$\tilde{b}_k^{(12)}$
1	0.042329125987	0.380960729687
2	0.026455529094	0.428583324556
3	0.019031312665	0.882368927951
4	0.026093503230	3.106884244014
5	0.036754295203	13.006466163819
6	-0.015099663650	73.274117576600
7	-0.742020898226	297.966084182852
8	-21.408004618587	2526.040556452494
9	-6.362414618305	4245.043677286622
10	-1177.354351044844	55725.065225312970

With $8 \leq x \leq 30$ holds

$$-1.0 \cdot 10^{-9} \ \leq \ F_{03}(x) \ \leq \ 4.5 \cdot 10^{-10} \quad .$$

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Power series for the modified Bessel function:

$$\int_0^x I_1^3(t)\,dt \;=\; \sum_{k=2}^{\infty} d_k^{(03)}\, x^{2k} = \frac{x^4}{32} + \frac{x^6}{128} + \frac{x^8}{1024} + \frac{19}{245760}\,x^{10} + \frac{101}{23592960}\,x^{12} + \frac{83}{471859200}\,x^{14} + \dots$$

With $n \geq 3$ the following recurrence relation holds:

$$d_n^{(03)} = \frac{16\,\sigma_1^{(03)}(n,d) \;+\; 36\,\sigma_2^{(03)}(n,d)}{3n(n-1)(n-2)}$$

with

$$\sigma_1^{(03)}(n,d) = \sum_{k=3}^{n-1} k(n-k+2)(2nk-n+7k-5k^2)\cdot d_k^{(03)}\cdot d_{n-k+2}^{(03)}$$

and

$$\sigma_2^{(03)}(n,d) = \sum_{k=2}^{n-1} k(n-k+1)\cdot d_k^{(03)}\cdot d_{n-k+1}^{(03)} \;.$$

Asymptotic formula for the modified Bessel function:

$$\int_0^x I_1^3(t)\,dt \sim \frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3 x}} \sum_{k=1}^{\infty} \frac{c_k^{(03)}}{x^k}$$

with the first values

k	$c_k^{(03)}$	$c_k^{(03)}$
1	1/12	0.08333 33333 33333 3
2	-5/96	-0.052083333333333
3	-173/4608	-0.037543402777778
4	-5735/110592	-0.051857277199074
5	-112415/1179648	-0.095295376247830
6	-6113875/28311552	-0.215949835600676
7	-790059305/1358954496	-0.581372891679222
8	-19738125085/10871635968	-1.815561626888352
9	-13496143234525/2087354105856	-6.465670197817436
10	-1298437733131525/50096498540544	-25.918732265902295

Let

$$\delta_n^{(03)}(x) = \left[\frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3 x}} \sum_{k=1}^{n} \frac{c_k^{(03)}}{x^k} \right] \cdot \left[\int_0^x I_1^3(t)\,dt \right]^{-1} - 1$$

be the relative error, then one has the following values of $\delta_n^{(03)}(x)$:

n	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
1	1.791E-01	7.271E-02	4.589E-02	3.355E-02	2.645E-02
2	3.170E-02	5.662E-03	2.315E-03	1.253E-03	7.841E-04
3	1.046E-02	8.293E-04	2.209E-04	8.875E-05	4.419E-05
4	4.587E-03	1.618E-04	2.801E-05	8.358E-06	3.311E-06
5	2.429E-03	3.912E-05	4.387E-06	9.712E-07	3.060E-07
6	1.451E-03	1.132E-05	8.179E-07	1.342E-07	3.362E-08
7	9.250E-04	3.840E-06	1.774E-07	2.154E-08	4.288E-09
8	5.962E-04	1.503E-06	4.399E-08	3.949E-09	6.244E-10
9	3.620E-04	6.710E-07	1.232E-08	8.161E-10	1.024E-10
10	1.742E-04	3.374E-07	3.863E-09	1.882E-10	1.874E-11

d) Basic integral $\int x^{-1} \cdot Z_0^3(x)\,dx$:

$$\int \frac{Z_0^3(x)\,dx}{x} \quad = \quad \ln|x| + \int_0^x \frac{Z_0^3(t)-1}{t}\,dt$$

Graph labeled $\int_0^x t^{-1}[J_0^3(t)-1]\,dt$
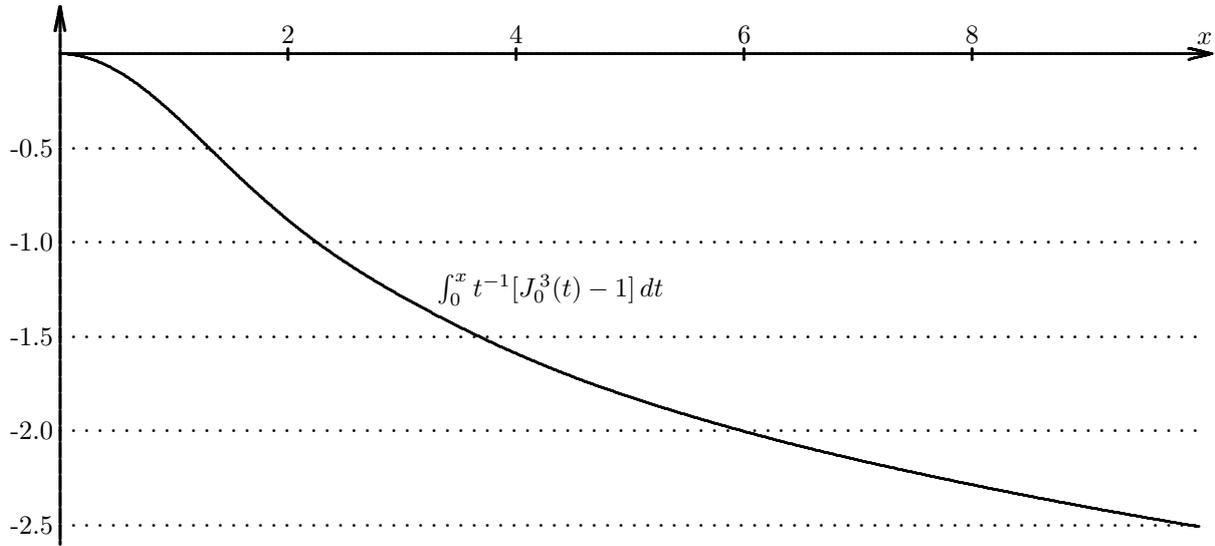
$$\int_0^x \frac{J_0^3(t)-1}{t}\,dt = -\frac{1}{4}\,x^3 + \frac{3}{64}\,x^5 - \frac{31}{5376}\,x^7 + \frac{71}{147456}\,x^9 - \frac{47}{1638400}\,x^{11} + \frac{11723}{9201254400}\,x^{13} - \frac{2021}{46242201600}\,x^{15} +$$

$$+ \frac{1567}{1315333734400}\,x^{17} - \frac{5773279}{218624250662092800}\,x^{19} + \frac{3125957}{6443662124777472000}\,x^{21} - \frac{1114457}{148511069923442688000}\,x^{23} + \dots$$

$$= -0.25x^3 + 0.046875x^5 - 0.00576\,63690\,47619x^7 + 0.00048\,14995\,65972x^9 -$$

$$-0.00002\,86865\,23438x^{11} + 0.00000\,12740\,65414x^{13} - 0.00000\,00437\,04667x^{15} + 0.00000\,00011\,91333x^{17} -$$

$$-0.00000\,00000\,26407x^{19} + 0.00000\,00000\,00485x^{21} - 0.00000\,00000\,00008x^{23} + \dots$$

With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x \frac{J_0^3(t)-1}{t}\,dt = \sum_{k=0}^{\infty} c_k^{(-)}\, T_{2k}\left(\frac{x}{8}\right) \ .$$

The first coefficients are

k	$c_k^{(-)}$	k	$c_k^{(-)}$
0	-1.66487772839693011416	15	-0.00000370566539255354
1	-0.85731855220711666821	16	0.00000057633802528230
2	0.35797423128412846686	17	-0.00000007814654021909
3	-0.18824568501714899999	18	0.00000000932468761665
4	0.10690149363496148802	19	-0.00000000098725238622
5	-0.06393636091495798502	20	0.00000000009342113704
6	0.03911001491993894893	21	-0.00000000000795204305
7	-0.02409344095904841252	22	0.00000000000061237408
8	0.01437795892387628520	23	-0.00000000000004288320
9	-0.00768362174411199843	24	0.00000000000000274344
10	0.00345368891420790184	25	-0.00000000000000016101
11	-0.00127440919485714551	26	0.00000000000000000870
12	0.00038620719490337840	27	-0.00000000000000000043
13	-0.00009714124269423777	28	0.00000000000000000002
14	0.00002055168104878868	-	-

The given approximation differs from the true function as shown in the following figure:

$$\bar{\Delta}^{(-)}(x) = \sum_{k=0}^{28} c_k^{(-)} T_{2k}(x/8) - \int_0^x t^{-1}[J_0^3(t) - 1]\, dt$$
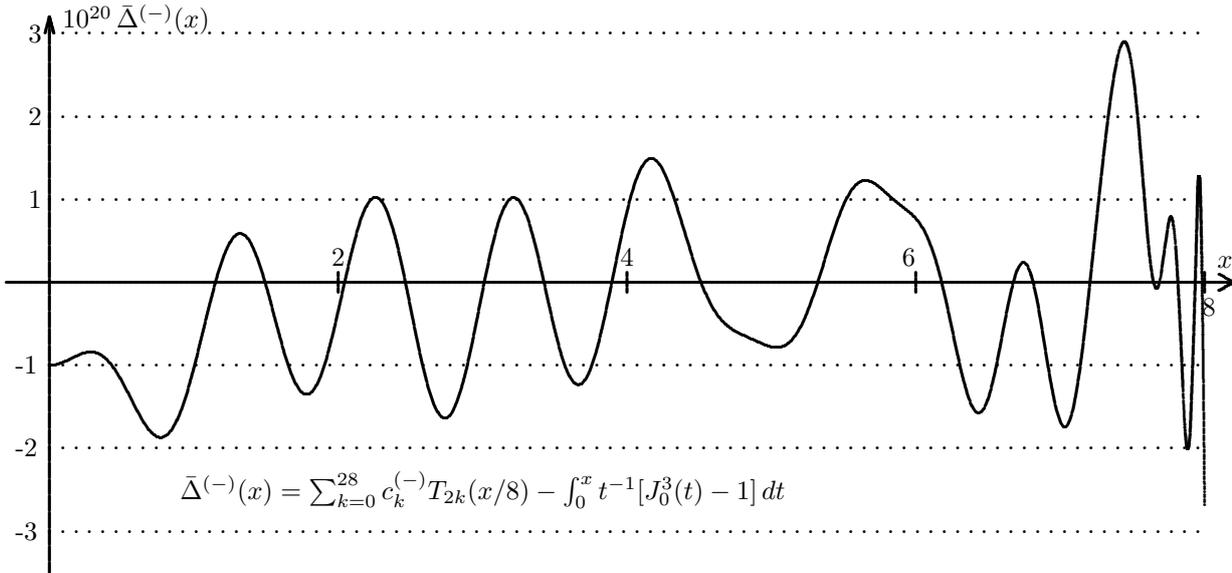
Asymptotic formula:

$$\int_1^x \frac{J_0^3(t)\, dt}{t} \sim 0.11548\ 77825\ 09057\ 98226\ \ldots\ +$$

$$+\sqrt{\frac{2}{\pi^3 x}} \sum_{k=1}^{\infty} \frac{1}{x^{k+1}} \left[a_k^{(01)} \sin\left(3x + \frac{7 - 2k}{4}\pi\right) + b_k^{(01)} \sin\left(x + \frac{1 - 2k}{4}\pi\right) \right]$$

with the first values

k	$a_k^{(01)}$	$a_k^{(01)}$
1	1/6	0.16666 66666 66666 66667
2	29/144	0.20138 88888 88888 88889
3	1921/6912	0.27792245370370370370
4	8527/18432	0.46261 93576 38888 88889
5	1621523/1769472	0.91638 80524 81192 12963
6	89993003/42467328	2.11911 14967 25200 13503
7	3821763071/679477248	5.62456 37101 89160 00555
8	275582100493/16307453952	16.89914 93892 39986 24700
9	177961856737289/3131031158784	56.83809 82852 95930 27586

k	$b_k^{(01)}$	$b_k^{(01)}$
1	3/2	1.50000 00000 00000 00000
2	63/16	3.93750 00000 00000 00000
3	3603/256	14.07421 87500 00000 00000
4	129981/2048	63.46728 51562 50000 00000
5	22909281/65536	349.56788 63525 39062 5000
6	1191489153/524288	2272.58520 69854 73632 813
7	143000089119/8388608	17046.93902 95743 94226 07
8	9724198215717/67108864	1 44901.84509 33247 80464 2
9	5912428624098201/4294967296	13 76594.56210 63469 44347

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The first consecutive maxima and minima of

$$\Delta_n^{(-1)}(x) = 0.115\ldots + \sqrt{\frac{2}{\pi^3 x}} \sum_{k=1}^{n} \frac{1}{x^{k+1}} \left[a_k^{(-1)} \sin(3x + \ldots) + b_k^{(-1)} \sin(x + \ldots) \right] - \int_1^x t^{-1} \cdot J_0^3(t)\, dt \quad :$$

i	x_i	$\Delta_1^{(-1)}(x_i)$	x_i	$\Delta_2^{(-1)}(x_i)$	x_i	$\Delta_3^{(-1)}(x_i)$	x_i	$\Delta_4^{(-1)}(x_i)$	x_i	$\Delta_5^{(-1)}(x_i)$
1	3.944	-5.284E-03	2.356	2.066E-02	3.929	3.477E-03	2.356	-5.340E-02	3.927	-5.296E-03
2	7.079	8.972E-04	5.498	-1.010E-03	7.070	-2.223E-04	5.498	6.258E-04	7.069	1.237E-04
3	10.217	-2.736E-04	8.639	1.668E-04	10.211	3.547E-05	8.639	-4.719E-05	10.210	-1.032E-05
4	13.357	1.120E-04	11.781	-4.561E-05	13.353	-8.883E-06	11.781	7.385E-06	13.352	1.589E-06
5	16.498	-5.477E-05	14.923	1.656E-05	16.494	2.923E-06	14.923	-1.732E-06	16.493	-3.531E-07
6	19.639	3.018E-05	18.064	-7.216E-06	19.635	-1.155E-06	18.064	5.264E-07	19.635	1.004E-07
7	22.780	-1.812E-05	21.206	3.571E-06	22.777	5.209E-07	21.206	-1.918E-07	22.777	-3.411E-08
8	25.921	1.160E-05	24.347	-1.941E-06	25.919	-2.594E-07	24.347	7.986E-08	25.918	1.324E-08
9	29.062	-7.806E-06	27.489	1.134E-06	29.060	1.396E-07	27.489	-3.685E-08	29.060	-5.709E-09
10	32.204	5.468E-06	30.631	-7.008E-07	32.202	-7.996E-08	30.631	1.844E-08	32.201	2.676E-09

i	x_i	$\Delta_6^{(-1)}(x_i)$	x_i	$\Delta_7^{(-1)}(x_i)$	x_i	$\Delta_8^{(-1)}(x_i)$	x_i	$\Delta_9^{(-1)}(x_i)$
1	2.356	2.837E-01	3.927	1.495E-02	2.356	-2.685E+00	3.927	-6.868E-02
2	5.498	-7.418E-04	7.069	-1.227E-04	5.498	1.470E-03	7.069	1.915E-04
3	8.639	2.526E-05	10.210	5.307E-06	8.639	-2.218E-05	10.210	-4.245E-06
4	11.781	-2.264E-06	13.352	-5.014E-07	11.781	1.134E-06	13.352	2.455E-07
5	14.923	3.441E-07	16.493	7.538E-08	14.923	-1.116E-07	16.493	-2.496E-08
6	18.064	-7.318E-08	19.635	-1.546E-08	18.064	1.663E-08	19.635	3.692E-09
7	21.206	1.968E-08	22.777	3.961E-09	21.206	-3.305E-09	22.777	-7.143E-10
8	24.347	-6.291E-09	25.918	-1.201E-09	24.347	8.122E-10	25.918	1.692E-10
9	27.489	2.297E-09	29.060	4.150E-10	27.489	-2.350E-10	29.060	-4.695E-11
10	30.631	-9.319E-10	32.201	-1.594E-10	30.631	7.734E-11	32.201	1.479E-11

In the case $x \geq 8$ one has $g_n^{(-1)} \leq \Delta_n^{(-1)}(x) \leq G_n^{(-1)}$ with the following values:

n	$g_n^{(-1)}$	$G_n^{(-1)}$	n	$g_n^{(-1)}$	$G_n^{(-1)}$
1	-2.736E-04	4.984E-04	6	-2.264E-06	2.526E-05
2	-4.561E-05	1.668E-04	7	-6.249E-05	5.307E-06
3	-1.301E-04	3.547E-05	8	-2.218E-05	1.134E-06
4	-4.719E-05	7.385E-06	9	-4.245E-06	8.772E-05
5	-1.032E-05	6.871E-05	-	-	-

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{-1}(x) = 0.115\ldots + \sum_{k=1}^{9} \frac{1}{x^{k+3/2}} \left[\tilde{a}_k^{(-1)} \sin\left(3x + \frac{7-2k}{4}\pi\right) + \tilde{b}_k^{(-1)} \sin\left(x + \frac{1-2k}{4}\pi\right) \right] \quad .$$

The values of the coefficients are

k	$\tilde{a}_k^{(-1)}$	$\tilde{b}_k^{(-1)}$
1	0.042328265593	0.380960857219
2	0.051124895140	0.999204230732
3	0.069839136706	3.573795779326
4	0.108172780150	15.423626931601
5	-0.035881146333	86.734624679653
6	-0.183965343668	370.667415902952
7	-39.511920554495	3145.668270195769
8	23.037919369082	5512.467112432406
9	-1996.507320270070	73698.065312438298

With $8 \leq x \leq 30$ holds

$$-2.0 \cdot 10^{-9} \;\leq\; F_{-1}(x) - \int_1^x t^{-1} \cdot J_0^3(t)\, dt \;\leq\; 3.2 \cdot 10^{-9} \quad .$$

Power series for the modified Bessel function:

$$\int_0^x \frac{I_0^3(t)-1}{t}\,dt = \sum_{k=1}^{\infty} d_k^{(-1)}\,x^{2k} = \frac{3}{8}\,x^2 + \frac{15}{256}\,x^4 + \frac{31}{4608}\,x^6 + \frac{71}{131072}\,x^8 + \frac{517}{16384000}\,x^{10} + \frac{11723}{8493465600}\,x^{12} + \cdots$$

From this:

$$\int_1^x I_0^3(t)\,dt = \ln|x| - 0.44089\ 58511\ 01198\ 85318 + \sum_{k=1}^{\infty} d_k^{(-1)}\,x^{2k}$$

With $n \geq 1$ the following recurrence relation holds:

$$d_{n+1}^{(-1)} = \frac{1}{24(n+1)^3}\left[16\sum_{k=1}^{n-1} k(n+1-k)(3n^2+5k^2-8kn+5n-6k+2)d_k^{(-1)}d_{n+1-k}^{(-1)} - \right.$$

$$\left. -36\sum_{k=1}^{n-1} k(n-k)d_k^{(-1)}d_{n-k}^{(-1)} - 36n\,d_n^{(-1)}\right]$$

Asymptotic formula for the modified Bessel function:

$$\int_1^x \frac{I_0^3(t)\,dt}{t} \sim \frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3\,x}}\sum_{k=1}^{\infty}\frac{c_k}{x^{k+1}}$$

with the first values

k	c_k	c_k
1	1/12	0.08333 33333 33333 3
2	29/288	0.10069 44444 44444 4
3	1921/13824	0.13896 12268 51851 9
4	8527/36864	0.2313 09678 81944 4
5	1621523/3538944	0.45819 40262 40596 1
6	89993003/84934656	1.05955 57483 62600 1
7	3821763071/1358954496	2.81228 18550 94580 0
8	275582100493/32614907904	8.44957 46946 19993 1
9	177961856737289/6262062317568	28.41904 91426 47965
10	5312592054074687/50096498540544	106.04717 31327 7115

Let

$$\delta_n(x) = \left[\frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3\,x}}\sum_{k=1}^{n}\frac{c_k}{x^{k+1}} - \frac{\sqrt{2}\,e^3}{\sqrt{\pi^3}}\sum_{k=1}^{n}c_k\right]\cdot\left[\int_1^x t^{-1}I_0^3(t)\,dt\right]^{-1} - 1\,,$$

then one has the following values of $\delta_n(x)$:

n	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
1	-2.591E-01	-1.236E-01	-8.166E-02	-6.101E-02	-4.870E-02
2	-8.036E-02	-1.768E-02	-7.680E-03	-4.277E-03	-2.722E-03
3	-3.135E-02	-3.069E-03	-8.738E-04	-3.624E-04	-1.838E-04
4	-1.560E-02	-6.368E-04	-1.185E-04	-3.658E-05	-1.478E-05
5	-1.048E-02	-1.549E-04	-1.881E-05	-4.316E-06	-1.388E-06
6	-1.070E-02	-4.350E-05	-3.431E-06	-5.850E-07	-1.498E-07
7	-1.769E-02	-1.394E-05	-7.097E-07	-8.985E-08	-1.831E-08
8	-4.254E-02	-5.103E-06	-1.647E-07	-1.547E-08	-2.507E-09
9	-1.287E-01	-2.292E-06	-4.254E-08	-2.958E-09	-3.813E-10
10	-4.521E-01	-1.839E-06	-1.214E-08	-6.242E-10	-6.393E-11

e) Integrals of the type $\int x^n Z_0^3(x)\,dx$:

$$\int x\, J_0^3(x)\,dx = x\, J_0^2(x)\, J_1(x) + \frac{2x}{3}\, J_1^3(x) + \frac{4}{3}\int J_1^3(x)\,dx$$

$$\int x\, I_0^3(x)\,dx = x\, I_0^2(x)\, I_1(x) - \frac{2x}{3}\, I_1^3(x) - \frac{4}{3}\int I_1^3(x)\,dx$$

$$\int x^2\, J_0^3(x)\,dx =$$

$$= -\frac{x}{9}\, J_0^3(x) + x^2\, J_0^2(x)\, J_1(x) - \frac{2x}{3}\, J_0(x)\, J_1^2(x) + \frac{2x^2}{3}\, J_1^3(x) + \frac{1}{9}\int J_0^3(x)\,dx - \frac{2}{3}\int J_0(x)\, J_1^2(x)\,dx$$

$$\int x^2\, I_0^3(x)\,dx =$$

$$= \frac{x}{9}\, I_0^3(x) + x^2\, I_0^2(x)\, I_1(x) - \frac{2x}{3}\, I_0(x)\, I_1^2(x) - \frac{2x^2}{3}\, I_1^3(x) - \frac{1}{9}\int I_0^3(x)\,dx - \frac{2}{3}\int I_0(x)\, I_1^2(x)\,dx$$

$$\int x^3\, J_0^3(x)\,dx = \frac{2x^2}{3}\, J_0^3(x) + \frac{3x^3 - 4x}{3}\, J_0^2(x)\, J_1(x) + \frac{6x^3 - 8x}{9}\, J_1^3(x) - \frac{16}{9}\int J_1^3(x)\,dx$$

$$\int x^3\, I_0^3(x)\,dx = -\frac{2x^2}{3}\, I_0^3(x) + \frac{3x^3 + 4x}{3}\, I_0^2(x)\, I_1(x) - \frac{6x^3 + 8x}{9}\, I_1^3(x) - \frac{16}{9}\int I_1^3(x)\,dx$$

Let

$$\int x^n\, J_0^3(x)\,dx = \mathcal{P}_n(x)\, J_0^3(x) + \mathcal{Q}_n(x)\, J_0^2(x)\, J_1(x) + \mathcal{R}_n(x)\, J_0(x)\, J_1^2(x) + \mathcal{S}_n(x)\, J_1^3(x) +$$

$$+ \mathcal{U}_n \int J_0^3(x)\,dx + \mathcal{V}_n \int J_0(x)\, J_1(x)^2\,dx + \mathcal{W}_n \int J_1^3(x)\,dx + \mathcal{X}_n \int \frac{J_0^3(x)\,dx}{x}$$

and

$$\int x^n\, I_0^3(x)\,dx = \mathcal{P}_n^*(x)\, I_0^3(x) + \mathcal{Q}_n^*(x)\, I_0^2(x)\, I_1(x) + \mathcal{R}_n^*(x)\, I_0(x)\, I_1^2(x) + \mathcal{S}_n^*(x)\, I_1^3(x) +$$

$$+ \mathcal{U}_n^* \int I_0^3(x)\,dx + \mathcal{V}_n^* \int I_0(x)\, I_1(x)^2\,dx + \mathcal{W}_n^* \int I_1^3(x)\,dx + \mathcal{X}_n^* \int \frac{I_0^3(x)\,dx}{x} \; .$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_4(x) = \frac{39\, x^3 + 17\, x}{27}\,, \quad \mathcal{Q}_4(x) = \frac{3\, x^4 - 13\, x^2}{3}\,, \quad \mathcal{R}_4(x) = \frac{6\, x^3 + 28\, x}{9}\,, \quad \mathcal{S}_4(x) = \frac{6\, x^4 - 28\, x^2}{9}\,,$$

$$\mathcal{U}_4 = -\frac{17}{24}\,, \quad \mathcal{V}_4 = \frac{28}{9}\,, \quad \mathcal{W}_4 = 0\,.$$

$$\mathcal{P}_4^*(x) = \frac{-39\, x^3 + 17\, x}{27}\,, \quad \mathcal{Q}_4^*(x) = \frac{3\, x^4 + 13\, x^2}{3}\,, \quad \mathcal{R}_4^*(x) = \frac{6\, x^3 - 28\, x}{9}\,, \quad \mathcal{S}_4^*(x) = -\frac{6\, x^4 + 28\, x^2}{9}\,,$$

$$\mathcal{U}_4^* = -\frac{17}{27}\,, \quad \mathcal{V}_4^* = -\frac{28}{9}\,, \quad \mathcal{W}_0^* = 0$$

$$\mathcal{P}_5(x) = \frac{60\, x^4 - 160\, x^2}{27}\,, \quad \mathcal{Q}_5(x) = \frac{27\, x^5 - 240\, x^3 + 320\, x}{27}\,, \quad \mathcal{R}_5(x) = \frac{4x^4}{3}\,,$$

$$\mathcal{S}_5(x) = \frac{54\, x^5 - 552\, x^3 + 640\, x}{81}\,, \quad \mathcal{U}_5 = 0\,, \quad \mathcal{V}_5 = 0\,, \quad \mathcal{W}_5 = \frac{1280}{81}$$

$$\mathcal{P}_5^*(x) = -\frac{60\, x^4 + 160\, x^2}{27}\,, \quad \mathcal{Q}_5^*(x) = \frac{27\, x^5 + 240\, x^3 + 320\, x}{27}\,, \quad \mathcal{R}_5^*(x) = \frac{4\, x^4}{3}\,,$$

$$\mathcal{S}_5^*(x) = -\frac{54\, x^5 + 552\, x^3 + 640\, x}{81}\,, \quad \mathcal{U}_5^* = 0\,, \quad \mathcal{V}_5^* = 0\,, \quad \mathcal{W}_5^* = -\frac{1280}{81}$$

$$\mathcal{P}_6(x) = \frac{9\, x^5 - 69\, x^3 - 31\, x}{3}\,, \quad \mathcal{Q}_6(x) = x^6 - 15\, x^4 + 69\, x^2\,, \quad \mathcal{R}_6(x) = 2\, x^5 - 12\, x^3 - 50\, x\,,$$

$$\mathcal{S}_6(x) = \frac{2\, x^6 - 36\, x^4 + 150\, x^2}{3}\,, \quad \mathcal{U}_6 = \frac{31}{3}\,, \quad \mathcal{V}_6 = -50\,, \quad \mathcal{W}_6 = 0$$

$$\mathcal{P}_6^*(x) = \frac{-9\,x^5 - 69\,x^3 + 31\,x}{3}\,, \quad \mathcal{Q}_6^*(x) = x^6 + 15\,x^4 + 69\,x^2\,, \quad \mathcal{R}_6^*(x) = 2\,x^5 + 12\,x^3 - 50\,x\,,$$

$$\mathcal{S}_6^*(x) = -\frac{2\,x^6 + 36\,x^4 + 150\,x^2}{3}\,, \quad \mathcal{U}_6^* = -\frac{31}{3}\,, \quad \mathcal{V}_6^* = -50\,, \quad \mathcal{W}_6^* = 0$$

$$\mathcal{P}_7(x) = \frac{102\,x^6 - 1488\,x^4 + 3968\,x^2}{27}\,, \quad \mathcal{Q}_7(x) = \frac{27\,x^7 - 612\,x^5 + 5952\,x^3 - 7936\,x}{27}\,,$$

$$\mathcal{R}_7(x) = \frac{8\,x^6 - 112\,x^4}{3}\,, \quad \mathcal{S}_7(x) = \frac{54\,x^7 - 1512\,x^5 + 13920\,x^3 - 15872\,x}{81}\,,$$

$$\mathcal{U}_7 = 0\,, \quad \mathcal{V}_7 = 0\,, \quad \mathcal{W}_7 = -\frac{31744}{81}$$

$$\mathcal{P}_7^*(x) = -\frac{102\,x^6 + 1488\,x^4 + 3968\,x^2}{27}\,, \quad \mathcal{Q}_7^*(x) = \frac{27\,x^7 + 612\,x^5 + 5952\,x^3 + 7936\,x}{27}\,,$$

$$\mathcal{R}_7^*(x) = \frac{8\,x^6 + 112\,x^4}{3}\,, \quad \mathcal{S}_7^*(x) = -\frac{54\,x^7 + 1512\,x^5 + 13920\,x^3 + 15872\,x}{81}\,,$$

$$\mathcal{U}_7^* = 0\,, \quad \mathcal{V}_7^* = 0\,, \quad \mathcal{W}_7^* = -\frac{31744}{81}\,,$$

$$\mathcal{P}_8(x) = \frac{1107\,x^7 - 25947\,x^5 + 200427\,x^3 + 90373\,x}{243}\,, \quad \mathcal{Q}_8(x) = \frac{27\,x^8 - 861\,x^6 + 14415\,x^4 - 66809\,x^2}{27}\,,$$

$$\mathcal{R}_8(x) = \frac{270\,x^7 - 6516\,x^5 + 35346\,x^3 + 145400\,x}{81}\,, \quad \mathcal{S}_8(x) = \frac{54\,x^8 - 2172\,x^6 + 35346\,x^4 - 145400\,x^2}{81}$$

$$\mathcal{U}_8 = -\frac{90373}{243}\,, \quad \mathcal{V}_8 = \frac{145400}{81}\,, \quad \mathcal{W}_8 = 0$$

$$\mathcal{P}_8^*(x) = \frac{-1107\,x^7 - 25947\,x^5 - 200427\,x^3 + 90373\,x}{243}\,, \quad \mathcal{Q}_8^*(x) = \frac{27\,x^8 + 861\,x^6 + 14415\,x^4 + 66809\,x^2}{27}\,,$$

$$\mathcal{R}_8^*(x) = \frac{270\,x^7 + 6516\,x^5 + 35346\,x^3 - 145400\,x}{81}\,, \quad \mathcal{S}_8^*(x) = -\frac{54\,x^8 + 2172\,x^6 + 35346\,x^4 + 145400\,x^2}{81}$$

$$\mathcal{U}_8^* = -\frac{90373}{243}\,, \quad \mathcal{V}_8^* = -\frac{145400}{81}\,, \quad \mathcal{W}_8^* = 0$$

$$\int \frac{J_0^3(x)\,dx}{x^2} = -\frac{J_0^3(x)}{x} + 3J_0^2(x)\,J_1(x) + 6\int J_0(x)\,J_1^2(x)\,dx - 3\int J_0^3(x)\,dx$$

$$\int \frac{I_0^3(x)\,dx}{x^2} = -\frac{I_0^3(x)}{x} - 3I_0^2(x)\,I_1(x) + 6\int I_0(x)\,I_1^2(x)\,dx + 3\int I_0^3(x)\,dx$$

$$\int \frac{J_0^3(x)\,dx}{x^3} = -\frac{x^2+1}{2x^2}\,J_0^3(x) + \frac{3}{4x}\,J_0^2(x)J_1(x) - \frac{3}{4}\,J_0(x)J_1^2(x) - \frac{3}{4}\int \frac{J_0^3(x)\,dx}{x} - \frac{3}{4}\int J_1^3(x)\,dx$$

$$\int \frac{I_0^3(x)\,dx}{x^3} = -\frac{x^2-1}{2x^2}\,I_0^3(x) - \frac{3}{4x}\,I_0^2(x)I_1(x) - \frac{3}{4}\,I_0(x)i_1^2(x) + \frac{3}{4}\int \frac{I_0^3(x)\,dx}{x} + \frac{3}{4}\int I_1^3(x)\,dx$$

$$\int \frac{J_0^3(x)\,dx}{x^4} = \frac{x^2-1}{3x^3}\,J_0^3(x) - \frac{13\,x^2-3}{9x^2}\,J_0^2(x)\,J_1(x) - \frac{2}{9x}\,J_0(x)\,J_1^2(x) + \frac{2}{27}\,J_1^3(x) +$$

$$+ \frac{13}{9}\int J_0^3(x)\,dx - \frac{28}{9}\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0^3(x)\,dx}{x^4} = -\frac{x^2+1}{3x^3}\,J_0^3(x) - \frac{13\,x^2+3}{9x^2}\,I_0^2(x)\,I_1(x) - \frac{2}{9x}\,I_0(x)\,I_1^2(x) - \frac{2}{27}\,I_1^3(x) +$$

$$+ \frac{13}{9}\int I_0^3(x)\,dx + \frac{28}{9}\int I_0(x)\,I_1^2(x)\,dx$$

$$\mathcal{P}_{-5}(x) = \frac{23\,x^4 + 12\,x^2 - 32}{128\,x^4}\,, \quad \mathcal{Q}_{-5}(x) = \frac{-15\,x^2 + 12}{64\,x^3}\,, \quad \mathcal{R}_{-5}(x) = \frac{69\,x^2 - 24}{256\,x^2}\,, \quad \mathcal{S}_{-5}(x) = \frac{3}{128x}$$

$$\mathcal{U}_{-5} = \frac{15}{64}\,, \quad \mathcal{V}_{-5} = 0\,, \quad \mathcal{W}_{-5} = 0\,, \quad \mathcal{X}_{-5} = \frac{69}{256}$$

$$\mathcal{P}^*_{-5}(x) = \frac{23\,x^4 - 12\,x^2 - 32}{128\,x^4}\,, \quad \mathcal{Q}^*_{-5}(x) = \frac{-15\,x^2 - 12}{64\,x^3}\,, \quad \mathcal{R}^*_{-5}(x) = \frac{-69\,x^2 - 24}{256\,x^2}\,, \quad \mathcal{S}^*_{-5}(x) = \frac{-3}{128x}$$

$$\mathcal{U}^*_{-5} = \frac{15}{64}\,, \quad \mathcal{V}^*_{-5} = 0\,, \quad \mathcal{W}^*_{-5} = 0\,, \quad \mathcal{X}^*_{-5} = \frac{69}{256}$$

$$\mathcal{P}_{-6}(x) = \frac{-9\,x^4 + 5\,x^2 - 25}{125\,x^5}\,, \quad \mathcal{Q}_{-6}(x) = \frac{207\,x^4 - 45\,x^2 + 75}{625\,x^4}\,, \quad \mathcal{R}_{-6}(x) = \frac{36\,x^2 - 30}{625\,x^3}\,,$$

$$\mathcal{S}_{-6}(x) = \frac{-12\,x^2 + 6}{625\,x^2}\,, \quad \mathcal{U}_{-6} = -\frac{207}{625}\,, \quad \mathcal{V}_{-6} = \frac{18}{25}\,, \quad \mathcal{W}_{-6} = 0$$

$$\mathcal{P}^*_{-6}(x) = -\frac{9\,x^4 + 5\,x^2 + 25}{125\,x^5}\,, \quad \mathcal{Q}^*_{-6}(x) = -\frac{207\,x^4 + 45\,x^2 + 75}{625\,x^4}\,, \quad \mathcal{R}^*_{-6}(x) = -\frac{36\,x^2 + 30}{625\,x^3}\,,$$

$$\mathcal{S}^*_{-6}(x) = -\frac{12\,x^2 + 6}{625\,x^2}\,, \quad \mathcal{U}^*_{-6} = \frac{207}{625}\,, \quad \mathcal{V}^*_{-6} = \frac{18}{25}\,, \quad \mathcal{W}^*_{-6} = 0$$

$$\mathcal{P}_{-7}(x) = \frac{-145\,x^6 - 68\,x^4 + 96\,x^2 - 768}{4608\,x^6}\,, \quad \mathcal{Q}_{-7}(x) = \frac{93\,x^4 - 68\,x^2 + 192}{2304\,x^5}\,,$$

$$\mathcal{R}_{-7}(x) = \frac{-435\,x^4 + 168\,x^2 - 256}{9216\,x^4}\,, \quad \mathcal{S}_{-7}(x) = \frac{-63\,x^2 + 64}{13824\,x^3}$$

$$\mathcal{U}_{-7} = -\frac{31}{768}\,, \quad \mathcal{V}_{-7} = 0\,, \quad \mathcal{W}_{-7} = 0\,, \quad \mathcal{X}_{-7} = -\frac{145}{3072}$$

$$\mathcal{P}^*_{-7}(x) = \frac{145\,x^6 - 68\,x^4 - 96\,x^2 - 768}{4608\,x^6}\,, \quad \mathcal{Q}^*_{-7}(x) = \frac{-93\,x^4 - 68\,x^2 - 192}{2304\,x^5}\,,$$

$$\mathcal{R}^*_{-7}(x) = \frac{-435\,x^4 - 168\,x^2 - 256}{9216\,x^4}\,, \quad \mathcal{S}^*_{-7}(x) = \frac{-63\,x^2 - 64}{13824\,x^3}$$

$$\mathcal{U}^*_{-7} = \frac{31}{768}\,, \quad \mathcal{V}^*_{-7} = 0\,, \quad \mathcal{W}^*_{-7} = 0\,, \quad \mathcal{X}^*_{-7} = \frac{145}{3072}$$

$$\mathcal{P}_{-8}(x) = \frac{2883\,x^6 - 1435\,x^4 + 3675\,x^2 - 42875}{300125\,x^7}\,, \quad \mathcal{Q}_{-8}(x) = \frac{-66809\,x^6 + 14415\,x^4 - 21525\,x^2 + 91875}{1500625\,x^6}\,,$$

$$\mathcal{R}_{-8}(x) = \frac{-11782\,x^4 + 10860\,x^2 - 26250}{1500625\,x^5}\,, \quad \mathcal{S}_{-8}(x) = \frac{11782\,x^4 - 6516\,x^2 + 11250}{4501875\,x^4}\,,$$

$$\mathcal{U}_{-8} = \frac{66809}{1500625}\,, \quad \mathcal{V}_{-8} = \frac{5816}{60025}\,, \quad \mathcal{W}_{-8} = 0$$

$$\mathcal{P}^*_{-8}(x) = -\frac{2883\,x^6 + 1435\,x^4 + 3675\,x^2 + 42875}{300125\,x^7}\,, \quad \mathcal{Q}^*_{-8}(x) = -\frac{66809\,x^6 + 14415\,x^4 + 21525\,x^2 + 91875}{1500625\,x^6}\,,$$

$$\mathcal{R}^*_{-8}(x) = -\frac{11782\,x^4 + 10860\,x^2 + 26250}{1500625\,x^5}\,, \quad \mathcal{S}^*_{-8}(x) = -\frac{11782\,x^4 + 6516\,x^2 + 11250}{4501875\,x^4}\,,$$

$$\mathcal{U}^*_{-8} = \frac{66809}{1500625}\,, \quad \mathcal{V}^*_{-8} = -\frac{5816}{60025}\,, \quad \mathcal{W}^*_{-8} = 0$$

<u>f) Integrals of the type $\int x^n Z_0^2(x) Z_1(x)\,dx$:</u>

$$\int J_0^2(x)\,J_1(x)\,dx = -\frac{1}{3}\,J_0^3(x)$$

$$\int I_0^2(x)\,I_1(x)\,dx = \frac{1}{3}\,I_0^3(x)$$

$$\int x\,J_0^2(x)\,J_1(x)\,dx = -\frac{x}{3}\,J_0^3(x) + \frac{1}{3}\int J_0^3(x)\,dx$$

$$\int x\,I_0^2(x)\,I_1(x)\,dx = \frac{x}{3}\,I_0^3(x) - \frac{1}{3}\int I_0^3(x)\,dx$$

$$\int x^2\,J_0^2(x)\,J_1(x)\,dx = -\frac{x^2}{3}\,J_0^3(x) + \frac{2x}{3}\,J_0^2(x)\,J_1(x) + \frac{4x}{9}\,J_1^3(x) + \frac{8}{9}\int J_1^3(x)\,dx$$

$$\int x^2\,I_0^2(x)\,I_1(x)\,dx = \frac{x^2}{3}\,I_0^3(x) - \frac{2x}{3}\,I_0^2(x)\,I_1(x) + \frac{4x}{9}\,I_1^3(x) + \frac{8}{9}\int I_1^3(x)\,dx$$

$$\int x^3\,J_0^2(x)\,J_1(x)\,dx = -\frac{3x^3 + x}{9}\,J_0^3(x) + x^2\,J_0^2(x)\,J_1(x) - \frac{2x}{3}\,J_0(x)\,J_1^2(x) + \frac{2x^2}{3}\,J_1^3(x) +$$

$$+ \frac{1}{9}\int J_0^3(x)\,dx - \frac{2}{3}\int J_0(x)\,J_1^2(x)\,dx$$

$$\int x^3\,I_0^2(x)\,I_1(x)\,dx = \frac{3x^3 - x}{9}\,I_0^3(x) - x^2\,I_0^2(x)\,I_1(x) + \frac{2x}{3}\,I_0(x)\,I_1^2(x) + \frac{2x^2}{3}\,I_1^3(x) +$$

$$+ \frac{1}{9}\int I_0^3(x)\,dx + \frac{2}{3}\int I_0(x)\,I_1^2(x)\,dx$$

$$\int x^4\,J_0^2(x)\,J_1(x)\,dx = \frac{-3x^4 + 8x^2}{9}\,J_0^3(x) + \frac{12x^3 - 16x}{9}\,J_0^2(x)\,J_1(x) + \frac{24x^3 - 32x}{27}\,J_1^3(x) - \frac{64}{27}\int J_1^3(x)\,dx$$

$$\int x^4\,I_0^2(x)\,I_1(x)\,dx = \frac{3x^4 + 8x^2}{9}\,I_0^3(x) - \frac{12x^3 + 16x}{9}\,I_0^2(x)\,I_1(x) + \frac{24x^3 + 32x}{27}\,I_1^3(x) + \frac{64}{27}\int I_1^3(x)\,dx$$

Let

$$\int x^n\,J_0^2(x)\,J_1(x)\,dx = \mathcal{P}_n(x)\,J_0^3(x) + \mathcal{Q}_n(x)\,J_0^2(x)\,J_1(x) + \mathcal{R}_n(x)\,J_0(x)\,J_1^2(x) + \mathcal{S}_n(x)\,J_1^3(x) +$$

$$+ \mathcal{U}_n\int J_0^3(x)\,dx + \mathcal{V}_n\int J_0(x)\,J_1(x)^2\,dx + \mathcal{W}_n\int J_1(x)^3\,dx + \mathcal{X}_n\int \frac{J_0^3(x)\,dx}{x}$$

and

$$\int x^n\,I_0^2(x)\,I_1(x)\,dx = \mathcal{P}_n^*(x)\,I_0^3(x) + \mathcal{Q}_n^*(x)\,I_0^2(x)\,I_1(x) + \mathcal{R}_n^*(x)\,I_0(x)\,I_1^2(x) + \mathcal{S}_n^*(x)\,I_1^3(x) +$$

$$+ \mathcal{U}_n^*\int I_0^3(x)\,dx + \mathcal{V}_n^*\int I_0(x)\,I_1(x)^2\,dx + \mathcal{W}_n^*\int I_1(x)^3\,dx + \mathcal{X}_n^*\int \frac{I_0^3(x)\,dx}{x}\;.$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_5(x) = \frac{-27\,x^5 + 195\,x^3 + 85\,x}{81}\;,\quad \mathcal{Q}_5(x) = \frac{15\,x^4 - 65\,x^2}{9}\;,\quad \mathcal{R}_5(x) = \frac{30\,x^3 + 140\,x}{27}\;,$$

$$\mathcal{S}_5(x) = \frac{30\,x^4 - 140\,x^2}{27}\;,\quad \mathcal{U}_5 = -\frac{85}{81}\;,\quad \mathcal{V}_5 = \frac{140}{27}\;,\quad \mathcal{W}_5 = 0$$

$$\mathcal{P}_5^*(x) = \frac{27\,x^5 + 195\,x^3 - 85\,x}{81}\;,\quad \mathcal{Q}_5^*(x) = -\frac{15\,x^4 + 65\,x^2}{9}\;,\quad \mathcal{R}_5^*(x) = \frac{-30\,x^3 + 140\,x}{27}\;,$$

$$\mathcal{S}_5^*(x) = \frac{30\,x^4 + 140\,x^2}{27}\;,\quad \mathcal{U}_5^* = \frac{85}{81}\;,\quad \mathcal{V}_5^* = \frac{140}{27}\;,\quad \mathcal{W}_5^* = 0$$

$$\mathcal{P}_6(x) = \frac{-9\,x^6 + 120\,x^4 - 320\,x^2}{27}\ , \quad \mathcal{Q}_6(x) = \frac{54\,x^5 - 480\,x^3 + 640\,x}{27}\ , \quad \mathcal{R}_6(x) = \frac{8\,x^4}{3}\ ,$$

$$\mathcal{S}_6(x) = \frac{108\,x^5 - 1104\,x^3 + 1280\,x}{81}\ , \quad \mathcal{U}_6 = 0\ , \quad \mathcal{V}_6 = 0\ , \quad \mathcal{W}_6 = \frac{2560}{81}$$

$$\mathcal{P}_6^*(x) = \frac{9\,x^6 + 120\,x^4 + 320\,x^2}{27}\ , \quad \mathcal{Q}_6^*(x) = -\frac{54\,x^5 + 480\,x^3 + 640\,x}{9}\ , \quad \mathcal{R}_6^*(x) = -\frac{8x^4}{3}\ ,$$

$$\mathcal{S}_6^*(x) = \frac{108\,x^5 + 1104\,x^3 + 1280\,x}{81}\ , \quad \mathcal{U}_6^* = 0\ , \quad \mathcal{V}_6^* = 0\ , \quad \mathcal{W}_6^* = \frac{2560}{81}$$

$$\mathcal{P}_7(x) = \frac{-3\,x^7 + 63\,x^5 - 483\,x^3 - 217\,x}{9}\ , \quad \mathcal{Q}_7(x) = \frac{7\,x^6 - 105\,x^4 + 483\,x^2}{3}\ ,$$

$$\mathcal{R}_7(x) = \frac{14\,x^5 - 84\,x^3 - 350\,x}{3}\ , \quad \mathcal{S}_7(x) = \frac{14\,x^6 - 252\,x^4 + 1050\,x^2}{9}\ ,$$

$$\mathcal{U}_7 = \frac{217}{9}\ , \quad \mathcal{V}_7 = -\frac{350}{3}\ , \quad \mathcal{W}_7 = 0$$

$$\mathcal{P}_7^*(x) = \frac{3\,x^7 + 63\,x^5 + 483\,x^3 - 217\,x}{9}\ , \quad \mathcal{Q}_7^*(x) = -\frac{7\,x^6 + 105\,x^4 + 483\,x^2}{3}\ ,$$

$$\mathcal{R}_7^*(x) = \frac{-14\,x^5 - 84\,x^3 + 350\,x}{3}\ , \quad \mathcal{S}_7^*(x) = \frac{14\,x^6 + 252\,x^4 + 1050\,x^2}{9}\ ,$$

$$\mathcal{U}_7^* = \frac{217}{9}\ , \quad \mathcal{V}_7^* = \frac{350}{3}\ , \quad \mathcal{W}_7^* = 0$$

$$\mathcal{P}_8(x) = \frac{-27\,x^8 + 816\,x^6 - 11904\,x^4 + 31744\,x^2}{81}\ , \quad \mathcal{Q}_8(x) = \frac{216\,x^7 - 4896\,x^5 + 47616\,x^3 - 63488\,x}{81}\ ,$$

$$\mathcal{R}_8(x) = \frac{64\,x^6 - 896\,x^4}{9}\ , \quad \mathcal{S}_8(x) = \frac{432\,x^7 - 12096\,x^5 + 111360\,x^3 - 126976\,x}{243}\ ,$$

$$\mathcal{U}_8 = 0\ , \quad \mathcal{V}_8 = 0\ , \quad \mathcal{W}_8 = -\frac{253952}{243}$$

$$\mathcal{P}_8^*(x) = \frac{27\,x^8 + 816\,x^6 + 11904\,x^4 + 31744\,x^2}{81}\ , \quad \mathcal{Q}_8^*(x) = -\frac{216\,x^7 + 4896\,x^5 + 47616\,x^3 + 63488\,x}{81}\ ,$$

$$\mathcal{R}_8^*(x) = -\frac{64\,x^6 + 896\,x^4}{9}\ , \quad \mathcal{S}_8^*(x) = \frac{432\,x^7 + 12096\,x^5 + 111360\,x^3 + 126976\,x}{243}\ ,$$

$$\mathcal{U}_8^* = 0\ , \quad \mathcal{V}_8^* = 0\ , \quad \mathcal{W}_8^* = \frac{253952}{243}$$

$$\int \frac{J_0^2(x)\,J_1(x)\,dx}{x} = -J_0^2(x)\,J_1(x) + \int J_0^3(x)\,dx - 2\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0^2(x)\,I_1(x)\,dx}{x} = -I_0^2(x)\,I_1(x) + \int I_0^3(x)\,dx + 2\int I_0(x)\,I_1^2(x)\,dx$$

$$\int \frac{J_0^2(x)J_1(x)\,dx}{x^2} = \frac{1}{3}\,J_0^3(x) - \frac{1}{2x}\,J_0^2(x)J_1(x) + \frac{1}{2}\,J_0(x)J_1^2(x) + \frac{1}{2}\int J_1^3(x)\,dx + \frac{1}{2}\int \frac{J_0^3(x)\,dx}{x}$$

$$\int \frac{I_0^2(x)I_1(x)\,dx}{x^2} = \frac{1}{3}\,I_0^3(x) - \frac{1}{2x}\,I_0^2(x)I_1(x) - \frac{1}{2}\,I_0(x)I_1^2(x) + \frac{1}{2}\int I_1^3(x)\,dx + \frac{1}{2}\int \frac{I_0^3(x)\,dx}{x}$$

$$\int \frac{J_0^2(x)\,J_1(x)\,dx}{x^3} = -\frac{1}{3x}\,J_0^3(x) + \frac{13x^2 - 3}{9x^2}\,J_0^2(x)\,J_1(x) + \frac{2}{9x}\,J_0(x)\,J_1^2(x) - \frac{2}{27}\,J_1^3(x) - $$
$$-\frac{13}{9}\int J_0^3(x)\,dx + \frac{28}{9}\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0^2(x)\,I_1(x)\,dx}{x^3} = -\frac{1}{3x}\,I_0^3(x) - \frac{13x^2 + 3}{9x^2}\,I_0^2(x)\,I_1(x) - \frac{2}{9x}\,I_0(x)\,I_1^2(x) - \frac{2}{27}\,I_1^3(x) + $$
$$+\frac{13}{9}\int I_0^3(x)\,dx + \frac{28}{9}\int I_0(x)\,I_1^2(x)\,dx$$

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$$\mathcal{P}_{-4}(x) = \frac{-23\,x^2 - 12}{96\,x^2} \;, \quad \mathcal{Q}_{-4}(x) = \frac{5\,x^2 - 4}{16\,x^3} \;, \quad \mathcal{R}_{-4}(x) = \frac{-23\,x^2 + 8}{64\,x^2} \;, \quad \mathcal{S}_{-4}(x) = -\frac{1}{32x} \;,$$

$$\mathcal{U}_{-4} = 0 \;, \quad \mathcal{V}_{-4} = 0 \;, \quad \mathcal{W}_{-4} = -\frac{23}{64} \;, \quad \mathcal{X}_{-4} = -\frac{5}{16}$$

$$\mathcal{P}^*_{-4}(x) = \frac{23\,x^2 - 12}{96\,x^2} \;, \quad \mathcal{Q}^*_{-4}(x) = \frac{-5\,x^2 - 4}{16\,x^3} \;, \quad \mathcal{R}^*_{-4}(x) = \frac{-23\,x^2 - 8}{64\,x^2} \;, \quad \mathcal{S}^*_{-4}(x) = -\frac{1}{32x} \;,$$

$$\mathcal{U}^*_{-4} = 0 \;, \quad \mathcal{V}^*_{-4} = 0 \;, \quad \mathcal{W}^*_{-4} = \frac{23}{64} \;, \quad \mathcal{X}^*_{-4} = \frac{5}{16}$$

$$\mathcal{P}_{-5}(x) = \frac{9\,x^2 - 5}{75\,x^3} \;, \quad \mathcal{Q}_{-5}(x) = \frac{-69\,x^4 + 15\,x^2 - 25}{125\,x^4} \;, \quad \mathcal{R}_{-5}(x) = \frac{-12\,x^2 + 10}{125\,x^3} \;,$$

$$\mathcal{S}_{-5}(x) = \frac{4\,x^2 - 2}{125\,x^2} \;, \quad \mathcal{U}_{-5} = \frac{69}{125} \;, \quad \mathcal{V}_{-5} = -\frac{6}{5} \;, \quad \mathcal{W}_{-5} = 0$$

$$\mathcal{P}^*_{-5}(x) = -\frac{9\,x^2 + 5}{75\,x^3} \;, \quad \mathcal{Q}^*_{-5}(x) = -\frac{69\,x^4 + 15\,x^2 + 25}{125\,x^4} \;, \quad \mathcal{R}^*_{-5}(x) = -\frac{12\,x^2 + 10}{125\,x^3} \;,$$

$$\mathcal{S}^*_{-5}(x) = -\frac{4\,x^2 + 2}{125\,x^2} \;, \quad \mathcal{U}^*_{-5} = \frac{69}{125} \;, \quad \mathcal{V}^*_{-5} = \frac{6}{5} \;, \quad \mathcal{W}^*_{-5} = 0$$

$$\mathcal{P}_{-6}(x) = \frac{145\,x^4 + 68\,x^2 - 96}{2304\,x^4} \;, \quad \mathcal{Q}_{-6}(x) = \frac{-93\,x^4 + 68\,x^2 - 192}{1152\,x^5} \;, \quad \mathcal{R}_{-6}(x) = \frac{435\,x^4 - 168\,x^2 + 256}{4608\,x^4} \;,$$

$$\mathcal{S}_{-6}(x) = \frac{63\,x^2 - 64}{6912\,x^3} \;, \quad \mathcal{U}_{-6} = 0 \;, \quad \mathcal{V}_{-6} = 0 \;, \quad \mathcal{W}_{-6} = \frac{145}{1536} \;, \quad \mathcal{X}_{-6} = \frac{31}{384}$$

$$\mathcal{P}^*_{-6}(x) = \frac{145\,x^4 - 68\,x^2 - 96}{2304\,x^4} \;, \quad \mathcal{Q}^*_{-6}(x) = \frac{-93\,x^4 - 68\,x^2 - 192}{1152\,x^5} \;, \quad \mathcal{R}^*_{-6}(x) = \frac{-435\,x^4 - 168\,x^2 - 256}{4608\,x^4} \;,$$

$$\mathcal{S}^*_{-6}(x) = \frac{-63\,x^2 - 64}{6912\,x^3} \;, \quad \mathcal{U}^*_{-6} = 0 \;, \quad \mathcal{V}^*_{-6} = 0 \;, \quad \mathcal{W}^*_{-6} = \frac{145}{1536} \;, \quad \mathcal{X}^*_{-6} = \frac{31}{384}$$

$$\mathcal{P}_{-7}(x) = \frac{-2883\,x^4 + 1435\,x^2 - 3675}{128625\,x^5} \;, \quad \mathcal{Q}_{-7}(x) = \frac{66809\,x^6 - 14415\,x^4 + 21525\,x^2 - 91875}{643125\,x^6} \;,$$

$$\mathcal{R}_{-7}(x) = \frac{11782\,x^4 - 10860\,x^2 + 26250}{643125\,x^5} \;, \quad \mathcal{S}_{-7}(x) = \frac{-11782\,x^4 + 6516\,x^2 - 11250}{1929375\,x^4} \;,$$

$$\mathcal{U}_{-7} = -\frac{66809}{643125} \;, \quad \mathcal{V}_{-7} = \frac{5816}{25725} \;, \quad \mathcal{W}_{-7} = 0$$

$$\mathcal{P}^*_{-7}(x) = -\frac{2883\,x^4 + 1435\,x^2 + 3675}{128625\,x^5} \;, \quad \mathcal{Q}^*_{-7}(x) = -\frac{66809\,x^6 + 14415\,x^4 + 21525\,x^2 + 91875}{643125\,x^6} \;,$$

$$\mathcal{R}^*_{-7}(x) = -\frac{11782\,x^4 + 10860\,x^2 + 26250}{643125\,x^5} \;, \quad \mathcal{S}^*_{-7}(x) = -\frac{11782\,x^4 + 6516\,x^2 + 11250}{1929375\,x^4} \;,$$

$$\mathcal{U}^*_{-7} = \frac{66809}{643125} \;, \quad \mathcal{V}^*_{-7} = \frac{5816}{25725} \;, \quad \mathcal{W}^*_{-7} = 0$$

$$\mathcal{P}_{-8}(x) = \frac{-1331\,x^6 - 616\,x^4 + 768\,x^2 - 3072}{147456\,x^6} \;, \quad \mathcal{Q}_{-8}(x) = \frac{213\,x^6 - 154\,x^4 + 384\,x^2 - 2304}{18432\,x^7} \;,$$

$$\mathcal{R}_{-8}(x) = \frac{-3993\,x^6 + 1560\,x^4 - 2624\,x^2 + 9216}{294912\,x^6} \;, \quad \mathcal{S}_{-8}(x) = \frac{-585\,x^4 + 656\,x^2 - 1728}{442368\,x^5} \;,$$

$$\mathcal{U}_{-8} = 0 \;, \quad \mathcal{V}_{-8} = 0 \;, \quad \mathcal{W}_{-8} = -\frac{1331}{98304} \;, \quad \mathcal{X}_{-8} = -\frac{71}{6144}$$

$$\mathcal{P}^*_{-8}(x) = \frac{1331\,x^6 - 616\,x^4 - 768\,x^2 - 3072}{147456\,x^6} \;, \quad \mathcal{Q}^*_{-8}(x) = \frac{-213\,x^6 - 154\,x^4 - 384\,x^2 - 2304}{18432\,x^7} \;,$$

$$\mathcal{R}^*_{-8}(x) = \frac{-3993\,x^6 - 1560\,x^4 - 2624\,x^2 - 9216}{294912\,x^6} \;, \quad \mathcal{S}^*_{-8}(x) = \frac{-585\,x^4 - 656\,x^2 - 1728}{442368\,x^5} \;,$$

$$\mathcal{U}^*_{-8} = 0 \;, \quad \mathcal{V}^*_{-8} = 0 \;, \quad \mathcal{W}^*_{-8} = \frac{1331}{98304} \;, \quad \mathcal{X}^*_{-8} = \frac{71}{6144}$$

g) Integrals of the type $\int x^n \, Z_0(x) \, Z_1^2(x) \, dx$:

$$\int x \, J_0(x) \, J_1^2(x) \, dx = \frac{x}{3} \, J_1^3(x) + \frac{2}{3} \int J_1^3(x) \, dx$$

$$\int x \, I_0(x) \, I_1^2(x) \, dx = \frac{x}{3} \, I_1^3(x) + \frac{2}{3} \int I_1^3(x) \, dx$$

$$\int x^2 \, J_0(x) \, J_1^2(x) \, dx = -\frac{2x}{9} \, J_0^3(x) - \frac{x}{3} \, J_0(x) \, J_1^2(x) + \frac{x^2}{3} \, J_0^3(x) + \frac{2}{9} \int J_0^3(x) \, dx - \frac{1}{3} \int J_0(x) \, J_1^2(x) \, dx$$

$$\int x^2 \, I_0(x) \, I_1^2(x) \, dx = -\frac{2x}{9} \, I_0^3(x) + \frac{x}{3} \, I_0(x) \, I_1^2(x) + \frac{x^2}{3} \, I_0^3(x) + \frac{2}{9} \int I_0^3(x) \, dx + \frac{1}{3} \int I_0(x) \, I_1^2(x) \, dx$$

$$\int x^3 \, J_0(x) \, J_1^2(x) \, dx = \frac{x^3}{3} \, J_1^3(x)$$

$$\int x^3 \, I_0(x) \, I_1^2(x) \, dx = \frac{x^3}{3} \, I_1^3(x)$$

$$\int x^4 \, J_0(x) \, J_1^2(x) \, dx = \frac{6\,x^3 + 4\,x}{27} \, J_0^3(x) - \frac{2x^2}{3} \, J_0^2(x) \, J_1(x) + \frac{3\,x^3 + 5\,x}{9} \, J_0(x) \, J_1^2(x) + \frac{3\,x^4 - 5\,x^2}{9} \, J_1^3(x) -$$

$$-\frac{4}{27} \int J_0^3(x) \, dx + \frac{5}{9} \int J_0(x) \, J_1^2(x) \, dx$$

$$\int x^4 \, I_0(x) \, I_1^2(x) \, dx = \frac{6\,x^3 - 4\,x}{27} \, I_0^3(x) - \frac{2x^2}{3} \, I_0^2(x) \, I_1(x) + \frac{-3\,x^3 + 5\,x}{9} \, I_0(x) \, I_1^2(x) + \frac{3\,x^4 + 5\,x^2}{9} \, I_1^3(x) +$$

$$+\frac{4}{27} \int I_0^3(x) \, dx + \frac{5}{9} \int I_0(x) \, I_1^2(x) \, dx$$

Let

$$\int x^n \, J_0(x) \, J_1^2(x) \, dx = \mathcal{P}_n(x) \, J_0^3(x) + \mathcal{Q}_n(x) \, J_0^2(x) \, J_1(x) + \mathcal{R}_n(x) \, J_0(x) \, J_1^2(x) + \mathcal{S}_n(x) \, J_1^3(x) +$$

$$+\mathcal{U}_n \int J_0^3(x) \, dx + \mathcal{V}_n \int J_0(x) \, J_1(x)^2 \, dx + \mathcal{W}_n \int J_1(x)^3 \, dx + \mathcal{X}_n \int \frac{J_0^3(x) \, dx}{x}$$

and

$$\int x^n \, I_0(x) \, I_1^2(x) \, dx = \mathcal{P}_n^*(x) \, I_0^3(x) + \mathcal{Q}_n^*(x) \, I_0^2(x) \, I_1(x) + \mathcal{R}_n^*(x) \, I_0(x) \, I_1^2(x) + \mathcal{S}_n^*(x) \, I_1^3(x) +$$

$$+\mathcal{U}_n^* \int I_0^3(x) \, dx + \mathcal{V}_n^* \int I_0(x) \, I_1(x)^2 \, dx + \mathcal{W}_n^* \int I_1(x)^3 \, dx + \mathcal{X}_n^* \int \frac{I_0^3(x) \, dx}{x}$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_5(x) = \frac{12\,x^4 - 32\,x^2}{27} \,, \quad \mathcal{Q}_5(x) = \frac{-48\,x^3 + 64\,x}{27} \,, \quad \mathcal{R}_5(x) = \frac{6\,x^4}{9} \,, \quad \mathcal{S}_5(x) = \frac{27\,x^5 - 132\,x^3 + 128\,x}{81} \,,$$

$$\mathcal{U}_5 = 0 \,, \quad \mathcal{V}_5 = 0 \,, \quad \mathcal{W}_5 = \frac{256}{81}$$

$$\mathcal{P}_5^*(x) = \frac{12\,x^4 + 32\,x^2}{27} \,, \quad \mathcal{Q}_5^*(x) = -\frac{48\,x^3 + 64\,x}{27} \,, \quad \mathcal{R}_5^*(x) = -\frac{6\,x^4}{9} \,, \quad \mathcal{S}_5^*(x) = \frac{27\,x^5 + 132\,x^3 + 128\,x}{81} \,,$$

$$\mathcal{U}_5^* = 0 \,, \quad \mathcal{V}_5^* = 0 \,, \quad \mathcal{W}_5^* = \frac{256}{81}$$

$$\mathcal{P}_6(x) = \frac{54\,x^5 - 444\,x^3 - 206\,x}{81} \,, \quad \mathcal{Q}_6(x) = \frac{-30\,x^4 + 148\,x^2}{9} \,, \quad \mathcal{R}_6(x) = \frac{27\,x^5 - 87\,x^3 - 325\,x}{27} \,,$$

$$\mathcal{S}_6(x) = \frac{9\,x^6 - 87\,x^4 + 325\,x^2}{27} \,, \quad \mathcal{U}_6 = \frac{206}{81} \,, \quad \mathcal{V}_6 = -\frac{325}{27} \,, \quad \mathcal{W}_6 = 0$$

$$\mathcal{P}_6^*(x) = \frac{54\,x^5 + 444\,x^3 - 206\,x}{81} \,, \quad \mathcal{Q}_6^*(x) = -\frac{30\,x^4 + 148\,x^2}{9} \,, \quad \mathcal{R}_6^*(x) = \frac{-27\,x^5 - 87\,x^3 + 325\,x}{27} \,,$$

$$\mathcal{S}_6^*(x) = \frac{9\,x^6 + 87\,x^4 + 325\,x^2}{27} \ , \quad \mathcal{U}_6^* = \frac{206}{81} \ , \quad \mathcal{V}_6^* = \frac{325}{27} \ , \quad \mathcal{W}_6^* = 0$$

$$\mathcal{P}_7(x) = \frac{24\,x^6 - 384\,x^4 + 1024\,x^2}{27} \ , \quad \mathcal{Q}_7(x) = \frac{-144\,x^5 + 1536\,x^3 - 2048\,x}{27} \ , \quad \mathcal{R}_7(x) = \frac{4\,x^6 - 32\,x^4}{3} \ ,$$

$$\mathcal{S}_7(x) = \frac{27\,x^7 - 432\,x^5 + 3648\,x^3 - 4096\,x}{81} \ , \quad \mathcal{U}_7 = 0 \ , \quad \mathcal{V}_7 = 0 \ , \quad \mathcal{W}_7 = -\frac{8192}{81}$$

$$\mathcal{P}_7^*(x) = \frac{24\,x^6 + 384\,x^4 + 1024\,x^2}{27} \ , \quad \mathcal{Q}_7^*(x) = -\frac{144\,x^5 + 1536\,x^3 + 2048\,x}{27} \ , \quad \mathcal{R}_7^*(x) = -\frac{4\,x^6 + 32\,x^4}{3} \ ,$$

$$\mathcal{S}_7^*(x) = \frac{27\,x^7 + 432\,x^5 + 3648\,x^3 + 4096\,x}{81} \ , \quad \mathcal{U}_7^* = 0 \ , \quad \mathcal{V}_7^* = 0 \ , \quad \mathcal{W}_7^* = \frac{8192}{81}$$

$$\mathcal{P}_8(x) = \frac{270\,x^7 - 7020\,x^5 + 54570\,x^3 + 24680\,x}{243} \ , \quad \mathcal{Q}_8(x) = \frac{-210\,x^6 + 3900\,x^4 - 18190\,x^2}{27} \ ,$$

$$\mathcal{R}_8(x) = \frac{135\,x^7 - 1935\,x^5 + 9735\,x^3 + 39625\,x}{81} \ , \quad \mathcal{S}_8(x) = \frac{27\,x^8 - 645\,x^6 + 9735\,x^4 - 39625\,x^2}{81} \ ,$$

$$\mathcal{U}_8 = -\frac{24680}{243} \ , \quad \mathcal{V}_8 = \frac{39625}{81} \ , \quad \mathcal{W}_8 = 0$$

$$\mathcal{P}_8^*(x) = \frac{270\,x^7 + 7020\,x^5 + 54570\,x^3 - 24680\,x}{243} \ , \quad \mathcal{Q}_8^*(x) = -\frac{210\,x^6 + 3900\,x^4 + 18190\,x^2}{27} \ ,$$

$$\mathcal{R}_8^*(x) = -\frac{135\,x^7 + 1935\,x^5 + 9735\,x^3 - 39625\,x}{81} \ , \quad \mathcal{S}_8^*(x) = \frac{27\,x^8 + 645\,x^6 + 9735\,x^4 + 39625\,x^2}{81} \ ,$$

$$\mathcal{U}_8^* = \frac{24680}{243} \ , \quad \mathcal{V}_n^* = \frac{39625}{81} \ , \quad \mathcal{W}_8^* = 0$$

$$\int \frac{J_0(x)\,J_1^2(x)\,dx}{x} = -\frac{1}{2}\,J_0(x)\,J_1^2(x) - \frac{1}{3}J_0^3(x) - \frac{1}{2}\int J_1^3(x)\,dx$$

$$\int \frac{I_0(x)\,I_1^2(x)\,dx}{x} = -\frac{1}{2}\,I_0(x)\,I_1^2(x) + \frac{1}{3}I_0^3(x) + \frac{1}{2}\int I_1^3(x)\,dx$$

$$\int \frac{J_0(x)\,J_1^2(x)\,dx}{x^2} = -\frac{2}{3}\,J_0^2(x)\,J_1(x) - \frac{1}{3x}\,J_0(x)\,J_1^2(x) + \frac{1}{9}\,J_1^3(x) - \frac{5}{3}\int J_0(x)\,J_1^2(x)\,dx + \frac{2}{3}\int J_0^3(x)\,dx$$

$$\int \frac{I_0(x)\,I_1^2(x)\,dx}{x^2} = -\frac{2}{3}\,I_0^2(x)\,I_1(x) - \frac{1}{3x}\,I_0(x)\,I_1^2(x) - \frac{1}{9}\,I_1^3(x) + \frac{5}{3}\int I_0(x)\,I_1^2(x)\,dx + \frac{2}{3}\int I_0^3(x)\,dx$$

$$\int \frac{J_0(x)J_1^2(x)\,dx}{x^3} =$$

$$= \frac{11}{48}\,J_0^3(x) - \frac{1}{4x}\,J_0^2(x)J_1(x) + \frac{11\,x^2 - 8}{32\,x^2}\,J_0(x)J_1^2(x) + \frac{1}{16}\,J_0(x)J_1^3(x) + \frac{11}{32}\int J_1^3(x)\,dx + \frac{1}{4}\int \frac{J_0^3(x)\,dx}{x}$$

$$\int \frac{I_0(x)I_1^2(x)\,dx}{x^3} =$$

$$= \frac{11}{48}\,I_0^3(x) - \frac{1}{4x}\,I_0^2(x)I_1(x) - \frac{11\,x^2 + 8}{32x^2}\,I_0(x)I_1^2(x) - \frac{1}{16x}\,I_0(x)I_1^3(x) + \frac{11}{32}\int I_3^3(x)\,dx + \frac{1}{4}\int \frac{I_0^3(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,J_1^2(x)\,dx}{x^4} = -\frac{2}{15x}\,J_0^3(x) + \frac{148\,x^2 - 30}{225\,x^2}\,J_0^2(x)\,J_1(x) + \frac{29\,x^2 - 45}{225\,x^3}\,J_0(x)\,J_1^2(x) + \frac{-29\,x^2 + 27}{675\,x^2}\,J_1^3(x) -$$

$$- \frac{148}{225}\int J_0^3(x)\,dx + \frac{13}{9}\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0(x)\,I_1^2(x)\,dx}{x^4} = -\frac{2}{15x}\,I_0^3(x) - \frac{148\,x^2 + 30}{225\,x^2}\,I_0^2(x)\,I_1(x) - \frac{29\,x^2 + 45}{225\,x^3}\,I_0(x)\,I_1^2(x) - \frac{29\,x^2 + 27}{675\,x^2}\,I_1^3(x) +$$

$$+ \frac{148}{225}\int I_0^3(x)\,dx + \frac{13}{9}\int I_0(x)\,I_1^2(x)\,dx$$

$$\mathcal{P}_{-5}(x) = \frac{-19\,x^2 - 8}{192\,x^2} \ , \quad \mathcal{Q}_{-5}(x) = \frac{3\,x^2 - 2}{24\,x^3} \ , \quad \mathcal{R}_{-5}(x) = \frac{-57\,x^4 + 24\,x^2 - 64}{384\,x^4} \ , \quad \mathcal{S}_{-5}(x) = \frac{-9\,x^2 + 16}{576\,x^3} \ ,$$

$$\mathcal{U}_{-5} = 0 \ , \quad \mathcal{V}_{-5} = 0 \ , \quad \mathcal{W}_{-5} = -\frac{19}{128} \ , \quad \mathcal{X}_{-5} = -\frac{1}{8}$$

$$\mathcal{P}^*_{-5}(x) = \frac{19\,x^2 - 8}{192\,x^2} \ , \quad \mathcal{Q}^*_{-5}(x) = \frac{-3\,x^2 - 2}{24\,x^3} \ , \quad \mathcal{R}^*_{-5}(x) = \frac{-57\,x^4 - 24\,x^2 - 64}{384\,x^4} \ , \quad \mathcal{S}^*_{-5}(x) = \frac{-9\,x^2 - 16}{576\,x^3} \ ,$$

$$\mathcal{U}^*_{-5} = 0 \ , \quad \mathcal{V}^*_{-5} = 0 \ , \quad \mathcal{W}^*_{-5} = \frac{19}{128} \ , \quad \mathcal{X}^*_{-5} = \frac{1}{8}$$

$$\mathcal{P}_{-6}(x) = \frac{156\,x^2 - 70}{3675\,x^3} \ , \quad \mathcal{Q}_{-6}(x) = \frac{-3638\,x^4 + 780\,x^2 - 1050}{18375\,x^4} \ , \quad \mathcal{R}_{-6}(x) = \frac{-649\,x^4 + 645\,x^2 - 2625}{18375\,x^5} \ ,$$

$$\mathcal{S}_{-6}(x) = \frac{649\,x^4 - 387\,x^2 + 1125}{55125\,x^4} \ , \quad \mathcal{U}_{-6} = \frac{3638}{18375} \ , \quad \mathcal{V}_{-6} = -\frac{317}{735} \ , \quad \mathcal{W}_{-6} = 0$$

$$\mathcal{P}^*_{-6}(x) = -\frac{156\,x^2 + 70}{3675\,x^3} \ , \quad \mathcal{Q}^*_{-6}(x) = -\frac{3638\,x^4 + 780\,x^2 + 1050}{18375\,x^4} \ , \quad \mathcal{R}^*_{-6}(x) = -\frac{649\,x^4 + 645\,x^2 + 2625}{18375\,x^5} \ ,$$

$$\mathcal{S}^*_{-6}(x) = -\frac{649\,x^4 + 387\,x^2 + 1125}{55125\,x^4} \ , \quad \mathcal{U}^*_{-6} = \frac{3638}{18375} \ , \quad \mathcal{V}^*_{-6} = \frac{317}{735} \ , \quad \mathcal{W}^*_{-6} = 0$$

$$\mathcal{P}_{-7}(x) = \frac{751\,x^4 + 344\,x^2 - 384}{36864\,x^4} \ , \quad \mathcal{Q}_{-7}(x) = \frac{-60\,x^4 + 43\,x^2 - 96}{2304\,x^5} \ ,$$

$$\mathcal{R}_{-7}(x) = \frac{2253\,x^6 - 888\,x^4 + 1600\,x^2 - 9216}{73728\,x^6} \ , \quad \mathcal{S}_{-7}(x) = \frac{333\,x^4 - 400\,x^2 + 1728}{110592\,x^5} \ ,$$

$$\mathcal{U}_{-7} = 0 \ , \quad \mathcal{V}_{-7} = 0 \ , \quad \mathcal{W}_{-7} = \frac{751}{24576} \ , \quad \mathcal{X}_{-7} = \frac{5}{192}$$

$$\mathcal{P}^*_{-7}(x) = \frac{751\,x^4 - 344\,x^2 - 384}{36864\,x^4} \ , \quad \mathcal{Q}^*_{-7}(x) = \frac{-60\,x^4 - 43\,x^2 - 96}{2304\,x^5} \ ,$$

$$\mathcal{R}^*_{-7}(x) = \frac{-2253\,x^6 - 888\,x^4 - 1600\,x^2 - 9216}{73728\,x^6} \ , \quad \mathcal{S}^*_{-7}(x) = \frac{-333\,x^4 - 400\,x^2 - 1728}{110592\,x^5} \ ,$$

$$\mathcal{U}^*_{-7} = 0 \ , \quad \mathcal{V}^*_{-7} = 0 \ , \quad \mathcal{W}^*_{-7} = \frac{751}{24576} \ , \quad \mathcal{X}^*_{-7} = \frac{5}{192}$$

$$\mathcal{P}_{-8}(x) = \frac{-22758\,x^4 + 11060\,x^2 - 22050}{3472875\,x^5} \ , \quad \mathcal{Q}_{-8}(x) = \frac{528184\,x^6 - 113790\,x^4 + 165900\,x^2 - 551250}{17364375\,x^6} \ ,$$

$$\mathcal{R}_{-8}(x) = \frac{93407\,x^6 - 87735\,x^4 + 249375\,x^2 - 1929375}{17364375\,x^7} \ , \quad \mathcal{S}_{-8}(x) = \frac{-93407\,x^6 + 52641\,x^4 - 106875\,x^2 + 643125}{52093125\,x^6} \ ,$$

$$\mathcal{U}_{-8} = -\frac{528184}{17364375} \ , \quad \mathcal{V}_{-8} = \frac{45991}{694575} \ , \quad \mathcal{W}_{-8} = 0$$

$$\mathcal{P}^*_{-8}(x) = -\frac{22758\,x^4 + 11060\,x^2 + 22050}{3472875\,x^5} \ , \quad \mathcal{Q}^*_{-8}(x) = -\frac{528184\,x^6 + 113790\,x^4 + 165900\,x^2 + 551250}{17364375\,x^6} \ ,$$

$$\mathcal{R}^*_{-8}(x) = -\frac{93407\,x^6 + 87735\,x^4 + 249375\,x^2 + 1929375}{17364375\,x^7} \ , \quad \mathcal{S}^*_{-8}(x) = -\frac{93407\,x^6 + 52641\,x^4 + 106875\,x^2 + 643125}{52093125\,x^6} \ ,$$

$$\mathcal{U}^*_{-8} = \frac{528184}{17364375} \ , \quad \mathcal{V}^*_{-8} = \frac{45991}{694575} \ , \quad \mathcal{W}^*_{-8} = 0$$

h) Integrals of the type $\int x^n\, Z_1^3(x)\, dx$:

$$\int x\, J_1^3(x)\, dx = -\frac{2x}{3}\, J_0^3(x) - x\, J_0(x)\, J_1^2(x) + \frac{2}{3} \int J_0^3(x)\, dx - \int J_0(x)\, J_1^2(x)\, dx$$

$$\int x\, I_1^3(x)\, dx = -\frac{2x}{3}\, I_0^3(x) + x\, I_0(x)\, I_1^2(x) + \frac{2}{3} \int I_0^3(x)\, dx + \int I_0(x)\, I_1^2(x)\, dx$$

$$\int x^2\, J_1^3(x)\, dx = -\frac{2x^2}{3}\, J_0^3(x) + \frac{4x}{3}\, J_0^2\, J_1(x) - x^2\, J_0(x)\, J_1^2(x) + \frac{8x}{9}\, J_1^3(x) + \frac{16}{9} \int J_1^3(x)\, dx$$

$$\int x^2\, I_1^3(x)\, dx = -\frac{2x^2}{3}\, I_0^3(x) + \frac{4x}{3}\, I_0^2\, I_1(x) + x^2\, I_0(x)\, I_1^2(x) - \frac{8x}{9}\, I_1^3(x) - \frac{16}{9} \int I_1^3(x)\, dx$$

