Werner Rosenheinrich
Ernst - Abbe - Hochschule Jena
University of Applied Sciences
Departement of Basic Sciences
Germany

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TABLES OF SOME INDEFINITE INTEGRALS

OF BESSEL FUNCTIONS OF INTEGER ORDER

Integrals of the type

$$\int x J_0^2(x)\, dx \qquad \text{or} \qquad \int x J_0(ax) J_0(bx)\, dx$$

are well-known.

Most of the following integrals are not found in the widely used tables of Gradstein/Ryshik, Bateman/Erdélyi, Abramowitz/ Stegun, Prudnikov/Brychkov/Marichev or Jahnke/Emde/Lösch. The goal of this table was to get tables for practicians. So the integrals should be expressed by Bessel and Struve functions. Indeed, there occured some exceptions. Generally, integrals of the type $\int x^\mu J_\nu(x)\, dx$ may be written with Lommel functions, see [8], 10 -74, or [3], III .
In many cases reccurence relations define more integrals in a simple way.

Partially the integrals may be found by MAPLE as well. In some cases MAPLE gives results with hypergeometric functions, see also [2], 9.6., or [4].
Some well-known integrals are included for completeness.

Here $Z_\nu(x)$ denotes some Bessel function or modified Bessel function of the first or second kind. Partially the functions $Y_\nu(x)$ [sometimes called Neumann's functions or Weber's functions and denoted by $N_\nu(x)$] and the Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are also considered. The same holds for the modified Bessel function of the second kind $K_\nu(x)$.

When a formula is continued in the next line, then the last sign '+' or '-' is repeated in the beginning of the new line.

On page 499 the used special functions and defined functions are described.

I wish to express my thanks to B. Eckstein, S. O. Zafra, Yao Sun, F. Nouguier, M. Carbonell, R. Oliver and Hunchul Jeong for their remarks.

The previous variant of this treatise was placed into the Web more then two years ago. Since then there was a reorganization of the servers of my university. So the address

http://www.eah-jena.de/rsh/Forschung/Stoer/besint.pdf

was lost. I am sorry for this. I am discharged from active service. Because of Covid 19 I had no access to my account for the university was closed for me. In the end the text was anew placed with the new address

https://www.eah-jena.de/fileadmin/user_upload/eah-jena.de/fachbereich/gw/Ehemalige/rosenheinrich/Rosenheinrich_2011_2012_.pdf

and I am going not to move it anymore as long as possible.

During this two years the integrals were revised from the beginning to the end. Errors and misprints were corrected and more formulas added. The chapter 2.2. was thorough done over again.
For some defined integrals with Bessel functions Gaussian integration formulas are given.

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Contents

1. Integrals with one Bessel function

3. Products of three Bessel Functions

Part I

Integrals with one Bessel Function

1. Integrals with one Bessel Function:

See also [10], 2. .

1.1. $x^n Z_\nu(x)$ with integer values of n

1.1.1. Integrals of the type $\int x^{2n} Z_0(x)\, dx$

Let

$$\Phi(x) = \frac{\pi x}{2}\left[J_1(x)\cdot\mathbf{H}_0(x) - J_0(x)\cdot\mathbf{H}_1(x)\right] ,$$

where $\mathbf{H}_\nu(x)$ denotes the Struve function, see [1], chapter 11.1.7, 11.1.8 and 12.
And let

$$\Psi(x) = \frac{\pi x}{2}\left[I_0(x)\cdot\mathbf{L}_1(x) - I_1(x)\cdot\mathbf{L}_0(x)\right]$$

be defined with the modified Struve function $\mathbf{L}_\nu(x)$.
Furthermore, let

$$\Phi_Y(x) = \frac{\pi x}{2}\left[Y_1(x)\cdot\mathbf{H}_0(x) - Y_0(x)\cdot\mathbf{H}_1(x)\right] ,$$

$$\Phi_H^{(1)}(x) = \frac{\pi x}{2}\left[H_1^{(1)}(x)\cdot\mathbf{H}_0(x) - H_0^{(1)}(x)\cdot\mathbf{H}_1(x)\right] ,$$

$$\Phi_H^{(2)}(x) = \frac{\pi x}{2}\left[H_1^{(2)}(x)\cdot\mathbf{H}_0(x) - H_0^{(2)}(x)\cdot\mathbf{H}_1(x)\right]$$

and

$$\Psi_K(x) = \frac{\pi x}{2}\left[K_0(x)\cdot\mathbf{L}_1(x) + K_1(x)\cdot\mathbf{L}_0(x)\right]$$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ and simultaneously $\Phi(x)$ by $\Phi_Y(x)$ or $H_\nu^{(p)}(x),\ p=1,2$ and $\Phi_H^{(p)}(x)$.

Well-known integrals:

$$\int J_0(x)\, dx = xJ_0(x) + \Phi(x) = \Lambda_0(x)$$

$$\int I_0(x)\, dx = xI_0(x) + \Psi(x) = \Lambda_0^*(x)$$

$$\int K_0(x)\, dx = xK_0(x) + \Psi_K(x)$$

The new-defined function $\Lambda_0(x)$ is discussed in 1.2.11 a) on page 119 and so is $\Lambda_0^*(x)$ on page 121.
See also [1], 11.1 .

$$\int Y_0(x)\, dx = xY_0(x) + \Phi_Y(x)$$

$$\int H_0^{(p)}(x)\, dx = xH_0^{(p)}(x) + \Phi_H^{(p)}(x) , \quad p = 1,\ 2$$

$$\int x^2\, J_0(x)\, dx = x^2 J_1(x) - \Phi(x)$$

$$\int x^2\, I_0(x)\, dx = x^2 I_1(x) + \Psi(x)$$

$$\int x^2\, K_0(x)\, dx = -x^2 K_1(x) + \Psi_K(x)$$

$$\int x^4\, J_0(x)\, dx = (x^4 - 9x^2)J_1(x) + 3x^3 J_0(x) + 9\Phi(x)$$

$$\int x^4\, I_0(x)\, dx = (x^4 + 9x^2)I_1(x) - 3x^3 I_0(x) + 9\Psi(x)$$

$$\int x^4\, K_0(x)\, dx = -(x^4 + 9x^2)K_1(x) - 3x^3 K_0(x) + 9\Psi_K(x)$$

$$\int x^6\, J_0(x)\, dx \;=\; (x^6 - 25x^4 + 225x^2)J_1(x) + (5x^5 - 75x^3)J_0(x) - 225\Phi(x)$$

$$\int x^6\, I_0(x)\, dx \;=\; (x^6 + 25x^4 + 225x^2)I_1(x) - (5x^5 + 75x^3)I_0(x) + 225\Psi(x)$$

$$\int x^6\, K_0(x)\, dx \;=\; -(x^6 + 25x^4 + 225x^2)K_1(x) - (5x^5 + 75x^3)K_0(x) + 225\Psi_K(x) \quad \text{and so on.}$$

$$\int x^8\, J_0(x)\, dx \;=\; (x^8 - 49x^6 + 1\,225x^4 - 11\,025x^2)J_1(x) + (7x^7 - 245x^5 + 3\,675x^3)J_0(x) + 11\,025\Phi(x)$$

$$\int x^8\, I_0(x)\, dx \;=\; (x^8 + 49x^6 + 1\,225x^4 + 11\,025x^2)I_1(x) - (7x^7 + 245x^5 + 3\,675x^3)I_0(x) + 11\,025\Psi(x)$$

$$\begin{aligned}\int x^{10}\, J_0(x)\, dx \;&=\; (x^{10} - 81x^8 + 3\,969x^6 - 99\,225x^4 + 893\,025)J_1(x) + \\ &+(9x^9 - 567x^7 + 19\,845x^5 - 297\,675x^3)J_0(x) - 893\,025\Phi(x)\end{aligned}$$

$$\begin{aligned}\int x^{10}\, I_0(x)\, dx \;&=\; (x^{10} + 81x^8 + 3\,969x^6 + 99\,225x^4 + 893\,025)I_1(x) - \\ &-(9x^9 + 567x^7 + 19\,845x^5 + 297\,675x^3)I_0(x) + 893\,025\Psi(x)\end{aligned}$$

$$\begin{aligned}\int x^{12}\, J_0(x)\, dx \;&=\; (11x^{11} - 1\,089x^9 + 68\,607x^7 - 2\,401\,245x^5 + 36\,018\,675x^3)J_0(x) + \\ &+(x^{12} - 121x^{10} + 9\,801x^8 - 480\,249x^6 + 12\,006\,225x^4 - 108\,056\,025x^2)J_1(x) + 108\,056\,025\Phi(x)\end{aligned}$$

$$\begin{aligned}\int x^{12}\, I_0(x)\, dx \;&=\; (x^{12} + 121x^{10} + 9\,801x^8 + 480\,249x^6 + 12\,006\,225x^4 + 108\,056\,025x^2)I_1(x) - \\ &-(11x^{11} + 1\,089x^9 + 68\,607x^7 + 2\,401\,245x^5 + 36\,018\,675x^3)I_0(x) + 108\,056\,025\Psi(x)\end{aligned}$$

Let

$$n!! \;=\; \begin{cases} 2 \cdot 4 \cdot \ldots \cdot (n-2) \cdot n & , \quad n = 2m \\ 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-2) \cdot n & , \quad n = 2m+1 \end{cases}$$

and $n!! = 1$ in the case $n \leq 0$.

<u>General formulas:</u>

$$\begin{aligned}\int x^{2n}\, J_0(x)\, dx &= \left(\sum_{k=0}^{n-2} (-1)^k \frac{[(2n-1)!!]^2\, x^{2n-2k-1}}{[(2n-1-2k)!!] \cdot [(2n-3-2k)!!]} \right) J_0(x) + \\ &+ \left(\sum_{k=0}^{n-1} (-1)^k \left[\frac{(2n-1)!!}{(2n-1-2k)!!} \right]^2 x^{2n-2k} \right) J_1(x) + (-1)^n\, [(2n-1)!!]^2\, \Phi(x) = \\ &= \left(\sum_{k=0}^{n-2} (-1)^k \frac{[(2n)!]^2 \cdot (n-k)! \cdot (n-k-1)! \cdot x^{2n-2k-1}}{2^{2k+1} \cdot (n!)^2 \cdot (2n-2k)! \cdot (2n-2-2k)!} \right) J_0(x) + \\ &+ \left(\sum_{k=0}^{n-1} (-1)^k \left[\frac{(2n)! \cdot (n-k)!}{2^k \cdot (n!) \cdot (2n-2k)!} \right]^2 x^{2n-2k} \right) J_1(x) + (-1)^n \left[\frac{(2n)!}{2^n \cdot n!} \right]^2 \Phi(x)\end{aligned}$$

and

$$\begin{aligned}\int x^{2n}\, I_0(x)\, dx &= \left(\sum_{k=0}^{n-1} \left[\frac{(2n-1)!!}{(2n-1-2k)!!} \right]^2 x^{2n-2k} \right) I_1(x) - \\ &- \left(\sum_{k=0}^{n-2} \frac{[(2n-1)!!]^2\, x^{2n-2k-1}}{[(2n-1-2k)!!] \cdot [(2n-3-2k)!!]} \right) I_0(x) + [(2n-1)!!]^2\, \Psi(x) =\end{aligned}$$

$$= \left(\sum_{k=0}^{n-1} \left[\frac{(2n)! \cdot (n-k)!}{2^k \cdot (n!) \cdot (2n-2k)!} \right]^2 x^{2n-2k} \right) I_1(x) -$$

$$- \left(\sum_{k=0}^{n-2} \frac{[(2n)!]^2 \cdot (n-k)! \cdot (n-k-1)! \cdot x^{2n-2k-1}}{2^{2k+1} \cdot (n!)^2 \cdot (2n-2k)! \cdot (2n-2-2k)!} \right) I_0(x) + \left[\frac{(2n)!}{2^n \cdot n!} \right]^2 \Psi(x)$$

Recurrence formulas:

$$\int x^{2n+2} J_0(x)\, dx = (2n+1) x^{2n+1} J_0(x) + x^{2n+2} J_1(x) - (2n+1)^2 \int x^{2n} J_0(x)\, dx$$

$$\int x^{2n+2} I_0(x)\, dx = -(2n+1) x^{2n+1} I_0(x) + x^{2n+2} I_1(x) + (2n+1)^2 \int x^{2n} I_0(x)\, dx$$

$$\int x^{2n+2} K_0(x)\, dx = -(2n+1) x^{2n+1} K_0(x) - x^{2n+2} K_1(x) + (2n+1)^2 \int x^{2n} K_0(x)\, dx$$