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$$\int x^3 J_1^3(x)\, dx = -\frac{6\,x^3 + 4\,x}{9}\, J_0^3(x) + 2x^2\, J_0^2\, J_1(x) - \frac{3\,x^3 + 5\,x}{3}\, J_0(x)\, J_1^2(x) + \frac{5x^2}{3}\, J_1^3(x) +$$

$$+\frac{4}{9}\int J_0^3(x)\, dx - \frac{5}{3}\int J_0(x)\, J_1^2(x)\, dx$$

$$\int x^3 I_1^3(x)\, dx = \frac{-6\,x^3 + 4\,x}{9}\, I_0^3(x) + 2x^2\, I_0^2\, I_1(x) + \frac{3\,x^3 - 5\,x}{3}\, I_0(x)\, I_1^2(x) - \frac{5x^2}{3}\, I_1^3(x) -$$

$$-\frac{4}{9}\int I_0^3(x)\, dx - \frac{5}{3}\int I_0(x)\, I_1^2(x)\, dx$$

$$\int x^4 J_1^3(x)\, dx =$$

$$=\frac{-6\,x^4 + 16\,x^2}{9}\, J_0^3(x) + \frac{24\,x^3 - 32\,x}{9}\, J_0^2\, J_1(x) - x^4\, J_0(x)\, J_1^2(x) + \frac{66\,x^3 - 64\,x}{27}\, J_1^3(x) - \frac{128}{27}\int J_1^3(x)\, dx$$

$$\int x^4 I_1^3(x)\, dx =$$

$$=-\frac{6\,x^4 + 16\,x^2}{9}\, I_0^3(x) + \frac{24\,x^3 + 32\,x}{9}\, I_0^2\, I_1(x) + x^4\, I_0(x)\, I_1^2(x) - \frac{66\,x^3 + 64\,x}{27}\, I_1^3(x) - \frac{128}{27}\int I_1^3(x)\, dx$$

Let

$$\int x^n J_1^3(x)\, dx = \mathcal{P}_n(x)\, J_0^3(x) + \mathcal{Q}_n(x)\, J_0^2(x)\, J_1(x) + \mathcal{R}_n(x)\, J_0(x)\, J_1^2(x) + \mathcal{S}_n(x)\, J_1^3(x) +$$

$$+\mathcal{U}_n \int J_0^3(x)\, dx + \mathcal{V}_n \int J_0(x)\, J_1(x)^2\, dx + \mathcal{W}_n \int J_1^3(x)\, dx + \mathcal{X}_n \int \frac{I_0^3(x)\, dx}{x}$$

and

$$\int x^n I_1^3(x)\, dx = \mathcal{P}_n^*(x)\, I_0^3(x) + \mathcal{Q}_n^*(x)\, I_0^2(x)\, I_1(x) + \mathcal{R}_n^*(x)\, I_0(x)\, I_1^2(x) + \mathcal{S}_n^*(x)\, I_1^3(x) +$$

$$+\mathcal{U}_n^* \int I_0^3(x)\, dx + \mathcal{V}_n^* \int I_0(x)\, I_1(x)^2\, dx + \mathcal{W}_n^* \int I_1^3(x)\, dx + \mathcal{X}_n^* \int \frac{I_0^3(x)\, dx}{x}$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_5(x) = \frac{-54\,x^5 + 444\,x^3 + 206\,x}{81}\ , \quad \mathcal{Q}_5(x) = \frac{30\,x^4 - 148\,x^2}{9}\ , \quad \mathcal{R}_5(x) = \frac{-27\,x^5 + 87\,x^3 + 325\,x}{27}\ ,$$

$$\mathcal{S}_5(x) = \frac{87\,x^4 - 325\,x^2}{27}\ , \quad \mathcal{U}_5 = -\frac{206}{81}\ , \quad \mathcal{V}_5 = \frac{325}{27}\ , \quad \mathcal{W}_5 = 0$$

$$\mathcal{P}_5^*(x) = \frac{-54\,x^5 - 444\,x^3 + 206\,x}{81}\ , \quad \mathcal{Q}_5^*(x) = \frac{30\,x^4 + 148\,x^2}{9}\ , \quad \mathcal{R}_5^*(x) = \frac{27\,x^5 + 87\,x^3 - 325\,x}{27}\ ,$$

$$\mathcal{S}_5^*(x) = -\frac{87\,x^4 + 325\,x^2}{27}\ , \quad \mathcal{U}_5^* = -\frac{206}{81}\ , \quad \mathcal{V}_5^* = -\frac{325}{27}\ , \quad \mathcal{W}_5^* = 0$$

$$\mathcal{P}_6(x) = \frac{-6\,x^6 + 96\,x^4 - 256\,x^2}{9}\ , \quad \mathcal{Q}_6(x) = \frac{36\,x^5 - 384\,x^3 + 512\,x}{9}\ , \quad \mathcal{R}_6(x) = -x^6 + 8\,x^4\ ,$$

$$\mathcal{S}_6(x) = \frac{108\,x^5 - 912\,x^3 + 1024\,x}{27}\ , \quad \mathcal{U}_6 = 0\ , \quad \mathcal{V}_6 = 0\ , \quad \mathcal{W}_6 = \frac{2048}{27}$$

$$\mathcal{P}_6^*(x) = -\frac{6\,x^6 + 96\,x^4 + 256\,x^2}{9}\ , \quad \mathcal{Q}_6^*(x) = \frac{36\,x^5 + 384\,x^3 + 512\,x}{9}\ , \quad \mathcal{R}_6^*(x) = x^6 + 8\,x^4\ ,$$

$$\mathcal{S}_6^*(x) = -\frac{108\,x^5 + 912\,x^3 + 1024\,x}{27}\ , \quad \mathcal{U}_6^* = 0\ , \quad \mathcal{V}_6^* = 0\ , \quad \mathcal{W}_6^* = -\frac{2048}{27}$$

$$\mathcal{P}_7(x) = \frac{-54\,x^7 + 1404\,x^5 - 10914\,x^3 - 4936\,x}{81}\ , \quad \mathcal{Q}_7(x) = \frac{42\,x^6 - 780\,x^4 + 3638\,x^2}{9}\ ,$$

$$\mathcal{R}_7(x) = \frac{-27\,x^7 + 387\,x^5 - 1947\,x^3 - 7925\,x}{27}\ , \quad \mathcal{S}_7(x) = \frac{129\,x^6 - 1947\,x^4 + 7925\,x^2}{27}\ ,$$

$$\mathcal{U}_7 = \frac{4936}{81}\ , \quad \mathcal{V}_7 = -\frac{7925}{27}\ , \quad \mathcal{W}_7 = 0$$

$$\mathcal{P}_7^*(x) = -\frac{54\,x^7 + 1404\,x^5 + 10914\,x^3 - 4936\,x}{81} \;, \quad \mathcal{Q}_7^*(x) = \frac{42\,x^6 + 780\,x^4 + 3638\,x^2}{9} \;,$$

$$\mathcal{R}_7^*(x) = \frac{27\,x^7 + 387\,x^5 + 1947\,x^3 - 7925\,x}{27} \;, \quad \mathcal{S}_7^*(x) = -\frac{129\,x^6 + 1947\,x^4 + 7925\,x^2}{27} \;,$$

$$\mathcal{U}_7^* = \frac{4936}{81} \;, \quad \mathcal{V}_7^* = \frac{7925}{27} \;, \quad \mathcal{W}_7^* = 0$$

$$\mathcal{P}_8(x) = \frac{-54\,x^8 + 2064\,x^6 - 30720\,x^4 + 81920\,x^2}{81} \;, \quad \mathcal{Q}_8(x) = \frac{432\,x^7 - 12384\,x^5 + 122880\,x^3 - 163840\,x}{81} \;,$$

$$\mathcal{R}_8(x) = \frac{-9\,x^8 + 200\,x^6 - 2368\,x^4}{9} \;, \quad \mathcal{S}_8(x) = \frac{1350\,x^7 - 31968\,x^5 + 288384\,x^3 - 327680\,x}{243} \;,$$

$$\mathcal{U}_8 = 0 \;, \quad \mathcal{V}_8 = 0 \;, \quad \mathcal{W}_8 = -\frac{655360}{243}$$

$$\mathcal{P}_8^*(x) = -\frac{54\,x^8 + 2064\,x^6 + 30720\,x^4 + 81920\,x^2}{81} \;, \quad \mathcal{Q}_8^*(x) = \frac{432\,x^7 + 12384\,x^5 + 122880\,x^3 + 163840\,x}{81} \;,$$

$$\mathcal{R}_8^*(x) = \frac{9\,x^8 + 200\,x^6 + 2368\,x^4}{9} \;, \quad \mathcal{S}_8^*(x) = -\frac{1350\,x^7 + 31968\,x^5 + 288384\,x^3 + 327680\,x}{243} \;,$$

$$\mathcal{U}_8^* = 0 \;, \quad \mathcal{V}_8^* = 0 \;, \quad \mathcal{W}_8^* = -\frac{655360}{243}$$

$$\int \frac{J_1^3(x)\,dx}{x} = -\frac{1}{3}\,J_1^3(x) + \int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_1^3(x)\,dx}{x} = -\frac{1}{3}\,I_1^3(x) + \int I_0(x)\,I_1^2(x)\,dx$$

$$\int \frac{J_1^3(x)\,dx}{x^2} = -\frac{1}{8}\,J_0^3(x) - \frac{3}{8}\,J_0(x)\,J_1^2(x) - \frac{1}{4x}\,J_1^3(x) + \frac{3}{4}\int J_0(x)\,J_1^2(x)\,dx - \frac{3}{8}\int J_1^3(x)\,dx$$

$$\int \frac{I_1^3(x)\,dx}{x^2} = \frac{1}{4}\,I_0^3(x) - \frac{3}{8}\,I_0(x)\,I_1^2(x) - \frac{1}{4x}\,I_1^3(x) + \frac{3}{4}\int I_0(x)\,I_1^2(x)\,dx + \frac{3}{8}\int I_1^3(x)\,dx$$

$$\int \frac{J_1^3(x)\,dx}{x^3} = -\frac{2}{5}\,J_0^2(x)\,J_1(x) - \frac{1}{5x}\,J_0(x)\,J_1^2(x) + \frac{x^2-3}{15x^2}\,J_1^3(x) + \frac{2}{5}\int J_0(x)^3\,dx - \int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_1^3(x)\,dx}{x^3} = -\frac{2}{5}\,I_0^2(x)\,I_1(x) - \frac{1}{5x}\,I_0(x)\,I_1^2(x) - \frac{x^2+3}{15x^2}\,I_1^3(x) + \frac{2}{5}\int I_0(x)^3\,dx + \int I_0(x)\,I_1^2(x)\,dx$$

$$\int \frac{J_1^3(x)\,dx}{x^4} =$$

$$= \frac{11}{96}\,J_0^3(x) - \frac{1}{8x}\,J_0^2(x)J_1(x) + \frac{11\,x^2 - 8}{64\,x^2}\,J_0(x)J_1^2(x) + \frac{3\,x^2 - 16}{96\,x^3}\,J_0(x)J_1^3(x) + \frac{11}{64}\int J_1^3(x)\,dx + \frac{1}{8}\int \frac{J_0^3(x)\,dx}{x}$$

$$\int \frac{I_1^3(x)\,dx}{x^4} =$$

$$= \frac{11}{96}\,I_0^3(x) - \frac{1}{8x}\,I_0^2(x)I_1(x) - \frac{11\,x^2 + 8}{64\,x^2}\,I_0(x)I_1^2(x) - \frac{3\,x^2 + 16}{96\,x^3}\,I_0(x)I_1^3(x) + \frac{11}{64}\int I_1^3(x)\,dx + \frac{1}{8}\int \frac{I_0^3(x)\,dx}{x}$$

$$\mathcal{P}_{-5}(x) = -\frac{2}{35x} \;, \quad \mathcal{Q}_{-5}(x) = \frac{148\,x^2 - 30}{525\,x^2} \;, \quad \mathcal{R}_{-5}(x) = \frac{29\,x^2 - 45}{525\,x^3} \;, \quad \mathcal{S}_{-5}(x) = \frac{-29\,x^4 + 27\,x^2 - 225}{1575\,x^4} \;,$$

$$\mathcal{U}_{-5} = -\frac{148}{525} \;, \quad \mathcal{V}_{-5} = \frac{13}{21} \;, \quad \mathcal{W}_{-5} = 0$$

$$\mathcal{P}_{-5}^*(x) = -\frac{2}{35x} \;, \quad \mathcal{Q}_{-5}^*(x) = -\frac{148\,x^2 + 30}{525x^2} \;, \quad \mathcal{R}_{-5}^*(x) = -\frac{29\,x^2 + 45}{525x^3} \;, \quad \mathcal{S}_{-5}^*(x) = -\frac{29\,x^4 + 27\,x^2 + 225}{1575\,x^4} \;,$$

$$\mathcal{U}_{-5}^* = \frac{148}{525} \;, \quad \mathcal{V}_{-5}^* = \frac{13}{21} \;, \quad \mathcal{W}_{-5}^* = 0$$

$$\mathcal{P}_{-6}(x) = \frac{-19\,x^2 - 8}{512\,x^2} \;, \quad \mathcal{Q}_{-6}(x) = \frac{3\,x^2 - 2}{64\,x^3} \;, \quad \mathcal{R}_{-6}(x) = \frac{-57\,x^4 + 24\,x^2 - 64}{1024\,x^4} \;,$$

$$\mathcal{S}_{-6}(x) = \frac{-9\,x^4 + 16\,x^2 - 192}{1536\,x^5}\ ,\quad \mathcal{U}_{-6} = 0\ ,\quad \mathcal{V}_{-6} = 0\ ,\quad \mathcal{W}_{-6} = -\frac{57}{1024}\ ,\quad \mathcal{X}_{-6} = -\frac{3}{64}$$

$$\mathcal{P}^*_{-6}(x) = \frac{19\,x^2 - 8}{512\,x^2}\ ,\quad \mathcal{Q}^*_{-6}(x) = \frac{-3\,x^2 - 2}{64\,x^3}\ ,\quad \mathcal{R}^*_{-6}(x) = \frac{-57\,x^4 - 24\,x^2 - 64}{1024\,x^4}\ ,$$

$$\mathcal{S}^*_{-6}(x) = \frac{-9\,x^4 - 16\,x^2 - 192}{1536\,x^5}\ ,\quad \mathcal{U}^*_{-6} = 0\ ,\quad \mathcal{V}^*_{-6} = 0\ ,\quad \mathcal{W}^*_{-6} = \frac{57}{1024}\ ,\quad \mathcal{X}^*_{-6} = \frac{3}{64}$$

$$\mathcal{P}_{-7}(x) = \frac{156\,x^2 - 70}{11025\,x^3}\ ,\quad \mathcal{Q}_{-7}(x) = \frac{-3638\,x^4 + 780\,x^2 - 1050}{55125\,x^4}\ ,\quad \mathcal{R}_{-7}(x) = \frac{-649\,x^4 + 645\,x^2 - 2625}{55125\,x^5}\ ,$$

$$\mathcal{S}_{-7}(x) = \frac{649x^6 - 387x^4 + 1125x^2 - 18375}{165375\,x^6}\ ,\quad \mathcal{U}_{-7} = \frac{3638}{55125}\ ,\quad \mathcal{V}_{-7} = -\frac{317}{2205}\ ,\quad \mathcal{W}_{-7} = 0$$

$$\mathcal{P}^*_{-7}(x) = -\frac{156\,x^2 + 70}{11025\,x^3}\ ,\quad \mathcal{Q}^*_{-7}(x) = -\frac{3638\,x^4 + 780\,x^2 + 1050}{55125\,x^4}\ ,\quad \mathcal{R}^*_{-7}(x) = -\frac{649\,x^4 + 645\,x^2 + 2625}{55125\,x^5}\ ,$$

$$\mathcal{S}^*_{-7}(x) = -\frac{649\,x^6 + 387\,x^4 + 1125\,x^2 + 18375}{165375\,x^6}\ ,\quad \mathcal{U}^*_{-7} = \frac{3638}{55125}\ ,\quad \mathcal{V}^*_{-7} = \frac{317}{2205}\ ,\quad \mathcal{W}^*_{-7} = 0$$

$$\mathcal{P}_{-8}(x) = \frac{751\,x^4 + 344\,x^2 - 384}{122880\,x^4}\ ,\quad \mathcal{Q}_{-8}(x) = \frac{-60\,x^4 + 43\,x^2 - 96}{7680\,x^5}\ ,$$

$$\mathcal{R}_{-8}(x) = \frac{2253\,x^6 - 888\,x^4 + 1600\,x^2 - 9216}{245760\,x^6}\ ,\quad \mathcal{S}_{-8}(x) = \frac{333\,x^6 - 400\,x^4 + 1728\,x^2 - 36864}{368640\,x^7}\ ,$$

$$\mathcal{U}_{-8} = 0\ ,\quad \mathcal{V}_{-8} = 0\ ,\quad \mathcal{W}_{-8} = \frac{751}{81920}\ ,\quad \mathcal{X}_{-8} = \frac{1}{128}$$

$$\mathcal{P}^*_{-8}(x) = \frac{751\,x^4 - 344\,x^2 - 384}{122880\,x^4}\ ,\quad \mathcal{Q}^*_{-8}(x) = \frac{-60\,x^4 - 43\,x^2 - 96}{7680\,x^5}\ ,$$

$$\mathcal{R}^*_{-8}(x) = \frac{-2253\,x^6 - 888\,x^4 - 1600\,x^2 - 9216}{245760\,x^6}\ ,\quad \mathcal{S}^*_{-8}(x) = \frac{-333\,x^6 - 400\,x^4 - 1728\,x^2 - 36864}{368640\,x^7}\ ,$$

$$\mathcal{U}^*_{-8} = 0\ ,\quad \mathcal{V}^*_{-8} = 0\ ,\quad \mathcal{W}^*_{-8} = \frac{751}{81920}\ ,\quad \mathcal{X}^*_{-8} = \frac{1}{128}$$

i) Recurrence Relations:

Let

$$\mathcal{J}_n^{(kl)} = \int x^n \, J_0^k(x) \, J_1^l(x) \, dx \qquad \text{and} \qquad \mathcal{I}_n^{(kl)} = \int x^n \, I_0^k(x) \, I_1^l(x) \, dx$$

with $k + l = 3$, $k, l \geq 0$. Then the following formulas hold:

Ascending recurrence:

$$\mathcal{J}_{n+1}^{(30)} = x^{n+1} \left[J_0^2(x) \, J_1(x) + \frac{2}{3} \, J_1^3(x) \right] - n \, \mathcal{J}_n^{(21)} - \frac{2}{3}(n-2) \, \mathcal{J}_n^{(03)}$$

$$\mathcal{J}_{n+1}^{(21)} = \frac{n+1}{3} \, \mathcal{J}_n^{(30)} - \frac{x^{n+1}}{3} \, J_0^3(x)$$

$$\mathcal{J}_{n+1}^{(12)} = \frac{x^{n+1}}{3} \, J_1^3(x) - \frac{n-2}{3} \, \mathcal{J}_n^{(03)}$$

$$\mathcal{J}_{n+1}^{(03)} = -x^{n+1} \left[J_0(x) \, J_1^2(x) + \frac{2}{3} \, J_0^3(x) \right] + (n-1)\mathcal{J}_n^{(12)} + \frac{2}{3}(n+1)\mathcal{J}_n^{(30)}$$

$$\mathcal{I}_{n+1}^{(30)} = x^{n+1} \left[I_0^2(x) \, I_1(x) - \frac{2}{3} \, I_1^3(x) \right] - n \, \mathcal{I}_n^{(21)} + \frac{2}{3}(n-2) \, \mathcal{I}_n^{(03)}$$

$$\mathcal{I}_{n+1}^{(21)} = -\frac{n+1}{3} \, \mathcal{I}_n^{(30)} + \frac{x^{n+1}}{3} \, I_0^3(x)$$

$$\mathcal{I}_{n+1}^{(12)} = \frac{x^{n+1}}{3} \, I_1^3(x) - \frac{n-2}{3} \, \mathcal{I}_n^{(03)}$$

$$\mathcal{I}_{n+1}^{(03)} = x^{n+1} \left[I_0(x) \, I_1^2(x) - \frac{2}{3} \, I_0^3(x) \right] - (n-1)\mathcal{I}_n^{(12)} + \frac{2}{3}(n+1)\mathcal{I}_n^{(30)}$$

Descending recurrence:

$$\mathcal{J}_n^{(30)} = \frac{x^{n+1}J_0^3(x) + 3\mathcal{J}_{n+1}^{(21)}}{n+1} \quad , \qquad \mathcal{J}_n^{(21)} = \frac{x^{n+1}J_0^2(x) \, J_1(x) + 2\mathcal{J}_{n+1}^{(12)} - \mathcal{J}_{n+1}^{(30)}}{n}$$

$$\mathcal{J}_n^{(12)} = \frac{2\mathcal{J}_{n+1}^{(21)} - x^{n+1}J_0(x) \, J_1^2(x) - \mathcal{J}_{n+1}^{(03)}}{n-1} \quad , \qquad \mathcal{J}_n^{(03)} = \frac{x^{n+1}J_1^3(x) - 3\mathcal{J}_{n+1}^{(12)}}{n-2}$$

Holds

$$J_0^3(x) + 3\mathcal{J}_0^{(21)} = xJ_0^2(x) \, J_1(x) + 2\mathcal{J}_1^{(12)} - \mathcal{J}_1^{(30)} = 2\mathcal{J}_2^{(21)} - x^2 J_0(x) \, J_1^2(x) - \mathcal{J}_2^{(03)} =$$

$$= x^3 J_1^3(x) - 3\mathcal{J}_3^{(12)} = 0 \; .$$

$$\mathcal{I}_n^{(30)} = \frac{x^{n+1}I_0^3(x) - 3\mathcal{I}_{n+1}^{(21)}}{n+1} \quad , \qquad \mathcal{I}_n^{(21)} = \frac{x^{n+1}I_0^2(x) \, I_1(x) - 2\mathcal{I}_{n+1}^{(12)} - \mathcal{I}_{n+1}^{(30)}}{n}$$

$$\mathcal{I}_n^{(12)} = \frac{2\mathcal{I}_{n+1}^{(21)} + \mathcal{I}_{n+1}^{(03)} - x^{n+1}I_0(x) \, I_1^2(x)}{n-1} \quad , \qquad \mathcal{I}_n^{(03)} = \frac{x^{n+1}I_1^3(x) - 3\mathcal{I}_{n+1}^{(12)}}{n-2}$$

Holds

$$I_0^3(x) - 3\mathcal{I}_0^{(21)} = xI_0^2(x) \, I_1(x) - 2\mathcal{I}_1^{(12)} - \mathcal{I}_1^{(30)} = x^2 I_0(x) \, I_1^2(x) - 2\mathcal{I}_2^{(21)} - \mathcal{I}_2^{(03)} =$$

$$= x^3 I_1^3(x) - 3\mathcal{I}_3^{(12)} = 0 \; .$$

j) Different Functions and Different Arguments

Let $Z_\nu(x)$, $Z_\nu^*(x) \in \{J_\nu(x), Y_\nu(x), H_\nu^{(1)}(x), H_\nu^{(2)}(x)\}$ with $Z_\nu(x) \neq Z_\nu^*(x)$ and $Z_\nu(x)$ and $Z_{1-\nu}(x)$ as well as $Z_\nu^*(x)$ and $Z_{1-\nu}^*(x)$ of the same type, then holds

$$\int x^3 Z_1^2(x) Z_0^*(x)\, dx =$$

$$= \frac{x^2}{3} \left\{ -2\, Z_0(x) Z_1(x)[x Z_0^*(x) + 2 Z_1^*(x)] + Z_1^2(x)[4 Z_0^*(x) + x Z_1^*(x)] + 2x\, Z_0^2(x) Z_1^*(x) \right\}$$

and

$$\int x^3 Z_0(x) Z_1(x) Z_1^*(x)\, dx =$$

$$= \frac{x^2}{3} \left\{ Z_0(x) Z_1(x)[x Z_0^*(x) + 2 Z_1^*(x)] - Z_1^2(x)[2 Z_0^*(x) - x Z_1^*(x)] - x Z_0^2(x) Z_1^*(x) \right\} \ .$$

$$\int x^3 I_1^2(x) K_0(x)\, dx =$$

$$= \frac{1}{3} \left[-4x^2\, I_0(x) I_1(x) K_1(x) + 2\, I_0(x) I_1(x) K_0(x) x^3 - x^3 I_1^2(x) K_1(x) + 2\, x^3 I_0^2(x) K_1(x) - 4\, x^2 I_1^2(x) K_0(x) \right]$$

$$\int x^3 K_1^2(x) I_0(x) =$$

$$= \frac{1}{3} \left[-2x^3\, I_0(x) K_0(x) K_1(x) - 4x^2\, I_1(x) K_0(x) K_1(x) + x^3 I_1(x) K_1^2(x) - 4\, x^2 I_0(x) K_1^2(x) - 2\, x^3 I_1(x) K_0^2(x) \right]$$

$$\int Z_1^2(x) Z_1^*(2x)\, dx = \frac{x}{2} \left\{ \left[Z_0^2(x) - Z_1^2(x) \right] Z_1^*(2x) - 2 Z_0(x) Z_1(x) Z_0^*(2x) \right\}$$

$$\int x Z_0^2(x) Z_0^*(2x)\, dx = \frac{x^2}{2} \left\{ \left[Z_0^2(x) - Z_1^2(x) \right] Z_0^*(2x) + 2 Z_0(x) Z_1(x) Z_1^*(2x) \right\}$$

$$\int x Z_1^2(x) Z_0^*(2x)\, dx =$$

$$= \frac{x}{2} \left\{ x \left[Z_1^2(x) - Z_0^2(x) \right] Z_0^*(2x) + 2 Z_0^2(x) Z_1^*(2x) - 2 Z_0(x) Z_1(x) \left[Z_0^*(2x) + x Z_1^*(2x) \right] \right\}$$

$$\int x Z_1^2(x) Z_0^* \left(\sqrt{2}x \right) dx = \frac{\sqrt{2}x}{2} Z_0^2(x) Z_1^* \left(\sqrt{2}x \right) - x Z_0(x) Z_1(x) Z_0^* \left(\sqrt{2}x \right)$$

$$\int x Z_0(x) Z_1(x) Z_1^*(2x)\, dx = \frac{x}{2} \left\{ x \left[Z_0^2(x) - Z_1^2(x) \right] Z_0^*(2x) + Z_0(x) Z_1^*(2x) \left[2x\, Z_1(x) - Z_0(x) \right] \right\}$$

$$\int x^2 Z_0^2(x) Z_1^*(2x)\, dx =$$

$$= \frac{x^2}{6} \left\{ x \left[Z_0^2(x) - Z_1^2(x) \right] Z_1^*(2x) - 2 Z_1^2(x) Z_0^*(2x) - 2 Z_0(x) Z_1(x) \left[x Z_0^*(2x) - 2 Z_1^*(2x) \right] \right\}$$

$$\int x^2 Z_0^2(x) Z_1^* \left(\sqrt{2}x \right) dx = x^2 Z_0(x) Z_1(x) Z_1^* \left(\sqrt{2}x \right) - \frac{\sqrt{2}x^2}{2} Z_1^2(x) Z_0^* \left(\sqrt{2}x \right)$$

$$\int x^2 Z_1^2(x) Z_1^*(2x)\, dx =$$

$$= \frac{x^2}{6} \left\{ x \left[Z_1^2(x) - Z_0^2(x) \right] Z_1^*(2x) + 2 Z_0(x) Z_1(x) \left[x Z_0^*(2x) + Z_1^*(2x) \right] - 4 Z_1^2(x) Z_0^*(2x) \right\}$$

$$\int x^2 Z_0(x) Z_1(x) Z_0^*(2x)\, dx =$$

$$= \frac{x^2}{6} \left\{ x \left[Z_1^2(x) - Z_0^2(x) \right] Z_1^*(2x) + 2 Z_0(x) Z_1(2x) \left[x Z_0^*(2x) + Z_1^*(2x) \right] - Z_1^2(x) Z_0^*(2x) \right\}$$

$$\int x^3 I_1^2(x) K_0(x)\, dx =$$

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$$= \frac{x^2}{3} \left\{ 2\, I_0(x)I_1(x)[xK_0(x) - 2K_1(x)] - I_1^2(x)[4K_0(x) + xK_1(x)] + 2xI_0^2(x)K_1(x) \right\}$$

$$\int x^3\, K_1^2(x)I_0(x)\, dx =$$

$$= -\frac{x^2}{3} \left\{ 2\, K_0(x)K_1(x)[xI_0(x) + 2I_1(x)] + K_1^2(x)[4I_0(x) - xI_1(x)] + 2xK_0^2(x)I_1(x) \right\}$$

$$\int x^3 I_0(x)I_1(x)K_1(x)\, dx =$$

$$= \frac{x^2}{3} \left\{ I_0(x)I_1(x)[xK_0(x) - 2K_1(x)] + I_1^2(x)[xK_1(x) - 2K_0(x)] + xI_0^2(x)K_1(x) \right\}$$

$$\int x^3 K_0(x)K_1(x)I_1(x)\, dx =$$

$$= -\frac{x^2}{3} \left\{ K_0(x)K_1(x)[xI_0(x) + 2I_1(x)] + K_1^2(x)[xI_1(x) + 2I_0(x)] + xK_0^2(x)I_1(x) \right\}$$

$$\int I_1^2(x)K_1(2x)\, dx = -\frac{x}{2}\left[I_0^2(x) + I_1^2(x)\right]\, K_1(2x) + 2\, I_0(x)I_1(x)K_0(2x)$$

$$\int K_1^2(x)I_1(2x)\, dx = -\frac{x}{2}\left[K_0^2(x) + K_1^2(x)\right]\, I_1(2x) + 2\, K_0(x)K_1(x)I_0(2x)$$

$$\int x\, I_0^2(x)K_0(2x)\, dx = \frac{x^2}{2}\left\{ \left[I_1^2(x) + I_0^2(x)\right]\, K_0(2x) + 2I_0(x)I_1(x)K_1(2x) \right\}$$

$$\int x\, K_0^2(x)I_0(2x)\, dx = \frac{x^2}{2}\left\{ \left[K_1^2(x) + K_0^2(x)\right]\, I_0(2x) + 2K_0(x)K_1(x)I_1(2x) \right\}$$

$$\int x\, I_1^2(x)K_0(2x)\, dx =$$

$$= \frac{x}{2}\left\{ x\left[I_0^2(x) + I_1^2(x)\right]\, K_0(2x) + 2I_0^2(x)K_1(2x) + 2I_0(x)I_1(x)\left[K_0(2x) + xK_1(2x)\right] \right\}$$

$$\int x\, K_1^2(x)I_0(2x)\, dx =$$

$$= \frac{x}{2}\left\{ x\left[K_0^2(x) + K_1^2(x)\right]\, I_0(2x) - 2K_0^2(x)I_1(2x) - 2K_0(x)K_1(x)\left[I_0(2x) - xI_1(2x)\right] \right\}$$

$$\int x\, I_1^2(x)K_0\left(\sqrt{2}\,x\right)\, dx = \frac{\sqrt{2}\,x}{2}I_0^2(x)K_1\left(\sqrt{2}\,x\right) + xI_0(x)I_1(x)K_0\left(\sqrt{2}\,x\right)$$

$$\int x\, K_1^2(x)I_0\left(\sqrt{2}\,x\right)\, dx = -\frac{\sqrt{2}\,x}{2}K_0^2(x)I_1\left(\sqrt{2}\,x\right) - xK_0(x)K_1(x)I_0\left(\sqrt{2}\,x\right)$$

$$\int x\, I_0(x)I_1(x)K_1(2x)\, dx = \frac{x}{2}\left\{ x\left[I_0^2(x) + I_1^2(x)\right]K_0(2x) + I_0(x)K_1(2x)\left[2x\,I_1(x) + I_0(x)\right] \right\}$$

$$\int x\, K_0(x)K_1(x)I_1(2x)\, dx = \frac{x}{2}\left\{ x\left[K_0^2(x) + K_1^2(x)\right]I_0(2x) + K_0(x)I_1(2x)\left[2x\,K_1(x) - K_0(x)\right] \right\}$$

$$\int x^2\, I_0^2(x)K_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{ x\left[I_0^2(x) + I_1^2(x)\right]\, K_1(2x) + 2I_1^2(x)K_0(2x) + 2I_0(x)I_1(x)\left[xK_0(2x) + 2K_1(2x)\right] \right\}$$

$$\int x^2\, K_0^2(x)I_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{ x\left[K_0^2(x) + K_1^2(x)\right]\, I_1(2x) - 2K_1^2(x)I_0(2x) + 2K_0(x)K_1(x)\left[xI_0(2x) - 2I_1(2x)\right] \right\}$$

$$\int x^2\, I_0^2(x)K_1\left(\sqrt{2}\,x\right)\, dx = x^2\, I_0(x)I_1(x)K_1\left(\sqrt{2}\,x\right) + \frac{\sqrt{2}\,x^2}{2}I_1^2(x)K_0\left(\sqrt{2}\,x\right)$$

$$\int x^2\, K_0^2(x) I_1\left(\sqrt{2}\,x\right)\, dx = -\,x^2\, K_0(x)K_1(x)I_1\left(\sqrt{2}\,x\right) - \frac{\sqrt{2}\,x^2}{2}K_1^2(x)I_0\left(\sqrt{2}\,x\right)$$

$$\int x^2\, I_1^2(x)K_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{ x\,\left[I_0^2(x) + I_1^2(x)\right]\, K_1(2x) + 2I_0(x)I_1(x)\left[xK_0(2x) - K_1(2x)\right] - 4I_1^2(x)K_0(2x)\right\}$$

$$\int x^2\, K_1^2(x)I_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{ x\,\left[K_1^2(x) + K_0^2(x)\right]\, I_1(2x) + 2K_0(x)K_1(x)\left[xI_0(2x) + I_1(2x)\right] + 4K_1^2(x)I_0(2x)\right\}$$

$$\int x^2\, I_0(x)I_1(x)K_0(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{ x\,\left[I_0^2(x) + I_1^2(x)\right]\, K_1(2x) + 2I_0(x)I_1(2x)\left[x\, K_0(2x) - K_1(2x)\right] - I_1^2(x)K_0(2x)\right\}$$

$$\int x^2\, K_0(x)K_1(x)I_0(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{ x\,\left[K_0^2(x) + K_1^2(x)\right]\, I_1(2x) + 2K_0(x)K_1(2x)\left[x\, I_0(2x) + I_1(2x)\right] + K_1^2(x)I_0(2x)\right\}$$

3.2. Integrals of the type $\int x^n Z_\kappa(\alpha x) Z_\mu(\beta x) Z_\nu(\gamma x)\, dx$

The general case is discussed in [12]. In the following some special solutions are given.

a) $x^n Z_\kappa(x) Z_\mu(x) Z_\nu(2x)$

With $\kappa, \mu, \nu \in \{0, 1\}$ the following integrals may be expressed by functions of the same kind:

$$\int x^{2n+1} Z_0^2(x) Z_0(2x)\, dx\ ,\quad \int x^{2n+1} Z_0(x) Z_1(x) Z_1(2x)\, dx\ ,\quad \int x^{2n+1} Z_1^2(x) Z_0(2x)\, dx\ ,\quad n \geq 0,$$

and

$$\int x^{2n} Z_0(x) Z_1(x) Z_0(2x)\, dx\ ,\quad \int x^{2n} Z_0^2(x) Z_1(2x)\, dx\ ,\quad \int x^{2n} Z_1^2(x) Z_1(x) Z_1(2x)\, dx\ ,\quad n \geq 1.$$

$$\int J_1^2(x) J_1(2x)\, dx = \frac{x}{2}\left[J_0^2(x) J_1(2x) - J_1^2(x) J_1(2x) - 2J_0(x) J_1(x) J_0(2x)\right]$$

$$\int I_1^2(x) I_1(2x)\, dx = -\frac{x}{2}\left[I_0^2(x) I_1(2x) + I_1^2(x) I_1(2x) - 2I_0(x) I_1(x) I_0(2x)\right]$$

$$\int K_1^2(x) K_1(2x)\, dx = -\frac{x}{2}\left[K_0^2(x) K_1(2x) + K_1^2(x) K_1(2x) - 2K_0(x) K_1(x) K_0(2x)\right]$$

$$\int x J_0^2(x) J_0(2x)\, dx = \frac{x^2}{2}\left[J_0^2(x) J_0(2x) - J_1^2(x) J_0(2x) + 2J_0(x) J_1(x) J_1(2x)\right]$$

$$\int x I_0^2(x) I_0(2x)\, dx = \frac{x^2}{2}\left[I_0^2(x) I_0(2x) + I_1^2(x) I_0(2x) - 2I_0(x) I_1(x) I_1(2x)\right]$$

$$\int x K_0^2(x) K_0(2x)\, dx = \frac{x^2}{2}\left[K_0^2(x) K_0(2x) + K_1^2(x) K_0(2x) - 2K_0(x) K_1(x) K_1(2x)\right]$$

$$\int x J_0(x) J_1(x) J_1(2x)\, dx = \frac{x}{2}\left[x\, J_0^2(x) J_0(2x) - J_0^2(x) J_1(2x) - x\, J_1^2(x) J_0(2x) + 2\, x J_0(x) J_1(x) J_1(2x)\right]$$

$$\int x I_0(x) I_1(x) I_1(2x)\, dx = \frac{x}{2}\left[-x\, I_0^2(x) I_0(2x) + I_0^2(x) I_1(2x) - x\, I_1^2(x) I_0(2x) + 2\, x I_0(x) I_1(x) I_1(2x)\right]$$

$$\int x K_0(x) K_1(x) K_1(2x)\, dx =$$

$$= \frac{x}{2}\left[-x\, K_0^2(x) K_0(2x) - K_0^2(x) K_1(2x) - x\, K_1^2(x) K_0(2x) + 2\, x K_0(x) K_1(x) K_1(2x)\right]$$

$$\int x J_1^2(x) J_0(2x)\, dx =$$

$$= \frac{x}{2}\left[-x\, J_0^2(x) J_0(2x) + 2J_0^2(x) J_1(2x) + x\, J_1^2(x) J_0(2x) - 2J_0(x) J_1(x) J_0(2x) - 2\, x J_0(x) J_1(x) J_1(2x)\right]$$

$$\int x I_1^2(x) I_0(2x)\, dx =$$

$$= \frac{x}{2}\left[x\, I_0^2(x) I_0(2x) - 2I_0^2(x) I_1(2x) + x\, I_1^2(x) I_0(2x) + 2I_0(x) I_1(x) I_0(2x) - 2\, x I_0(x) I_1(x) I_1(2x)\right]$$

$$\int x K_1^2(x) K_0(2x)\, dx =$$

$$= \frac{x}{2}\left[x\, K_0^2(x) K_0(2x) + 2K_0^2(x) K_1(2x) + x\, K_1^2(x) K_0(2x) - 2K_0(x) K_1(x) K_0(2x) - 2\, x K_0(x) K_1(x) K_1(2x)\right]$$

$$\int x^2 J_0(x)\, J_1(x)\, J_0(2x)\, dx =$$

$$= \frac{x^2}{6}\left[-x\, J_0^2(x)J_1(2x) - J_1^2(x)J_0(2x) + x\, J_1^2(x)J_1(2x) + 2\, xJ_0(x)J_1(x)J_0(2x) + 2J_0(x)J_1(x)J_1(2x)\right]$$

$$\int x^2 I_0(x)\, I_1(x)\, I_0(2x)\, dx =$$

$$= \frac{x^2}{6}\left[-x\, I_0^2(x)I_1(2x) - I_1^2(x)I_0(2x) - x\, I_1^2(x)I_1(2x) + 2\, xI_0(x)I_1(x)I_0(2x) + 2I_0(x)I_1(x)I_1(2x)\right]$$

$$\int x^2 K_0(x)\, K_1(x)\, K_0(2x)\, dx =$$

$$= \frac{x^2}{6}\left[-x\, K_0^2(x)K_1(2x) + K_1^2(x)K_0(2x) - x\, K_1^2(x)K_1(2x) + 2\, xK_0(x)K_1(x)K_0(2x) - 2K_0(x)K_1(x)K_1(2x)\right]$$

$$\int x^2\, J_0^2(x)\, J_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left[x\, J_0^2(x)J_1(2x) - 2\, J_1^2(x)J_0(2x) - x\, J_1^2(x)J_1(2x) - 2\, xJ_0(x)J_1(x)J_0(2x) + 4J_0(x)J_1(x)J_1(2x)\right]$$

$$\int x^2\, I_0^2(x)\, I_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left[x\, I_0^2(x)I_1(2x) - 2\, I_1^2(x)I_0(2x) + x\, I_1^2(x)I_1(2x) - 2\, xI_0(x)I_1(x)I_0(2x) + 4I_0(x)I_1(x)I_1(2x)\right]$$

$$\int x^2\, K_0^2(x)\, K_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left[x\, K_0^2(x)K_1(2x) + 2\, K_1^2(x)K_0(2x) + x\, K_1^2(x)K_1(2x) - 2\, xK_0(x)K_1(x)K_0(2x) - 4K_0(x)K_1(x)K_1(2x)\right]$$

$$\int x^2\, J_1^2(x)\, J_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left[-xJ_0^2(x)J_1(2x) - 4\, J_1^2(x)\, J_0(2x) + x\, J_1^2(x)\, J_1(2x) + 2\, xJ_0(x)\, J_1(x)J_0(2x) + 2J_0(x)\, J_1(x)J_1(2x)\right]$$

$$\int x^2\, I_1^2(x)\, I_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left[xI_0^2(x)I_1(2x) + 4\, I_1^2(x)\, I_0(2x) + x\, I_1^2(x)\, I_1(2x) - 2\, xI_0(x)\, I_1(x)I_0(2x) - 2I_0(x)\, I_1(x)I_1(2x)\right]$$

$$\int x^2\, K_1^2(x)\, K_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left[xK_0^2(x)K_1(2x) - 4\, K_1^2(x)\, K_0(2x) + x\, K_1^2(x)\, K_1(2x) - 2\, xK_0(x)\, K_1(x)K_0(2x) + 2K_0(x)\, K_1(x)K_1(2x)\right]$$

$$\int x^3 J_0^2(x)\, J_0(2x)\, dx = \frac{x^2}{10}[x^2\, J_0^2(x)\, J_0(2x) + 2x\, J_0(x)\, J_1(x)\, J_0(2x) + (2-x^2)\, J_1^2(x)\, J_0(2x) + 2x\, J_0^2(x)\, J_1(2x) +$$

$$+ (2x^2 - 4)\, J_0(x)\, J_1(x)\, J_1(2x) + 2x\, J_1^2(x)\, J_1(2x)]$$

$$\int x^3 I_0^2(x)\, I_0(2x)\, dx = \frac{x^2}{10}[x^2\, I_0^2(x)\, I_0(2x) + 2x\, I_0(x)\, I_1(x)\, I_0(2x) + (x^2+2)\, I_1^2(x)\, I_0(2x) + 2x\, I_0^2(x)\, I_1(2x) -$$

$$- (2x^2 + 4)\, I_0(x)\, I_1(x)\, I_1(2x) - 2x\, I_1^2(x)\, I_1(2x)]$$

$$\int x^3 K_0^2(x)\, K_0(2x)\, dx = \frac{x^2}{10}[x^2\, K_0^2(x)\, K_0(2x) - 2x\, K_0(x)\, K_1(x)\, K_0(2x) + (x^2+2)\, K_1^2(x)\, K_0(2x) -$$

$$- 2x\, K_0^2(x)\, K_1(2x) - (2x^2+4)\, K_0(x)\, K_1(x)\, K_1(2x) + 2x\, K_1^2(x)\, K_1(2x)]$$

$$\int x^3 J_1^2(x) J_0(2x)\, dx = \frac{x^2}{10}[-x^2 J_0^2(x) J_0(2x) + 2x J_0(x) J_1(x) J_0(2x) + (2 - x^2) J_1^2(x) J_0(2x) +$$

$$+ 2x J_0^2(x) J_1(2x) + (2x^2 - 4) J_0(x) J_1(x) J_1(2x) + 2x J_1^2(x) J_1(2x)]$$

$$\int x^3 I_1^2(x) I_0(2x)\, dx = \frac{x^2}{10}[x^2 I_0^2(x) I_0(2x) + 2x I_0(x) I_1(x) I_0(2x) + (x^2 + 2) I_1^2(x) I_0(2x) +$$

$$+ 2x I_0^2(x) I_1(2x) - (2x^2 + 4) I_0(x) I_1(x) I_1(2x) - 2x I_1^2(x) I_1(2x)]$$

$$\int x^3 K_1^2(x) K_0(2x)\, dx = \frac{x^2}{10}[x^2 K_0^2(x) K_0(2x) - 2x K_0(x) K_1(x) K_0(2x) + (x^2 + 2) K_1^2(x) K_0(2x) -$$

$$-2x K_0^2(x) K_1(2x) - (2x^2 + 4) K_0(x) K_1(x) K_1(2x) + 2x K_1^2(x) K_1(2x)]$$

$$\int x^3 J_0(x) J_1(x) J_1(2x)\, dx = \frac{x^2}{30}[3x^2 J_0^2(x) J_0(2x) - 4x J_0(x) J_1(x) J_0(2x) - (3x^2 + 4) J_1^2(x) J_0(2x) -$$

$$-4x J_0^2(x) J_1(2x) + (6x^2 + 8) J_0(x) J_1(x) J_1(2x) + x J_1^2(x) J_1(2x)]$$

$$\int x^3 I_0(x) I_1(x) I_1(2x)\, dx = \frac{x^2}{30}[-3x^2 I_0^2(x) I_0(2x) + 4x I_0(x) I_1(x) I_0(2x) + (4 - 3x^2) I_1^2(x) I_0(2x) +$$

$$+4x I_0^2(x) I_1(2x) + (6x^2 - 8) I_0(x) I_1(x) I_1(2x) + x I_1^2(x) I_1(2x)]$$

$$\int x^3 K_0(x) K_1(x) K_1(2x)\, dx = \frac{x^2}{30}[-3x^2 K_0^2(x) K_0(2x) - 4x K_0(x) K_1(x) K_0(2x) + (4 - 3x^2) K_1^2(x) K_0(2x) -$$

$$-4x K_0^2(x) K_1(2x) + (6x^2 - 8) K_0(x) K_1(x) K_1(2x) - x K_1^2(x) K_1(2x)]$$

$$\int x^4 J_0(x) J_1(x) J_0(2x)\, dx = \frac{x^2}{42}[-3x^2 J_0^2(x) J_0(2x) + (6x^3 + 4x) J_0(x) J_1(x) J_0(2x) + 4 J_1^2(x) J_0(2x) +$$

$$+(4x - 3x^3) J_0^2(x) J_1(2x) + (6x^2 - 8) J_0(x) J_1(x) J_1(2x) + (3x^3 + 2x) J_1^2(x) J_1(2x)]$$

$$\int x^4 I_0(x) I_1(x) I_0(2x)\, dx = \frac{x^2}{42}[3x^2 I_0^2(x) I_0(2x) + (6x^3 - 4x) I_0(x) I_1(x) I_0(2x) - 4 I_1^2(x) I_0(2x) -$$

$$-(3x^3 + 4x) I_0^2(x) I_1(2x) + (6x^2 + 8) I_0(x) I_1(x) I_1(2x) + (2x - 3x^3) I_1^2(x) I_1(2x)]$$

$$\int x^4 K_0(x) K_1(x) K_0(2x)\, dx = \frac{x^2}{42}[-3x^2 K_0^2(x) K_0(2x) + (6x^3 - 4x) K_0(x) K_1(x) K_0(2x) + 4 K_1^2(x) K_0(2x) -$$

$$-(3x^3 + 4x) K_0^2(x) K_1(2x) - (6x^2 + 8) K_0(x) K_1(x) K_1(2x) + (2x - 3x^3) K_1^2(x) K_1(2x)]$$

$$\int x^4 J_0^2(x) J_1(2x)\, dx = \frac{x^2}{210}[-48x^2 J_0^2(x) J_0(2x) + (-30x^3 + 64x) J_0(x) J_1(x) J_0(2x) +$$

$$+(-42x^2 + 64) J_1^2(x) J_0(2x) + (15x^3 + 64x) J_0^2(x) J_1(2x) + (54x^2 - 128) J_0(x) J_1(x) J_1(2x) +$$

$$+(-15x^3 + 74x) J_1^2(x) J_1(2x)]$$

$$\int x^4 I_0^2(x) I_1(2x)\, dx = \frac{x^2}{210}[48x^2 I_0^2(x) I_0(2x) - (30x^3 + 64x) I_0(x) I_1(x) I_0(2x) - (42x^2 + 64) I_1^2(x) I_0(2x) +$$

$$+(15x^3 - 64x) I_0^2(x) I_1(2x) + (54x^2 + 128) I_0(x) I_1(x) I_1(2x) + (15x^3 + 74x) I_1^2(x) I_1(2x)]$$

$$\int x^4 K_0^2(x) K_1(2x)\, dx = \frac{x^2}{210}[-48x^2 K_0^2(x) K_0(2x) - (30x^3 + 64x) K_0(x) K_1(x) K_0(2x) +$$

$$+(42x^2 + 64) K_1^2(x) K_0(2x) + (15x^3 - 64x) K_0^2(x) K_1(2x) - (54x^2 + 128) K_0(x) K_1(x) K_1(2x) +$$

$$+(15x^3 + 74x) K_1^2(x) K_1(2x)]$$

$$\int x^4 J_1^2(x)\, J_1(2x)\, dx = \frac{x^2}{70}\big[-12x^2 J_0^2(x)\, J_0(2x)+(10\,x^3+16\,x)\, J_0(x)\, J_1(x)\, J_0(2x)+(-28\,x^2+16)\, J_1^2(x)\, J_0(2x)+$$

$$+(-5\,x^3+16\,x)\, J_0^2(x)\, J_1(2x)-(4\,x^2+32)\, J_0(x)\, J_1(x)\, J_1(2x)+(5\,x^3+36\,x)\, J_1^2(x)\, J_1(2x)\big]$$

$$\int x^4 I_1^2(x)\, I_1(2x)\, dx = \frac{x^2}{70}\big[-12x^2 I_0^2(x)\, I_0(2x)+(-10\,x^3+16\,x)\, I_0(x)\, I_1(x)\, I_0(2x)+(28\,x^2+16)\, I_1^2(x)\, I_0(2x)+$$

$$+(5\,x^3+16\,x)\, I_0^2(x)\, I_1(2x)+(4\,x^2-32)\, I_0(x)\, I_1(x)\, I_1(2x)+(5\,x^3-36\,x)\, I_1^2(x)\, I_1(2x)\big]$$

$$\int x^4 K_1^2(x)\, K_1(2x)\, dx = \frac{x^2}{70}\big[12x^2 K_0^2(x)\, K_0(2x)+(-10\,x^3+16\,x)\, K_0(x)\, K_1(x)\, K_0(2x)-$$

$$-(28\,x^2+16)\, K_1^2(x)\, K_0(2x)+(5\,x^3+16\,x)\, K_0^2(x)\, K_1(2x)+(-4\,x^2+32)\, K_0(x)\, K_1(x)\, K_1(2x)+$$

$$+(5\,x^3-36\,x)\, K_1^2(x)\, K_1(2x)\big]$$

$$\int x^5 J_0^2(x)\, J_0(2x)\, dx = \frac{x^2}{630}\big[(35\,x^4+216\,x^2)\, J_0^2(x)\, J_0(2x)+(100\,x^3-288\,x)\, J_0(x)\, J_1(x)\, J_0(2x)+$$

$$+(-35\,x^4+224\,x^2-288)\, J_1^2(x)\, J_0(2x)+(160\,x^3-288\,x)\, J_0^2(x)\, J_1(2x)+(70\,x^4-208\,x^2+576)\, J_0(x)\, J_1(x)\, J_1(2x)+$$

$$+(120\,x^3-368\,x)\, J_1^2(x)\, J_1(2x)\big]$$

$$\int x^5 I_0^2(x)\, I_0(2x)\, dx = \frac{x^2}{630}\big[(35\,x^4-216\,x^2)\, I_0^2(x)\, I_0(2x)+(100\,x^3+288\,x)\, I_0(x)\, I_1(x)\, I_0(2x)+$$

$$+(35\,x^4+224\,x^2+288)\, I_1^2(x)\, I_0(2x)+(160x^3+288x)\, I_0^2(x)\, I_1(2x)-(70\,x^4+208\,x^2+576)\, I_0(x)\, I_1(x)\, I_1(2x)-$$

$$-(120\,x^3+368\,x)\, I_1^2(x)\, I_1(2x)\big]$$

$$\int x^5 K_0^2(x)\, K_0(2x)\, dx = \frac{x^2}{630}\big[(35\,x^4-216\,x^2)\, K_0^2(x)\, K_0(2x)-(100\,x^3+288\,x)\, K_0(x)\, K_1(x)\, K_0(2x)+$$

$$+(35\,x^4+224\,x^2+288)\, K_1^2(x)\, K_0(2x)-(160\,x^3+288\,x)\, K_0^2(x)\, K_1(2x)-$$

$$-(70\,x^4+208\,x^2+576)\, K_0(x)\, K_1(x)\, K_1(2x)+(120\,x^3+368\,x)\, K_1^2(x)\, K_1(2x)\big]$$

$$\int x^5 J_1^2(x)\, J_0(2x)\, dx = \frac{x^2}{630}\big[(-35\,x^4+180\,x^2)\, J_0^2(x)\, J_0(2x)+(100\,x^3-288\,x)\, J_0(x)\, J_1(x)\, J_0(2x)+$$

$$+(-35\,x^4+244\,x^2-288)\, J_1^2(x)\, J_0(2x)+(160\,x^3-288\,x)\, J_0^2(x)\, J_1(2x)+(70\,x^4-208\,x^2+576)\, J_0(x)\, J_1(x)\, J_1(2x)+$$

$$+(120\,x^3-368\,x)\, J_1^2(x)\, J_1(2x)\big]$$

$$\int x^5 I_1^2(x)\, I_0(2x)\, dx = \frac{x^2}{630}\big[(35x^4+180x^2)\, I_0^2(x)\, I_0(2x)+(100\,x^3+288\,x)\, I_0(x)\, I_1(x)\, I_0(2x)+$$

$$+(35\,x^4+224\,x^2+288)\, I_1^2(x)\, I_0(2x)+(160\,x^3+288\,x)\, I_0^2(x)\, I_1(2x)-(70\,x^4+208\,x^2+576)\, I_0(x)\, I_1(x)\, I_1(2x)-$$

$$-(120\,x^3+368\,x)\, I_1^2(x)\, I_1(2x)\big]$$

$$\int x^5 K_1^2(x)\, K_0(2x)\, dx = \frac{x^2}{630}\big[(35\,x^4+180\,x^2)\, K_0^2(x)\, K_0(2x)-(100\,x^3+288\,x)\, K_0(x)\, K_1(x)\, K_0(2x)+$$

$$+(35\,x^4+224\,x^2+288)\, K_1^2(x)\, K_0(2x)-(160\,x^3+288\,x)\, K_0^2(x)\, K_1(2x)-$$

$$-(70\,x^4+208\,x^2+576)\, K_0(x)\, K_1(x)\, K_1(2x)+(120\,x^3+368\,x)\, K_1^2(x)\, K_1(2x)\big]$$

$$\int x^5 J_0(x)\, J_1(x)\, J_1(2x)\, dx = \frac{x^2}{630}\big[(35\,x^4-72\,x^2)\, J_0^2(x)\, J_0(2x)+(-80\,x^3+96\,x)\, J_0(x)\, J_1(x)\, J_0(2x)+$$

$$+(-35\,x^4-28\,x^2+96)\, J_1^2(x)\, J_0(2x)+(-65\,x^3+96\,x)\, J_0^2(x)\, J_1(2x)+(70\,x^4+116\,x^2-192)\, J_0(x)\, J_1(x)\, J_1(2x)+$$

$$+(30\,x^3 + 76\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^5 I_0(x)\,I_1(x)\,I_1(2x)\,dx = \frac{x^2}{630}[-(35\,x^4 + 72\,x^2)\,I_0^2(x)\,I_0(2x) + (80\,x^3 + 96\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+$$

$$+(-35\,x^4 + 28\,x^2 + 96)\,I_1^2(x)\,I_0(2x) + (65\,x^3 + 96\,x)\,I_0^2(x)\,I_1(2x) + (70\,x^4 - 116\,x^2 - 192)\,I_0(x)\,I_1(x)\,I_1(2x)+$$

$$+(30\,x^3 - 76\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^5 K_0(x)\,K_1(x)\,K_1(2x)\,dx = \frac{x^2}{630}[-(35\,x^4 + 72\,x^2)\,K_0^2(x)\,K_0(2x) - (80\,x^3 + 96\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+$$

$$+(-35\,x^4 + 28\,x^2 + 96)\,K_1^2(x)\,K_0(2x) - (65\,x^3 + 96\,x)\,K_0^2(x)\,K_1(2x)+$$

$$+(70\,x^4 - 116\,x^2 - 192)\,K_0(x)\,K_1(x)\,K_1(2x) + (-30\,x^3 + 76\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^6 J_0(x)\,J_1(x)\,J_0(2x)\,dx =$$

$$= \frac{x^2}{6930}[(-490\,x^4 + 1224\,x^2)\,J_0^2(x)\,J_0(2x) + (630\,x^5 + 940\,x^3 - 1632\,x)\,J_0(x)\,J_1(x)\,J_0(2x)+$$

$$+(175\,x^4 + 896\,x^2 - 1632)\,J_1^2(x)\,J_0(2x) + (-315\,x^5 + 1000\,x^3 - 1632\,x)\,J_0^2(x)\,J_1(2x)+$$

$$+(910\,x^4 - 1552\,x^2 + 3264)\,J_0(x)\,J_1(x)\,J_1(2x) + (315\,x^5 + 120\,x^3 - 1712\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^6 I_0(x)\,I_1(x)\,I_0(2x)\,dx =$$

$$= \frac{x^2}{6930}[(490\,x^4 + 1224\,x^2)\,I_0^2(x)\,I_0(2x) + (630\,x^5 - 940\,x^3 - 1632\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+$$

$$+(175\,x^4 - 896\,x^2 - 1632)\,I_1^2(x)\,I_0(2x) - (315\,x^5 + 1000\,x^3 + 1632\,x)\,I_0^2(x)\,I_1(2x)+$$

$$+(910\,x^4 + 1552\,x^2 + 3264)\,I_0(x)\,I_1(x)\,I_1(2x) + (-315\,x^5 + 120\,x^3 + 1712\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^6 K_0(x)\,K_1(x)\,K_0(2x)\,dx =$$

$$= \frac{x^2}{6930}[-(490\,x^4 + 1224\,x^2)\,K_0^2(x)\,K_0(2x) + (630\,x^5 - 940\,x^3 - 1632\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+$$

$$+(-175\,x^4 + 896\,x^2 + 1632)\,K_1^2(x)\,K_0(2x) - (315\,x^5 + 1000\,x^3 + 1632\,x)\,K_0^2(x)\,K_1(2x)-$$

$$-(910\,x^4 + 1552\,x^2 + 3264)\,K_0(x)\,K_1(x)\,K_1(2x) + (-315\,x^5 + 120\,x^3 + 1712\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^6 J_0^2(x)\,J_1(2x)\,dx = \frac{x^2}{6930}[(-1820\,x^4 + 5904\,x^2)\,J_0^2(x)\,J_0(2x)+$$

$$+(-630\,x^5 + 2360\,x^3 - 7872\,x)\,J_0(x)\,J_1(x)\,J_0(2x) + (-1330\,x^4 + 6496\,x^2 - 7872)\,J_1^2(x)\,J_0(2x)+$$

$$+(315\,x^5 + 4280\,x^3 - 7872\,x)\,J_0^2(x)\,J_1(2x) + (1400\,x^4 - 5312\,x^2 + 15744)\,J_0(x)\,J_1(x)\,J_1(2x)+$$

$$+(-315\,x^5 + 3840\,x^3 - 10432\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^6 I_0^2(x)\,I_1(2x)\,dx = \frac{x^2}{6930}[(1820\,x^4 + 5904\,x^2)\,I_0^2(x)\,I_0(2x)-$$

$$-(630\,x^5 + 2360\,x^3 + 7872\,x)\,I_0(x)\,I_1(x)\,I_0(2x) - (1330\,x^4 + 6496\,x^2 + 7872)\,I_1^2(x)\,I_0(2x)+$$

$$+(315\,x^5 - 4280\,x^3 - 7872\,x)\,I_0^2(x)\,I_1(2x) + (1400\,x^4 + 5312\,x^2 + 15744)\,I_0(x)\,I_1(x)\,I_1(2x)+$$

$$+(315\,x^5 + 3840\,x^3 + 10432\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^6 K_0^2(x)\,K_1(2x)\,dx = \frac{x^2}{6930}[-(1820\,x^4 + 5904\,x^2)\,K_0^2(x)\,K_0(2x)-$$

$$-(630\,x^5 + 2360\,x^3 + 7872\,x)\,K_0(x)\,K_1(x)\,I_0(2x) + (1330\,x^4 + 6496\,x^2 + 7872)\,K_1^2(x)\,K_0(2x)+$$

$$+(315\,x^5 - 4280\,x^3 - 7872\,x)\,K_0^2(x)\,K_1(2x) - (1400\,x^4 + 5312\,x^2 + 15744)\,K_0(x)\,K_1(x)\,K_1(2x)+$$

$$+(315\,x^5 + 3840\,x^3 + 10432\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^6\,J_1^2(x)\,J_1(2x)\,dx = \frac{x^2}{770}[(-140\,x^4 + 576\,x^2)\,J_0^2(x)\,J_0(2x)+$$

$$+(70\,x^5 + 80\,x^3 - 768\,x)\,J_0(x)\,J_1(x)\,J_0(2x) + (-280\,x^4 + 784\,x^2 - 768)\,J_1^2(x)\,J_0(2x)+$$

$$+(-35\,x^5 + 380\,x^3 - 768\,x)\,J_0^2(x)\,J_1(2x) + (-70\,x^4 - 368\,x^2 + 1536)\,J_0(x)\,J_1(x)\,J_1(2x)+$$

$$+(35\,x^5 + 600\,x^3 - 1168\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^6\,I_1^2(x)\,I_1(2x)\,dx = \frac{x^2}{770}[-(140\,x^4 + 576\,x^2)\,I_0^2(x)\,I_0(2x)+$$

$$+(-70\,x^5 + 80\,x^3 + 768\,x)\,I_0(x)\,I_1(x)\,I_0(2x) + (280\,x^4 + 784\,x^2 + 768)\,I_1^2(x)\,I_0(2x)+$$

$$+(35\,x^5 + 380\,x^3 + 768\,x)\,I_0^2(x)\,I_1(2x) + (70\,x^4 - 368\,x^2 - 1536)\,I_0(x)\,I_1(x)\,I_1(2x)+$$

$$+(35\,x^5 - 600\,x^3 - 1168\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^6\,K_1^2(x)\,K_1(2x)\,dx = \frac{x^2}{770}[(140\,x^4 + 576\,x^2)\,K_0^2(x)\,K_0(2x)+$$

$$+(-70\,x^5 + 80\,x^3 + 768\,x)\,K_0(x)\,K_1(x)\,K_0(2x) - (280\,x^4 + 784\,x^2 + 768)\,K_1^2(x)\,K_0(2x)+$$

$$+(35\,x^5 + 380\,x^3 + 768\,x)\,K_0^2(x)\,K_1(2x) + (-70\,x^4 + 368\,x^2 + 1536)\,K_0(x)\,K_1(x)\,K_1(2x)+$$

$$+(35\,x^5 - 600\,x^3 - 1168\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^7 J_0^2(x)\,J_0(2x)\,dx = \frac{x^2}{1430}[(55\,x^6 + 900\,x^4 - 3024\,x^2)\,J_0^2(x)\,J_0(2x)+$$

$$+(210\,x^5 - 1080\,x^3 + 4032\,x)\,J_0(x)\,J_1(x)\,J_0(2x) + (-55\,x^6 + 810\,x^4 - 3456\,x^2 + 4032)\,J_1^2(x)\,J_0(2x)+$$

$$+(390\,x^5 - 2160\,x^3 + 4032\,x)\,J_0^2(x)\,J_1(2x) + (110\,x^6 - 540\,x^4 + 2592\,x^2 - 8064)\,J_0(x)\,J_1(x)\,J_1(2x)+$$

$$+(270\,x^5 - 2160\,x^3 + 5472\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^7 I_0^2(x)\,I_0(2x)\,dx = \frac{x^2}{1430}[(55\,x^6 - 900\,x^4 - 3024\,x^2)\,I_0^2(x)\,I_0(2x)+$$

$$+(210\,x^5 + 1080\,x^3 + 4032\,x)\,I_0(x)\,I_1(x)\,I_0(2x) + (55\,x^6 + 810\,x^4 + 3456\,x^2 + 4032)\,I_1^2(x)\,I_0(2x)+$$

$$+(390\,x^5 + 2160\,x^3 + 4032\,x)\,I_0^2(x)\,I_1(2x) - (110\,x^6 + 540\,x^4 + 2592\,x^2 + 8064)\,I_0(x)\,I_1(x)\,I_1(2x)-$$

$$-(270\,x^5 + 2160\,x^3 + 5472\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^7 K_0^2(x)\,K_0(2x)\,dx = \frac{x^2}{1430}[(55\,x^6 - 900\,x^4 - 3024\,x^2)\,K_0^2(x)\,K_0(2x)-$$

$$-(210\,x^5 + 1080\,x^3 + 4032\,x)\,K_0(x)\,K_1(x)\,K_0(2x) + (55\,x^6 + 810\,x^4 + 3456\,x^2 + 4032)\,K_1^2(x)\,K_0(2x)-$$

$$-(390\,x^5 + 2160\,x^3 + 4032\,x)\,K_0^2(x)\,K_1(2x) - (110\,x^6 + 540\,x^4 + 2592\,x^2 + 8064)\,K_0(x)\,K_1(x)\,K_1(2x)+$$

$$+(270\,x^5 + 2160\,x^3 + 5472\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^7 J_1^2(x)\,J_0(2x)\,dx = \frac{x^2}{30030}[(-1155\,x^6 + 15680\,x^4 - 58176\,x^2)\,J_0^2(x)\,J_0(2x)+$$

$$+(4410\,x^5 - 22680\,x^3 + 84672\,x)\,J_0(x)\,J_1(x)\,J_0(2x) + (-1155\,x^6 + 17010\,x^4 - 72576\,x^2 + 84672)\,J_1^2(x)\,J_0(2x)+$$

$$+(8190\,x^5 - 45360\,x^3 + 84672\,x)\,J_0^2(x)\,J_1(2x) + (2310\,x^6 - 11340\,x^4 + 54432\,x^2 - 169344)\,J_0(x)\,J_1(x)\,J_1(2x)+$$

$$+(5670\,x^5 - 45360\,x^3 + 114912\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^7 I_1^2(x)\,I_0(2x)\,dx = \frac{x^2}{30030}[(1155\,x^6 + 15680\,x^4 + 58176\,x^2)\,I_0^2(x)\,I_0(2x)+$$

$$+(4410\,x^5 + 22680\,x^3 + 84672\,x)\,I_0(x)\,I_1(x)\,I_0(2x) + (1155\,x^6 + 17010\,x^4 + 72576\,x^2 + 84672)\,I_1^2(x)\,I_0(2x)+$$

$$+(8190\,x^5 + 45360\,x^3 + 84672\,x)\,I_0^2(x)\,I_1(2x) - (2310\,x^6 + 11340\,x^4 + 54432\,x^2 + 169344)\,I_0(x)\,I_1(x)\,I_1(2x)-$$

$$-(5670\,x^5 + 45360\,x^3 + 114912\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^7 K_1^2(x)\,K_0(2x)\,dx = \frac{x^2}{30030}[(1155\,x^6 + 15680\,x^4 + 58176\,x^2)\,K_0^2(x)\,K_0(2x)-$$

$$-(4410\,x^5+22680\,x^3+84672\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+(1155\,x^6+17010\,x^4+72576\,x^2+84672)\,K_1^2(x)\,K_0(2x)-$$

$$-(8190\,x^5+45360\,x^3+84672\,x)\,K_0^2(x)\,K_1(2x)-(2310\,x^6+11340\,x^4+54432\,x^2+169344)\,K_0(x)\,K_1(x)\,K_1(2x)+$$

$$+(5670\,x^5 + 45360\,x^3 + 114912\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^7 J_0(x)\,J_1(x)\,J_1(2x)\,dx = \frac{x^2}{30030}[(1155\,x^6 - 4760\,x^4 + 13248\,x^2)\,J_0^2(x)\,J_0(2x)+$$

$$+(-3780\,x^5+8000\,x^3-17664\,x)\,J_0(x)\,J_1(x)\,J_0(2x)+(-1155\,x^6-280\,x^4+11872\,x^2-17664)\,J_1^2(x)\,J_0(2x)+$$

$$+(-2730\,x^5+10280\,x^3-17664\,x)\,J_0^2(x)\,J_1(2x)+(2310\,x^6+6860\,x^4-14624\,x^2+35328)\,J_0(x)\,J_1(x)\,J_1(2x)+$$

$$+(1575\,x^5 + 4560\,x^3 - 20704\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^7 I_0(x)\,I_1(x)\,I_1(2x)\,dx = \frac{x^2}{30030}[-(1155\,x^6 + 4760\,x^4 + 13248\,x^2)\,I_0^2(x)\,I_0(2x)+$$