In the case $n < 0$ the previous formulas give

$$\int \frac{J_0(x)}{x^2}\, dx = J_1(x) - \frac{x^2+1}{x} J_0(x) - \Phi(x)$$

$$\int \frac{I_0(x)}{x^2}\, dx = \frac{x^2-1}{x} I_0(x) - I_1(x) + \Psi(x)$$

$$\int \frac{K_0(x)}{x^2}\, dx = \frac{x^2-1}{x} K_0(x) + K_1(x) + \Psi_K(x)$$

$$\int \frac{J_0(x)}{x^4}\, dx = \frac{1}{9} \left[\frac{x^4+x^2-3}{x^3} J_0(x) - \frac{x^2-1}{x^2} J_1(x) + \Phi(x) \right]$$

$$\int \frac{I_0(x)}{x^4}\, dx = \frac{1}{9} \left[\frac{x^4-x^2-3}{x^3} I_0(x) - \frac{x^2+1}{x^2} I_1(x) + \Psi(x) \right]$$

$$\int \frac{K_0(x)}{x^4}\, dx = \frac{1}{9} \left[\frac{x^4-x^2-3}{x^3} K_0(x) + \frac{x^2+1}{x^2} K_1(x) + \Psi_K(x) \right]$$

$$\int \frac{J_0(x)}{x^6}\, dx = \frac{1}{225} \left[\frac{x^4-x^2+9}{x^4} J_1(x) - \frac{x^6+x^4-3x^2+45}{x^5} J_0(x) - \Phi(x) \right]$$

$$\int \frac{I_0(x)}{x^6}\, dx = \frac{1}{225} \left[\frac{x^6-x^4-3x^2-45}{x^5} I_0(x) - \frac{x^4+x^2+9}{x^4} I_1(x) + \Psi(x) \right]$$

$$\int \frac{K_0(x)}{x^6}\, dx = \frac{1}{225} \left[\frac{x^6-x^4-3x^2-45}{x^5} K_0(x) + \frac{x^4+x^2+9}{x^4} K_1(x) + \Psi_K(x) \right] \quad \text{and so on.}$$

$$\int \frac{J_0(x)}{x^8}\, dx = \frac{1}{11\,025} \left[\frac{x^8+x^6-3x^4+45x^2-1\,575}{x^7} J_0(x) - \frac{x^6-x^4+9x^2-225}{x^6} J_1(x) + \Phi(x) \right]$$

$$\int \frac{I_0(x)}{x^8}\, dx = \frac{1}{11\,025} \left[\frac{x^8-x^6-3x^4-45x^2-1\,575}{x^7} I_0(x) - \frac{x^6+x^4+9x^2+225}{x^6} I_1(x) + \Psi(x) \right]$$

$$\int \frac{J_0(x)}{x^{10}}\, dx = \frac{1}{893\,025} \left[\frac{x^8-x^6+9x^4-225x^2+11\,025}{x^8} J_1(x) - \right.$$

$$\left. - \frac{x^{10}+x^8-3x^6+45x^4-1\,575x^2+99\,225}{x^9} J_0(x) - \Phi(x) \right]$$

$$\int \frac{I_0(x)}{x^{10}}\, dx = \frac{1}{893\,025} \left[\frac{x^{10}-x^8-3x^6-45x^4-1\,575x^2-99\,225}{x^9} I_0(x) - \right.$$

$$\left. - \frac{x^8+x^6+9x^4+225x^2+11\,025}{x^8} I_1(x) + \Psi(x) \right]$$

$$\int \frac{J_0(x)}{x^{12}}\,dx \;=\; \frac{1}{108\,056\,025}\left[\frac{x^{12}+x^{10}-3x^8+45x^6-1\,575x^4+99\,225x^2-9\,823\,275}{x^{11}}\,J_0(x)\,-\right.$$

$$\left.-\;\frac{x^{10}-x^8+9x^6-225x^4+11\,025x^2-893\,025}{x^{10}}J_1(x)+\Phi(x)\right]$$

$$\int \frac{I_0(x)}{x^{12}}\,dx \;=\; \frac{1}{108\,056\,025}\left[\frac{x^{12}-x^{10}-3x^8-45x^6-1\,575x^4-99\,225x^2-9\,823\,275}{x^{11}}\,I_0(x)\,-\right.$$

$$\left.-\;\frac{x^{10}+x^8+9x^6+225x^4+11\,025x^2+893\,025}{x^{10}}I_1(x)+\Psi(x)\right]$$

General formula: With $n!!$ as defined on page 10 holds

$$\int \frac{J_0(x)\,dx}{x^{2n}} = \frac{(-1)^n}{[(2n-1)!!]^2}\left[\left(x+\sum_{k=0}^{n-1}(-1)^k\cdot(2k+1)!!\cdot(2k-1)!!\cdot x^{-2k-1}\right)J_0(x)-\right.$$

$$\left.-\left(1-\sum_{k=0}^{n-2}(-1)^k\cdot[(2k+1)!!]^2\,x^{-2k-2}\right)J_1(x)+\Phi(x)\right]=$$

$$=\frac{(-1)^n\cdot 2^{2n}\cdot(n!)^2}{(2n)!}\left\{\left(x+\sum_{k=0}^{n-1}(-1)^k\frac{(2k+2)!\cdot(2k)!}{2^{2k+1}\cdot(k+1)!\cdot k!\cdot x^{2k+1}}\right)J_0(x)-\right.$$

$$\left.-\left(1-\sum_{k=0}^{n-2}\frac{(-1)^k}{x^{2k+2}}\left[\frac{(2k+2)!}{2^{k+1}\cdot(k+1)!}\right]^2\right)J_1(x)+\Phi(x)\right\}$$

With obviously modifications one gets the the formulas for the integrals $\int x^{-2n}I_0(x)\,dx$ and $\int x^{-2n}K_0(x)\,dx$.

1.1.2. Integrals of the type $\int x^{2n+1} Z_0(x)\,dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x),\ p = 1, 2.$

$$\int x\,J_0(x)\,dx \;=\; xJ_1(x)$$

$$\int x\,I_0(x)\,dx \;=\; x\,I_1(x)$$

$$\int x\,K_0(x)\,dx \;=\; -x\,K_1(x)$$

$$\int x^3\,J_0(x)\,dx \;=\; x\left[2x\,J_0(x) + (x^2-4)\,J_1(x)\right]$$

$$\int x^3\,I_0(x)\,dx \;=\; x\left[(x^2+4)\,I_1(x) - 2x\,I_0(x)\right]$$

$$\int x^3\,K_0(x)\,dx \;=\; -x\left[(x^2+4)\,K_1(x) + 2x\,K_0(x)\right]$$

$$\int x^5\,J_0(x)\,dx \;=\; x\left[(4x^3-32x)\,J_0(x) + (x^4-16x^2+64)\,J_1(x)\right]$$

$$\int x^5\,I_0(x)\,dx \;=\; x\left[(x^4+16x^2+64)\,I_1(x) - (4x^3+32x)\,I_0(x)\right]$$

$$\int x^5\,K_0(x)\,dx \;=\; -x\left[(x^4+16x^2+64)\,K_1(x) + (4x^3+32x)\,K_0(x)\right]$$

$$\int x^7\,J_0(x)\,dx \;=\; x\left[(6x^5-144x^3+1\,152x)\,J_0(x) + (x^6-36x^4+576x^2-2\,304)\,J_1(x)\right]$$

$$\int x^7\,I_0(x)\,dx \;=\; x\left[(x^6+36x^4+576x^2+2\,304)\,I_1(x) - (6x^5+144x^3+1\,152x)\,I_0(x)\right]$$

$$\int x^7\,K_0(x)\,dx \;=\; -x\left[(x^6+36x^4+576x^2+2\,304)\,K_1(x) + (6x^5+144x^3+1\,152x)\,K_0(x)\right]$$

$$\int x^9\,J_0(x)\,dx \;=$$

$$=\; x\left[(8x^7-384x^5+9\,216x^3-73\,728x)\,J_0(x) + (x^8-64x^6+2\,304x^4-36\,864x^2+147\,456)\,J_1(x)\right]$$

$$\int x^9\,I_0(x)\,dx \;=$$

$$=\; x\left[(x^8+64x^6+2\,304x^4+36\,864x^2+147\,456)\,I_1(x) - (8x^7+384x^5+9\,216x^3+73\,728x)\,I_0(x)\right]$$

$$\int x^9\,K_0(x)\,dx \;=$$

$$=\; -x\left[(x^8+64x^6+2\,304x^4+36\,864x^2+147\,456)\,K_1(x) + (8x^7+384x^5+9\,216x^3+73\,728x)\,K_0(x)\right]$$

Let

$$\int x^m J_0(x)\,dx = x[P_m(x)J_0(x) + Q_m(x)J_1(x)] \quad \text{and} \quad \int x^m I_0(x)\,dx = x[Q_m^*(x)I_1(x) - P_m^*(x)I_0(x)]\,,$$

$$\int x^m K_0(x)\,dx = -x[Q_m^*(x)K_1(x) + P_m^*(x)K_0(x)]\,,$$

then holds

$P_{11}(x) = 10\,x^9 - 800\,x^7 + 38400\,x^5 - 921600\,x^3 + 7372800\,x$
$Q_{11}(x) = x^{10} - 100\,x^8 + 6400\,x^6 - 230400\,x^4 + 3686400\,x^2 - 14745600$
$P_{11}^*(x) = 10\,x^9 + 800\,x^7 + 38400\,x^5 + 921600\,x^3 + 7372800\,x$
$Q_{11}^*(x) = x^{10} + 100\,x^8 + 6400\,x^6 + 230400\,x^4 + 3686400\,x^2 + 14745600$

$P_{13}(x) = 12\,x^{11} - 1440\,x^9 + 115200\,x^7 - 5529600\,x^5 + 132710400\,x^3 - 1061683200\,x$
$Q_{13}(x) = x^{12} - 144\,x^{10} + 14400\,x^8 - 921600\,x^6 + 33177600\,x^4 - 530841600\,x^2 + 2123366400$
$P^*_{13}(x) = 12\,x^{11} + 1440\,x^9 + 115200\,x^7 + 5529600\,x^5 + 132710400\,x^3 + 1061683200\,x$
$Q^*_{13}(x) = x^{12} + 144\,x^{10} + 14400\,x^8 + 921600\,x^6 + 33177600\,x^4 + 530841600\,x^2 + 2123366400$

$P_{15}(x) = 14\,x^{13} - 2352\,x^{11} + 282240\,x^9 - 22579200\,x^7 + 1083801600\,x^5 - 26011238400\,x^3 + 208089907200\,x$
$Q_{15}(x) =$
$x^{14} - 196\,x^{12} + 28224\,x^{10} - 2822400\,x^8 + 180633600\,x^6 - 6502809600\,x^4 + 104044953600\,x^2 - 416179814400$
$P^*_{15}(x) =$
$= 14\,x^{13} - 2352\,x^{11} + 282240\,x^9 + 22579200\,x^7 + 1083801600\,x^5 + 26011238400\,x^3 + 208089907200\,x$
$Q^*_{15}(x) =$
$x^{14} + 196\,x^{12} + 28224\,x^{10} + 2822400\,x^8 + 180633600\,x^6 + 6502809600\,x^4 + 104044953600\,x^2 + 416179814400$

Recurrence formulas:

$$\int x^{2n+1}\,J_0(x)\,dx \;=\; 2nx^{2n}\,J_0(x) + x^{2n+1}\,J_1(x) - 4n^2 \int x^{2n-1}\,J_0(x)\,dx$$

$$\int x^{2n+1}\,I_0(x)\,dx \;=\; -2nx^{2n}\,I_0(x) + x^{2n+1}\,I_1(x) + 4n^2 \int x^{2n-1}\,I_0(x)\,dx$$

$$\int x^{2n+1}\,K_0(x)\,dx \;=\; -2nx^{2n}\,K_0(x) - x^{2n+1}\,K_1(x) + 4n^2 \int x^{2n-1}\,K_0(x)\,dx$$

General formula: With $n!!$ as defined on page 10 holds

$$\int x^{2n+1}\,J_0(x)\,dx = \left(\sum_{k=0}^{n-1} (-1)^k\,\frac{[(2n)!!]^2\,x^{2n-2k}}{[(2n-2k)!!]\cdot[(2n-2k-2)!!]}\right) J_0(x) +$$

$$+\left(\sum_{k=0}^{n} (-1)^k \left[\frac{(2n)!!}{(2n-2k)!!}\right]^2 x^{2n+1-2k}\right) J_1(x) =$$

$$= \left(\sum_{k=0}^{n-1} (-1)^k\,\frac{2^{2k+1}\cdot(n!)^2\,x^{2n-2k}}{(n-k)!\cdot(n-k-1)!}\right) J_0(x) + \left(\sum_{k=0}^{n} (-1)^k \left[\frac{2^k\cdot n!}{(n-k)!}\right]^2 x^{2n+1-2k}\right) J_1(x)\;.$$

With obviously modifications one gets the the formulas for the integrals $\int x^{2n+1}\,I_0(x)\,dx$ and $\int x^{2n+1}\,K_0(x)\,dx$.