$$+(3780\,x^5 + 8000\,x^3 + 17664\,x)\,I_0(x)\,I_1(x)\,I_0(2x) + (-1155\,x^6 + 280\,x^4 + 11872\,x^2 + 17664)\,I_1^2(x)\,I_0(2x)+$$

$$+(2730\,x^5 + 10280\,x^3 + 17664\,x)\,I_0^2(x)\,I_1(2x) + (2310\,x^6 - 6860\,x^4 - 14624\,x^2 - 35328)\,I_0(x)\,I_1(x)\,I_1(2x)+$$

$$+(1575\,x^5 - 4560\,x^3 - 20704\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^7 K_0(x)\,K_1(x)\,K_1(2x)\,dx = \frac{x^2}{30030}[-(1155\,x^6 + 4760\,x^4 + 13248\,x^2)\,K_0^2(x)\,K_0(2x)-$$

$$-(3780\,x^5+8000\,x^3+17664\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+(-1155\,x^6+280\,x^4+11872\,x^2+17664)\,K_1^2(x)\,K_0(2x)-$$

$$-(2730\,x^5+10280\,x^3+17664\,x)\,K_0^2(x)\,K_1(2x)+(2310\,x^6-6860\,x^4-14624\,x^2-35328)\,K_0(x)\,K_1(x)\,K_1(2x)+$$

$$+(-1575\,x^5 + 4560\,x^3 + 20704\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^8 J_0(x)\,J_1(x)\,J_0(2x)\,dx = \frac{x^2}{30030}[(-2079\,x^6 + 10752\,x^4 - 32832\,x^2)\,J_0^2(x)\,J_0(2x)+$$

$$+(2002\,x^7+5712\,x^5-15648\,x^3+43776\,x)\,J_0(x)\,J_1(x)\,J_0(2x)+(1078\,x^6+4872\,x^4-33600\,x^2+43776)\,J_1^2(x)\,J_0(2x)+$$

$$+(-1001\,x^7 + 5460\,x^5 - 24432\,x^3 + 43776\,x)\,J_0^2(x)\,J_1(2x)+$$

$$+(3850\,x^6 - 11256\,x^4 + 32064\,x^2 - 87552)\,J_0(x)\,J_1(x)\,J_1(2x)+$$

$$+(1001\,x^7 - 378\,x^5 - 17568\,x^3 + 55488\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^8 I_0(x)\,I_1(x)\,I_0(2x)\,dx = \frac{x^2}{30030}[(2079\,x^6 + 10752\,x^4 + 32832\,x^2)\,I_0^2(x)\,I_0(2x)+$$

$$+(2002\,x^7-5712\,x^5-15648\,x^3-43776\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+(1078\,x^6-4872\,x^4-33600\,x^2-43776)\,I_1^2(x)\,I_0(2x)-$$

$$-(1001\,x^7+5460\,x^5+24432\,x^3+43776\,x)\,I_0^2(x)\,I_1(2x)+(3850\,x^6+11256\,x^4+32064\,x^2+87552)\,I_0(x)\,I_1(x)\,I_1(2x)+$$

$$+(-1001\,x^7 - 378\,x^5 + 17568\,x^3 + 55488\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^8 K_0(x)\,K_1(x)\,K_0(2x)\,dx = \frac{x^2}{30030}[-(2079\,x^6 + 10752\,x^4 + 32832\,x^2)\,K_0^2(x)\,K_0(2x)+$$

$$+(2002\,x^7 - 5712\,x^5 - 15648\,x^3 - 43776\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+$$

$$+(-1078\,x^6 + 4872\,x^4 + 33600\,x^2 + 43776)\,K_1^2(x)\,K_0(2x)-$$

$$-(1001\,x^7 + 5460\,x^5 + 24432\,x^3 + 43776\,x)\,K_0^2(x)\,K_1(2x)-$$

$$-(3850\,x^6 + 11256\,x^4 + 32064\,x^2 + 87552)\,K_0(x)\,K_1(x)\,K_1(2x)+$$

$$+(-1001\,x^7 - 378\,x^5 + 17568\,x^3 + 55488\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^8 J_0^2(x)\, J_1(2x)\, dx = \frac{x^2}{30030}[(-8316\,x^6 + 64848\,x^4 - 221184\,x^2)\, J_0^2(x)\, J_0(2x)+$$

$$+(-2002\,x^7 + 11928\,x^5 - 75072\,x^3 + 294912\,x)\, J_0(x)\, J_1(x)\, J_0(2x)+$$

$$+(-5698\,x^6 + 63168\,x^4 - 256704\,x^2 + 294912)\, J_1^2(x)\, J_0(2x)+$$

$$+(1001\,x^7 + 27300\,x^5 - 157008\,x^3 + 294912\,x)\, J_0^2(x)\, J_1(2x)+$$

$$+(5390\,x^6 - 34104\,x^4 + 185664\,x^2 - 589824)\, J_0(x)\, J_1(x)\, J_1(2x)+$$

$$+(-1001\,x^7 + 23058\,x^5 - 163872\,x^3 + 404160\,x)\, J_1^2(x)\, J_1(2x)]$$

$$\int x^8 I_0^2(x)\, I_1(2x)\, dx = \frac{x^2}{30030}[(8316\,x^6 + 64848\,x^4 + 221184\,x^2)\, I_0^2(x)\, I_0(2x)-$$

$$-(2002\,x^7 + 11928\,x^5 + 75072\,x^3 + 294912\,x)\, I_0(x)\, I_1(x)\, I_0(2x)-$$

$$-(5698\,x^6 + 63168\,x^4 + 256704\,x^2 + 294912)\, I_1^2(x)\, I_0(2x)+$$

$$+(1001\,x^7 - 27300\,x^5 - 157008\,x^3 - 294912\,x)\, I_0^2(x)\, I_1(2x)+$$

$$+(5390\,x^6 + 34104\,x^4 + 185664\,x^2 + 589824)\, I_0(x)\, I_1(x)\, I_1(2x)+$$

$$+(1001\,x^7 + 23058\,x^5 + 163872\,x^3 + 404160\,x)\, I_1^2(x)\, I_1(2x)]$$

$$\int x^8 K_0^2(x)\, K_1(2x)\, dx = \frac{x^2}{30030}[-(8316\,x^6 + 64848\,x^4 + 221184\,x^2)\, K_0^2(x)\, K_0(2x)-$$

$$-(2002\,x^7 + 11928\,x^5 + 75072\,x^3 + 294912\,x)\, K_0(x)\, K_1(x)\, K_0(2x)+$$

$$+(5698\,x^6 + 63168\,x^4 + 256704\,x^2 + 294912)\, K_1^2(x)\, K_0(2x)+$$

$$+(1001\,x^7 - 27300\,x^5 - 157008\,x^3 - 294912\,x)\, K_0^2(x)\, K_1(2x)-$$

$$-(5390\,x^6 + 34104\,x^4 + 185664\,x^2 + 589824)\, K_0(x)\, K_1(x)\, K_1(2x)+$$

$$+(1001\,x^7 + 23058\,x^5 + 163872\,x^3 + 404160\,x)\, K_1^2(x)\, K_1(2x)]$$

$$\int x^8 J_1^2(x)\, J_1(2x)\, dx = \frac{x^2}{30030}[(-5544\,x^6 + 57792\,x^4 - 207360\,x^2)\, J_0^2(x)\, J_0(2x)+$$

$$+(2002\,x^7 + 672\,x^5 - 58368\,x^3 + 276480\,x)\, J_0(x)\, J_1(x)\, J_0(2x)+$$

$$+(-10472\,x^6 + 71232\,x^4 - 252672\,x^2 + 276480)\, J_1^2(x)\, J_0(2x)+$$

$$+(-1001\,x^7 + 21840\,x^5 - 144192\,x^3 + 276480\,x)\, J_0^2(x)\, J_1(2x)+$$

$$+(-3080\,x^6 - 15456\,x^4 + 162048\,x^2 - 552960)\, J_0(x)\, J_1(x)\, J_1(2x)+$$

$$+(1001\,x^7 + 31752\,x^5 - 171648\,x^3 + 390912\,x)\, J_1^2(x)\, J_1(2x)]$$

$$\int x^8 I_1^2(x)\, I_1(2x)\, dx = \frac{x^2}{30030}[-(5544\,x^6 + 57792\,x^4 + 207360\,x^2)\, I_0^2(x)\, I_0(2x)+$$

$$+(-2002\,x^7 + 672\,x^5 + 58368\,x^3 + 276480\,x)\, I_0(x)\, I_1(x)\, I_0(2x)+$$

$$+(10472\,x^6 + 71232\,x^4 + 252672\,x^2 + 276480)\, I_1^2(x)\, I_0(2x)+$$

$$+(1001\,x^7 + 21840\,x^5 + 144192\,x^3 + 276480\,x)\, I_0^2(x)\, I_1(2x)+$$

$$+(3080\,x^6 - 15456\,x^4 - 162048\,x^2 - 552960)\, I_0(x)\, I_1(x)\, I_1(2x)+$$

$$+(1001\,x^7 - 31752\,x^5 - 171648\,x^3 - 390912\,x)\, I_1^2(x)\, I_1(2x)]$$

$$\int x^8 K_1^2(x)\, K_1(2x)\, dx = \frac{x^2}{30030}[(5544\,x^6 + 57792\,x^4 + 207360\,x^2)\, K_0^2(x)\, K_0(2x)+$$

$$+(-2002\,x^7 + 672\,x^5 + 58368\,x^3 + 276480\,x)\, K_0(x)\, K_1(x)\, K_0(2x)-$$

$$-(10472\,x^6 + 71232\,x^4 + 252672\,x^2 + 276480)\, K_1^2(x)\, K_0(2x)+$$

$$+(1001\,x^7 + 21840\,x^5 + 144192\,x^3 + 276480\,x)\, K_0^2(x)\, K_1(2x)+$$

$$+(-3080\,x^6 + 15456\,x^4 + 162048\,x^2 + 552960)\, K_0(x)\, K_1(x)\, K_1(2x)+$$

$$+(1001\,x^7 - 31752\,x^5 - 171648\,x^3 - 390912\,x)\, K_1^2(x)\, K_1(2x)]$$

Recurrence relations:

$$\int x^{2n+1} J_0^2(x) J_0(2x)\, dx = -\frac{n}{4n+1} \int x^{2n} J_1(2x) \left[3n J_0^2(x) + (n-1) J_1^2(x)\right]\, dx +$$

$$+\frac{x^{2n+1}}{2(4n+1)}[x J_0^2(x) J_0(2x) - x J_1^2(x) J_0(2x) + 3n J_0^2(x) J_1(2x) + 2x J_0(x) J_1(x) J_1(2x) + n J_1^2(x) J_1(2x)]$$

$$\int x^{2n+1} J_0(x) J_1(x) J_1(2x)\, dx =$$

$$= \frac{1}{2(4n+1)} \int x^{2n} \left[-n(2n-1) J_0^2(x) J_1(2x) + 2n(4n+1) J_0(x) J_1(x) J_0(2x) + (n-1)(2n+1) J_1^2(x) J_1(2x)\right]\, dx +$$

$$+\frac{x^{2n+1}}{4(4n+1)}[(2n-1) J_0^2(x) J_1(2x) - 2(4n+1) J_0(x) J_1(x) J_0(2x) - (2n+1) J_1^2(x) J_1(2x) + 2x J_0^2(x) J_0(2x) +$$

$$+4x J_0(x) J_1(x) J_1(2x) - 2x J_1^2(x) J_0(2x)]$$

$$\int x^{2n+1} J_1^2(x) J_0(2x)\, dx = -\frac{1}{4n+1} \int x^{2n} \left[n(n+1) J_0^2(x) J_1(2x) + (3n+1)(n-1) J_1^2(x) J_1(2x)\right]\, dx +$$

$$+\frac{x^{2n+1}}{2(4n+1)}[(n+1) J_0^2(x) J_1(2x) + (3n+1) J_1^2(x) J_1(2x) - x J_0^2(x) J_0(2x) - 2x J_0(x) J_1(x) J_1(2x) + x J_1^2(x) J_0(2x)]$$

$$\int x^{2n+2} J_0(x) J_1(x) J_0(2x)\, dx =$$

$$= \frac{1}{4(4n+3)} \left\{-2(n+1)(2n+1) \int x^{2n+1} J_0^2(x) J_0(2x)\, dx - 4(4n+3) \int x^{2n+1} J_0(x) J_1(x) J_1(2x)\, dx +$$

$$+2n(2n+3) \int x^{2n+1} J_1^2(x) J_0(2x)\, dx + x^{2n+2}[(2n+1) J_0^2(x) J_0(2x) - 2x J_0^2(x) J_1(2x) + 4x J_0(x) J_1(x) J_0(2x) -$$

$$-(2n+3) J_1^2(x) J_0(2x) + 2(4n+3) J_0(x) J_1(x) J_1(2x) + 2x J_1^2(x) J_1(2x)]\right\}$$

$$\int x^{2n+2} J_0^2(x) J_1(2x)\, dx =$$

$$= \frac{1}{2(4n+3)} \left\{2(n+1)(3n+2) \int x^{2n+1} J_0^2(x) J_0(2x)\, dx + 2n^2 \int x^{2n+1} J_1^2(x) J_0(2x)\, dx +$$

$$+x^{2n+2} \left[-(3n+2) J_0^2(x) J_0(2x) + x J_0^2(x) J_1 - 2x J_0(x) J_1(x) J_0(2x) - n J_1^2(x) J_0(2x) - x J_1^2(x) J_1(2x)\right]\right\}$$

$$\int x^{2n+2} J_1^2(x) J_1(2x)\, dx =$$

$$= \frac{1}{2(4n+3)} \left\{2(n+1)^2 \int x^{2n+1} J_0^2(x) J_0(2x)\, dx + 6n(n+1) \int x^{2n+1} J_1^2(x) J_0(2x)\, dx +$$

$$+x^{2n+2}[-(n+1) J_0^2(x) J_0(2x) - x J_0^2(x) J_1(2x) + 2x J_0(x) J_1(x) J_0(2x) - 3(n+1) J_1^2(x) J_0(2x) + x J_0(x) J_1(x) J_0(2x)]\right\}$$

$$\int x^{2n+1} I_0^2(x) I_0(2x)\, dx = \frac{1}{4n+1} \int x^{2n}[-3n^2 I_0^2(x)I_1(2x) + n(n-1)I_1^2(x)I_1(2x)]\, dx +$$

$$+\frac{x^{2n+1}}{2(4n+1)}[xI_0^2(x) I_0(2x) + xI_1^2(x) I_0(2x) - 2xI_0(x) I_1(x) I_1(2x) + 3nI_0^2(x) I_1(2x) - nI_1^2(x) I_1(2x)]$$

$$\int x^{2n+1} I_0(x) I_1(x) I_1(2x)\, dx =$$

$$= \frac{1}{2(4n+1)} \int x^{2n} [n(2n-1) I_0^2(x)I_1(2x) - 2n(4n+1) I_0(x)I_1(x)I_0(2x) + (n-1)(2n+1)I_1^2(x)I_1(2x)]\, dx +$$

$$+\frac{x^{2n+1}}{4(4n+1)}[-(2n-1)I_0^2(x) I_1(2x) + 2(4n+1)I_0(x)I_1(x)I_0(2x) - (2n+1)I_1^2(x) I_1(2x) -$$

$$-2xI_0^2(x) I_0(2x) + 4xI_0(x)I_1(x)I_1(2x) - 2xI_1^2(x) I_0(2x)]$$

$$\int x^{2n+1} I_1^2(x) I_0(2x)\, dx = \frac{1}{4n+1} \int x^{2n} \left[n(n+1)I_0^2(x)I_1(2x) - (3n+1)(n-1) I_1^2(x) I_1(2x)\right]\, dx +$$

$$+\frac{x^{2n+1}}{2(4n+1)}[-(n+1)I_0^2(x)I_1(2x) + (3n+1)I_1^2(x)I_1(2x) + xI_0^2(x) I_0(2x) - 2xI_0(x)I_1(x)I_1(2x) + xI_1^2(x)I_0(2x)]$$

$$\int x^{2n+2} I_0(x) I_1(x) I_0(2x)\, dx =$$

$$= \frac{1}{2(4n+3)} \int x^{2n+1} [(n+1)(2n+1)I_0^2(x) I_0(2x) - 2n(4n+3) I_0(x) I_1(x) I_1(2x) + n(2n+3)I_1^2(x)I_0(2x)]\, dx +$$

$$+x^{2n+2}[-(2n+1) I_0^2(x) I_0(2x) + 2(4n+3) I_0(x) I_1(x) I_1(2x) - (2n+3)x I_1^2(x) I_0(2x) -$$

$$-2x I_0^2(x) I_1(2x) + 4xI_0(x) I_1(x) I_0(2x) - 3x I_1^2(x) I_1(2x)\}$$

$$\int x^{2n+2} I_0^2(x) I_1(2x)\, dx =$$

$$= \frac{1}{4n+3} \int x^{2n+1} [-(n+1)(3n+2)I_0^2(x)I_0(2x) + n^2 I_1^2(x)I_0(2x)]\, dx +$$

$$+x^{2n+2}[(3n+2) I_0^2(x) I_0(2x) - n I_1^2(x) I_0(2x) + x\, I_0^2(x) I_1(2x) - 2x\, I_0(x)I_1(x) I_0(2x) + x I_1^2(x) I_1(2x)]$$

$$\int x^{2n+2} I_1^2(x) I_1(2x)\, dx =$$

$$= \frac{1}{4n+3} \int x^{2n+1} [(n+1)^2 I_0^2(x) I_0(2x) - 3n(n+1) I_1^2(x) I_0(2x)]\, dx +$$

$$+x^{2n+2}[-(n+1) I_0^2(x) I_0(2x) + 3(n+1) I_1^2(x) I_0(2x) + x\, I_0^2(x) I_1(2x) - 2x\, I_0(x) I_1(x) I_0(2x) + x\, I_1^2(x) I_1(2x)]$$

$$\int x^{2n+1}\, K_0^2(x)\, K_0(2x)\, dx = \frac{1}{4n+1}\int x^{2n}[3n^2\, K_0^2(x)K_1(2x) - n(n-1)K_1^2(x)K_1(2x)]\, dx +$$

$$+\frac{x^{2n+1}}{2(4n+1)}[xK_0^2(x)\, K_0(2x)+xK_1^2(x)\, K_0(2x)-2xK_0(x)\, K_1(x)\, K_1(2x)-3nK_0^2(x)\, K_1(2x)+nK_1^2(x)\, K_1(2x)]$$

$$\int x^{2n+1}\, K_0(x)\, K_1(x)\, K_1(2x)\, dx =$$

$$= \frac{1}{2(4n+1)}\int x^{2n}\left[-n(2n-1)\, K_0^2(x)K_1(2x)+2n(4n+1)\, K_0(x)K_1(x)K_0(2x)-(n-1)(2n+1)K_1^2(x)K_1(2x)\right]dx +$$

$$+\frac{x^{2n+1}}{4(4n+1)}[(2n-1)K_0^2(x)\, K_1(2x) - 2(4n+1)K_0(x)K_1(x)K_0(2x) + (2n+1)K_1^2(x)\, K_1(2x)-$$

$$-2xK_0^2(x)\, K_0(2x) + 4xK_0(x)K_1(x)K_1(2x) - 2xK_1^2(x)\, K_0(2x)]$$

$$\int x^{2n+1}\, K_1^2(x)\, K_0(2x)\, dx = \frac{1}{4n+1}\int x^{2n}\left[-n(n+1)K_0^2(x)K_1(2x) + (3n+1)(n-1)\, K_1^2(x)\, K_1(2x)\right]dx +$$

$$+\frac{x^{2n+1}}{2(4n+1)}\left[(n+1)K_0^2(x)K_1(2x) - (3n+1)K_1^2(x)K_1(2x) + xK_0^2(x)\, K_0(2x) - 2xK_0(x)K_1(x)K_1(2x)+\right.$$

$$\left.+ xK_1^2(x)K_0(2x)\right]$$

$$\int x^{2n+2}\, K_0(x)\, K_1(x)\, K_0(2x)\, dx = \frac{1}{2(4n+3)}\cdot$$

$$\cdot \int x^{2n+1}\left[(n+1)(2n+1)K_0^2(x)\, K_0(2x) - 2n(4n+3)\, K_0(x)\, K_1(x)\, K_1(2x) + n(2n+3)K_1^2(x)K_0(2x)\right]dx +$$

$$+\frac{x^{2n+2}}{4(4n+3)}[-(2n+1)\, K_0^2(x)\, K_0(2x) + 2(4n+3)\, K_0(x)\, K_1(x)\, K_1(2x) - (3n+2)x\, K_1^2(x)\, K_0(2x)-$$

$$-2x\, K_0^2(x)\, K_1(2x) + 4xK_0(x)\, K_1(x)\, K_0(2x) - 3x\, K_1^2(x)\, K_1(2x)$$

$$\int x^{2n+2}\, K_0^2(x)\, K_1(2x)\, dx =$$

$$= \frac{1}{4n+3}\int x^{2n+1}\left[(n+1)(3n+2)K_0^2(x)K_0(2x) - n^2 K_1^2(x)K_0(2x)\right]dx +$$

$$+\frac{x^{2n+2}}{2(4n+3)}\left[-(3n+2)\, K_0^2(x)\, K_0(2x) + n\, K_1^2(x)\, K_0(2x) + x\, K_0^2(x)\, K_1(2x) - 2x\, K_0(x)K_1(x)\, K_0(2x)+\right.$$

$$\left.+x\, K_1^2(x)\, K_1(2x)\right]$$

$$\int x^{2n+2}\, K_1^2(x)\, K_1(2x)\, dx =$$

$$= \frac{1}{4n+3}\int x^{2n+1}\left[-(n+1)^2\, K_0^2(x)\, K_0(2x) + 3n(n+1)\, K_1^2(x)\, K_0(2x)\right]dx +$$

$$+\frac{x^{2n+2}}{2(4n+3)}\left[(n+1)\, K_0^2(x)\, K_0(2x) - 3(n+1)\, K_1^2(x)\, K_0(2x) + x\, K_0^2(x)\, K_1(2x) - 2x\, K_0(x)\, K_1(x)\, K_0(2x)+\right.$$

$$\left.+x\, K_1^2(x)\, K_1(2x)\right]$$

b) $x^n \, Z_\kappa(\alpha x) \, Z_\mu(\beta x) \, Z_\nu((\alpha + \beta)x)$

Formulas were found for the following integrals only:

$$\int x^{2n+1} \, Z_0(\alpha x) \, Z_0(\beta x) \, Z_0((\alpha + \beta)x) \, dx \ , \quad \int x^{2n+1} \, Z_0(\alpha x) \, Z_1(\beta x) \, Z_1((\alpha + \beta)x) \, dx \ ,$$

$$\int x^{2n+1} \, Z_1(\alpha x) \, Z_1(\beta x) \, Z_0((\alpha + \beta)x) \, dx \ , \quad n \geq 0,$$

and

$$\int x^{2n} \, Z_0(\alpha x) \, Z_0(\beta x) \, Z_1((\alpha + \beta)x) \, dx \ , \quad \int x^{2n} \, Z_0(\alpha x) \, Z_1(\beta x) \, Z_0((\alpha + \beta)x) \, dx \ ,$$

$$\int x^{2n} \, Z_1(\alpha x) \, Z_1(\beta x) \, Z_1((\alpha + \beta)x) \, dx \ , \quad n \geq 1.$$

The integrals $\int x^n \, Z_\nu(\alpha x) \, Z_\nu(\beta x) \, Z_{1-\nu}((\alpha + \beta)x) \, dx$ and $\int x^n \, Z_{1-\nu}(\alpha x) \, Z_\nu(\beta x) \, Z_\nu((\alpha + \beta)x) \, dx$ may be expressed by each other. Nevertheless, they are both listed.

$$\int J_1(\alpha x) \, J_1(\beta x) \, J_1((\alpha + \beta)x) \, dx = \frac{x}{2} \, [\, J_0(\alpha x)J_0(\beta x)J_1((\alpha + \beta)x) -$$

$$- J_0(\alpha x)J_1(\beta x)J_0((\alpha + \beta)x) - J_1(\alpha x)J_0(\beta x)J_0((\alpha + \beta)x) - J_1(\alpha x)J_1(\beta x)J_1((\alpha + \beta)x)]$$

$$\int I_1(\alpha x) \, I_1(\beta x) \, I_1((\alpha + \beta)x) \, dx = \frac{x}{2} \, [-I_0(\alpha x)I_0(\beta x)I_1((\alpha + \beta)x) +$$

$$+ I_0(\alpha x)I_1(\beta x)I_0((\alpha + \beta)x) + I_1(\alpha x)I_0(\beta x)I_0((\alpha + \beta)x) - I_1(\alpha x)I_1(\beta x)I_1((\alpha + \beta)x)]$$

$$\int K_1(\alpha x) \, K_1(\beta x) \, K_1((\alpha + \beta)x) \, dx = \frac{x}{2} \, [-K_0(\alpha x)K_0(\beta x)K_1((\alpha + \beta)x) +$$

$$+ K_0(\alpha x)K_1(\beta x)K_0((\alpha + \beta)x) + K_1(\alpha x)K_0(\beta x)K_0((\alpha + \beta)x) - K_1(\alpha x)K_1(\beta x)K_1((\alpha + \beta)x)]$$

$$\int x \, J_0(\alpha x) \, J_0(\beta x) \, J_0((\alpha + \beta)x) \, dx = \frac{x^2}{2} \, [J_0(\alpha x)J_0(\beta x)J_0((\alpha + \beta)x) +$$

$$+ J_0(\alpha x)J_1(\beta x)J_1((\alpha + \beta)x) + J_1(\alpha x)J_0(\beta x)J_1((\alpha + \beta)x) - J_1(\alpha x)J_1(\beta x)J_0((\alpha + \beta)x)]$$

$$\int x \, I_0(\alpha x) \, I_0(\beta x) \, I_0((\alpha + \beta)x) \, dx = \frac{x^2}{2} \, [I_0(\alpha x)I_0(\beta x)I_0((\alpha + \beta)x) -$$

$$- I_0(\alpha x)I_1(\beta x)I_1((\alpha + \beta)x) - I_1(\alpha x)I_0(\beta x)I_1((\alpha + \beta)x) + I_1(\alpha x)I_1(\beta x)I_0((\alpha + \beta)x)]$$

$$\int x \, K_0(\alpha x) \, K_0(\beta x) \, K_0((\alpha + \beta)x) \, dx = \frac{x^2}{2} \, [K_0(\alpha x)K_0(\beta x)K_0((\alpha + \beta)x) -$$

$$- K_0(\alpha x)K_1(\beta x)K_1((\alpha + \beta)x) - K_1(\alpha x)K_0(\beta x)K_1((\alpha + \beta)x) + K_1(\alpha x)K_1(\beta x)K_0((\alpha + \beta)x)]$$

$$\int x \, J_0(\alpha x) \, J_1(\beta x) \, J_1((\alpha + \beta)x) \, dx = \frac{x}{2\beta(\alpha + \beta)} \, \big[(\beta \, \alpha \, x + \beta^2 x) \, J_0(\alpha x)J_0(\beta x)J_0((\alpha + \beta)x) -$$

$$- (\alpha + \beta) \, J_0(\alpha x)J_0(\beta x)J_1((\alpha + \beta)x) - \beta \, J_0(\alpha x)J_1(\beta x)J_0((\alpha + \beta)x) +$$

$$+ (\beta \, \alpha \, x + \beta^2 x) \, J_0(\alpha x)J_1(\beta x)J_1((\alpha + \beta)x) + \alpha \, J_1(\alpha x)J_0(\beta x)J_0((\alpha + \beta)x) +$$

$$+ (\beta \, \alpha \, x + \beta^2 x) \, J_1(\alpha x)J_0(\beta x)J_1((\alpha + \beta)x) - (\beta \, \alpha \, x + \beta^2 x) \, J_1(\alpha x)J_1(\beta x)J_0((\alpha + \beta)x) \big]$$

$$\int x \, I_0(\alpha x) \, I_1(\beta x) \, I_1((\alpha + \beta)x) \, dx = \frac{x}{2\beta(\alpha + \beta)} \, \big[-(\beta \, \alpha \, x + \beta^2 x) \, I_0(\alpha x)I_0(\beta x)I_0((\alpha + \beta)x) +$$

$$+ (\alpha + \beta) \, I_0(\alpha x)I_0(\beta x)I_1((\alpha + \beta)x) + \beta \, I_0(\alpha x)I_1(\beta x)I_0((\alpha + \beta)x) +$$

$$+(\beta\,\alpha\,x+\beta^2x)\,I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)-\alpha\,I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)+$$
$$+(\beta\,\alpha\,x+\beta^2x)\,I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-(\beta\,\alpha\,x+\beta^2x)\,I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)\big]$$

$$\int x\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx=\frac{x}{2\beta(\alpha+\beta)}\;\big[-(\beta\,\alpha\,x+\beta^2x)\,K_0(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)-$$
$$-(\alpha+\beta)K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-\beta K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)+$$
$$+(\beta\,\alpha\,x+\beta^2x)K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)+\alpha K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)+$$
$$+(\beta\,\alpha\,x+\beta^2x)K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-(\beta\,\alpha\,x+\beta^2x)K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)\big]$$

$$\int x\,J_1(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)\,dx=\frac{x}{2\alpha\beta}\;\big[-\alpha\beta x\,J_0(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x)+$$
$$+(\alpha+\beta)\,J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)-\beta\,J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)-\alpha\,J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)+\alpha\beta x\,J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)\big]$$

$$\int x\,I_1(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x)\,dx=\frac{x}{2\alpha\beta}\;\big[\alpha\beta x\,I_0(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)-$$
$$-(\alpha+\beta)\,I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)+\beta\,I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)+\alpha\,I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)+\alpha\beta x\,I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)\big]$$

$$\int x\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx=\frac{x}{2\alpha\beta}\;\big[\alpha\beta x\,K_0(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)+$$
$$+(\alpha+\beta)\,K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-\beta\,K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)-\alpha\,K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)-$$
$$-\alpha\beta x\,K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)+\alpha\beta x\,K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)\big]$$

$$\int x^2\,J_0(\alpha x)\,J_0(\beta x)\,J_1((\alpha+\beta)x)\,dx=\frac{x^2}{6\alpha\beta(\alpha+\beta)}\;\big[\alpha\beta\,x\,(\alpha+\beta)\,J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)-$$
$$-\alpha\beta\,x\,(\alpha+\beta)\,J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)+\alpha\,(\alpha+3\beta)\,J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)-$$
$$-\alpha\beta\,(\alpha+\beta)\,x\,J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x)+\beta\,(3\alpha+\beta)\,J_1(\alpha x)J_0(\beta x)\,J_1((\alpha+\beta)x)-$$
$$-(\alpha+\beta)^2\,J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)-\alpha\beta\,(\alpha+\beta)\,x\,J_1(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)\big]$$

$$\int x^2\,I_0(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x)\,dx=\frac{x^2}{6\alpha\beta(\alpha+\beta)}\;\big[\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-$$
$$-\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)+\alpha(\alpha+3\beta)\,I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)-$$
$$-\alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)+\beta(3\alpha+\beta)\,I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-$$
$$-(\alpha+\beta)^2\,I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)\big]$$

$$\int x^2\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)\,dx=\frac{x^2}{6\alpha\beta(\alpha+\beta)}\;\big[\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-$$
$$-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)-\alpha(\alpha+3\beta)\,K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x) - \beta(3\alpha+\beta)\,K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)+$$
$$+(\alpha+\beta)^2\,K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)\big]$$

$$\int x^2\,J_0(\alpha x)\,J_1(\beta x)\,J_0((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\,\big[-\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)+$$
$$+\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) + \alpha(2\alpha+3\beta)\,J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)+$$
$$+\alpha\beta(\alpha+\beta)x\,J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x) - \beta^2\,J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)-$$
$$-(\alpha+\beta)(2\alpha-\beta)\,J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,J_1(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)\big]$$

$$\int x^2\,I_0(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\,\big[-\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)+$$
$$+\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) + \alpha(2\alpha+3\beta)\,I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)+$$
$$+\alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x) - \beta^2\,I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-$$
$$-(\alpha+\beta)(2\alpha-\beta)\,I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)\big]$$

$$\int x^2\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\,\big[-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)+$$
$$+\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) - \alpha(2\alpha+3\beta)\,K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)+$$
$$+\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x) + \beta^2\,K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)+$$
$$+(\alpha+\beta)(2\alpha-\beta)\,K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)\big]$$

$$\int x^2\,J_1(\alpha x)\,J_1(\beta x)\,J_1((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\,\big[-\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)+$$
$$+\alpha\beta(\alpha+\beta)x\,J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) + 2\alpha^2\,J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)+$$
$$+\alpha\beta(\alpha+\beta)x\,J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x) + 2\beta^2\,J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)-$$
$$-2(\alpha+\beta)^2\,J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,J_1(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)\big]$$

$$\int x^2\,I_1(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\,\big[\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-$$
$$-\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) - 2\alpha^2\,I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)-$$
$$-\alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x) - 2\beta^2\,I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)+$$
$$+\,2(\alpha+\beta)^2\,I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,I_1(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)\big]$$

$$\int x^2\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\,\big[\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-$$
$$-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) + 2\alpha^2\,K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)-$$
$$-\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x) + 2\beta^2\,K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-$$
$$-2(\alpha+\beta)^2\,K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,K_1(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)\big]$$

Let

$$\int x^n \, Z_\kappa(\alpha x) \, Z_\mu(\beta x) \, Z_\nu((\alpha+\beta)x) \, dx = {}^{\kappa\mu\nu}V_n \sum_{\kappa',\mu',\nu'=0}^{1} {}^{\kappa\mu\nu}P_{n,Z}^{\kappa'\mu'\nu'}(x) \, Z_{\kappa'}(\alpha x) \, Z_{\mu'}(\beta x) \, Z_{\nu'}((\alpha+\beta)x) \,,$$

then holds

$$^{000}V_3 = \frac{x^2}{30\,\alpha^2\beta^2\,(\alpha+\beta)^2}$$

$$^{000}P_{3,J}^{000}(x) = 3\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2$$

$$^{000}P_{3,J}^{001}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\,(\alpha^2+4\,\alpha\,\beta+\beta^2)\,x$$

$$^{000}P_{3,J}^{010}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\,(2\,\alpha^2+2\,\alpha\,\beta-\beta^2)\,x$$

$$^{000}P_{3,J}^{011}(x) = \alpha^2\,[3\,\beta^2\,(\alpha+\beta)^2\,x^2-4\,\alpha^2-10\,\alpha\,\beta-10\,\beta^2]$$

$$^{000}P_{3,J}^{100}(x) = -2\,\alpha\,\beta\,(\alpha+\beta)\,(\alpha^2-2\,\alpha\,\beta-2\,\beta^2)\,x$$

$$^{000}P_{3,J}^{101}(x) = \beta^2\,[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-4\,\beta^2-10\,\alpha\,\beta-10\,\alpha^2]$$

$$^{000}P_{3,J}^{110}(x) = -\,(\alpha+\beta)^2\,[3\,\alpha^2\beta^2x^2-4\,\alpha^2+2\,\alpha\,\beta-4\,\beta^2]$$

$$^{000}P_{3,J}^{111}(x) = 4\,\alpha\,\beta\,(\alpha+\beta)\,(\alpha^2+\alpha\,\beta+\beta^2)\,x$$

$$^{000}P_{3,I}^{000}(x) = 3\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2$$

$$^{000}P_{3,I}^{001}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\,(\alpha^2+4\,\alpha\,\beta+\beta^2)\,x$$

$$^{000}P_{3,I}^{010}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\,(2\,\alpha^2+2\,\alpha\,\beta-\beta^2)\,x$$

$$^{000}P_{3,I}^{011}(x) = -\alpha^2\,[3\,\beta^2\,(\alpha+\beta)^2\,x^2+4\,\alpha^2+10\,\alpha\,\beta+10\,\beta^2]$$

$$^{000}P_{3,I}^{100}(x) = -2\,\alpha\,\beta\,(\alpha+\beta)\,(\alpha^2-2\,\alpha\,\beta-2\,\beta^2)\,x$$

$$^{000}P_{3,I}^{101}(x) = -\beta^2\,[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+10\,\alpha^2+10\,\alpha\,\beta+4\,\beta^2]$$

$$^{000}P_{3,I}^{110}(x) = (\alpha+\beta)^2\,[3\,\alpha^2\beta^2x^2+4\,\alpha^2-2\,\alpha\,\beta+4\,\beta^2]$$

$$^{000}P_{3,I}^{111}(x) = -4\,\alpha\,\beta\,(\alpha+\beta)\,(\alpha^2+\alpha\,\beta+\beta^2)\,x$$

$$^{000}P_{3,K}^{000}(x) = 3\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2$$

$$^{000}P_{3,K}^{001}(x) = -2\,\alpha\,\beta\,(\alpha+\beta)\,(\alpha^2+4\,\alpha\,\beta+\beta^2)\,x$$

$$^{000}P_{3,K}^{010}(x) = -2\,\alpha\,\beta\,(\alpha+\beta)\,(2\,\alpha^2+2\,\alpha\,\beta-\beta^2)\,x$$

$$^{000}P_{3,K}^{011}(x) = -\alpha^2\,[3\,\beta^2\,(\alpha+\beta)^2\,x^2+4\,\alpha^2+10\,\alpha\,\beta+10\,\beta^2]$$

$$^{000}P_{3,K}^{100}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\,(\alpha^2-2\,\alpha\,\beta-2\,\beta^2)\,x$$

$$^{000}P_{3,K}^{101}(x) = -\beta^2\,[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+10\,\alpha^2+10\,\alpha\,\beta+4\,\beta^2]$$

$$^{000}P_{3,K}^{110}(x) = (\alpha+\beta)^2\,[3\,\alpha^2\beta^2x^2+4\,\alpha^2-2\,\alpha\,\beta+4\,\beta^2]$$

$$^{000}P_{3,K}^{111}(x) = 4\,\alpha\,\beta\,(\alpha+\beta)\,(\alpha^2+\alpha\,\beta+\beta^2)\,x$$

$$^{011}V_3 = \frac{x^2}{30\alpha^2\beta^2\,(\alpha+\beta)^2}$$

$$^{011}P_{3,J}^{000}(x) = 3\,\beta^2\alpha^2\,(\alpha+\beta)^2\,x^2$$

$$^{011}P_{3,J}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\,(3\,\alpha^2+7\,\alpha\,\beta-2\,\beta^2)\,x$$

$$^{011}P_{3,J}^{010}(x) = -\alpha\beta\,(\alpha+\beta)\left(6\,\alpha^2 + 11\,\alpha\,\beta + 2\,\beta^2\right)x$$

$$^{011}P_{3,J}^{011}(x) = \alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2 + 6\,\alpha^2 + 20\,\alpha\,\beta + 20\,\beta^2\right]$$

$$^{011}P_{3,J}^{100}(x) = \alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 + 4\,\alpha\,\beta + 4\,\beta^2\right)x$$

$$^{011}P_{3,J}^{101}(x) = \beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2 - 2\,\beta\,(5\,\alpha + 2\,\beta)\right]$$

$$^{011}P_{3,J}^{110}(x) = -\,(\alpha+\beta)^2\left[3\,\alpha^2\beta^2 x^2 + 2\,(\alpha+\beta)(3\,\alpha - 2\,\beta)\right]$$

$$^{011}P_{3,J}^{111}(x) = -2\,\alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 - 2\,\alpha\,\beta - 2\,\beta^2\right)x$$

$$^{011}P_{3,I}^{000}(x) = -3\,\beta^2\alpha^2\,(\alpha+\beta)^2\,x^2$$

$$^{011}P_{3,I}^{001}(x) = \alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 + 7\,\alpha\,\beta - 2\,\beta^2\right)x$$

$$^{011}P_{3,I}^{010}(x) = \alpha\beta\,(\alpha+\beta)\left(6\,\alpha^2 + 11\,\alpha\,\beta + 2\,\beta^2\right)x$$

$$^{011}P_{3,I}^{011}(x) = \alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2 - 6\,\alpha^2 - 20\,\alpha\,\beta - 20\,\beta^2\right]$$

$$^{011}P_{3,I}^{100}(x) = -\alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 + 4\,\alpha\,\beta + 4\,\beta^2\right)$$

$$^{011}P_{3,I}^{101}(x) = \beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2 + 2\,\beta\,(5\,\alpha + 2\,\beta)\right]$$

$$^{011}P_{3,I}^{110}(x) = -\,(\alpha+\beta)^2\left[3\,\alpha^2\beta^2 x^2 - 2\,(\alpha+\beta)(3\,\alpha - 2\,\beta)\right]$$

$$^{011}P_{3,I}^{111}(x) = -2\,\alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 - 2\,\alpha\,\beta - 2\,\beta^2\right)x$$

$$^{011}P_{3,K}^{000}(x) = -3\,\beta^2\alpha^2\,(\alpha+\beta)^2\,x^2$$

$$^{011}P_{3,K}^{001}(x) = -\alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 + 7\,\alpha\,\beta - 2\,\beta^2\right)x$$

$$^{011}P_{3,K}^{010}(x) = -\alpha\beta\,(\alpha+\beta)\left(6\,\alpha^2 + 11\,\alpha\,\beta + 2\,\beta^2\right)x$$

$$^{011}P_{3,K}^{011}(x) = \alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2 - 6\,\alpha^2 - 20\,\alpha\,\beta - 20\,\beta^2\right]$$

$$^{011}P_{3,K}^{100}(x) = \alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 + 4\,\alpha\,\beta + 4\,\beta^2\right)x$$

$$^{011}P_{3,K}^{101}(x) = \beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2 + 2\,\beta\,(5\,\alpha + 2\,\beta)\right]$$

$$^{011}P_{3,K}^{110}(x) = -\,(\alpha+\beta)^2\left[3\,\alpha^2\beta^2 x^2 - 2\,(\alpha+\beta)(3\,\alpha - 2\,\beta)\right]$$

$$^{011}P_{3,K}^{111}(x) = 2\,\alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 - 2\,\alpha\,\beta - 2\,\beta^2\right)x$$

$$^{110}V_3 = \frac{x^2}{30\,\alpha^2\beta^2\,(\alpha+\beta)^2}$$

$$^{110}P_{3,J}^{000}(x) = -3\,\beta^2\alpha^2\,(\alpha+\beta)^2\,x^2$$

$$^{110}P_{3,J}^{001}(x) = \alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 + 2\,\alpha\,\beta + 3\,\beta^2\right)x$$

$$^{110}P_{3,J}^{010}(x) = \alpha\beta\,(\alpha+\beta)\left(6\,\alpha^2 + \alpha\,\beta - 3\,\beta^2\right)x$$

$$^{110}P_{3,J}^{011}(x) = -\alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2 + 2\,\alpha\,(3\,\alpha + 5\,\beta)\right]$$

$$^{110}P_{3,J}^{100}(x) = -\alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 - \alpha\,\beta - 6\,\beta^2\right)x$$

$$^{110}P_{3,J}^{101}(x) = -\beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2 + 2\,\beta\,(5\,\alpha + 3\,\beta)\right]$$

$$^{110}P_{3,J}^{110}(x) = (\alpha+\beta)^2\left[3\,\alpha^2\beta^2 x^2 + 6\,\alpha^2 - 8\,\alpha\,\beta + 6\,\beta^2\right]$$

$$^{110}P_{3,J}^{111}(x) = 2\,\alpha\beta\,(\alpha+\beta)\left(3\,\alpha^2 + 8\,\alpha\,\beta + 3\,\beta^2\right)x$$

$$^{110}P_{3,I}^{000}(x) = 3\,\beta^2\alpha^2\,(\alpha+\beta)^2\,x^2$$

$$^{110}P_{3,I}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+2\,\alpha\,\beta+3\,\beta^2\right)x$$

$$^{110}P_{3,I}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(6\,\alpha^2+\alpha\,\beta-3\,\beta^2\right)x$$

$$^{110}P_{3,I}^{011}(x) = -\alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2-2\,\alpha\,(3\,\alpha+5\,\beta)\right]$$

$$^{110}P_{3,I}^{100}(x) = \alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2-\alpha\,\beta-6\,\beta^2\right)x$$

$$^{110}P_{3,I}^{101}(x) = -\beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-2\,\beta\,(5\,\alpha+3\,\beta)\right]$$

$$^{110}P_{3,I}^{110}(x) = (\alpha+\beta)^2\left[3\,\alpha^2\beta^2x^2-6\,\alpha^2+8\,\alpha\,\beta-6\,\beta^2\right]$$

$$^{110}P_{3,I}^{111}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)x$$

$$^{110}P_{3,K}^{000}(x) = 3\,\beta^2\alpha^2\,(\alpha+\beta)^2\,x^2$$

$$^{110}P_{3,K}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+2\,\alpha\,\beta+3\,\beta^2\right)x$$

$$^{110}P_{3,K}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\left(6\,\alpha^2+\alpha\,\beta-3\,\beta^2\right)x$$

$$^{110}P_{3,K}^{011}(x) = -\alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2-2\,\alpha\,(3\,\alpha+5\,\beta)\right]$$

$$^{110}P_{3,K}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2-\alpha\,\beta-6\,\beta^2\right)x$$

$$^{110}P_{3,K}^{101}(x) = -\beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-2\,\beta\,(5\,\alpha+3\,\beta)\right]$$

$$^{110}P_{3,K}^{110}(x) = (\alpha+\beta)^2\left[3\,\alpha^2\beta^2x^2-6\,\alpha^2+8\,\alpha\,\beta-6\,\beta^2\right]$$

$$^{110}P_{3,K}^{111}(x) = -2\,\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)x$$

$$^{001}V_4 = \frac{x^2}{210\,\alpha^3\beta^3\,(\alpha+\beta)^3}$$

$$^{001}P_{4,J}^{000}(x) = -6\,\alpha^2\beta^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2$$

$$^{001}P_{4,J}^{001}(x) = \alpha\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+18\,\alpha^4+52\,\alpha^3\beta+116\,\alpha^2\beta^2+52\,\alpha\,\beta^3+18\,\beta^4\right]$$

$$^{001}P_{4,J}^{010}(x) = -\alpha\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-36\,\alpha^4-86\,\alpha^3\beta-58\,\alpha^2\beta^2+34\,\alpha\,\beta^3+18\,\beta^4\right]$$

$$^{001}P_{4,J}^{011}(x) = \alpha^2\left[9\,\beta^2\,(\alpha+2\,\beta)\,(3\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2-4\,\alpha\,\left(9\,\alpha^3+35\,\beta\,\alpha^2+49\,\alpha\,\beta^2+35\,\beta^3\right)\right]$$

$$^{001}P_{4,J}^{100}(x) = -\alpha\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+18\,\alpha^4+34\,\alpha^3\beta-58\,\alpha^2\beta^2-86\,\alpha\,\beta^3-36\,\beta^4\right]$$

$$^{001}P_{4,J}^{101}(x) = -\beta^2\left[9\,\alpha^2\,(2\,\alpha+\beta)\,(\alpha-3\,\beta)\,(\alpha+\beta)^2\,x^2+4\,\beta\,\left(35\,\alpha^3+49\,\beta\,\alpha^2+35\,\alpha\,\beta^2+9\,\beta^3\right)\right]$$

$$^{001}P_{4,J}^{110}(x) = -\,(\alpha+\beta)^2\,\left[3\,\alpha^2\beta^2\,\left(9\,\alpha^2+10\,\alpha\,\beta+9\,\beta^2\right)x^2-4\,\left(9\,\alpha^2-10\,\alpha\,\beta+9\,\beta^2\right)(\alpha+\beta)^2\right]$$

$$^{001}P_{4,J}^{111}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-36\,\alpha^4-86\,\alpha^3\beta-52\,\alpha^2\beta^2-86\,\alpha\,\beta^3-36\,\beta^4\right]$$

$$^{001}P_{4,I}^{000}(x) = 6\,\alpha^2\beta^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2$$

$$^{001}P_{4,I}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-18\,\alpha^4-52\,\alpha^3\beta-116\,\alpha^2\beta^2-52\,\alpha\,\beta^3-18\,\beta^4\right]$$

$$^{001}P_{4,I}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+36\,\alpha^4+86\,\alpha^3\beta+58\,\alpha^2\beta^2-34\,\alpha\,\beta^3-18\,\beta^4\right]$$

$$^{001}P_{4,I}^{011}(x) = \alpha^2\left[9\,\beta^2\,(\alpha+2\,\beta)\,(3\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2+4\,\alpha\,\left(9\,\alpha^3+35\,\beta\,\alpha^2+49\,\alpha\,\beta^2+35\,\beta^3\right)\right]$$

$$^{001}P_{4,I}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-18\,\alpha^4-34\,\alpha^3\beta+58\,\alpha^2\beta^2+86\,\alpha\,\beta^3+36\,\beta^4\right]$$

$$^{001}P_{4,I}^{101}(x) = -\beta^2\left[9\,\alpha^2\,(2\,\alpha+\beta)\,(\alpha-3\,\beta)\,(\alpha+\beta)^2\,x^2-4\,\beta\,\left(35\,\alpha^3+49\,\beta\,\alpha^2+35\,\alpha\,\beta^2+9\,\beta^3\right)\right]$$

$$^{001}P_{4,I}^{110}(x) = -\,(\alpha+\beta)^2\,\left[3\,\alpha^2\beta^2\,\left(9\,\alpha^2+10\,\alpha\,\beta+9\,\beta^2\right)x^2+4\,\left(9\,\alpha^2-10\,\alpha\,\beta+9\,\beta^2\right)(\alpha+\beta)^2\right]$$

$$^{001}P_{4,I}^{111}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+36\,\alpha^4+86\,\alpha^3\beta+52\,\alpha^2\beta^2+86\,\alpha\,\beta^3+36\,\beta^4\right]$$

$$^{001}P_{4,K}^{000}(x) = -6\,\alpha^2\beta^2\,(\alpha + 3\,\beta)\,(3\,\alpha + \beta)\,(\alpha + \beta)^2\,x^2$$

$$^{001}P_{4,K}^{001}(x) = \alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 - 18\,\alpha^4 - 52\,\alpha^3\beta - 116\,\alpha^2\beta^2 - 52\,\alpha\beta^3 - 18\,\beta^4]$$

$$^{001}P_{4,K}^{010}(x) = -\alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 + 36\,\alpha^4 + 86\,\alpha^3\beta + 58\,\alpha^2\beta^2 - 34\,\alpha\beta^3 - 18\,\beta^4]$$

$$^{001}P_{4,K}^{011}(x) = -\alpha^2\,[9\,\beta^2\,(\alpha + 2\,\beta)\,(3\,\alpha - \beta)\,(\alpha + \beta)^2\,x^2 + 4\,\alpha\,(9\,\alpha^3 + 35\,\beta\alpha^2 + 49\,\alpha\beta^2 + 35\,\beta^3)]$$

$$^{001}P_{4,K}^{100}(x) = -\alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 - 18\,\alpha^4 - 34\,\alpha^3\beta + 58\,\alpha^2\beta^2 + 86\,\alpha\beta^3 + 36\,\beta^4]$$

$$^{001}P_{4,K}^{101}(x) = \beta^2\,[9\,\alpha^2\,(2\,\alpha + \beta)\,(\alpha - 3\,\beta)\,(\alpha + \beta)^2\,x^2 - 4\,\beta\,(35\,\alpha^3 + 49\,\beta\alpha^2 + 35\,\alpha\beta^2 + 9\,\beta^3)]$$

$$^{001}P_{4,K}^{110}(x) = (\alpha + \beta)^2\,[3\,\alpha^2\beta^2\,(9\,\alpha^2 + 10\,\alpha\beta + 9\,\beta^2)\,x^2 + 4\,(9\,\alpha^2 - 10\,\alpha\beta + 9\,\beta^2)\,(\alpha + \beta)^2]$$

$$^{001}P_{4,K}^{111}(x) = \alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 + 36\,\alpha^4 + 86\,\alpha^3\beta + 52\,\alpha^2\beta^2 + 86\,\alpha\beta^3 + 36\,\beta^4]$$

$$^{010}V_4 = \frac{x^2}{210\,\alpha^3\beta^3\,(\alpha + \beta)^3}$$

$$^{010}P_{4,J}^{000}(x) = -6\,\alpha^2\beta^2\,(2\,\alpha + 3\,\beta)\,(2\,\alpha - \beta)\,(\alpha + \beta)^2\,x^2$$

$$^{010}P_{4,J}^{001}(x) = -\alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 - 24\,\alpha^4 - 60\,\alpha^3\beta - 52\,\alpha^2\beta^2 + 38\,\alpha\beta^3 + 18\,\beta^4]$$

$$^{010}P_{4,J}^{010}(x) = \alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 + 48\,\alpha^4 + 96\,\alpha^3\beta + 68\,\alpha^2\beta^2 + 20\,\alpha\beta^3 + 18\,\beta^4]$$

$$^{010}P_{4,J}^{011}(x) = \alpha^2\,[9\,\beta^2\,(\alpha + 2\,\beta)\,(4\,\alpha + \beta)\,(\alpha + \beta)^2\,x^2 - 4\,\alpha\,(12\,\alpha^3 + 42\,\beta\alpha^2 + 56\,\alpha\beta^2 + 35\,\beta^3)]$$

$$^{010}P_{4,J}^{100}(x) = \alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 - 24\,\alpha^4 - 36\,\alpha^3\beta - 16\,\alpha^2\beta^2 - 58\,\alpha\beta^3 - 36\,\beta^4]$$

$$^{010}P_{4,J}^{101}(x) = -\beta^2\,[3\,\alpha^2\,(8\,\alpha^2 + 8\,\alpha\beta + 9\,\beta^2)\,(\alpha + \beta)^2\,x^2 - 4\,\beta^2\,(28\,\alpha^2 + 28\,\alpha\beta + 9\,\beta^2)]$$

$$^{010}P_{4,J}^{110}(x) =$$

$$= -\,(\alpha + \beta)^2\,[9\,\alpha^2\beta^2\,(4\,\alpha + 3\,\beta)\,(\alpha - \beta)\,x^2 - 4\,(\alpha + \beta)\,(12\,\alpha^3 - 6\,\beta\alpha^2 + 8\,\alpha\beta^2 - 9\,\beta^3)]$$

$$^{010}P_{4,J}^{111}(x) = \alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 + 48\,\alpha^4 + 96\,\alpha^3\beta - 10\,\alpha^2\beta^2 - 58\,\alpha\beta^3 - 36\,\beta^4]$$

$$^{010}P_{4,I}^{000}(x) = 6\,\alpha^2\beta^2\,(2\,\alpha + 3\,\beta)\,(2\,\alpha - \beta)\,(\alpha + \beta)^2\,x^2$$

$$^{010}P_{4,I}^{001}(x) = -\alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 + 24\,\alpha^4 + 60\,\alpha^3\beta + 52\,\alpha^2\beta^2 - 38\,\alpha\beta^3 - 18\,\beta^4]$$

$$^{010}P_{4,I}^{010}(x) = \alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 - 48\,\alpha^4 - 96\,\alpha^3\beta - 68\,\alpha^2\beta^2 - 20\,\alpha\beta^3 - 18\,\beta^4]$$

$$^{010}P_{4,I}^{011}(x) = \alpha^2\,[9\,\beta^2\,(\alpha + 2\,\beta)\,(4\,\alpha + \beta)\,(\alpha + \beta)^2\,x^2 + 4\,\alpha\,(12\,\alpha^3 + 42\,\beta\alpha^2 + 56\,\alpha\beta^2 + 35\,\beta^3)]$$

$$^{010}P_{4,I}^{100}(x) = \alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 + 24\,\alpha^4 + 36\,\alpha^3\beta + 16\,\alpha^2\beta^2 + 58\,\alpha\beta^3 + 36\,\beta^4]$$

$$^{010}P_{4,I}^{101}(x) = -\beta^2\,[3\,\alpha^2\,(8\,\alpha^2 + 8\,\alpha\beta + 9\,\beta^2)\,(\alpha + \beta)^2\,x^2 + 4\,\beta^2\,(28\,\alpha^2 + 28\,\alpha\beta + 9\,\beta^2)]$$

$$^{010}P_{4,I}^{110}(x) =$$

$$= -\,(\alpha + \beta)^2\,[9\,\alpha^2\beta^2\,(4\,\alpha + 3\,\beta)\,(\alpha - \beta)\,x^2 + 4\,(\alpha + \beta)\,(12\,\alpha^3 - 6\,\beta\alpha^2 + 8\,\alpha\beta^2 - 9\,\beta^3)]$$

$$^{010}P_{4,I}^{111}(x) = -\alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 - 48\,\alpha^4 - 96\,\alpha^3\beta + 10\,\alpha^2\beta^2 + 58\,\alpha\beta^3 + 36\,\beta^4]$$

$$^{010}P_{4,K}^{000}(x) = -6\,\alpha^2\beta^2\,(2\,\alpha + 3\,\beta)\,(2\,\alpha - \beta)\,(\alpha + \beta)^2\,x^2$$

$$^{010}P_{4,K}^{001}(x) = -\alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 + 24\,\alpha^4 + 60\,\alpha^3\beta + 52\,\alpha^2\beta^2 - 38\,\alpha\beta^3 - 18\,\beta^4]$$

$$^{010}P_{4,K}^{010}(x) = \alpha\,\beta\,(\alpha + \beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha + \beta)^2\,x^2 - 48\,\alpha^4 - 96\,\alpha^3\beta - 68\,\alpha^2\beta^2 - 20\,\alpha\beta^3 - 18\,\beta^4]$$

$$^{010}P_{4,K}^{011}(x) =$$

$$= -\alpha^2\,[9\,\beta^2\,(\alpha + 2\,\beta)\,(4\,\alpha + \beta)\,(\alpha + \beta)^2\,x^2 + 4\,\alpha\,(12\,\alpha^3 + 42\,\beta\alpha^2 + 56\,\alpha\beta^2 + 35\,\beta^3)]$$

$$^{010}P_{4,K}^{100}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 24\,\alpha^4 + 36\,\alpha^3\beta + 16\,\alpha^2\beta^2 + 58\,\alpha\,\beta^3 + 36\,\beta^4]$$

$$^{010}P_{4,K}^{101}(x) = \beta^2\,[3\,\alpha^2\,\left(8\,\alpha^2 + 8\,\alpha\,\beta + 9\,\beta^2\right)(\alpha+\beta)^2\,x^2 + 4\,\beta^2\,\left(28\,\alpha^2 + 28\,\alpha\,\beta + 9\,\beta^2\right)]$$

$$^{010}P_{4,K}^{110}(x) =$$

$$= (\alpha+\beta)^2\,[9\,\alpha^2\beta^2\,(4\,\alpha+3\,\beta)(\alpha-\beta)\,x^2 + 4\,(\alpha+\beta)\left(12\,\alpha^3 - 6\,\beta\,\alpha^2 + 8\,\alpha\,\beta^2 - 9\,\beta^3\right)]$$

$$^{010}P_{4,K}^{111}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[-15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 48\,\alpha^4 + 96\,\alpha^3\beta - 10\,\alpha^2\beta^2 - 58\,\alpha\,\beta^3 - 36\,\beta^4]$$

$$^{111}V_4 = \frac{x^2}{70\,\alpha^3\beta^3\,(\alpha+\beta)^3}$$

$$^{111}P_{4,J}^{000}(x) = -8\,\alpha^2\beta^2\,\left(\alpha^2 + \alpha\,\beta + \beta^2\right)(\alpha+\beta)^2\,x^2$$

$$^{111}P_{4,J}^{001}(x) = -\alpha\,x\,\beta\,(\alpha+\beta)\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 8\,\alpha^4 - 20\,\alpha^3\beta - 8\,\alpha^2\beta^2 - 20\,\beta^3\alpha - 8\,\beta^4]$$

$$^{111}P_{4,J}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 16\,\alpha^4 + 32\,\alpha^3\beta + 4\,\alpha^2\beta^2 - 12\,\beta^3\alpha - 8\,\beta^4]$$

$$^{111}P_{4,J}^{011}(x) = 4\alpha^2\,[\beta^2\,\left(3\,\alpha^2 - 2\,\alpha\,\beta - 2\,\beta^2\right)(\alpha+\beta)^2\,x^2 - 2\,\alpha^2\,\left(2\,\alpha^2 + 7\,\alpha\,\beta + 7\,\beta^2\right)]$$

$$^{111}P_{4,J}^{100}(x) = \alpha\,x\,\beta\,(\alpha+\beta)\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 8\,\alpha^4 - 12\,\alpha^3\beta + 4\,\alpha^2\beta^2 + 32\,\beta^3\alpha + 16\,\beta^4]$$

$$^{111}P_{4,J}^{101}(x) = -4\,\beta^2\,[\alpha^2\,\left(2\,\alpha^2 + 2\,\alpha\,\beta - 3\,\beta^2\right)(\alpha+\beta)^2\,x^2 + 2\,\beta^2\,\left(7\,\alpha^2 + 7\,\alpha\,\beta + 2\,\beta^2\right)]$$

$$^{111}P_{4,J}^{110}(x) = -4\,(\alpha+\beta)^2\,[\alpha^2\beta^2\,\left(3\,\alpha^2 + 8\,\alpha\,\beta + 3\,\beta^2\right)x^2 - 2\,\left(2\,\alpha^2 - 3\,\alpha\,\beta + 2\,\beta^2\right)(\alpha+\beta)^2]$$

$$^{111}P_{4,J}^{111}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 16\,\left(\alpha^2 + \alpha\,\beta + \beta^2\right)^2]$$

$$^{111}P_{4,I}^{000}(x) = -8\,\alpha^2\beta^2\,\left(\alpha^2 + \alpha\,\beta + \beta^2\right)(\alpha+\beta)^2\,x^2$$

$$^{111}P_{4,I}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 8\,\alpha^4 + 20\,\alpha^3\beta + 8\,\alpha^2\beta^2 + 20\,\beta^3\alpha + 8\,\beta^4]$$

$$^{111}P_{4,I}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 16\,\alpha^4 - 32\,\alpha^3\beta - 4\,\alpha^2\beta^2 + 12\,\beta^3\alpha + 8\,\beta^4]$$

$$^{111}P_{4,I}^{011}(x) = -4\,\alpha^2\,[\beta^2\,\left(3\,\alpha^2 - 2\,\alpha\,\beta - 2\,\beta^2\right)(\alpha+\beta)^2\,x^2 + 2\,\alpha^2\,\left(2\,\alpha^2 + 7\,\alpha\,\beta + 7\,\beta^2\right)]$$

$$^{111}P_{4,I}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 8\,\alpha^4 + 12\,\alpha^3\beta - 4\,\alpha^2\beta^2 - 32\,\beta^3\alpha - 16\,\beta^4]$$

$$^{111}P_{4,I}^{101}(x) = 4\,\beta^2\,[\alpha^2\,\left(2\,\alpha^2 + 2\,\alpha\,\beta - 3\,\beta^2\right)(\alpha+\beta)^2\,x^2 - 2\,\beta^2\,\left(7\,\alpha^2 + 7\,\alpha\,\beta + 2\,\beta^2\right)]$$

$$^{111}P_{4,I}^{110}(x) = 4\,(\alpha+\beta)^2\,[\alpha^2\beta^2\,\left(3\,\alpha^2 + 8\,\alpha\,\beta + 3\,\beta^2\right)x^2 + 2\,\left(2\,\alpha^2 - 3\,\alpha\,\beta + 2\,\beta^2\right)(\alpha+\beta)^2]$$

$$^{111}P_{4,I}^{111}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 16\,\left(\alpha^2 + \alpha\,\beta + \beta^2\right)^2]$$

$$^{111}P_{4,K}^{000}(x) = 8\,\alpha^2\beta^2\,\left(\alpha^2 + \alpha\,\beta + \beta^2\right)(\alpha+\beta)^2\,x^2$$

$$^{111}P_{4,K}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 8\,\alpha^4 + 20\,\alpha^3\beta + 8\,\alpha^2\beta^2 + 20\,\beta^3\alpha + 8\,\beta^4]$$

$$^{111}P_{4,K}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 16\,\alpha^4 - 32\,\alpha^3\beta - 4\,\alpha^2\beta^2 + 12\,\beta^3\alpha + 8\,\beta^4]$$

$$^{111}P_{4,K}^{011}(x) = 4\,\alpha^2\,[\beta^2\,\left(3\,\alpha^2 - 2\,\alpha\,\beta - 2\,\beta^2\right)(\alpha+\beta)^2\,x^2 + 2\,\alpha^2\,\left(2\,\alpha^2 + 7\,\alpha\,\beta + 7\,\beta^2\right)]$$

$$^{111}P_{4,K}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 8\,\alpha^4 + 12\,\alpha^3\beta - 4\,\alpha^2\beta^2 - 32\,\beta^3\alpha - 16\,\beta^4]$$

$$^{111}P_{4,K}^{101}(x) = -4\,\beta^2\,[\alpha^2\,\left(2\,\alpha^2 + 2\,\alpha\,\beta - 3\,\beta^2\right)(\alpha+\beta)^2\,x^2 - 2\,\beta^2\,\left(7\,\alpha^2 + 7\,\alpha\,\beta + 2\,\beta^2\right)]$$

$$^{111}P_{4,K}^{110}(x) = -4\,(\alpha+\beta)^2\,[\alpha^2\beta^2\,\left(3\,\alpha^2 + 8\,\alpha\,\beta + 3\,\beta^2\right)x^2 + 2\,\left(2\,\alpha^2 - 3\,\alpha\,\beta + 2\,\beta^2\right)(\alpha+\beta)^2]$$

$$^{111}P_{4,K}^{111}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 16\,\left(\alpha^2 + \alpha\,\beta + \beta^2\right)^2]$$

$$^{000}V_5 = \frac{x^2}{630\,\alpha^4\beta^4\,(\alpha+\beta)^4}$$

$$^{000}P_{5,J}^{000}(x) = [35\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^2 + 96\,\alpha^2\beta^2\,(\alpha+\beta)^2\,(\alpha^2+\alpha\,\beta+\beta^2)^2]\,x^2$$

$$^{000}P_{5,J}^{001}(x) = 4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2-$$
$$-12\,(\alpha^2+\alpha\,\beta+\beta^2)\,(2\,\beta^4+5\,\beta^3\alpha+2\,\alpha^2\beta^2+5\,\alpha^3\beta+2\,\alpha^4)]$$

$$^{000}P_{5,J}^{010}(x) = 4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(2\,\alpha+3\,\beta)\,(2\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2-$$
$$-12\,(\alpha^2+\alpha\,\beta+\beta^2)\,(-2\,\beta^4-3\,\beta^3\alpha+\alpha^2\beta^2+8\,\alpha^3\beta+4\,\alpha^4)]$$

$$^{000}P_{5,J}^{011}(x) =$$
$$= \alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4 - 8\,\beta^2\,(18\,\alpha^4+41\,\alpha^3\beta+29\,\alpha^2\beta^2-24\,\beta^3\alpha-12\,\beta^4)\,(\alpha+\beta)^2\,x^2+$$
$$+96\,\alpha^2\,(7\,\beta^2+7\,\alpha\,\beta+2\,\alpha^2)\,(\alpha^2+\alpha\,\beta+\beta^2)]$$

$$^{000}P_{5,J}^{100}(x) = -4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(3\,\alpha+2\,\beta)\,(\alpha-2\,\beta)\,(\alpha+\beta)^2\,x^2-$$
$$-12\,(\alpha^2+\alpha\,\beta+\beta^2)\,(-4\,\beta^4-8\,\beta^3\alpha-\alpha^2\beta^2+3\,\alpha^3\beta+2\,\alpha^4)]$$

$$^{000}P_{5,J}^{101}(x) =$$
$$= \beta^2\,[35\,\beta^2\alpha^4\,(\alpha+\beta)^4\,x^4 + 8\,\alpha^2\,(12\,\alpha^4+24\,\alpha^3\beta-29\,\alpha^2\beta^2-41\,\beta^3\alpha-18\,\beta^4)\,(\alpha+\beta)^2\,x^2+$$
$$+96\,\beta^2\,(\alpha^2+\alpha\,\beta+\beta^2)\,(2\,\beta^2+7\,\alpha\,\beta+7\,\alpha^2)]$$

$$^{000}P_{5,J}^{110}(x) =$$
$$= -\,(\alpha+\beta)^2\,[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4 - 8\,\alpha^2\beta^2\,(18\,\alpha^4+31\,\alpha^3\beta+14\,\alpha^2\beta^2+31\,\beta^3\alpha+18\,\beta^4)\,x^2+$$
$$+96\,(2\,\beta^2-3\,\alpha\,\beta+2\,\alpha^2)\,(\alpha^2+\alpha\,\beta+\beta^2)\,(\alpha+\beta)^2]$$

$$^{000}P_{5,J}^{111}(x) = 16\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha^2+\alpha\,\beta+\beta^2)\,(\alpha+\beta)^2\,x^2-$$
$$-12\,\alpha^6-36\,\alpha^5\beta-37\,\beta^2\alpha^4-14\,\alpha^3\beta^3-37\,\beta^4\alpha^2-36\,\beta^5\alpha-12\,\beta^6]$$

$$^{000}P_{5,I}^{000}(x) = 35\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^2 - 96\,\alpha^2\beta^2\,(\alpha+\beta)^2\,(\alpha^2+\alpha\,\beta+\beta^2)^2\,x^2$$

$$^{000}P_{5,I}^{001}(x) = 4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2+$$
$$+12\,(\alpha^2+\alpha\,\beta+\beta^2)\,(2\,\beta^4+5\,\beta^3\alpha+2\,\alpha^2\beta^2+5\,\alpha^3\beta+2\,\alpha^4)]$$

$$^{000}P_{5,I}^{010}(x) = 4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(2\,\alpha+3\,\beta)\,(2\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2+$$
$$+12\,(\alpha^2+\alpha\,\beta+\beta^2)\,(-2\,\beta^4-3\,\beta^3\alpha+\alpha^2\beta^2+8\,\alpha^3\beta+4\,\alpha^4)]$$

$$^{000}P_{5,I}^{011}(x) = -\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4+$$
$$+8\,\beta^2\,(18\,\alpha^4+41\,\alpha^3\beta+29\,\alpha^2\beta^2-24\,\beta^3\alpha-12\,\beta^4)\,(\alpha+\beta)^2\,x^2+$$
$$+96\,\alpha^2\,(7\,\beta^2+7\,\alpha\,\beta+2\,\alpha^2)\,(\alpha^2+\alpha\,\beta+\beta^2)]$$

$$^{000}P_{5,I}^{100}(x) = -4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\,(3\,\alpha+2\,\beta)\,(\alpha-2\,\beta)\,(\alpha+\beta)^2\,x^2+$$
$$+12\,(\alpha^2+\alpha\,\beta+\beta^2)\,(-4\,\beta^4-8\,\beta^3\alpha-\alpha^2\beta^2+3\,\alpha^3\beta+2\,\alpha^4)]$$

$$^{000}P_{5,I}^{101}(x) = -\beta^2\,[35\,\beta^2\alpha^4\,(\alpha+\beta)^4\,x^4-$$
$$-8\,\alpha^2\,(12\,\alpha^4+24\,\alpha^3\beta-29\,\alpha^2\beta^2-41\,\beta^3\alpha-18\,\beta^4)\,(\alpha+\beta)^2\,x^2+$$
$$+96\,\beta^2\,(\alpha^2+\alpha\,\beta+\beta^2)\,(2\,\beta^2+7\,\alpha\,\beta+7\,\alpha^2)]$$

$$^{000}P_{5,I}^{110}(x) = (\alpha+\beta)^2\,[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4+$$
$$+8\,\alpha^2\beta^2\,(18\,\alpha^4+31\,\alpha^3\beta+14\,\alpha^2\beta^2+31\,\beta^3\alpha+18\,\beta^4)\,x^2+$$

$$+96\left(\alpha^2+\alpha\beta+\beta^2\right)\left(2\beta^2-3\alpha\beta+2\alpha^2\right)(\alpha+\beta)^2]$$

$$^{000}P_{5,I}^{111}(x)=-16\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(\alpha^2+\alpha\beta+\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+12\,\alpha^6+36\,\alpha^5\beta+37\,\beta^2\alpha^4+14\,\alpha^3\beta^3+37\,\beta^4\alpha^2+36\,\beta^5\alpha+12\,\beta^6]$$

$$^{000}P_{5,K}^{000}(x)=35\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-96\,\alpha^2\beta^2\,(\alpha+\beta)^2\left(\beta\alpha+\alpha^2+\beta^2\right)^2\,x^2$$

$$^{000}P_{5,K}^{001}(x)=-4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\beta^2\alpha^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2+$$

$$+12\left(\beta\alpha+\alpha^2+\beta^2\right)\left(2\,\beta^4+5\,\alpha\,\beta^3+2\,\beta^2\alpha^2+5\,\beta\,\alpha^3+2\,\alpha^4\right)]$$

$$^{000}P_{5,K}^{010}(x)=-4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\beta^2\alpha^2\,(2\,\alpha+3\,\beta)\,(2\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2+$$

$$+12\left(\beta\alpha+\alpha^2+\beta^2\right)\left(-2\,\beta^4-3\,\alpha\,\beta^3+\beta^2\alpha^2+8\,\beta\,\alpha^3+4\,\alpha^4\right)]$$

$$^{000}P_{5,K}^{011}(x)=-\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4+$$

$$+8\,\beta^2\left(18\,\alpha^4+41\,\beta\,\alpha^3+29\,\beta^2\alpha^2-24\,\alpha\,\beta^3-12\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^2\left(7\,\beta^2+7\,\beta\,\alpha+2\,\alpha^2\right)\left(\beta\alpha+\alpha^2+\beta^2\right)]$$

$$^{000}P_{5,K}^{100}(x)=4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\beta^2\alpha^2\,(3\,\alpha+2\,\beta)\,(\alpha-2\,\beta)\,(\alpha+\beta)^2\,x^2+$$

$$+12\left(\beta\alpha+\alpha^2+\beta^2\right)\left(-4\,\beta^4-8\,\alpha\,\beta^3-\beta^2\alpha^2+3\,\beta\,\alpha^3+2\,\alpha^4\right)]$$

$$^{000}P_{5,K}^{101}(x)=-\beta^2\,[35\,\beta^2\alpha^4\,(\alpha+\beta)^4\,x^4-$$

$$-8\,\alpha^2\left(12\,\alpha^4+24\,\beta\,\alpha^3-29\,\beta^2\alpha^2-41\,\alpha\,\beta^3-18\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\beta^2\left(2\,\beta^2+7\,\beta\,\alpha+7\,\alpha^2\right)\left(\beta\alpha+\alpha^2+\beta^2\right)]$$

$$^{000}P_{5,K}^{110}(x)=$$

$$=(\alpha+\beta)^2\,[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4+8\,\beta^2\alpha^2\left(18\,\alpha^4+31\,\beta\,\alpha^3+14\,\beta^2\alpha^2+31\,\alpha\,\beta^3+18\,\beta^4\right)x^2+$$

$$+96\left(2\,\beta^2-3\,\beta\,\alpha+2\,\alpha^2\right)\left(\beta\alpha+\alpha^2+\beta^2\right)(\alpha+\beta)^2]$$