1.1.3. Integrals of the type $\int x^{-2n-1} \cdot Z_0(x)\,dx$

The basic integral
$$\int \frac{J_0(x)\,dx}{x} \quad \text{can be expressed by} \quad \int_0^x \frac{1-J_0(t)}{t}\,dt \quad \text{or} \quad -\int_x^\infty \frac{J_0(t)\,dt}{t} = Ji_0(x)\;,$$
see [1], equation 11.1.19 and the following formulas. There are given asymptotic expansions and polynomial approximations as well. Tables of these functions may be found by [1], [11.13] or [11.22]. The function $Ji_0(x)$ is introduced and discussed in [9].
For fast computations of this integrals one should use approximations with Chebyshev polynomials, see [2], tables 9.3 .
I got the information from F. Nouguier, that there is an error in a formula in [9], p. 278.
He found the following true formula:
$$Ji_0(x) - \ln x = \frac{\sin \pi x}{\pi x}(\gamma - \ln 2) + \frac{2x \sin \pi x}{\pi} \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s^2 - x^2}\,[Ji_0(s) - \ln s]\;.$$
The power series in
$$\int \frac{I_0(x)\,dx}{x} \quad = \quad \ln x \;+\; \sum_{k=1}^{\infty} \frac{1}{2k \cdot (k!)^2} \left(\frac{x}{2}\right)^{2k}$$
can be used without numerical problems.

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$.
$$\int \frac{J_0(x)\,dx}{x^3} = -\frac{J_0(x)}{2x^2} + \frac{J_1(x)}{4x} - \frac{1}{4} \int \frac{J_0(x)\,dx}{x}$$
$$\int \frac{I_0(x)\,dx}{x^3} = -\frac{I_0(x)}{2x^2} - \frac{I_1(x)}{4x} + \frac{1}{4} \int \frac{I_0(x)\,dx}{x}$$
$$\int \frac{J_0(x)\,dx}{x^5} = \left(\frac{1}{32x^2} - \frac{1}{4x^4}\right) J_0(x) + \left(-\frac{1}{64x} + \frac{1}{16x^3}\right) J_1(x) + \frac{1}{64} \int \frac{J_0(x)\,dx}{x}$$
$$\int \frac{I_0(x)\,dx}{x^5} = -\left(\frac{1}{32x^2} + \frac{1}{4x^4}\right) I_0(x) - \left(\frac{1}{64x} + \frac{1}{16x^3}\right) I_1(x) + \frac{1}{64} \int \frac{I_0(x)\,dx}{x}$$
$$\int \frac{J_0(x)\,dx}{x^7} = \frac{-x^4 + 8\,x^2 - 192}{1152\,x^6}\,J_0(x) + \frac{x^4 - 4\,x^2 + 64}{2304\,x^5}\,J_1(x) - \frac{1}{2304} \int \frac{J_0(x)\,dx}{x}$$
$$\int \frac{I_0(x)\,dx}{x^7} = -\frac{x^4 + 8\,x^2 + 192}{1152\,x^6}\,I_0(x) - \frac{x^4 + 4\,x^2 + 64}{2304\,x^5}\,I_1(x) + \frac{1}{2304} \int \frac{I_0(x)\,dx}{x}$$
$$\int \frac{J_0(x)\,dx}{x^9} =$$
$$= \frac{x^6 - 8\,x^4 + 192\,x^2 - 9216}{73728\,x^8}\,J_0(x) + \frac{-x^6 + 4\,x^4 - 64\,x^2 + 2304}{147456\,x^7}\,J_1(x) + \frac{1}{147456} \int \frac{J_0(x)\,dx}{x}$$
$$\int \frac{I_0(x)\,dx}{x^9} =$$
$$= -\frac{x^6 + 8\,x^4 + 192\,x^2 + 9216}{73728\,x^8}\,I_0(x) - \frac{x^6 + 4\,x^4 + 64\,x^2 + 2304}{147456\,x^7}\,I_1(x) + \frac{1}{147456} \int \frac{I_0(x)\,dx}{x}$$
$$\int \frac{J_0(x)\,dx}{x^{11}} = \frac{-x^8 + 8\,x^6 - 192\,x^4 + 9216\,x^2 - 737280}{7372800\,x^{10}}\,J_0(x) +$$
$$+ \frac{x^8 - 4\,x^6 + 64\,x^4 - 2304\,x^2 + 147456}{14745600\,x^9}\,J_1(x) - \frac{1}{14745600} \int \frac{J_0(x)\,dx}{x}$$
$$\int \frac{I_0(x)\,dx}{x^{11}} = -\frac{x^8 + 8\,x^6 + 192\,x^4 + 9216\,x^2 + 737280}{7372800\,x^{10}}\,I_0(x) -$$
$$- \frac{x^8 + 4\,x^6 + 64\,x^4 + 2304\,x^2 + 147456}{14745600\,x^9}\,I_1(x) + \frac{1}{14745600} \int \frac{I_0(x)\,dx}{x}$$

Descending recurrence formulas:

$$\int x^{-2n-1}\,J_0(x)\,dx = \frac{1}{4n^2}\,\left[x^{-2n+1}J_1(x) - 2nx^{-2n}\,J_0(x) - \int x^{-2n+1}\,J_0(x)\,dx\right]$$

$$\int x^{-2n-1}\,I_0(x)\,dx = \frac{1}{4n^2}\,\left[-x^{-2n+1}I_1(x) - 2nx^{-2n}\,I_0(x) + \int x^{-2n+1}\,I_0(x)\,dx\right]$$

General formula: With $n!!$ as defined on page 10 holds

$$\int \frac{J_0(x)\,dx}{x^{2n+1}} =$$

$$= \frac{(-1)^n}{[(2n)!!]^2}\left\{\left(\sum_{k=0}^{n-1}(-1)^k\frac{(2k+2)!!\cdot(2k)!!}{x^{2k+2}}\right)J_0(x) - \left(\sum_{k=0}^{n-1}(-1)^k\frac{[(2k)!!]^2}{x^{2k+1}}\right)J_1(x) + \int\frac{J_0(x)\,dx}{x}\right\} =$$

$$= \frac{(-1)^n}{2^{2n}\cdot(n!)^2}\left\{\left(\sum_{k=0}^{n-1}(-1)^k\frac{2^{2k+1}\cdot(k+1)!\cdot k!}{x^{2k+2}}\right)J_0(x) - \left(\sum_{k=0}^{n-1}(-1)^k\frac{2^{2k}\cdot(k!)^2}{x^{2k+1}}\right)J_1(x) + \int\frac{J_0(x)\,dx}{x}\right\}$$

With obviously modifications one gets the the formula for the integral $\int x^{-2n-1}\,I_0(x)\,dx$.

1.1.4. Integrals of the type $\int x^{2n} Z_1(x)\,dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)},\ p=1,2$.

$$\int J_1(x)\,dx \;=\; -\,J_0(x)$$

$$\int I_1(x)\,dx \;=\; I_0(x)$$

$$\int K_1(x)\,dx \;=\; -K_0(x)$$

$$\int x^2\,J_1(x)\,dx \;=\; x\,[2J_1(x) - x\,J_0(x)]$$

$$\int x^2\,I_1(x)\,dx \;=\; x\,[x\,I_0(x) - 2I_1(x)]$$

$$\int x^2\,K_1(x)\,dx \;=\; -x\,[x\,K_0(x) + 2K_1(x)]$$

$$\int x^4\,J_1(x)\,dx \;=\; x\left[(4x^2-16)\,J_1(x) - (x^3-8x)\,J_0(x)\right]$$

$$\int x^4\,I_1(x)\,dx \;=\; x\left[(x^3+8x)\,I_0(x) - (4x^2+16)\,I_1(x)\right]$$

$$\int x^4\,K_1(x)\,dx \;=\; -x\left[(x^3+8x)\,K_0(x) + (4x^2+16)\,K_1(x)\right]$$

$$\int x^6\,J_1(x)\,dx \;=\; x\left[(6x^4-96x^2+384)\,J_1(x) - (x^5-24x^3+192x)\,J_0(x)\right]$$

$$\int x^6\,I_1(x)\,dx \;=\; x\left[(x^5+24x^3+192x)\,I_0(x) - (6x^4+96x^2+384)\,I_1(x)\right]$$

$$\int x^6\,K_1(x)\,dx \;=\; -x\left[(x^5+24x^3+192x)\,K_0(x) + (6x^4+96x^2+384)\,K_1(x)\right]$$

$$\int x^8\,J_1(x)\,dx \;=$$
$$= x\left[(8x^6-288x^4+4\,608x^2-18\,432)\,J_1(x) - (x^7-48x^5+1\,152x^3-9\,216x)\,J_0(x)\right]$$

$$\int x^8\,I_1(x)\,dx \;=$$
$$= x\left[(x^7+48x^5+1\,152x^3+9\,216x)\,I_0(x) - (8x^6+288x^4+4\,608x^2+18\,432)\,I_1(x)\right]$$

$$\int x^8\,K_1(x)\,dx \;=$$
$$= -x\left[(x^7+48x^5+1\,152x^3+9\,216x)\,K_0(x) + (8x^6+288x^4+4\,608x^2+18\,432)\,K_1(x)\right]$$

$$\int x^{10}\,J_1(x)\,dx \;=\; x\left[(10x^8-640x^6+23\,040x^4-368\,640x^2+1\,474\,560)\,J_1(x)-\right.$$
$$\left.-\,(x^9-80x^7+3\,840x^5-92\,160x^3+737\,280x)\,J_0(x)\right]$$

$$\int x^{10}\,I_1(x)\,dx \;=\; x\left[(x^9+80x^7+3\,840x^5+92\,160x^3+737\,280x)\,I_0(x)-\right.$$
$$\left.-(10x^8+640x^6+23\,040x^4+368\,640x^2+1\,474\,560)\,I_1(x)\right]$$

$$\int x^{10}\,K_1(x)\,dx \;=\; -x\left[(x^9+80x^7+3\,840x^5+92\,160x^3+737\,280x)\,K_0(x)+\right.$$

$$+(10x^8+640x^6+23\,040x^4+368\,640x^2+1\,474\,560)\,K_1(x)\big]$$

Let

$$\int x^m J_1(x)\,dx = x[Q_m(x)J_1(x)-P_m(x)J_0(x)] \quad \text{and} \quad \int x^m I_1(x)\,dx = x[P_m^*(x)I_0(x)-Q_m^*(x)I_1(x)]\ ,$$

$$\int x^m K_1(x)\,dx = -x[P_m^*(x)I_0(x)+Q_m^*(x)I_1(x)]\ ,$$

then holds

$P_{12}(x) = x^{11} - 120\,x^9 + 9600\,x^7 - 460800\,x^5 + 11059200\,x^3 - 88473600\,x$
$Q_{12}(x) = 12\,x^{10} - 1200\,x^8 + 76800\,x^6 - 2764800\,x^4 + 44236800\,x^2 - 176947200$
$P_{12}^*(x) = x^{11} + 120\,x^9 + 9600\,x^7 + 460800\,x^5 + 11059200\,x^3 + 88473600\,x$
$Q_{12}^*(x) = 12\,x^{10} + 1200\,x^8 + 76800\,x^6 + 2764800\,x^4 + 44236800\,x^2 + 176947200$

$P_{14}(x) = x^{13} - 168\,x^{11} + 20160\,x^9 - 1612800\,x^7 + 77414400\,x^5 - 1857945600\,x^3 + 14863564800\,x$
$Q_{14}(x) = 14\,x^{12} - 2016\,x^{10} + 201600\,x^8 - 12902400\,x^6 + 464486400\,x^4 - 7431782400\,x^2 + 29727129600$
$P_{14}^*(x) = x^{13} + 168\,x^{11} + 20160\,x^9 + 1612800\,x^7 + 77414400\,x^5 + 1857945600\,x^3 + 14863564800\,x$
$Q_{14}^*(x) = 14\,x^{12} + 2016\,x^{10} + 201600\,x^8 + 12902400\,x^6 + 464486400\,x^4 + 7431782400\,x^2 + 29727129600$

Recurrence formulas:

$$\int x^{2n+2}\,J_1(x)\,dx = -x^{2n+2}\,J_0(x) + (2n+2)x^{2n+1}\,J_1(x) - 4n(n+1)\int x^{2n}\,J_1(x)\,dx$$

$$\int x^{2n+2}\,I_1(x)\,dx = x^{2n+2}\,I_0(x) - (2n+2)x^{2n+1}\,I_1(x) + 4n(n+1)\int x^{2n}\,I_1(x)\,dx$$

$$\int x^{2n+2}\,K_1(x)\,dx = -x^{2n+2}\,K_0(x) - (2n+2)x^{2n+1}\,K_1(x) + 4n(n+1)\int x^{2n}\,K_1(x)\,dx$$

General formula: With $n!!$ as defined on page 10 holds

$$\int x^{2n}\,J_1(x)\,dx = \left(\sum_{k=0}^{n-1}(-1)^k\,\frac{[(2n)!!]\,\cdot[(2n-2)!!]\,\cdot x^{2n-1-2k}}{[(2n-2-2k)!!]^2}\right)J_1(x) -$$

$$-\left(\sum_{k=0}^{n-1}(-1)^k\,\frac{(2n)!!\cdot(2n-2)!!\cdot x^{2n-2k}}{[(2n-2k)!!]\cdot[(2n-2-2k)!!]}\right)J_0(x) =$$

$$= \left(\sum_{k=0}^{n-1}(-1)^k\,\frac{2^{2k+1}\cdot(n!)\cdot(n-1)!\cdot x^{2n-1-2k}}{[(n-1-k)!]^2}\right)J_1(x) -$$

$$-\left(\sum_{k=0}^{n-1}(-1)^k\,\frac{2^{2k}\cdot n!\cdot(n-1)!!\cdot x^{2n-2k}}{(n-k)!\cdot(n-1-k)!}\right)J_0(x)$$

With obviously modifications one gets the the formulas for the integrals $\int x^{2n}\,I_1(x)\,dx$ and $\int x^{2n}\,K_1(x)\,dx$.