$$^{000}P_{5,K}^{111}(x)=16\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\beta^2\alpha^2\left(\beta\alpha+\alpha^2+\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+12\,\alpha^6+36\,\alpha^5\beta+37\,\beta^2\alpha^4+14\,\beta^3\alpha^3+37\,\beta^4\alpha^2+36\,\beta^5\alpha+12\,\beta^6]$$

$$^{011}V_5=\frac{x^2}{630\alpha^4\beta^4(\alpha+\beta)^4}$$

$$^{011}P_{5,J}^{000}(x)=\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^4-276\,\alpha^3\beta-180\,\alpha^2\beta^2+192\,\beta^3\alpha+96\,\beta^4]$$

$$^{011}P_{5,J}^{001}(x)=-\alpha\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+23\,\alpha\beta-12\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^6-456\,\alpha^5\beta-624\,\alpha^4\beta^2-384\,\beta^3\alpha^3+384\,\beta^4\alpha^2+336\,\beta^5\alpha+96\,\beta^6]$$

$$^{011}P_{5,J}^{010}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(20\,\alpha^2+47\,\alpha\beta+12\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-240\,\alpha^6-792\,\alpha^5\beta-912\,\alpha^4\beta^2-480\,\beta^3\alpha^3-144\,\beta^4\alpha^2-240\,\beta^5\alpha-96\,\beta^6]$$

$$^{011}P_{5,J}^{011}(x)=\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4+$$

$$+4\,\beta^2\left(45\,\alpha^4+116\,\alpha^3\beta+140\,\alpha^2\beta^2+48\,\beta^3\alpha+24\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-48\,\alpha^2\left(5\,\alpha^4+24\,\alpha^3\beta+45\,\alpha^2\beta^2+42\,\beta^3\alpha+21\,\beta^4\right)]$$

$$^{011}P_{5,J}^{100}(x)=$$

$$=\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2+16\,\alpha\beta+16\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^6-336\,\alpha^5\beta-288\,\alpha^4\beta^2-96\,\beta^3\alpha^3-528\,\beta^4\alpha^2-576\,\beta^5\alpha-192\,\beta^6]$$

$$^{011}P_{5,J}^{101}(x) = \beta^2 \left[35\,\alpha^4\beta^2\left(\alpha+\beta\right)^4 x^4 - \right.$$

$$-4\,\alpha^2\left(30\,\alpha^4 + 69\,\alpha^3\beta + 40\,\alpha^2\beta^2 + 82\,\beta^3\alpha + 36\,\beta^4\right)\left(\alpha+\beta\right)^2 x^2 +$$

$$\left. +48\,\beta^3\left(4\,\beta^3 + 21\,\alpha^3 + 30\,\alpha^2\beta + 18\,\beta^2\alpha\right) \right]$$

$$^{011}P_{5,J}^{110}(x) =$$

$$= -\left(\alpha+\beta\right)^2 \left[35\,\alpha^4\beta^4\left(\alpha+\beta\right)^2 x^4 + 4\,\alpha^2\beta^2\left(45\,\alpha^4 + 91\,\alpha^3\beta - 10\,\alpha^2\beta^2 - 62\,\beta^3\alpha - 36\,\beta^4\right) x^2 - \right.$$

$$\left. -48\left(5\,\alpha^3 - 6\,\alpha^2\beta + 6\,\beta^2\alpha - 4\,\beta^3\right)\left(\alpha+\beta\right)^3 \right]$$

$$^{011}P_{5,J}^{111}(x) = -4\,\alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(5\,\alpha^2 - 4\,\alpha\,\beta - 4\,\beta^2\right)\left(\alpha+\beta\right)^2 x^2 - \right.$$

$$\left. -60\,\alpha^6 - 198\,\alpha^5\beta - 218\,\alpha^4\beta^2 + 8\,\beta^3\alpha^3 + 124\,\beta^4\alpha^2 + 144\,\beta^5\alpha + 48\,\beta^6 \right]$$

$$^{011}P_{5,I}^{000}(x) = -\alpha^2\beta^2\left(\alpha+\beta\right)^2 x^2 \left[35\,\alpha^2\beta^2\left(\alpha+\beta\right)^2 x^2 + \right.$$

$$\left. +120\,\alpha^4 + 276\,\alpha^3\beta + 180\,\alpha^2\beta^2 - 192\,\beta^3\alpha - 96\,\beta^4 \right]$$

$$^{011}P_{5,I}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(15\,\alpha^2 + 23\,\alpha\,\beta - 12\,\beta^2\right)\left(\alpha+\beta\right)^2 x^2 + \right.$$

$$\left. +120\,\alpha^6 + 456\,\alpha^5\beta + 624\,\beta^2\alpha^4 + 384\,\beta^3\alpha^3 - 384\,\beta^4\alpha^2 - 336\,\beta^5\alpha - 96\,\beta^6 \right]$$

$$^{011}P_{5,I}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(20\,\alpha^2 + 47\,\alpha\,\beta + 12\,\beta^2\right)\left(\alpha+\beta\right)^2 x^2 + \right.$$

$$\left. +240\,\alpha^6 + 792\,\alpha^5\beta + 912\,\beta^2\alpha^4 + 480\,\beta^3\alpha^3 + 144\,\beta^4\alpha^2 + 240\,\beta^5\alpha + 96\,\beta^6 \right]$$

$$^{011}P_{5,I}^{011}(x) = \alpha^2 \left[35\,\beta^4\alpha^2\left(\alpha+\beta\right)^4 x^4 - \right.$$

$$-4\,\beta^2\left(45\,\alpha^4 + 116\,\alpha^3\beta + 140\,\alpha^2\beta^2 + 48\,\beta^3\alpha + 24\,\beta^4\right)\left(\alpha+\beta\right)^2 x^2 -$$

$$\left. -48\,\alpha^2\left(21\,\beta^4 + 45\,\alpha^2\beta^2 + 24\,\alpha^3\beta + 42\,\beta^3\alpha + 5\,\alpha^4\right) \right]$$

$$^{011}P_{5,I}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(15\,\alpha^2 + 16\,\alpha\,\beta + 16\,\beta^2\right)\left(\alpha+\beta\right)^2 x^2 + \right.$$

$$\left. +120\,\alpha^6 + 336\,\alpha^5\beta + 288\,\beta^2\alpha^4 + 96\,\beta^3\alpha^3 + 528\,\beta^4\alpha^2 + 576\,\beta^5\alpha + 192\,\beta^6 \right]$$

$$^{011}P_{5,I}^{101}(x) =$$

$$= \beta^2 \left[35\,\beta^2\alpha^4\left(\alpha+\beta\right)^4 x^4 + 4\,\alpha^2\left(30\,\alpha^4 + 69\,\alpha^3\beta + 40\,\alpha^2\beta^2 + 82\,\beta^3\alpha + 36\,\beta^4\right)\left(\alpha+\beta\right)^2 x^2 + \right.$$

$$\left. +48\,\beta^3\left(21\,\alpha^3 + 30\,\alpha^2\beta + 18\,\beta^2\alpha + 4\,\beta^3\right) \right]$$

$$^{011}P_{5,I}^{110}(x) =$$

$$= -\left(\alpha+\beta\right)^2 \left[35\,\alpha^4\beta^4\left(\alpha+\beta\right)^2 x^4 - 4\,\alpha^2\beta^2\left(45\,\alpha^4 + 91\,\alpha^3\beta - 10\,\alpha^2\beta^2 - 62\,\beta^3\alpha - 36\,\beta^4\right) x^2 - \right.$$

$$\left. -48\left(5\,\alpha^3 - 6\,\alpha^2\beta + 6\,\beta^2\alpha - 4\,\beta^3\right)\left(\alpha+\beta\right)^3 \right]$$

$$^{011}P_{5,I}^{111}(x) = -4\,\alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(5\,\alpha^2 - 4\,\alpha\,\beta - 4\,\beta^2\right)\left(\alpha+\beta\right)^2 x^2 + \right.$$

$$\left. +60\,\alpha^6 + 198\,\alpha^5\beta + 218\,\beta^2\alpha^4 - 8\,\beta^3\alpha^3 - 124\,\beta^4\alpha^2 - 144\,\beta^5\alpha - 48\,\beta^6 \right]$$

$$^{011}P_{5,K}^{000}(x) =$$

$$= -\alpha^2\beta^2\left(\alpha+\beta\right)^2 x^2 \left[35\,\alpha^2\beta^2\left(\alpha+\beta\right)^2 x^2 + 120\,\alpha^4 + 276\,\alpha^3\beta + 180\,\alpha^2\beta^2 - 192\,\beta^3\alpha - 96\,\beta^4 \right]$$

$$^{011}P_{5,K}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(15\,\alpha^2 + 23\,\alpha\,\beta - 12\,\beta^2\right)\left(\alpha+\beta\right)^2 x^2 + \right.$$

$$\left. +120\,\alpha^6 + 456\,\alpha^5\beta + 624\,\beta^2\alpha^4 + 384\,\beta^3\alpha^3 - 384\,\beta^4\alpha^2 - 336\,\beta^5\alpha - 96\,\beta^6 \right]$$

$$^{011}P_{5,K}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[5\,\alpha^2\beta^2\left(20\,\alpha^2 + 47\,\alpha\,\beta + 12\,\beta^2\right)\left(\alpha+\beta\right)^2 x^2 + \right.$$

$$\left. +240\,\alpha^6 + 792\,\alpha^5\beta + 912\,\beta^2\alpha^4 + 480\,\beta^3\alpha^3 + 144\,\beta^4\alpha^2 + 240\,\beta^5\alpha + 96\,\beta^6 \right]$$

$$^{011}P_{5,K}^{011}(x) =$$

$$= \alpha^2 \left[35\,\beta^4\alpha^2\left(\alpha+\beta\right)^4 x^4 - 4\,\beta^2\left(45\,\alpha^4 + 116\,\alpha^3\beta + 140\,\alpha^2\beta^2 + 48\,\beta^3\alpha + 24\,\beta^4\right)\left(\alpha+\beta\right)^2 x^2 - \right.$$

$$-48\,\alpha^2\left(21\,\beta^4 + 45\,\alpha^2\beta^2 + 24\,\alpha^3\beta + 42\,\beta^3\alpha + 5\,\alpha^4\right)]$$

$$^{011}P_{5,K}^{100}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2 + 16\,\alpha\,\beta + 16\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^6 + 336\,\alpha^5\beta + 288\,\beta^2\alpha^4 + 96\,\beta^3\alpha^3 + 528\,\beta^4\alpha^2 + 576\,\beta^5\alpha + 192\,\beta^6]$$

$$^{011}P_{5,K}^{101}(x) = \beta^2\,[35\,\beta^2\alpha^4\,(\alpha+\beta)^4\,x^4+$$

$$+4\,\alpha^2\left(30\,\alpha^4 + 69\,\alpha^3\beta + 40\,\alpha^2\beta^2 + 82\,\beta^3\alpha + 36\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+48\,\beta^3\left(21\,\alpha^3 + 30\,\alpha^2\beta + 18\,\beta^2\alpha + 4\,\beta^3\right)]$$

$$^{011}P_{5,K}^{110}(x) - (\alpha+\beta)^2\,[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4-$$

$$-4\,\alpha^2\beta^2\left(45\,\alpha^4 + 91\,\alpha^3\beta - 10\,\alpha^2\beta^2 - 62\,\beta^3\alpha - 36\,\beta^4\right)x^2-$$

$$-48\left(5\,\alpha^3 - 6\,\alpha^2\beta + 6\,\beta^2\alpha - 4\,\beta^3\right)(\alpha+\beta)^3]$$

$$^{011}P_{5,K}^{111}(x) = 4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(5\,\alpha^2 - 4\,\alpha\,\beta - 4\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+60\,\alpha^6 + 198\,\alpha^5\beta + 218\,\beta^2\alpha^4 - 8\,\beta^3\alpha^3 - 124\,\beta^4\alpha^2 - 144\,\beta^5\alpha - 48\,\beta^6]$$

$$^{110}V_5 = \frac{x^2}{630\alpha^4\beta^4(\alpha+\beta)^4}$$

$$^{110}P_{5,J}^{000}(x) = -\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^4 - 204\,\beta\,\alpha^3 - 72\,\alpha^2\beta^2 - 204\,\alpha\,\beta^3 - 120\,\beta^4]$$

$$^{110}P_{5,J}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2 + 14\,\alpha\,\beta + 15\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^6 - 384\,\alpha^5\beta - 408\,\alpha^4\beta^2 - 96\,\alpha^3\beta^3 - 408\,\beta^4\alpha^2 - 384\,\beta^5\alpha - 120\,\beta^6]$$

$$^{110}P_{5,J}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(20\,\alpha^2 - 7\,\alpha\,\beta - 15\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-240\,\alpha^6 - 648\,\alpha^5\beta - 552\,\alpha^4\beta^2 - 48\,\alpha^3\beta^3 + 144\,\beta^4\alpha^2 + 264\,\beta^5\alpha + 120\,\beta^6]$$

$$^{110}P_{5,J}^{011}(x) = -\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4+$$

$$+4\,\beta^2\left(45\,\alpha^4 + 89\,\beta\,\alpha^3 - 13\,\alpha^2\beta^2 - 51\,\alpha\,\beta^3 - 30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-48\,\alpha^3\left(21\,\beta^3 + 5\,\alpha^3 + 21\,\beta\,\alpha^2 + 33\,\alpha\,\beta^2\right)]$$

$$^{110}P_{5,J}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2 + 7\,\alpha\,\beta - 20\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-120\,\alpha^6 - 264\,\alpha^5\beta - 144\,\alpha^4\beta^2 + 48\,\alpha^3\beta^3 + 552\,\beta^4\alpha^2 + 648\,\beta^5\alpha + 240\,\beta^6]$$

$$^{110}P_{5,J}^{101}(x) = -\beta^2\,[35\,\alpha^4\beta^2\,(\alpha+\beta)^4\,x^4-$$

$$-4\,\alpha^2\left(30\,\alpha^4 + 51\,\beta\,\alpha^3 + 13\,\alpha^2\beta^2 - 89\,\alpha\,\beta^3 - 45\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-48\,\beta^3\left(21\,\alpha^3 + 33\,\beta\,\alpha^2 + 21\,\alpha\,\beta^2 + 5\,\beta^3\right)]$$

$$^{110}P_{5,J}^{110}(x) = (\alpha+\beta)^2\,[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4+$$

$$+4\,\alpha^2\beta^2\left(45\,\alpha^4 + 64\,\beta\,\alpha^3 + 62\,\alpha^2\beta^2 + 64\,\alpha\,\beta^3 + 45\,\beta^4\right)x^2-$$

$$-48\left(5\,\alpha^4 - 4\,\beta\,\alpha^3 + 3\,\alpha^2\beta^2 - 4\,\alpha\,\beta^3 + 5\,\beta^4\right)(\alpha+\beta)^2]$$

$$^{110}P_{5,J}^{111}(x) = 4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(5\,\alpha^2 + 14\,\alpha\,\beta + 5\,\beta^2\right)(\alpha+\beta)^2\,x^2-$$

$$-60\,\alpha^6 - 162\,\alpha^5\beta - 128\,\alpha^4\beta^2 - 100\,\alpha^3\beta^3 - 128\,\beta^4\alpha^2 - 162\,\beta^5\alpha - 60\,\beta^6]$$

$$^{110}P_{5,I}^{000}(x) = \alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^4 + 204\,\beta\,\alpha^3 + 72\,\alpha^2\beta^2 + 204\,\alpha\,\beta^3 + 120\,\beta^4]$$

$$^{110}P_{5,I}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2 + 14\,\alpha\,\beta + 15\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^6 + 384\,\alpha^5\beta + 408\,\alpha^4\beta^2 + 96\,\alpha^3\beta^3 + 408\,\beta^4\alpha^2 + 384\,\beta^5\alpha + 120\,\beta^6]$$

$$^{110}P_{5,I}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(20\,\alpha^2 - 7\,\alpha\,\beta - 15\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$
$$+240\,\alpha^6 + 648\,\alpha^5\beta + 552\,\alpha^4\beta^2 + 48\,\alpha^3\beta^3 - 144\,\beta^4\alpha^2 - 264\,\beta^5\alpha - 120\,\beta^6]$$

$$^{110}P_{5,I}^{011}(x) = -\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4-$$
$$-4\,\beta^2\left(45\,\alpha^4 + 89\,\beta\,\alpha^3 - 13\,\alpha^2\beta^2 - 51\,\alpha\,\beta^3 - 30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-48\,\alpha^3\left(21\,\beta^3 + 5\,\alpha^3 + 21\,\beta\,\alpha^2 + 33\,\alpha\,\beta^2\right)]$$

$$^{110}P_{5,I}^{100}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2 + 7\,\alpha\,\beta - 20\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$
$$+120\,\alpha^6 + 264\,\alpha^5\beta + 144\,\alpha^4\beta^2 - 48\,\alpha^3\beta^3 - 552\,\beta^4\alpha^2 - 648\,\beta^5\alpha - 240\,\beta^6]$$

$$^{110}P_{5,I}^{101}(x) = -\beta^2\,[35\,\alpha^4\beta^2\,(\alpha+\beta)^4\,x^4+$$
$$+4\,\alpha^2\left(30\,\alpha^4 + 51\,\beta\,\alpha^3 + 13\,\alpha^2\beta^2 - 89\,\alpha\,\beta^3 - 45\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-48\,\beta^3\left(21\,\alpha^3 + 33\,\beta\,\alpha^2 + 21\,\alpha\,\beta^2 + 5\,\beta^3\right)]$$

$$^{110}P_{5,I}^{110}(x) = (\alpha+\beta)^2\,[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4-$$
$$-4\,\alpha^2\beta^2\left(45\,\alpha^4 + 64\,\beta\,\alpha^3 + 62\,\alpha^2\beta^2 + 64\,\alpha\,\beta^3 + 45\,\beta^4\right)x^2-$$
$$-48\left(5\,\alpha^4 - 4\,\beta\,\alpha^3 + 3\,\alpha^2\beta^2 - 4\,\alpha\,\beta^3 + 5\,\beta^4\right)(\alpha+\beta)^2]$$

$$^{110}P_{5,I}^{111}(x) = 4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(5\,\alpha^2 + 14\,\alpha\,\beta + 5\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$
$$+60\,\alpha^6 + 162\,\alpha^5\beta + 128\,\alpha^4\beta^2 + 100\,\alpha^3\beta^3 + 128\,\beta^4\alpha^2 + 162\,\beta^5\alpha + 60\,\beta^6]$$

$$^{110}P_{5,K}^{000}(x) = \alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+$$
$$+120\,\alpha^4 + 204\,\beta\,\alpha^3 + 72\,\alpha^2\beta^2 + 204\,\alpha\,\beta^3 + 120\,\beta^4]$$

$$^{110}P_{5,K}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2 + 14\,\alpha\,\beta + 15\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$
$$+120\,\alpha^6 + 384\,\alpha^5\beta + 408\,\alpha^4\beta^2 + 96\,\alpha^3\beta^3 + 408\,\beta^4\alpha^2 + 384\,\beta^5\alpha + 120\,\beta^6]$$

$$^{110}P_{5,K}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(20\,\alpha^2 - 7\,\alpha\,\beta - 15\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$
$$+240\,\alpha^6 + 648\,\alpha^5\beta + 552\,\alpha^4\beta^2 + 48\,\alpha^3\beta^3 - 144\,\beta^4\alpha^2 - 264\,\beta^5\alpha - 120\,\beta^6]$$

$$^{110}P_{5,K}^{011}(x) = -\alpha^2\,[35\,\beta^4\alpha^2\,(\alpha+\beta)^4\,x^4-$$
$$-4\,\beta^2\left(45\,\alpha^4 + 89\,\beta\,\alpha^3 - 13\,\alpha^2\beta^2 - 51\,\alpha\,\beta^3 - 30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-48\,\alpha^3\left(21\,\beta^3 + 5\,\alpha^3 + 21\,\beta\,\alpha^2 + 33\,\alpha\,\beta^2\right)]$$

$$^{110}P_{5,K}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(15\,\alpha^2 + 7\,\alpha\,\beta - 20\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$
$$+120\,\alpha^6 + 264\,\alpha^5\beta + 144\,\alpha^4\beta^2 - 48\,\alpha^3\beta^3 - 552\,\beta^4\alpha^2 - 648\,\beta^5\alpha - 240\,\beta^6]$$

$$^{110}P_{5,K}^{101}(x) = -\beta^2\,[35\,\alpha^4\beta^2\,(\alpha+\beta)^4\,x^4+$$
$$+4\,\alpha^2\left(30\,\alpha^4 + 51\,\beta\,\alpha^3 + 13\,\alpha^2\beta^2 - 89\,\alpha\,\beta^3 - 45\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$
$$-48\,\beta^3\left(21\,\alpha^3 + 33\,\beta\,\alpha^2 + 21\,\alpha\,\beta^2 + 5\,\beta^3\right)]$$

$$^{110}P_{5,K}^{110}(x) = (\alpha+\beta)^2\,[35\,\alpha^4\beta^4\,(\alpha+\beta)^2\,x^4-$$
$$-4\,\alpha^2\beta^2\left(45\,\alpha^4 + 64\,\beta\,\alpha^3 + 62\,\alpha^2\beta^2 + 64\,\alpha\,\beta^3 + 45\,\beta^4\right)x^2-$$
$$-48\left(5\,\alpha^4 - 4\,\beta\,\alpha^3 + 3\,\alpha^2\beta^2 - 4\,\alpha\,\beta^3 + 5\,\beta^4\right)(\alpha+\beta)^2]$$

$$^{110}P_{5,K}^{111}(x) = -4\,\alpha\,\beta\,(\alpha+\beta)\,x\,[5\,\alpha^2\beta^2\left(5\,\alpha^2 + 14\,\alpha\,\beta + 5\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$
$$+60\,\alpha^6 + 162\,\alpha^5\beta + 128\,\alpha^4\beta^2 + 100\,\alpha^3\beta^3 + 128\,\beta^4\alpha^2 + 162\,\beta^5\alpha + 60\,\beta^6]$$

$$^{001}V_6 = \frac{x^2}{6930\,\alpha^5\,\beta^5\,(\alpha+\beta)^5}$$

$$^{001}P_{6,J}^{000}(x) = -4\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\,(5\,\alpha^2+16\,\alpha\,\beta+5\,\beta^2)\,(\alpha+\beta)^2\,x^2-$$

$$-600\,\alpha^6-1932\,\alpha^5\beta-2172\,\alpha^4\beta^2-2400\,\alpha^3\beta^3-2172\,\beta^4\alpha^2-1932\,\beta^5\alpha-600\,\beta^6]$$

$$^{001}P_{6,J}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\alpha^2\beta^2\,(75\,\alpha^4+184\,\beta\,\alpha^3+338\,\alpha^2\beta^2+184\,\alpha\,\beta^3+75\,\beta^4)\,(\alpha+\beta)^2\,x^2-$$

$$-2400\,\alpha^8-11328\,\beta\,\alpha^7-20880\,\beta^2\alpha^6-18720\,\alpha^5\beta^3-19296\,\alpha^4\beta^4-18720\,\alpha^3\beta^5-20880\,\alpha^2\beta^6-$$

$$-11328\,\alpha\,\beta^7-2400\,\beta^8]$$

$$^{001}P_{6,J}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-20\,\alpha^2\beta^2\,(100\,\alpha^4+222\,\beta\,\alpha^3+138\,\alpha^2\beta^2-149\,\alpha\,\beta^3-75\,\beta^4)\,(\alpha+\beta)^2\,x^2+$$

$$+4800\,\alpha^8+20256\,\beta\,\alpha^7+32832\,\beta^2\alpha^6+25488\,\alpha^5\beta^3+9648\,\alpha^4\beta^4-6768\,\alpha^3\beta^5-11952\,\alpha^2\beta^6-$$

$$-8928\,\alpha\,\beta^7-2400\,\beta^8]$$

$$^{001}P_{6,J}^{011}(x) = \alpha^2\,[175\,\beta^4\alpha^2\,(5\,\alpha^2+7\,\alpha\,\beta-4\,\beta^2)\,(\alpha+\beta)^4\,x^4-$$

$$-16\,\beta^2\,(225\,\alpha^6+787\,\alpha^5\beta+972\,\alpha^4\beta^2+495\,\alpha^3\beta^3-518\,\beta^4\alpha^2-483\,\beta^5\alpha-150\,\beta^6)\,(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^3\,(671\,\beta^2\alpha^3+286\,\beta\,\alpha^4+825\,\alpha^2\beta^3+561\,\beta^4\alpha+50\,\alpha^5+231\,\beta^5)]$$

$$^{001}P_{6,J}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\alpha^2\beta^2\,(75\,\alpha^4+149\,\beta\,\alpha^3-138\,\alpha^2\beta^2-222\,\alpha\,\beta^3-100\,\beta^4)\,(\alpha+\beta)^2\,x^2-$$

$$-2400\,\alpha^8-8928\,\beta\,\alpha^7-11952\,\beta^2\alpha^6-6768\,\alpha^5\beta^3+9648\,\alpha^4\beta^4+25488\,\alpha^3\beta^5+32832\,\alpha^2\beta^6+$$

$$+20256\,\alpha\,\beta^7+4800\,\beta^8]$$

$$^{001}P_{6,J}^{101}(x) = -\beta^2\,[175\,\alpha^4\beta^2\,(4\,\alpha^2-7\,\alpha\,\beta-5\,\beta^2)\,(\alpha+\beta)^4\,x^4-$$

$$-16\,\alpha^2\,(150\,\alpha^6+483\,\alpha^5\beta+518\,\alpha^4\beta^2-495\,\alpha^3\beta^3-972\,\beta^4\alpha^2-787\,\beta^5\alpha-225\,\beta^6)\,(\alpha+\beta)^2\,x^2-$$

$$-96\,\beta^3\,(825\,\beta^2\alpha^3+671\,\alpha^2\beta^3+231\,\alpha^5+561\,\beta\,\alpha^4+286\,\beta^4\alpha+50\,\beta^5)]$$

$$^{001}P_{6,J}^{110}(x) = -(\alpha+\beta)^2\,[35\,\alpha^4\beta^4\,(25\,\alpha^2+26\,\alpha\,\beta+25\,\beta^2)\,(\alpha+\beta)^2\,x^4-$$

$$-16\,\alpha^2\beta^2\,(225\,\alpha^6+662\,\alpha^5\beta+632\,\alpha^4\beta^2+210\,\alpha^3\beta^3+632\,\beta^4\alpha^2+662\,\beta^5\alpha+225\,\beta^6)\,x^2+$$

$$+96\,(50\,\alpha^4-64\,\beta\,\alpha^3+69\,\alpha^2\beta^2-64\,\alpha\,\beta^3+50\,\beta^4)\,(\alpha+\beta)^4]$$

$$^{001}P_{6,J}^{111}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-40\,\alpha^2\beta^2\,(50\,\alpha^4+111\,\beta\,\alpha^3+62\,\alpha^2\beta^2+111\,\alpha\,\beta^3+50\,\beta^4)\,(\alpha+\beta)^2\,x^2+$$

$$+4800\,\alpha^8+20256\,\beta\,\alpha^7+32032\,\beta^2\alpha^6+22912\,\alpha^5\beta^3+6912\,\alpha^4\beta^4+22912\,\alpha^3\beta^5+32032\,\alpha^2\beta^6+$$

$$+20256\,\alpha\,\beta^7+4800\,\beta^8]$$

$$^{001}P_{6,I}^{000}(x) = 4\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\,(5\,\alpha^2+16\,\alpha\,\beta+5\,\beta^2)\,(\alpha+\beta)^2\,x^2+$$

$$+600\,\alpha^6+1932\,\alpha^5\beta+2172\,\alpha^4\beta^2+2400\,\beta^3\alpha^3+2172\,\beta^4\alpha^2+1932\,\alpha\,\beta^5+600\,\beta^6]$$

$$^{001}P_{6,I}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-20\,\alpha^2\beta^2\,(75\,\alpha^4+184\,\alpha^3\beta+338\,\alpha^2\beta^2+184\,\beta^3\alpha+75\,\beta^4)\,(\alpha+\beta)^2\,x^2-$$

$$-2400\,\alpha^8-11328\,\beta\,\alpha^7-20880\,\beta^2\alpha^6-18720\,\alpha^5\beta^3-19296\,\alpha^4\beta^4-18720\,\beta^5\alpha^3-20880\,\beta^6\alpha^2-$$

$$-11328\,\beta^7\alpha-2400\,\beta^8]$$

$$^{001}P_{6,I}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\alpha^2\beta^2\left(100\,\alpha^4 + 222\,\alpha^3\beta + 138\,\alpha^2\beta^2 - 149\,\beta^3\alpha - 75\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+4800\,\alpha^8 + 20256\,\beta\,\alpha^7 + 32832\,\beta^2\alpha^6 + 25488\,\alpha^5\beta^3 + 9648\,\alpha^4\beta^4 - 6768\,\beta^5\alpha^3 - 11952\,\beta^6\alpha^2-$$

$$-8928\,\beta^7\alpha - 2400\,\beta^8]$$

$$^{001}P_{6,I}^{011}(x) = \alpha^2\left[175\,\beta^4\alpha^2\left(5\,\alpha^2 + 7\,\alpha\beta - 4\,\beta^2\right)(\alpha+\beta)^4\,x^4+\right.$$

$$+16\,\beta^2\left(225\,\alpha^6 + 787\,\alpha^5\beta + 972\,\alpha^4\beta^2 + 495\,\beta^3\alpha^3 - 518\,\beta^4\alpha^2 - 483\,\alpha\beta^5 - 150\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^3\left(825\,\beta^3\alpha^2 + 231\,\beta^5 + 286\,\alpha^4\beta + 561\,\beta^4\alpha + 671\,\alpha^3\beta^2 + 50\,\alpha^5\right)]$$

$$^{001}P_{6,I}^{100}(x) = -\alpha\beta\,(\alpha+\beta)\,x\left[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-\right.$$

$$-20\,\alpha^2\beta^2\left(75\,\alpha^4 + 149\,\alpha^3\beta - 138\,\alpha^2\beta^2 - 222\,\beta^3\alpha - 100\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-2400\,\alpha^8 - 8928\,\beta\,\alpha^7 - 11952\,\beta^2\alpha^6 - 6768\,\alpha^5\beta^3 + 9648\,\alpha^4\beta^4 + 25488\,\beta^5\alpha^3 + 32832\,\beta^6\alpha^2+$$

$$+20256\,\beta^7\alpha + 4800\,\beta^8]$$

$$^{001}P_{6,I}^{101}(x) = -\beta^2\left[175\,\alpha^4\beta^2\left(4\,\alpha^2 - 7\,\alpha\beta - 5\,\beta^2\right)(\alpha+\beta)^4\,x^4+\right.$$

$$+16\,\alpha^2\left(150\,\alpha^6 + 483\,\alpha^5\beta + 518\,\alpha^4\beta^2 - 495\,\beta^3\alpha^3 - 972\,\beta^4\alpha^2 - 787\,\alpha\beta^5 - 225\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$

$$-96\,\beta^3\left(825\,\alpha^3\beta^2 + 231\,\alpha^5 + 286\,\beta^4\alpha + 561\,\alpha^4\beta + 671\,\beta^3\alpha^2 + 50\,\beta^5\right)]$$

$$^{001}P_{6,I}^{110}(x) = -(\alpha+\beta)^2\left[35\,\alpha^4\beta^4\left(25\,\alpha^2 + 26\,\alpha\beta + 25\,\beta^2\right)(\alpha+\beta)^2\,x^4+\right.$$

$$+16\,\alpha^2\beta^2\left(225\,\alpha^6 + 662\,\alpha^5\beta + 632\,\alpha^4\beta^2 + 210\,\beta^3\alpha^3 + 632\,\beta^4\alpha^2 + 662\,\alpha\beta^5 + 225\,\beta^6\right)x^2+$$

$$+96\left(50\,\alpha^4 - 64\,\alpha^3\beta + 69\,\alpha^2\beta^2 - 64\,\beta^3\alpha + 50\,\beta^4\right)(\alpha+\beta)^4]$$

$$^{001}P_{6,I}^{111}(x) = \alpha\beta\,(\alpha+\beta)\,x\left[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+\right.$$

$$+40\,\alpha^2\beta^2\left(50\,\alpha^4 + 111\,\alpha^3\beta + 62\,\alpha^2\beta^2 + 111\,\beta^3\alpha + 50\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+4800\,\alpha^8 + 20256\,\beta\,\alpha^7 + 32032\,\beta^2\alpha^6 + 22912\,\alpha^5\beta^3 + 6912\,\alpha^4\beta^4 + 22912\,\beta^5\alpha^3 + 32032\,\beta^6\alpha^2+$$

$$+20256\,\beta^7\alpha + 4800\,\beta^8]$$

$$^{001}P_{6,K}^{000}(x) = -4\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\left[35\,\alpha^2\beta^2\left(5\,\alpha^2 + 16\,\alpha\beta + 5\,\beta^2\right)(\alpha+\beta)^2\,x^2+\right.$$

$$+600\,\alpha^6 + 1932\,\alpha^5\beta + 2172\,\alpha^4\beta^2 + 2400\,\beta^3\alpha^3 + 2172\,\beta^4\alpha^2 + 1932\,\alpha\beta^5 + 600\,\beta^6]$$

$$^{001}P_{6,K}^{001}(x) = \alpha\beta\,(\alpha+\beta)\,x\left[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-\right.$$

$$-20\,\alpha^2\beta^2\left(75\,\alpha^4 + 184\,\alpha^3\beta + 338\,\alpha^2\beta^2 + 184\,\beta^3\alpha + 75\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-2400\,\alpha^8 - 11328\,\beta\,\alpha^7 - 20880\,\beta^2\alpha^6 - 18720\,\alpha^5\beta^3 - 19296\,\alpha^4\beta^4 - 18720\,\beta^5\alpha^3 - 20880\,\beta^6\alpha^2-$$

$$-11328\,\beta^7\alpha - 2400\,\beta^8]$$

$$^{001}P_{6,K}^{010}(x) = -\alpha\beta\,(\alpha+\beta)\,x\left[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+\right.$$

$$+20\,\alpha^2\beta^2\left(100\,\alpha^4 + 222\,\alpha^3\beta + 138\,\alpha^2\beta^2 - 149\,\beta^3\alpha - 75\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+4800\,\alpha^8 + 20256\,\beta\,\alpha^7 + 32832\,\beta^2\alpha^6 + 25488\,\alpha^5\beta^3 + 9648\,\alpha^4\beta^4 - 6768\,\beta^5\alpha^3 - 11952\,\beta^6\alpha^2-$$

$$-8928\,\beta^7\alpha - 2400\,\beta^8]$$

$$^{001}P_{6,K}^{011}(x) = -\alpha^2\left[175\,\beta^4\alpha^2\left(5\,\alpha^2 + 7\,\alpha\beta - 4\,\beta^2\right)(\alpha+\beta)^4\,x^4+\right.$$

$$+16\,\beta^2\left(225\,\alpha^6 + 787\,\alpha^5\beta + 972\,\alpha^4\beta^2 + 495\,\beta^3\alpha^3 - 518\,\beta^4\alpha^2 - 483\,\alpha\beta^5 - 150\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^3\left(825\,\beta^3\alpha^2 + 231\,\beta^5 + 286\,\alpha^4\beta + 561\,\beta^4\alpha + 671\,\alpha^3\beta^2 + 50\,\alpha^5\right)]$$

$$^{001}P_{6,K}^{100}(x) = -\alpha\beta\,(\alpha+\beta)\,x\left[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-\right.$$

$$-20\,\alpha^2\beta^2\left(75\,\alpha^4 + 149\,\alpha^3\beta - 138\,\alpha^2\beta^2 - 222\,\beta^3\alpha - 100\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-2400\,\alpha^8 - 8928\,\beta\,\alpha^7 - 11952\,\beta^2\alpha^6 - 6768\,\alpha^5\beta^3 + 9648\,\alpha^4\beta^4 + 25488\,\beta^5\alpha^3 + 32832\,\beta^6\alpha^2+$$

$$+20256\,\beta^7\alpha + 4800\,\beta^8$$

$$^{001}P_{6,K}^{101}(x) = \beta^2\left[175\,\alpha^4\beta^2\left(4\,\alpha^2 - 7\,\alpha\beta - 5\,\beta^2\right)(\alpha+\beta)^4\,x^4+\right.$$

$$+16\,\alpha^2\left(150\,\alpha^6 + 483\,\alpha^5\beta + 518\,\alpha^4\beta^2 - 495\,\beta^3\alpha^3 - 972\,\beta^4\alpha^2 - 787\,\alpha\beta^5 - 225\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$

$$\left.-96\,\beta^3\left(825\,\alpha^3\beta^2 + 231\,\alpha^5 + 286\,\beta^4\alpha + 561\,\alpha^4\beta + 671\,\beta^3\alpha^2 + 50\,\beta^5\right)\right]$$

$$^{001}P_{6,K}^{110}(x) = (\alpha+\beta)^2\left[35\,\alpha^4\beta^4\left(25\,\alpha^2 + 26\,\alpha\beta + 25\,\beta^2\right)(\alpha+\beta)^2\,x^4+\right.$$

$$+16\,\alpha^2\beta^2\left(225\,\alpha^6 + 662\,\alpha^5\beta + 632\,\alpha^4\beta^2 + 210\,\beta^3\alpha^3 + 632\,\beta^4\alpha^2 + 662\,\alpha\beta^5 + 225\,\beta^6\right)x^2+$$

$$\left.+96\left(50\,\alpha^4 - 64\,\alpha^3\beta + 69\,\alpha^2\beta^2 - 64\,\beta^3\alpha + 50\,\beta^4\right)(\alpha+\beta)^4\right]$$

$$^{001}P_{6,K}^{111}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[315\,\alpha^4\beta^4(\alpha+\beta)^4\,x^4+\right.$$

$$+40\,\alpha^2\beta^2\left(50\,\alpha^4 + 111\,\alpha^3\beta + 62\,\alpha^2\beta^2 + 111\,\beta^3\alpha + 50\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+4800\,\alpha^8 + 20256\,\beta\,\alpha^7 + 32032\,\beta^2\alpha^6 + 22912\,\alpha^5\beta^3 + 6912\,\alpha^4\beta^4 + 22912\,\beta^5\alpha^3 + 32032\,\beta^6\alpha^2+$$

$$\left.+20256\,\beta^7\alpha + 4800\,\beta^8\right]$$

$$^{010}V_6 = \frac{x^2}{6930\,\alpha^5\beta^5(\alpha+\beta)^5}$$

$$^{010}P_{6,J}^{000}(x) = -4\,\alpha^2\beta^2(\alpha+\beta)^2\,x^2\left[35\,\alpha^2\beta^2\left(6\,\alpha^2 + 6\,\alpha\beta - 5\,\beta^2\right)(\alpha+\beta)^2\,x^2-\right.$$

$$\left.-720\,\alpha^6 - 2160\,\beta\,\alpha^5 - 2316\,\beta^2\alpha^4 - 1032\,\beta^3\alpha^3 + 1512\,\beta^4\alpha^2 + 1668\,\beta^5\alpha + 600\,\beta^6\right]$$

$$^{010}P_{6,J}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\left[315\,\alpha^4\beta^4(\alpha+\beta)^4\,x^4-\right.$$

$$-20\,\alpha^2\beta^2\left(90\,\alpha^4 + 201\,\alpha^3\beta + 135\,\alpha^2\beta^2 - 151\,\beta^3\alpha - 75\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+2880\,\alpha^8 + 12960\,\alpha^7\beta + 22944\,\alpha^6\beta^2 + 19824\,\alpha^5\beta^3 + 8688\,\alpha^4\beta^4 - 11856\,\alpha^3\beta^5 - 16656\,\alpha^2\beta^6-$$

$$\left.-10272\,\beta^7\alpha - 2400\,\beta^8\right]$$

$$^{010}P_{6,J}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[315\,\alpha^4\beta^4(\alpha+\beta)^4\,x^4+\right.$$

$$+20\,\alpha^2\beta^2\left(120\,\alpha^4 + 240\,\alpha^3\beta + 236\,\alpha^2\beta^2 + 116\,\beta^3\alpha + 75\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-5760\,\alpha^8 - 23040\,\alpha^7\beta - 35808\,\alpha^6\beta^2 - 26784\,\alpha^5\beta^3 - 10416\,\alpha^4\beta^4 - 3072\,\alpha^3\beta^5-$$

$$\left.-8784\,\alpha^2\beta^6 - 7872\,\beta^7\alpha - 2400\,\beta^8\right]$$

$$^{010}P_{6,J}^{011}(x) = \alpha^2\left[175\,\beta^4\alpha^2\left(6\,\alpha^2 + 15\,\alpha\beta + 4\,\beta^2\right)(\alpha+\beta)^4\,x^4-\right.$$

$$-16\,\beta^2\left(270\,\alpha^6 + 885\,\beta\,\alpha^5 + 1041\,\beta^2\alpha^4 + 737\,\beta^3\alpha^3 + 353\,\beta^4\alpha^2 + 417\,\beta^5\alpha + 150\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$\left.+96\,\alpha^3\left(60\,\alpha^5 + 748\,\alpha^3\beta^2 + 330\,\alpha^4\beta + 594\,\alpha\beta^4 + 891\,\beta^3\alpha^2 + 231\,\beta^5\right)\right]$$

$$^{010}P_{6,J}^{100}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\left[315\,\alpha^4\beta^4(\alpha+\beta)^4\,x^4-\right.$$

$$-20\,\alpha^2\beta^2\left(90\,\alpha^4 + 159\,\alpha^3\beta + 72\,\alpha^2\beta^2 + 178\,\beta^3\alpha + 100\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+2880\,\alpha^8 + 10080\,\alpha^7\beta + 12864\,\alpha^6\beta^2 + 6960\,\alpha^5\beta^3 + 1728\,\alpha^4\beta^4 + 14928\,\alpha^3\beta^5 + 25440\,\alpha^2\beta^6+$$

$$\left.+18144\,\beta^7\alpha + 4800\,\beta^8\right]$$

$$^{010}P_{6,J}^{101}(x) = -\beta^2\left[35\,\beta^2\alpha^4\left(24\,\alpha^2 + 24\,\alpha\beta + 25\,\beta^2\right)(\alpha+\beta)^4\,x^4-\right.$$

$$-16\,\alpha^2\left(180\,\alpha^6 + 540\,\beta\,\alpha^5 + 549\,\beta^2\alpha^4 + 198\,\beta^3\alpha^3 + 697\,\beta^4\alpha^2 + 688\,\beta^5\alpha + 225\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$\left.+96\,\beta^4\left(561\,\alpha^2\beta^2 + 264\,\beta^3\alpha + 297\,\alpha^4 + 594\,\alpha^3\beta + 50\,\beta^4\right)\right]$$

$$^{010}P_{6,J}^{110}(x) = -(\alpha+\beta)^2\left[175\,\alpha^4\beta^4\left(6\,\alpha^2 - 3\,\alpha\beta - 5\,\beta^2\right)(\alpha+\beta)^2\,x^4-\right.$$

$$-16\,\alpha^2\beta^2\left(270\,\alpha^6 + 735\,\beta\,\alpha^5 + 666\,\beta^2\alpha^4 - 23\,\beta^3\alpha^3 - 412\,\beta^4\alpha^2 - 563\,\beta^5\alpha - 225\,\beta^6\right)x^2+$$

$$\left.+96\left(60\,\alpha^5 - 30\,\alpha^4\beta + 28\,\alpha^3\beta^2 - 27\,\beta^3\alpha^2 + 36\,\alpha\beta^4 - 50\,\beta^5\right)(\alpha+\beta)^3\right]$$

$$^{010}P_{6,J}^{111}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+40\,\alpha^2\beta^2\left(60\,\alpha^4+120\,\alpha^3\beta-29\,\alpha^2\beta^2-89\,\beta^3\alpha-50\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-5760\,\alpha^8-23040\,\alpha^7\beta-34848\,\alpha^6\beta^2-23904\,\alpha^5\beta^3-128\,\alpha^4\beta^4+12704\,\alpha^3\beta^5+24640\,\alpha^2\beta^6+$$

$$+18144\,\beta^7\alpha+4800\,\beta^8]$$

$$^{010}P_{6,I}^{000}(x) = 4\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\left(6\,\alpha^2+6\,\alpha\beta-5\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+720\,\alpha^6+2160\,\alpha^5\beta+2316\,\alpha^4\beta^2+1032\,\beta^3\alpha^3-1512\,\beta^4\alpha^2-1668\,\beta^5\alpha-600\,\beta^6]$$

$$^{010}P_{6,I}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\alpha^2\beta^2\left(90\,\alpha^4+201\,\alpha^3\beta+135\,\alpha^2\beta^2-151\,\beta^3\alpha-75\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+2880\,\alpha^8+12960\,\alpha^7\beta+22944\,\beta^2\alpha^6+19824\,\alpha^5\beta^3+8688\,\alpha^4\beta^4-11856\,\beta^5\alpha^3-16656\,\beta^6\alpha^2-$$

$$-10272\,\beta^7\alpha-2400\,\beta^8]$$

$$^{010}P_{6,I}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-20\,\alpha^2\beta^2\left(120\,\alpha^4+240\,\alpha^3\beta+236\,\alpha^2\beta^2+116\,\beta^3\alpha+75\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-5760\,\alpha^8-23040\,\alpha^7\beta-35808\,\beta^2\alpha^6-26784\,\alpha^5\beta^3-10416\,\alpha^4\beta^4-3072\,\beta^5\alpha^3-8784\,\beta^6\alpha^2-$$

$$-7872\,\beta^7\alpha-2400\,\beta^8]$$

$$^{010}P_{6,I}^{011}(x) = \alpha^2\,[175\,\beta^4\alpha^2\left(6\,\alpha^2+15\,\alpha\beta+4\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$

$$+16\,\beta^2\left(270\,\alpha^6+885\,\alpha^5\beta+1041\,\alpha^4\beta^2+737\,\beta^3\alpha^3+353\,\beta^4\alpha^2+417\,\beta^5\alpha+150\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^3\left(594\,\alpha\beta^4+330\,\alpha^4\beta+891\,\beta^3\alpha^2+748\,\alpha^3\beta^2+231\,\beta^5+60\,\alpha^5\right)]$$

$$^{010}P_{6,I}^{100}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\alpha^2\beta^2\left(90\,\alpha^4+159\,\alpha^3\beta+72\,\alpha^2\beta^2+178\,\beta^3\alpha+100\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+2880\,\alpha^8+10080\,\alpha^7\beta+12864\,\beta^2\alpha^6+6960\,\alpha^5\beta^3+1728\,\alpha^4\beta^4+14928\,\beta^5\alpha^3+25440\,\beta^6\alpha^2+$$

$$+18144\,\beta^7\alpha+4800\,\beta^8]$$

$$^{010}P_{6,I}^{101}(x) = -\beta^2\,[35\,\alpha^4\beta^2\left(24\,\alpha^2+24\,\alpha\beta+25\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$

$$+16\,\alpha^2\left(180\,\alpha^6+540\,\alpha^5\beta+549\,\alpha^4\beta^2+198\,\beta^3\alpha^3+697\,\beta^4\alpha^2+688\,\beta^5\alpha+225\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\beta^4\left(561\,\alpha^2\beta^2+264\,\beta^3\alpha+297\,\alpha^4+594\,\alpha^3\beta+50\,\beta^4\right)]$$

$$^{010}P_{6,I}^{110}(x) = -(\alpha+\beta)^2\,[175\,\alpha^4\beta^4\left(6\,\alpha^2-3\,\alpha\beta-5\,\beta^2\right)(\alpha+\beta)^2\,x^4+$$

$$+16\,\alpha^2\beta^2\left(270\,\alpha^6+735\,\alpha^5\beta+666\,\alpha^4\beta^2-23\,\beta^3\alpha^3-412\,\beta^4\alpha^2-563\,\beta^5\alpha-225\,\beta^6\right)x^2+$$

$$+96\left(60\,\alpha^5-30\,\alpha^4\beta+28\,\alpha^3\beta^2-27\,\beta^3\alpha^2+36\,\alpha\beta^4-50\,\beta^5\right)(\alpha+\beta)^3]$$

$$^{010}P_{6,I}^{111}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-40\,\alpha^2\beta^2\left(60\,\alpha^4+120\,\alpha^3\beta-29\,\alpha^2\beta^2-89\,\beta^3\alpha-50\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-5760\,\alpha^8-23040\,\alpha^7\beta-34848\,\beta^2\alpha^6-23904\,\alpha^5\beta^3-128\,\alpha^4\beta^4+12704\,\beta^5\alpha^3+24640\,\beta^6\alpha^2+$$

$$+18144\,\beta^7\alpha+4800\,\beta^8]$$

$$^{010}P_{6,K}^{000}(x) = -4\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\left(6\,\alpha^2+6\,\alpha\beta-5\,\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+720\,\alpha^6+2160\,\alpha^5\beta+2316\,\alpha^4\beta^2+1032\,\beta^3\alpha^3-1512\,\beta^4\alpha^2-1668\,\beta^5\alpha-600\,\beta^6]$$

$$^{010}P_{6,K}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\alpha^2\beta^2\left(90\,\alpha^4+201\,\alpha^3\beta+135\,\alpha^2\beta^2-151\,\beta^3\alpha-75\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+2880\,\alpha^8 + 12960\,\alpha^7\beta + 22944\,\beta^2\alpha^6 + 19824\,\alpha^5\beta^3 + 8688\,\alpha^4\beta^4 - 11856\,\beta^5\alpha^3 - 16656\,\beta^6\alpha^2 -$$
$$-10272\,\beta^7\alpha - 2400\,\beta^8$$

$$^{010}P_{6,K}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4 -$$
$$-20\,\alpha^2\beta^2\,(120\,\alpha^4 + 240\,\alpha^3\beta + 236\,\alpha^2\beta^2 + 116\,\beta^3\alpha + 75\,\beta^4)\,(\alpha+\beta)^2\,x^2 -$$
$$-5760\,\alpha^8 - 23040\,\alpha^7\beta - 35808\,\beta^2\alpha^6 - 26784\,\alpha^5\beta^3 - 10416\,\alpha^4\beta^4 - 3072\,\beta^5\alpha^3 - 8784\,\beta^6\alpha^2 -$$
$$-7872\,\beta^7\alpha - 2400\,\beta^8]$$

$$^{010}P_{6,K}^{011}(x) = -\alpha^2\,[175\,\beta^4\alpha^2\,(6\,\alpha^2 + 15\,\alpha\beta + 4\,\beta^2)\,(\alpha+\beta)^4\,x^4 +$$
$$+16\,\beta^2\,(270\,\alpha^6 + 885\,\alpha^5\beta + 1041\,\alpha^4\beta^2 + 737\,\beta^3\alpha^3 + 353\,\beta^4\alpha^2 + 417\,\beta^5\alpha + 150\,\beta^6)\,(\alpha+\beta)^2\,x^2 +$$
$$+96\,\alpha^3\,(594\,\alpha\beta^4 + 330\,\alpha^4\beta + 891\,\beta^3\alpha^2 + 748\,\alpha^3\beta^2 + 231\,\beta^5 + 60\,\alpha^5)]$$

$$^{010}P_{6,K}^{100}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4 +$$
$$+20\,\alpha^2\beta^2\,(90\,\alpha^4 + 159\,\alpha^3\beta + 72\,\alpha^2\beta^2 + 178\,\beta^3\alpha + 100\,\beta^4)\,(\alpha+\beta)^2\,x^2 +$$
$$+2880\,\alpha^8 + 10080\,\alpha^7\beta + 12864\,\beta^2\alpha^6 + 6960\,\alpha^5\beta^3 + 1728\,\alpha^4\beta^4 + 14928\,\beta^5\alpha^3 + 25440\,\beta^6\alpha^2 +$$
$$+18144\,\beta^7\alpha + 4800\,\beta^8]$$

$$^{010}P_{6,K}^{101}(x) = \beta^2\,[35\,\alpha^4\beta^2\,(24\,\alpha^2 + 24\,\alpha\beta + 25\,\beta^2)\,(\alpha+\beta)^4\,x^4 +$$
$$+16\,\alpha^2\,(180\,\alpha^6 + 540\,\alpha^5\beta + 549\,\alpha^4\beta^2 + 198\,\beta^3\alpha^3 + 697\,\beta^4\alpha^2 + 688\,\beta^5\alpha + 225\,\beta^6)\,(\alpha+\beta)^2\,x^2 +$$
$$+96\,\beta^4\,(561\,\alpha^2\beta^2 + 264\,\beta^3\alpha + 297\,\alpha^4 + 594\,\alpha^3\beta + 50\,\beta^4)]$$

$$^{010}P_{6,K}^{110}(x) = (\alpha+\beta)^2\,[175\,\alpha^4\beta^4\,(6\,\alpha^2 - 3\,\alpha\beta - 5\,\beta^2)\,(\alpha+\beta)^2\,x^4 +$$
$$+16\,\alpha^2\beta^2\,(270\,\alpha^6 + 735\,\alpha^5\beta + 666\,\alpha^4\beta^2 - 23\,\beta^3\alpha^3 - 412\,\beta^4\alpha^2 - 563\,\beta^5\alpha - 225\,\beta^6)\,x^2 +$$
$$+96\,(60\,\alpha^5 - 30\,\alpha^4\beta + 28\,\alpha^3\beta^2 - 27\,\beta^3\alpha^2 + 36\,\alpha\beta^4 - 50\,\beta^5)\,(\alpha+\beta)^3]$$

$$^{010}P_{6,K}^{111}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[315\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4 -$$
$$-40\,\alpha^2\beta^2\,(60\,\alpha^4 + 120\,\alpha^3\beta - 29\,\alpha^2\beta^2 - 89\,\beta^3\alpha - 50\,\beta^4)\,(\alpha+\beta)^2\,x^2 -$$
$$-5760\,\alpha^8 - 23040\,\alpha^7\beta - 34848\,\beta^2\alpha^6 - 23904\,\alpha^5\beta^3 - 128\,\alpha^4\beta^4 + 12704\,\beta^5\alpha^3 + 24640\,\beta^6\alpha^2 +$$
$$+18144\,\beta^7\alpha + 4800\,\beta^8$$

$$^{111}V_6 = \frac{x^2}{2310\,\alpha^5\beta^5(\alpha+\beta)^5}$$

$$^{111}P_{6,I}^{000}(x) = -8\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\alpha^2\beta^2\,(\alpha^2 + \alpha\beta + \beta^2)\,(\alpha+\beta)^2\,x^2 -$$
$$-120\,\alpha^6 - 360\,\alpha^5\beta - 342\,\alpha^4\beta^2 - 84\,\beta^3\alpha^3 - 342\,\beta^4\alpha^2 - 360\,\beta^5\alpha - 120\,\beta^6$$

$$^{111}P_{6,J}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4 -$$
$$-20\,\beta^2\alpha^2\,(30\,\alpha^4 + 67\,\alpha^3\beta + 34\,\beta^2\alpha^2 + 67\,\alpha\beta^3 + 30\,\beta^4)\,(\alpha+\beta)^2\,x^2 +$$
$$+960\,\alpha^8 + 4320\,\alpha^7\beta + 7296\,\alpha^6\beta^2 + 5376\,\alpha^5\beta^3 + 960\,\alpha^4\beta^4 + 5376\,\beta^5\alpha^3 + 7296\,\beta^6\alpha^2 +$$
$$+4320\,\beta^7\alpha + 960\,\beta^8]$$

$$^{111}P_{6,J}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4 +$$
$$+20\,\beta^2\alpha^2\,(40\,\alpha^4 + 80\,\alpha^3\beta - 13\,\beta^2\alpha^2 - 53\,\alpha\beta^3 - 30\,\beta^4)\,(\alpha+\beta)^2\,x^2 -$$
$$-1920\,\alpha^8 - 7680\,\alpha^7\beta - 11232\,\alpha^6\beta^2 - 6816\,\alpha^5\beta^3 - 480\,\alpha^4\beta^4 + 1440\,\beta^5\alpha^3 + 3936\,\beta^6\alpha^2 +$$
$$+3360\,\beta^7\alpha + 960\,\beta^8]$$

$$^{111}P_{6,J}^{011}(x) = 2\,\alpha^2\,[35\,\beta^4\alpha^2\,(5\,\alpha^2 - 4\,\alpha\beta - 4\,\beta^2)\,(\alpha+\beta)^4\,x^4 -$$

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$$-8\,\beta^2\left(90\,\alpha^6+295\,\alpha^5\beta+314\,\beta^2\alpha^4-22\,\alpha^3\beta^3-161\,\beta^4\alpha^2-180\,\alpha\,\beta^5-60\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^4\left(10\,\alpha^4+121\,\beta^2\alpha^2+55\,\alpha^3\beta+132\,\alpha\,\beta^3+66\,\beta^4\right)]$$

$${}^{111}P_{6,J}^{100}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-20\,\beta^2\alpha^2\left(30\,\alpha^4+53\,\alpha^3\beta+13\,\beta^2\alpha^2-80\,\alpha\,\beta^3-40\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+960\,\alpha^8+3360\,\alpha^7\beta+3936\,\alpha^6\beta^2+1440\,\alpha^5\beta^3-480\,\alpha^4\beta^4-6816\,\beta^5\alpha^3-11232\,\beta^6\alpha^2-$$

$$-7680\,\beta^7\alpha-1920\,\beta^8]$$

$${}^{111}P_{6,J}^{101}(x)=-2\,\beta^2\,[35\,\beta^2\alpha^4\left(4\,\alpha^2+4\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^4\,x^4-$$

$$-8\,\alpha^2\left(60\,\alpha^6+180\,\alpha^5\beta+161\,\beta^2\alpha^4+22\,\alpha^3\beta^3-314\,\beta^4\alpha^2-295\,\alpha\,\beta^5-90\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$

$$-96\,\beta^4\left(66\,\alpha^4+132\,\alpha^3\beta+121\,\beta^2\alpha^2+55\,\alpha\,\beta^3+10\,\beta^4\right)]$$

$${}^{111}P_{6,J}^{110}(x)=-2\,(\alpha+\beta)^2\,[35\,\alpha^4\beta^4\left(5\,\alpha^2+14\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^4-$$

$$-8\,\beta^2\alpha^2\left(90\,\alpha^6+245\,\alpha^5\beta+189\,\beta^2\alpha^4+128\,\alpha^3\beta^3+189\,\beta^4\alpha^2+245\,\alpha\,\beta^5+90\,\beta^6\right)x^2+$$

$$+96\left(10\,\alpha^4-15\,\alpha^3\beta+16\,\beta^2\alpha^2-15\,\alpha\,\beta^3+10\,\beta^4\right)(\alpha+\beta)^4]$$

$${}^{111}P_{6,J}^{111}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+800\,\beta^2\alpha^2\,(\alpha+\beta)^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2\,x^2-$$

$$-32\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(60\,\beta^6+180\,\alpha\,\beta^5+101\,\beta^4\alpha^2-98\,\alpha^3\beta^3+101\,\beta^2\alpha^4+180\,\alpha^5\beta+60\,\alpha^6\right)]$$

$${}^{111}P_{6,I}^{000}(x)=-8\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\beta^2\alpha^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^6+360\,\alpha^5\beta+342\,\beta^2\alpha^4+84\,\alpha^3\beta^3+342\,\beta^4\alpha^2+360\,\alpha\,\beta^5+120\,\beta^6]$$

$${}^{111}P_{6,I}^{001}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\beta^2\alpha^2\left(30\,\alpha^4+67\,\alpha^3\beta+34\,\beta^2\alpha^2+67\,\alpha\,\beta^3+30\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+960\,\alpha^8+4320\,\alpha^7\beta+7296\,\alpha^6\beta^2+5376\,\alpha^5\beta^3+960\,\alpha^4\beta^4+5376\,\beta^5\alpha^3+7296\,\beta^6\alpha^2+$$

$$+4320\,\beta^7\alpha+960\,\beta^8]$$

$${}^{111}P_{6,I}^{010}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-20\,\beta^2\alpha^2\left(40\,\alpha^4+80\,\alpha^3\beta-13\,\beta^2\alpha^2-53\,\alpha\,\beta^3-30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-1920\,\alpha^8-7680\,\alpha^7\beta-11232\,\alpha^6\beta^2-6816\,\alpha^5\beta^3-480\,\alpha^4\beta^4+1440\,\beta^5\alpha^3+3936\,\beta^6\alpha^2+$$

$$+3360\,\beta^7\alpha+960\,\beta^8]$$

$${}^{111}P_{6,I}^{011}(x)=-2\,\alpha^2\,[35\,\beta^4\alpha^2\left(5\,\alpha^2-4\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$

$$+8\,\beta^2\left(90\,\alpha^6+295\,\alpha^5\beta+314\,\beta^2\alpha^4-22\,\alpha^3\beta^3-161\,\beta^4\alpha^2-180\,\alpha\,\beta^5-60\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^4\left(10\,\alpha^4+121\,\beta^2\alpha^2+55\,\alpha^3\beta+132\,\alpha\,\beta^3+66\,\beta^4\right)]$$

$${}^{111}P_{6,I}^{100}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\beta^2\alpha^2\left(30\,\alpha^4+53\,\alpha^3\beta+13\,\beta^2\alpha^2-80\,\alpha\,\beta^3-40\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+960\,\alpha^8+3360\,\alpha^7\beta+3936\,\alpha^6\beta^2+1440\,\alpha^5\beta^3-480\,\alpha^4\beta^4-6816\,\beta^5\alpha^3-11232\,\beta^6\alpha^2-$$

$$-7680\,\beta^7\alpha-1920\,\beta^8]$$

$${}^{111}P_{6,I}^{101}(x)=2\,\beta^2\,[35\,\beta^2\alpha^4\left(4\,\alpha^2+4\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$

$$+8\,\alpha^2\left(60\,\alpha^6+180\,\alpha^5\beta+161\,\beta^2\alpha^4+22\,\alpha^3\beta^3-314\,\beta^4\alpha^2-295\,\alpha\,\beta^5-90\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$

$$-96\,\beta^4\left(66\,\alpha^4+132\,\alpha^3\beta+121\,\beta^2\alpha^2+55\,\alpha\,\beta^3+10\,\beta^4\right)]$$

$${}^{111}P_{6,I}^{110}(x)=2\,(\alpha+\beta)^2\,[35\,\alpha^4\beta^4\left(5\,\alpha^2+14\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^4+$$

$$+8\,\beta^2\alpha^2\left(90\,\alpha^6+245\,\alpha^5\beta+189\,\beta^2\alpha^4+128\,\alpha^3\beta^3+189\,\beta^4\alpha^2+245\,\alpha\,\beta^5+90\,\beta^6\right)x^2+$$

$$+96\left(10\,\alpha^4-15\,\alpha^3\beta+16\,\beta^2\alpha^2-15\,\alpha\beta^3+10\,\beta^4\right)(\alpha+\beta)^4]$$

$$^{111}P_{6,I}^{111}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-800\,\beta^2\alpha^2\,(\alpha+\beta)^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2x^2-$$

$$-32\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(60\,\beta^6+180\,\alpha\,\beta^5+101\,\beta^4\alpha^2-98\,\alpha^3\beta^3+101\,\beta^2\alpha^4+180\,\alpha^5\beta+60\,\alpha^6\right)]$$

$$^{111}P_{6,K}^{000}(x)=8\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2\,[35\,\beta^2\alpha^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)(\alpha+\beta)^2\,x^2+$$

$$+120\,\alpha^6+360\,\alpha^5\beta+342\,\beta^2\alpha^4+84\,\alpha^3\beta^3+342\,\beta^4\alpha^2+360\,\alpha\,\beta^5+120\,\beta^6]$$

$$^{111}P_{6,K}^{001}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\beta^2\alpha^2\left(30\,\alpha^4+67\,\alpha^3\beta+34\,\beta^2\alpha^2+67\,\alpha\,\beta^3+30\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+960\,\alpha^8+4320\,\alpha^7\beta+7296\,\alpha^6\beta^2+5376\,\alpha^5\beta^3+960\,\alpha^4\beta^4+5376\,\beta^5\alpha^3+7296\,\beta^6\alpha^2+$$

$$+4320\,\beta^7\alpha+960\,\beta^8]$$

$$^{111}P_{6,K}^{010}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-$$

$$-20\,\beta^2\alpha^2\left(40\,\alpha^4+80\,\alpha^3\beta-13\,\beta^2\alpha^2-53\,\alpha\,\beta^3-30\,\beta^4\right)(\alpha+\beta)^2\,x^2-$$

$$-1920\,\alpha^8-7680\,\alpha^7\beta-11232\,\alpha^6\beta^2-6816\,\alpha^5\beta^3-480\,\alpha^4\beta^4+1440\,\beta^5\alpha^3+3936\,\beta^6\alpha^2+$$

$$+3360\,\beta^7\alpha+960\,\beta^8]$$

$$^{111}P_{6,K}^{011}(x)=2\,\alpha^2\,[35\,\beta^4\alpha^2\left(5\,\alpha^2-4\,\alpha\,\beta-4\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$

$$+8\,\beta^2\left(90\,\alpha^6+295\,\alpha^5\beta+314\,\beta^2\alpha^4-22\,\alpha^3\beta^3-161\,\beta^4\alpha^2-180\,\alpha\,\beta^5-60\,\beta^6\right)(\alpha+\beta)^2\,x^2+$$

$$+96\,\alpha^4\left(10\,\alpha^4+121\,\beta^2\alpha^2+55\,\alpha^3\beta+132\,\alpha\,\beta^3+66\,\beta^4\right)]$$

$$^{111}P_{6,K}^{100}(x)=-\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4+$$

$$+20\,\beta^2\alpha^2\left(30\,\alpha^4+53\,\alpha^3\beta+13\,\beta^2\alpha^2-80\,\alpha\,\beta^3-40\,\beta^4\right)(\alpha+\beta)^2\,x^2+$$

$$+960\,\alpha^8+3360\,\alpha^7\beta+3936\,\alpha^6\beta^2+1440\,\alpha^5\beta^3-480\,\alpha^4\beta^4-6816\,\beta^5\alpha^3-11232\,\beta^6\alpha^2-$$

$$-7680\,\beta^7\alpha-1920\,\beta^8]$$

$$^{111}P_{6,K}^{101}(x)=-2\,\beta^2\,[35\,\beta^2\alpha^4\left(4\,\alpha^2+4\,\alpha\,\beta-5\,\beta^2\right)(\alpha+\beta)^4\,x^4+$$

$$+8\,\alpha^2\left(60\,\alpha^6+180\,\alpha^5\beta+161\,\beta^2\alpha^4+22\,\alpha^3\beta^3-314\,\beta^4\alpha^2-295\,\alpha\,\beta^5-90\,\beta^6\right)(\alpha+\beta)^2\,x^2-$$

$$-96\,\beta^4\left(66\,\alpha^4+132\,\alpha^3\beta+121\,\beta^2\alpha^2+55\,\alpha\,\beta^3+10\,\beta^4\right)]$$

$$^{111}P_{6,K}^{110}(x)=-2\,(\alpha+\beta)^2\,[35\,\alpha^4\beta^4\left(5\,\alpha^2+14\,\alpha\,\beta+5\,\beta^2\right)(\alpha+\beta)^2\,x^4+$$

$$+8\,\beta^2\alpha^2\left(90\,\alpha^6+245\,\alpha^5\beta+189\,\beta^2\alpha^4+128\,\alpha^3\beta^3+189\,\beta^4\alpha^2+245\,\alpha\,\beta^5+90\,\beta^6\right)x^2+$$

$$+96\left(10\,\alpha^4-15\,\alpha^3\beta+16\,\beta^2\alpha^2-15\,\alpha\,\beta^3+10\,\beta^4\right)(\alpha+\beta)^4]$$

$$^{111}P_{6,K}^{111}(x)=\alpha\,\beta\,(\alpha+\beta)\,x\,[105\,\alpha^4\beta^4\,(\alpha+\beta)^4\,x^4-800\,\beta^2\alpha^2\,(\alpha+\beta)^2\left(\alpha^2+\alpha\,\beta+\beta^2\right)^2x^2-$$

$$-32\left(\alpha^2+\alpha\,\beta+\beta^2\right)\left(60\,\beta^6+180\,\alpha\,\beta^5+101\,\beta^4\alpha^2-98\,\alpha^3\beta^3+101\,\beta^2\alpha^4+180\,\alpha^5\beta+60\,\alpha^6\right)]$$