1.1.5. Integrals of the type $\int x^{-2n} \cdot Z_1(x)\,dx$

About the integrals

$$\int \frac{J_0(x)\,dx}{x} \qquad \text{and} \qquad \int \frac{I_0(x)\,dx}{x}$$

see 1.1.3, page 15.

In the following formulas $J_0(x)$ may be substituted by $Y_0(x)$ and simultaneously $J_1(x)$ by $Y_1(x)$.

$$\int \frac{J_1(x)\,dx}{x^2} = -\frac{1}{2x}\,J_1(x) + \frac{1}{2}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^2} = -\frac{1}{2x}\,I_1(x) + \frac{1}{2}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^4} = -\frac{1}{8\,x^2}\,J_0(x) + \frac{x^2-4}{16\,x^3}\,J_1(x) - \frac{1}{16}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^4} = -\frac{1}{8\,x^2}\,I_0(x) - \frac{x^2+4}{16\,x^3}\,I_1(x) + \frac{1}{16}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^6} =$$

$$= \frac{x^2-8}{192\,x^4}\,J_0(x) + \frac{-x^4+4\,x^2-64}{384\,x^5}\,J_1(x) + \frac{1}{384}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^6} = -\frac{x^2+8}{192\,x^4}\,I_0(x) - \frac{x^4+4\,x^2+64}{384\,x^5}\,I_1(x) + \frac{1}{384}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^8} =$$

$$= \frac{-x^4+8\,x^2-192}{9216\,x^6}\,J_0(x) + \frac{x^6-4\,x^4+64\,x^2-2304}{18432\,x^7}\,J_1(x) - \frac{1}{18432}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^8} = -\frac{x^4+8\,x^2+192}{9216\,x^6}\,I_0(x) - \frac{x^6+4\,x^4+64\,x^2+2304}{18432\,x^7}\,I_1(x) + \frac{1}{18432}\int \frac{I_0(x)\,dx}{x}$$

$$\int \frac{J_1(x)\,dx}{x^{10}} =$$

$$= \frac{x^6-8\,x^4+192\,x^2-9216}{737280\,x^8}\,J_0(x) + \frac{-x^8+4\,x^6-64\,x^4+2304\,x^2-147456}{1474560\,x^9}\,J_1(x) +$$

$$+ \frac{1}{1474560}\int \frac{J_0(x)\,dx}{x}$$

$$\int \frac{I_1(x)\,dx}{x^{10}} =$$

$$= -\frac{x^6+8\,x^4+192\,x^2+9216}{737280\,x^8}\,I_0(x) - \frac{x^8+4\,x^6+64\,x^4+2304\,x^2+147456}{1474560\,x^9}\,I_1(x) +$$

$$+ \frac{1}{1474560}\int \frac{I_0(x)\,dx}{x}$$

Recurrence formulas:

$$\int \frac{J_1(x)\,dx}{x^{2n+2}} = -\frac{J_0(x)}{4n(n+1)x^{2n}} - \frac{J_1(x)}{(2n+2)x^{2n+1}} - \frac{1}{4n(n+1)}\int \frac{J_1(x)\,dx}{x^{2n}}$$

$$\int \frac{I_1(x)\,dx}{x^{2n+2}} = -\frac{I_0(x)}{4n(n+1)x^{2n}} - \frac{I_1(x)}{(2n+2)x^{2n+1}} + \frac{1}{4n(n+1)}\int \frac{I_1(x)\,dx}{x^{2n}}$$

General formula: With $n!!$ as defined on page 10 holds

$$\int \frac{J_1(x)\,dx}{x^{2n}} = \frac{(-1)^{n+1}}{(2n)!!\cdot(2n-2)!!}\cdot$$

$$\cdot\left\{\left(\sum_{k=0}^{n-2}(-1)^k\,\frac{(2k+2)!!\cdot(2k)!!}{x^{2k+2}}\right)J_0(x)-\left(\sum_{k=0}^{n-1}(-1)^k\,\frac{[(2k)!!]^2}{x^{2k+1}}\right)J_1(x)+\int\frac{J_0(x)\,dx}{x}\right\}=$$

$$=\frac{(-1)^{n+1}}{2^{2n-1}\cdot n!\cdot(n-1)!}\cdot$$

$$\cdot\left[\left(\sum_{k=0}^{n-2}(-1)^k\,\frac{2^{2k+1}\cdot(k+1)!\cdot k!}{x^{2k+2}}\right)J_0(x)-\left(\sum_{k=0}^{n-1}(-1)^k\,\frac{2^{2k}\cdot(k!)^2}{x^{2k+1}}\right)J_1(x)+\int\frac{J_0(x)\,dx}{x}\right]$$

With obviously modifications one gets the the formula for the integral $\int x^{-2n}\,I_1(x)\,dx$.

1.1.6. Integrals of the type $\int x^{2n+1} Z_1(x)\,dx$

$\Phi(x),\ \Phi_Y(x),\ \Psi(x)$ and $\Psi_K(x)$ are the same as in 1.1.1, page 9 .

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ and simultaneously $\Phi(x)$ by $\Phi_Y(x)$ or $H_\nu^{(p)}(x),\ p=1,2$ and $\Phi_H^{(p)}(x)$.

$$\int x\,J_1(x)\,dx \;=\; \Phi(x)$$

$$\int x\,I_1(x)\,dx \;=\; -\Psi(x)$$

$$\int x\,K_1(x)\,dx \;=\; \Psi_K(x)$$

$$\int x^3\,J_1(x)\,dx \;=\; 3x^2\,J_1(x) - x^3\,J_0(x) - 3\Phi(x)$$

$$\int x^3\,I_1(x)\,dx \;=\; -3x^2\,I_1(x) + x^3\,I_0(x) - 3\Psi(x)$$

$$\int x^3\,K_1(x)\,dx \;=\; -3x^2\,K_1(x) - x^3\,K_0(x) + 3\Psi_K(x)$$

$$\int x^5\,J_1(x)\,dx \;=\; (5x^4-45x^2)\,J_1(x) - (x^5-15x^3)\,J_0(x) + 45\Phi(x)$$

$$\int x^5\,I_1(x)\,dx \;=\; -(5x^4+45x^2)\,I_1(x) + (x^5+15x^3)\,I_0(x) - 45\Psi(x)$$

$$\int x^5\,K_1(x)\,dx \;=\; -(5x^4+45x^2)\,K_1(x) - (x^5+15x^3)\,K_0(x) + 45\Psi_K(x)$$

$$\int x^7\,J_1(x)\,dx \;=\; (7x^6-175x^4+1\,575x^2)\,J_1(x) - (x^7-35x^5+525x^3)\,J_0(x) - 1\,575\Phi(x)$$

$$\int x^7\,I_1(x)\,dx \;=\; -(7x^6+175x^4+1\,575x^2)\,I_1(x) + (x^7+35x^5+525x^3)\,I_0(x) - 1\,575\Psi(x)$$

$$\int x^7\,K_1(x)\,dx \;=\; -(7x^6+175x^4+1\,575x^2)\,K_1(x) - (x^7+35x^5+525x^3)\,K_0(x) + 1\,575\Psi_K(x)$$

$$\int x^9\,J_1(x)\,dx =$$
$$= (9x^8-441x^6+11\,025x^4-99\,225x^2)\,J_1(x) - (x^9-63x^7+2\,205x^5-33\,075x^3)\,J_0(x) + 99\,225\,\Phi(x)$$

$$\int x^9\,I_1(x)\,dx =$$
$$= -(9x^8+441x^6+11\,025x^4+99\,225x^2)\,I_1(x) + (x^9+63x^7+2\,205x^5+33\,075x^3)\,I_0(x) - 99\,225\,\Psi(x)$$

$$\int x^9\,K_1(x)\,dx =$$
$$= -(9x^8+441x^6+11\,025x^4+99\,225x^2)\,K_1(x) - (x^9+63x^7+2\,205x^5+33\,075x^3)\,K_0(x) + 99\,225\,\Psi(x)$$

General formula: With $n!!$ as defined on page 10 holds

$$\int x^{2n+1}\,J_1(x)\,dx = \left(\sum_{k=0}^{n-1} (-1)^k\,\frac{(2n+1)!!\cdot(2n-1)!!\cdot x^{2n-2k}}{[(2n-1-2k)!!]^2}\right) J_1(x) -$$
$$-\left(\sum_{k=0}^{n-1} (-1)^k\,\frac{(2n+1)!!\cdot(2n-1)!!\cdot x^{2n+1-2k}}{(2n+1-2k)!!\cdot(2n-1-2k)!!}\right) J_0(x) + (-1)^n\cdot(2n+1)!!\cdot(2n-1)!!\,\Phi(x) =$$

$$= \left(\sum_{k=0}^{n-1} (-1)^k \, \frac{(2n+2)! \cdot (2n)! \cdot [(n-k)!]^2 \cdot x^{2n-2k}}{2^{2k+1} \cdot (n+1)! \cdot n! \cdot [(2n-2k)!]^2} \right) J_1(x)$$

$$- \left(\sum_{k=0}^{n-1} (-1)^k \, \frac{(2n+2)! \cdot (2n)! \cdot (n+1-k)! \cdot (n-k)! \cdot x^{2n+1-2k}}{2^{2k} \cdot (n+1)! \cdot n! \cdot (2n+2-2k)! \cdot (2n-2k)!} \right) J_0(x) +$$

$$+ (-1)^n \frac{(2n+2)! \cdot (2n)!}{2^{2n+1} \cdot (n+1)! \cdot n!} \, \Phi(x)$$

With obviously modifications one gets the the formulas for the integrals $\int x^{2n+1} I_1(x)\,dx$ and $\int x^{2n+1} K_1(x)\,dx$.

<u>Recurrence formulas:</u>

$$\int x^{2n+1} \, J_1(x)\,dx \;=\; -x^{2n+1} \, J_0(x) + (2n+1)x^{2n} \, J_1(x) - (2n-1)(2n+1) \int x^{2n-1} \, J_1(x)\,dx$$

$$\int x^{2n+1} \, I_1(x)\,dx \;=\; x^{2n+1} \, I_0(x) - (2n+1)x^{2n} \, I_1(x) + (2n-1)(2n+1) \int x^{2n-1} \, I_1(x)\,dx$$

$$\int x^{2n+1} \, K_1(x)\,dx \;=\; -x^{2n+1} \, K_0(x) - (2n+1)x^{2n} \, K_1(x) + (2n-1)(2n+1) \int x^{2n-1} \, K_1(x)\,dx$$

Descending:

$$\int \frac{J_1(x)\,dx}{x^{2n+1}} = -\frac{J_0(x)}{(4n^2-1)x^{2n-1}} - \frac{J_1(x)}{(2n+1)x^{2n}} - \frac{1}{4n^2-1} \int \frac{J_1(x)\,dx}{x^{2n-1}}$$

$$\int \frac{I_1(x)\,dx}{x^{2n+1}} = -\frac{I_0(x)}{(4n^2-1)x^{2n-1}} - \frac{I_1(x)}{(2n+1)x^{2n}} + \frac{1}{4n^2-1} \int \frac{I_1(x)\,dx}{x^{2n-1}}$$

$$\int \frac{K_1(x)\,dx}{x^{2n+1}} = \frac{K_0(x)}{(4n^2-1)x^{2n-1}} - \frac{K_1(x)}{(2n+1)x^{2n}} + \frac{1}{4n^2-1} \int \frac{K_1(x)\,dx}{x^{2n-1}}$$

$$\int \frac{J_1(x)}{x}\,dx \;=\; x \cdot J_0(x) - J_1(x) + \Phi(x)$$

$$\int \frac{I_1(x)}{x}\,dx \;=\; x \cdot I_0(x) - I_1(x) + \Psi(x)$$

$$\int \frac{K_1(x)}{x}\,dx \;=\; -x \cdot K_0(x) - K_1(x) - \Psi_K(x)$$

$$\int \frac{J_1(x)}{x^3}\,dx \;=\; \frac{1}{3}\left[\frac{x^2-1}{x^2} J_1(x) - \frac{x^2+1}{x} J_0(x) - \Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^3}\,dx \;=\; \frac{1}{3}\left[-\frac{x^2+1}{x^2} I_1(x) + \frac{x^2-1}{x} I_0(x) + \Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^3}\,dx \;=\; \frac{1}{3}\left[-\frac{x^2+1}{x^2} K_1(x) - \frac{x^2-1}{x} K_0(x) - \Psi_K(x)\right]$$

$$\int \frac{J_1(x)}{x^5}\,dx \;=\; \frac{1}{45}\left[\frac{x^4+x^2-3}{x^3} J_0(x) - \frac{x^4-x^2+9}{x^4} J_1(x) + \Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^5}\,dx \;=\; \frac{1}{45}\left[\frac{x^4-x^2-3}{x^3} I_0(x) - \frac{x^4+x^2+9}{x^4} I_1(x) + \Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^5}\,dx \;=\; \frac{1}{45}\left[-\frac{x^4-x^2-3}{x^3} K_0(x) - \frac{x^4+x^2+9}{x^4} K_1(x) - \Psi_K(x)\right]$$

$$\int \frac{J_1(x)}{x^7}\,dx \;=\; \frac{1}{1\,575}\left[\frac{x^6-x^4+9x^2-225}{x^6} J_1(x) - \frac{x^6+x^4-3x^2+45}{x^5} J_0(x) - \Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^7}\,dx \;=\; \frac{1}{1\,575}\left[-\frac{x^6+x^4+9x^2+225}{x^6}I_1(x)+\frac{x^6-x^4-3x^2-45}{x^5}I_0(x)+\Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^7}\,dx \;=\; \frac{1}{1\,575}\left[-\frac{x^6+x^4+9x^2+225}{x^6}K_1(x)-\frac{x^6-x^4-3x^2-45}{x^5}K_0(x)-\Psi_k(x)\right]$$

$$\int \frac{J_1(x)}{x^9}\,dx \;=$$

$$=\frac{1}{99\,225}\left[\frac{x^8+x^6-3x^4+45x^2-1\,575}{x^7}J_0(x)-\frac{x^8-x^6+9x^4-225x^2+11\,025}{x^8}J_1(x)+\Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^9}\,dx \;=$$

$$=\frac{1}{99\,225}\left[\frac{x^8-x^6-3x^4-45x^2-1\,575}{x^7}I_0(x)-\frac{x^8+x^6+9x^4+225x^2+11\,025}{x^8}I_1(x)+\Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^9}\,dx \;=$$

$$=\frac{1}{99\,225}\left[-\frac{x^8-x^6-3x^4-45x^2-1\,575}{x^7}K_0(x)-\frac{x^8+x^6+9x^4+225x^2+11\,025}{x^8}I_1(x)-\Psi_K(x)\right]$$

$$\int \frac{J_1(x)}{x^{11}}\,dx \;=\; \frac{1}{9\,823\,275}\left[\frac{x^{10}-x^8+9x^6-225x^4+11\,025x^2-893\,025}{x^{10}}J_1(x)-\right.$$
$$\left.-\frac{x^{10}+x^8-3x^6+45x^4-1\,575x^2+99\,225}{x^9}J_0(x)-\Phi(x)\right]$$

$$\int \frac{I_1(x)}{x^{11}}\,dx \;=\; \frac{1}{9\,823\,275}\left[-\frac{x^{10}+x^8+9x^6+225x^4+11\,025x^2+893\,025}{x^{10}}I_1(x)+\right.$$
$$\left.+\frac{x^{10}-x^8-3x^6-45x^4-1\,575x^2-99\,225}{x^9}I_0(x)+\Psi(x)\right]$$

$$\int \frac{K_1(x)}{x^{11}}\,dx \;=\; \frac{1}{9\,823\,275}\left[-\frac{x^{10}+x^8+9x^6+225x^4+11\,025x^2+893\,025}{x^{10}}K_1(x)-\right.$$
$$\left.-\frac{x^{10}-x^8-3x^6-45x^4-1\,575x^2-99\,225}{x^9}K_0(x)+\Psi_K(x)\right]$$

General formula: With $n!!$ as defined on page 10 holds

$$\int \frac{J_1(x)\,dx}{x^{2n+1}}=\frac{(-1)^n}{(2n+1)!!\cdot(2n-1)!!}\left\{\left(x+\sum_{k=0}^{n-1}\frac{(-1)^k\cdot(2k+1)!!\cdot(2k-1)!!}{x^{2k+1}}\right)J_0(x)-\right.$$
$$\left.-\left(1-\sum_{k=0}^{n-1}(-1)^k\frac{[(2k+1)!!]^2}{x^{2k+2}}\right)J_1(x)+\Phi(x)\right\}=$$
$$=\frac{2^{2n+1}\cdot(n+1)!\cdot n!}{(2n+2)!\cdot(2n)!}\left\{\left(x-\sum_{k=0}^{n-1}(-1)^k\frac{(2k+2)!\cdot(2k)!}{2^{2k+1}\cdot(k+1)!\cdot k!\cdot x^{2k+1}}\right)J_0(x)-\right.$$
$$\left.-\left(1-\sum_{k=0}^{n-1}(-1)^k\frac{[(2k+2)!]^2}{2^{2k+2}\cdot[(k+1)!]^2\cdot x^{2k+2}}\right)J_1(x)+\Phi(x)\right\}$$

With obviously modifications one gets the the formulas for the integrals $\int x^{-2n-1}I_1(x)\,dx$ and $\int x^{-2n-1}K_1(x)\,dx$.

1.1.7. Integrals of the type $\int x^n Z_\nu(x)\,dx,\ \nu>1$:

From the well-known recurrence relations one gets immadiately

$$\int J_{\nu+1}(x)\,dx \;=\; -2J_\nu(x)+\int J_{\nu-1}(x)\,dx \quad \text{and} \quad \int I_{\nu+1}(x)\,dx \;=\; 2I_\nu(x)-\int I_{\nu-1}(x)\,dx\;.$$

With this formulas follows

$$\int_0^x J_{2\nu}(t)\,dt=\Lambda_0(x)-2\sum_{\kappa=1}^{n}J_{2\kappa-1}(x)\;,\qquad \int_0^x J_{2\nu+1}(t)\,dt=1-J_0(x)-2\sum_{\kappa=1}^{n}J_{2\kappa}(x)$$

$$\int_0^x I_{2\nu}(t)\,dt=(-1)^n\Lambda_0^*(x)+2\sum_{\kappa=1}^{n}(-1)^{n+\kappa}I_{2\kappa-1}(x)\;,\quad \int_0^x I_{2\nu+1}(t)\,dt=(-1)^n[I_0(x)-1]+2\sum_{\kappa=1}^{n}(-1)^{n+\kappa}I_{2\kappa}(x)$$

The integrals $\Lambda_0(x)$ and $\Lambda_0^*(x)$ are defined on page 9 and discussed on page 119 and 121.
Holds

$$\int Y_{2\nu}(x)\,dx=xY_0(x)+\Phi_Y(x)-2\sum_{\kappa=1}^{n}Y_{2\kappa-1}(x)\;,\quad \int Y_{2\nu+1}(x)\,dx=-Y_0(x)-2\sum_{\kappa=1}^{n}Y_{2\kappa-1}(x)$$

$$\int H_{2\nu}^{(1)}(x)\,dx=xH_0^{(1)}(x)+\Phi_H^{(1)}(x)-2\sum_{\kappa=1}^{n}H_{2\kappa-1}^{(1)}(x)\;,\quad \int H_{2\nu+1}^{(1)}(x)\,dx=-H_0^{(1)}(x)-2\sum_{\kappa=1}^{n}H_{2\kappa-1}^{(1)}(x)$$

$$\int H_{2\nu}^{(2)}(x)\,dx=xH_0^{(2)}(x)+\Phi_H^{(2)}(x)-2\sum_{\kappa=1}^{n}H_{2\kappa-1}^{(2)}(x)\;,\quad \int H_{2\nu+1}^{(2)}(x)\,dx=-H_0^{(2)}(x)-2\sum_{\kappa=1}^{n}H_{2\kappa-1}^{(2)}(x)$$

$$\int K_{2\nu}(x)\,dx=(-1)^n\left\{xK_0(x)+\frac{\pi x}{2}\left[K_0(x)\mathbf{L}_1(x)+K_1(x)\mathbf{L}_0(x)\right]\right\}+2\sum_{\kappa=1}^{n}(-1)^{n+\kappa}K_{2\kappa-1}(x)\;,$$

$$\int K_{2\nu+1}(x)\,dx=(-1)^{n+1}K_0(x)+2\sum_{\kappa=1}^{n}(-1)^{n+\kappa+1}K_{2\kappa}(x)$$

About the functions $\Phi_Y(x)$, $\Phi_H^{(1)}(x)$, $\Phi_H^{(2)}(x)$ see page 9.
Further on, holds

$$\int_0^x t\,J_{2\nu+1}(t)\,dt=(2\nu+1)\Lambda_0(x)-x\left[J_0(x)+2\sum_{\kappa=1}^{\nu}J_{2\kappa}(x)\right]-4\sum_{\kappa=0}^{\nu-1}(\nu-\kappa)J_{2\kappa+1}(x)$$

$$\int_0^x t\,J_{2\nu}(t)\,dt=-x\left[J_1(x)+2\sum_{\kappa=1}^{\nu-1}J_{2\kappa+1}(x)\right]+2\nu[1-J_0(x)]-4\sum_{\kappa=1}^{\nu-1}(\nu-\kappa)J_{2\kappa}(x)$$

$$\int_0^x t\,I_{2\nu+1}(t)\,dt=(-1)^{\nu+1}\left[(2\nu+1)\Lambda_0^*(x)-xI_0(x)-2x\sum_{\kappa=1}^{\nu}(-1)^\kappa I_{2\kappa}(x)-4\sum_{\kappa=0}^{\nu-1}(-1)^\kappa(\nu-\kappa)I_{2\kappa+1}(x)\right]$$

$$\int_0^x t\,I_{2\nu}(t)\,dt=(-1)^{\nu+1}\left[xI_1(x)+2x\sum_{\kappa=1}^{\nu-1}(-1)^\kappa I_{2\kappa+1}(x)+2\nu[1-I_0(x)]-4\sum_{\kappa=1}^{\nu-1}(-1)^\kappa(\nu-\kappa)I_{2\kappa}(x)\right]$$

Some of the previous sums may cause numerical problems, if x is located near 0. For instance, the sum

$$\int_0^x t\,I_6(t)\,dt=xJ_1(x)-2xJ_3(x)+2xJ_5(x)+6-6J_0(x)+8J_2(x)-4J_4(x)$$

gives with $x=0.3$

$$0.045\,508\,152\,001-0.000\,339\,402\,714+0.000\,000\,381\,114+6-6.135\,761\,276\,110+0.090\,676\,901\,288-$$

$$-0.000\,084\,755\,400=6.136\,185\,434\,403-6.136\,185\,434\,224=0.000\,000\,000\,179\;,$$

which means the loss of 10 decimal digits.
For that reason the value of such integrals should be computed by the power series or other formulas. See also the following remark.