Recurrence relations:

$$\int x^{2n+1}\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ -4n^2\beta(2\alpha+\beta)\int x^{2n} J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x)\, dx - \right.$$

$$-4n^2(\alpha^2-\beta^2)\int x^{2n} J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx - 4n(n-1)\alpha^2\int x^{2n} J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) + 2n\beta(2\alpha+\beta)\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x) +$$

$$+2n(\alpha^2-\beta^2)\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) +$$

$$\left. + 2n\alpha^2\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+1}\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ 2n\beta(\alpha-2n\beta)\int x^{2n} J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x)\, dx + \right.$$

$$+2n[(2n+1)\alpha+2n\beta](\alpha+\beta)\int x^{2n} J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx +$$

$$+2(2n+1)(n-1)\alpha^2\int x^{2n} J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) +$$

$$+\beta(2n\beta-\alpha)\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x) - (\alpha+\beta)((2n+1)\alpha+2n\beta)\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x) -$$

$$\left. - \alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) - (2n+1)\alpha^2\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+1}\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ -2n\beta[(2n+1)\beta+\alpha]\int x^{2n} J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x)\, dx - \right.$$

$$-2n(2n+1)(\alpha^2-\beta^2)\int x^{2n} J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx -$$

$$-2(n-1)\alpha[(2n+1)\alpha+(4n+1)\beta]\int x^{2n+1} J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[-\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) + \beta[(2n+1)\beta+\alpha]\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x) +$$

$$+(2n+1)(\alpha^2-\beta^2)\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) +$$

$$\left. +\alpha[(4n+1)\beta+(2n+1)\alpha]\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+2} J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ 2(n+1)\beta[(4n+3)\alpha + (2n+1)\beta] \int x^{2n+1} J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x)\, dx - \right.$$

$$-2n(2n+1)(\alpha^2-\beta^2) \int x^{2n+1} J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\, dx +$$

$$+2\alpha n[(2n+1)\alpha - \beta] \int x^{2n+1} J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\big[-\beta[(4n+3)\alpha + (2n+1)\beta] J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) +$$

$$+(2n+1)(\alpha^2-\beta^2) J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x J_1(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) -$$

$$\left. -\alpha[(2n+1)\alpha - \beta] J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta) x J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\big] \right\}$$

$$\int x^{2n+2} J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ -2\beta^2(n+1)(2n+1) \int x^{2n+1} J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x)\, dx - \right.$$

$$-2n(\alpha+\beta)[2(n+1)\alpha + (2n+1)\beta] \int x^{2n+1} J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\, dx +$$

$$+2n\alpha[2(n+1)\alpha + \beta] \int x^{2n+1} J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\big[(2n+1)\beta^2 J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) + (\alpha+\beta)[2(n+1)\alpha + (2n+1)\beta] J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x J_1(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) - \alpha[2(n+1)\alpha + \beta] J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) +$$

$$\left. +\alpha\beta(\alpha+\beta) J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\big] \right\}$$

$$\int x^{2n+2} J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ 4\beta^2(n+1)^2 \int x^{2n+1} J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x)\, dx - \right.$$

$$-4n(n+1)(\alpha^2-\beta^2) \int x^{2n+1} J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\, dx +$$

$$+4n(n+1)\alpha(\alpha+2\beta) \int x^{2n+1} J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\big[-2(n+1)\beta^2 J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) + 2(n+1)(\alpha^2-\beta^2) J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x J_1(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) - 2(n+1)\alpha(\alpha+2\beta) J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) +$$

$$\left. +\alpha\beta(\alpha+\beta)x J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\big] \right\}$$

$$\int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)}\left\{-4n^2\beta(2\alpha+\beta)\int x^{2n}\,I_0(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x)\,dx -\right.$$

$$-4n^2(\alpha^2-\beta^2)\int x^{2n}\,I_0(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x)\,dx + 4n(n-1)\alpha^2\int x^{2n}\,I_1(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x)\,dx+$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)\,I_0(\beta x)\,I_0((\alpha+\beta)x) + 2n\beta(2\alpha+\beta)\,I_0(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x)+$$

$$+2n(\alpha^2-\beta^2)\,I_0(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\,I_0(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\,I_1(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,I_1(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x)-$$

$$\left.- 2n\alpha^2\,I_1(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x)]\right\}$$

$$\int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)}\left\{-2n\beta(\alpha-2n\beta)\int x^{2n}\,I_0(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x)\,dx-\right.$$

$$-2n[(2n+1)\alpha+2n\beta](\alpha+\beta)\int x^{2n}\,I_0(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x)\,dx+$$

$$+2(2n+1)(n-1)\alpha^2\int x^{2n}\,I_1(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x)\,dx+$$

$$+x^{2n+1}[-\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)\,I_0(\beta x)\,I_0((\alpha+\beta)x)-$$

$$-\beta(2n\beta-\alpha)\,I_0(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x) + (\alpha+\beta)((2n+1)\alpha+2n\beta)\,I_0(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,I_1(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x)-$$

$$\left.- \alpha\beta(\alpha+\beta)x\,I_1(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x) - (2n+1)\alpha^2\,I_1(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x)]\right\}$$

$$\int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)}\left\{2n\beta[(2n+1)\beta+\alpha]\int x^{2n}\,I_0(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x)\,dx+\right.$$

$$+2n(2n+1)(\alpha^2-\beta^2)\int x^{2n}\,I_0(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x)\,dx-$$

$$-2(n-1)\alpha[(2n+1)\alpha+(4n+1)\beta]\int x^{2n+1}\,I_1(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x)\,dx+$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\,I_0(\alpha x)\,I_0(\beta x)\,I_0((\alpha+\beta)x) - \beta[(2n+1)\beta+\alpha]\,I_0(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x)-$$

$$-(2n+1)(\alpha^2-\beta^2)\,I_0(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\,I_0(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\,I_1(\alpha x)\,I_0(\beta x)\,I_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\,I_1(\alpha x)\,I_1(\beta x)\,I_0((\alpha+\beta)x)+$$

$$\left.+\alpha[(4n+1)\beta+(2n+1)\alpha]\,I_1(\alpha x)\,I_1(\beta x)\,I_1((\alpha+\beta)x)]\right\}$$

$$\int x^{2n+2}\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ -2(n+1)\beta[(4n+3)\alpha+(2n+1)\beta] \int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx - \right.$$

$$-2n(2n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx +$$

$$+2\alpha n[(2n+1)\alpha-\beta] \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}[\beta[(4n+3)\alpha+(2n+1)\beta]\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) +$$

$$+(2n+1)(\alpha^2-\beta^2)\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) -$$

$$\left. -\alpha[(2n+1)\alpha-\beta]\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) +\alpha\beta(\alpha+\beta)\, x\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+2}\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ 2\beta^2(n+1)(2n+1) \int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx - \right.$$

$$-2n(\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta] \int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx +$$

$$+2n\alpha[2(n+1)\alpha+\beta] \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}[-(2n+1)\beta^2\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) + (\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta]\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) - \alpha[2(n+1)\alpha+\beta]\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) -$$

$$\left. -\alpha\beta(\alpha+\beta)\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+2}\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ 4\beta^2(n+1)^2 \int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx + \right.$$

$$+4n(n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx -$$

$$-4n(n+1)\alpha(\alpha+2\beta) \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}[-2(n+1)\beta^2\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) - 2(n+1)(\alpha^2-\beta^2)\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) + 2(n+1)\alpha(\alpha+2\beta)\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) +$$

$$\left. +\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+1} K_0(\alpha x) K_0(\beta x) K_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ 4n^2\beta(2\alpha+\beta) \int x^{2n} K_0(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x)\, dx + \right.$$

$$+4n^2(\alpha^2-\beta^2) \int x^{2n} K_0(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x)\, dx - 4n(n-1)\alpha^2 \int x^{2n} K_1(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, K_0(\alpha x) K_0(\beta x) K_0((\alpha+\beta)x) - 2n\beta(2\alpha+\beta) K_0(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x) -$$

$$-2n(\alpha^2-\beta^2) K_0(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, K_0(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, K_1(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, K_1(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x) +$$

$$\left. + 2n\alpha^2\, K_1(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+1} K_0(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ 2n\beta(\alpha-2n\beta) \int x^{2n} K_0(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x)\, dx + \right.$$

$$+2n[(2n+1)\alpha + 2n\beta](\alpha+\beta) \int x^{2n} K_0(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x)\, dx -$$

$$-2(2n+1)(n-1)\alpha^2 \int x^{2n} K_1(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[-\alpha\beta(\alpha+\beta)x\, K_0(\alpha x) K_0(\beta x) K_0((\alpha+\beta)x) +$$

$$+\beta(2n\beta-\alpha) K_0(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x) - (\alpha+\beta)((2n+1)\alpha + 2n\beta) K_0(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, K_0(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, K_1(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x) -$$

$$\left. - \alpha\beta(\alpha+\beta)x\, K_1(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x) + (2n+1)\alpha^2\, K_1(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+1} K_1(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ -2n\beta[(2n+1)\beta + \alpha] \int x^{2n} K_0(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x)\, dx - \right.$$

$$-2n(2n+1)(\alpha^2-\beta^2) \int x^{2n} K_0(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x)\, dx +$$

$$+2(n-1)\alpha[(2n+1)\alpha + (4n+1)\beta] \int x^{2n+1} K_1(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, K_0(\alpha x) K_0(\beta x) K_0((\alpha+\beta)x) + \beta[(2n+1)\beta + \alpha] K_0(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x) +$$

$$+(2n+1)(\alpha^2-\beta^2) K_0(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, K_0(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, K_1(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, K_1(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x) -$$

$$\left. -\alpha[(4n+1)\beta + (2n+1)\alpha] K_1(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+2} \, K_0(\alpha x) \, K_0(\beta x) \, K_1((\alpha+\beta)x) \, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ 2(n+1)\beta[(4n+3)\alpha + (2n+1)\beta] \int x^{2n+1} \, K_0(\alpha x) \, K_0(\beta x) \, K_0((\alpha+\beta)x) \, dx + \right.$$

$$+2n(2n+1)(\alpha^2 - \beta^2) \int x^{2n+1} \, K_0(\alpha x) \, K_1(\beta x) \, K_1((\alpha+\beta)x) \, dx -$$

$$-2\alpha n[(2n+1)\alpha - \beta] \int x^{2n+1} \, K_1(\alpha x) \, K_1(\beta x) \, K_0((\alpha+\beta)x) \, dx +$$

$$+x^{2n+2}[-\beta[(4n+3)\alpha + (2n+1)\beta] \, K_0(\alpha x) \, K_0(\beta x) \, K_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x \, K_0(\alpha x) \, K_0(\beta x) \, K_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x \, K_0(\alpha x) \, K_1(\beta x) \, K_0((\alpha+\beta)x) -$$

$$-(2n+1)(\alpha^2 - \beta^2) \, K_0(\alpha x) \, K_1(\beta x) \, K_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x \, K_1(\alpha x) \, K_0(\beta x) \, K_0((\alpha+\beta)x) +$$

$$\left. +\alpha[(2n+1)\alpha - \beta] \, K_1(\alpha x) \, K_1(\beta x) \, K_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)\, x \, K_1(\alpha x) \, K_1(\beta x) \, K_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+2} \, K_0(\alpha x) \, K_1(\beta x) \, K_0((\alpha+\beta)x) \, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ -2\beta^2(n+1)(2n+1) \int x^{2n+1} \, K_0(\alpha x) \, K_0(\beta x) \, K_0((\alpha+\beta)x) \, dx + \right.$$

$$+2n(\alpha+\beta)[2(n+1)\alpha + (2n+1)\beta] \int x^{2n+1} \, K_0(\alpha x) \, K_1(\beta x) \, K_1((\alpha+\beta)x) \, dx -$$

$$-2n\alpha[2(n+1)\alpha + \beta] \int x^{2n+1} \, K_1(\alpha x) \, K_1(\beta x) \, K_0((\alpha+\beta)x) \, dx +$$

$$+x^{2n+2}[(2n+1)\beta^2 \, K_0(\alpha x) \, K_0(\beta x) \, K_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x \, K_0(\alpha x) \, K_0(\beta x) \, K_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x \, K_0(\alpha x) \, K_1(\beta x) \, K_0((\alpha+\beta)x) - (\alpha+\beta)[2(n+1)\alpha + (2n+1)\beta] \, K_0(\alpha x) \, K_1(\beta x) \, K_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x \, K_1(\alpha x) \, K_0(\beta x) \, K_0((\alpha+\beta)x) + \alpha[2(n+1)\alpha + \beta] \, K_1(\alpha x) \, K_1(\beta x) \, K_0((\alpha+\beta)x) -$$

$$\left. -\alpha\beta(\alpha+\beta) \, K_1(\alpha x) \, K_1(\beta x) \, K_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+2} \, K_1(\alpha x) \, K_1(\beta x) \, K_1((\alpha+\beta)x) \, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ -4\beta^2(n+1)^2 \int x^{2n+1} \, K_0(\alpha x) \, K_0(\beta x) \, K_0((\alpha+\beta)x) \, dx - \right.$$

$$-4n(n+1)(\alpha^2 - \beta^2) \int x^{2n+1} \, K_0(\alpha x) \, K_1(\beta x) \, K_1((\alpha+\beta)x) \, dx +$$

$$+4n(n+1)\alpha(\alpha+2\beta) \int x^{2n+1} \, K_1(\alpha x) \, K_1(\beta x) \, K_0((\alpha+\beta)x) \, dx +$$

$$+x^{2n+2} \, [2(n+1)\beta^2 \, K_0(\alpha x) \, K_0(\beta x) \, K_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x \, K_0(\alpha x) \, K_0(\beta x) \, K_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x \, K_0(\alpha x) \, K_1(\beta x) \, K_0((\alpha+\beta)x) + 2(n+1)(\alpha^2 - \beta^2) \, K_0(\alpha x) \, K_1(\beta x) \, K_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x \, K_1(\alpha x) \, K_0(\beta x) \, K_0((\alpha+\beta)x) - 2(n+1)\alpha(\alpha+2\beta) \, K_1(\alpha x) \, K_1(\beta x) \, K_0((\alpha+\beta)x) +$$

$$\left. +\alpha\beta(\alpha+\beta)x \, K_1(\alpha x) \, K_1(\beta x) \, K_1((\alpha+\beta)x)] \right\}$$

c) $x^n Z_\kappa(\alpha x) Z_\mu(\beta x) Z_\nu(\sqrt{\alpha^2 \pm \beta^2}\, x)$

Formulas with $\sqrt{\alpha^2 + \beta^2}$ were found for the following integrals.
Replacing β by $\beta\, i$ one gets some modifications.

$$\int x\, J_1(\alpha x)\, J_1(\beta x)\, J_0(\sqrt{\alpha^2 + \beta^2}\, x)\, dx =$$

$$= \frac{x}{2\alpha\beta} \left[\sqrt{\alpha^2 + \beta^2}\, J_0(\alpha x)\, J_0(\beta x)\, J_1(\sqrt{\alpha^2 + \beta^2}\, x) - \beta\, J_0(\alpha x)\, J_1(\beta x)\, J_0(\sqrt{\alpha^2 + \beta^2}\, x) - \right.$$

$$\left. - \alpha\, J_1(\alpha x)\, J_0(\beta x)\, J_0(\sqrt{\alpha^2 + \beta^2}\, x) \right]$$

$$\int x\, I_1(\alpha x)\, I_1(\beta x)\, I_0(\sqrt{\alpha^2 + \beta^2}\, x)\, dx =$$

$$= \frac{x}{2\alpha\beta} \left[-\sqrt{\alpha^2 + \beta^2}\, I_0(\alpha x)\, I_0(\beta x)\, I_1(\sqrt{\alpha^2 + \beta^2}\, x) + \beta\, I_0(\alpha x)\, I_1(\beta x)\, I_0(\sqrt{\alpha^2 + \beta^2}\, x) + \right.$$

$$\left. + \alpha\, I_1(\alpha x)\, I_0(\beta x)\, I_0(\sqrt{\alpha^2 + \beta^2}\, x) \right]$$

$$\int x\, K_1(\alpha x)\, K_1(\beta x)\, K_0(\sqrt{\alpha^2 + \beta^2}\, x)\, dx =$$

$$= \frac{x}{2\alpha\beta} \left[\sqrt{\alpha^2 + \beta^2}\, K_0(\alpha x)\, K_0(\beta x)\, K_1(\sqrt{\alpha^2 + \beta^2}\, x) - \beta\, K_0(\alpha x)\, K_1(\beta x)\, K_0(\sqrt{\alpha^2 + \beta^2}\, x) - \right.$$

$$\left. - \alpha\, K_1(\alpha x)\, K_0(\beta x)\, K_0(\sqrt{\alpha^2 + \beta^2}\, x) \right]$$

$$\int x\, J_1(\alpha x)\, I_1(\beta x)\, J_0(\sqrt{\alpha^2 - \beta^2}\, x)\, dx =$$

$$= \frac{x}{2\alpha\beta} \left[-\sqrt{\alpha^2 - \beta^2}\, J_0(\alpha x)\, I_0(\beta x)\, J_1(\sqrt{\alpha^2 - \beta^2}\, x) - \beta\, J_0(\alpha x)\, I_1(\beta x)\, J_0(\sqrt{\alpha^2 - \beta^2}\, x) + \right.$$

$$\left. + \alpha\, J_1(\alpha x)\, I_0(\beta x)\, J_0(\sqrt{\alpha^2 - \beta^2}\, x) \right]$$

$$\int x\, I_1(\alpha x)\, J_1(\beta x)\, I_0(\sqrt{\alpha^2 - \beta^2}\, x)\, dx =$$

$$= \frac{x}{2\alpha\beta} \left[\sqrt{\alpha^2 - \beta^2}\, I_0(\alpha x)\, J_0(\beta x)\, I_1(\sqrt{\alpha^2 - \beta^2}\, x) + \beta\, I_0(\alpha x)\, J_1(\beta x)\, I_0(\sqrt{\alpha^2 - \beta^2}\, x) - \right.$$

$$\left. - \alpha\, I_1(\alpha x)\, J_0(\beta x)\, I_0(\sqrt{\alpha^2 - \beta^2}\, x) \right]$$

$$\int x^2\, J_0(\alpha x)\, J_0(\beta x)\, J_1(\sqrt{\alpha^2 + \beta^2}\, x)\, dx = \frac{x^2}{2\alpha\beta} \left[\alpha\, J_0(\alpha x) J_1(\beta x) J_1(\sqrt{\alpha^2 + \beta^2} x) + \right.$$

$$\left. + \beta\, J_1(\alpha x) J_0(\beta x) J_1(\sqrt{\alpha^2 + \beta^2} x) - \sqrt{\alpha^2 + \beta^2} J_1(\alpha x) J_1(\beta x) J_0(\sqrt{\alpha^2 + \beta^2} x) \right]$$

$$\int x^2\, I_0(\alpha x)\, I_0(\beta x)\, I_1(\sqrt{\alpha^2 + \beta^2}\, x)\, dx = \frac{x^2}{2\alpha\beta} \left[\alpha\, I_0(\alpha x) I_1(\beta x) I_1(\sqrt{\alpha^2 + \beta^2} x) + \right.$$

$$\left. + \beta\, I_1(\alpha x) I_0(\beta x) I_1(\sqrt{\alpha^2 + \beta^2} x) - \sqrt{\alpha^2 + \beta^2} I_1(\alpha x) I_1(\beta x) I_0(\sqrt{\alpha^2 + \beta^2} x) \right]$$

$$\int x^2\, K_0(\alpha x)\, K_0(\beta x)\, K_1(\sqrt{\alpha^2 + \beta^2}\, x)\, dx = -\frac{x^2}{2\alpha\beta} \left[\alpha\, K_0(\alpha x) K_1(\beta x) K_1(\sqrt{\alpha^2 + \beta^2} x) + \right.$$

$$\left. + \beta\, K_1(\alpha x) K_0(\beta x) K_1(\sqrt{\alpha^2 + \beta^2} x) - \sqrt{\alpha^2 + \beta^2} K_1(\alpha x) K_1(\beta x) K_0(\sqrt{\alpha^2 + \beta^2} x) \right]$$

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$$\int x^2 \, J_0(\alpha x) \, I_0(\beta x) \, J_1(\sqrt{\alpha^2 - \beta^2}\, x)\, dx = \frac{x^2}{2\alpha\beta} \left[\alpha \, J_0(\alpha\, x) I_1(\beta\, x) J_1(\sqrt{\alpha^2 - \beta^2}x) + \right.$$

$$\left. + \beta \, J_1(\alpha\, x) I_0(\beta\, x) J_1(\sqrt{\alpha^2 - \beta^2}x) - \sqrt{\alpha^2 - \beta^2}\, J_1(\alpha\, x) I_1(\beta\, x) J_0(\sqrt{\alpha^2 - \beta^2}x) \right]$$

$$\int x^2 \, I_0(\alpha x) \, J_0(\beta x) \, I_1(\sqrt{\alpha^2 - \beta^2}\, x)\, dx = \frac{x^2}{2\alpha\beta} \left[\alpha \, I_0(\alpha\, x) J_1(\beta\, x) I_1(\sqrt{\alpha^2 - \beta^2}x) + \right.$$

$$\left. + \beta \, I_1(\alpha\, x) J_0(\beta\, x) I_1(\sqrt{\alpha^2 - \beta^2}x) - \sqrt{\alpha^2 - \beta^2}\, I_1(\alpha\, x) J_1(\beta\, x) I_0(\sqrt{\alpha^2 - \beta^2}x) \right]$$

$$\int x \, J_0(\alpha x) \, J_1(\beta x) \, J_1(\sqrt{\alpha^2 - \beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2 - \beta^2}} \left[-\sqrt{\alpha^2 - \beta^2}\, J_0(\alpha\, x) J_0(\beta\, x) J_1(\sqrt{\alpha^2 - \beta^2}x) - \right.$$

$$\left. - \beta \, J_0(\alpha\, x) J_1(\beta\, x) J_0(\sqrt{\alpha^2 - \beta^2}x) + \alpha \, J_1(\alpha\, x) J_0(\beta\, x) J_0(\sqrt{\alpha^2 - \beta^2}x) \right]$$

$$\int x \, I_0(\alpha x) \, I_1(\beta x) \, I_1(\sqrt{\alpha^2 - \beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2 - \beta^2}} \left[\sqrt{\alpha^2 - \beta^2}\, I_0(\alpha\, x) I_0(\beta\, x) I_1(\sqrt{\alpha^2 - \beta^2}x) + \right.$$

$$\left. + \beta \, I_0(\alpha\, x) I_1(\beta\, x) I_0(\sqrt{\alpha^2 - \beta^2}x) - \alpha \, I_1(\alpha\, x) I_0(\beta\, x) I_0(\sqrt{\alpha^2 - \beta^2}x) \right]$$

$$\int x \, K_0(\alpha x) \, K_1(\beta x) \, K_1(\sqrt{\alpha^2 - \beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2 - \beta^2}} \left[-\sqrt{\alpha^2 - \beta^2}\, K_0(\alpha\, x) K_0(\beta\, x) K_1(\sqrt{\alpha^2 - \beta^2}x) - \right.$$

$$\left. - \beta \, K_0(\alpha\, x) K_1(\beta\, x) K_0(\sqrt{\alpha^2 - \beta^2}x) + \alpha \, K_1(\alpha\, x) K_0(\beta\, x) K_0(\sqrt{\alpha^2 - \beta^2}x) \right]$$

$$\int x \, J_0(\alpha x) \, I_1(\beta x) \, J_1(\sqrt{\alpha^2 + \beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2 + \beta^2}} \left[\sqrt{\alpha^2 + \beta^2}\, J_0(\alpha\, x) I_0(\beta\, x) J_1(\sqrt{\alpha^2 + \beta^2}x) - \right.$$

$$\left. - \beta \, J_0(\alpha\, x) I_1(\beta\, x) J_0(\sqrt{\alpha^2 + \beta^2}x) - \alpha \, J_1(\alpha\, x) I_0(\beta\, x) J_0(\sqrt{\alpha^2 + \beta^2}x) \right]$$

$$\int x \, I_0(\alpha x) \, J_1(\beta x) \, I_1(\sqrt{\alpha^2 + \beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2 + \beta^2}} \left[-\sqrt{\alpha^2 + \beta^2}\, I_0(\alpha\, x) J_0(\beta\, x) I_1(\sqrt{\alpha^2 + \beta^2}x) + \right.$$

$$\left. + \beta \, I_0(\alpha\, x) J_1(\beta\, x) I_0(\sqrt{\alpha^2 + \beta^2}x) + \alpha \, I_1(\alpha\, x) J_0(\beta\, x) I_0(\sqrt{\alpha^2 + \beta^2}x) \right]$$

$$\int x^2 \, J_1(\alpha x) \, J_0(\beta x) \, J_0(\sqrt{\alpha^2 - \beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2 - \beta^2}} \left[\sqrt{\alpha^2 - \beta^2}\, J_1(\alpha\, x) J_1(\beta\, x) J_0(\sqrt{\alpha^2 - \beta^2}x) + \right.$$

$$\left. + \beta \, J_1(\alpha\, x) J_0(\beta\, x) J_1(\sqrt{\alpha^2 - \beta^2}x) - \alpha \, J_0(\alpha\, x) J_1(\beta\, x) J_1(\sqrt{\alpha^2 - \beta^2}x) \right]$$

$$\int x^2 \, I_1(\alpha x) \, I_0(\beta x) \, I_0(\sqrt{\alpha^2 - \beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2 - \beta^2}} \left[\sqrt{\alpha^2 - \beta^2}\, I_1(\alpha\, x) I_1(\beta\, x) I_0(\sqrt{\alpha^2 - \beta^2}x) + \right.$$

$$\left. + \beta \, I_1(\alpha\, x) I_0(\beta\, x) I_1(\sqrt{\alpha^2 - \beta^2}x) - \alpha \, I_0(\alpha\, x) I_1(\beta\, x) I_1(\sqrt{\alpha^2 - \beta^2}x) \right]$$

$$\int x^2 \, K_1(\alpha x) \, K_0(\beta x) \, K_0(\sqrt{\alpha^2 - \beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2 - \beta^2}} \left[-\sqrt{\alpha^2 - \beta^2}\, K_1(\alpha\, x) K_1(\beta\, x) K_0(\sqrt{\alpha^2 - \beta^2}x) - \right.$$

$$\left. - \beta \, K_1(\alpha\, x) K_0(\beta\, x) K_1(\sqrt{\alpha^2 - \beta^2}x) + \alpha \, K_0(\alpha\, x) K_1(\beta\, x) K_1(\sqrt{\alpha^2 - \beta^2}x) \right]$$

$$\int x^2 \, J_1(\alpha x) \, I_0(\beta x) \, J_0(\sqrt{\alpha^2 + \beta^2} \, x) \, dx = \frac{x^2}{2\beta \sqrt{\alpha^2 + \beta^2}} \left[\sqrt{\alpha^2 + \beta^2} \, J_1(\alpha x) I_1(\beta x) J_0(\sqrt{\alpha^2 + \beta^2} x) + \right.$$

$$\left. + \beta \, J_1(\alpha x) I_0(\beta x) J_1(\sqrt{\alpha^2 + \beta^2} x) - \alpha \, J_0(\alpha x) I_1(\beta x) J_1(\sqrt{\alpha^2 + \beta^2} x) \right]$$

$$\int x^2 \, I_1(\alpha x) \, J_0(\beta x) \, I_0(\sqrt{\alpha^2 + \beta^2} \, x) \, dx = \frac{x^2}{2\beta \sqrt{\alpha^2 + \beta^2}} \left[\sqrt{\alpha^2 + \beta^2} \, I_1(\alpha x) J_1(\beta x) I_0(\sqrt{\alpha^2 + \beta^2} x) + \right.$$

$$\left. + \beta \, I_1(\alpha x) J_0(\beta x) I_1(\sqrt{\alpha^2 + \beta^2} x) - \alpha \, I_0(\alpha x) J_1(\beta x) I_1(\sqrt{\alpha^2 + \beta^2} x) \right]$$

4. Products of four Bessel Functions

4.1. Integrals of the type $\int x^m Z_0^n(x) Z_1^{4-n}(x)\, dx$

4.1. a) Explicit Integrals

$$\int J_0^3(x)\, J_1(x)\, dx = -\frac{1}{4}\, J_0^4(x)$$

$$\int I_0^3(x)\, I_1(x)\, dx = \frac{1}{4}\, I_0^4(x)$$

$$\int K_0^3(x)\, K_1(x)\, dx = -\frac{1}{4}\, K_0^4(x)$$

$$\int J_0(x) J_1^3(x)\, dx = \frac{1}{4}\left[x^2 J_0^4(x) - 2x J_0^3(x) J_1(x) + 2x^2 J_0^2(x) J_1^2(x) - 2x J_0(x) J_1^3(x) + x^2 J_1^4(x)\right]$$

$$\int I_0(x) I_1^3(x)\, dx = \frac{1}{4}\left[-x^2 I_0^4(x) + 2x I_0^3(x) I_1(x) + 2x^2 I_0^2(x) I_1^2(x) - 2x I_0(x) I_1^3(x) - x^2 I_1^4(x)\right]$$

$$\int K_0(x) K_1^3(x)\, dx = \frac{1}{4}\left[x^2 K_0^4(x) + 2x K_0^3(x) K_1(x) - 2x^2 K_0^2(x) K_1^2(x) - 2x K_0(x) K_1^3(x) + x^2 K_1^4(x)\right]$$

$$\int x^4\, J_0(x)\, J_1^3(x)\, dx = \frac{x^4}{4}\, J_1^4(x) \qquad \text{(See also p. 508.)}$$

$$\int x^4\, I_0(x)\, I_1^3(x)\, dx = \frac{x^4}{4}\, I_1^4(x)$$

$$\int x^4\, K_0(x)\, K_1^3(x)\, dx = -\frac{x^4}{4}\, K_1^4(x)$$

$$\int \frac{J_0^2(x) J_1^2(x)}{x}\, dx = -\frac{x^2+1}{4} J_0^4(x) + \frac{x}{2} J_0^3(x) J_1(x) - \frac{x^2+1}{2} J_0^2(x) J_1^2(x) + \frac{x}{2} J_0(x) J_1^3(x) - \frac{x^2}{4} J_1^4(x)$$

$$\int \frac{I_0^2(x) I_1^2(x)}{x}\, dx = -\frac{x^2-1}{4} I_0^4(x) + \frac{x}{2} I_0^3(x) I_1(x) + \frac{x^2-1}{2} I_0^2(x) I_1^2(x) - \frac{x}{2} I_0(x) I_1^3(x) - \frac{x^2}{4} I_1^4(x)$$

$$\int \frac{K_0^2(x) K_1^2(x)}{x}\, dx = -\frac{x^2-1}{4} K_0^4(x) - \frac{x}{2} K_0^3(x) K_1(x) + \frac{x^2-1}{2} K_0^2(x) K_1^2(x) + \frac{x}{2} K_0(x) K_1^3(x) - \frac{x^2}{4} K_1^4(x)$$

$$\int \frac{J_0(x) J_1^3(x)}{x^2}\, dx = -\frac{4x^2+3}{16} J_0^4(x) + \frac{x}{2} J_0^3(x) J_1(x) - \frac{4x^2+3}{8} J_0^2(x) J_1^2(x) + \frac{2x^2-1}{4x} J_0(x) J_1^3(x) - \frac{4x^2-1}{16} J_1^4(x)$$

$$\int \frac{I_0(x) I_1^3(x)}{x^2}\, dx = -\frac{4x^2-3}{16} I_0^4(x) + \frac{x}{2} I_0^3(x) I_1(x) + \frac{4x^2-3}{8} I_0^2(x) I_1^2(x) - \frac{2x^2+1}{4x} I_0(x) I_1^3(x) - \frac{4x^2+1}{16} I_1^4(x)$$

$$\int \frac{K_0(x) K_1^3(x)}{x^2}\, dx =$$

$$= \frac{4x^2-3}{16} K_0^4(x) + \frac{x}{2} K_0^3(x) K_1(x) - \frac{4x^2-3}{8} K_0^2(x) K_1^2(x) - \frac{2x^2+1}{4x} K_0(x) K_1^3(x) + \frac{4x^2+1}{16} K_1^4(x)$$

$$\int \frac{J_1^4(x)}{x}\, dx = \frac{1}{4}\left[x^2 J_0^4(x) - 2x J_0^3(x) J_1(x) + 2x^2 J_0^2(x) J_1^2(x) - 2x J_0(x) J_1^3(x) + (x^2-1) J_1^4(x)\right]$$

$$\int \frac{I_1^4(x)}{x}\, dx = \frac{1}{4}\left[-x^2 I_0^4(x) + 2x J_0^3(x) I_1(x) + 2x^2 I_0^2(x) I_1^2(x) - 2x I_0(x) I_1^3(x) - (x^2+1) I_1^4(x)\right]$$

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$$\int \frac{K_1^4(x)}{x}\,dx =$$

$$= \frac{1}{4}\left[-x^2 K_0^4(x) - 2x K_I^3(x)K_1(x) + 2x^2 K_0^2(x)K_1^2(x) + 2x K_0(x)K_1^3(x) - (x^2+1)K_1^4(x)\right]$$

$$\int \frac{J_1^4(x)}{x^3}\,dx = -\frac{4x^2+3}{24}J_0^4(x) + \frac{x}{3}J_0^3(x)J_1(x) - \frac{4x^2+3}{12}J_0^2(x)J_1^2(x) + \frac{2x^2-1}{6x}J_0(x)J_1^3(x) - \frac{4x^4-x^2+4}{24x^2}J_1^4(x)$$

$$\int \frac{I_1^4(x)}{x^3}\,dx = -\frac{4x^2-3}{24}I_0^4(x) + \frac{x}{3}I_0^3(x)I_1(x) + \frac{4x^2-3}{12}I_0^2(x)I_1^2(x) - \frac{2x^2+1}{6x}I_0(x)I_1^3(x) - \frac{4x^4+x^2+4}{24x^2}I_1^4(x)$$

$$\int \frac{K_1^4(x)}{x^3}\,dx =$$

$$= -\frac{4x^2-3}{24}K_0^4(x) - \frac{x}{3}K_0^3(x)K_1(x) + \frac{4x^2-3}{12}K_0^2(x)K_1^2(x) + \frac{2x^2+1}{6x}K_0(x)K_1^3(x) - \frac{4x^4+x^2+4}{24x^2}K_1^4(x)$$

4.1. b) Basic Integral $Z_0^4(x)$
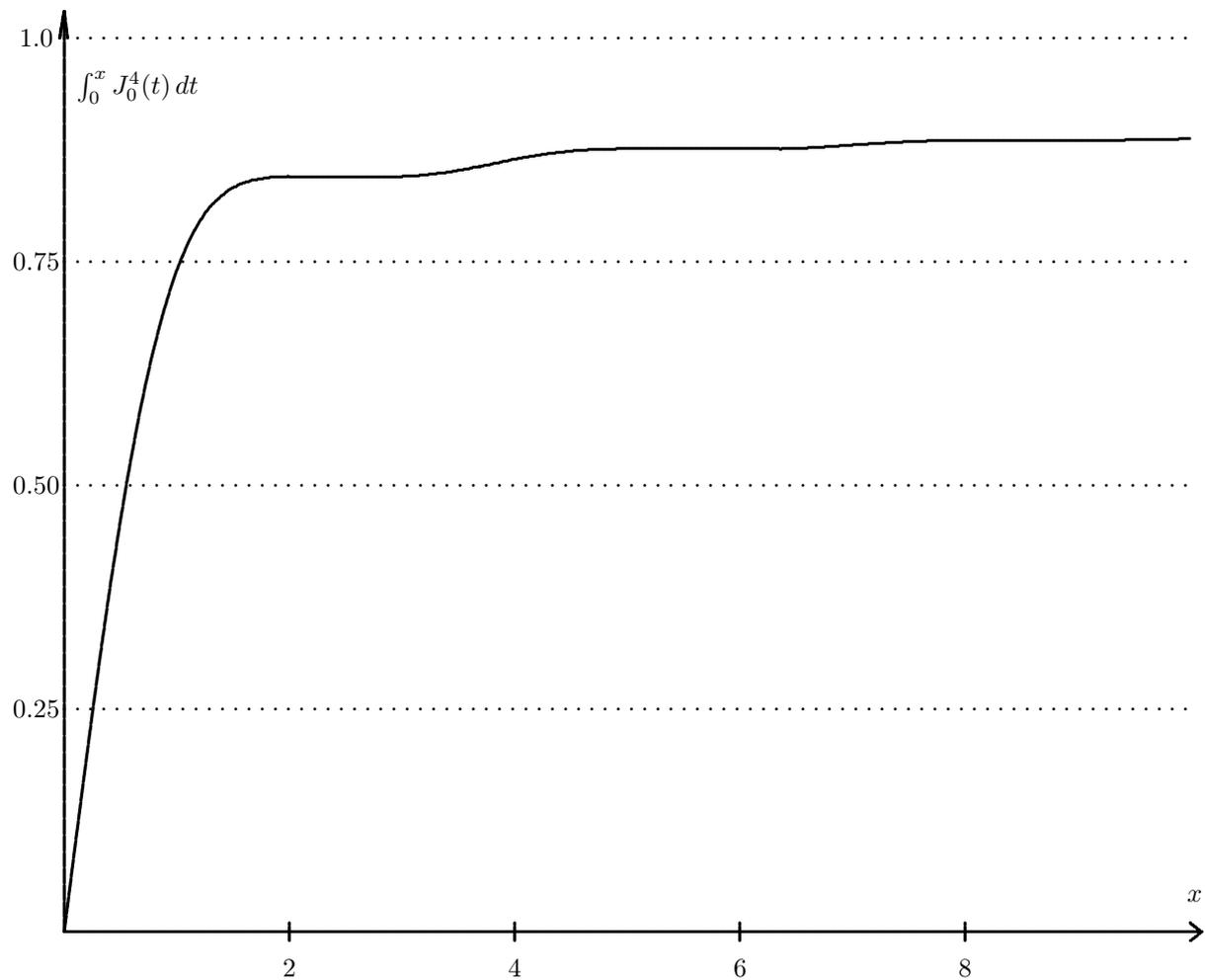
Power series:

$$\int_0^x J_0^4(t)\,dt = \sum_{k=0}^{\infty}(-1)^k a_k\, x^{2k+1} = x - \frac{1}{3}x^3 + \frac{7}{80}x^5 - \frac{1}{63}x^7 + \frac{679}{331776}x^9 - \frac{179}{921600}x^{11} + \frac{6049}{431308800}x^{13} - \cdots$$

$$\int_0^x I_0^4(t)\,dt = \sum_{k=0}^{\infty} a_k\, x^{2k+1} = x + \frac{1}{3}x^3 + \frac{7}{80}x^5 + \frac{1}{63}x^7 + \frac{679}{331776}x^9 + \frac{179}{921600}x^{11} + \frac{6049}{431308800}x^{13} + \cdots$$

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k	a_k	a_k
0	1	1.00000 00000 00000 00000
1	$\dfrac{1}{3}$	0.33333 33333 33333 33333
2	$\dfrac{7}{80}$	0.08750 00000 00000 00000
3	$\dfrac{1}{63}$	0.01587 30158 73015 87302
4	$\dfrac{679}{331776}$	0.00204 65615 35493 82716
5	$\dfrac{179}{921600}$	0.00019 42274 30555 55556
6	$\dfrac{6049}{431308800}$	0.00001 40247 54421 88984
7	$\dfrac{9671}{12192768000}$	0.00000 07931 75101 83086
8	$\dfrac{16304551}{452803638067200}$	0.00000 00360 07994 70074
9	$\dfrac{7844077}{5856006714163200}$	0.00000 00013 39492 48744
10	$\dfrac{752932783}{18122799725936640000}$	0.00000 00000 41546 16253
11	$\dfrac{93524251}{85775087818506240000}$	0.00000 00000 01090 34282
12	$\dfrac{36868956721}{150367458297485721600000}$	0.00000 00000 00024 51924
13	$\dfrac{131084576323}{27445068488457093906432000}$	0.00000 00000 00000 47763
14	$\dfrac{134309549357}{165077004537978964829798400000}$	0.00000 00000 00000 00814
15	$\dfrac{242618760673}{198519328733172979256524800000}$	0.00000 00000 00000 00012

Approximation by Chebyshev polynomials: For $|x| \le 8$ holds (based on [2], 9.7)

$$\int_0^x J_0^4(t)\,dt \;=\; \sum_{k=0}^{33} \alpha_k\, T_{2n+1}\left(\frac{x}{8}\right) + \Delta^{(40)}(x)$$

with the coefficients

k	α_k	k	α_k
0	1.11216 87232 77997 70374	17	-0.00004 87217 67090 45658
1	-0.34519 17415 46385 57003	18	0.00001 20357 15738 11243
2	0.19338 82074 76973 67514	19	-0.00000 25907 87311 63049
3	-0.12647 08183 95063 85502	20	0.00000 04907 99998 13736
4	0.09004 56892 27881 33025	21	-0.00000 00825 41454 11576
5	-0.06770 17817 94869 79248	22	0.00000 00124 18007 27445
6	0.04909 46023 85233 15906	23	-0.00000 00016 82526 91798
7	-0.03372 40353 97394 50397	24	0.00000 00002 06529 29192
8	0.02266 69978 75674 80247	25	-0.00000 00000 23088 90409
9	-0.01515 53930 28431 47066	26	0.00000 00000 02361 98127
10	0.01031 44234 66796 38411	27	-0.00000 00000 00222 04627
11	-0.00711 67527 37994 13547	28	0.00000 00000 00019 25615
12	0.00463 50732 52595 22669	29	-0.00000 00000 00001 54585
13	-0.00263 71375 49146 56248	30	0.00000 00000 00000 11524
14	0.00126 22032 52082 49660	31	-0.00000 00000 00000 00800
15	-0.00050 48484 79216 82394	32	0.00000 00000 00000 00052
16	0.00016 99635 21655 61120	33	-0.00000 00000 00000 00003

The derivation $\Delta^{(40)}(x)$:
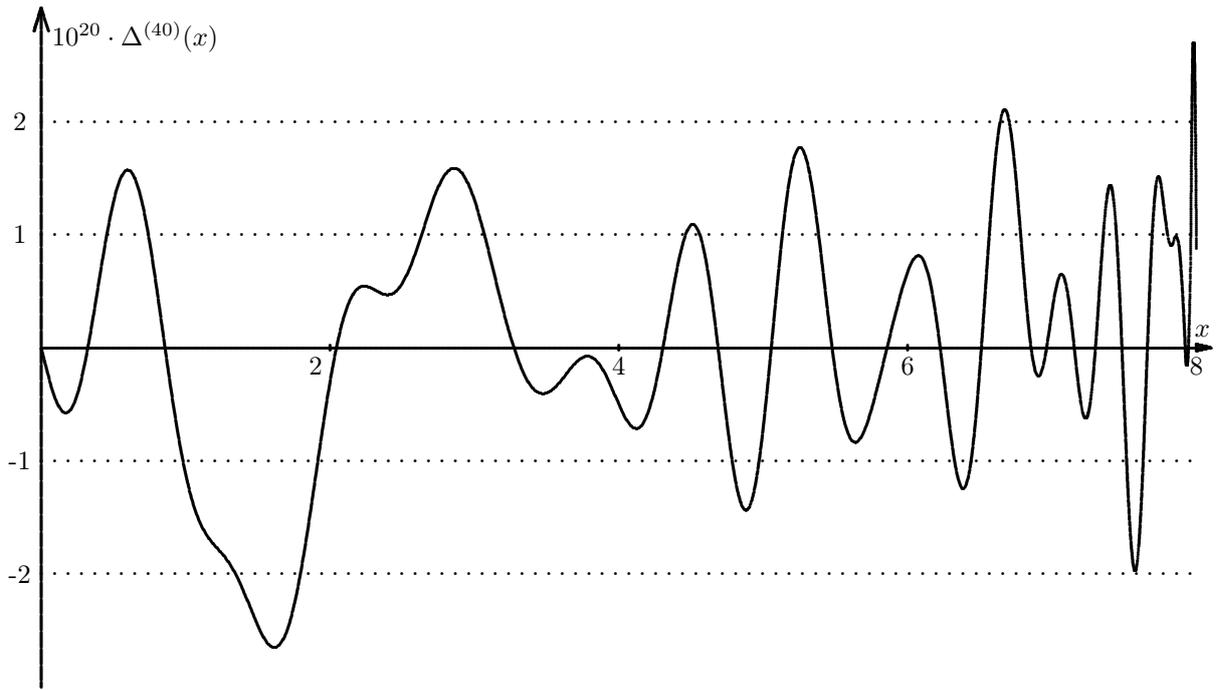
Asymptotic formula:

$$\int_0^\infty J_0^4(x)\,dx \quad = \quad 0.90272\ 85783\ 23834\ 82419\ \ldots$$

$$\int_0^x J_0^4(t)\,dt \sim 0.90272\ \ldots - \frac{1}{\pi^2}\left[\frac{3}{2x} + \frac{8\,\cos{(2\,x)} + \sin{(4\,x)}}{8x^2} - \frac{1 - 10\,\sin{(2\,x)} + \cos{(4\,x)}}{8\,x^3} + \ldots\right] =$$

$$= 0.90272\ \ldots - \frac{1}{\pi^2}\sum_{k=1}^\infty \frac{p_k + q_k\,\sin 2x + r_k\,\sin 4x + s_k\,\cos 2x + t_k\,\cos 4x}{n_k\cdot x^k} = 0.90272\ \ldots - \frac{1}{\pi^2}\sum_{k=1}^\infty \frac{\sigma_k(x)}{x^k}$$

k	p_k	q_k	r_k	s_k	t_k	n_k
1	3	0	0	0	0	2
2	0	0	1	8	0	8
3	-1	10	0	0	-1	8
4	0	0	-9	-138	0	64
5	168	-5770	0	0	245	1280
6	0	0	644	24095	0	2048
7	-27648	2053401	0	0	-35364	57344
8	0	0	-93636	-93636	0	65536
9	1042944	-134972229	0	0	1015836	262144
10	0	0	102333888	19644534099	0	8388608
11	-1979596800	394714074735	0	0	-1482416640	33554432
12	0	0	-48926574720	-17487139338315	0	268435456

Let

$$D_n(x) = 0.90272\ \ldots - \frac{1}{\pi^2}\sum_{k=1}^n \frac{\sigma_k(x)}{x^k} - \int_0^x J_0^4(t)\,dt$$

denote the derivation of the partial sums of the asymptotic series. The following table shows consecutive maxima and minima of this functions:

x_k	1.505	3.298	4.622	6.418	7.755	9.552	10.893
$D_1(x_k)$	-3.113E-02	8.502E-03	-4.513E-03	2.446E-03	-1.678E-03	1.126E-03	-8.628E-04
x_k	2.356	3.964	5.498	7.091	8.639	10.226	11.781
$D_2(x_k)$	-6.617E-03	1.749E-03	-6.893E-04	3.361E-04	-1.879E-04	1.152E-04	-7.558E-05

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x_k	3.221	4.656	6.355	7.796	9.494	10.936	12.634
$D_3(x_k)$	-1.514E-03	3.813E-04	-1.230E-04	5.450E-05	-2.601E-05	1.463E-05	-8.457E-06
x_k	3.945	5.498	7.080	8.639	10.219	11.781	13.358
$D_4(x_k)$	-3.508E-04	7.283E-05	-2.295E-05	8.509E-06	-3.893E-06	1.881E-06	-1.046E-06
x_k	3.176	4.688	6.314	7.830	9.454	10.971	12.595
$D_5(x_k)$	6.526E-04	-7.890E-05	1.534E-05	-4.427E-06	1.511E-06	-6.268E-07	2.819E-07
x_k	3.937	5.498	7.076	8.639	10.216	11.781	13.356
$D_6(x_k)$	1.452E-04	-1.689E-05	3.288E-06	-8.541E-07	2.789E-07	-1.044E-07	4.469E-08
x_k	3.157	4.702	6.297	7.844	9.438	10.985	12.579
$D_7(x_k)$	-5.521E-04	3.208E-05	-3.752E-06	7.106E-07	-1.734E-07	5.326E-08	-1.868E-08
x_k	3.933	5.498	7.074	8.639	10.214	11.781	13.355
$D_8(x_k)$	-1.077E-04	6.921E-06	-8.432E-07	1.521E-07	-3.594E-08	1.033E-08	-3.490E-09
x_k	3.150	4.708	6.290	7.849	9.431	10.991	12.573
$D_9(x_k)$	7.860E-04	-2.183E-05	1.525E-06	-1.904E-07	3.320E-08	-7.650E-09	2.080E-09
x_k	3.931	5.498	7.072	8.639	10.213	11.781	13.354
$D_{10}(x_k)$	1.271E-04	-4.450E-06	3.409E-07	**-4.254E-08**	7.270E-09	-1.600E-09	4.100E-10

Holds $D_9(8) = $ -1.798E-7 and

$$\min \{D_n(x_k)\,|\, 8 \leq x \} = |D_{10}(8.639)| = 4.254E-8\ .$$

Therefore using the partial sum of the asymptotic series with $n = 10$ means to get the best uniform approximation of the integral with $x \geq 8$. It is the best way to continue the representation by the sum of Chebyshev polynomials given before.

$$\int_0^x I_0^4(t)\,dt \sim \frac{e^{4x}}{16\pi^2\,x^2}\left[1 + \frac{1}{x} + \frac{9}{8x^2} + \frac{49}{32\,x^3} + \frac{161}{64\,x^4} + \frac{1263}{256\,x^5} + \frac{23409}{2048\,x^6} + \cdots\right] = \frac{e^{4x}}{16\pi^2\,x^2}\sum_{k=0}^{\infty}\frac{\mu_k}{x^k}$$

k	μ_k	μ_k	μ_k/μ_{k-1}
0	1	1.000 000 000 000	-
1	1	1.000 000 000 000	1.000
2	$\dfrac{9}{8}$	1.125 000 000 000	1.125
3	$\dfrac{49}{32}$	1.531 250 000 000	1.361
4	$\dfrac{161}{64}$	2.515 625 000 000	1.643
5	$\dfrac{1263}{256}$	4.933 593 750 000	1.961
6	$\dfrac{23409}{2048}$	11.430 175 781 25	2.317
7	$\dfrac{253959}{8192}$	31.000 854 492 19	2.712
8	$\dfrac{1598967}{16384}$	97.593 200 683 59	3.148
9	$\dfrac{2895345}{8192}$	353.435 668 945 3	3.622
10	$\dfrac{382238865}{262144}$	1 458.125 553 131	4.126
11	$\dfrac{7110791145}{1048576}$	6 781.378 884 315	4.651
12	$\dfrac{295087625775}{8388608}$	35 177.186 223 864	5.187

For a given $x \gg 0$ the series can be used while $\mu_k/\mu_{k-1} \leq x$.

<u>4.1. c) Basic Integral $x\,Z_0^2(x)\,Z_1^2(x)$</u>
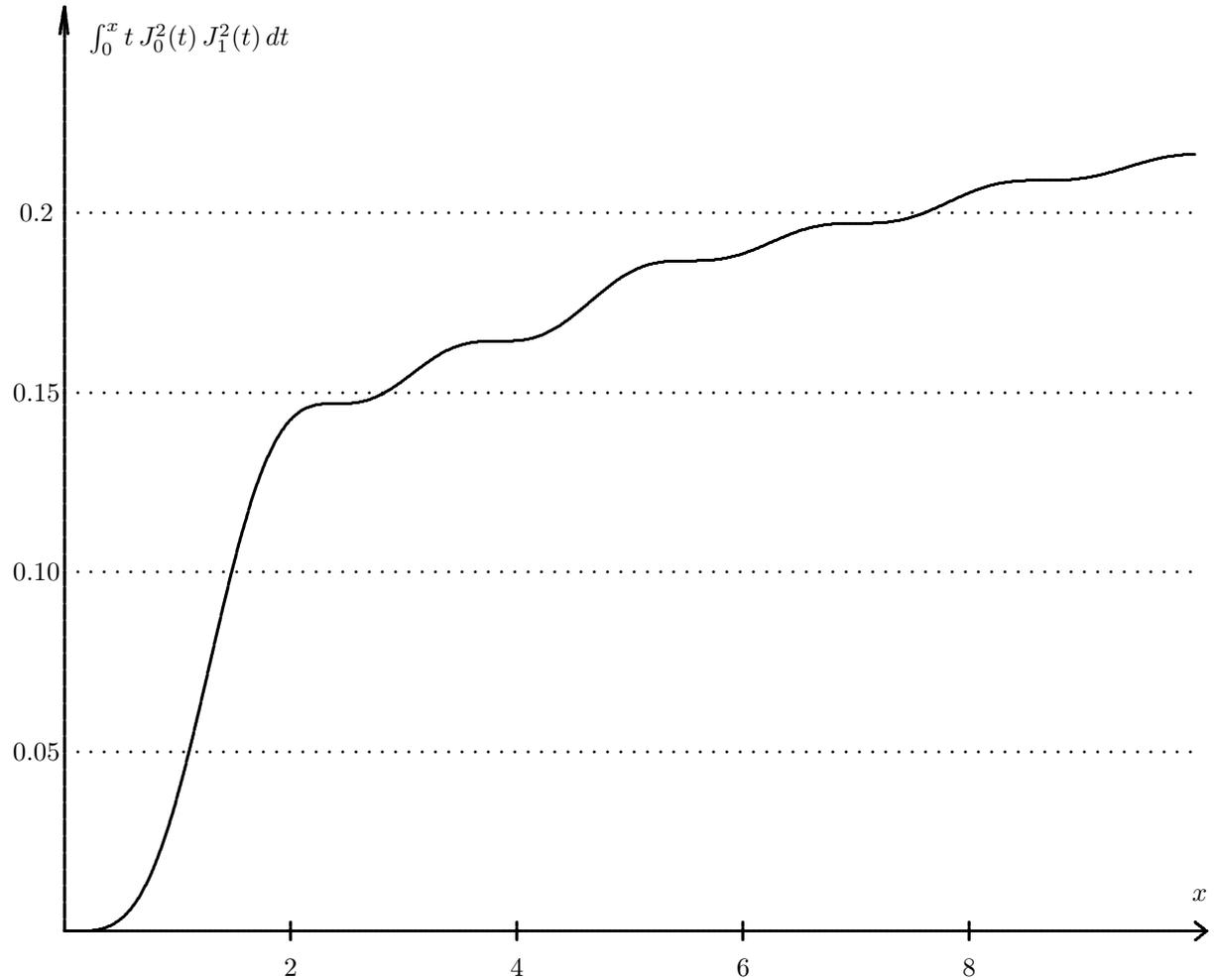
Power series:

$$\int_0^x t\,J_0^2(t)\,J_1^2(t)\,dt = \sum_{k=2}^{\infty} (-1)^k b_k\,x^{2k} = \frac{x^4}{16} - \frac{x^6}{32} + \frac{47}{6144}\,x^8 - \frac{43}{36864}\,x^{10} + \frac{17}{138240}\,x^{12} - \frac{211}{22118400}\,x^{14} + \ldots$$

$$\int_0^x t\,I_0^2(t)\,I_1^2(t)\,dt = \sum_{k=2}^{\infty} b_k\,x^{2k} = \frac{x^4}{16} + \frac{x^6}{32} + \frac{47}{6144}\,x^8 + \frac{43}{36864}\,x^{10} + \frac{17}{138240}\,x^{12} + \frac{211}{22118400}\,x^{14} + \ldots$$

k	b_k	b_k
2	$\dfrac{1}{4}$	0.25000 00000 00000 00000
3	$\dfrac{1}{32}$	0.03125 00000 00000 00000
4	$\dfrac{47}{6144}$	0.00764 97395 83333 33333
5	$\dfrac{43}{36864}$	0.00116 64496 52777 77778
6	$\dfrac{17}{138240}$	0.00012 29745 37037 03704
7	$\dfrac{211}{22118400}$	0.00000 95395 68865 74074
8	$\dfrac{540619}{951268147200}$	0.00000 05683 13993 89465

k	b_k	b_k
9	$\dfrac{1072333}{39953262182400}$	0.00000 00268 39685 70838
10	$\dfrac{19751801}{19177565847552000}$	0.00000 00010 29943 06770
11	$\dfrac{11307553}{345196185255936000}$	0.00000 00000 32756 88864
12	$\dfrac{88869497}{101257547675074560000}$	0.00000 00000 00877 65800
13	$\dfrac{402630853}{20048994439664762880000}$	0.00000 00000 00020 08235
14	$\dfrac{17384556227}{43787003856227842129920000}$	0.00000 00000 00000 39703
15	$\dfrac{16710855809}{24395616434184083472384000000}$	0.00000 00000 00000 00685
16	$\dfrac{58219427293829}{55957689152074083305615523840000}$	0.00000 00000 00000 00010

Approximation by Chebyshev polynomials: For $|x| \leq 8$ holds (based on [2], 9.7)

$$\int_0^x t\, J_0^2(t)\, J_1^2(t)\, dt \;=\; x^4 \sum_{k=0}^{33} \beta_k\, T_{2n}\left(\frac{x}{8}\right) + \Delta^{(22)}(x)$$

with the coefficients

k	β_k	k	β_k
0	0.00654 92504 91232 31929	16	0.00000 13488 21251 71102
1	-0.01239 97369 78264 31272	17	-0.00000 03066 11870 00840
2	0.01094 53569 42769 10753	18	0.00000 00617 07706 64541
3	-0.00914 38524 55194 87209	19	-0.00000 00110 63931 41290
4	0.00727 27632 79880 71248	20	0.00000 00017 77850 51057
5	-0.00551 50237 23344 53306	21	-0.00000 00002 57472 65556
6	0.00398 42908 14902 62113	22	0.00000 00000 33780 93775
7	-0.00273 36991 93851 90098	23	-0.00000 00000 04034 52317
8	0.00177 13047 44123 60885	24	0.00000 00000 00440 55264
9	-0.00107 36173 62206 65294	25	-0.00000 00000 00044 16078
10	0.00059 89651 07157 49244	26	0.00000 00000 00004 07863
11	-0.00030 12533 67887 99958	27	-0.00000 00000 00000 34826
12	0.00013 40719 51669 75647	28	0.00000 00000 00000 02758
13	-0.00005 21809 55261 86499	29	-0.00000 00000 00000 00203
14	0.00001 76814 63326 68133	30	0.00000 00000 00000 00014
15	-0.00000 52208 90537 98925	31	-0.00000 00000 00000 00001

The derivation $\Delta^{(22)(x)}$:
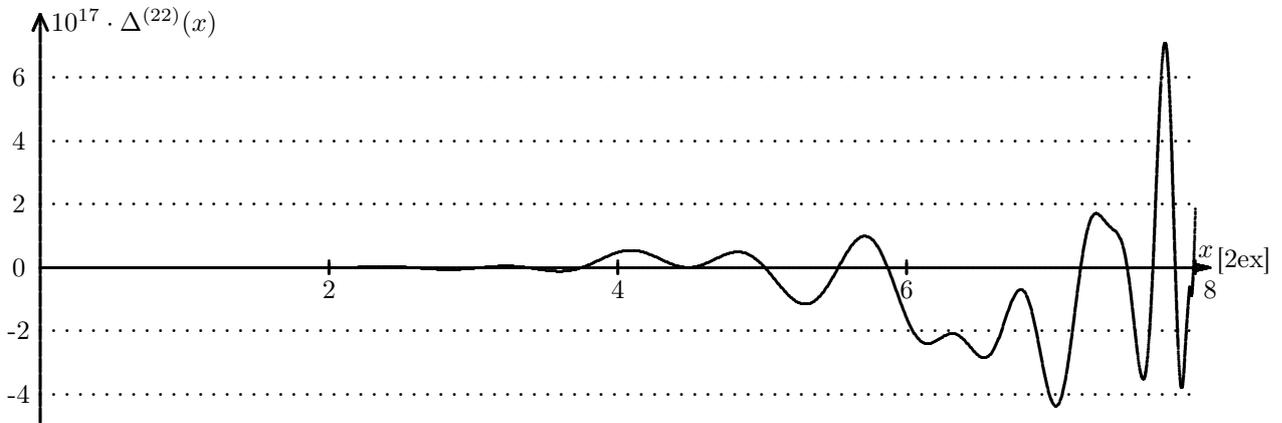
Asymptotic formula:

$$\int_0^x t\, J_0^2(t)\, J_1^2(t)\, dt \sim 0.09947\ 25799\ 65044\ 03230\ \ldots +$$

$$+\frac{1}{\pi^2}\left[\frac{\ln x}{2}+\frac{\sin 4x}{8x}+\frac{\cos 4x-16\sin 2x-6}{8x^2}+\frac{\sin 4x+12\cos 2x}{32\,x^3}+\ldots\right]=$$

$$=0.09947\ \ldots +\frac{1}{\pi^2}\left[\frac{\ln x}{2}+\sum_{k=1}^{\infty}\frac{p_k+q_k\sin 2x+r_k\sin 4x+s_k\cos 2x+t_k\cos 4x}{n_k\cdot x^k}\right]=$$

$$=0.09947\ \ldots +\frac{1}{\pi^2}\left[\sigma_0(x)+\sum_{k=1}^{\infty}\frac{\sigma_k(x)}{x^k}\right]$$

k	p_k	q_k	r_k	s_k	t_k	n_k
1	0	0	1	0	0	8
2	-6	-16	0	0	1	32
3	0	0	1	12	0	32
4	18	180	0	0	-9	256
5	0	0	-51	-1356	0	1024
6	-576	-15483	0	0	357	4096
7	0	0	3015	182385	0	16384
8	99360	99360	0	0	-60300	131072
9	0	0	-703530	-88668135	0	524288
10	-137687040	-13633922835	0	0	75666960	16777216
11	0	0	1159159680	271539997785	0	67108864
12	67108864	12571439587875	0	0	-40044715200	536870912

Let

$$D_n(x)=0.09947\ \ldots +\frac{1}{\pi^2}\left[\sigma_0(x)+\sum_{k=1}^{n}\frac{\sigma_k(x)}{x^k}\right]-\int_0^x t\, J_0^2(t)\, J_1^2(t)\, dt$$

denote the derivation of the partial sums of the asymptotic series. The following table shows consecutive maxima and minima of this functions:

x_k	3.391	4.211	5.147	5.931	6.602	7.398	8.273
$D_1(x_k)$	-5.002E-04	6.375E-03	-2.955E-03	1.615E-03	-8.607E-04	2.793E-03	-1.744E-03
x_k	3.870	5.505	7.037	8.644	10.188	11.784	13.335
$D_2(x_k)$	4.475E-03	-8.864E-04	1.430E-03	-3.717E-04	6.919E-04	-2.023E-04	4.062E-04
x_k	3.248	4.676	6.384	7.800	9.520	10.934	12.658
$D_3(x_k)$	-1.021E-03	3.135E-04	-1.471E-04	7.402E-05	-4.504E-05	2.779E-05	-1.922E-05
x_k	3.901	5.502	7.053	8.642	10.199	11.783	13.343
$D_4(x_k)$	-2.742E-04	5.650E-05	-3.011E-05	1.011E-05	-7.214E-06	3.019E-06	-2.511E-06
x_k	3.201	4.711	6.340	7.840	9.477	10.975	12.616
$D_5(x_k)$	2.732E-04	-4.339E-05	1.169E-05	-3.970E-06	1.674E-06	-7.807E-07	4.107E-07
x_k	3.175	4.724	6.316	7.857	9.455	10.994	12.594
$D_6(x_k)$	-1.747E-04	1.386E-05	-2.198E-06	4.983E-07	-1.481E-07	5.176E-08	-2.095E-08
x_k	3.921	5.499	7.064	8.640	10.207	11.782	13.349
$D_7(x_k)$	-3.990E-05	3.253E-06	-5.342E-07	1.093E-07	-3.219E-08	9.988E-09	-3.994E-09
x_k	3.161	4.726	6.305	7.863	9.444	11.001	12.584
$D_8(x_k)$	2.032E-04	-7.880E-06	7.394E-07	-1.115E-07	2.348E-08	-6.142E-09	1.922E-09
x_k	3.924	5.499	7.066	8.640	10.208	11.782	13.350
$D_9(x_k)$	3.864E-05	-1.772E-06	1.785E-07	**-2.587E-08**	5.416E-09	-1.316E-09	4.030E-10
x_k	3.154	4.725	6.299	7.864	9.439	11.003	12.579
$D_{10}(x_k)$	-3.766E-04	7.014E-06	-3.886E-07	3.885E-08	-5.815E-09	1.138E-09	-2.759E-10

Holds
$$\min \{D_n(x_k) \,|\, 8 \leq x \,\} = |D_9(9.439)| = 2.587E - 08 \,.$$

Therefore using the partial sum of the asymptotic series with $n = 9$ means to get the best uniform approximation of the integral with $x \geq 8$. It is the best way to continue the representation by the sum of Chebyshev polynomials given before.

$$\int_0^x t\, I_0^2(t)\, I_1^2(t)\, dt \sim \frac{e^{4x}}{16\pi^2\, x} \left[1 - \frac{1}{4x} - \frac{1}{4x^2} - \frac{9}{32\, x^3} - \frac{51}{128\, x^4} - \frac{357}{512\, x^5} - \frac{3015}{2048\, x^6} - \dots \right] =$$

$$= \frac{e^{4x}}{16\pi^2\, x} \left(1 - \sum_{k=1}^{\infty} \frac{\mu_k}{x^k} \right)$$

k	μ_k	μ_k	μ_k/μ_{k-1}
1	$\dfrac{1}{4}$	0.250 000 000 000	-
2	$\dfrac{1}{4}$	0.250 000 000 000	1.000
3	$\dfrac{9}{32}$	0.281 250 000 000	1.125
4	$\dfrac{51}{128}$	0.398 437 500 000	1.417
5	$\dfrac{357}{512}$	0.697 265 625 000	1.750
6	$\dfrac{3015}{2048}$	1.472 167 968 750	2.111
7	$\dfrac{15075}{4096}$	3.680 419 921 875	2.500
8	$\dfrac{351765}{32768}$	10.735 015 869 14	2.917
9	$\dfrac{4729185}{131072}$	36.080 818 176 27	3.361
10	$\dfrac{9055935}{65536}$	138.182 601 928 7	3.830
11	$\dfrac{625698675}{1048576}$	596.712 756 156 9	4.318
12	$\dfrac{24131137275}{8388608}$	2 876.655 730 605	4.821

For a given $x >> 0$ the series can be used while $\mu_k/\mu_{k-1} \leq x$.
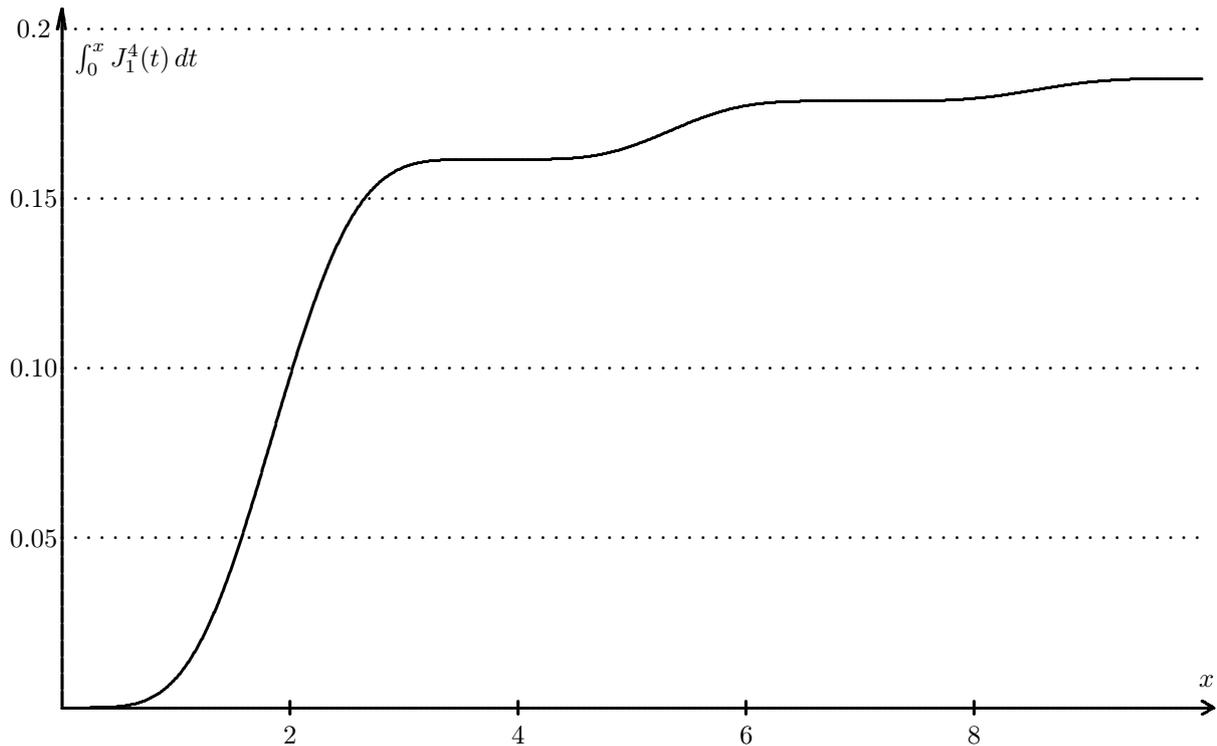
$\int_0^x J_1^4(t)\,dt$

Power series:

$$\int_0^x J_1^4(t)\,dt = \sum_{k=2}^{\infty}(-1)^k c_k\, x^{2k+1} = \frac{1}{80}\,x^5 - \frac{1}{224}\,x^7 + \frac{11}{13824}\,x^9 - \frac{37}{405504}\,x^{11} + \frac{11}{1474560}\,x^{13} - \frac{1223}{2654208000}\,x^{15} + \ldots$$

$$\int_0^x I_1^4(t)\,dt = \sum_{k=2}^{\infty} c_k\, x^{2k+1} = \frac{1}{80}\,x^5 + \frac{1}{224}\,x^7 + \frac{11}{13824}\,x^9 + \frac{37}{405504}\,x^{11} + \frac{11}{1474560}\,x^{13} + \frac{1223}{2654208000}\,x^{15} + \ldots$$

k	c_k	c_k
2	$\dfrac{1}{80}$	0.01250 00000 00000 00000
3	$\dfrac{1}{224}$	0.00446 42857 14285 71429
4	$\dfrac{11}{13824}$	0.00079 57175 92592 59259
5	$\dfrac{37}{405504}$	0.00009 12444 76010 10101
6	$\dfrac{11}{1474560}$	0.00000 74598 52430 55556
7	$\dfrac{1223}{2654208000}$	0.00000 04607 77753 66512
8	$\dfrac{45173}{2021444812800}$	0.00000 00223 46887 58949
9	$\dfrac{221467}{253037327155200}$	0.00000 00008 75234 50587
10	$\dfrac{2278819}{80545776559718400}$	0.00000 00000 28292 22210
11	$\dfrac{1524667}{1984878065221632000}$	0.00000 00000 00768 14139

k	c_k	c_k
12	$\dfrac{33739889}{1898579018907648000000}$	0.00000 00000 00017 77113
13	$\dfrac{191964463}{541322849870948597760000}$	0.00000 00000 00000 35462
14	$\dfrac{639303779}{103659029537192442593280000}$	0.00000 00000 00000 00617
15	$\dfrac{1383459431}{146669403046730974573363200000}$	0.00000 00000 00000 00009

Approximation by Chebyshev polynomials: For $|x| \leq 8$ holds (based on [2], 9.7)

$$\int_0^x J_1^4(t)\, dt \;=\; x^5 \sum_{k=0}^{33} \gamma_k\, T_{2n}\left(\frac{x}{8}\right) + \Delta^{(04)}(x)$$

with the coefficients

k	γ_k	k	γ_k
0	0.00153 17807 42787 63788	15	-0.00000 01214 49955 80695
1	-0.00288 55180 23979 94997	16	0.00000 00263 11808 30629
2	0.00246 63655 58097 67279	17	-0.00000 00050 87634 73551
3	-0.00194 01230 51814 45392	18	0.00000 00008 82115 64716
4	0.00141 41566 95709 63740	19	-0.00000 00001 37795 20459
5	-0.00095 77666 19168 38344	20	0.00000 00000 19482 53784
6	0.00060 29805 69397 05353	21	-0.00000 00000 02504 22692
7	-0.00035 21217 05640 59655	22	0.00000 00000 00293 84879
8	0.00018 96461 12672 32841	23	-0.00000 00000 00031 59998
9	-0.00009 33179 14008 29393	24	0.00000 00000 00003 12564
10	0.00004 14766 56943 42490	25	-0.00000 00000 00000 28533
11	-0.00001 64773 77688 70038	26	0.00000 00000 00000 02411
12	0.00000 58075 26406 73172	27	-0.00000 00000 00000 00189
13	-0.00000 18093 92101 58874	28	0.00000 00000 00000 00014
14	0.00000 04981 61958 19526	29	-0.00000 00000 00000 00001

The derivation $\Delta^{(04)(x)}$:
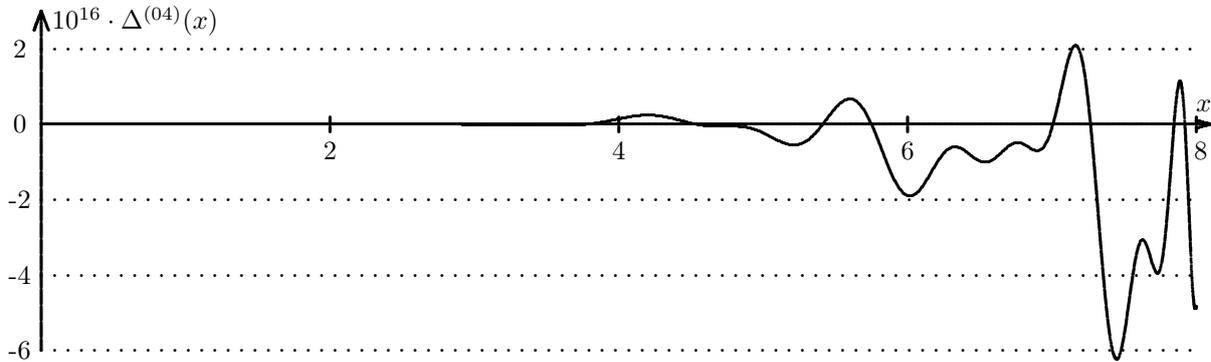
Asymptotic formula:

$$\int_0^\infty J_1^4(x)\, dx \;=\; 0.20025\ 27575\ 82806\ 70455\ \ldots$$

$$\int_0^x J_1^4(t)\, dt \sim 0.20025 \ldots - \frac{1}{\pi^2}\left[\frac{3}{2x} + \frac{8\cos(2x) - \sin(4x)}{8x^2} - \frac{3 - 2\sin(2x) + \cos(4x)}{8x^3} + \ldots\right] =$$

$$= 0.20025 \ldots - \frac{1}{\pi^2}\sum_{k=1}^\infty \frac{p_k + q_k \sin 2x + r_k \sin 4x + s_k \cos 2x + t_k \cos 4x}{n_k \cdot x^k} = 0.20025 \ldots - \frac{1}{\pi^2}\sum_{k=1}^\infty \frac{\sigma_k(x)}{x^k}$$

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k	p_k	q_k	r_k	s_k	t_k	n_k
1	3	0	0	0	0	2
2	0	0	1	-8	0	8
3	3	-2	0	0	1	8
4	0	0	3	-6	0	64
5	-216	-30	0	0	-75	1280
6	0	0	-204	1173	0	2048
7	34560	65536	0	0	12348	57344
8	0	0	37116	-586503	0	65536
9	-1267200	-9916623	0	0	-9916623	262144
10	0	0	-52716096	1650714957	0	8388608
11	25778995200	376332256575	0	0	9405918720	369098752
12	0	0	369098752	-1633144646925	0	-1633144646925

Let

$$D_n(x) = 0.20025\ \ldots - \frac{1}{\pi^2} \sum_{k=1}^{n} \frac{\sigma_k(x)}{x^k} - \int_0^x J_0^4(t)\, dt$$

denote the derivation of the partial sums of the asymptotic series. The following table shows consecutive maxima and minima of this functions:

x_k	2.930	4.738	6.115	7.917	9.274	11.073	12.424
$D_1(x_k)$	-9.815E-03	5.061E-03	-2.533E-03	1.764E-03	-1.140E-03	8.883E-04	-6.457E-04
x_k	3.927	4.804	7.069	8.003	10.210	11.962	11.962
$D_2(x_k)$	-6.708E-06	5.006E-04	-5.760E-07	1.111E-04	-1.053E-07	3.014E-05	3.014E-05
x_k	4.118	4.967	5.948	7.057	7.182	8.102	9.122
$D_3(x_k)$	-7.413E-06	1.140E-05	-1.107E-05	-5.756E-07	-5.905E-07	2.135E-06	-1.971E-06
x_k	4.567	5.842	6.168	7.055	7.737	8.858	9.382
$D_4(x_k)$	-1.253E-05	-1.396E-06	-1.515E-06	-5.756E-07	-9.571E-07	-1.679E-07	-2.281E-07
x_k	4.932	6.114	8.037	9.272	11.165	12.418	14.300
$D_5(x_k)$	-2.582E-06	9.433E-07	-1.787E-07	8.754E-08	-2.737E-08	1.592E-08	-6.480E-09
x_k	3.932	5.630	7.072	8.735	10.212	11.855	13.353
$D_6(x_k)$	7.787E-06	-6.017E-07	1.910E-07	-3.239E-08	1.673E-08	-4.050E-09	2.700E-09
x_k	3.044	4.823	6.215	7.943	9.364	11.075	12.507
$D_7(x_k)$	-4.681E-05	1.729E-06	-2.866E-07	4.340E-08	-1.287E-08	3.440E-09	-1.340E-09
x_k	3.927	5.571	7.069	8.696	10.210	11.826	13.352
$D_8(x_k)$	-7.839E-06	4.320E-07	-6.420E-08	9.910E-09	-2.750E-09	7.500E-10	-2.800E-10
x_k	3.092	4.775	6.248	7.905	9.394	11.040	12.537
$D_9(x_k)$	7.406E-05	-1.525E-06	1.346E-07	-1.447E-08	**2.910E-09**	-5.800E-10	2.100E-10
x_k	3.927	5.542	7.069	8.677	10.210	11.812	13.352
$D_{10}(x_k)$	1.082E-05	-3.483E-07	2.989E-08	-3.330E-09	6.900E-10	-1.500E-10	2.000E-11

Note that $D_2(x)$, $D_3(x)$ and $D_4(x)$ do not have the regular behaviour of the other functions.
Holds

$$\max\{|D_{10}(x)| \mid x \geq 8\} = 4.00E-9\ .$$

Using the partial sum of the asymptotic series with $n = 10$ means to get the best uniform approximation of the integral with $x \geq 8$. It is the best way to continue the representation by the sum of Chebyshev polynomials given before.

$$\int_0^x I_1^4(t)\, dt \sim \frac{e^{4x}}{16\pi^2\, x^2} \left[1 - \frac{1}{x} - \frac{3}{8x^2} - \frac{15}{32\, x^3} - \frac{51}{64\, x^4} - \frac{441}{256\, x^5} - \frac{9279}{2048\, x^6} - \ldots \right] = \frac{e^{4x}}{16\pi^2\, x^2} \left(1 - \sum_{k=1}^{\infty} \frac{\mu_k}{x^k}\right)$$

k	μ_k	μ_k	μ_k/μ_{k-1}
1	1	1.000 000 000 000	-
2	$\dfrac{3}{8}$	0.375 000 000 000	0.375
3	$\dfrac{15}{32}$	0.468 750 000 000	1.250
4	$\dfrac{51}{64}$	0.796 875 000 000	1.700
5	$\dfrac{441}{256}$	1.722 656 250 000	2.162
6	$\dfrac{9279}{2048}$	4.530 761 718 750	2.630
7	$\dfrac{115137}{8192}$	14.054 809 570 31	3.102
8	$\dfrac{823689}{16384}$	50.273 986 816 41	3.577
9	$\dfrac{1670085}{8192}$	203.867 797 851 6	4.055
10	$\dfrac{242464455}{262144}$	924.928 493 499 8	4.537
11	$\dfrac{4871010735}{1048576}$	4 645.357 832 909	5.022
12	$\dfrac{214767448785}{8388608}$	25 602.274 988 294 5.511	

For a given $x \gg 0$ the series can be used while $\mu_k/\mu_{k-1} \le x$.