In the following the integrals are expressed by $Z_0(x)$ and $Z_1(x)$.
Integrals with $-2 \le n \le 4$ are written explicitely: at first $n = 0,\ 1,\ 2,\ 3,\ 4$, after them $n = -1,\ -2$. In the other cases the functions $\mathcal{P}_\nu^{(n)}(x)$, $\mathcal{Q}_\nu^{(n)}(x)$ and the coefficients $\mathcal{R}_\nu^{(n)}$, $\mathcal{S}_\nu^{(n)}$ describe the integral

$$\int x^n \cdot J_\nu(x)\,dx = \mathcal{P}_\nu^{(n)}(x)\,J_0(x) + \mathcal{Q}_\nu^{(n)}(x) J_1(x) + \mathcal{R}_\nu^{(n)}\,\Lambda_0(x) + \mathcal{S}_\nu^{(n)} \int \frac{J_0(x)\,dx}{x}\ .$$

Furthermore, let

$$\int x^n \cdot I_\nu^*(x)\,dx = \mathcal{P}_\nu^{(n),*}(x)\,I_0(x) + \mathcal{Q}_\nu^{(n),*}(x) I_1(x) + \mathcal{R}_\nu^{(n),*}\,\Lambda_0^*(x) + \mathcal{S}_\nu^{(n),*} \int \frac{I_0(x)\,dx}{x}\ .$$

Concerning $\int x^{-1} \cdot Z_0(x)\,dx$ see 1.1.3., page 15.

Simple recurrence formula:

$$\int x^n \cdot J_{\nu+1}(x)\,dx \quad = \quad 2\nu \int x^{n-1} \cdot J_\nu(x)\,dx - \int x^n \cdot J_{\nu-1}(x)\,dx$$

$$\int x^n \cdot I_{\nu+1}(x)\,dx \quad = \quad -2\nu \int x^{n-1} \cdot J_\nu(x)\,dx + \int x^n \cdot J_{\nu-1}(x)\,dx$$

The integrals of $x^n\,Z_0(x)$ and $x^n\,Z_1(x)$ to start this recurrences are already described.

Remark:
Let $F_\nu^{(m)}(x)$ denote the antiderivative of $x^m\,Z_\nu(x)$ as given in the following tables. They do not exist in the point $x = 0$ in the case $\nu + m < 0$. However, even if $\nu + m \ge 0$ the value of $F_\nu^{(m)}(0)$ sometimes turns out to be a limit of the type $\infty - \infty$. For instance, holds

$$\int \frac{J_3(x)\,dx}{x^2} = \frac{J_0(x)}{x^2} - \frac{2J_1(x)}{x^3} = F_3^{(-2)}(x) \quad \text{with} \quad \lim_{x\to 0} F_3^{(-2)}(x) = -\frac{1}{8}\ .$$

With $L_{\nu,m} = \lim_{x\to 0} F_\nu^{(m)}(x)$ for the Bessel functions $J_\nu(x)$ and $L_{\nu,m}^*$ for the modified Bessel functions $I_\nu(x)$ one has the following limits in the tables of integrals (The values $L_{\nu,m} = 0$ are omitted.):

$L_{2,-1} = -1/2,\ L_{2,-1}^* = 1/2$

$L_{3,0} = -1,\ L_{3,-2} = -1/8;\ L_{3,0}^* = -1,\ L_{3,-2}^* = 1/8$

$L_{4,1} = -4,\ L_{4,-1} = -1/4,\ L_{4,-3} = -1/48;\ L_{4,1}^* = 4,\ L_{4,-1}^* = -1/4,\ L_{4,-3}^* = 1/48$

$L_{5,2} = -24,\ L_{5,0} = -1,\ L_{5,-2} = -1/24,\ L_{5,-4} = -1/384;$
$L_{5,2}^* = -24,\ L_{5,0}^* = 1,\ L_{5,-2}^* = -1/24,\ L_{5,-4}^* = 1/384$

$L_{6,3} = -192,\ L_{6,1} = -6,\ L_{6,-1} = -1/6,\ L_{6,-3} = -1/192,\ L_{6,-5} = -1/3840;$
$L_{6,3}^* = 192,\ L_{6,1}^* = -6,\ L_{6,-1}^* = 1/6,\ L_{6,-3}^* = -1/192,\ L_{6,-5}^* = 1/3840$

$L_{7,4} = -1920,\ L_{7,2} = -48,\ L_{7,0} = -1,\ L_{7,-2} = -1/48,\ L_{7,-4} = -1/1920,\ L_{7,-6} = -1/46080;$
$L_{7,4}^* = -1920,\ L_{7,2}^* = 48,\ L_{7,0}^* = -1,\ L_{7,-2}^* = 1/48,\ L_{7,-4}^* = -1/1920,\ L_{7,-6}^* = 1/46080$

$L_{8,5} = -23040,\ L_{8,3} = -480,\ L_{8,1} = -8,\ L_{8,-1} = -1/8,\ L_{8,-3} = -1/480,\ L_{8,-5} = -1/23040;$
$L_{8,5}^* = 23040,\ L_{8,3}^* = -480,\ L_{8,1}^* = 8,\ L_{8,-1}^* = -1/8,\ L_{8,-3}^* = 1/480,\ L_{8,-5}^* = -1/23040$

$L_{9,6} = -322560,\ L_{9,4} = -5760,\ L_{9,2} = -80,\ L_{9,0} = -1,\ L_{9,-2} = -1/80,\ L_{9,-4} = -1/5760,$
$L_{9,-6} = -1/322560;$
$L_{9,6}^* = -322560,\ L_{9,4}^* = 5760,\ L_{9,2}^* = -80,\ L_{9,0}^* = 1,\ L_{9,-2}^* = -1/80,\ L_{9,-4}^* = 1/5760,$
$L_{9,-6}^* = -1/322560$

$L_{10,7} = -5160960,\ L_{10,5} = -80640,\ L_{10,3} = -960,\ L_{10,1} = -10,\ L_{10,-1} = -1/10,\ L_{10,-3} = -1/960,$
$L_{10,-5} = -1/80640;$
$L_{10,7}^* = 5160960,\ L_{10,5}^* = -80640,\ L_{10,3}^* = 960,\ L_{10,1}^* = -10,\ L_{10,-1}^* = 1/10,\ L_{10,-3}^* = -1/960,$
$L_{10,-5}^* = 1/80640$

In the described cases of limits of the type $\infty - \infty$ the numerical computation of $F_\nu^{(m)}(x)$ causes difficulties, if $0 < x << 1$. Then it is preferable to use the power series, which has a fast convergengence for such values of x. With $m + \nu \geq 0$ holds

$$\int_0^x t^m\, J_\nu(t)\, dt = \frac{x^{m+\nu+1}}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k! \cdot (\nu+k)! \cdot 4^k \cdot (m+\nu+1+2k)}$$

and

$$\int_0^x t^m\, I_\nu(t)\, dt = \frac{x^{m+\nu+1}}{2^\nu} \sum_{k=0}^{\infty} \frac{x^{2k}}{k! \cdot (\nu+k)! \cdot 4^k \cdot (m+\nu+1+2k)}\ .$$

From this one has

$$F_\nu^{(m)}(x) = L_{\nu,m} + \int_0^x t^m\, J_\nu(t)\, dt \quad \text{and} \quad F_\nu^{*,(m)}(x) = L_{\nu,m}^* + \int_0^x t^m\, I_\nu(t)\, dt\ .$$

For instance,

$$\int_{0.002}^3 \frac{J_3(x)}{x^2}\, dx = \frac{J_0(x)}{x^2} - \frac{2J_1(x)}{x^3}\bigg|_{0.002}^3 =$$

$$= (-0.0288946616557703820 - 0.0251154784093286266) - (249999.750000062500 - 249999.875000020834) =$$

$$= -0.0540101400650990086 - (-0.124999958334^*) = 0.070989818269$$

It was a loss of seven decimal digits at $x = 0.002$. This value may be found without problems by the power series:

$$F_3^{(-2)}(0.002) =$$

$$= -\frac{1}{8} + 5\cdot 10^{-7}\, [0.0833333333333333335 - 1.04166666666666667\cdot 10^{-8} + 6.94444444444444445\cdot 10^{-16} - \ldots] =$$

$$= -\frac{1}{8} + 4.16666614583336806 \cdot 10^{-8} = -0.124999958333338542$$

In the previous value, signed by *, the last digit should be 3 instead of 4 and the result had to finish with 8.

The integrals with $I_\nu(x)$ may be computed in the same way.

This method can be used even if $\nu + m < 0$. For instance,

$$\int_{0.002}^3 \frac{J_4(x)}{x^7}\, dx = \int_{0.002}^1 \frac{J_4(x)}{x^7}\, dx + \int_1^3 \frac{J_4(x)}{x^7}\, dx$$

and the second integral is given in the following tables. For the first one holds with the power series of the function $J_4(x)$

$$\int_{0.002}^1 \frac{J_4(x)}{x^7}\, dx =$$

$$= \int_{0.002}^1 \frac{1}{x^7} \left(\frac{1}{384}\, x^4 - \frac{1}{7680}\, x^6 + \frac{1}{368640}\, x^8 - \frac{1}{30965760}\, x^{10} + \frac{1}{3963617280}\, x^{12} - \frac{1}{713451110400}\, x^{14} + \ldots \right) dx =$$

$$= \int_{0.002}^1 \left(\frac{1}{384\, x^3} - \frac{1}{7680\, x} + \frac{1}{368640}\, x - \frac{1}{30965760}\, x^3 + \frac{1}{3963617280}\, x^5 - \frac{1}{713451110400}\, x^7 + \ldots \right) dx =$$

$$= -\frac{1}{768\, x^2} - \frac{\ln x}{7680} + \frac{1}{737280}\, x^2 - \frac{1}{123863040}\, x^4 + \frac{1}{23781703680}\, x^6 - \frac{1}{5707608883200}\, x^8 + \ldots \bigg|_{0.002}^1 =$$

$$= (-0.001302083 - 0.0 + 1.3563 \cdot 10^{-6} - 8.07343 \cdot 10^{-9} + 4.20491 \cdot 10^{-11} - 0.17520 \cdot 10^{-13} + \ldots)$$

$$-(-325.520833333333333 + 0.000809193762815389549 + 5.4253472\cdot 10^{-12} - 1.2917\cdot 10^{-19} + 0.26911\cdot 10^{-27} - \ldots) =$$

$$= -0.00130073502808721678 - (-325.520024139565093) = 325.518723404537006\ .$$

Here are no differences of nearly the same values.

$\mathbf{Z_2(x)}$:

$$\int J_2(x)\,dx = -2J_1(x) + \Lambda_0(x)$$

$$\int I_2(x)\,dx = 2I_1(x) - \Lambda_0^*(x)$$

$$\int x\,J_2(x)\,dx = -2J_0(x) - xJ_1(x)$$

$$\int x\,I_2(x)\,dx = -2I_0(x) + xI_1(x)$$

$$\int x^2\,J_2(x)\,dx = -3xJ_0(x) - x^2J_1(x) + 3\Lambda_0(x)$$

$$\int x^2\,I_2(x)\,dx = -3xI_0(x) + x^2I_1(x) + 3\Lambda_0^*(x)$$

$$\int x^3\,J_2(x)\,dx = -4x^2J_0(x) - \left(x^2-8\right)xJ_1(x)$$

$$\int x^3\,I_2(x)\,dx = -4x^2I_0(x) + \left(x^2+8\right)xI_1(x)$$

$$\int x^4\,J_2(x)\,dx = -5x\,\left(x^2-3\right)J_0(x) - \left(x^2-15\right)x^2J_1(x) - 15\Lambda_0(x)$$

$$\int x^4\,I_2(x)\,dx = -5x\,\left(x^2+3\right)I_0(x) + \left(x^2+15\right)x^2I_1(x) + 15\Lambda_0^*(x)$$

$$\int \frac{J_2(x)\,dx}{x} = -\frac{J_1(x)}{x}$$

$$\int \frac{I_2(x)\,dx}{x} = \frac{I_1(x)}{x}$$

$$\int \frac{J_2(x)\,dx}{x^2} = \frac{1}{3x}\,J_0(x) - \frac{x^2+2}{3x^2}\,J_1(x) + \frac{1}{3}\,\Lambda_0(x)$$

$$\int \frac{I_2(x)\,dx}{x^2} = -\frac{1}{3x}\,I_0(x) - \frac{x^2-2}{3x^2}\,I_1(x) + \frac{1}{3}\,\Lambda_0^*(x)$$

$$\mathcal{P}_2^{(5)}(x) = -6\,\left(x^2-8\right)x^2\,, \quad \mathcal{Q}_2^{(5)}(x) = -\left(x^4-24\,x^2+96\right)x\,, \quad \mathcal{R}_2^{(5)} = 0\,, \quad \mathcal{S}_2^{(5)} = 0$$

$$\mathcal{P}_2^{(5),*}(x) = -6\,\left(x^2+8\right)x^2\,, \quad \mathcal{Q}_2^{(5),*}(x) = x^5+24\,x^3+96x\,, \quad \mathcal{R}_2^{(5),*} = 0\,, \quad \mathcal{S}_2^{(5),*} = 0$$

$$\mathcal{P}_2^{(6)}(x) = -7\,\left(x^4-15\,x^2+45\right)x\,, \quad \mathcal{Q}_2^{(6)}(x) = -\left(x^4-35\,x^2+315\right)x^2\,, \quad \mathcal{R}_2^{(6)} = 315\,, \quad \mathcal{S}_2^{(6)} = 0$$

$$\mathcal{P}_2^{(6),*}(x) = -7(x^5+15\,x^3+45\,x)\,, \quad \mathcal{Q}_2^{(6),*}(x) = x^6+35\,x^4+315\,x^2\,, \quad \mathcal{R}_2^{(6),*} = 315\,, \quad \mathcal{S}_2^{(6),*} = 0$$

$$\mathcal{P}_2^{(7)}(x) = -8\,\left(x^4-24\,x^2+192\right)x^2\,, \quad \mathcal{Q}_2^{(7)}(x) = -\left(x^6-48\,x^4+768\,x^2-3072\right)x\,, \quad \mathcal{R}_2^{(7)} = 0\,, \quad \mathcal{S}_2^{(7)} = 0$$

$$\mathcal{P}_2^{(7),*}(x) = -(8\,x^4+192\,x^2+1536)\,x^2\,, \quad \mathcal{Q}_2^{(7),*}(x) = x^7+48\,x^5+768\,x^3+3072\,x\,, \quad \mathcal{R}_2^{(7),*} = 0\,, \quad \mathcal{S}_2^{(7),*} = 0$$

$$\mathcal{P}_2^{(8)}(x) = -9\,\left(x^6-35\,x^4+525\,x^2-1575\right)x\,, \quad \mathcal{Q}_2^{(8)}(x) = -\left(x^6-63\,x^4+1575\,x^2-14175\right)x^2\,,$$

$$\mathcal{R}_2^{(8)} = -14175\,, \quad \mathcal{S}_2^{(8)} = 0$$

$$\mathcal{P}_2^{(8),*}(x) = -(9\,x^7+315\,x^5+4725\,x^3+14175\,x)\,, \quad \mathcal{Q}_2^{(8),*}(x) = x^8+63\,x^6+1575\,x^4+14175\,x^2\,,$$

$$\mathcal{R}_2^{(8),*} = 14175\,, \quad \mathcal{S}_2^{(8),*} = 0$$

$$\mathcal{P}_2^{(9)}(x) = -10\,\left(x^6-48\,x^4+1152\,x^2-9216\right)x^2\,, \quad \mathcal{Q}_2^{(9)}(x) = -\left(x^8-80\,x^6+2880\,x^4-46080\,x^2+184320\right)\,,$$

$$\mathcal{R}_2^{(9)} = 0\,, \quad \mathcal{S}_2^{(9)} = 0$$

$$\mathcal{P}_2^{(9),*}(x) = -(10\,x^8+480\,x^6+11520\,x^4+92160\,x^2)\ , \quad \mathcal{Q}_2^{(9),*}(x) = x^9+80\,x^7+2880\,x^5+46080\,x^3+184320\,x\ ,$$

$$\mathcal{R}_2^{(9),*} = 0\ , \quad \mathcal{S}_2^{(9),*} = 0$$

$$\mathcal{P}_2^{(10)}(x) = -11\ \left(x^8 - 63\,x^6 + 2205\,x^4 - 33075\,x^2 + 99225\right)x\ ,$$

$$\mathcal{Q}_2^{(10)}(x) = -\left(x^8 - 99\,x^6 + 4851\,x^4 - 121275\,x^2 + 1091475\right)x^2\ , \quad \mathcal{R}_2^{(10)} = 1091475\ , \quad \mathcal{S}_2^{(10)} = 0$$

$$\mathcal{P}_2^{(10),*}(x) = -(11\,x^9 + 693\,x^7 + 24255\,x^5 + 363825\,x^3 + 1091475\,x)\ ,$$

$$\mathcal{Q}_2^{(10),*}(x) = x^{10} + 99\,x^8 + 4851\,x^6 + 121275\,x^4 + 1091475\,x^2\ , \quad \mathcal{R}_2^{(10),*} = 1091475\ , \quad \mathcal{S}_2^{(10),*} = 0$$

$$\mathcal{P}_2^{(-3)}(x) = \frac{1}{4\,x^2}\ , \quad \mathcal{Q}_2^{(-3)}(x) = -\frac{x^2+4}{8\,x^3}\ , \quad \mathcal{R}_2^{(-3)} = 0\ , \quad \mathcal{S}_2^{(-3)} = \frac{1}{8}$$

$$\mathcal{P}_2^{(-3),*}(x) = -\frac{1}{4x^2}\ , \quad \mathcal{Q}_2^{(-3),*}(x) = -\frac{x^2-4}{8\,x^3}\ , \quad \mathcal{R}_2^{(-3),*} = 0\ , \quad \mathcal{S}_2^{(-3),*} = \frac{1}{8}$$

$$\mathcal{P}_2^{(-4)}(x) = -\frac{x^2-3}{15\,x^3}\ , \quad \mathcal{Q}_2^{(-4)}(x) = \frac{x^4-x^2-6}{15\,x^4}\ , \quad \mathcal{R}_2^{(-4)} = -\frac{1}{15}\ , \quad \mathcal{S}_2^{(-4)} = 0$$

$$\mathcal{P}_2^{(-4),*}(x) = -\frac{x^2+3}{15\,x^3}\ , \quad \mathcal{Q}_2^{(-4),*}(x) = -\frac{x^4+x^2-6}{15\,x^4}\ , \quad \mathcal{R}_2^{(-4),*} = \frac{1}{15}\ , \quad \mathcal{S}_2^{(-4),*} = 0$$

$$\mathcal{P}_2^{(-5)}(x) = -\frac{x^2-8}{48\,x^4}\ , \quad \mathcal{Q}_2^{(-5)}(x) = \frac{x^4-4\,x^2-32}{96\,x^5}\ , \quad \mathcal{R}_2^{(-5)} = 0\ , \quad \mathcal{S}_2^{(-5)} = -\frac{1}{96}$$

$$\mathcal{P}_2^{(-5),*}(x) = -\frac{x^2+8}{48\,x^4}\ , \quad \mathcal{Q}_2^{(-5),*}(x) = -\frac{x^4+4\,x^2-32}{96\,x^5}\ , \quad \mathcal{R}_2^{(-5),*} = 0\ , \quad \mathcal{S}_2^{(-5),*} = \frac{1}{96}$$

$$\mathcal{P}_2^{(-6)}(x) = \frac{x^4-3\,x^2+45}{315\,x^5}\ , \quad \mathcal{Q}_2^{(-6)}(x) = -\frac{x^6-x^4+9\,x^2+90}{315\,x^6}\ , \quad \mathcal{R}_2^{(-6)} = \frac{1}{315}\ , \quad \mathcal{S}_2^{(-6)} = 0$$

$$\mathcal{P}_2^{(-6),*}(x) = -\frac{x^4+3\,x^2+45}{315x^5}\ , \quad \mathcal{Q}_2^{(-6),*}(x) = -\frac{x^6+x^4+9\,x^2-90}{315x^6}\ , \quad \mathcal{R}_2^{(-6),*} = \frac{1}{315}\ , \quad \mathcal{S}_2^{(-6),*} = 0$$

$\mathbf{Z_3(x)}$:

$$\int J_3(x)\,dx = J_0(x) - \frac{4}{x}\,J_1(x)$$

$$\int I_3(x)\,dx = I_0(x) - \frac{4}{x}\,I_1(x)$$

$$\int x\,J_3(x)\,dx = xJ_0(x) - 8J_1(x) + 3\Lambda_0(x)$$

$$\int x\,I_3(x)\,dx = xI_0(x) - 8I_1(x) + 3\Lambda_0^*(x)$$

$$\int x^2\,J_3(x)\,dx = (x^2-8)J_0(x) - 6xJ_1(x)$$

$$\int x^2\,I_3(x)\,dx = (x^2+8)I_0(x) - 6xI_1(x)$$

$$\int x^3\,J_3(x)\,dx = \left(x^2-15\right)x\,J_0(x) - 7x^2J_1(x) + 15\Lambda_0(x)$$

$$\int x^3\,I_3(x)\,dx = (x^2+15)x\,I_0(x) - 7x^2\,I_1(x) - 15\Lambda_0^*(x)$$

$$\int x^4\,J_3(x)\,dx = \left(x^2-24\right)x^2J_0(x) - 8\ \left(x^2-6\right)xJ_1(x)$$

$$\int x^4\,I_3(x)\,dx = \left(x^2+24\right)x^2I_0(x) - 8\ \left(x^2+6\right)xI_1(x)$$

$$\int \frac{J_3(x)\,dx}{x} = \frac{4}{3x}\,J_0(x) - \frac{x^2+8}{3x^2}\,J_1(x) + \frac{1}{3}\,\Lambda_0(x)$$

$$\int \frac{I_3(x)\,dx}{x} = \frac{4}{3x}\,I_0(x) + \frac{x^2-8}{3x^2}\,I_1(x) - \frac{1}{3}\Lambda_0^*(x)$$

$$\int \frac{J_3(x)\,dx}{x^2} = \frac{J_0(x)}{x^2} - \frac{2J_1(x)}{x^3}$$

$$\int \frac{I_3(x)\,dx}{x^2} = \frac{I_0(x)}{x^2} - \frac{2I_1(x)}{x^3}$$

$$\mathcal{P}_3^{(5)}(x) = x^5 - 35\,x^3 + 105\,x\;, \quad \mathcal{Q}_3^{(5)}(x) = -(9\,x^4 - 105\,x^2)\;, \quad \mathcal{R}_3^{(5)} = -105\;, \quad \mathcal{S}_3^{(5)} = 0$$

$$\mathcal{P}_3^{(5),*}(x) = x^5 + 35\,x^3 + 105\,x\;, \quad \mathcal{Q}_3^{(5),*}(x) = -(9\,x^4 + 105\,x^2)\;, \quad \mathcal{R}_3^{(5),*} = -105\;, \quad \mathcal{S}_3^{(5),*} = 0$$

$$\mathcal{P}_3^{(6)}(x) = x^6 - 48\,x^4 + 384\,x^2\;, \quad \mathcal{Q}_3^{(6)}(x) = -(10\,x^5 - 192\,x^3 + 768\,x)\;, \quad \mathcal{R}_3^{(6)} = 0\;, \quad \mathcal{S}_3^{(6)} = 0$$

$$\mathcal{P}_3^{(6),*}(x) = x^6 + 48\,x^4 + 384\,x^2\;, \quad \mathcal{Q}_3^{(6),*}(x) = -(10\,x^5 + 192\,x^3 + 768\,x)\;, \quad \mathcal{R}_3^{(6),*} = 0\;, \quad \mathcal{S}_3^{(6),*} = 0$$

$$\mathcal{P}_3^{(7)}(x) = x^7 - 63\,x^5 + 945\,x^3 - 2835\,x\;, \quad \mathcal{Q}_3^{(7)}(x) = -(11\,x^6 - 315\,x^4 + 2835\,x^2)\;, \quad \mathcal{R}_3^{(7)} = 2835\;, \quad \mathcal{S}_3^{(7)} = 0$$

$$\mathcal{P}_3^{(7),*}(x) = x^7 + 63\,x^5 + 945\,x^3 + 2835\,x\;, \quad \mathcal{Q}_3^{(7),*}(x) = -(11\,x^6 + 315\,x^4 + 2835\,x^2)\;, \quad \mathcal{R}_3^{(7),*} = -2835\;, \quad \mathcal{S}_3^{(7),*} = 0$$

$$\mathcal{P}_3^{(8)}(x) = x^8 - 80\,x^6 + 1920\,x^4 - 15360\,x^2\;, \quad \mathcal{Q}_3^{(8)}(x) = -(12\,x^7 - 480\,x^5 + 7680\,x^3 - 30720\,x)\;,$$

$$\mathcal{R}_3^{(8)} = 0\;, \quad \mathcal{S}_3^{(8)} = 0$$

$$\mathcal{P}_3^{(8),*}(x) = x^8 + 80\,x^6 + 1920\,x^4 + 15360\,x^2\;, \quad \mathcal{Q}_3^{(8),*}(x) = -(12\,x^7 + 480\,x^5 + 7680\,x^3 + 30720\,x)\;,$$

$$\mathcal{R}_3^{(8),*} = 0\;, \quad \mathcal{S}_3^{(8),*} = 0$$

$$\mathcal{P}_3^{(9)}(x) = x^9 - 99\,x^7 + 3465\,x^5 - 51975\,x^3 + 155925\,x\;,$$