4.1. e) Integrals of $x^m Z_0^4(x)$

With the basic integrals

$$\mathcal{I}_0^{(40)}(x) = \int J_0^4(x)\,dx\,, \quad \mathcal{I}_1^{(22)}(x) = \int x\,J_0^2(x)\,J_1{}^2(x)dx\,, \quad \mathcal{I}_0^{(04)}(x) = \int J_1^4(x)\,dx$$

and

$$\mathcal{I}_0^{*(40)}(x) = \int I_0^4(x)\,dx\,, \quad \mathcal{I}_1^{*(22)}(x) = \int x\,I_0^2(x)\,I_1{}^2(x)dx\,, \quad \mathcal{I}_0^{*(04)}(x) = \int I_1^4(x)\,dx$$

holds

$$\int x\,J_0^4(x)\,dx = x\,J_0^3(x)\,J_1(x) + 3\mathcal{I}_1^{(22)}(x)$$

$$\int x\,I_0^4(x)\,dx = x\,I_0^3(x)\,I_1(x) - 3\mathcal{I}_1^{*(22)}(x)$$

$$\int x^2\,J_0^4(x)\,dx = \frac{12x^3 - x}{32}\,J_0^4(x) - \frac{x^2}{8}\,J_0^3(x)\,J_1(x) + \frac{3x^3}{4}\,J_0^2(x)\,J_1^2(x) - \frac{3x^2}{8}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{12x^3 - 3x}{32}\,J_1^4(x) + \frac{1}{32}\,\mathcal{I}_0^{(40)}(x) - \frac{9}{32}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^2\,I_0^4(x)\,dx = \frac{12x^3 + x}{32}\,I_0^4(x) - \frac{x^2}{8}\,I_0^3(x)\,I_1(x) - \frac{3x^3}{4}\,I_0^2(x)\,I_1^2(x) + \frac{3x^2}{8}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{12x^3 + 3x}{32}\,I_1^4(x) - \frac{1}{32}\,\mathcal{I}_0^{*(40)}(x) + \frac{9}{32}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^3\,J_0^4(x)\,dx = \frac{3x^4 + 2x^2}{16}\,J_0^4(x) + \frac{x^3 - x}{4}\,J_0^3(x)\,J_1(x) + \frac{3x^4}{8}\,J_0^2(x)\,J_1^2(x) + \frac{3x^4}{16}\,J_1^4(x) - \frac{3}{4}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^3\,I_0^4(x)\,dx = \frac{3\,x^4 - 2\,x^2}{16}\,I_0^4(x) + \frac{x^3 + x}{4}\,I_0^3(x)\,I_1(x) - \frac{3x^4}{8}\,I_0^2(x)\,I_1^2(x) + \frac{3x^4}{16}\,I_1^4(x) - \frac{3}{4}\,\mathcal{I}_1^{*(22)}(x)$$

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$$\int x^4\, J_0^4(x)\, dx = \frac{32\,x^5 - 12\,x^3 + 9\,x}{256}\, J_0^4(x) + \frac{24\,x^4 + 9\,x^2}{64}\, J_0^3(x)\, J_1(x) + \frac{8\,x^5 - 21\,x^3}{32}\, J_0^2(x)\, J_1^2(x) +$$

$$+ \frac{8\,x^4 + 23\,x^2}{64}\, J_0(x)\, J_1^3(x) + \frac{32\,x^5 - 92\,x^3 + 23\,x}{256}\, J_1^4(x) - \frac{9}{256}\, \mathcal{I}_0^{(40)}(x) + \frac{69}{256}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^4\, I_0^4(x)\, dx = \frac{32\,x^5 + 12\,x^3 + 9\,x}{256}\, I_0^4(x) + \frac{24\,x^4 - 9\,x^2}{64}\, I_0^3(x)\, I_1(x) - \frac{8\,x^5 + 21\,x^3}{32}\, I_0^2(x)\, I_1^2(x) -$$

$$- \frac{8\,x^4 - 23\,x^2}{64}\, I_0(x)\, I_1^3(x) + \frac{32\,x^5 + 92\,x^3 + 23\,x}{256}\, I_1^4(x) - \frac{9}{256}\, \mathcal{I}_0^{*(40)}(x) + \frac{69}{256}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^5\, J_0^4(x)\, dx = \frac{6\,x^6 + 7\,x^4 - 14\,x^2}{64}\, J_0^4(x) + \frac{7\,x^5 - 7\,x^3 + 7\,x}{16}\, J_0^3(x)\, J_1(x) + \frac{6\,x^6 - 21\,x^4}{32}\, J_0^2(x)\, J_1^2(x) +$$

$$+ \frac{3x^5}{16}\, J_0(x)\, J_1^3(x) + \frac{6\,x^6 - 27\,x^4}{64}\, J_1^4(x) + \frac{21}{16}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^5\, I_0^4(x)\, dx = \frac{6\,x^6 - 7\,x^4 - 14\,x^2}{64}\, I_0^4(x) + \frac{7\,x^5 + 7\,x^3 + 7\,x}{16}\, I_0^3(x)\, I_1(x) - \frac{6\,x^6 + 21\,x^4}{32}\, I_0^2(x)\, I_1^2(x) -$$

$$- \frac{3x^5}{16}\, I_0(x)\, I_1^3(x) + \frac{6\,x^6 + 27\,x^4}{64}\, I_1^4(x) - \frac{21}{16}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^6\, J_0^4(x)\, dx = \frac{768\,x^7 + 2496\,x^5 + 1668\,x^3 - 1251\,x}{10240}\, J_0^4(x) + \frac{1216\,x^6 - 3120\,x^4 - 1251\,x^2}{2560}\, J_0^3(x)\, J_1(x) +$$

$$+ \frac{192\,x^7 - 896\,x^5 + 2757\,x^3}{1280}\, J_0^2(x)\, J_1^2(x) + \frac{576\,x^6 - 1328\,x^4 - 3089\,x^2}{2560}\, J_0(x)\, J_1^3(x) +$$

$$+ \frac{768\,x^7 - 5312\,x^5 + 12356\,x^3 - 3089\,x}{10240}\, J_1^4(x) + \frac{1251}{10240}\, \mathcal{I}_0^{(40)}(x) - \frac{9267}{10240}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^6\, I_0^4(x)\, dx = \frac{768\,x^7 - 2496\,x^5 + 1668\,x^3 + 1251\,x}{10240}\, I_0^4(x) + \frac{1216\,x^6 + 3120\,x^4 - 1251\,x^2}{2560}\, I_0^3(x)\, I_1(x) -$$

$$- \frac{192\,x^7 + 896\,x^5 + 2757\,x^3}{1280}\, I_0^2(x)\, I_1^2(x) - \frac{576\,x^6 + 1328\,x^4 - 3089\,x^2}{2560}\, I_0(x)\, I_1^3(x) +$$

$$+ \frac{768\,x^7 + 5312\,x^5 + 12356\,x^3 + 3089\,x}{10240}\, I_1^4(x) - \frac{1251}{10240}\, \mathcal{I}_0^{*(40)}(x) + \frac{9267}{10240}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^7\, J_0^4(x)\, dx = \frac{x^8 + 6\,x^6 - 9\,x^4 + 18\,x^2}{16}\, J_0^4(x) + \frac{2\,x^7 - 9\,x^5 + 9\,x^3 - 9\,x}{4}\, J_0^3(x)\, J_1(x) +$$

$$+ \frac{x^8 - 6\,x^6 + 27\,x^4}{8}\, J_0^2(x)\, J_1^2(x) + \frac{x^7 - 5\,x^5}{4}\, J_0(x)\, J_1^3(x) + \frac{x^8 - 10\,x^6 + 37\,x^4}{16}\, J_1^4(x) - \frac{27}{4}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^7\, I_0^4(x)\, dx = \frac{x^8 - 6\,x^6 - 9\,x^4 - 18\,x^2}{16}\, I_0^4(x) + \frac{2\,x^7 + 9\,x^5 + 9\,x^3 + 9\,x}{4}\, I_0^3(x)\, I_1(x) -$$

$$- \frac{x^8 + 6\,x^6 + 27\,x^4}{8}\, I_0^2(x)\, I_1^2(x) - \frac{x^7 + 5\,x^5}{4}\, I_0(x)\, I_1^3(x) + \frac{x^8 + 10\,x^6 + 37\,x^4}{16}\, I_1^4(x) - \frac{27}{4}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^8\, J_0^4(x)\, dx = \frac{15360\,x^9 + 145024\,x^7 - 535152\,x^5 - 364116\,x^3 + 273087\,x}{286720}\, J_0^4(x) +$$

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$$+\frac{37120\,x^8 - 253792\,x^6 + 668940\,x^4 + 273087\,x^2}{71680}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{1920\,x^9 - 14352\,x^7 + 91726\,x^5 - 296367\,x^3}{17920}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{19200\,x^8 - 158112\,x^6 + 302036\,x^4 + 668243\,x^2}{71860}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{15360\,x^9 - 210816\,x^7 + 1208144\,x^5 - 2672972\,x^3 + 668243\,x}{286720}\,J_1^4(x) - \frac{273087}{286720}\,\mathcal{I}_0^{(40)}(x) + \frac{2004729}{286720}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^8\,I_0^4(x)\,dx = \frac{15360\,x^9 - 145024\,x^7 - 535152\,x^5 + 364116\,x^3 + 273087\,x}{286720}\,I_0^4(x)+$$

$$+\frac{37120\,x^8 + 253792\,x^6 + 668940\,x^4 - 273087\,x^2}{71680}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{1920\,x^9 + 14352\,x^7 + 91726\,x^5 + 296367\,x^3}{17920}\,I_0^2(x)\,I_1^2(x)-$$

$$-\frac{19200\,x^8 + 158112\,x^6 + 302036\,x^4 - 668243\,x^2}{71680}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{15360\,x^9 + 210816\,x^7 + 1208144\,x^5 + 2672972\,x^3 + 668243\,x}{286720}\,I_1^4(x)-$$

$$-\frac{273087}{286720}\,\mathcal{I}_0^{*(40)}(x) + \frac{2004729}{286720}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^9\,J_0^4(x)\,dx = \frac{48\,x^{10} + 652\,x^8 - 4074\,x^6 + 6111\,x^4 - 12222\,x^2}{1024}\,J_0^4(x)+$$

$$+\frac{136\,x^9 - 1304\,x^7 + 6111\,x^5 - 6111\,x^3 + 6111\,x}{256}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{48\,x^{10} - 436\,x^8 + 3750\,x^6 - 18333\,x^4}{512}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{72\,x^9 - 868\,x^7 + 3611\,x^5}{256}\,J_0(x)\,J_1^3(x) + \frac{48\,x^{10} - 868\,x^8 + 7222\,x^6 - 25555\,x^4}{1024}\,J_1^4(x) + \frac{18333}{256}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^9\,I_0^4(x)\,dx = \frac{48\,x^{10} - 652\,x^8 - 4074\,x^6 - 6111\,x^4 - 12222\,x^2}{1024}\,I_0^4(x)+$$

$$+\frac{136\,x^9 + 1304\,x^7 + 6111\,x^5 + 6111\,x^3 + 6111\,x}{256}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{48\,x^{10} + 436\,x^8 + 3750\,x^6 + 18333\,x^4}{512}\,I_0^2(x)\,I_1^2(x)-$$

$$-\frac{72\,x^9 + 868\,x^7 + 3611\,x^5}{256}\,I_0(x)\,I_1^3(x) + \frac{48\,x^{10} + 868\,x^8 + 7222\,x^6 + 25555\,x^4}{1024}\,I_1^4(x) - \frac{18333}{256}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^{10}\,J_0^4(x)\,dx =$$

$$= \frac{573440\,x^{11} + 10567680\,x^9 - 97664256\,x^7 + 363060288\,x^5 + 248212404\,x^3 - 186159303\,x}{13762560}\,J_0^4(x)+$$

$$+\frac{1863680\,x^{10} - 23777280\,x^8 + 170912448\,x^6 - 453825360\,x^4 - 186159303\,x^2}{3440640}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{143360\,x^{11} - 1551360\,x^9 + 17194176\,x^7 - 122875488\,x^5 + 402422121\,x^3}{1720320}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{1003520\,x^{10} - 16537600\,x^8 + 113598528\,x^6 - 208074384\,x^4 - 454440717\,x^2}{3440640}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{573440\,x^{11} - 13230080\,x^9 + 151464704\,x^7 - 832297536\,x^5 + 1817762868\,x^3 - 454440717\,x}{13762560}\,J_1^4(x)-$$

$$+ \frac{62053101}{4587520}\,\mathcal{I}_0^{(40)}(x) - \frac{454440717}{4587520}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^{10}\,I_0^4(x)\,dx =$$

$$= \frac{573440\,x^{11} - 10567680\,x^9 - 97664256\,x^7 - 363060288\,x^5 + 248212404\,x^3 + 186159303\,x}{13762560}\,I_0^4(x)+$$

$$+ \frac{1863680\,x^{10} + 23777280\,x^8 + 170912448\,x^6 + 453825360\,x4 - 186159303\,x^2}{3440640}\,I_0^3(x)\,I_1(x)-$$

$$- \frac{143360\,x^{11} + 1551360\,x^9 + 17194176\,x^7 + 122875488\,x^5 + 402422121\,x^3}{1720320}\,I_0^2(x)\,I_1^2(x)-$$

$$- \frac{1003520\,x^{10} + 16537600\,x^8 + 113598528\,x^6 + 208074384\,x^4 - 454440717\,x^2}{3440640}\,I_0(x)\,I_1^3(x)+$$

$$+ \frac{573440\,x^{11} + 13230080\,x^9 + 151464704\,x^7 + 832297536\,x^5 + 1817762868\,x^3 + 454440717\,x}{13762560}\,I_1^4(x)+$$

$$- \frac{62053101}{4587520}\,\mathcal{I}_0^{*(40)}(x) + \frac{454440717}{4587520}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^4(x)\,dx}{x^2} = -\frac{6x^2+1}{x}\,J_0^4(x) + 4\,J_0^3(x)\,J_1(x) - 12x\,J_0^2(x)\,J_1^2(x) - 6x\,J_1^4(x) + 2\mathcal{I}_0^{(40)}(x) - 18\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^4(x)\,dx}{x^2} = \frac{6x^2-1}{x}\,I_0^4(x) - 4\,I_0^3(x)\,I_1(x) - 12x\,I_0^2(x)\,I_1^2(x) + 6x\,I_1^4(x) - 2\mathcal{I}_0^{*(40)}(x) + 18\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^4(x)\,dx}{x^4} = \frac{40\,x^4 + 4\,x^2 - 3}{9x^3}\,J_0^4(x) - \frac{24\,x^2 - 4}{9x^2}\,J_0^3(x)\,J_1(x) + \frac{80\,x^2 - 4}{9x}\,J_0^2(x)\,J_1^2(x)+$$

$$+ \frac{8}{27}\,J_0(x)\,J_1^3(x) + \frac{40x}{9}\,J_1^4(x) - \frac{16}{9}\,\mathcal{I}_0^{(40)}(x) + \frac{368}{27}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^4(x)\,dx}{x^4} = \frac{40\,x^4 - 4\,x^2 - 3}{9x^3}\,I_0^4(x) - \frac{24\,x^2 + 4}{9x^2}\,I_0^3(x)\,I_1(x) - \frac{80\,x^2 + 4}{9x}\,I_0^2(x)\,I_1^2(x)-$$

$$- \frac{8}{27}\,I_0(x)\,I_1^3(x) + \frac{40x}{9}\,I_1^4(x) - \frac{16}{9}\,\mathcal{I}_0^{*(40)}(x) + \frac{368}{27}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^4(x)\,dx}{x^6} = -\frac{42752\,x^6 + 3800\,x^4 - 1500\,x^2 + 5625}{28125\,x^5}\,J_0^4(x) + \frac{4992\,x^4 - 760\,x^2 + 900}{5625\,x^4}\,J_0^3(x)\,J_1(x)-$$

$$- \frac{85504\,x^4 - 4880\,x^2 + 2700}{28125\,x^3}\,J_0^2(x)\,J_1^2(x) - \frac{10624\,x^2 - 3240}{84375\,x^2}\,J_0(x)\,J_1^3(x) - \frac{42752\,x^2 + 216}{28125\,x}\,J_1^4(x)+$$

$$+ \frac{17792}{28125}\,\mathcal{I}_0^{(40)}(x) - \frac{395392}{84375}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^4(x)\,dx}{x^6} = \frac{42752\,x^6 - 3800\,x^4 - 1500\,x^2 - 5625}{28125\,x^5}\,I_0^4(x) - \frac{4992\,x^4 + 760\,x^2 + 900}{5625\,x^4}\,I_0^3(x)\,I_1(x)-$$

$$- \frac{85504\,x^4 + 4880\,x^2 + 2700}{28125\,x^3}\,I_0^2(x)\,I_1^2(x) - \frac{10624\,x^2 + 3240}{84375\,x^2}\,I_0(x)\,I_1^3(x) + \frac{42752\,x^2 - 216}{28125\,x}\,I_1^4(x)+$$

$$+ \frac{17792}{28125}\,\mathcal{I}_0^{*(40)}(x) - \frac{395392}{84375}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. f) Integrals of $x^m Z_0^3(x) Z_1(x)$

Explicit and basic integrals are omitted.

With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 465 holds

$$\int x\, J_0^3(x)\, J_1(x)\, dx = -\frac{x}{4}\, J_0^4(x) + \frac{1}{4}\, \mathcal{I}_0^{(40)}(x)$$

$$\int x\, I_0^3(x)\, I_1(x)\, dx = \frac{x}{4}\, I_0^4(x) - \frac{1}{4}\, \mathcal{I}_0^{*(40)}(x)$$

$$\int x^2\, J_0^3(x)\, J_1(x)\, dx = -\frac{x^2}{4}\, J_0^4(x) + \frac{x}{2}\, J_0^3(x)\, J_1(x) + \frac{3}{2}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^2\, I_0^3(x)\, I_1(x)\, dx = \frac{x^2}{4}\, I_0^4(x) - \frac{x}{2}\, I_0^3(x)\, I_1(x) + \frac{3}{2}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^3\, J_0^3(x)\, J_1(x)\, dx = \frac{4\,x^3 - 3\,x}{128}\, J_0^4(x) - \frac{3x^2}{32}\, J_0^3(x)\, J_1(x) + \frac{9x^3}{16}\, J_0^2(x)\, J_1^2(x) - \frac{9x^2}{32}\, J_0(x)\, J_1^3(x) +$$

$$+\frac{36\,x^3 - 9\,x}{128}\, J_1^4(x) + \frac{3}{128}\, \mathcal{I}_0^{(40)}(x) - \frac{27}{128}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^3\, I_0^3(x)\, I_1(x)\, dx = -\frac{4\,x^3 + 3\,x}{128}\, I_0^4(x) + \frac{3x^2}{32}\, I_0^3(x)\, I_1(x) + \frac{9x^3}{16}\, I_0^2(x)\, I_1^2(x) - \frac{9x^2}{32}\, I_0(x)\, I_1^3(x) -$$

$$-\frac{36\,x^3 + 9\,x}{128}\, I_1^4(x) + \frac{3}{128}\, \mathcal{I}_0^{*(40)}(x) - \frac{27}{128}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^4\, J_0^3(x)\, J_1(x)\, dx = -\frac{x^4 - 2\,x^2}{16}\, J_0^4(x) + \frac{x^3 - x}{4}\, J_0^3(x)\, J_1(x) + \frac{3x^4}{8}\, J_0^2(x)\, J_1^2(x) + \frac{3x^4}{16}\, J_1^4(x) - \frac{3}{4}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^4\, I_0^3(x)\, I_1(x)\, dx = \frac{x^4 + 2\,x^2}{16}\, I_0^4(x) - \frac{x^3 + x}{4}\, I_0^3(x)\, I_1(x) + \frac{3x^4}{8}\, I_0^2(x)\, I_1^2(x) - \frac{3x^4}{16}\, I_1^4(x) + \frac{3}{4}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^5\, J_0^3(x)\, J_1(x)\, dx = -\frac{96\,x^5 + 60\,x^3 - 45\,x}{1024}\, J_0^4(x) + \frac{120\,x^4 + 45\,x^2}{256}\, J_0^3(x)\, J_1(x) +$$

$$+\frac{40\,x^5 - 105\,x^3}{128}\, J_0^2(x)\, J_1^2(x) + \frac{40\,x^4 + 115\,x^2}{256}\, J_0(x)\, J_1^3(x) + \frac{160\,x^5 - 460\,x^3 + 115\,x}{1024}\, J_1^4(x) -$$

$$-\frac{45}{1024}\, \mathcal{I}_0^{(40)}(x) + \frac{345}{1024}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^5\, I_0^3(x)\, I_1(x)\, dx = \frac{96\,x^5 - 60\,x^3 - 45\,x}{1024}\, I_0^4(x) - \frac{120\,x^4 - 45\,x^2}{256}\, I_0^3(x)\, I_1(x) +$$

$$+\frac{40\,x^5 + 105\,x^3}{128}\, I_0^2(x)\, I_1^2(x) + \frac{40\,x^4 - 115\,x^2}{256}\, I_0(x)\, I_1^3(x) - \frac{160\,x^5 + 460\,x^3 + 115\,x}{1024}\, I_1^4(x) +$$

$$+\frac{45}{1024}\, \mathcal{I}_0^{*(40)}(x) - \frac{345}{1024}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^6\, J_0^3(x)\, J_1(x)\, dx = -\frac{14\,x^6 - 21\,x^4 + 42\,x^2}{128}\, J_0^4(x) + \frac{21\,x^5 - 21\,x^3 + 21\,x}{32}\, J_0^3(x)\, J_1(x) +$$

$$+\frac{18\,x^6 - 63\,x^4}{64}\, J_0^2(x)\, J_1^2(x) + \frac{9x^5}{32}\, J_0(x)\, J_1^3(x) + \frac{18\,x^6 - 81\,x^4}{128}\, J_1^4(x) + \frac{63}{32}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^6\, I_0^3(x)\, I_1(x)\, dx = \frac{14\,x^6 + 21\,x^4 + 42\,x^2}{128}\, I_0^4(x) - \frac{21\,x^5 + 21\,x^3 + 21\,x}{32}\, I_0^3(x)\, I_1(x) +$$

$$+\frac{18\,x^6+63\,x^4}{64}\,I_0^2(x)\,I_1^2(x)+\frac{9x^5}{32}\,I_0(x)\,I_1^3(x)-\frac{18\,x^6+81\,x^4}{128}\,I_1^4(x)+\frac{63}{32}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^7\,J_0^3(x)\,J_1(x)\,dx=-\frac{4864\,x^7-17472\,x^5-11676\,x^3+8757\,x}{40960}\,J_0^4(x)+$$

$$+\frac{8512\,x^6-21840\,x^4-8757\,x^2}{10240}\,J_0^3(x)\,J_1(x)+\frac{1344\,x^7-6272\,x^5+19299\,x^3}{5120}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{4032\,x^6-9296\,x^4-21623\,x^2}{10240}\,J_0(x)\,J_1^3(x)+\frac{5376\,x^7-37184\,x^5+86492\,x^3-21623\,x}{40960}\,J_1^4(x)+$$

$$+\frac{8757}{40960}\,\mathcal{I}_0^{(40)}(x)-\frac{64869}{40960}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^7\,I_0^3(x)\,I_1(x)\,dx=\frac{4864\,x^7+17472\,x^5-11676\,x^3-8757\,x}{40960}\,I_0^4(x)-$$

$$-\frac{8512\,x^6+21840\,x^4-8757\,x^2}{10240}\,I_0^3(x)\,I_1(x)+\frac{1344\,x^7+6272\,x^5+19299\,x^3}{5120}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{4032\,x^6+9296\,x^4-21623\,x^2}{10240}\,I_0(x)\,I_1^3(x)-\frac{5376\,x^7+37184\,x^5+86492\,x^3+21623\,x}{40960}\,I_1^4(x)+$$

$$+\frac{8757}{40960}\,\mathcal{I}_0^{*(40)}(x)-\frac{64869}{40690}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^8\,J_0^3(x)\,J_1(x)\,dx=-\frac{x^8-6\,x^6+9\,x^4-18\,x^2}{8}\,J_0^4(x)+\frac{2\,x^7-9\,x^5+9\,x^3-9\,x}{2}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{x^8-6\,x^6+27\,x^4}{4}\,J_0^2(x)\,J_1^2(x)+\frac{x^7-5\,x^5}{2}\,J_0(x)\,J_1^3(x)+\frac{x^8-10\,x^6+37\,x^4}{8}\,J_1^4(x)-\frac{27}{2}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^8\,I_0^3(x)\,I_1(x)\,dx=\frac{x^8+6\,x^6+9\,x^4+18\,x^2}{8}\,I_0^4(x)-\frac{2\,x^7+9\,x^5+9\,x^3+9\,x}{2}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{x^8+6\,x^6+27\,x^4}{4}\,I_0^2(x)\,I_1^2(x)+\frac{x^7+5\,x^5}{2}\,I_0(x)\,I_1^3(x)-\frac{x^8+10\,x^6+37\,x^4}{8}\,I_1^4(x)+\frac{27}{2}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^9\,J_0^3(x)\,J_1(x)\,dx=-\frac{148480\,x^9-1305216\,x^7+4816368\,x^5+3277044\,x^3-2457783\,x}{1146880}\,J_0^4(x)+$$

$$+\frac{334080\,x^8-2284128\,x^6+6020460\,x^4+2457783\,x^2}{286720}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{17280\,x^9-129168\,x^7+825534\,x^5-2667303\,x^3}{71680}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{172800\,x^8-1423008\,x^6+2718324\,x^4+6014187\,x^2}{286720}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{138240\,x^9-1897344\,x^7+10873296\,x^5-24056748\,x^3+6014187\,x}{1146880}\,J_1^4(x)-$$

$$-\frac{2457783}{1146880}\,\mathcal{I}_0^{(40)}(x)+\frac{18042561}{1146880}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^9\,I_0^3(x)\,I_1(x)\,dx=\frac{148480\,x^9+1305216\,x^7+4816368\,x^5-3277044\,x^3-2457783\,x}{1146880}\,I_0^4(x)-$$

$$-\frac{334080\,x^8+2284128\,x^6+6020460\,x^4-2457783\,x^2}{286720}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{17280\,x^9 + 129168\,x^7 + 825534\,x^5 + 2667303\,x^3}{71680}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{172800\,x^8 + 1423008\,x^6 + 2718324\,x^4 - 6014187\,x^2}{286720}\,I_0(x)\,I_1^3(x)-$$

$$-\frac{138240\,x^9 + 1897344\,x^7 + 10873296\,x^5 + 24056748\,x^3 + 6014187\,x}{1146880}\,I_1^4(x)+$$

$$+\frac{2457783}{1146880}\,\mathcal{I}_0^{*(40)}(x) - \frac{18042561}{1146880}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^{10}\,J_0^3(x)\,J_1(x)\,dx = -\frac{272\,x^{10} - 3260\,x^8 + 20370\,x^6 - 30555\,x^4 + 61110\,x^2}{2048}\,J_0^4(x)+$$

$$+\frac{680\,x^9 - 6520\,x^7 + 30555\,x^5 - 30555\,x^3 + 30555\,x}{512}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{240\,x^{10} - 2180\,x^8 + 18750\,x^6 - 91665\,x^4}{1024}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{360\,x^9 - 4340\,x^7 + 18055\,x^5}{512}\,J_0(x)\,J_1^3(x) + \frac{240\,x^{10} - 4340\,x^8 + 36110\,x^6 - 127775\,x^4}{2048}\,J_1^4(x) + \frac{91665}{512}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^{10}\,I_0^3(x)\,I_1(x)\,dx = \frac{272\,x^{10} + 3260\,x^8 + 20370\,x^6 + 30555\,x^4 + 61110\,x^2}{2048}\,I_0^4(x)-$$

$$-\frac{680\,x^9 + 6520\,x^7 + 30555\,x^5 + 30555\,x^3 + 30555\,x}{512}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{240\,x^{10} + 2180\,x^8 + 18750\,x^6 + 91665\,x^4}{1024}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{360\,x^9 + 4340\,x^7 + 18055\,x^5}{512}\,I_0(x)\,I_1^3(x) - \frac{240\,x^{10} + 4340\,x^8 + 36110\,x^6 + 127775\,x^4}{2048}\,I_1^4(x) + \frac{91665}{512}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int \frac{J_0^3(x)\,J_1(x)\,dx}{x} = \frac{3x}{2}\,J_0^4(x) - J_0^3(x)\,I_1(x) + 3x\,J_0^2(x)\,J_1^2(x) + \frac{3x}{2}\,J_1^4(x) - \frac{1}{2}\,\mathcal{I}_0^{(40)}(x) + \frac{9}{2}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^3(x)\,I_1(x)\,dx}{x} = \frac{3x}{2}\,I_0^4(x) - I_0^3(x)\,I_1(x) - 3x\,I_0^2(x)\,I_1^2(x) + \frac{3x}{2}\,I_1^4(x) - \frac{1}{2}\,\mathcal{I}_0^{*(40)}(x) + \frac{9}{2}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^3(x)\,J_1(x)\,dx}{x^3} = -\frac{10\,x^2 + 1}{3x}\,J_0^4(x) + \frac{6\,x^2 - 1}{3x^2}\,J_0^3(x)\,I_1(x) - \frac{20\,x^2 - 1}{3x}\,J_0^2(x)\,J_1^2(x) - \frac{2}{9}\,J_0(x)\,J_1^3(x)-$$

$$-\frac{10x}{3}\,J_1^4(x) + \frac{4}{3}\,\mathcal{I}_0^{(40)}(x) - \frac{92}{9}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^3(x)\,I_1(x)\,dx}{x^3} = \frac{10\,x^2 - 1}{3x}\,I_0^4(x) - \frac{6\,x^2 + 1}{3x^2}\,I_0^3(x)\,I_1(x) - \frac{20\,x^2 + 1}{3x}\,I_0^2(x)\,I_1^2(x) - \frac{2}{9}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{10x}{3}\,I_1^4(x) - \frac{4}{3}\,\mathcal{I}_0^{*(40)}(x) + \frac{92}{9}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^3(x)\,J_1(x)\,dx}{x^5} = \frac{10688\,x^4 + 950\,x^2 - 375}{5625\,x^3}\,J_0^4(x) - \frac{1248\,x^4 - 190\,x^2 + 225}{1125\,x^4}\,J_0^3(x)\,I_1(x)+$$

$$+\frac{21376\,x^4 - 1220\,x^2 + 675}{5625\,x^3}\,J_0^2(x)\,J_1^2(x) + \frac{2656\,x^2 - 810}{16875\,x^2}\,J_0(x)\,J_1^3(x) + \frac{10688\,x^2 + 54}{5625\,x}\,J_1^4(x)-$$

$$-\frac{4448}{5625}\,\mathcal{I}_0^{(40)}(x) + \frac{98848}{16875}\,\mathcal{I}_0^{(04)}(x)$$

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$$\int \frac{I_0^3(x)\,I_1(x)\,dx}{x^5} = \frac{10688\,x^4 - 950\,x^2 - 375}{5625\,x^3}\,I_0^4(x) - \frac{1248\,x^4 + 190\,x^2 + 225}{1125\,x^4}\,I_0^3(x)\,I_1(x) -$$

$$-\frac{21376\,x^4 + 1220\,x^2 + 675}{5625\,x^3}\,I_0^2(x)\,I_1^2(x) - \frac{2656\,x^2 + 810}{16875\,x^2}\,I_0(x)\,I_1^3(x) + \frac{10688\,x^2 - 54}{5625\,x}\,I_1^4(x) -$$

$$-\frac{4448}{5625}\,\mathcal{I}_0^{*(40)}(x) + \frac{98848}{16875}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. g) Integrals of $x^m\,Z_0^2(x)\,Z_1^2(x)$

Explicit and basic integrals are omitted.

With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 465 holds

$$\int J_0^2(x)\,J_1^2(x)\,dx = -\frac{x}{2}\,J_0^4(x) - x\,J_0^2(x)\,J_1^2(x) - \frac{x}{2}\,J_1^4(x) + \frac{1}{2}\,\mathcal{I}_0^{(40)}(x) - \frac{3}{2}\,\mathcal{I}_0^{(04)}(x)$$

$$\int I_0^2(x)\,I_1^2(x)\,dx = \frac{x}{2}\,I_0^4(x) - x\,I_0^2(x)\,I_1^2(x) + \frac{x}{2}\,I_1^4(x) - \frac{1}{2}\,\mathcal{I}_0^{*(40)}(x) + \frac{3}{2}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^2\,J_0^2(x)\,J_1^2(x)\,dx = \frac{4\,x^3 - 3\,x}{32}\,J_0^4(x) - \frac{3x^2}{8}\,J_0^3(x)\,J_1(x) + \frac{x^3}{4}\,J_0^2(x)\,J_1^2(x) - \frac{x^2}{8}\,J_0(x)\,J_1^3(x) +$$

$$+\frac{4\,x^3 - x}{32}\,J_1^4(x) + \frac{3}{32}\,\mathcal{I}_0^{(40)}(x) - \frac{3}{32}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^2\,I_0^2(x)\,I_1^2(x)\,dx = -\frac{4\,x^3 + 3\,x}{32}\,I_0^4(x) + \frac{3x^2}{8}\,I_0^3(x)\,I_1(x) + \frac{x^3}{4}\,I_0^2(x)\,I_1^2(x) - \frac{x^2}{8}\,I_0(x)\,I_1^3(x) -$$

$$-\frac{4\,x^3 + x}{32}\,I_1^4(x) + \frac{3}{32}\,\mathcal{I}_0^{*(40)}(x) - \frac{3}{32}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^3\,J_0^2(x)\,J_1^2(x)\,dx = \frac{x^4 - 2\,x^2}{16}\,J_0^4(x) - \frac{x^3 - x}{4}\,J_0^3(x)\,J_1(x) + \frac{x^4}{8}\,J_0^2(x)\,J_1^2(x) + \frac{x^4}{16}\,J_1^4(x) + \frac{3}{4}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^3\,I_0^2(x)\,I_1^2(x)\,dx = -\frac{x^4 + 2\,x^2}{16}\,I_0^4(x) + \frac{x^3 + x}{4}\,I_0^3(x)\,I_1(x) + \frac{x^4}{8}\,I_0^2(x)\,I_1^2(x) - \frac{x^4}{16}\,I_1^4(x) - \frac{3}{4}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^4\,J_0^2(x)\,J_1^2(x)\,dx = \frac{32\,x^5 + 12\,x^3 - 9\,x}{768}\,J_0^4(x) - \frac{40\,x^4 + 9\,x^2}{192}\,J_0^3(x)\,J_1(x) + \frac{8\,x^5 + 33\,x^3}{96}\,J_0^2(x)\,J_1^2(x) +$$

$$+\frac{8\,x^4 - 31\,x^2}{192}\,J_0(x)\,J_1^3(x) + \frac{32\,x^5 + 124\,x^3 - 31\,x}{768}\,J_1^4(x) + \frac{3}{256}\,\mathcal{I}_0^{(40)}(x) - \frac{31}{256}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^4\,I_0^2(x)\,I_1^2(x)\,dx = -\frac{32\,x^5 - 12\,x^3 - 9\,x}{768}\,I_0^4(x) + \frac{40\,x^4 - 9\,x^2}{192}\,I_0^3(x)\,I_1(x) + \frac{8\,x^5 - 33\,x^3}{96}\,I_0^2(x)\,I_1^2(x) +$$

$$+\frac{8\,x^4 + 31\,x^2}{192}\,I_0(x)\,I_1^3(x) - \frac{32\,x^5 - 124\,x^3 - 31\,x}{768}\,I_1^4(x) - \frac{3}{256}\,\mathcal{I}_0^{*(40)}(x) + \frac{31}{256}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^5\,J_0^2(x)\,J_1^2(x)\,dx = \frac{2\,x^6 - 3\,x^4 + 6\,x^2}{64}\,J_0^4(x) - \frac{3\,x^5 - 3\,x^3 + 3\,x}{16}\,J_0^3(x)\,J_1(x) + \frac{2\,x^6 + 9\,x^4}{32}\,J_0^2(x)\,J_1^2(x) +$$

$$+\frac{x^5}{16}\,J_0(x)\,J_1^3(x)+\frac{2\,x^6+7\,x^4}{64}\,J_1^4(x)-\frac{9}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^5\,I_0^2(x)\,I_1^2(x)\,dx=-\frac{2\,x^6+3\,x^4+6\,x^2}{64}\,I_0^4(x)+\frac{3\,x^5+3\,x^3+3\,x}{16}\,I_0^3(x)\,I_1(x)+\frac{2\,x^6-9\,x^4}{32}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{x^5}{16}\,I_0(x)\,I_1^3(x)-\frac{2\,x^6-7\,x^4}{64}\,I_1^4(x)-\frac{9}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^6\,J_0^2(x)\,J_1^2(x)\,dx=\frac{256\,x^7-768\,x^5-444\,x^3+333\,x}{10240}\,J_0^4(x)-\frac{448\,x^6-960\,x^4-333\,x^2}{2560}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{64\,x^7+368\,x^5-831\,x^3}{1280}\,J_0^2(x)\,J_1^2(x)+\frac{192\,x^6+224\,x^4+887\,x^2}{2560}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{256\,x^7+896\,x^5-3548\,x^3+887\,x}{10240}\,J_1^4(x)-\frac{333}{10240}\,\mathcal{I}_0^{(40)}(x)+\frac{2661}{10240}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^6\,I_0^2(x)\,I_1^2(x)\,dx=-\frac{256\,x^7+768\,x^5-444\,x^3-333\,x}{10240}\,I_0^4(x)+\frac{448\,x^6+960\,x^4-333\,x^2}{2560}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{64\,x^7-368\,x^5-831\,x^3}{1280}\,I_0^2(x)\,I_1^2(x)+\frac{192\,x^6-224\,x^4+887\,x^2}{2560}\,I_0(x)\,I_1^3(x)-$$

$$-\frac{256\,x^7-896\,x^5-3548\,x^3-887\,x}{10240}\,I_1^4(x)-\frac{333}{10240}\,\mathcal{I}_0^{*(40)}(x)+\frac{2661}{10240}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^7\,J_0^2(x)\,J_1^2(x)\,dx=\frac{4\,x^8-18\,x^6+27\,x^4-54\,x^2}{192}\,J_0^4(x)-\frac{8\,x^7-27\,x^5+27\,x^3-27\,x}{48}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{4\,x^8+30\,x^6-81\,x^4}{96}\,J_0^2(x)\,J_1^2(x)+\frac{4\,x^7+7\,x^5}{48}\,J_0(x)\,J_1^3(x)\frac{4\,x^8+14\,x^6-95\,x^4}{192}\,J_1^4(x)+\frac{27}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^7\,I_0^2(x)\,I_1^2(x)\,dx=-\frac{4\,x^8+18\,x^6+27\,x^4+54\,x^2}{192}\,I_0^4(x)+\frac{8\,x^7+27\,x^5+27\,x^3+27\,x}{48}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{4\,x^8-30\,x^6-81\,x^4}{96}\,I_0^2(x)\,I_1^2(x)+\frac{4\,x^7-7\,x^5}{48}\,I_0(x)\,I_1^3(x)-\frac{4\,x^8-14\,x^6-95\,x^4}{192}\,I_1^4(x)-\frac{27}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^8\,J_0^2(x)\,J_1^2(x)\,dx=\frac{2560\,x^9-15552\,x^7+53496\,x^5+34668\,x^3-26001\,x}{143360}\,J_0^4(x)-$$

$$-\frac{5760\,x^8-27216\,x^6+66870\,x^4+26001\,x^2}{35840}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{1280\,x^9+12384\,x^7-41292\,x^5+117639\,x^3}{35840}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{1600\,x^8+3288\,x^6-12789\,x^4-32607\,x^2}{17920}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{1280\,x^9+4384\,x^7-51156\,x^5+130428\,x^3-32607\,x}{71680}\,J_1^4(x)-$$

$$+\frac{26001}{143360}\,\mathcal{I}_0^{(40)}(x)-\frac{97821}{71680}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^8\,I_0^2(x)\,I_1^2(x)\,dx=-\frac{2560\,x^9+15552\,x^7+53496\,x^5-34668\,x^3-26001\,x}{143360}\,I_0^4(x)+$$

$$+\frac{5760\,x^8+27216\,x^6+66870\,x^4-26001\,x^2}{35840}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{1280\,x^9-12384\,x^7-41292\,x^5-117639\,x^3}{35840}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{1600\,x^8-3288\,x^6-12789\,x^4+32607\,x^2}{17920}\,I_0(x)\,I_1^3(x)-$$

$$-\frac{1280\,x^9-4384\,x^7-51156\,x^5-130428\,x^3-32607\,x}{71680}\,I_1^4(x)-\frac{26001}{143360}\,\mathcal{I}_0^{*(40)}(x)+\frac{97821}{71680}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^9\,J_0^2(x)\,J_1^2(x)\,dx=\frac{16\,x^{10}-124\,x^8+690\,x^6-1035\,x^4+2070\,x^2}{1024}\,J_0^4(x)-$$

$$-\frac{40\,x^9-248\,x^7+1035\,x^5-1035\,x^3+1035\,x}{256}\,J_0^3(x)\,J_1(x)+\frac{16\,x^{10}+196\,x^8-798\,x^6+3105\,x^4}{512}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{24\,x^9+52\,x^7-503\,x^5}{256}\,J_0(x)\,J_1^3(x)+\frac{16\,x^{10}+52\,x^8-1006\,x^6+4111\,x^4}{1024}\,J_1^4(x)-\frac{3105}{256}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^9\,I_0^2(x)\,I_1^2(x)\,dx=-\frac{16\,x^{10}+124\,x^8+690\,x^6+1035\,x^4+2070\,x^2}{1024}\,I_0^4(x)+$$

$$+\frac{40\,x^9+248\,x^7+1035\,x^5+1035\,x^3+1035\,x}{256}\,I_0^3(x)\,I_1(x)+\frac{16\,x^{10}-196\,x^8-798\,x^6-3105\,x^4}{512}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{24\,x^9-52\,x^7-503\,x^5}{256}\,I_0(x)\,I_1^3(x)-\frac{16\,x^{10}-52\,x^8-1006\,x^6-4111\,x^4}{1024}\,I_1^4(x)-\frac{3105}{256}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^{10}\,J_0^2(x)\,J_1^2(x)\,dx=$$

$$=\frac{573440\,x^{11}-5468160\,x^9+43299072\,x^7-157107456\,x^5-105708348\,x^3+79281261\,x}{41287680}\,J_0^4(x)-$$

$$-\frac{1576960\,x^{10}-12303360\,x^8+75773376\,x^6-196384320\,x^4-79281261\,x^2}{10321920}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{143360\,x^{11}+2181120\,x^9-10706112\,x^7+55439856\,x^5-173715327\,x^3}{5160960}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{1003520\,x^{10}+2124800\,x^8-40086336\,x^6+85504608\,x^4+195091479\,x^2}{10321920}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{573440\,x^{11}+1699840\,x^9-53448448\,x^7+342018432\,x^5-780365916\,x^3+195091479\,x}{41287680}\,J_1^4(x)-$$

$$-\frac{8809029}{4587520}\,\mathcal{I}_0^{(40)}(x)+\frac{65030493}{4587520}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^{10}\,I_0^2(x)\,I_1^2(x)\,dx=$$

$$=-\frac{573440\,x^{11}+5468160\,x^9+43299072\,x^7+157107456\,x^5-105708348\,x^3-79281261\,x}{41287680}\,I_0^4(x)+$$

$$+\frac{1576960\,x^{10}+12303360\,x^8+75773376\,x^6+196384320\,x^4-79281261\,x^2}{10321920}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{143360\,x^{11}-2181120\,x^9-10706112\,x^7-55439856\,x^5-173715327\,x^3}{5160960}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{1003520\,x^{10}-2124800\,x^8-40086336\,x^6-85504608\,x^4+195091479\,x^2}{10321920}\,I_0(x)\,I_1^3(x)-$$

$$-\frac{573440\,x^{11}-1699840\,x^9-53448448\,x^7-342018432\,x^5-780365916\,x^3-195091479\,x}{41287680}\,I_1^4(x)-$$

$$-\frac{8809029}{4587520}\,\mathcal{I}_0^{*(40)}(x)+\frac{65030493}{4587520}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^2(x)\,J_1^2(x)\,dx}{x^2} = \frac{4x}{3}\,J_0^4(x) - \frac{2}{3}\,J_0^3(x)\,J_1(x) + \frac{8\,x^2-1}{3x}\,J_0^2(x)\,J_1^2(x) + \frac{2}{9}\,J_0(x)\,J_1^3(x) + \frac{4x}{3}\,J_1^4(x) -$$
$$-\frac{2}{3}\,\mathcal{I}_0^{(40)}(x) + \frac{38}{9}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^2(x)\,I_1^2(x)\,dx}{x^2} = \frac{4x}{3}\,I_0^4(x) - \frac{2}{3}\,I_0^3(x)\,I_1(x) - \frac{8\,x^2+1}{3x}\,I_0^2(x)\,I_1^2(x) - \frac{2}{9}\,I_0(x)\,I_1^3(x) + \frac{4x}{3}\,I_1^4(x) -$$
$$-\frac{2}{3}\,\mathcal{I}_0^{*(40)}(x) + \frac{38}{9}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^2(x)\,J_1^2(x)\,dx}{x^4} = -\frac{632\,x^2+50}{375\,x}\,J_0^4(x) + \frac{72\,x^2-10}{75\,x^2}\,J_0^3(x)\,J_1(x) - \frac{1264\,x^4-80\,x^2+75}{375\,x^3}\,J_0^2(x)\,J_1^2(x) -$$
$$-\frac{184\,x^2-90}{1125x^2}\,J_0(x)\,J_1^3(x) - \frac{632\,x^2+6}{375\,x}\,J_1^4(x) + \frac{272}{375}\,\mathcal{I}_0^{(40)}(x) - \frac{5872}{1125}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^2(x)\,I_1^2(x)\,dx}{x^4} = \frac{632\,x^2-50}{375\,x}\,I_0^4(x) - \frac{72\,x^2+10}{75x^2}\,I_0^3(x)\,I_1(x) - \frac{1264\,x^4+80\,x^2+75}{375x^3}\,I_0^2(x)\,I_1^2(x) -$$
$$-\frac{184\,x^2+90}{1125\,x^2}\,I_0(x)\,I_1^3(x) + \frac{632\,x^2-6}{375\,x}\,I_1^4(x) - \frac{272}{375}\,\mathcal{I}_0^{*(40)}(x) + \frac{5872}{1125}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^2(x)\,J_1^2(x)\,dx}{x^6} = \frac{1485184\,x^4+124600\,x^2-36750}{1929375\,x^3}\,J_0^4(x) - \frac{171264\,x^4-24920\,x^2+22050}{385875\,x^4}\,J_0^3(x)\,J_1(x) +$$
$$+\frac{2970368\,x^6-178960\,x^4+113400\,x^2-275625}{1929375\,x^5}\,J_0^2(x)\,J_1^2(x) + \frac{401408\,x^4-163080\,x^2+236250}{5788125\,x^4}\,J_0(x)\,J_1^3(x) +$$
$$+\frac{1485184\,x^4+10872\,x^2-11250}{1929375\,x^3}\,J_1^4(x) - \frac{628864}{1929375}\,\mathcal{I}_0^{(40)}(x) + \frac{13768064}{5788125}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^2(x)\,I_1^2(x)\,dx}{x^6} = \frac{1485184\,x^4-124600\,x^2-36750}{1929375\,x^3}\,I_0^4(x) - \frac{171264\,x^4+24920\,x^2+22050}{385875\,x^4}\,I_0^3(x)\,I_1(x) -$$
$$-\frac{2970368\,x^6+178960\,x^4+113400\,x^2+275625}{1929375\,x^5}\,I_0^2(x)\,I_1^2(x) \frac{401408\,x^4+163080\,x^2+236250}{5788125\,x^4}\,I_0(x)\,I_1^3(x) +$$
$$+\frac{1485184\,x^4-10872\,x^2-11250}{1929375\,x^3}\,I_1^4(x) - \frac{628864}{1929375}\,\mathcal{I}_0^{*(40)}(x) + \frac{13768064}{5788125}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. h) Integrals of $x^m\,Z_0(x)\,Z_1^3(x)$

Explicit and basic integrals are omitted.
With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 465 holds

$$\int x\,J_0(x)\,J_1^3(x)\,dx = \frac{x}{4}\,J_1^4(x) + \frac{3}{4}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x\,J_0(x)\,I_1^3(x)\,dx = \frac{x}{4}\,I_1^4(x) + \frac{3}{4}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^2\,J_0(x)\,J_1^3(x)\,dx = -\frac{x^2}{4}\,J_0^4(x) + \frac{x}{2}\,J_0^3(x)\,J_1(x) - \frac{x^2}{2}\,J_0^2(x)\,J_1^2(x) + \frac{3}{2}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^2 \, I_0(x) \, I_1^3(x) \, dx = -\frac{x^2}{4} I_0^4(x) + \frac{x}{2} I_0^3(x) I_1(x) + \frac{x^2}{2} I_0^2(x) I_1^2(x) - \frac{3}{2} \mathcal{I}_1^{*(22)}(x)$$

$$\int x^3 \, J_0(x) \, J_1^3(x) \, dx = \frac{12\,x^3 - 9\,x}{128} J_0^4(x) - \frac{9x^2}{32} J_0^3(x) J_1(x) + \frac{3x^3}{16} J_0^2(x) J_1^2(x) - \frac{11x^2}{32} J_0(x) J_1^3(x) +$$

$$+ \frac{44\,x^3 - 11\,x}{128} J_1^4(x) + \frac{9}{128} \mathcal{I}_0^{(40)}(x) - \frac{33}{128} \mathcal{I}_0^{(04)}(x)$$

$$\int x^3 \, I_0(x) \, I_1^3(x) \, dx = \frac{12\,x^3 + 9\,x}{128} I_0^4(x) - \frac{9x^2}{32} I_0^3(x) I_1(x) - \frac{3x^3}{16} I_0^2(x) I_1^2(x) + \frac{11x^2}{32} I_0(x) I_1^3(x) +$$

$$+ \frac{44\,x^3 + 11\,x}{128} I_1^4(x) - \frac{9}{128} \mathcal{I}_0^{*(40)}(x) + \frac{33}{128} \mathcal{I}_0^{*(04)}(x)$$

$$\int x^5 \, J_0(x) \, J_1^3(x) \, dx = -\frac{32\,x^5 + 36\,x^3 - 27\,x}{1024} J_0^4(x) + \frac{40\,x^4 + 27\,x^2}{256} J_0^3(x) J_1(x) - \frac{8\,x^5 + 39\,x^3}{128} J_0^2(x) J_1^2(x) +$$

$$+ \frac{56\,x^4 + 53\,x^2}{256} J_0(x) J_1^3(x) + \frac{224\,x^5 - 212\,x^3 + 53\,x}{1024} J_1^4(x) - \frac{27}{1024} \mathcal{I}_0^{(40)}(x) + \frac{159}{1024} \mathcal{I}_0^{(04)}(x)$$

$$\int x^5 \, I_0(x) \, I_1^3(x) \, dx = -\frac{32\,x^5 - 36\,x^3 - 27\,x}{1024} I_0^4(x) + \frac{40\,x^4 - 27\,x^2}{256} I_0^3(x) I_1(x) + \frac{8\,x^5 - 39\,x^3}{128} I_0^2(x) I_1^2(x) -$$

$$- \frac{56\,x^4 - 53\,x^2}{256} I_0(x) I_1^3(x) + \frac{224\,x^5 + 212\,x^3 + 53\,x}{1024} I_1^4(x) - \frac{27}{1024} \mathcal{I}_0^{*(40)}(x) + \frac{159}{1024} \mathcal{I}_0^{*(04)}(x)$$

$$\int x^6 \, J_0(x) \, J_1^3(x) \, dx = -\frac{6\,x^6 - 9\,x^4 + 18\,x^2}{128} J_0^4(x) + \frac{9\,x^5 - 9\,x^3 + 9\,x}{32} J_0^3(x) J_1(x) -$$

$$- \frac{6\,x^6 + 27\,x^4}{64} J_0^2(x) J_1^2(x) + \frac{13x^5}{32} J_0(x) J_1^3(x) + \frac{26\,x^6 - 53\,x^4}{128} J_1^4(x) + \frac{27}{32} \mathcal{I}_1^{(22)}(x)$$

$$\int x^6 \, I_0(x) \, I_1^3(x) \, dx = -\frac{6\,x^6 + 9\,x^4 + 18\,x^2}{128} I_0^4(x) + \frac{9\,x^5 + 9\,x^3 + 9\,x}{32} I_0^3(x) I_1(x) +$$

$$+ \frac{6\,x^6 - 27\,x^4}{64} I_0^2(x) I_1^2(x) - \frac{13x^5}{32} I_0(x) I_1^3(x) + \frac{26\,x^6 + 53\,x^4}{128} I_1^4(x) - \frac{27}{32} \mathcal{I}_1^{*(22)}(x)$$

$$\int x^7 \, J_0(x) \, J_1^3(x) \, dx = -\frac{2304\,x^7 - 9792\,x^5 - 7236\,x^3 + 5427\,x}{40960} J_0^4(x) +$$

$$+ \frac{4032\,x^6 - 12240\,x^4 - 5427\,x^2}{10240} J_0^3(x) J_1(x) - \frac{576\,x^7 + 2592\,x^5 - 10989\,x^3}{5120} J_0^2(x) J_1^2(x) +$$

$$+ \frac{5952\,x^6 - 7056\,x^4 - 12753\,x^2}{10240} J_0(x) J_1^3(x) + \frac{7936\,x^7 - 28224\,x^5 + 51012\,x^3 - 12753\,x}{40960} J_1^4(x) +$$

$$+ \frac{5427}{40960} \mathcal{I}_0^{(40)}(x) - \frac{38259}{40960} \mathcal{I}_0^{(04)}(x)$$

$$\int x^7 \, I_0(x) \, I_1^3(x) \, dx = -\frac{2304\,x^7 + 9792\,x^5 - 7236\,x^3 - 5427\,x}{40960} I_0^4(x) +$$

$$+ \frac{4032\,x^6 + 12240\,x^4 - 5427\,x^2}{10240} I_0^3(x) I_1(x) + \frac{576\,x^7 - 2592\,x^5 - 10989\,x^3}{5120} I_0^2(x) I_1^2(x) -$$

$$- \frac{5952\,x^6 + 7056\,x^4 - 12753\,x^2}{10240} I_0(x) I_1^3(x) + \frac{7936\,x^7 + 28224\,x^5 + 51012\,x^3 + 12753\,x}{40960} I_1^4(x) -$$

$$-\frac{5427}{40960}\,\mathcal{I}_0^{*(40)}(x)+\frac{38259}{40960}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^8\,J_0(x)\,J_1^3(x)\,dx=-\frac{4\,x^8-30\,x^6+45\,x^4-90\,x^2}{64}\,J_0^4(x)+\frac{8\,x^7-45\,x^5+45\,x^3-45\,x}{16}\,J_0^3(x)\,J_1(x)-$$

$$-\frac{4\,x^8+18\,x^6-135\,x^4}{32}\,J_0^2(x)\,J_1^2(x)+\frac{12\,x^7-33\,x^5}{16}\,J_0(x)\,J_1^3(x)+\frac{12\,x^8-66\,x^6+201\,x^4}{64}\,J_1^4(x)-\frac{135}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^8\,I_0(x)\,I_1^3(x)\,dx=-\frac{4\,x^8+30\,x^6+45\,x^4+90\,x^2}{64}\,I_0^4(x)+\frac{8\,x^7+45\,x^5+45\,x^3+45\,x}{16}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{4\,x^8-18\,x^6-135\,x^4}{32}\,I_0^2(x)\,I_1^2(x)-\frac{12\,x^7+33\,x^5}{16}\,I_0(x)\,I_1^3(x)+\frac{12\,x^8+66\,x^6+201\,x^4}{64}\,I_1^4(x)-\frac{135}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^9\,J_0(x)\,J_1^3(x)\,dx=-\frac{15360\,x^9-173952\,x^7+663696\,x^5+461268\,x^3-345951\,x}{229376}\,J_0^4(x)+$$

$$+\frac{34560\,x^8-304416\,x^6+829620\,x^4+345951\,x^2}{57344}\,J_0^3(x)\,J_1(x)-$$

$$-\frac{960\,x^9+4248\,x^7-53649\,x^5+184383\,x^3}{7168}\,J_0^2(x)\,J_1^2(x)+$$

$$+\frac{52480\,x^8-247776\,x^6+400428\,x^4+837639\,x^2}{57344}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{41984\,x^9-330368\,x^7+1601712\,x^5-3350556\,x^3+837639\,x}{229376}\,J_1^4(x)-\frac{345951}{229376}\,\mathcal{I}_0^{(40)}(x)+\frac{2512917}{229376}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^9\,I_0(x)\,I_1^3(x)\,dx=-\frac{15360\,x^9+173952\,x^7+663696\,x^5-461268\,x^3-345951\,x}{229376}\,I_0^4(x)+$$

$$+\frac{34560\,x^8+304416\,x^6+829620\,x^4-345951\,x^2}{57344}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{3840\,x^9-16992\,x^7-214596\,x^5-737532\,x^3}{28762}\,I_0^2(x)\,I_1^2(x)-$$

$$-\frac{52480\,x^8+247776\,x^6+400428\,x^4-837639\,x^2}{57344}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{41984\,x^9+330368\,x^7+1601712\,x^5+3350556\,x^3+837639\,x}{229376}\,I_1^4(x)-\frac{345951}{229376}\,\mathcal{I}_0^{*(40)}(x)+\frac{2512917}{229376}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^{10}\,J_0(x)\,J_1^3(x)\,dx=-\frac{144\,x^{10}-2268\,x^8+14850\,x^6-22275\,x^4+44550\,x^2}{2048}\,J_0^4(x)+$$

$$+\frac{360\,x^9-4536\,x^7+22275\,x^5-22275\,x^3+22275\,x}{512}\,J_0^3(x)\,J_1(x)-$$

$$-\frac{144\,x^{10}+612\,x^8-12366\,x^6+66825\,x^4}{1024}\,J_0^2(x)\,J_1^2(x)+\frac{552\,x^9-3924\,x^7+14031\,x^5}{512}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{368\,x^{10}-3924\,x^8+28062\,x^6-94887\,x^4}{2048}\,J_1^4(x)+\frac{66825}{512}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^{10}\,I_0(x)\,I_1^3(x)\,dx=-\frac{144\,x^{10}+2268\,x^8+14850\,x^6+22275\,x^4+44550\,x^2}{2048}\,I_0^4(x)+$$

$$+\frac{360\,x^9+4536\,x^7+22275\,x^5+22275\,x^3+22275\,x}{512}\,I_0^3(x)\,I_1(x)+$$

$$+\frac{144\,x^{10} - 612\,x^8 - 12366\,x^6 - 66825\,x^4}{1024}\,I_0^2(x)\,I_1^2(x)-$$

$$-\frac{552\,x^9 + 3924\,x^7 + 14031\,x^5}{512}\,I_0(x)\,I_1^3(x)+\frac{368\,x^{10} + 3924\,x^8 + 28062\,x^6 + 94887\,x^4}{2048}\,I_1^4(x)-\frac{66825}{512}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int \frac{J_0(x)\,J_1^3(x)\,dx}{x} = -\frac{x}{2}\,J_0^4(x) - x\,J_0^2(x)\,J_1^2(x) - \frac{1}{3}\,J_0(x)\,J_1^3(x) - \frac{x}{2}\,J_1^4(x) + \frac{1}{2}\,\mathcal{I}_0^{(40)}(x) - \frac{11}{6}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0(x)\,I_1^3(x)\,dx}{x} = \frac{x}{2}\,I_0^4(x) - x\,I_0^2(x)\,I_1^2(x) - \frac{1}{3}\,I_0(x)\,I_1^3(x) + \frac{x}{2}\,I_1^4(x) - \frac{1}{2}\,\mathcal{I}_0^{*(40)}(x) + \frac{11}{6}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0(x)\,J_1^3(x)\,dx}{x^3} = \frac{22x}{25}\,J_0^4(x) - \frac{2}{5}\,J_0^3(x)\,J_1(x) + \frac{44\,x^2 - 5}{25x}\,J_0^2(x)\,J_1^2(x) + \frac{14\,x^2 - 15}{75x^2}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{22\,x^2 + 1}{25x}\,J_1^4(x) - \frac{12}{25}\,\mathcal{I}_0^{(40)}(x) + \frac{212}{75}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0(x)\,I_1^3(x)\,dx}{x^3} = \frac{22x}{25}\,I_0^4(x) - \frac{2}{5}\,I_0^3(x)\,I_1(x) - \frac{44\,x^2 + 5}{25x}\,I_0^2(x)\,I_1^2(x) - \frac{14\,x^2 + 15}{75x^2}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{22\,x^2 - 1}{25x}\,I_1^4(x) - \frac{12}{25}\,\mathcal{I}_0^{*(40)}(x) + \frac{212}{75}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0(x)\,J_1^3(x)\,dx}{x^5} = -\frac{4864\,x^2 + 350}{6125\,x}\,J_0^4(x) + \frac{544\,x^2 - 70}{1225\,x^2}\,J_0^3(x)\,J_1(x) - \frac{9728\,x^4 - 660\,x^2 + 525}{6125\,x^3}\,J_0^2(x)\,J_1^2(x)-$$

$$-\frac{1568\,x^4 - 930\,x^2 + 2625}{18375\,x^4}\,J_0(x)\,J_1^3(x) - \frac{4864\,x^4 + 62\,x^2 - 125}{6125\,x^3}\,J_1^4(x) + \frac{2144}{6125}\,\mathcal{I}_0^{(40)}(x) - \frac{45344}{18375}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0(x)\,I_1^3(x)\,dx}{x^5} = \frac{4864\,x^2 - 350}{6125\,x}\,I_0^4(x) - \frac{544\,x^2 + 70}{1225\,x^2}\,I_0^3(x)\,I_1(x) - \frac{9728\,x^4 + 660\,x^2 + 525}{6125\,x^3}\,I_0^2(x)\,I_1^2(x)-$$

$$-\frac{1568\,x^4 + 930\,x^2 + 2625}{18375\,x^4}\,I_0(x)\,I_1^3(x) + \frac{4864\,x^4 - 62\,x^2 - 125}{6125\,x^3}\,I_1^4(x) - \frac{2144}{6125}\,\mathcal{I}_0^{*(40)}(x) + \frac{45344}{18375}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. i) Integrals of $x^m\,Z_1^4(x)$

Explicit and basic integrals are omitted.
With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 465 holds

$$\int x\,J_1^4(x)\,dx = -\frac{x^2}{2}\,J_0^4(x) + x\,J_0^3(x)\,J_1(x) - x^2\,J_0^2(x)\,J_1^2(x) - \frac{x^2}{2}\,J_1^4(x) + 3\mathcal{I}_1^{(22)}(x)$$

$$\int x\,I_1^4(x)\,dx = -\frac{x^2}{2}\,I_0^4(x) + x\,I_0^3(x)\,I_1(x) + x^2\,I_0^2(x)\,I_1^2(x) - \frac{x^2}{2}\,I_1^4(x) - 3\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^2\,J_1^4(x)\,dx = \frac{12\,x^3 - 9\,x}{32}\,J_0^4(x) - \frac{9x^2}{8}\,J_0^3(x)\,J_1(x) + \frac{3x^3}{4}\,J_0^2(x)\,J_1^2(x) - \frac{11x^2}{8}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{12\,x^3 - 11\,x}{32}\,J_1^4(x) + \frac{9}{32}\,\mathcal{I}_0^{(40)}(x) - \frac{33}{32}\,\mathcal{I}_0^{(04)}(x)$$

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$$\int x^2\, I_1^4(x)\, dx = \frac{12\,x^3 + 9\,x}{32}\, I_0^4(x) - \frac{9x^2}{8}\, I_0^3(x)\, I_1(x) - \frac{3x^3}{4}\, I_0^2(x)\, I_1^2(x) + \frac{11x^2}{8}\, I_0(x)\, I_1^3(x) +$$

$$+ \frac{12\,x^3 + 11\,x}{32}\, I_1^4(x) - \frac{9}{32}\, \mathcal{I}_0^{*(40)}(x) + \frac{33}{32}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^3\, J_1^4(x)\, dx =$$

$$= \frac{3\,x^4 - 6\,x^2}{16}\, J_0^4(x) - \frac{3\,x^3 - 3\,x}{4}\, J_0^3(x)\, J_1(x) + \frac{3x^4}{8}\, J_0^2(x)\, J_1^2(x) - x^3\, J_0(x)\, J_1^3(x) + \frac{3x^4}{16}\, J_1^4(x) + \frac{9}{4}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^3\, I_1^4(x)\, dx =$$

$$= \frac{3\,x^4 + 6\,x^2}{16}\, I_0^4(x) - \frac{3\,x^3 + 3\,x}{4}\, I_0^3(x)\, I_1(x) - \frac{3x^4}{8}\, I_0^2(x)\, I_1^2(x) + x^3\, I_0(x)\, I_1^3(x) + \frac{3x^4}{16}\, I_1^4(x) + \frac{9}{4}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^4\, J_1^4(x)\, dx = \frac{32\,x^5 + 36\,x^3 - 27\,x}{256}\, J_0^4(x) - \frac{40\,x^4 + 27\,x^2}{64}\, J_0^3(x)\, J_1(x) + \frac{8\,x^5 + 39\,x^3}{32}\, J_0^2(x)\, J_1^2(x) -$$

$$- \frac{56\,x^4 + 53\,x^2}{64}\, J_0(x)\, J_1^3(x) + \frac{32\,x^5 + 212\,x^3 - 53\,x}{256}\, J_1^4(x) + \frac{27}{256}\, \mathcal{I}_0^{(40)}(x) - \frac{159}{256}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^4\, I_1^4(x)\, dx = \frac{32\,x^5 - 36\,x^3 - 27\,x}{256}\, I_0^4(x) - \frac{40\,x^4 - 27\,x^2}{64}\, I_0^3(x)\, I_1(x) - \frac{8\,x^5 - 39\,x^3}{32}\, I_0^2(x)\, I_1^2(x) +$$

$$+ \frac{56\,x^4 - 53\,x^2}{64}\, I_0(x)\, I_1^3(x) + \frac{32\,x^5 - 212\,x^3 - 53\,x}{256}\, I_1^4(x) + \frac{27}{256}\, \mathcal{I}_0^{*(40)}(x) - \frac{159}{256}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^5\, J_1^4(x)\, dx = \frac{6\,x^6 - 9\,x^4 + 18\,x^2}{64}\, J_0^4(x) - \frac{9\,x^5 - 9\,x^3 + 9\,x}{16}\, J_0^3(x)\, J_1(x) + \frac{6\,x^6 + 27\,x^4}{32}\, J_0^2(x)\, J_1^2(x) -$$

$$- \frac{13x^5}{16}\, J_0(x)\, J_1^3(x) + \frac{6\,x^6 + 53\,x^4}{64}\, J_1^4(x) - \frac{27}{16}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^5\, I_1^4(x)\, dx = \frac{6\,x^6 + 9\,x^4 + 18\,x^2}{64}\, I_0^4(x) - \frac{9\,x^5 + 9\,x^3 + 9\,x}{16}\, I_0^3(x)\, I_1(x) - \frac{6\,x^6 - 27\,x^4}{32}\, I_0^2(x)\, I_1^2(x) +$$

$$+ \frac{13x^5}{16}\, I_0(x)\, I_1^3(x) + \frac{6\,x^6 - 53\,x^4}{64}\, I_1^4(x) + \frac{27}{16}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^6\, J_1^4(x)\, dx = \frac{768\,x^7 - 3264\,x^5 - 2412\,x^3 + 1809\,x}{10240}\, J_0^4(x) - \frac{1344\,x^6 - 4080\,x^4 - 1809\,x^2}{2560}\, J_0^3(x)\, J_1(x) +$$

$$+ \frac{192\,x^7 + 864\,x^5 - 3663\,x^3}{1280}\, J_0^2(x)\, J_1^2(x) - \frac{1984\,x^6 - 2352\,x^4 - 4251\,x^2}{2560}\, J_0(x)\, J_1^3(x) +$$

$$+ \frac{768\,x^7 + 9408\,x^5 - 17004\,x^3 + 4251\,x}{10240}\, J_1^4(x) - \frac{1809}{10240}\, \mathcal{I}_0^{(40)}(x) + \frac{12753}{10240}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^6\, I_1^4(x)\, dx = \frac{768\,x^7 + 3264\,x^5 - 2412\,x^3 - 1809\,x}{10240}\, I_0^4(x) - \frac{1344\,x^6 + 4080\,x^4 - 1809\,x^2}{2560}\, I_0^3(x)\, I_1(x) -$$

$$- \frac{192\,x^7 - 864\,x^5 - 3663\,x^3}{1280}\, I_0^2(x)\, I_1^2(x) + \frac{1984\,x^6 + 2352\,x^4 - 4251\,x^2}{2560}\, I_0(x)\, I_1^3(x) +$$

$$+\frac{768\,x^7 - 9408\,x^5 - 17004\,x^3 - 4251\,x}{10240}\,I_1^4(x) + \frac{1809}{10240}\,\mathcal{I}_0^{*(40)}(x) - \frac{12753}{10240}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^7\,J_1^4(x)\,dx = \frac{4\,x^8 - 30\,x^6 + 45\,x^4 - 90\,x^2}{64}\,J_0^4(x) - \frac{8\,x^7 - 45\,x^5 + 45\,x^3 - 45\,x}{16}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{4\,x^8 + 18\,x^6 - 135\,x^4}{32}\,J_0^2(x)\,J_1^2(x) - \frac{12\,x^7 - 33\,x^5}{16}\,J_0(x)\,J_1^3(x) + \frac{4\,x^8 + 66\,x^6 - 201\,x^4}{64}\,J_1^4(x) + \frac{135}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^7\,I_1^4(x)\,dx = \frac{4\,x^8 + 30\,x^6 + 45\,x^4 + 90\,x^2}{64}\,I_0^4(x) - \frac{8\,x^7 + 45\,x^5 + 45\,x^3 + 45\,x}{16}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{4\,x^8 - 18\,x^6 - 135\,x^4}{32}\,I_0^2(x)\,I_1^2(x) + \frac{12\,x^7 + 33\,x^5}{16}\,I_0(x)\,I_1^3(x) + \frac{4\,x^8 - 66\,x^6 - 201\,x^4}{64}\,I_1^4(x)\frac{135}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^8\,J_1^4(x)\,dx = \frac{15360\,x^9 - 173952\,x^7 + 663696\,x^5 + 461268\,x^3 - 345951\,x}{286720}\,J_0^4(x)-$$

$$-\frac{34560\,x^8 - 304416\,x^6 + 829620\,x^4 + 345951\,x^2}{71680}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{960\,x^9 + 4248\,x^7 - 53649\,x^5 + 184383\,x^3}{8960}\,J_0^2(x)\,J_1^2(x)-$$

$$-\frac{52480\,x^8 - 247776\,x^6 + 400428\,x^4 + 837639\,x^2}{71680}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{15360\,x^9 + 330368\,x^7 - 1601712\,x^5 + 3350556\,x^3 - 837639\,x}{286720}\,J_1^4(x)+$$

$$+\frac{345951}{286720}\,\mathcal{I}_0^{(40)}(x) - \frac{2512917}{286720}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^8\,I_1^4(x)\,dx = \frac{15360\,x^9 + 173952\,x^7 + 663696\,x^5 - 461268\,x^3 - 345951\,x}{286720}\,I_0^4(x)-$$

$$-\frac{34560\,x^8 + 304416\,x^6 + 829620\,x^4 - 345951\,x^2}{71680}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{960\,x^9 - 4248\,x^7 - 53649\,x^5 - 184383\,x^3}{8960}\,I_0^2(x)\,I_1^2(x)+$$

$$+\frac{52480\,x^8 + 247776\,x^6 + 400428\,x^4 - 837639\,x^2}{71680}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{15360\,x^9 - 330368\,x^7 - 1601712\,x^5 - 3350556\,x^3 - 837639\,x}{286720}\,I_1^4(x)+$$

$$+\frac{345951}{286720}\,\mathcal{I}_0^{*(40)}(x) - \frac{2512917}{286720}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^9\,J_1^4(x)\,dx = \frac{48\,x^{10} - 756\,x^8 + 4950\,x^6 - 7425\,x^4 + 14850\,x^2}{1024}\,J_0^4(x)-$$

$$-\frac{120\,x^9 - 1512\,x^7 + 7425\,x^5 - 7425\,x^3 + 7425\,x}{256}\,J_0^3(x)\,J_1(x)+$$

$$+\frac{48\,x^{10} + 204\,x^8 - 4122\,x^6 + 22275\,x^4}{512}\,J_0^2(x)\,J_1^2(x) - \frac{184\,x^9 - 1308\,x^7 + 4677\,x^5}{256}\,J_0(x)\,J_1^3(x)+$$

$$+\frac{48\,x^{10} + 1308\,x^8 - 9354\,x^6 + 31629\,x^4}{1024}\,J_1^4(x) - \frac{22275}{256}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^9\,I_1^4(x)\,dx = \frac{48\,x^{10} + 756\,x^8 + 4950\,x^6 + 7425\,x^4 + 14850\,x^2}{1024}\,I_0^4(x)-$$

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$$-\frac{120\,x^9 + 1512\,x^7 + 7425\,x^5 + 7425\,x^3 + 7425\,x}{256}\, I_0^3(x)\, I_1(x) -$$

$$-\frac{48\,x^{10} - 204\,x^8 - 4122\,x^6 - 22275\,x^4}{512}\, I_0^2(x)\, I_1^2(x) + \frac{184\,x^9 + 1308\,x^7 + 4677\,x^5}{256}\, I_0(x)\, I_1^3(x) +$$

$$+\frac{48\,x^{10} - 1308\,x^8 - 9354\,x^6 - 31629\,x^4}{1024}\, I_1^4(x) + \frac{22275}{256}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^{10}\, J_1^4(x)\, dx =$$

$$= \frac{573440\,x^{11} - 11919360\,x^9 + 116358912\,x^7 - 435859776\,x^5 - 299440908\,x^3 + 224580681\,x}{13762560}\, J_0^4(x) -$$

$$-\frac{1576960\,x^{10} - 26818560\,x^8 + 203628096\,x^6 - 544824720\,x^4 - 224580681\,x^2}{3440640}\, J_0^3(x)\, J_1(x) +$$

$$+\frac{143360\,x^{11} + 568320\,x^9 - 17842752\,x^7 + 145570176\,x^5 - 483478767\,x^3}{1720320}\, J_0^2(x)\, J_1^2(x) -$$

$$-\frac{2437120\,x^{10} - 24166400\,x^8 + 144152256\,x^6 - 253684368\,x^4 - 546899859\,x^2}{3440640}\, J_0(x)\, J_1^3(x) +$$

$$+\frac{573440\,x^{11} + 19333120\,x^9 - 192203008\,x^7 + 1014737472\,x^5 - 2187599436\,x^3 + 546899859\,x}{13762560}\, J_1^4(x) -$$

$$-\frac{74860227}{4587520}\, \mathcal{I}_0^{(40)}(x) + \frac{546899859}{4587520}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^{10}\, I_1^4(x)\, dx = \frac{573440\,x^{11} + 11919360\,x^9 + 116358912\,x^7 + 435859776\,x^5 - 299440908\,x^3 - 224580681\,x}{13762560}\, I_0^4(x) -$$

$$-\frac{1576960\,x^{10} + 26818560\,x^8 + 203628096\,x^6 + 544824720\,x^4 - 224580681\,x^2}{3440640}\, I_0^3(x)\, I_1(x) -$$

$$-\frac{143360\,x^{11} - 568320\,x^9 - 17842752\,x^7 - 145570176\,x^5 - 483478767\,x^3}{1720320}\, I_0^2(x)\, I_1^2(x) +$$

$$+\frac{2437120\,x^{10} + 24166400\,x^8 + 144152256\,x^6 + 253684368\,x^4 - 546899859\,x^2}{3440640}\, I_0(x)\, I_1^3(x) +$$

$$+\frac{573440\,x^{11} - 19333120\,x^9 - 192203008\,x^7 - 1014737472\,x^5 - 2187599436\,x^3 - 546899859\,x}{13762560}\, I_1^4(x) +$$

$$+\frac{74860227}{4587520}\, \mathcal{I}_0^{*(40)}(x) - \frac{546899859}{4587520}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_1^4(x)\, dx}{x^2} =$$

$$= -\frac{2x}{5}\, J_0^4(x) - \frac{4x}{5}\, J_0^2(x)\, J_1^2(x) - \frac{4}{15}\, J_0(x)\, J_1^3(x) - \frac{2\,x^2 + 1}{5x}\, J_1^4(x) + \frac{2}{5}\, \mathcal{I}_0^{(40)}(x) - \frac{22}{15}\, \mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_1^4(x)\, dx}{x^2} =$$

$$= \frac{2x}{5}\, I_0^4(x) - \frac{4x}{5}\, I_0^2(x)\, I_1^2(x) - \frac{4}{15}\, I_0(x)\, I_1^3(x) + \frac{2\,x^2 - 1}{5x}\, I_1^4(x) - \frac{2}{5}\, \mathcal{I}_0^{*(40)}(x) + \frac{22}{15}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_1^4(x)\, dx}{x^4} = \frac{88x}{175}\, J_0^4(x) - \frac{8}{35}\, J_0^3(x)\, J_1(x) + \frac{176\,x^2 - 20}{175x}\, J_0^2(x)\, J_1^2(x) + \frac{56\,x^2 - 60}{525\,x^2}\, J_0(x)\, J_1^3(x) +$$

$$+\frac{88\,x^4 + 4\,x^2 - 25}{175\,x^3}\, J_1^4(x) - \frac{48}{175}\, \mathcal{I}_0^{(40)}(x) + \frac{848}{525}\, \mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_1^4(x)\,dx}{x^4} = \frac{88x}{175}\,I_0^4(x) - \frac{8}{35}\,I_0^3(x)\,I_1(x) - \frac{176\,x^2+20}{175x}\,I_0^2(x)\,I_1^2(x) - \frac{56\,x^2+60}{525\,x^2}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{88\,x^4-4\,x^2-25}{175\,x^3}\,I_1^4(x) - \frac{48}{175}\,\mathcal{I}_0^{*(40)}(x) + \frac{848}{525}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_1^4(x)\,dx}{x^6} = -\frac{19456\,x^2+1400}{55125\,x}\,J_0^4(x) + \frac{2176\,x^2-280}{11025\,x^2}\,J_0^3(x)\,J_1(x)-$$

$$-\frac{38912\,x^4-2640\,x^2+2100}{55125\,x^3}\,J_0^2(x)\,J_1^2(x) - \frac{6272\,x^4-3720\,x^2+10500}{165375\,x^4}\,J_0(x)\,J_1^3(x)-$$

$$-\frac{19456\,x^6+248\,x^4-500\,x^2+6125}{55125\,x^5}\,J_1^4(x) + \frac{8576}{55125}\,\mathcal{I}_0^{(40)}(x) - \frac{181376}{165375}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_1^4(x)\,dx}{x^6} = \frac{19456\,x^2-1400}{55125\,x}\,I_0^4(x) - \frac{2176\,x^2+280}{11025\,x^2}\,I_0^3(x)\,I_1(x)-$$

$$-\frac{38912\,x^4+2640\,x^2+2100}{55125\,x^3}\,I_0^2(x)\,I_1^2(x) - \frac{6272\,x^4+3720\,x^2+10500}{165375\,x^4}\,I_0(x)\,I_1^3(x)+$$

$$+\frac{19456\,x^6-248\,x^4-500\,x^2-6125}{55125\,x^5}\,I_1^4(x) - \frac{8576}{55125}\,\mathcal{I}_0^{*(40)}(x) + \frac{181376}{165375}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. j) Recurrence relations

With the integrals

$$\mathcal{I}_n^{(pq)}(x) \quad = \quad \int x^n\,J_0^p(x)\,J_1^q(x)\,dx \ , \qquad p+q=4$$

and

$$\mathcal{I}_n^{*(pq)}(x) \quad = \quad \int x^n\,I_0^p(x)\,I_1^q(x)\,dx \ , \qquad p+q=4$$

holds

Ascending recurrence relations:

$$\mathcal{I}_{n+1}^{(40)}(x) = \frac{x^{n+1}}{8n}\left[3x\,J_0^4(x) + (5n-6)\,J_0^3(x)\,J_1(x) + 6x\,J_0^2(x)\,J_1^2(x) + 3(n-2)\,J_0(x)\,J_1^3(x) + 3x\,J_1^4(x)\right]+$$

$$-\frac{5n-6}{8}\,\mathcal{I}_n^{(31)}(x) - \frac{3(n-2)^2}{8n}\,\mathcal{I}_n^{(13)}(x)$$

$$\mathcal{I}_{n+1}^{(31)}(x) = -\frac{x^{n+1}}{4}\,J_0^4(x) + \frac{n+1}{4}\,\mathcal{I}_n^{(40)}(x)$$

$$\mathcal{I}_{n+1}^{(22)}(x) = \frac{x^{n+1}}{8n}\left[x\,J_0^4(x) - (n+2)\,J_0^3(x)\,J_1(x) + 2x\,J_0^2(x)\,J_1^2(x) + (n-2)\,J_0(x)\,J_1^3(x) + x\,J_1^4(x)\right]+$$

$$+\frac{n+2}{8}\,\mathcal{I}_{n+1}^{(31)}(x) - \frac{(n-2)^2}{8n}\,\mathcal{I}_{n+1}^{(13)}(x)$$

$$\mathcal{I}_{n+1}^{(13)}(x) = -\frac{x^{n+1}}{4}\left[J_0^4(x) + 2\,J_0^2(x)\,J_1^2(x)\right] + \frac{n+1}{4}\,\mathcal{I}_{n+1}^{(40)}(x) + \frac{n-1}{2}\,\mathcal{I}_{n+1}^{(22)}(x)$$

$$\mathcal{I}_{n+1}^{(04)}(x) = \frac{x^{n+1}}{8n}\left[3x\,J_0^4(x) - 3(n+2)\,J_0^3(x)\,J_1(x) + 6x\,J_0^2(x)\,J_1^2(x) - (5n+6)\,J_0(x)\,J_1^3(x) + 3x\,J_1^4(x)\right]+$$

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$$+\frac{3(n+2)}{8}\,\mathcal{I}_{n+1}^{(31)}(x)+\frac{(5n+6)(n-2)}{8n}\,\mathcal{I}_{n+1}^{(13)}(x)$$

$$\mathcal{I}_{n+1}^{*(40)}(x)=\frac{x^{n+1}}{8n}\left[3x\,I_0^4(x)+(5n-6)\,I_0^3(x)\,I_1(x)-6x\,I_0^2(x)\,I_1^2(x)-3(n-2)\,I_0(x)\,I_1^3(x)+3x\,I_1^4(x)\right]-$$

$$-\frac{5n-6}{8}\,\mathcal{I}_n^{*(31)}(x)+\frac{3(n-2)^2}{8n}\,\mathcal{I}_n^{*(13)}(x)$$

$$\mathcal{I}_{n+1}^{*(31)}(x)=\frac{x^{n+1}}{4}\,I_0^4(x)-\frac{n+1}{4}\,\mathcal{I}_n^{*(40)}(x)$$

$$\mathcal{I}_{n+1}^{*(22)}(x)=\frac{x^{n+1}}{8n}\left[-x\,I_0^4(x)+(n+2)\,I_0^3(x)\,I_1(x)+2x\,I_0^2(x)\,I_1^2(x)+(n-2)\,I_0(x)\,I_1^3(x)-x\,I_1^4(x)\right]-$$

$$-\frac{n+2}{8}\,\mathcal{I}_{n+1}^{*(31)}(x)-\frac{(n-2)^2}{8n}\mathcal{I}_{n+1}^{*(13)}(x)$$

$$\mathcal{I}_{n+1}^{*(13)}(x)=\frac{x^{n+1}}{4}\left[-I_0^4(x)+2\,I_0^2(x)\,I_1^2(x)\right]+\frac{n+1}{4}\,\mathcal{I}_{n+1}^{*(40)}(x)-\frac{n-1}{2}\,\mathcal{I}_{n+1}^{*(22)}(x)$$

$$\mathcal{I}_{n+1}^{*(04)}(x)=\frac{x^{n+1}}{8n}\left[3x\,I_0^4(x)-3(n+2)\,I_0^3(x)\,I_1(x)-6x\,I_0^2(x)\,I_1^2(x)+(5n+6)\,I_0(x)\,I_1^3(x)+3x\,I_1^4(x)\right]+$$

$$+\frac{3(n+2)}{8}\,\mathcal{I}_{n+1}^{*(31)}(x)-\frac{(5n+6)(n-2)}{8n}\,\mathcal{I}_{n+1}^{*(13)}(x)$$

4.2. Different Functions and Different Arguments

a) $x^m Z_0^p(x) Z_1^{3-p}(x) Z_\nu^*\alpha x)$:

$$\int J_1^3(x)\, J_0(3x)\, dx = \frac{1}{4}\, \big\{\, [\,-3x^2\, J_0^3(x) - 3x\, J_0^2(x)J_1(x) + 9x^2\, J_0(x)J_1^2(x) - 2x\, J_1^3(x)\,]\, J_0(3x) +$$

$$+ [3x\, J_0^3(x) - 9x^2\, J_0^2(x)J_1(x) + 3x^2\, J_1^3(x)\,]\, J_1(3x)\, \big\}$$

$$\int I_1^3(x)\, I_0(3x)\, dx = \frac{1}{4}\, \big\{\, [\, 3x^2\, I_0^3(x) + 3x\, I_0^2(x)I_1(x) + 9x^2\, I_0(x)I_1^2(x) - 2x\, I_1^3(x)\,]\, I_0(3x) -$$

$$- [3x\, I_0^3(x) + 9x^2\, I_0^2(x)I_1(x) + 3x^2\, I_1^3(x)\,]\, I_1(3x)\, \big\}$$

$$\int K_1^3(x)\, K_0(3x)\, dx = \frac{1}{4}\, \big\{\, [\,-3x^2\, K_0^3(x) + 3x\, K_0^2(x)K_1(x) - 9x^2\, K_0(x)K_1^2(x) - 2x\, K_1^3(x)\,]\, K_0(3x) -$$

$$- [3x\, K_0^3(x) - 9x^2\, K_0^2(x)K_1(x) - 3x^2\, K_1^3(x)\,]\, K_1(3x)\, \big\}$$

$$\int \frac{J_1^3(x)\, J_1(3x)\, dx}{x} = \frac{1}{8}\, \big\{\, [\,-6x^2\, J_0^3(x) - 9x\, J_0^2(x)\, J_1(x) + 18x^2\, J_0(x)\, J_1^2(x) - 3x\, J_1^3(x)\,]\, J_0(3x) +$$

$$+ [\, 7x\, J_0^3(x) - 18x^2\, J_0^2(x)\, J_1(x) - 3x\, J_0(x)\, J_1^2(x)\, J_0(x) + (6x^2 - 2)\, J_1^3(x)\,]\, J_1(3x)\, \big\}$$

$$\int \frac{I_1^3(x)\, I_1(3x)\, dx}{x} = \frac{1}{8}\, \big\{\, [6x^2\, I_0^3(x) + 9x\, I_0^2(x)\, I_1(x) + 18x^2\, I_0(x)\, I_1^2(x) - 3x\, I_1^3(x)\,]\, I_0(3x) -$$

$$- [\, 7x\, I_0^3(x) + 18x^2\, I_0^2(x)\, I_1(x) + 3x\, I_0(x)\, I_1^2(x)\, I_0(x) + (6x^2 + 2)\, K_1^3(x)\,]\, K_1(3x)\, \big\}$$

$$\int \frac{K_1^3(x)\, K_1(3x)\, dx}{x} = \frac{1}{8}\, \big\{\, [6x^2\, K_0^3(x) - 9x\, K_0^2(x)\, K_1(x) + 18x^2\, K_0(x)\, K_1^2(x) + 3x\, K_1^3(x)\,]\, K_0(3x) +$$

$$+ [\, 7x\, K_0^3(x) - 18x^2\, K_0^2(x)\, K_1(x) + 3x\, K_0(x)\, K_1^2(x)\, K_0(x) - (6x^2 + 2)\, I_1^3(x)\,]\, I_1(3x)\, \big\}$$

$$\int J_0(x)\, J_1^2(x)\, J_1(3x)\, dx = \frac{x}{12}\, \big\{\, [\,-3x\, J_0^3(x) - 9\, J_0^2(x)J_1(x) + 9x\, J_0(x)J_1^2(x)\,]\, J_0(3x) +$$

$$+ [5\, J_0^3(x) - 9x\, J_0^2(x)J_1(x) - 6\, J_0(x)J_1^2(x) + 3x\, J_1^3(x)\,]\, J_1(3x)\, \big\}$$

$$\int I_0(x)\, I_1^2(x)\, I_1(3x)\, dx = \frac{x}{12}\, \big\{\, [3x\, I_0^3(x) + 9\, I_0^2(x)I_1(x) + 9x\, I_0(x)I_1^2(x)\,]\, I_0(3x) -$$

$$- [5\, I_0^3(x) + 9x\, I_0^2(x)I_1(x) + 6\, I_0(x)I_1^2(x) + 3x\, I_1^3(x)\,]\, I_1(3x)\, \big\}$$

$$\int K_0(x)\, K_1^2(x)\, K_1(3x)\, dx = \frac{x}{12}\, \big\{\, [\,-3x\, K_0^3(x) + 9\, K_0^2(x)K_1(x) - 9x\, K_0(x)K_1^2(x)\,]\, K_0(3x) -$$

$$- [5\, K_0^3(x) - 9x\, K_0^2(x)K_1(x) + 6\, K_0(x)K_1^2(x) - 3x\, K_1^3(x)\,]\, K_1(3x)\, \big\}$$

$$\int x\, J_0(x)\, J_1^2(x)\, J_0(\sqrt{3}\, x)\, dx = \frac{J_0^2(x)\, x}{6}\, \Big[\sqrt{3}\, J_0(x)\, J_1(\sqrt{3}\, x) - 3\, J_1(x)\, J_0(\sqrt{3}\, x)\Big]$$

$$\int x\, I_0(x)\, I_1^2(x)\, I_0(\sqrt{3}\, x)\, dx = \frac{I_0^2(x)\, x}{6}\, \Big[3\, I_1(x)\, I_0(\sqrt{3}\, x) - \sqrt{3}\, I_0(x)\, I_1(\sqrt{3}\, x)\Big]$$

$$\int x\, I_0(x)\, I_1^2(x)\, K_0(\sqrt{3}\, x)\, dx = \frac{I_0^2(x)\, x}{6}\, \Big[3\, I_1(x)\, K_0(\sqrt{3}\, x) + \sqrt{3}\, I_0(x)\, K_1(\sqrt{3}\, x)\Big]$$

$$\int x\, K_0(x)\, K_1^2(x)\, K_0(\sqrt{3}\, x)\, dx = \frac{K_0^2(x)\, x}{6}\, \Big[\sqrt{3}\, K_0(x)\, K_1(\sqrt{3}\, x) - 3\, K_1(x)\, K_0(\sqrt{3}\, x)\Big]$$

$$\int x\, K_0(x)\, K_1^2(x)\, I_0(\sqrt{3}\, x)\, dx = -\frac{K_0^2(x)\, x}{6}\, \Big[\sqrt{3}\, K_0(x)\, I_1(\sqrt{3}\, x) + 3\, K_1(x)\, I_0(\sqrt{3}\, x)\Big]$$

$$\int x^3\, J_0^2(x)\, J_1(x)\, J_1(\sqrt{3}\, x)\, dx = \frac{J_1^2(x)\, x^3}{6}\, \Big[3\, J_0(x)\, J_1(\sqrt{3}\, x) - \sqrt{3}\, J_1(x)\, J_0(\sqrt{3}\, x)\Big]$$

$$\int x^3\, I_0^2(x)\, I_1(x)\, I_1(\sqrt{3}\, x)\, dx = \frac{I_1^2(x)\, x^3}{6}\, \Big[3\, I_0(x)\, I_1(\sqrt{3}\, x) - \sqrt{3}\, I_1(x)\, I_0(\sqrt{3}\, x)\Big]$$

$$\int x^3\, I_0^2(x)\, I_1(x)\, K_1(\sqrt{3}\,x)\, dx = \frac{I_1^2(x)\, x^3}{6}\left[3\, I_0(x)\, K_1(\sqrt{3}\,x) + \sqrt{3}\, I_1(x)\, K_0(\sqrt{3}\,x)\right]$$

$$\int x^3\, K_0^2(x)\, K_1(x)\, K_1(\sqrt{3}\,x)\, dx = \frac{K_1^2(x)\, x^3}{6}\left[\sqrt{3}\, K_1(x)\, K_0(\sqrt{3}\,x) - 3\, K_0(x)\, K_1(\sqrt{3}\,x)\right]$$

$$\int x^3\, K_0^2(x)\, K_1(x)\, I_1(\sqrt{3}\,x)\, dx = -\frac{K_1^2(x)\, x^3}{6}\left[\sqrt{3}\, K_1(x)\, I_0(\sqrt{3}\,x) + 3\, K_0(x)\, I_1(\sqrt{3}\,x)\right]$$

b) $\underline{x^m Z_0^{\,p}(x) Z_1^{3-p}(x)\, Z_\nu^*\alpha x):}$

Holds (see [1], 9.6.14)

$$x^2\left[I_0^2(x)K_1^2(x) + 2I_0(x)I_1(x)K_0(x)K_1(x) + I_1^2(x)K_0^2(x)\right] \;=\; 1\;.$$

Therefore each multiple of this expression may be added to the antiderivatives, concerning to this functions.

$$\int \frac{I_1^2(x)\, K_1^2(x)\, dx}{x} = \frac{1}{4}\left\{-x^2\, I_0^2(x)\, K_0^2(x) - x\, I_0^2(x)\, K_0(x)K_1(x) + x^2\, I_0^2(x)K_1^2(x) + \right.$$

$$+x\, I_0(x)I_1(x)[K_0^2(x) - K_1^2(x)] + x^2\, I_1^2(x)K_0^2(x) + x\, I_1^2(x)K_0(x)K_1(x) - (x^2 + 1)I_1^2(x)K_1^2(x)\}$$

$$\int \frac{I_0(x)\, I_1(x)\, K_0(x)\, K_1(x)\, dx}{x} = \frac{1}{4}\left\{(x^2 - 1)\, I_0^2(x)K_0^2(x) + x\, I_0^2(x)K_0(x)K_1(x) - \right.$$

$$-x^2\, I_0^2(x)K_1^2(x) - x\, I_0(x)I_1(x)K_0^2(x) - 2\, I_0(x)I_1(x)K_0(x)K_1(x) + x\, I_0(x)I_1(x)K_1^2(x) - $$

$$-x^2\, I_1^2(x)K_0^2(x) - x\, I_1^2(x)K_0(x)K_1(x) + x^2\, I_1^2(x)K_1^2(x)\}$$

$$\int I_1^2(x)\, K_0(x)\, K_1(x)\, dx = \frac{x}{4}\left[\, x\, I_0^2(x)K_0^2(x) - 2\, I_0^2(x)K_0(x)K_1(x) + \right.$$

$$+ 2x\, I_0(x)I_1(x)K_0(x)K_1(x) - 2\, I_0^2(x)K_0(x)K_1(x) + x\, I_1^2(x)K_1^2(x)]$$

$$\int I_0(x)I_1(x)K_1^2(x)\, dx = -\frac{x}{4}\left[\, x\, I_0^2(x)K_0^2(x) + \right.$$

$$+ 2I_0^2(x)K_0(x)K_1(x) + 2x\, I_0(x)I_1(x)K_0(x)K_1(x) + 2I_0(x)I_1(x)K_1^2(x) + x\, I_1^2(x)K_1^2(x)]$$

$$\int x^3\, J_0^2(x)I_0^2(x)\, dx = \frac{x^2}{8}[\, x^2\, J_0^2(x)I_0^2(x) + 2x\, J_0^2(x)I_0(x)I_1(x) - (x^2 + 1)\, J_0^2(x)I_1^2(x) + $$

$$+ 2x\, J_0(x)J_1(x)I_0^2(x) - 2J_0(x)J_1(x)I_0(x)I_1(x) + (x^2 - 1)\, J_1^2(x)I_0^2(x) - x^2\, J_1^2(x)I_1^2(x)\,]$$

$$\int x^3\, J_0^2(x)K_0^2(x)\, dx = \frac{x^2}{8}[\, x^2\, J_0^2(x)K_0^2(x) - 2x\, J_0^2(x)K_0(x)K_1(x) - (x^2 + 1)\, J_0^2(x)K_1^2(x) + $$

$$+ 2x\, J_0(x)J_1(x)K_0^2(x) + 2J_0(x)J_1(x)K_0(x)K_1(x) + (x^2 - 1)\, J_1^2(x)K_0^2(x) - x^2\, J_1^2(x)K_1^2(x)\,]$$

$$\int x^3\, J_1^2(x)J_0^2\left(\frac{x}{\sqrt{3}}\right) dx = \frac{x^2}{8}\left[\, x^2\, J_0^2(x)J_0^2\left(\frac{x}{\sqrt{3}}\right) + 2\sqrt{3}\, x\, J_0^2(x)J_0\left(\frac{x}{\sqrt{3}}\right)J_1\left(\frac{x}{\sqrt{3}}\right) + \right.$$

$$+(x^2 + 3)\, J_0^2(x)J_1^2\left(\frac{x}{\sqrt{3}}\right) - 6x\, J_0(x)J_1(x)J_0^2\left(\frac{x}{\sqrt{3}}\right) - 6\sqrt{3}\, J_0(x)J_1(x)J_0\left(\frac{x}{\sqrt{3}}\right)J_1\left(\frac{x}{\sqrt{3}}\right) + $$

$$+(x^2 + 9)\, J_1^2(x)J_0^2\left(\frac{x}{\sqrt{3}}\right) + x^2\, J_1^2(x)J_1^2\left(\frac{x}{\sqrt{3}}\right)\right]$$

$$\int x^3\, I_0^2(x)K_1^2(\sqrt{3}x)\, dx = \frac{x^2}{8}\left[-x^2\, I_0^2(x)K_0^2(\sqrt{3}x) - 2\sqrt{3}x\, I_0^2(x)K_0(\sqrt{3}x)K_1(\sqrt{3}x) + \right.$$

$$+(x^2 - 3)\, I_0^2(x)K_1^2(\sqrt{3}x) - 2x\, I_0(x)I_1(x)K_0^2(\sqrt{3}x) - 2\sqrt{3}\, I_0(x)I_1(x)K_0(\sqrt{3}x)K_1(\sqrt{3}x) + $$

$$+(x^2 - 1)\, I_1^2(x)K_0^2(\sqrt{3}x) - x^2\, I_1^2(x)K_1^2(\sqrt{3}x)]$$

$$\int x^3\, K_0^2(x)I_1^2(\sqrt{3}x)\, dx = \frac{x^2}{8}\left[-x^2\, K_0^2(x)I_0^2(\sqrt{3}x) + 2\sqrt{3}x\, K_0^2(x)I_0(\sqrt{3}x)K_1(\sqrt{3}x) + \right.$$

$$+(x^2 - 3)\, K_0^2(x)I_1^2(\sqrt{3}x) + 2x\, K_0(x)K_1(x)I_0^2(\sqrt{3}x) - 2\sqrt{3}\, K_0(x)K_1(x)I_0(\sqrt{3}x)I_1(\sqrt{3}x) + $$

$$+(x^2 - 1)\, K_1^2(x)I_0^2(\sqrt{3}x) - x^2\, K_1^2(x)I_1^2(\sqrt{3}x)]$$

$$\int x^3\, J_1^2(x)I_1^2(x)\, dx = \frac{x^2}{8}[-x^2\, J_0^2(x)I_0^2(x) + 2x\, J_0^2(x)I_0(x)I_1(x) + (x^2 - 1)\, J_0^2(x)I_1^2(x)+$$

$$+2x\, J_0(x)J_1(x)I_0^2(x) - 2J_0(x)J_1(x)I_0(x)I_1(x) - 4x\, J_0(x)J_1(x)I_1^2(x) - (x^2 + 1)\, J_1^2(x)I_0^2(x)+$$

$$+4x\, J_1^2(x)I_0(x)I_1(x) + x^2\, J_1^2(x)I_1^2(x)\,]$$

$$\int x^3\, J_1^2(x)K_1^2(x)\, dx = \frac{x^2}{8}[-x^2\, J_0^2(x)K_0^2(x) - 2x\, J_0^2(x)K_0(x)I_1(x) + (x^2 - 1)\, J_0^2(x)K_1^2(x)+$$

$$+2x\, J_0(x)J_1(x)K_1^2(x) + 2J_0(x)J_1(x)K_0(x)K_1(x) - 4x\, J_0(x)J_1(x)K_1^2(x) - (x^2 + 1)\, J_1^2(x)K_0^2(x)-$$

$$-4x\, J_1^2(x)K_0(x)K_1(x) + x^2\, J_1^2(x)K_1^2(x)\,]$$

$$\int x^4\, I_1^2(x)\, K_0(x)\, K_1(x)\, dx = \frac{x^3}{12}\,[x^2\, I_0^2(x)K_0(x)K_1(x) + 2x\, I_0^2(x)K_1^2(x) + x^2\, I_0(x)I_1(x)K_0^2(x)+$$

$$+(x^2 - 4)\, I_0(x)I_1(x)K_1^2(x) - 2x\, I_1^2(x)K_0^2(x) + (x^2 - 4)\, I_1^2(x)K_0(x)K_1(x) - 3x\, I_1^2(x)K_1^2(x)\,]$$

$$\int x^4\, I_0(x)\, I_1(x)\, K_1^2(x)\, dx = \frac{x^3}{12}\,[x^2\, I_0^2(x)K_0(x)K_1(x) + 2x\, I_0^2(x)K_1^2(x) + x^2\, I_0(x)I_1(x)K_0^2(x)+$$

$$+(x^2 - 4)\, I_0(x)I_1(x)K_1^2(x) - 2x\, I_1^2(x)K_0^2(x) + (x^2 - 4)\, I_1^2(x)K_0(x)K_1(x) + 3x^2\, I_1^2(x)K_1^2(x)\,]$$

$$\int x^4\, J_0^2(x)I_0\left(\sqrt{3}\,x\right)I_1\left(\sqrt{3}\,x\right)\, dx = \frac{x^2}{384}\,\Big[20\sqrt{3}\,x^2\, J_0^2(x)I_0^2\left(\sqrt{3}\,x\right) - 60x\, J_0^2(x)I_0\left(\sqrt{3}\,x\right)I_1\left(\sqrt{3}\,x\right)+$$

$$+3\sqrt{3}\,(12x^2 + 5)\, J_0^2(x)I_1^2\left(\sqrt{3}\,x\right) - 20\sqrt{3}\,x\, J_0(x)J_1(x)I_0^2\left(\sqrt{3}\,x\right)+$$

$$+6(8x^2 + 5)\, J_0(x)J_1(x)I_0\left(\sqrt{3}\,x\right)I_1\left(\sqrt{3}\,x\right) - 12\sqrt{3}\,x\, J_0(x)J_1(x)I_1^2\left(\sqrt{3}\,x\right)-$$

$$-\sqrt{3}\,(4x^2 - 5)\, J_1^2(x)I_0^2\left(\sqrt{3}\,x\right) - 12x\, J_1^2(x)I_0\left(\sqrt{3}\,x\right)I_1\left(\sqrt{3}\,x\right) + 12\sqrt{3}\,x^2\, J_1^2(x)I_1^2\left(\sqrt{3}\,x\right)\Big]$$

$$\int x^4\, J_0^2(x)K_0\left(\sqrt{3}\,x\right)K_1\left(\sqrt{3}\,x\right)\, dx = \frac{x^2}{384}\,\Big[-20\sqrt{3}\,x^2\, J_0^2(x)K_0^2\left(\sqrt{3}\,x\right) - 60x\, J_0^2(x)K_0\left(\sqrt{3}\,x\right)K_1\left(\sqrt{3}\,x\right)-$$

$$-3\sqrt{3}\,(12x^2 + 5)\, J_0^2(x)K_1^2\left(\sqrt{3}\,x\right) + 20\sqrt{3}\,x\, J_0(x)J_1(x)K_0^2\left(\sqrt{3}\,x\right)+$$

$$+6(8x^2 + 5)\, J_0(x)J_1(x)K_0\left(\sqrt{3}\,x\right)K_1\left(\sqrt{3}\,x\right) + 12\sqrt{3}\,x\, J_0(x)J_1(x)K_1^2\left(\sqrt{3}\,x\right)+$$

$$+\sqrt{3}\,(4x^2 - 5)\, J_1^2(x)K_0^2\left(\sqrt{3}\,x\right) - 12x\, J_1^2(x)K_0\left(\sqrt{3}\,x\right)K_1\left(\sqrt{3}\,x\right) - 12\sqrt{3}\,x^2\, J_1^2(x)K_1^2\left(\sqrt{3}\,x\right)\Big]$$

$$\int x^5\, J_0(x)J_1(x)I_0(x)I_1(x)\, dx = \frac{x^2}{8}\,[x^2\, J_0^2(x)I_0^2(x) - x(x^2 + 2)\, J_0^2(x)I_0(x)I_1(x) + (x^2 + 1)\, J_0^2(x)I_1^2(x)+$$

$$+x(x^2 - 2)\, J_0(x)J_1(x)I_0^2(x) + 2J_0(x)J_1(x)I_0(x)I_1(x) + x^3\, J_0(x)J_1(x)I_1^2(x) - (x^2 - 1)\, J_1^2(x)I_0^2(x)+$$

$$+x^3\, J_1^2(x)I_0(x)I_1(x)\,]$$

$$\int x^5\, J_0(x)J_1(x)K_0(x)K_1(x)\, dx = \frac{x^2}{8}\,[-x^2\, J_0^2(x)K_0^2(x) - x(x^2 + 2)\, J_0^2(x)K_0(x)K_1(x) - (x^2 + 1)\, J_0^2(x)K_1^2(x)-$$

$$-x(x^2 - 2)\, J_0(x)J_1(x)K_0^2(x) + 2J_0(x)J_1(x)K_0(x)K_1(x) - x^3\, J_0(x)J_1(x)K_1^2(x) + (x^2 - 1)\, J_1^2(x)K_0^2(x)+$$

$$+x^3\, J_1^2(x)K_0(x)K_1(x)\,]$$

$$\eta \;=\; \frac{\sqrt{30\sqrt{34} - 15}}{15} \;=\; 0.84308\,57524$$

$$\int x^5\, I_1^2(x)\, K_0^2(\eta x)\, dx = -\frac{1}{32}\left(2\,x^2 + 73 + 11\,\sqrt{34}\right) x^4\, I_0^2(x)K_0^2(\eta x)+$$

$$+\frac{1}{1440}\,\sqrt{30\,\sqrt{34} - 15}\left(\sqrt{34} + 5\right)\left(10\,x^2 + 228 + 51\,\sqrt{34}\right) x^3\, I_0^2(x)K_0(\eta x)K_1(\eta x)-$$

$$-\frac{1}{64}\left(-4\,x^4 + 18\,\sqrt{34}x^2 + 114\,x^2 + 801\,\sqrt{34} + 4653\right) x^2\, I_0^2(x)K_1^2(\eta x)+$$

$$+\frac{1}{864}\left(\sqrt{34}+13\right)\left(54\,x^2+1396+227\,\sqrt{34}\right)x^3\,I_0(x)I_1(x)K_0^2(\eta x)-$$

$$-\frac{1}{1440}\,\sqrt{30\,\sqrt{34}-15}\left(8\,\sqrt{34}+49\right)\left(20\,x^2+219+33\,\sqrt{34}\right)x^2\,I_0(x)I_1(x)K_0(\eta x)K_1(\eta x)+$$

$$+\frac{1}{32}\left(\sqrt{34}+3\right)\left(2\,x^2+15\,\sqrt{34}+90\right)x^3\,I_0(x)I_1(x)K_1^2(\eta x)-$$

$$-\frac{1}{64}\left(-4\,x^4+178\,x^2+26\,\sqrt{34}x^2+1123\,\sqrt{34}+6569\right)x^2\,I_1^2(x)K_0^2(\eta x)+$$

$$+\frac{1}{15840}\,\sqrt{30\,\sqrt{34}-15}\left(\sqrt{34}+23\right)\left(22\,x^2+215\,\sqrt{34}+1270\right)x^3\,I_1^2(x)K_0(\eta x)K_1(\eta x)-$$

$$-\frac{1}{32}\left(2\,x^2+59+13\,\sqrt{34}\right)x^4\,I_1^2(x)K_1^2(\eta x)$$

$$\int x^5\,K_1^2(x)\,I_0^2(\eta x)\,dx=-\frac{1}{32}\left(2\,x^2+73+11\,\sqrt{34}\right)x^4\,K_0^2(x)I_0^2(\eta x)-$$

$$-\frac{1}{1440}\,\sqrt{30\,\sqrt{34}-15}\left(\sqrt{34}+5\right)\left(10\,x^2+228+51\,\sqrt{34}\right)x^3\,K_0^2(x)I_0(\eta x)I_1(\eta x)-$$

$$-\frac{1}{64}\left(-4\,x^4+18\,\sqrt{34}x^2+114\,x^2+801\,\sqrt{34}+4653\right)x^2\,K_0^2(x)I_1^2(\eta x)-$$

$$-\frac{1}{864}\left(\sqrt{34}+13\right)\left(54\,x^2+1396+227\,\sqrt{34}\right)x^3\,K_0(x)K_1(x)I_0^2(\eta x)-$$

$$-\frac{1}{1440}\,\sqrt{30\,\sqrt{34}-15}\left(8\,\sqrt{34}+49\right)\left(20\,x^2+219+33\,\sqrt{34}\right)x^2\,K_0(x)K_1(x)I_0(\eta x)I_1(\eta x)-$$

$$-\frac{1}{32}\left(\sqrt{34}+3\right)\left(2\,x^2+15\,\sqrt{34}+90\right)x^3\,K_0(x)K_1(x)I_1^2(\eta x)-$$

$$-\frac{1}{64}\left(-4\,x^4+178\,x^2+26\,\sqrt{34}x^2+1123\,\sqrt{34}+6569\right)x^2\,K_1^2(x)I_0^2(\eta x)-$$

$$-\frac{1}{15840}\,\sqrt{30\,\sqrt{34}-15}\left(\sqrt{34}+23\right)\left(22\,x^2+215\,\sqrt{34}+1270\right)x^3\,K_1^2(x)I_0(\eta x)I_1(\eta x)-$$

$$-\frac{1}{32}\left(2\,x^2+59+13\,\sqrt{34}\right)x^4\,K_1^2(x)I_1^2(\eta x)$$

$$\zeta=\frac{\sqrt{2\sqrt{34}+1}}{3}=1.18611\,89649$$

$$\int x^5\,I_0^2(x)\,K_1^2(\zeta x)\,dx=-\frac{1}{32}\left(2\,x^2+9\,\sqrt{34}+45\right)x^4\,I_0^2(x)K_0^2(\zeta x)-$$

$$-\frac{1}{864}\,\sqrt{1+2\,\sqrt{34}}\left(5\,\sqrt{34}+11\right)\left(6\,x^2+19\,\sqrt{34}+104\right)x^3\,I_0^2(x)K_0(\zeta x)K_1(\zeta x)-$$

$$-\frac{1}{64}\left(-4\,x^4+22\,\sqrt{34}x^2+106\,x^2+3321+567\,\sqrt{34}\right)x^2\,I_0^2(x)K_1^2(\zeta x)-$$

$$-\frac{1}{160}\left(7+\sqrt{34}\right)\left(10\,x^2+27\,\sqrt{34}+216\right)x^3\,I_0(x)I_1(x)K_0^2(\zeta x)-$$

$$-\frac{1}{288}\,\sqrt{1+2\,\sqrt{34}}\left(4\,\sqrt{34}+25\right)\left(20\,x^2+27\,\sqrt{34}+135\right)x^2\,I_0(x)I_1(x)K_0(\zeta x)K_1(\zeta x)-$$

$$-\frac{1}{352}\left(1+\sqrt{34}\right)\left(22\,x^2+155\,\sqrt{34}+890\right)x^3\,I_0(x)I_1(x)K_1^2(\zeta x)-$$

$$-\frac{1}{64}\left(-4\,x^4+74\,x^2+14\,\sqrt{34}x^2+2349+405\,\sqrt{34}\right)x^2\,I_1^2(x)K_0^2(\zeta x)-$$

$$-\frac{1}{288}\,\sqrt{1+2\,\sqrt{34}}\left(\sqrt{34}+13\right)\left(2\,x^2+11\,\sqrt{34}+62\right)x^3\,I_1^2(x)K_0(\zeta x)K_1(\zeta x)-$$

$$-\frac{1}{32}\left(2\,x^2+55+7\,\sqrt{34}\right)x^4\,I_1^2(x)K_1^2(\zeta x)$$

$$\int x^5\, K_0^2(x)\, I_1^2(\zeta x)\, dx = -\frac{1}{32}\left(2\,x^2 + 9\,\sqrt{34} + 45\right) x^4\, K_0^2(x)I_0^2(\zeta x)+$$

$$+\frac{1}{864}\sqrt{1+2\sqrt{34}}\left(5\sqrt{34}+11\right)\left(6\,x^2 + 19\sqrt{34} + 104\right) x^3\, K_0^2(x)I_0(\zeta x)I_1(\zeta x)-$$

$$-\frac{1}{64}\left(-4\,x^4 + 22\sqrt{34}x^2 + 106\,x^2 + 3321 + 567\sqrt{34}\right) x^2\, K_0^2(x)I_1^2(\zeta x)+$$

$$+\frac{1}{160}\left(7+\sqrt{34}\right)\left(10\,x^2 + 27\sqrt{34} + 216\right) x^3\, K_0(x)K_1(x)I_0^2(\zeta x)-$$

$$-\frac{1}{288}\sqrt{1+2\sqrt{34}}\left(4\sqrt{34}+25\right)\left(20\,x^2 + 27\sqrt{34} + 135\right) x^2\, K_0(x)K_1(x)I_0(\zeta x)I_1(\zeta x)+$$

$$+\frac{1}{352}\left(1+\sqrt{34}\right)\left(22\,x^2 + 155\sqrt{34} + 890\right) x^3\, K_0(x)K_1(x)I_1^2(\zeta x)-$$

$$-\frac{1}{64}\left(-4\,x^4 + 74\,x^2 + 14\sqrt{34}x^2 + 2349 + 405\sqrt{34}\right) x^2\, K_1^2(x)I_0^2(\zeta x)+$$

$$+\frac{1}{288}\sqrt{1+2\sqrt{34}}\left(\sqrt{34}+13\right)\left(2\,x^2 + 11\sqrt{34} + 62\right) x^3\, K_1^2(x)I_0(\zeta x)I_1(\zeta x)-$$

$$-\frac{1}{32}\left(2\,x^2 + 55 + 7\sqrt{34}\right) x^4\, K_1^2(x)I_1^2(\zeta x)$$

$$\lambda \;=\; \frac{\sqrt{10\sqrt{19}-5}}{5} \;=\; 1.24240\,07314$$

$$\int x^6\, J_1^2(x)\, J_0(\lambda x)\, J_1(\lambda x)\, dx = \frac{\sqrt{10\sqrt{19}-5}\,\left(3\sqrt{19}+14\right) x^4\left(2\,x^2 - 18\sqrt{19} - 39\right)}{1600}\, J_0^2(x)J_0^2(\lambda x)-$$

$$-\frac{\left(14\sqrt{19}+67\right) x^3\left(102\,x^2 - 1357 - 349\sqrt{19}\right)}{3264}\, J_0^2(x)J_0(\lambda x)J_1(\lambda x)-$$

$$-\frac{\sqrt{10\sqrt{19}-5}\,\left(\sqrt{19}+8\right) x^2\left(12\,x^4 - 124\,x^2\sqrt{19} - 502\,x^2 + 9696 + 2217\sqrt{19}\right)}{5760}\, J_0^2(x)J_1^2(\lambda x)+$$

$$+\frac{\sqrt{10\sqrt{19}-5}\,\left(52\sqrt{19}+251\right) x^3\left(62\,x^2 - 167\sqrt{19} - 791\right)}{49600}\, J_0(x)J_1(x)J_0^2(\lambda x)+$$

$$+\frac{\left(3+\sqrt{19}\right) x^2\left(16\,x^4 - 100\,x^2\sqrt{19} - 420\,x^2 + 3603 + 831\sqrt{19}\right)}{128}\, J_0(x)J_1(x)J_0(\lambda x)J_1(\lambda x)-$$

$$-\frac{\sqrt{10\sqrt{19}-5}\,\left(38\sqrt{19}+169\right) x^3\left(6\,x^2 - 73 - 19\sqrt{19}\right)}{2880}\, J_0(x)J_1(x)J_1^2(\lambda x)-$$

$$-\frac{\sqrt{10\sqrt{19}-5}\,\left(7\sqrt{19}+16\right)\left(108\,x^4 - 356\,x^2\sqrt{19} - 1678\,x^2 + 5727\sqrt{19} + 25026\right) x^2}{86400}\, J_1^2(x)J_1^2(\lambda x)+$$

$$+\frac{\left(16\sqrt{19}+63\right) x^3\left(358\,x^2 - 885\sqrt{19} - 3955\right)}{11456}\, J_1^2(x)J_0(\lambda x)J_1(\lambda x)+$$

$$+\frac{\sqrt{10\sqrt{19}-5}\,\left(\sqrt{19}+2\right) x^4\left(10\,x^2 - 36\sqrt{19} - 143\right)}{960}\, J_1^2(x)J_1^2(\lambda x)$$

$$\int x^6\, I_1^2(x)\, K_0(\lambda x)\, K_1(\lambda x)\, dx = -\frac{1}{1600}\sqrt{10\sqrt{19}-5}\,\left(3\sqrt{19}+14\right)\left(2\,x^2 + 18\sqrt{19} + 39\right) x^4\, I_0^2(x)K_0^2(\lambda x)-$$

$$-\frac{1}{3264}\left(14\sqrt{19}+67\right)\left(102\,x^2 + 349\sqrt{19} + 1357\right) x^3\, I_0^2(x)K_0(\lambda x)K_1(\lambda x)-$$

$$-\frac{1}{5760}\sqrt{10\sqrt{19}-5}\,\left(\sqrt{19}+8\right)\left(12\,x^4 + 124\sqrt{19}x^2 + 502\,x^2 + 2217\sqrt{19} + 9696\right) x^2\, I_0^2(x)K_1^2(\lambda x)-$$

$$-\frac{1}{49600}\sqrt{10\sqrt{19}-5}\,\left(251+52\sqrt{19}\right)\left(62\,x^2 + 791 + 167\sqrt{19}\right) x^3\, I_0(x)I_1(x)K_0^2(\lambda x)-$$

$$-\frac{1}{128}\left(3+\sqrt{19}\right)\left(16\,x^4 + 420\,x^2 + 100\sqrt{19}x^2 + 3603 + 831\sqrt{19}\right) x^2\, I_0(x)I_1(x)K_0(\lambda x)K_1(\lambda x)-$$

$$-\frac{1}{2880}\sqrt{10\sqrt{19}-5}\left(38\sqrt{19}+169\right)\left(6x^2+19\sqrt{19}+73\right)x^3\,I_0(x)I_1(x)K_1^2(\lambda x)-$$

$$-\frac{1}{86400}\sqrt{10\sqrt{19}-5}\left(7\sqrt{19}+16\right)\left(108x^4+356\sqrt{19}x^2+1678x^2+5727\sqrt{19}+25026\right)x^2\,I_1^2(x)K_0^2(\lambda x)-$$

$$-\frac{1}{11456}\left(16\sqrt{19}+63\right)\left(358x^2+3955+885\sqrt{19}\right)x^3\,I_1^2(x)K_0(\lambda x)K_1(\lambda x)-$$

$$-\frac{1}{960}\sqrt{10\sqrt{19}-5}\left(\sqrt{19}+2\right)\left(10x^2+36\sqrt{19}+143\right)x^4\,I_1^2(x)K_1^2(\lambda x)$$

$$\int x^6\,K_1^2(x)\,I_0(\lambda x)\,I_1(\lambda x)\,dx=\frac{1}{1600}\sqrt{10\sqrt{19}-5}\left(3\sqrt{19}+14\right)\left(2x^2+18\sqrt{19}+39\right)x^4\,K_0^2(x)I_0^2(\lambda x)-$$

$$-\frac{1}{3264}\left(14\sqrt{19}+67\right)\left(102x^2+349\sqrt{19}+1357\right)x^3\,K_0^2(x)I_0(\lambda x)I_1(\lambda x)+$$

$$+\frac{1}{5760}\sqrt{10\sqrt{19}-5}\left(\sqrt{19}+8\right)\left(12x^4+124\sqrt{19}x^2+502x^2+2217\sqrt{19}+9696\right)x^2\,K_0^2(x)I_1^2(\lambda x)-$$

$$-\frac{1}{49600}\sqrt{10\sqrt{19}-5}\left(251+52\sqrt{19}\right)\left(62x^2+791+167\sqrt{19}\right)x^3\,K_0(x)K_1(x)I_0^2(\lambda x)+$$

$$+\frac{1}{128}\left(3+\sqrt{19}\right)\left(16x^4+420x^2+100\sqrt{19}x^2+3603+831\sqrt{19}\right)x^2\,K_0(x)K_1(x)I_0(\lambda x)I_1(\lambda x)-$$

$$-\frac{1}{2880}\sqrt{10\sqrt{19}-5}\left(38\sqrt{19}+169\right)\left(6x^2+19\sqrt{19}+73\right)x^3\,K_0(x)K_1(x)I_1^2(\lambda x)+$$

$$+\frac{1}{86400}\sqrt{10\sqrt{19}-5}\left(7\sqrt{19}+16\right)\left(108x^4+356\sqrt{19}x^2+1678x^2+5727\sqrt{19}+25026\right)x^2\,K_1^2(x)I_0^2(\lambda x)-$$

$$-\frac{1}{11456}\left(16\sqrt{19}+63\right)\left(358x^2+3955+885\sqrt{19}\right)x^3\,K_1^2(x)I_0(\lambda x)I_1(\lambda x)+$$

$$+\frac{1}{960}\sqrt{10\sqrt{19}-5}\left(\sqrt{19}+2\right)\left(10x^2+36\sqrt{19}+143\right)x^4\,K_1^2(x)I_1^2(\lambda x)$$

$$\mu\;=\;\frac{\sqrt{10\sqrt{19}+5}}{5}\;=\;1.39411\,60559$$

$$\int x^6\,J_1^2(x)\,I_0(\mu x)\,I_1(\mu x)\,dx=-\frac{\sqrt{10\sqrt{19}+5}\left(3\sqrt{19}-14\right)\left(2x^2+18\sqrt{19}-39\right)x^4}{1600}J_0^2(x)I_0^2(\mu x)+$$

$$+\frac{\left(14\sqrt{19}-67\right)\left(102x^2-1357+349\sqrt{19}\right)x^3}{3264}J_0^2(x)I_0(\mu x)I_1(\mu x)-$$

$$-\frac{\sqrt{10\sqrt{19}+5}\left(\sqrt{19}-8\right)\left(12x^4-502x^2+124\sqrt{19}x^2+9696-2217\sqrt{19}\right)x^2}{5760}J_0^2(x)I_1^2(\mu x)-$$

$$-\frac{\sqrt{10\sqrt{19}+5}\left(52\sqrt{19}-251\right)\left(62x^2+167\sqrt{19}-791\right)x^3}{49600}J_0(x)J_1(x)I_0^2(\mu x)-$$

$$-\frac{\left(\sqrt{19}-3\right)\left(16x^4+100\sqrt{19}x^2-420x^2+3603-831\sqrt{19}\right)x^2}{128}J_0(x)J_1(x)I_0(\mu x)I_1(\mu x)-$$

$$-\frac{\sqrt{10\sqrt{19}+5}\left(38\sqrt{19}-169\right)\left(6x^2+19\sqrt{19}-73\right)x^3}{2880}J_0(x)J_1(x)I_1^2(\mu x)+$$

$$+\frac{\sqrt{10\sqrt{19}+5}\left(7\sqrt{19}-16\right)\left(108x^4+356\sqrt{19}x^2-1678x^2+25026-5727\sqrt{19}\right)x^2}{86400}J_1^2(x)I_0^2(\mu x)-$$

$$-\frac{\left(16\sqrt{19}-63\right)\left(358x^2-3955+885\sqrt{19}\right)x^3}{11456}J_1^2(x)I_0(\mu x)I_1(\mu x)+$$

$$+\frac{\sqrt{10\sqrt{19}+5}\left(\sqrt{19}-2\right)\left(10x^2-143+36\sqrt{19}\right)x^4}{960}J_1^2(x)I_1^2(\mu x)$$

$$\int x^6\, J_1^2(x)\, K_0(\mu x)\, K_1(\mu x)\, dx = \frac{\sqrt{10\sqrt{19}+5}\,\left(3\sqrt{19}-14\right)\left(2\,x^2+18\sqrt{19}-39\right)x^4}{1600} J_0^2(x)K_0^2(x)+$$

$$+\frac{\left(14\sqrt{19}-67\right)\left(102\,x^2-1357+349\sqrt{19}\right)x^3}{3264} J_0^2(x)K_0(x)K_1(x)+$$

$$+\frac{\sqrt{10\sqrt{19}+5}\,\left(\sqrt{19}-8\right)\left(12\,x^4-502\,x^2+124\sqrt{19}x^2+9696-2217\sqrt{19}\right)x^2}{5760} J_0^2(x)K_1^2(x)+$$

$$+\frac{\sqrt{10\sqrt{19}+5}\,\left(52\sqrt{19}-251\right)\left(62\,x^2+167\sqrt{19}-791\right)x^3}{49600} J_0(x)J_1(x)K_0^2(x)-$$

$$-\frac{\left(\sqrt{19}-3\right)\left(16\,x^4+100\sqrt{19}x^2-420\,x^2+3603-831\sqrt{19}\right)x^2}{128} J_0(x)J_1(x)K_0(x)K_1(x)+$$

$$+\frac{\sqrt{10\sqrt{19}+5}\,\left(38\sqrt{19}-169\right)\left(6\,x^2+19\sqrt{19}-73\right)x^3}{2880} J_0(x)J_1(x)K_1^2(x)-$$

$$-\frac{\sqrt{10\sqrt{19}+5}\,\left(7\sqrt{19}-16\right)\left(108\,x^4+356\sqrt{19}x^2-1678\,x^2+25026-5727\sqrt{19}\right)x^2}{86400} J_1^2(x)K_0^2(x)-$$

$$-\frac{\left(16\sqrt{19}-63\right)\left(358\,x^2-3955+885\sqrt{19}\right)x^3}{11456} J_1^2(x)K_0(x)K_1(x)-$$

$$-\frac{\sqrt{10\sqrt{19}+5}\,\left(\sqrt{19}-2\right)\left(10\,x^2-143+36\sqrt{19}\right)x^4}{960} J_1^2(x)K_1^2(x)$$

$$\nu = \frac{\sqrt{30\sqrt{19}+15}}{15} = 0.80489\,32802$$

$$\int x^6\, J_0(x)\, J_1(x)\, J_1^2(\nu x)\, dx = \frac{\left(\sqrt{19}+4\right)\left(2\,x^2-129-12\sqrt{19}\right)x^4}{64} J_0^2(x)J_0^2(\nu x)+$$

$$+\frac{\sqrt{15+30\sqrt{19}}\,\left(41+8\sqrt{19}\right)\left(62\,x^2-1111-283\sqrt{19}\right)x^3}{9920} J_0^2(x)J_0(\nu x)J_1(\nu x)-$$

$$-\frac{\left(10+\sqrt{19}\right)\left(36\,x^4-790\,x^2-200\sqrt{19}x^2+4545\sqrt{19}+19890\right)x^2}{1152} J_0^2(x)J_1^2(\nu x)-$$

$$-\frac{\left(31+8\sqrt{19}\right)\left(102\,x^2-2381-473\sqrt{19}\right)x^3}{1088} J_0(x)J_1(x)J_1^2(\nu x)+$$

$$+\frac{\sqrt{15+30\sqrt{19}}\,\left(7+\sqrt{19}\right)\left(16\,x^4-148\sqrt{19}x^2-676\,x^2+1983\sqrt{19}+8571\right)x^2}{1920} J_0(x)J_1(x)J_0(\nu x)J_1(\nu x)+$$

$$+\frac{\left(22\sqrt{19}+109\right)\left(358\,x^2-5935-1405\sqrt{19}\right)x^3}{11456} J_0(x)J_1(x)J_1^2(\nu x)-$$

$$-\frac{\left(\sqrt{19}+2\right)\left(12\,x^4-176\sqrt{19}x^2-842\,x^2+5277\sqrt{19}+23124\right)x^2}{384} J_1^2(x)J_0^2(\nu x)-$$

$$-\frac{\sqrt{15+30\sqrt{19}}\,\left(9+2\sqrt{19}\right)\left(30\,x^2-649-127\sqrt{19}\right)x^3}{960} J_1^2(x)J_0(\nu x)J_1(\nu x)+$$

$$+\frac{\left(\sqrt{19}+12\right)\left(2\,x^2-10\sqrt{19}-49\right)x^4}{64} J_1^2(x)J_1^2(\nu x)$$

$$\int x^6\, I_0(x)\, I_1(x)\, K_1^2(\nu x)\, dx = \frac{1}{64}\left(\sqrt{19}+4\right)\left(2\,x^2+129+12\sqrt{19}\right)x^4\, I_0^2(x)K_0^2(\nu x)-$$

$$-\frac{1}{9920}\sqrt{15+30\sqrt{19}}\left(41+8\sqrt{19}\right)\left(62\,x^2+283\sqrt{19}+1111\right)x^3\, I_0^2(x)K_0(\nu x)K_1(\nu x)+$$

$$+\frac{1}{1152}\left(\sqrt{19}+10\right)\left(36\,x^4+790\,x^2+200\sqrt{19}x^2+4545\sqrt{19}+19890\right)x^2\, I_0^2(x)K_1^2(\nu x)-$$

$$-\frac{1}{1088}\left(31+8\sqrt{19}\right)\left(102\,x^2+2381+473\sqrt{19}\right)x^3\, I_0(x)I_1(x)K_0^2(\nu x)+$$

$$+\frac{\sqrt{19}+7}{1920}\sqrt{15+30\sqrt{19}}\left(16\,x^4+676\,x^2+148\,\sqrt{19}x^2+1983\,\sqrt{19}+8571\right)x^2\,I_0(x)I_1(x)K_0(\nu x)K_1(\nu x)-$$

$$-\frac{1}{11456}\left(22\,\sqrt{19}+109\right)\left(358\,x^2+5935+1405\,\sqrt{19}\right)x^3\,I_0(x)I_1(x)K_1^2(\nu x)+$$

$$+\frac{\sqrt{19}+2}{384}\left(12\,x^4+176\,\sqrt{19}x^2+842\,x^2+5277\,\sqrt{19}+23124\right)x^2\,I_1^2(x)K_0^2(\nu x)-$$

$$-\frac{1}{960}\sqrt{15+30\sqrt{19}}\left(9+2\,\sqrt{19}\right)\left(30\,x^2+127\,\sqrt{19}+649\right)x^3\,I_1^2(x)K_0(\nu x)K_1(\nu x)+$$

$$+\frac{1}{64}\left(\sqrt{19}+12\right)\left(2\,x^2+10\,\sqrt{19}+49\right)x^4\,I_1^2(x)K_1^2(\nu x)$$

$$\xi\;=\;\frac{\sqrt{30\sqrt{19}-15}}{15}\;=\;0.71730\,03967$$

$$\int x^6\,J_0(x)\,J_1(x)\,I_1^2(\xi x)\,dx=\frac{\left(\sqrt{19}-4\right)\left(2\,x^2-129+12\,\sqrt{19}\right)x^4}{64}\,J_0^2(x)I_0^2(\xi x)+$$

$$+\frac{\sqrt{30\,\sqrt{19}-15}\left(41-8\,\sqrt{19}\right)\left(62\,x^2+283\,\sqrt{19}-1111\right)x^3}{9920}\,J_0^2(x)I_0(\xi x)I_1(\xi x)+$$

$$+\frac{\left(\sqrt{19}-10\right)\left(36\,x^4-790\,x^2+200\,\sqrt{19}x^2+19890-4545\,\sqrt{19}\right)x^2}{1152}\,J_0^2(x)I_1^2(\xi x)+$$

$$+\frac{\left(31-8\,\sqrt{19}\right)\left(102\,x^2-2381+473\,\sqrt{19}\right)x^3}{1088}\,J_0(x)J_1(x)I_0^2(\xi x)+$$

$$+\frac{\sqrt{30\,\sqrt{19}-15}\left(7-\sqrt{19}\right)\left(16\,x^4+148\,\sqrt{19}x^2-676\,x^2+8571-1983\,\sqrt{19}\right)x^2}{1920}\,J_0(x)J_1(x)I_0(\xi x)I_1(\xi x)+$$

$$+\frac{\left(109-22\,\sqrt{19}\right)\left(358\,x^2-5935+1405\,\sqrt{19}\right)x^3}{11456}\,J_0(x)J_1(x)I_1^2(\xi x)-$$

$$-\frac{\left(\sqrt{19}-2\right)\left(12\,x^4+176\,\sqrt{19}x^2-842\,x^2-5277\,\sqrt{19}+23124\right)x^2}{384}\,J_1^2(x)I_0^2(\xi x)-$$

$$-\frac{\sqrt{30\,\sqrt{19}-15}\left(9-2\,\sqrt{19}\right)\left(30\,x^2+127\,\sqrt{19}-649\right)x^3}{960}\,J_1^2(x)I_0(\xi x)I_1(\xi x)-$$

$$-\frac{\left(\sqrt{19}-12\right)\left(2\,x^2+10\,\sqrt{19}-49\right)x^4}{64}\,J_1^2(x)I_1^2(\xi x)$$

$$\int x^6\,J_0(x)\,J_1(x)\,K_1^2(\xi x)\,dx=\frac{\left(\sqrt{19}-4\right)\left(2\,x^2-129+12\,\sqrt{19}\right)x^4}{64}\,J_0^2(x)K_1^2(\xi x)-$$

$$-\frac{\sqrt{30\,\sqrt{19}-15}\left(41-8\,\sqrt{19}\right)\left(62\,x^2+283\,\sqrt{19}-1111\right)x^3}{9920}\,J_0^2(x)K_0(\xi x)K_1(\xi x)+$$

$$+\frac{\left(\sqrt{19}-10\right)\left(36\,x^4-790\,x^2+200\,\sqrt{19}x^2+19890-4545\,\sqrt{19}\right)x^2}{1152}\,J_0^2(x)K_1^2(\xi x)+$$

$$+\frac{\left(31-8\,\sqrt{19}\right)\left(102\,x^2-2381+473\,\sqrt{19}\right)x^3}{1088}\,J_0(x)J_1(x)K_0^2(\xi x)-$$

$$-\frac{\sqrt{30\,\sqrt{19}-15}\left(7-\sqrt{19}\right)\left(16\,x^4-676\,x^2+148\,\sqrt{19}x^2-1983\,\sqrt{19}+8571\right)x^2}{1920}\,J_0(x)J_1(x)K_0(\xi x)K_1(\xi x)+$$

$$+\frac{\left(109-22\,\sqrt{19}\right)\left(358\,x^2-5935+1405\,\sqrt{19}\right)x^3}{11456}\,J_0(x)J_1(x)K_1^2(\xi x)-$$

$$-\frac{\left(\sqrt{19}-2\right)\left(12\,x^4+176\,\sqrt{19}x^2-842\,x^2-5277\,\sqrt{19}+23124\right)x^2}{384}\,J_1^2(x)K_0^2(\xi x)+$$

$$+\frac{\sqrt{30\,\sqrt{19}-15}\left(9-2\,\sqrt{19}\right)\left(30\,x^2+127\,\sqrt{19}-649\right)x^3}{960}\,J_1^2(x)K_0(\xi x)K_1(\xi x)-$$

$$-\frac{\left(\sqrt{19}-12\right)\left(2\,x^2+10\,\sqrt{19}-49\right)x^4}{64}\,J_1^2(x)K_1^2(\xi x)$$

5. Quotients

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$.

Integrals are omitted, when $f(x)$ turns out to be of a very special kind or when the antiderivative is expressed by Whittaker or other hypergeometric functions.

5.1. Denominator $p(x)\, Z_0(x) + q(x)\, Z_1(x)$

a) Typ $f(x)\, Z_\nu(x) / [p(x)\, Z_0(x) + q(x)\, Z_1(x)]$:

$$\int \frac{J_1(x)\, dx}{J_0(x)} = -\ln|J_0(x)| \,, \quad \int \frac{I_1(x)\, dx}{I_0(x)} = \ln|I_0(x)| \,, \quad \int \frac{K_1(x)\, dx}{K_0(x)} = -\ln|K_0(x)|$$

$$\int \frac{J_0(x)\, dx}{J_1(x)} = \ln|x\, J_1(x)| \,, \quad \int \frac{I_0(x)\, dx}{I_1(x)} = \ln|x\, I_1(x)| \,, \quad \int \frac{K_0(x)\, dx}{K_1(x)} = -\ln|x\, K_1(x)|$$

$$\int \frac{(x^2 + a^2 - 2a)\, J_1(x)\, dx}{x[x\, J_0(x) - a\, J_1(x)]} = -\ln|x^a\, J_0(x) - ax^{a-1}\, J_1(x)|$$

$$\int \frac{(x^2 - a^2 + 2a)\, I_1(x)\, dx}{x[x\, I_0(x) - a\, I_1(x)]} = \ln|x^a\, I_0(x) - ax^{a-1}\, I_1(x)|$$

$$\int \frac{(x^2 - a^2 + 2a)\, K_1(x)\, dx}{x[x\, K_0(x) + a\, K_1(x)]} = -\ln|x^a\, K_0(x) + ax^{a-1}\, K_1(x)|$$

$$\int \frac{\cos x\, J_1(x)\, dx}{x[\sin x\, J_0(x) - \cos x\, J_1(x)]} = \ln|\sin x\, J_0(x) - \cos x\, J_1(x)|$$

$$\int \frac{(2x\,\sin x + \cos x)\, I_1(x)\, dx}{x[\sin x\, I_0(x) - \cos x\, I_1(x)]} = \ln|\sin x\, I_0(x) - \cos x\, I_1(x)|$$

$$\int \frac{(2x\,\sin x + \cos x)\, K_1(x)\, dx}{x[\sin x\, K_0(x) + \cos x\, K_1(x)]} = -\ln|\sin x\, K_0(x) + \cos x\, K_1(x)|$$

$$\int \frac{\sin x\, J_1(x)\, dx}{x[\cos x\, J_0(x) + \sin x\, J_1(x)]} = -\ln|\cos x\, J_0(x) + \sin x\, J_1(x)|$$

$$\int \frac{(2x\,\cos x - \sin x)\, I_1(x)\, dx}{x[\cos x\, I_0(x) + \sin x\, I_1(x)]} = \ln|\cos\, I_0(x) + \sin x\, I_1(x)|$$

$$\int \frac{(2x\,\cos x - \sin x)\, K_1(x)\, dx}{x[\cos x\, K_0(x) - \sin x\, K_1(x)]} = -\ln|\cos\, K_0(x) - \sin x\, K_1(x)|$$

5.2. Denominator $[p(x)\, Z_0(x) + q(x)\, Z_1(x)]^2$

a) Typ $f(x)\, Z_\mu(x)/[p(x)\, Z_0(x) + q(x)\, Z_1(x)]^2$:

$$\int \frac{[(a^2 + b^2)x + ab]\exp(-\frac{ax}{b})\, J_0(x)\, dx}{x^2\, [a\, J_0(x) + b\, J_1(x)]^2} = -\frac{b\exp(-\frac{ax}{b})}{x\, [a\, J_0(x) + b\, J_1(x)]}$$

$$\int \frac{[(a^2 - b^2)x - ab]\exp(\frac{ax}{b})\, I_0(x)\, dx}{x^2\, [a\, I_0(x) + b\, I_1(x)]^2} = \frac{b\exp(\frac{ax}{b})}{x\, [a\, I_0(x) + b\, I_1(x)]}$$

$$\int \frac{[(a^2 - b^2)x + ab]\exp(-\frac{ax}{b})\, K_0(x)\, dx}{x^2\, [a\, K_0(x) + b\, K_1(x)]^2} = -\frac{b\exp(-\frac{ax}{b})}{x\, [a\, I_0(x) + b\, I_1(x)]}$$

$$\int \frac{[(a^2 + b^2)x + ab]\exp(\frac{bx}{a})\, J_1(x)\, dx}{x\, [a\, J_0(x) + b\, J_1(x)]^2} = \frac{a\exp(\frac{bx}{a})}{a\, J_0(x) + b\, J_1(x)}$$

$$\int \frac{[(a^2 - b^2)x - ab]\exp(\frac{bx}{a})\, I_1(x)\, dx}{x\, [a\, I_0(x) + b\, I_I(x)]^2} = -\frac{a\exp(\frac{bx}{a})}{a\, I_0(x) + b\, I_1(x)}$$

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$$\int \frac{[(a^2 - b^2)x + ab] \exp(-\frac{bx}{a}) \, K_1(x) \, dx}{x \, [a \, K_0(x) + b \, K_I(x)]^2} = \frac{a \exp(-\frac{bx}{a})}{a \, K_0(x) + b \, K_1(x)}$$

$$\int \frac{[a^2 x^2 + b^2 + 2ab] \exp(-\frac{ax^2}{2b}) \, J_0(x) \, dx}{x[ax \, J_0(x) + b \, J_1(x)]^2} = -\frac{b \exp(\frac{-ax^2}{2b})}{x \, [ax \, J_0(x) + b \, J_1(x)]}$$

$$\int \frac{[a^2 x^2 - b^2 - 2ab] \exp(\frac{ax^2}{2b}) \, I_0(x) \, dx}{x[ax \, I_0(x) + b \, I_1(x)]^2} = \frac{b \exp(\frac{ax^2}{2b})}{x \, [ax \, I_0(x) + b \, I_1(x)]}$$

$$\int \frac{[a^2 x^2 - b^2 + 2ab] \exp(-\frac{ax^2}{2b}) \, K_0(x) \, dx}{x[ax \, K_0(x) + b \, K_1(x)]^2} = -\frac{b \exp(\frac{-ax^2}{2b})}{x \, [ax \, K_0(x) + b \, K_1(x)]}$$

$$\int \frac{x^{b/a} \, [a^2 x^2 + b^2 + 2ab] \, J_1(x) \, dx}{[ax \, J_0(x) + b \, J_1(x)]^2} = \frac{a \, x^{1+b/a}}{ax \, J_0(x) + b \, J_1(x)}$$

$$\int \frac{x^{b/a} \, [a^2 x^2 - b^2 - 2ab] \, I_1(x) \, dx}{[ax \, I_0(x) + b \, I_1(x)]^2} = -\frac{a \, x^{1+b/a}}{ax \, I_0(x) + b \, I_1(x)}$$

$$\int \frac{x^{-b/a} \, [a^2 x^2 - b^2 + 2ab] \, K_1(x) \, dx}{[ax \, K_0(x) + b \, K_1(x)]^2} = \frac{a \, x^{1-b/a}}{ax \, K_0(x) + b \, K_1(x)}$$

$$\int \frac{x^{-a/b} \, [a^2 + b^2 x^2] \, J_0(x) \, dx}{x \, [a \, J_0(x) + bx \, J_1(x)]^2} = -\frac{b \, x^{-a/b}}{a \, J_0(x) + bx \, J_1(x)}$$

$$\int \frac{x^{a/b} \, [a^2 - b^2 x^2] \, I_0(x) \, dx}{x \, [a \, I_0(x) + bx \, I_1(x)]^2} = \frac{b \, x^{a/b}}{a \, I_0(x) + bx \, I_1(x)}$$

$$\int \frac{x^{-a/b} \, [a^2 - b^2 x^2] \, K_0(x) \, dx}{x \, [a \, K_0(x) + bx \, K_1(x)]^2} = -\frac{b \, x^{-a/b}}{a \, K_0(x) + bx \, K_1(x)}$$

$$\int \frac{[a^2 + b^2 x^2] \exp(\frac{bx^2}{2a}) \, J_1(x) \, dx}{[a \, J_0(x) + bx \, J_1(x)]^2} = \frac{a \exp(\frac{bx^2}{2a})}{a \, J_0(x) + bx \, J_1(x)}$$

$$\int \frac{[b^2 x^2 - a^2] \exp(\frac{bx^2}{2a}) \, I_1(x) \, dx}{[a \, I_0(x) + bx \, I_1(x)]^2} = \frac{a \exp(\frac{bx^2}{2a})}{a \, I_0(x) + bx \, I_1(x)}$$

$$\int \frac{[b^2 x^2 - a^2] \exp(-\frac{bx^2}{2a}) \, K_1(x) \, dx}{[a \, K_0(x) + bx \, K_1(x)]^2} = -\frac{a \exp(-\frac{bx^2}{2a})}{a \, K_0(x) + bx \, K_1(x)}$$

$$\int \frac{[(a^2 + c^2)x^3 + (2ab + ac + 2cd)x^2 + (b^2 + 2ad + d^2)x + bd] \, (cx + d)^{(da-cb)/c^2} \exp(-\frac{ax}{c}) \, J_0(x) \, dx}{x^2 \, [(ax + b) \, J_0(x) + (cx + d) \, J_1(x)]^2} =$$

$$= -\frac{(cx + d)^{1+(da-cb)/c^2} \exp(-\frac{ax}{c})}{x \, [(ax + b) \, J_0(x) + (cx + d) \, J_1(x)]}$$

$$\int \frac{[(a^2 - c^2)x^3 + (2ab - ac - 2cd)x^2 + (b^2 - 2ad - d^2)x - bd] \, (cx + d)^{(cb-da)/c^2} \exp(\frac{ax}{c}) \, I_0(x) \, dx}{x^2 \, [(ax + b) \, I_0(x) + (cx + d) \, I_1(x)]^2} =$$

$$= \frac{(cx + d)^{1+(cb-da)/c^2} \exp(\frac{ax}{c})}{x \, [(ax + b) \, I_0(x) + (cx + d) \, I_1(x)]}$$

$$\int \frac{[(a^2 - c^2)x^3 + (2ab + ac - 2cd)x^2 + (b^2 + 2ad - d^2)x + bd] \, (cx + d)^{(da-cb)/c^2} \exp(-\frac{ax}{c}) \, K_0(x) \, dx}{x^2 \, [(ax + b) \, K_0(x) + (cx + d) \, K_1(x)]^2} =$$

$$= -\frac{(cx + d)^{1+(da-cb)/c^2} \exp(-\frac{ax}{c})}{x \, [(ax + b) \, K_0(x) + (cx + d) \, K_1(x)]}$$

$$\int \frac{[(a^2+c^2)x^3 + (2ab+ac+2cd)x^2 + (b^2+2ad+d^2)x + bd]\,(ax+b)^{(da-cb)/a^2}\exp(\frac{cx}{a})\,J_1(x)\,dx}{x\,[(ax+b)\,J_0(x) + (cx+d)\,J_1(x)]^2} =$$

$$= \frac{(ax+b)^{1+(da-cb)/a^2}\exp(\frac{cx}{a})}{(ax+b)\,J_0(x) + (cx+d)\,J_1(x)}$$

$$\int \frac{[(a^2-c^2)x^3 + (2ab-ac-2cd)x^2 + (b^2-2ad-d^2)x - bd]\,(ax+b)^{(da-cb)/a^2}\exp(\frac{cx}{a})\,I_1(x)\,dx}{x\,[(ax+b)\,I_0(x) + (cx+d)\,I_1(x)]^2} =$$

$$= -\frac{(ax+b)^{1+(da-cb)/c^2}\exp(\frac{cx}{a})}{(ax+b)\,I_0(x) + (cx+d)\,I_1(x)}$$

$$\int \frac{[(a^2-c^2)x^3 + (2ab+ac-2cd)x^2 + (b^2+2ad-d^2)x + bd]\,(ax+b)^{(cb-da)/c^2}\exp(-\frac{cx}{a})\,K_1(x)\,dx}{x\,[(ax+b)\,K_0(x) + (cx+d)\,K_1(x)]^2} =$$

$$= \frac{(ax+b)^{1+(cb-da)/c^2}\exp(-\frac{cx}{a})}{(ax+b)\,K_0(x) + (cx+d)\,K_1(x)}$$

$$\int \frac{[2a^2\,x^{3/2} + 2b^2\sqrt{x} + 3ab]\exp(-\frac{2a}{3b}\,x^{3/2})\,J_0(x)\,dx}{x^{3/2}\,[a\sqrt{x}\,J_0(x) + b\,J_1(x)]^2} = -\frac{2b\exp(-\frac{2a}{3b}\,x^{3/2})}{x[a\sqrt{x}\,J_0(x) + b\,J_1(x)]}$$

$$\int \frac{[2a^2\,x^{3/2} - 2b^2\sqrt{x} - 3ab]\exp(\frac{2ax}{3b}\,x^{3/2})\,I_0(x)\,dx}{x^{3/2}\,[a\sqrt{x}\,I_0(x) + b\,I_1(x)]^2} = \frac{2b\exp(\frac{2a}{3b}\,x^{3/2})}{x[a\sqrt{x}\,I_0(x) + b\,I_1(x)]}$$

$$\int \frac{[2a^2\,x^{3/2} - 2b^2\sqrt{x} + 3ab]\exp(-\frac{2a}{3b}\,x^{3/2})\,K_0(x)\,dx}{x^{3/2}\,[a\sqrt{x}\,K_0(x) + b\,K_1(x)]^2} = -\frac{2b\exp(-\frac{2a}{3b}\,x^{3/2})}{x[a\sqrt{x}\,K_0(x) + b\,K_1(x)]}$$

$$\int \frac{[2a^2\,x^{3/2} + 2b^2\sqrt{x} + 3ab]\exp(\frac{2b}{a}\,\sqrt{x})\,J_1(x)\,dx}{\sqrt{x}\,[a\sqrt{x}\,J_0(x) + b\,J_1(x)]^2} = \frac{2a\,\sqrt{x}\exp(\frac{2b}{a}\,\sqrt{x})}{a\sqrt{x}\,J_0(x) + b\,J_1(x)}$$

$$\int \frac{[2a^2\,x^{3/2} - 2b^2\sqrt{x} - 3ab]\exp(\frac{2b}{a}\,\sqrt{x})\,I_1(x)\,dx}{\sqrt{x}\,[a\sqrt{x}\,I_0(x) + b\,I_1(x)]^2} = -\frac{2a\,\sqrt{x}\exp(\frac{2b}{a}\,\sqrt{x})}{a\sqrt{x}\,I_0(x) + b\,I_1(x)}$$

$$\int \frac{[2a^2\,x^{3/2} - 2b^2\sqrt{x} + 3ab]\exp(-\frac{2b}{a}\,\sqrt{x})\,K_1(x)\,dx}{\sqrt{x}\,[a\sqrt{x}\,K_0(x) + b\,K_1(x)]^2} = \frac{2a\,\sqrt{x}\exp(-\frac{2b}{a}\,\sqrt{x})}{a\sqrt{x}\,K_0(x) + b\,K_1(x)}$$

$$\int \frac{[2b^2\,x^{3/2} + 2a^2\sqrt{x} + ab]\exp(-\frac{2a}{b}\,\sqrt{x})\,J_0(x)\,dx}{x^{3/2}\,[a\,J_0(x) + b\sqrt{x}\,J_1(x)]^2} = -\frac{2b\exp(-\frac{2a}{b}\,\sqrt{x})}{\sqrt{x}\,[a\,J_0(x) + b\sqrt{x}\,J_1(x)]}$$

$$\int \frac{[2b^2\,x^{3/2} - 2a^2\sqrt{x} + ab]\exp(\frac{2a}{b}\,\sqrt{x})\,I_0(x)\,dx}{x^{3/2}\,[a\,I_0(x) + b\sqrt{x}\,I_1(x)]^2} = -\frac{2b\exp(\frac{2a}{b}\,\sqrt{x})}{\sqrt{x}\,[a\,I_0(x) + b\sqrt{x}\,I_1(x)]}$$

$$\int \frac{[2b^2\,x^{3/2} - 2a^2\sqrt{x} - ab]\exp(-\frac{2a}{b}\,\sqrt{x})\,K_0(x)\,dx}{x^{3/2}\,[a\,K_0(x) + b\sqrt{x}\,K_1(x)]^2} = \frac{2b\exp(-\frac{2a}{b}\,\sqrt{x})}{\sqrt{x}\,[a\,K_0(x) + b\sqrt{x}\,K_1(x)]}$$

$$\int \frac{[2b^2\,x^{3/2} + 2a^2\sqrt{x} + ab]\exp(\frac{2b}{3a}\,x^{3/2})\,J_1(x)\,dx}{\sqrt{x}\,[a\,J_0(x) + b\sqrt{x}\,J_1(x)]^2} = \frac{2a\exp(\frac{2b}{3a}\,x^{3/2})}{a\,J_0(x) + b\sqrt{x}\,J_1(x)}$$

$$\int \frac{[2b^2\,x^{3/2} - 2a^2\sqrt{x} + ab]\exp(\frac{2b}{3a}\,x^{3/2})\,I_1(x)\,dx}{\sqrt{x}\,[a\,I_0(x) + b\sqrt{x}\,I_1(x)]^2} = \frac{2a\exp(\frac{2b}{3a}\,x^{3/2})}{a\,I_0(x) + b\sqrt{x}\,I_1(x)}$$

$$\int \frac{[2b^2\,x^{3/2} - 2a^2\sqrt{x} - ab]\exp(-\frac{2b}{3a}\,x^{3/2})\,K_1(x)\,dx}{\sqrt{x}\,[a\,K_0(x) + b\sqrt{x}\,K_1(x)]^2} = -\frac{2a\exp(-\frac{2b}{3a}\,x^{3/2})}{a\,K_0(x) + b\sqrt{x}\,K_1(x)}$$

$$\int \frac{[2a^2\,x^3 + 5ab\,\sqrt{x} + 2b^2]\exp(-\frac{2a}{5b}\,x^{5/2})\,J_0(x)\,dx}{x\,[ax^{3/2}\,J_0(x) + b\,J_1(x)]^2} = -\frac{2b\exp(-\frac{2a}{5b}\,x^{5/2})}{x\,[ax^{3/2}\,J_0(x) + b\,J_1(x)]}$$

$$\int \frac{[2a^2\,x^3 - 5ab\,\sqrt{x} - 2b^2]\exp(\frac{2a}{5b}\,x^{5/2})\,I_0(x)\,dx}{x\,[ax^{3/2}\,I_0(x) + b\,I_1(x)]^2} = \frac{2b\exp(\frac{2a}{5b}\,x^{5/2})}{x\,[ax^{3/2}\,I_0(x) + b\,I_1(x)]}$$

$$\int \frac{[2a^2\,x^3 + 5ab\,\sqrt{x} - 2b^2]\exp(-\frac{2a}{5b}\,x^{5/2})\,K_0(x)\,dx}{x\,[ax^{3/2}\,K_0(x) + b\,K_1(x)]^2} = -\frac{2b\exp(-\frac{2a}{5b}\,x^{5/2})}{x\,[ax^{3/2}\,K_0(x) + b\,K_1(x)]}$$

$$\int \frac{[2a^2\,x^3 + 5ab\,\sqrt{x} + 2b^2]\exp(-2b/a\sqrt{x}\,)\,J_1(x)\,dx}{[ax^{3/2}\,J_0(x) + b\,J_1(x)]^2} = \frac{2a\,x^{3/2}\exp(-2b/a\sqrt{x}\,)}{ax^{3/2}\,J_0(x) + b\,J_1(x)}$$

$$\int \frac{[2a^2\,x^3 - 5ab\,\sqrt{x} - 2b^2]\exp(-2b/a\sqrt{x}\,)\,I_1(x)\,dx}{[ax^{3/2}\,I_0(x) + b\,I_1(x)]^2} = -\frac{2a\,x^{3/2}\exp(-2b/a\sqrt{x}\,)}{ax^{3/2}\,I_0(x) + b\,I_1(x)}$$

$$\int \frac{[2a^2\,x^3 + 5ab\,\sqrt{x} - 2b^2]\exp(2b/a\sqrt{x}\,)\,K_1(x)\,dx}{[ax^{3/2}\,K_0(x) + b\,K_1(x)]^2} = \frac{2a\,x^{3/2}\exp(2b/a\sqrt{x}\,)}{ax^{3/2}\,K_0(x) + b\,K_1(x)}$$

$$\int \frac{[2b^2\,x^3 - ab\,\sqrt{x} + 2a^2]\exp(2a/b\sqrt{x}\,)\,J_0(x)\,dx}{x\,[a\,J_0(x) + bx^{3/2}\,J_1(x)]^2} = -\frac{2b\,\sqrt{x}\exp(2a/b\sqrt{x}\,)}{a\,J_0(x) + bx^{3/2}\,J_1(x)}$$

$$\int \frac{[2b^2\,x^3 - ab\,\sqrt{x} - 2a^2]\exp(-2a/b\sqrt{x}\,)\,I_0(x)\,dx}{x\,[a\,I_0(x) + bx^{3/2}\,I_1(x)]^2} = -\frac{2b\,\sqrt{x}\exp(-2a/b\sqrt{x}\,)}{a\,I_0(x) + bx^{3/2}\,I_1(x)}$$

$$\int \frac{[2b^2\,x^3 + ab\,\sqrt{x} - 2a^2]\exp(2a/b\sqrt{x}\,)\,K_0(x)\,dx}{x\,[a\,K_0(x) + bx^{3/2}\,K_1(x)]^2} = \frac{2b\,\sqrt{x}\exp(2a/b\sqrt{x}\,)}{a\,K_0(x) + bx^{3/2}\,K_1(x)}$$

$$\int \frac{[2b^2\,x^3 - ab\,\sqrt{x} + 2a^2]\exp(2bx^{5/2}/5a)\,J_1(x)\,dx}{[a\,J_0(x) + bx^{3/2}\,J_1(x)]^2} = \frac{2a\exp(2bx^{5/2}/5a)}{a\,J_0(x) + bx^{3/2}\,J_1(x)}$$

$$\int \frac{[2b^2\,x^3 - ab\,\sqrt{x} - 2a^2]\exp(2bx^{5/2}/5a)\,I_1(x)\,dx}{[a\,I_0(x) + bx^{3/2}\,I_1(x)]^2} = \frac{2a\exp(2bx^{5/2}/5a)}{a\,I_0(x) + bx^{3/2}\,I_1(x)}$$

$$\int \frac{[2b^2\,x^3 + ab\,\sqrt{x} - 2a^2]\exp(-2bx^{5/2}/5a)\,K_1(x)\,dx}{[a\,K_0(x) + bx^{3/2}\,K_1(x)]^2} = -\frac{2a\exp(-2bx^{5/2}/5a)}{a\,K_0(x) + bx^{3/2}\,K_1(x)}$$

b) Typ $f(x)\,Z_0^{\,n}(x)\,Z_1^{2-n}(x)/[p(x)\,Z_0(x) + q(x)\,Z_1(x)]^2$, $n = 0,1,2$:

$$\int \frac{[(a^2 + b^2)x + ab]\cdot\exp\left(-\frac{2ax}{b}\right)\cdot J_0^2(x)\,dx}{x^2\,[a\,J_0(x) + b\,J_1(x)]^2} = E_1\left(\frac{2ax}{b}\right) - \frac{b\,J_0(x)}{x\,[a\,J_0(x) + b\,J_1(x)]}\cdot\exp\left(-\frac{2ax}{b}\right)$$

with the exponential integral $E_1(x)$ (see page 509).

$$\int \frac{[(a^2 - b^2)x - ab]\cdot\exp\left(\frac{2ax}{b}\right)\cdot I_0^2(x)\,dx}{x^2\,[a\,I_0(x) + b\,I_1(x)]^2} = E_1\left(-\frac{2ax}{b}\right) + \frac{b\,I_0(x)}{x\,[a\,I_0(x) + b\,I_1(x)]}\cdot\exp\left(\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2 - b^2)x + ab]\cdot\exp\left(-\frac{2ax}{b}\right)\cdot K_0^2(x)\,dx}{x^2\,[a\,K_0(x) + b\,K_1(x)]^2} = E_1\left(\frac{2ax}{b}\right) - \frac{b\,K_0(x)}{x\,[a\,K_0(x) + b\,K_1(x)]}\cdot\exp\left(-\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2 + b^2)x + ab]\cdot\exp\left(-\frac{a^2-b^2}{ab}x\right)\cdot J_0(x)\cdot J_1(x)\,dx}{x\,[a\,J_0(x) + b\,J_1(x)]^2} = -\frac{ab\,[b\,J_0(x) + a\,J_1(x)]}{(a^2 - b^2)\,[a J_0(x) + b J_1(x)]}\cdot\exp\left(-\frac{a^2 - b^2}{ab}x\right)$$

$$\int \frac{[(a^2 - b^2)x - ab]\cdot\exp\left(\frac{a^2+b^2}{ab}x\right)\cdot I_0(x)\cdot I_1(x)\,dx}{x\,[a\,I_0(x) + b\,I_1(x)]^2} = -\frac{ab\,[b\,J_0(x) - a\,J_1(x)]}{(a^2 + b^2)\,[a\,I_0(x) + b\,I_1(x)]}\cdot\exp\left(\frac{a^2 + b^2}{ab}x\right)$$

$$\int \frac{[(a^2 - b^2)x + ab] \cdot \exp\left(-\frac{a^2+b^2}{ab}\,x\right) \cdot K_0(x) \cdot K_1(x)\, dx}{x\,[a\,K_0(x) + b\,K_1(x)]^2} = -\,\frac{ab\,[b\,K_0(x) - a\,K_1(x)]}{(a^2 + b^2)\,[a\,K_0(x) + b\,K_1(x)]}\cdot\exp\left(-\frac{a^2 + b^2}{ab}\,x\right)$$

$$\int \frac{[(a^2 + b^2)x + ab] \cdot \exp\left(\frac{2bx}{a}\right) \cdot J_1^2(x)\, dx}{[a\,J_0(x) + b\,J_1(x)]^2} = \frac{a\,[a(a - 2bx)\,J_0(x) + b(a + 2bx)\,J_1(x)]}{4b^2\,[a\,J_0(x) + b\,J_1(x)]} \cdot \exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2 - b^2)x - ab] \cdot \exp\left(\frac{2bx}{a}\right) \cdot I_1^2(x)\, dx}{[a\,I_0(x) + b\,I_1(x)]^2} = -\,\frac{a\,[a(a - 2bx)\,I_0(x) + b(a + 2bx)\,I_1(x)]}{4b^2\,[a\,I_0(x) + b\,I_1(x)]} \cdot \exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2 - b^2)x + ab] \cdot \exp\left(-\frac{2bx}{a}\right) \cdot K_1^2(x)\, dx}{[a\,K_0(x) + b\,K_1(x)]^2} = -\,\frac{a\,[a(a + 2bx)\,K_0(x) + b(a - 2bx)\,K_1(x)]}{4b^2\,[a\,K_0(x) + b\,K_1(x)]} \cdot \exp\left(-\frac{2bx}{a}\right)$$

$$\int \frac{[a^2x^2 + b^2 + 2ab] \cdot \exp\left(-\frac{ax^2}{b}\right) \cdot J_0^2(x)\, dx}{x\,[ax\,J_0(x) + b\,J_1(x)]^2} = \frac{1}{2}\,E_1\left(\frac{ax^2}{b}\right) - \frac{b\,J_0(x)}{x\,[ax\,J_0(x) + b\,J_1(x)]} \cdot \exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2 - b^2 - 2ab] \cdot \exp\left(\frac{ax^2}{b}\right) \cdot I_0^2(x)\, dx}{x\,[ax\,I_0(x) + b\,I_1(x)]^2} = \frac{1}{2}\,E_1\left(-\frac{ax^2}{b}\right) + \frac{b\,I_0(x)}{x\,[ax\,I_0(x) + b\,I_1(x)]} \cdot \exp\left(\frac{ax^2}{b}\right)$$

$$\int \frac{[(a^2x^2 - b^2 + 2ab] \cdot \exp\left(-\frac{ax^2}{b}\right) \cdot K_0^2(x)\, dx}{x\,[ax\,K_0(x) + b\,K_1(x)]^2} = \frac{1}{2}\,E_1\left(\frac{ax^2}{b}\right) - \frac{b\,K_0(x)}{x\,[ax\,K_0(x) + b\,K_1(x)]} \cdot \exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{(a^2(a + b)x^2 + 2a^2b + 3ab^2 + b^3)\,x^{1+2b/a}\,J_1^2(x)\, dx}{[ax\,J_0(x) + b\,J_1(x)]^2} = -\,\frac{a\,x^{2+2b/a}\,[ax\,J_0(x) - (2a + b)\,J_1(x)]}{2\,[ax\,J_0(x) + b\,J_1(x)]}$$

$$\int \frac{(a^2(a + b)x^2 - 2a^2b - 3ab^2 - b^3)\,x^{1+2b/a}\,I_1^2(x)\, dx}{[ax\,I_0(x) + b\,I_1(x)]^2} = \frac{a\,x^{2+2b/a}\,[ax\,I_0(x) - (2a + b)\,I_1(x)]}{2\,[ax\,I_0(x) + b\,I_1(x)]}$$

$$\int \frac{(a^2(a - b)x^2 + 2a^2b - 3ab^2 + b^3)\,x^{1+2b/a}\,K_1^2(x)\, dx}{[ax\,K_0(x) + b\,K_1(x)]^2} = \frac{a\,x^{2-2b/a}\,[ax\,K_0(x) + (2a - b)\,K_1(x)]}{2\,[ax\,K_0(x) + b\,K_1(x)]}$$

$$\int \frac{(a^2 + b^2x^2)\,x^{-1-2a/b}\,J_0^2(x)\, dx}{[a\,J_0(x) + bx\,J_1(x)]^2} = -\,\frac{b\,x^{-2a/b}\,[a\,J_0(x) - bx\,J_1(x)]}{2a\,[a\,J_0(x) + bx\,J_1(x)]}$$

$$\int \frac{(a^2 - b^2x^2)\,x^{-1+2a/b}\,I_0^2(x)\, dx}{[a\,I_0(x) + bx\,I_1(x)]^2} = \frac{b\,x^{2a/b}\,[a\,I_0(x) - bx\,I_1(x)]}{2a\,[a\,I_0(x) + bx\,I_1(x)]}$$

$$\int \frac{(a^2 - b^2x^2)\,x^{-1-2a/b}\,K_0^2(x)\, dx}{[a\,K_0(x) + bx\,K_1(x)]^2} = -\,\frac{b\,x^{-2a/b}\,[a\,K_0(x) - bx\,K_1(x)]}{2a\,[a\,K_0(x) + bx\,K_1(x)]}$$

$$\int \frac{x(a^2 + b^2x^2) \cdot \exp\left(\frac{bx^2}{a}\right) \cdot J_1^2(x)\, dx}{[a\,J_0(x) + bx\,J_1(x)]^2} = -\,\frac{a\,[a\,J_0(x) - bx\,J_1(x)]}{2b\,[a\,J_0(x) + bx\,J_1(x)]} \cdot \exp\left(\frac{bx^2}{a}\right)$$

$$\int \frac{x(a^2 - b^2x^2) \cdot \exp\left(\frac{bx^2}{a}\right) \cdot I_1^2(x)\, dx}{[a\,I_0(x) + bx\,I_1(x)]^2} = \frac{a\,[a\,I_0(x) - bx\,I_1(x)]}{2b\,[a\,I_0(x) + bx\,I_1(x)]} \cdot \exp\left(\frac{bx^2}{a}\right)$$

$$\int \frac{x(a^2 - b^2x^2) \cdot \exp\left(-\frac{bx^2}{a}\right) \cdot K_1^2(x)\, dx}{[a\,K_0(x) + bx\,K_1(x)]^2} = -\,\frac{a\,[a\,K_0(x) - bx\,K_1(x)]}{2b\,[a\,K_0(x) + bx\,K_1(x)]} \cdot \exp\left(-\frac{bx^2}{a}\right)$$

5.3. Denominator $[p\,(x)\,Z_0(x) + q(x)\,Z_1(x)]^3$

b) Typ $f(x)\,Z_\nu/[p\,(x)\,Z_0(x) + q(x)\,Z_1(x)]^3,\ :$

$$\int \frac{[(a^2 + b^2)x + ab]\exp(-\frac{2ax}{b})\,J_0(x)\, dx}{x^3\,[a\,J_0(x) + b\,J_1(x)]^3} = -\,\frac{b}{2x^2\,[a\,J_0(x) + b\,J_1(x)]^2}\exp\left(-\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2 - b^2)x - ab]\exp(\frac{2ax}{b})\, I_0(x)\, dx}{x^3\, [a\, I_0(x) + b\, I_1(x)]^3} = \frac{b}{2x^2\, [a\, I_0(x) + b\, I_1(x)]^2}\exp\left(\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2 - b^2)x + ab]\exp(-\frac{2ax}{b})\, K_0(x)\, dx}{x^3\, [a\, K_0(x) + b\, K_1(x)]^3} = -\frac{b}{2x^2\, [a\, K_0(x) + b\, K_1(x)]^2}\exp\left(-\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2 + b^2)x + ab]\exp(\frac{2bx}{a})\, J_1(x)\, dx}{x\, [a\, J_0(x) + b\, J_1(x)]^3} = \frac{a}{2\, [a\, J_0(x) + b\, J_1(x)]^2}\exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2 - b^2)x - ab]\exp(\frac{2bx}{a})\, I_1(x)\, dx}{x\, [a\, I_0(x) + b\, I_1(x)]^3} = -\frac{a}{2\, [a\, I_0(x) + b\, I_1(x)]^2}\exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2 - b^2)x + ab]\exp(-\frac{2bx}{a})\, K_1(x)\, dx}{x\, [a\, K_0(x) + b\, K_1(x)]^3} = \frac{a}{2\, [a\, K_0(x) + b\, K_1(x)]^2}\exp\left(-\frac{2bx}{a}\right)$$

$$\int \frac{[a^2x^2 + b^2 + 2ab]\exp(-\frac{ax^2}{b})\, J_0(x)\, dx}{x^2\, [ax\, J_0(x) + b\, J_1(x)]^3} = -\frac{b}{2x^2\, [ax\, J_0(x) + b\, J_1(x)]^2}\exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2 - b^2 - 2ab]\exp(\frac{ax^2}{b})\, I_0(x)\, dx}{x^2\, [ax\, I_0(x) + b\, I_1(x)]^3} = \frac{b}{2x^2\, [ax\, I_0(x) + b\, I_1(x)]^2}\exp\left(\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2 - b^2 + 2ab]\exp(-\frac{ax^2}{b})\, K_0(x)\, dx}{x^2\, [ax\, K_0(x) + b\, K_1(x)]^3} = -\frac{b}{2x^2\, [ax\, J_0(x) + b\, J_1(x)]^2}\exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2 + b^2 + 2ab]\, x^{1+2b/a}\, J_1(x)\, dx}{[ax\, J_0(x) + b\, J_1(x)]^3} = \frac{ax^{2+2b/a}}{2\, [ax\, J_0(x) + b\, J_1(x)]^2}$$

$$\int \frac{[a^2x^2 - b^2 - 2ab]\, x^{1+2b/a}\, I_1(x)\, dx}{[ax\, I_0(x) + b\, I_1(x)]^3} = -\frac{ax^{2+2b/a}}{2\, [ax\, I_0(x) + b\, I_1(x)]^2}$$

$$\int \frac{[a^2x^2 - b^2 + 2ab]\, x^{1-2b/a}\, K_1(x)\, dx}{[ax\, K_0(x) + b\, K_1(x)]^3} = \frac{ax^{2-2b/a}}{2\, [ax\, K_0(x) + b\, K_1(x)]^2}$$

$$\int \frac{[a^2 + b^2x^2]\, x^{-1-2a/b}\, J_0(x)\, dx}{[a\, J_0(x) + bx\, J_1(x)]^3} = -\frac{bx^{-2a/b}}{2\, [a\, J_0(x) + bx\, J_1(x)]^2}$$

$$\int \frac{[a^2 - b^2x^2]\, x^{-1+2a/b}\, I_0(x)\, dx}{[a\, J_0(x) + bx\, J_1(x)]^3} = \frac{bx^{2a/b}}{2\, [a\, I_0(x) + bx\, I_1(x)]^2}$$

$$\int \frac{[a^2 - b^2x^2]\, x^{-1-2a/b}\, K_0(x)\, dx}{[a\, K_0(x) + bx\, K_1(x)]^3} = \frac{bx^{-2a/b}}{2\, [a\, K_0(x) + bx\, K_1(x)]^2}$$

$$\int \frac{[a^2 + b^2x^2]\exp(\frac{b}{a}\, x^2)\, J_1(x)\, dx}{[a\, J_0(x) + bx\, J_1(x)]^3} = \frac{a}{2\, [a\, J_0(x) + bx\, J_1(x)]^2}\exp\left(\frac{b\, x^2}{a}\right)$$

$$\int \frac{[a^2 - b^2x^2]\exp(\frac{b}{a}\, x^2)\, I_1(x)\, dx}{[a\, I_0(x) + bx\, I_1(x)]^3} = -\frac{a}{2\, [a\, I_0(x) + bx\, I_1(x)]^2}\exp\left(\frac{b\, x^2}{a}\right)$$

$$\int \frac{[a^2 - b^2x^2]\exp(-\frac{b}{a}\, x^2)\, K_1(x)\, dx}{[a\, K_0(x) + bx\, K_1(x)]^3} = \frac{a}{2\, [a\, K_0(x) + bx\, K_1(x)]^2}\exp\left(-\frac{b\, x^2}{a}\right)$$

5.4. Denominator $[p\,(x)\, Z_0(x) + q\,(x)\, Z_1(x)]^4$

a) Typ $f(x)\, Z_\nu / [p\,(x)\, Z_0(x) + q\,(x)\, Z_1(x)]^4$:

$$\int \frac{[(a^2 + b^2)x + ab]\exp(-\frac{3ax}{b})\, J_0(x)\, dx}{x^4\, [a\, J_0(x) + b\, J_1(x)]^4} = -\frac{b}{3x^3\, [a\, J_0(x) + b\, J_1(x)]^3}\exp\left(-\frac{3ax}{b}\right)$$

$$\int \frac{[(a^2 - b^2)x - ab]\exp(\frac{3ax}{b})\,I_0(x)\,dx}{x^4\,[a\,I_0(x) + b\,I_1(x)]^4} = \frac{b}{3x^3\,[a\,I_0(x) + b\,I_1(x)]^3}\exp\left(\frac{3ax}{b}\right)$$

$$\int \frac{[(a^2 - b^2)x + ab]\exp(-\frac{3ax}{b})\,K_0(x)\,dx}{x^4\,[a\,K_0(x) + b\,K_1(x)]^4} = -\frac{b}{3x^3\,[a\,J_0(x) + b\,J_1(x)]^3}\exp\left(-\frac{3ax}{b}\right)$$

$$\int \frac{[(a^2 + b^2)x + ab]\exp(\frac{3bx}{a})\,J_1(x)\,dx}{x\,[a\,J_0(x) + b\,J_1(x)]^4} = -\frac{a}{3\,[a\,J_0(x) + b\,J_1(x)]^3}\exp\left(\frac{3bx}{a}\right)$$

$$\int \frac{[(a^2 - b^2)x - ab]\exp(\frac{3bx}{a})\,I_1(x)\,dx}{x\,[a\,I_0(x) + b\,I_1(x)]^4} = -\frac{a}{3\,[a\,I_0(x) + b\,I_1(x)]^3}\exp\left(\frac{3bx}{a}\right)$$

$$\int \frac{[(a^2 - b^2)x + ab]\exp(-\frac{3bx}{a})\,K_1(x)\,dx}{x\,[a\,K_0(x) + b\,K_1(x)]^4} = \frac{a}{3\,[a\,K_0(x) + b\,K_1(x)]^3}\exp\left(-\frac{3bx}{a}\right)$$

$$\int \frac{[a^2x^2 + b^2 + 2ab]\exp(-\frac{3ax^2}{2b})\,J_0(x)\,dx}{x^3\,[ax\,J_0(x) + b\,J_1(x)]^4} = -\frac{b}{3x^3\,[ax\,J_0(x) + b\,J_1(x)]^3}\exp\left(-\frac{3ax^2}{2b}\right)$$

$$\int \frac{[a^2x^2 - b^2 - 2ab]\exp(\frac{3ax^2}{2b})\,I_0(x)\,dx}{x^3\,[ax\,I_0(x) + b\,I_1(x)]^4} = \frac{b}{3x^3\,[ax\,I_0(x) + b\,I_1(x)]^3}\exp\left(\frac{3ax^2}{2b}\right)$$

$$\int \frac{[a^2x^2 - b^2 + 2ab]\exp(-\frac{3ax^2}{2b})\,K_0(x)\,dx}{x^3\,[ax\,K_0(x) + b\,K_1(x)]^4} = -\frac{b}{3x^3\,[ax\,K_0(x) + b\,K_1(x)]^3}\exp\left(-\frac{3ax^2}{2b}\right)$$

$$\int \frac{[a^2x^2 + b^2 + 2ab]\,x^{2+3b/a}\,J_1(x)\,dx}{[ax\,J_0(x) + b\,J_1(x)]^4} = \frac{a\,x^{3+3b/a}}{3\,[ax\,J_0(x) + b\,J_1(x)]^3}$$

$$\int \frac{[a^2x^2 - b^2 - 2ab]\,x^{2+3b/a}\,I_1(x)\,dx}{[ax\,I_0(x) + b\,I_1(x)]^4} = -\frac{a\,x^{3+3b/a}}{3\,[ax\,I_0(x) + b\,I_1(x)]^3}$$

$$\int \frac{[a^2x^2 - b^2 + 2ab]\,x^{2-3b/a}\,K_1(x)\,dx}{[ax\,K_0(x) + b\,K_1(x)]^4} = \frac{a\,x^{3-3b/a}}{3\,[ax\,K_0(x) + b\,K_1(x)]^3}$$

$$\int \frac{(a^2 + b^2x^2)\,x^{-1-3a/b}\,J_0(x)\,dx}{[a\,J_0(x) + bx\,J_1(x)]^4} = -\frac{b\,x^{-3a/b}}{3\,[a\,J_0(x) + bx\,J_1(x)]^3}$$

$$\int \frac{(a^2 - b^2x^2)\,x^{-1+3a/b}\,I_0(x)\,dx}{[a\,I_0(x) + bx\,I_1(x)]^4} = \frac{b\,x^{3a/b}}{3\,[a\,I_0(x) + bx\,I_1(x)]^3}$$

$$\int \frac{(a^2 - b^2x^2)\,x^{-1-3a/b}\,K_0(x)\,dx}{[a\,K_0(x) + bx\,K_1(x)]^4} = \frac{b\,x^{-3a/b}}{3\,[a\,K_0(x) + bx\,K_1(x)]^3}$$

$$\int \frac{[a^2 + b^2x^2]\exp(\frac{3bx^2}{2a})\,J_1(x)\,dx}{[a\,J_0(x) + bx\,J_1(x)]^4} = \frac{a}{3\,[a\,J_0(x) + bx\,J_1(x)]^3}\exp\left(\frac{3bx^2}{2a}\right)$$

$$\int \frac{[a^2 - b^2x^2]\exp(\frac{3bx^2}{2a})\,I_1(x)\,dx}{[a\,I_0(x) + bx\,I_1(x)]^4} = -\frac{a}{3\,[a\,I_0(x) + bx\,I_1(x)]^3}\exp\left(\frac{3bx^2}{2a}\right)$$

$$\int \frac{[a^2 - b^2x^2]\exp(-\frac{3bx^2}{2a})\,K_1(x)\,dx}{[a\,K_0(x) + bx\,K_1(x)]^4} = \frac{a}{3\,[a\,K_0(x) + bx\,K_1(x)]^3}\exp\left(-\frac{3bx^2}{2a}\right)$$

5.5. Denominator $p(x) Z_0^2(x) + q(x) Z_0(x) Z_1(x) + r(x) Z_1^2(x)$

a) Typ $f(x) Z_0^n(x) Z_1^{2-n}(x)/[p(x) Z_0^2(x) + q(x) Z_1^2(x)]$, $n = 0, 1, 2$:

$$\int \frac{J_0^2(x)\, dx}{x\,[J_0^2(x) + J_1^2(x)]} = \frac{1}{2}\,\ln\{x^2\,[J_0^2(x) + J_1^2(x)]\}$$

$$\int \frac{I_0^2(x)\, dx}{x\,[I_0^2(x) - I_1^2(x)]} = \frac{1}{2}\,\ln\{x^2\,[I_0^2(x) - I_1^2(x)]\}$$

$$\int \frac{K_0^2(x)\, dx}{x\,[K_0^2(x) - K_1^2(x)]} = \frac{1}{2}\,\ln\{x^2\,[K_1^2(x) - K_0^2(x)]\}$$

$$\int \frac{(x^2 - a)\, J_0(x)\, J_1(x)\, dx}{a\, J_0^2(x) + x^2\, J_1^2(x)} = \frac{1}{2}\,\ln[a\, J_0^2(x) + x^2\, J_1^2(x)]$$

$$\int \frac{(x^2 + a)\, I_0(x)\, I_1(x)\, dx}{a\, I_0^2(x) + x^2\, I_1^2(x)} = \frac{1}{2}\,\ln[a\, I_0^2(x) + x^2\, I_1^2(x)]$$

$$\int \frac{(x^2 + a)\, K_0(x)\, K_1(x)\, dx}{a\, K_0^2(x) + x^2\, K_1^2(x)} = -\frac{1}{2}\,\ln[a\, K_0^2(x) + x^2\, K_1^2(x)]$$

$$\int \frac{J_1^2(x)\, dx}{x\,[J_0^2(x) + J_1^2(x)]} = -\frac{1}{2}\,\ln[J_0^2(x) + J_1^2(x)]$$

$$\int \frac{I_1^2(x)\, dx}{x\,[I_0^2(x) - I_1^2(x)]} = -\frac{1}{2}\,\ln[I_0^2(x) - I_1^2(x)]$$

$$\int \frac{K_1^2(x)\, dx}{x\,[K_1^2(x) - K_0^2(x)]} = -\frac{1}{2}\,\ln[K_1^2(x) - K_0^2(x)]$$

b) Typ $f(x) Z_0^n(x) Z_1^{2-n}(x)/[p(x) Z_0^2(x) + q(x) Z_0(x) Z_1(x) + r(x) Z_1^2(x)]$, $n = 0, 1, 2$:

$$\int \frac{x\, J_0^2(x)\, dx}{x^2\, J_0^2(x) + 2x\, J_0(x)\, J_1(x) + (x^2 - 2)\, J_1^2(x)} = \frac{1}{6}\,\ln\left|x^4\, J_0^2(x) + 2x^3\, J_0(x)\, J_1(x) + (x^4 - 2x^3)\, J_1^2(x)\right|$$

$$\int \frac{x\, I_0^2(x)\, dx}{x^2\, I_0^2(x) + 2x\, I_0(x)\, I_1(x) - (x^2 + 2)\, I_1^2(x)} = \frac{1}{6}\,\ln\left|x^4\, I_0^2(x) + 2x^3\, I_0(x)\, I_1(x) - (x^4 + 2x^2)\, I_1^2(x)\right|$$

$$\int \frac{x\, K_0^2(x)\, dx}{x^2\, K_0^2(x) - 2x\, K_0(x)\, K_1(x) - (x^2 + 2)\, K_1^2(x)} = \frac{1}{6}\,\ln\left|x^4\, K_0^2(x) - 2x^3\, K_0(x)\, K_1(x) - (x^2 + 2)\, K_1^2(x)\right|$$

$$\int \frac{J_0(x) J_1(x)\, dx}{x[x J_0^2(x) x - J_0(x) J_1(x) + x J_1^2(x)]} = \ln\left|x J_0^2(x) x - J_0(x) J_1(x) + x J_1^2(x)\right|$$

$$\int \frac{I_0(x) I_1(x)\, dx}{x[x I_0^2(x) x - I_0(x) I_1(x) - x I_1^2(x)]} = \ln\left|x I_0^2(x) x - I_0(x) I_1(x) - x I_1^2(x)\right|$$

$$\int \frac{K_0(x) K_1(x)\, dx}{x[x K_0^2(x) x + K_0(x) K_1(x) + x K_1^2(x)]} = -\ln\left|x K_0^2(x) x + K_0(x) K_1(x) - x K_1^2(x)\right|$$

$$\int \frac{(1 + 2\ln x)x\, J_0(x)\, J_1(x)\, dx}{x J_0^2(x) - 2 J_0(x) J_1(x) - 2x \ln x J_1^2(x)} = -\frac{1}{2}\,\ln\left|x\,\left[x J_0^2(x) - 2 J_0(x) J_1(x) - 2x \ln x J_1^2(x)\right]\right|$$

$$\int \frac{(1 + 2\ln x)x\, I_0(x)\, I_1(x)\, dx}{x I_0^2(x) - 2 I_0(x) I_1(x) + 2x \ln x I_1^2(x)} = \frac{1}{2}\,\ln\left|x\,\left[x I_0^2(x) - 2 I_0(x) I_1(x) + 2x \ln x I_1^2(x)\right]\right|$$

$$\int \frac{(1 + 2\ln x)x\, K_0(x)\, K_1(x)\, dx}{x K_0^2(x) + 2 K_0(x) K_1(x) - 2x \ln x K_1^2(x)} = -\frac{1}{2}\,\ln\left|x\,\left[x K_0^2(x) + 2 K_0(x) K_1(x) + 2x \ln x K_1^2(x)\right]\right|$$

$$\int \frac{(8x^2+3)\,J_0(x)J_1(x)\,dx}{x[xJ_0^2(x)-3J_0(x)J_1(x)-3xJ_1^2(x)]} = -\ln\left[x^2\left(xJ_0^2(x)-3J_0(x)J_1(x)-3xJ_1^2(x)\right)\right]\Big|$$

$$\int \frac{(8x^2-3)\,I_0(x)I_1(x)\,dx}{x[xI_0^2(x)-3I_0(x)I_1(x)+3xI_1^2(x)]} = \ln\left[x^2\left(xI_0^2(x)-3I_0(x)I_1(x)+3xI_1^2(x)\right)\right]\Big|$$

$$\int \frac{(8x^2-3)\,K_0(x)K_1(x)\,dx}{x[xK_0^2(x)+3K_0(x)K_1(x)+3xK_1^2(x)]} = -\ln\left[x^2\left(xK_0^2(x)+3K_0(x)K_1(x)+3xK_1^2(x)\right)\right]\Big|$$

$$\int \frac{x^2\,J_0(x)\,J_1(x)\,dx}{x^2\,J_0^2(x)-4x\,J_0(x)\,J_1(x)-(2x^2-4)\,J_1^2(x)} = -\frac{1}{6}\ln\left|x^4\,J_0^2(x)-4x^3\,J_0(x)\,J_1(x)-(x^4-2x^2)\,J_1^2(x)\right|$$

$$\int \frac{x^2\,I_0(x)\,I_1(x)\,dx}{x^2\,I_0^2(x)-4x\,I_0(x)\,I_1(x)+(2x^2+4)\,I_1^2(x)} = \frac{1}{6}\ln\left|x^4\,I_0^2(x)-2x^3\,I_0(x)\,I_1(x)+(2x^4+4x^2)\,I_1^2(x)\right|$$

$$\int \frac{x^2\,K_0(x)\,K_1(x)\,dx}{x^2\,K_0^2(x)+4x\,K_0(x)\,K_1(x)+(2x^2+4)\,K_1^2(x)} = -\frac{1}{6}\ln\left|x^4\,K_0^2(x)+4x^3\,K_0(x)\,K_1(x)+(2x^4+4x^2)\,K_1^2(x)\right|$$

$$\int \frac{(3x^3+4)\,J_0(x)\,J_1(x)\,dx}{x^2\,J_0^2(x)-4x\,J_0(x)\,J_1(x)-2x^2\,J_1^2(x)} = -\frac{1}{2}\ln\left|x^4\,J_0^2(x)-4x^3\,J_0(x)\,J_1(x)-2x^4\,J_1^2(x)\right|$$

$$\int \frac{(3x^2-4)\,I_0(x)\,I_1(x)\,dx}{x^2\,I_0^2(x)-4x\,I_0(x)\,I_1(x)+2x^2\,I_1^2(x)} = \frac{1}{2}\ln\left|x^4\,I_0^2(x)-4x^3\,I_0(x)\,I_1(x)+2x^4\,I_1^2(x)\right|$$

$$\int \frac{(3x^2-4)\,K_0(x)\,K_1(x)\,dx}{x^2\,K_0^2(x)+4x\,K_0(x)\,K_1(x)+2x^2\,K_1^2(x)} = -\frac{1}{2}\ln\left|x^4\,K_0^2(x)+4x^3\,K_0(x)\,K_1(x)+2x^4\,K_1^2(x)\right|$$

$$\int \frac{J_1^2(x)\,dx}{xJ_0^2(x)-2J_0(x)J_1(x)+xJ_1^2(x)} = \frac{1}{2}\ln\left|x^2\,J_0^2(x)-2x\,J_0(x)J_1(x)+x^2\,J_1^2(x)\right|$$

$$\int \frac{I_1^2(x)\,dx}{xI_0^2(x)-2I_0(x)I_1(x)-xI_1^2(x)} = -\frac{1}{2}\ln\left|x^2\,I_0^2(x)-2x\,I_0(x)I_1(x)-x^2\,I_1^2(x)\right|$$

$$\int \frac{K_1^2(x)\,dx}{xK_0^2(x)+2K_0(x)K_1(x)-xK_1^2(x)} = -\frac{1}{2}\ln\left|x^2\,K_0^2(x)+2x\,K_0(x)K_1(x)-x^2\,K_1^2(x)\right|$$

$$\int \frac{x\,J_1^2(x)\,dx}{x^2\,J_0^2(x)-4x\,J_0(x)\,J_1(x)+(x^2+4)\,J_1^2(x)} = \frac{1}{6}\ln\left|x^4\,J_0^2(x)-4x^3\,J_0(x)\,J_1(x)+(x^4+4x^2)\,J_1^2(x)\right|$$

$$\int \frac{x\,I_1^2(x)\,dx}{x^2\,I_0^2(x)-4x\,I_0(x)\,I_1(x)-(x^2-4)\,I_1^2(x)} = -\frac{1}{6}\ln\left|x^4\,I_0^2(x)-4x^3\,I_0(x)\,I_1(x)-(x^2-4)\,I_1^2(x)\right|$$

$$\int \frac{x\,K_1^2(x)\,dx}{x^2\,K_0^2(x)+4x\,K_0(x)\,K_1(x)-(x^2-4)\,K_1^2(x)} = -\frac{1}{6}\ln\left|x^4\,K_0^2(x)+4x^3\,K_0(x)\,K_1(x)-(x^4-4x^2)\,K_1^2(x)\right|$$

5.6. Denominator $\sqrt{a(x)\,Z_0(x) + b(x)\,Z_1(x) + p\,(x)\,Z_0^2(x) + q(x)\,Z_0(x)\,Z_1(x) + r(x)\,Z_1^2(x)}$

Generally: From

$$\int \varphi(x)\,Z_0^m(x)\,Z_1^n(x)\,dx \quad = \quad a(x)\,Z_0(x) + b(x)\,Z_1(x) + p\,(x)\,Z_0^2(x) + q(x)\,Z_0(x)\,Z_1(x) + r(x)\,Z_1^2(x)$$

follows

$$\int \frac{\varphi(x)\,Z_0^m(x)\,Z_1^n(x)\,dx}{\sqrt{a(x)\,Z_0(x) + b(x)\,Z_1(x) + p\,(x)\,Z_0^2(x) + q(x)\,Z_0(x)\,Z_1(x) + r(x)\,Z_1^2(x)}} =$$

$$= 2\,\sqrt{a(x)\,Z_0(x) + b(x)\,Z_1(x) + p\,(x)\,Z_0^2(x) + q(x)\,Z_0(x)\,Z_1(x) + r(x)\,Z_1^2(x)} \;.$$

Therefore the formulas from 1. and 2. give a lot of integrals of this kind.
Some special cases:

$$\int \frac{J_1(x)\,dx}{\sqrt{J_0(x)}} = -2\,\sqrt{J_0(x)}\,, \qquad \int \frac{\sqrt{x}\,J_0(x)\,dx}{\sqrt{J_1(x)}} = 2\,\sqrt{x\,J_1(x)}$$

$$\int \frac{x^2\,J_1(x)\,dx}{\sqrt{2x\,J_1(x) - x^2\,J_0(x)}} = 2\,\sqrt{2x\,J_1(x) - x^2\,J_0(x)} \quad \text{(at least for } 0 < x < 5.1356\text{)}$$

$$\int \frac{x^3\,J_0(x)\,dx}{\sqrt{2x^2\,J_0(x) + (x^3 - 4x)\,J_1(x)}} = 2\,\sqrt{2x^2\,J_0(x) + (x^3 - 4x)\,J_1(x)} \quad \text{(at least for } 0 < x < 3.0542\text{)}$$

$$\int \frac{x\,\ln x\,J_0(x)\,dx}{\sqrt{J_0(x) + x\,\ln x\,J_1(x)}} = 2\,\sqrt{J_0(x) + x\,\ln x\,J_1(x)} \quad \text{(at least for } 0 < x < 3.6265\text{)}$$

$$\int \frac{\sin x\,J_0(x)\,dx}{\sqrt{x\,\sin x\,J_0(x) - x\,\cos x\,J_1(x)}} = 2\,\sqrt{x\,\sin x\,J_0(x) - x\,\cos x\,J_1(x)}$$

$$\int \frac{\cos x\,J_0(x)\,dx}{\sqrt{x\,\cos x\,J_0(x) + x\,\sin x\,J_1(x)}} = 2\,\sqrt{x\,\cos x\,J_0(x) + x\,\sin x\,J_1(x)}$$

6. Some Gaussian Quadrature Formulas for special Defined Integrals

About the origin of the formulas see [15], 7.1.

$$\int_0^a \varrho(x)\, f(x)\, dx = \sum_{k=1}^n A_k\, f(x_k) + R_n(f)$$

Assuming that the continous derivative $d^{2n}f/dx^{2n}$ exists one has with some $\eta \in [0, a]$

$$R_n(f) = r_n \cdot f^{(2n)}(\eta)\ .$$

The given values of r_n are rounded to up.

Let $z(0) = 2.40482\ 55576\ 95772\ 76862$ denote the first positive zero of $J_0(x)$ and $z(1) = 3.83170\ 59702\ 07512\ 31561$ the first positive zero of $J_1(x)$.

1. $\varrho(x) = J_0(x)$, $a = z(0)$:

n	k	x_k	A_k
1	1	0.84911 86375 95859 14440	1.47030 00433 84178 98213
2	1	0.39882 92282 29946 50719	0.90108 88277 42624 27265
	2	1.56194 85061 59024 25690	0.56921 12156 41554 70947
3	1	0.22634 80927 64316 44350	0.54778 48193 39959 52854
	2	1.00516 39594 78741 19902	0.70092 58605 26759 67124
	3	1.89505 35380 34339 27541	0.22158 93635 17459 78235
4	1	0.14473 82282 55696 21841	0.35943 22951 47741 12189
	2	0.68411 14453 73497 09001	0.60043 63758 36197 57058
	3	1.41258 23653 31484 08156	0.41175 50421 69162 76320
	4	2.06769 20930 02856 90479	0.09867 63302 31077 52645
5	1	0.10021 38764 64392 85496	0.25188 30914 84562 78217
	2	0.49030 87468 48216 27800	0.47696 66584 93069 36682
	3	1.06603 72772 05082 94676	0.45505 16639 82458 95258
	4	1.67579 09335 27784 59726	0.23687 69005 56173 29393
	5	2.16668 25320 32995 02545	0.04952 17288 67914 58662
6	1	0.07339 09066 03133 18519	0.18567 01891 22829 58772
	2	0.36659 94270 55653 86888	0.37603 00607 89693 99979
	3	0.82318 89118 45400 56078	0.42592 25681 22988 96567
	4	1.35083 69247 90597 31458	0.31454 44042 01812 98980
	5	1.85067 67803 21263 94621	0.14082 51792 75388 66431
	6	2.22814 70218 71287 87049	0.02730 76418 71464 77484
7	1	0.05602 45260 34280 52744	0.14228 77583 04715 82762
	2	0.28359 36021 81776 57522	0.29980 85241 57316 96860
	3	0.65055 43979 28046 67146	0.37494 72554 54351 71927
	4	1.09825 71874 06190 64557	0.33591 91612 03562 33066
	5	1.56062 92196 48622 52565	0.21370 34786 04027 46305
	6	1.97116 65464 72516 80454	0.08742 58793 99324 47881
	7	2.26873 98995 48321 17386	0.01620 79862 60880 19412
8	1	0.04415 16822 70185 67264	0.11241 45281 58959 86690
	2	0.22550 46441 44908 65760	0.24287 21983 42402 78068
	3	0.52500 76324 59107 67743	0.32298 95029 80519 02245
	4	0.90401 79209 12731 39428	0.32551 60301 08802 50497
	5	1.31711 94427 36933 38632	0.25319 54398 14890 36268
	6	1.71683 41508 13325 77892	0.14651 48161 74850 95088
	7	2.05706 85469 40771 30526	0.05660 02199 54023 77307
	8	2.29687 90390 82795 50272	0.01019 73078 49729 72049

n	r_n
1	2.360E-01
2	6.834E-03
3	8.069E-05
4	5.145E-07
5	2.050E-09
6	5.578E-12
7	1.103E-14
8	1.656E-17
9	1.950E-20
10	1.851E-23

n	k	x_k	A_k
9	1	0.03568 24385 35339 63793	0.09100 54880 68636 79312
	2	0.18339 95455 20560 72825	0.19994 03942 89885 35994
	3	0.43153 27709 86789 28064	0.27682 48640 75388 09817
	4	0.75378 95449 42819 65529	0.30111 60165 22402 67682
	5	1.11817 72467 97612 79458	0.26594 41351 42704 29766
	6	1.49018 46734 69815 92962	0.18838 00567 60179 29507
	7	1.83510 82326 65444 36893	0.10231 86185 89166 42240
	8	2.12018 77212 80538 58048	0.03804 60533 61272 04146
	9	2.31715 56799 89430 39540	0.00672 44165 74543 99748
10	1	0.02943 20801 70948 03192	0.07515 49753 94236 45992
	2	0.15196 93001 27994 54755	0.16706 34344 69764 88751
	3	0.36038 97622 40754 97575	0.23778 12425 00351 80276
	4	0.63628 72652 14219 79340	0.27240 08422 34623 59420
	5	0.95662 39779 19819 83159	0.26197 96107 07124 04222
	6	1.29588 08007 77129 56440	0.21151 59005 73181 40869
	7	1.62769 77336 04089 43949	0.14039 29837 65469 47691
	8	1.92626 72552 05830 23170	0.07297 54521 24144 36484
	9	2.16779 36949 02068 73559	0.02642 69230 91999 99596
	10	2.33223 52003 55296 43725	0.00460 86785 23282 94911

2. $\varrho(x) = J_1(x)$, $a = z(1)$:

n	k	x_k	A_k
1	1	1.87876 52796 87720 77862	1.40275 93957 02552 97210
2	1	1.06490 35971 77076 73955	0.71352 73057 09344 42983
	2	2.72131 53320 91136 68635	0.68923 20899 93208 54226
3	1	0.67234 43602 14129 71914	0.32906 60341 43142 50045
	2	1.89442 60157 28575 44303	0.76668 74947 14353 05601
	3	3.13276 49538 77173 30814	0.30700 58668 45057 41563
4	1	0.45894 55203 26475 31235	0.16288 77917 03040 35012
	2	1.36622 66845 01665 12141	0.55523 92326 02955 25023
	3	2.43260 54162 85590 13131	0.53670 44151 79575 17804
	4	3.35654 13570 69680 31486	0.14792 79562 16982 19371
5	1	0.33180 20337 36165 55678	0.08770 25071 91829 47300
	2	1.02142 19523 37440 45833	0.36410 09808 94805 82001
	3	1.90084 44312 86510 28562	0.53080 78354 68009 88152
	4	2.78606 68373 94159 50371	0.34208 41472 38969 07093
	5	3.48948 15673 50310 05946	0.07806 39249 08938 72664
6	1	0.25050 63052 28711 97552	0.05081 72006 85742 58212
	2	0.78800 95790 83595 25210	0.23625 18296 43115 36822
	3	1.50976 00320 21313 92188	0.43333 57977 70745 12177
	4	2.29683 53747 01485 20854	0.42085 76435 00112 46757
	5	3.02585 79200 18626 93204	0.21694 12820 42452 78091
	6	3.57417 42088 49931 43235	0.04455 56420 60384 65150
7	1	0.19558 40010 41249 50480	0.03128 48720 61092 70863
	2	0.62427 48418 07360 43163	0.15598 42462 27833 87320
	3	1.22041 72865 27573 05119	0.32922 48592 89470 36004
	4	1.90430 89567 09341 85832	0.40649 88779 46399 56710
	5	2.59089 22472 42858 90095	0.31201 30064 45854 17579
	6	3.19412 16451 13656 70328	0.14063 79161 58331 59436
	7	3.63118 89959 66345 37392	0.02711 56175 73570 69297

n	r_n
1	4.810E-01
2	3.317E-02
3	9.615E-04
4	1.525E-05
5	1.520E-07
6	1.040E-09
7	5.177E-12
8	1.960E-14
9	5.833E-17
10	1.400E-19

n	k	x_k	A_k
8	1	0.15682 33177 08037 94485	0.02024 22670 07056 04918
	2	0.50569 47441 49987 41001	0.10565 63995 87912 13878
	3	1.00301 79183 33490 80019	0.24426 85534 74077 52574
	4	1.59383 56846 69951 78381	0.35160 77800 21817 38025
	5	2.21766 91829 95139 45587	0.34288 84302 64882 18094
	6	2.81224 79584 90927 04488	0.22683 43760 90276 89708
	7	3.31591 27752 75352 15183	0.09387 42676 40371 70507
	8	3.67129 80952 12594 29179	0.01738 73216 16159 09505
9	1	0.12848 64993 58978 72698	0.01364 69714 56644 03898
	2	0.41739 43521 28271 20996	0.07350 42895 51720 43702
	3	0.83678 02396 08545 23386	0.18078 17286 84913 48106
	4	1.34780 40848 24007 81937	0.28859 28656 90951 59125
	5	1.90647 50319 97646 23886	0.32952 79063 95977 39406
	6	2.46660 60964 63479 03024	0.27546 79821 25950 98159
	7	2.98165 22148 56115 11947	0.16506 19568 17501 99304
	8	3.40651 53800 04878 64863	0.06453 73457 74434 07420
	9	3.70053 85543 16270 69345	0.01163 83492 04458 98090
10	1	0.10716 10473 47871 95337	0.00952 20553 84942 54239
	2	0.35003 94839 50296 02288	0.05244 74001 75843 56752
	3	0.70744 95101 06619 77663	0.13474 30147 12181 03556
	4	1.15132 70571 55910 48530	0.23129 32570 81504 53353
	5	1.64907 06313 31381 24496	0.29468 70039 45311 30494
	6	2.16577 21273 78569 27320	0.28831 22736 46923 12496
	7	2.66569 90324 63164 53996	0.21677 46332 60589 06045
	8	3.11347 56523 36544 98403	0.12130 15002 95402 68207
	9	3.47554 69151 98466 54666	0.04560 49082 00260 04102
	10	3.72249 03211 68481 97545	0.00807 33489 99595 07965

3. $\varrho(x) = 1/I_0(x),\ a = +\infty$:

n	k	x_k	A_k
1	1	1.47310 83641 91783 60064	2.08323 32771 13127 95218
2	1	0.85994 11440 15675 54142	1.68527 96108 11361 25049
	2	4.06978 80826 72085 70164	0.39795 36663 01766 70170
3	1	0.59755 34675 56868 22131	1.31493 75299 05595 50844
	2	2.78391 18750 36898 84897	0.73424 13347 08061 85724
	3	7.01875 89825 32937 25977	0.03405 44124 99470 58650
4	1	0.45331 39149 12805 13818	1.05306 26097 75956 07647
	2	2.13558 95289 43486 97404	0.90251 20817 24169 45700
	3	5.12185 75626 39841 77113	0.12563 37834 53695 37282
	4	10.16728 24266 09517 1802	0.00202 48021 59307 04589
5	1	0.36298 25783 15813 27329	0.86853 56833 36953 59899
	2	1.73485 64987 24408 50688	0.96129 35895 64597 42781
	3	4.08840 02959 78222 62846	0.23987 45810 26644 27473
	4	7.73113 12361 47051 67671	0.01343 17967 91856 15687
	5	13.44227 21087 98979 8510	0.00009 76263 93076 49378
6	1	0.30151 68788 09093 54645	0.73449 87050 55025 51775
	2	1.46024 60077 14868 15583	0.96103 53245 61001 79325
	3	3.41763 04991 38670 51478	0.34845 63866 58389 09438
	4	6.33346 62336 46446 50357	0.03815 67869 92360 35923
	5	10.52504 11270 02343 6353	0.00108 19749 48290 46170
	6	16.80563 48913 20209 8367	0.00000 40988 98060 72587

n	r_n
1	1.659E+00
2	7.008E-01
3	2.463E-01
4	7.923E-02
5	2.427E-02
6	7.159E-03
7	2.065E-03
8	5.849E-04
9	1.634E-04
10	4.515E-05

504

n	k	x_k	A_k
7	1	0.25719 28130 60929 86946	0.63388 14421 79722 85943
	2	1.25963 30870 75378 31156	0.93148 83147 06274 65778
	3	2.94166 10663 04726 06097	0.43893 34884 50473 10899
	4	5.39356 68489 20765 95279	0.07444 72415 92360 02467
	5	8.78880 43993 86755 55756	0.00441 07658 14508 73668
	6	13.45335 60054 46439 2050	0.00007 18685 05049 87430
	7	20.23479 55816 33220 2576	0.00000 01558 64738 69033
8	1	0.22382 55043 26641 84575	0.55609 91065 39113 92192
	2	1.10647 82752 34951 19485	0.88878 78296 83998 36066
	3	2.58436 85632 76880 28626	0.50855 35314 40214 59842
	4	4.70885 05701 98697 06913	0.11803 55528 25536 67137
	5	7.59021 39321 60880 65158	0.01134 48734 35362 17610
	6	11.40223 93494 86021 5306	0.00040 82335 53367 93791
	7	16.48452 76170 86259 2225	0.00000 41441 36194 91737
	8	23.71516 09184 23344 8341	0.00000 00054 99339 36845
9	1	0.19786 00369 14523 66909	0.49444 05162 73108 20774
	2	0.98569 93319 61811 69138	0.84142 32147 27086 81061
	3	2.30538 20387 32673 00735	0.55894 08015 70505 76109
	4	4.18429 63508 70013 91124	0.16460 33980 34094 18270
	5	6.69919 82101 70259 59420	0.02243 19355 14115 12146
	6	9.95733 80255 39896 41979	0.00136 11916 62248 90737
	7	14.13944 41319 09732 9650	0.00003 20048 45227 02075
	8	19.59733 34118 88961 3254	0.00000 02143 03798 89280
	9	27.23669 59380 24852 2412	0.00000 00001 82943 04765
10	1	0.17711 56423 97524 12307	0.44451 30313 63133 30267
	2	0.88803 66578 43806 85033	0.79375 38890 34493 66663
	3	2.08104 76942 91994 03659	0.59320 76369 61425 61019
	4	3.76801 32238 68663 82652	0.21078 37160 81969 02376
	5	6.00541 80730 87554 18790	0.03751 18505 28639 45894
	6	8.86595 94139 19861 31376	0.00332 46628 58770 68241
	7	12.46050 83159 44928 1065	0.00013 62761 73233 19971
	8	16.97656 44317 08342 6912	0.00000 22039 45477 48061
	9	22.77674 86309 55387 9507	0.00000 00101 60182 75463
	10	30.79217 32438 36959 0984	0.00000 00000 05802 77263

Beside of this, one can change the integrand

$$\int_0^\infty \frac{f(x)\,dx}{I_0(x)} = \int_0^\infty e^{-x} \cdot \frac{e^x\,f(x)\,dx}{I_0(x)} = \int_0^\infty e^{-x}\,g(x)\,dx$$

and use the Laguerre quadrature formula, see [1], table 25.9, or [15], 7, §5.

4. $\varrho(x) = K_0(x)$, $a = +\infty$:

n	k	x_k	A_k
1	1	0.63661 97723 67581 34308	1.57079 63267 94896 61923E+00
2	1	0.36721 86882 16808 67238	1.39995 12372 93972 84811E+00
	2	2.84416 56970 81442 72853	1.70845 08950 09237 71119E-01
3	1	0.26096 12883 46442 72853	1.22944 21664 78323 37814E+00
	2	1.88024 25951 92307 23720	3.31389 46564 20531 53151E-01
	3	5.62692 59843 49789 79049	9.96469 46745 20087 94482E-03
4	1	0.20346 78616 08216 51366	1.09483 09833 31706 91533E+00
	2	1.42025 85051 16463 38592	4.37739 19226 58012 04252E-01
	3	4.01398 76795 63093 10778	3.77769 61179 76693 12798E-02
	4	8.67787 18602 95781 85143	4.49190 01762 15683 70873E-04

n	r_n
1	4.6709E-01
2	1.2218E-01
3	3.1239E-02
4	7.9193E-03
5	1.9991E-03
6	5.0338E-04
7	1.2656E-04
8	3.1783E-05
9	7.9758E-06
10	2.0003E-06
11	5.0146E-07
12	1.2567E-07
13	3.1483E-08
14	7.8855E-09
15	1.9747E-09

n	k	x_k	A_k
5	1	0.16724 81117 75635 19008	9.88881 05969 949954 3270E-01
	2	1.14562 94187 88403 73001	5.03750 00939 28931 22366E-01
	3	3.16287 39844 90265 02804	7.51092 39652 71542 59028E-02
	4	6.49392 78146 52040 42058	3.03857 89524 58977 43499E-03
	5	11.88748 74318 93429 0386	1.74390 97329 55025 71627E-05
6	1	0.14226 07913 50015 95177	9.03772 753858 39543 9448E-01
	2	0.96187 99865 97519 53802	5.43482 69975 45733 51846E-01
	3	2.62201 32377 19443 00296	1.14485 47202 58102 84508E-01
	4	5.26789 04874 99131 39894	8.85829 44081 57636 25731E-03
	5	9.19710 80556 60770 25097	1.96493 93094 09694 52219E-04
	6	15.20345 83591 31339 3193	6.12817 01893 77195 58116E-07
7	1	0.12394 43636 01194 07490	8.33919 89568 75937 18633E-01
	2	0.82988 83221 59694 48432	5.66346 20956 66190 73155E-01
	3	2.24414 54746 20568 87466	1.51866 97126 90954 23456E-01
	4	4.45570 42365 48512 14226	1.78361 62975 61136 50738E-02
	5	7.62115 63463 58316 49173	8.16158 13396 80016 96570E-04
	6	12.05770 47887 46960 7366	1.09091 25532 15799 63666E-05
	7	18.59639 89087 07247 5628	2.00364 76879 22110 73556E-08
8	1	0.10992 12217 79063 47820	7.75487 94104 82396 66325E-01
	2	0.73030 99184 96626 27276	5.78280 43327 62226 41733E-01
	3	1.96392 13601 62659 60774	1.85527 69447 78725 48967E-01
	4	3.87067 52344 22834 48264	2.92897 69299 20942 43596E-02
	5	6.54552 79256 07758 08568	2.14705 89526 54715 50129E-03
	6	10.15637 02498 02973 1618	6.28886 01672 33874 87484E-05
	7	15.03608 50747 98943 6149	5.40518 99097 91903 55114E-07
	8	22.04801 25531 10893 9881	6.20034 30440 66445 76386E-10
9	1	0.09882 78556 23114 63690	7.25811 60449 17844 26489E-01
	2	0.65242 44716 71066 36745	5.83041 21905 83546 68687E-01
	3	1.74726 54508 81345 84395	2.14974 71416 28028 82723E-01
	4	3.42646 28324 00874 32952	4.23961 25905 71691 85370E-02
	5	5.75292 01007 91584 35353	4.35508 12282 78554 50495E-03
	6	8.82893 56824 62934 01474	2.13330 88358 08940 90902E-04
	7	12.83142 52914 87200 0793	4.22653 88832 24398 80839E-06
	8	18.10638 59719 76917 8300	2.45071 26720 89884 24315E-08
	9	25.54606 35566 72559 7562	1.83683 28901 32509 76716E-11
10	1	0.08982 51394 65073 51273	6.82994 89619 94860 36260E-01
	2	0.58979 23377 43165 66686	5.83042 63419 81469 41508E-01
	3	1.57448 14702 83904 92456	2.40305 65626 39476 58196E-01
	4	3.07648 51680 49643 48192	5.64166 06837 67699 52586E-02
	5	5.14011 28674 71344 69787	7.48668 81075 14947 72824E-03
	6	7.83334 36939 12404 28440	5.31342 50451 01792 72256E-04
	7	11.26446 66379 46739 0096	1.82468 37982 20892 53018E-05
	8	15.61778 56548 08941 9316	2.54810 60245 23016 36845E-07
	9	21.25071 58718 07071 5525	1.03450 40580 33080 17366E-09
	10	29.08192 75401 85443 7448	5.25141 74849 18088 84324E-13
11	1	0.08236 77885 11932 84267	6.45656 12245 15324 32620E-01
	2	0.53830 42174 60810 24499	5.79866 59479 16994 90525E-01
	3	1.43332 57406 93156 32340	2.61869 96123 31840 38039E-01
	4	2.79303 70825 51615 23874	7.07683 11295 16245 38708E-02
	5	4.65010 81703 82186 44747	1.14955 85174 84005 90127E-02
	6	7.05248 30651 19581 69108	1.08295 47372 78113 37447E-03
	7	10.07243 67672 01059 8710	5.54001 06526 25586 03069E-05
	8	13.82312 70627 43583 2047	1.38290 62069 38244 65083E-06
	9	18.49514 52561 69282 3437	1.40572 85846 63155 07890E-08
	10	24.45613 73102 10957 4690	4.11664 16272 01518 94937E-11
	11	32.64927 52185 13629 2464	1.45747 80918 09705 44138E-14

n	k	x_k	A_k
12	1	0.07608 57882 50235 63758	6.12765 46804 50956 12505E-01
	2	0.49521 11095 73286 20413	5.74569 06792 87910 51939E-01
	3	1.31576 02596 15865 36831	2.80099 99156 55699 22646E-01
	4	2.55847 37878 47117 54569	8.50248 36738 95814 23795E-02
	5	4.24831 19117 23435 69224	1.62781 15740 64017 67714E-02
	6	6.42060 11559 58385 67849	1.92028 95070 27545 46607E-03
	7	9.12651 33155 54329 64769	1.33382 38197 61008 91425E-04
	8	12.44182 68474 11704 5934	5.07933 68891 68808 01790E-06
	9	16.48378 26734 16144 0524	9.48282 82781 20804 58381E-08
	10	21.44849 55480 31779 1532	7.20106 97808 86132 95653E-10
	11	27.71296 43975 44468 9734	1.55874 40769 12198 79104E-12
	12	36.24331 06958 33002 4413	3.94450 18083 93934 48913E-16
13	1	0.07071 90377 02890 09607	5.83539 67509 87336 90331E-01
	2	0.45860 28154 46093 47768	5.67866 58120 70556 01925E-01
	3	1.21627 76917 77038 65095	2.95430 91088 04601 42422E-01
	4	2.36096 77216 74187 01875	9.88917 14822 79633 78337E-02
	5	3.91229 73216 52154 83253	2.17030 79294 34218 88261E-02
	6	5.89718 96444 74715 05026	3.07679 16187 92118 60211E-03
	7	8.35347 14273 44963 05193	2.72769 30595 49304 57578E-04
	8	11.33497 56978 23805 8848	1.43803 46693 84918 11724E-05
	9	14.92039 85019 19729 1158	4.18209 18105 32645 01122E-07
	10	19.23051 24236 51812 2933	5.97619 29749 14475 02083E-09
	11	24.46640 19811 69534 7053	3.46371 56342 79385 79301E-11
	12	31.01373 96474 88543 7010	5.65646 84480 67588 05362E-14
	13	39.86030 43545 88647 3104	1.04462 88975 82268 15012E-17
14	1	0.06607 92542 87197 05588	5.57372 16924 85212 42542E-01
	2	0.42710 98238 23474 02059	5.60252 11422 07917 17142E-01
	3	1.13097 22321 82868 98669	3.08265 61805 57273 89040E-01
	4	2.19226 73382 20556 28418	1.12176 92995 23107 03927E-01
	5	3.62679 43065 21450 26432	2.76325 24758 20712 50276E-02
	6	5.45563 27696 16088 09515	4.56709 46057 15565 19263E-03
	7	7.70768 37932 07981 76937	4.94626 45737 19349 95842E-04
	8	10.42286 91489 38411 1168	3.38322 27050 15478 35954E-05
	9	13.65723 57500 28335 6339	1.38549 60545 13256 27717E-06
	10	17.49197 68176 73898 5242	3.14212 15796 72591 01046E-08
	11	22.05099 76317 16098 9700	3.50350 47684 73894 10118E-10
	12	27.53993 42961 55962 9685	1.57802 10894 89395 47540E-12
	13	34.35258 73659 97028 3468	1.97838 98315 53400 48757E-15
	14	43.49729 27023 45869 9242	2.71466 96074 78691 64974E-19
15	1	0.06202 67216 56829 93108	5.33785 61428 41967 65974E-01
	2	0.39972 41701 83036 76645	5.52068 65251 72263 02545E-01
	3	1.05699 38427 49686 81902	3.18962 03512 84810 42727E-01
	4	2.04642 73190 92256 48638	1.24764 35585 43809 99258E-01
	5	3.38100 07046 29827 80444	3.39344 92783 74848 32671E-02
	6	5.07758 78900 45886 99813	6.38953 27442 89064 45733E-03
	7	7.15885 71243 61857 47109	8.18421 70743 05650 14719E-04
	8	9.65540 04705 05974 30396	6.93599 04527 53483 27028E-05
	9	12.60888 84012 86229 6717	3.73845 99172 19009 07461E-06
	10	16.07725 14867 41133 4419	1.21209 96048 29684 18562E-07
	11	20.14387 34572 83771 8962	2.18138 00928 92865 22647E-09
	12	24.93549 37021 98155 1603	1.92894 31807 61151 09915E-11
	13	30.66197 25884 29149 5671	6.85674 74694 94063 76532E-14
	14	37.72478 78416 60054 6250	6.69959 94575 86869 67315E-17
	15	47.15187 79364 92118 7526	6.93798 54584 51528 91456E-21

6. Miscellaneous:

$$\int x^n \cdot J_0(x) \cdot J_1^{n-1}(x)\, dx = \frac{x^n}{n}\, J_1^n(x)\,, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots\ldots$$

$$\int \frac{dx}{x^n \cdot J_0(x) \cdot J_1^{n+1}(x)} = -\frac{1}{n\, x^n\, J_1^n(x}\,, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots\ldots$$

$$\int x^n \cdot I_0(x) \cdot I_1^{n-1}(x)\, dx = \frac{x^n}{n}\, I_1^n(x)\,, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots\ldots$$

$$\int \frac{dx}{x^n \cdot I_0(x) \cdot I_1^{n+1}(x)} = -\frac{1}{n\, x^n\, I_1^n(x)}\,, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots\ldots$$

$$\int \frac{\Phi(x)}{x}\, dx = \Lambda_1(x) - x\, J_0(x) - \Phi(x)$$

$$\int \frac{x^5\, J_1(px^2 + i)\, dx}{px^2 + i} = \frac{1}{2p^3}\left[J_1(px^2 + i) - (px^2 - i)J_0(px^2 + i)\right]$$

$$\int \frac{x^7\, J_1(px^2 + \sqrt{3}\, i)\, dx}{px^2 + \sqrt{3}\, i} = \frac{1}{2p^4}\left[(2px^2 - \sqrt{3}\, i)\, J_1(px^2 + \sqrt{3}\, i) - (p^2 x^4 - \sqrt{3}\, px^2\, i - 3)J_0(px^2 + \sqrt{3}\, i)\right]$$

7. Used special functions and defined functions:

Used functions:

$J_\nu(x)$	Bessel function of the first kind	
$I_\nu(x)$	Modified Bessel function (of the first kind)	
$Y_\nu(x)$	Bessel function of the second kind, Neumann's function, Weber's function	
$K_\nu(x)$	Modified Bessel function (of the second [third] kind), MacDonald Function	
$H_\nu^{(p)}(x)$, $p = 1, 2$	Bessel function of the third kind, Hankel function	
$\Gamma(x)$	Gamma function $\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt$	[1] 6.1, [2] 1.1, [4] II.1.,[5] V, [7]
$\Gamma(x, z)$	Upper incomplete Gamma function $\Gamma(x, z) = \int_z^\infty t^{x-1}\, e^{-t}\, dt$	[1] 6.5, [6] 8.35
$E_1(x)$	Exponential integral $E_1(x) = \int_x^\infty \frac{e^{-t}\, dt}{t} = \int_1^\infty \frac{e^{-xt}\, dt}{t}$	[1] 5.1, [4] II.5.
$\mathbf{H}_\nu(x)$	Struve functions	[1] 12.1, [2] 10.1, [5] XIII 2., [7] 8.55
$\mathbf{L}_\nu(x)$	Modified Struve functions	[1] 12.2, [2] 10.1, [7] 8.55
$Ji_0(x)$	$Ji_0(x) = \int_x^\infty \frac{J_0(t)\, dt}{t}$	[1] 11.1.19, [9]
$s_{\mu,\nu}(x)$	Lommel functions	[2] 10.1, [7] 8.57
$P_n^m(x)$	(Associated) Legendre functions of the first kind	[1] 8, [5] XII 3.1.
$P_n(x)$	Legendre polynom	$w(x) \equiv 1$
$T_n(x)$	Chebyshev polynom, first kind	$w(x) = 1/\sqrt{1 - x^2}$
$U_n(x)$	Chebyshev polynom, second kind	$w(x) = \sqrt{1 - x^2}$
$C_n^{(\alpha)}(x)$	Ultraspherical (Gegenbauer) Polynomials	$w(x) = (1 - x^2)^{\alpha - 1/2}$
$C_n^{(\alpha,\beta)}(x)$	Jacobi Polynomials	$w(x) = (1 - x)^\alpha (1 + x)^\beta$
$L_n(x)$	Laguerre polynom	$w(x) = e^{-x}$
$L_n^{(\alpha)}(x)$	Generalized Laguerre polynom	$w(x) = x^\alpha\, e^{-x}$
$H_n(x)$	Hermite polynom	$w(x) = \exp(-x^2)$
$\mathbf{E}(x)$, $\mathbf{K}(x)$	Complete elliptic integrals	[1], 17.3., [5] IX 30., [7] 8.11

Defined functions:

Function	Page
$\Phi(x) = \frac{\pi x}{2}[J_1(x) \cdot \mathbf{H}_0(x) - J_0(x) \cdot \mathbf{H}_1(x)]$	7
$\Phi_Y(x) = \frac{\pi x}{2}[Y_1(x) \cdot \mathbf{H}_0(x) - Y_0(x) \cdot \mathbf{H}_1(x)]$	7
$\Phi_H^{(1)} = \frac{\pi x}{2}[H_1^{(1)}(x) \cdot \mathbf{H}_0(x) - H_0^{(1)}(x) \cdot \mathbf{H}_1(x)]$	7
$\Phi_H^{(2)} = \frac{\pi x}{2}[H_1^{(2)}(x) \cdot \mathbf{H}_0(x) - H_0^{(2)}(x) \cdot \mathbf{H}_1(x)]$	7
$\Psi(x) = \frac{\pi x}{2}[I_0(x) \cdot \mathbf{L}_1(x) - I_1(x) \cdot \mathbf{L}_0(x)]$	7
$\Psi_K(x) = \frac{\pi x}{2}[K_0(x) \cdot \mathbf{L}_1(x) + K_1(x) \cdot \mathbf{L}_0(x)]$	7
$\Theta(x) = \int_0^x J_0^2(t)\,dt$	278
$\Omega(x) = \int_0^x I_0^2(t)\,dt$	278
$\Lambda_0(x) = \int_0^x J_0(t)\,dt$	122
$\Lambda_0^* = \int_0^x I_0(t)\,dt(x)$	124
$\Lambda_1(x) = \int_0^x t^{-1} \cdot \Lambda_0(t)\,dt$	124
$\Lambda_1^*(x) = \int_0^x t^{-1} \cdot \Lambda_0^*(t)\,dt$	126
$\Theta_0(x;\gamma) = \int_0^x J_0(t)\,J_0(\gamma t)\,dt$	310
$\Omega_0(x;\gamma) = \int_0^x I_0(t)\,I_0(\gamma t)\,dt$	310
$\Theta_1(x;\gamma) = \int_0^x J_1(t)\,J_1(\gamma t)\,dt$	312
$\Omega_1(x;\gamma) = \int_0^x I_1(t)\,I_1(\gamma t)\,dt$	312
$\mathfrak{H}_p(x,a),\ p = 0,1$	79
$\mathfrak{H}_p^*(x,a),\ p = 0,1$	85