$$\mathcal{Q}_3^{(9)}(x) = -(13\,x^8 - 693\,x^6 + 17325\,x^4 - 155925\,x^2)\;, \quad \mathcal{R}_3^{(9)} = -155925\;, \quad \mathcal{S}_3^{(9)} = 0$$

$$\mathcal{P}_3^{(9),*}(x) = x^9 + 99\,x^7 + 3465\,x^5 + 51975\,x^3 + 155925\,x\;,$$

$$\mathcal{Q}_3^{(9),*}(x) = -(13\,x^8 + 693\,x^6 + 17325\,x^4 + 155925\,x^2)\;, \quad \mathcal{R}_3^{(9),*} = -155925\;, \quad \mathcal{S}_3^{(9),*} = 0$$

$$\mathcal{P}_3^{(10)}(x) = x^{10} - 120\,x^8 + 5760\,x^6 - 138240\,x^4 + 1105920\,x^2\;,$$

$$\mathcal{Q}_3^{(10)}(x) = -(14\,x^9 - 960\,x^7 + 34560\,x^5 - 552960\,x^3 + 2211840\,x)\;, \quad \mathcal{R}_3^{(10)} = 0\;, \quad \mathcal{S}_3^{(10)} = 0$$

$$\mathcal{P}_3^{(10),*}(x) = x^{10} + 120\,x^8 + 5760\,x^6 + 138240\,x^4 + 1105920\,x^2\;,$$

$$\mathcal{Q}_3^{(10),*}(x) = -(14\,x^9 + 960\,x^7 + 34560\,x^5 + 552960\,x^3 + 2211840\,x)\;, \quad \mathcal{R}_3^{(10),*} = 0\;, \quad \mathcal{S}_3^{(10),*} = 0$$

$$\mathcal{P}_3^{(-3)}(x) = \frac{x^2+12}{15\,x^3}\;, \quad \mathcal{Q}_3^{(-3)}(x) = -\frac{x^4-x^2+24}{15\,x^4}\;, \quad \mathcal{R}_3^{(-3)} = \frac{1}{15}\;, \quad \mathcal{S}_3^{(-3)} = 0$$

$$\mathcal{P}_3^{(-3),*}(x) = -\frac{x^2-12}{15\,x^3}\;, \quad \mathcal{Q}_3^{(-3),*}(x) = -\frac{x^4+x^2+24}{15\,x^4}\;, \quad \mathcal{R}_3^{(-3),*} = \frac{1}{15}\;, \quad \mathcal{S}_3^{(-3),*} = 0$$

$$\mathcal{P}_3^{(-4)}(x) = \frac{x^2+16}{24\,x^4}\;, \quad \mathcal{Q}_3^{(-4)}(x) = -\frac{x^4-4\,x^2+64}{48\,x^5}\;, \quad \mathcal{R}_3^{(-4)} = 0\;, \quad \mathcal{S}_3^{(-4)} = \frac{1}{48}$$

$$\mathcal{P}_3^{(-4),*}(x) = -\frac{x^2-16}{24\,x^4}\;, \quad \mathcal{Q}_3^{(-4),*}(x) = -\frac{x^4+4\,x^2+64}{48\,x^5}\;, \quad \mathcal{R}_3^{(-4),*} = 0\;, \quad \mathcal{S}_3^{(-4),*} = \frac{1}{48}$$

$$\mathcal{P}_3^{(-5)}(x) = -\frac{x^4-3\,x^2-60}{105\,x^5}\;, \quad \mathcal{Q}_3^{(-5)}(x) = \frac{x^6-x^4+9\,x^2-120}{105\,x^6}\;, \quad \mathcal{R}_3^{(-5)} = -\frac{1}{105}\;, \quad \mathcal{S}_3^{(-5)} = 0$$

$$\mathcal{P}_3^{(-5),*}(x) = -\frac{x^4+3\,x^2-60}{105\,x^5}\;, \quad \mathcal{Q}_3^{(-5),*}(x) = \frac{x^6+x^4+9\,x^2+120}{105\,x^6}\;, \quad \mathcal{R}_3^{(-5),*} = -\frac{1}{105}\;, \quad \mathcal{S}_3^{(-5),*} = 0$$

$$\mathcal{P}_3^{(-6)}(x) = -\frac{x^4-8\,x^2-192}{384\,x^6}\;, \quad \mathcal{Q}_3^{(-6)}(x) = \frac{x^6-4\,x^4+64\,x^2-768}{768\,x^7}\;, \quad \mathcal{R}_3^{(-6)} = 0\;, \quad \mathcal{S}_3^{(-6)} = -\frac{1}{768}$$

$$\mathcal{P}_3^{(-6),*}(x) = -\frac{x^4+8\,x^2-192}{384\,x^6}\;, \quad \mathcal{Q}_3^{(-6),*}(x) = -\frac{x^6+4\,x^4+64\,x^2+768}{768\,x^7}\;, \quad \mathcal{R}_3^{(-6),*} = 0\;, \quad \mathcal{S}_3^{(-6),*} = \frac{1}{768}$$

$\underline{\mathbf{Z_4(x)}}$:

$$\int J_4(x)\,dx = \frac{8J_0(x)}{x} - \frac{16J_1(x)}{x^2} + \Lambda_0(x)$$
$$\int I_4(x)\,dx = -\frac{8I_0(x)}{x} + \frac{16I_1(x)}{x^2} + \Lambda_0^*(x)$$
$$\int x\,J_4(x)\,dx = 8J_0(x) + \frac{x^2-24}{x}J_1(x)$$
$$\int x\,I_4(x)\,dx = -8I_0(x) + \frac{x^2+24}{x}I_1(x)$$
$$\int x^2\,J_4(x)\,dx = 9xJ_0(x) + (x^2-48)J_1(x) + 15\Lambda_0(x)$$
$$\int x^2\,I_4(x)\,dx = -9xI_0(x) + (x^2+48)I_1(x) - 15\Lambda_0^*(x)$$
$$\int x^3\,J_4(x)\,dx = (10\,x^2-48)J_0(x) + \left(x^2-44\right)xJ_1(x)$$
$$\int x^3\,I_4(x)\,dx = -(10\,x^2+48)I_0(x) + \left(x^2+44\right)xI_1(x)$$
$$\int x^4\,J_4(x)\,dx = \left(11\,x^2-105\right)xJ_0(x) + \left(x^2-57\right)x^2J_1(x) + 105\Lambda_0(x)$$
$$\int x^4\,I_4(x)\,dx = -\left(11\,x^2+105\right)xI_0(x) + \left(x^2+57\right)x^2I_1(x) + 105\Lambda_0^*(x)$$
$$\int \frac{J_4(x)\,dx}{x} = \frac{6J_0(x)}{x^2} + \frac{x^2-12}{x^3}J_1(x)$$
$$\int \frac{I_4(x)\,dx}{x} = -\frac{6J_0(x)}{x^2} + \frac{x^2+12}{x^3}J_1(x)$$
$$\int \frac{J_4(x)\,dx}{x^2} = \frac{x^2+72}{15x^3}J_0(x) - \frac{x^4-16\,x^2+144}{15x^4}J_1(x) + \frac{1}{15}\Lambda_0(x)$$
$$\int \frac{I_4(x)\,dx}{x^2} = \frac{x^2-72}{15x^3}I_0(x) + \frac{x^4+16\,x^2+144}{15x^4}I_1(x) - \frac{1}{15}\Lambda_0^*(x)$$
$$\mathcal{P}_4^{(5)}(x) = 12\,x^4 - 192\,x^2\ , \quad \mathcal{Q}_4^{(5)}(x) = x^5 - 72\,x^3 + 384\,x\ , \quad \mathcal{R}_4^{(5)} = 0\ , \quad \mathcal{S}_4^{(5)} = 0$$
$$\mathcal{P}_4^{(5),*}(x) = -(12\,x^4 + 192\,x^2)\ , \quad \mathcal{Q}_4^{(5),*}(x) = x^5 + 72\,x^3 + 384\,x\ , \quad \mathcal{R}_4^{(5),*} = 0\ , \quad \mathcal{S}_4^{(5),*} = 0$$
$$\mathcal{P}_4^{(6)}(x) = 13\,x^5 - 315\,x^3 + 945\,x\ , \quad \mathcal{Q}_4^{(6)}(x) = x^6 - 89\,x^4 + 945\,x^2\ ,$$
$$\mathcal{R}_4^{(6)} = -945\ , \quad \mathcal{S}_4^{(6)} = 0$$
$$\mathcal{P}_4^{(6),*}(x) = -(13\,x^5 + 315\,x^3 + 945\,x)\ , \quad \mathcal{Q}_4^{(6),*}(x) = x^6 + 89\,x^4 + 945\,x^2\ ,$$
$$\mathcal{R}_4^{(6),*} = 945\ , \quad \mathcal{S}_4^{(6),*} = 0$$
$$\mathcal{P}_4^{(7)}(x) = 14\,x^6 - 480\,x^4 + 3840\,x^2\ , \quad \mathcal{Q}_4^{(7)}(x) = x^7 - 108\,x^5 + 1920\,x^3 - 7680\,x\ ,$$
$$\mathcal{R}_4^{(7)} = 0\ , \quad \mathcal{S}_4^{(7)} = 0$$
$$\mathcal{P}_4^{(7),*}(x) = -(14\,x^6 + 480\,x^4 + 3840\,x^2)\ , \quad \mathcal{Q}_4^{(7),*}(x) = x^7 + 108\,x^5 + 1920\,x^3 + 7680\,x\ ,$$
$$\mathcal{R}_4^{(7),*} = 0\ , \quad \mathcal{S}_4^{(7),*} = 0$$
$$\mathcal{P}_4^{(8)}(x) = 15\,x^7 - 693\,x^5 + 10395\,x^3 - 31185\,x\ , \quad \mathcal{Q}_4^{(8)}(x) = x^8 - 129\,x^6 + 3465\,x^4 - 31185\,x^2\ ,$$
$$\mathcal{R}_4^{(8)} = 31185\ , \quad \mathcal{S}_4^{(8)} = 0$$

$$\mathcal{P}_4^{(8),*}(x) = -(15\,x^7 + 693\,x^5 + 10395\,x^3 + 31185\,x)\;, \quad \mathcal{Q}_4^{(8),*}(x) = x^8 + 129\,x^6 + 3465\,x^4 + 31185\,x^2\;,$$
$$\mathcal{R}_4^{(8),*} = 31185\;, \quad \mathcal{S}_4^{(8),*} = 0$$
$$\mathcal{P}_4^{(9)}(x) = 16\,x^8 - 960\,x^6 + 23040\,x^4 - 184320\,x^2\;,$$
$$\mathcal{Q}_4^{(9)}(x) = x^9 - 152\,x^7 + 5760\,x^5 - 92160\,x^3 + 368640\,x\;, \quad \mathcal{R}_4^{(9)} = 0\;, \quad \mathcal{S}_4^{(9)} = 0$$
$$\mathcal{P}_4^{(9),*}(x) = -(16\,x^8 + 960\,x^6 + 23040\,x^4 + 184320\,x^2)\;,$$
$$\mathcal{Q}_4^{(9),*}(x) = x^9 + 152\,x^7 + 5760\,x^5 + 92160\,x^3 + 368640\,x\;, \quad \mathcal{R}_4^{(9),*} = 0\;, \quad \mathcal{S}_4^{(9),*} = 0$$
$$\mathcal{P}_4^{(10)}(x) = 17\,x^9 - 1287\,x^7 + 45045\,x^5 - 675675\,x^3 + 2027025\,x\;,$$
$$\mathcal{Q}_4^{(10)}(x) = x^{10} - 177\,x^8 + 9009\,x^6 - 225225\,x^4 + 2027025\,x^2\;,$$
$$\mathcal{R}_4^{(10)} = -2027025\;, \quad \mathcal{S}_4^{(10)} = 0$$
$$\mathcal{P}_4^{(10),*}(x) = -(17\,x^9 + 1287\,x^7 + 45045\,x^5 + 675675\,x^3 + 2027025\,x)\;,$$
$$\mathcal{Q}_4^{(10),*}(x) = x^{10} + 177\,x^8 + 9009\,x^6 + 225225\,x^4 + 2027025\,x^2\;,$$
$$\mathcal{R}_4^{(10),*} = 2027025\;, \quad \mathcal{S}_4^{(10),*} = 0$$
$$\mathcal{P}_4^{(-3)}(x) = \frac{4}{x^4}\;, \quad \mathcal{Q}_4^{(-3)}(x) = \frac{x^2 - 8}{x^5}\;, \quad \mathcal{R}_4^{(-3)} = 0\;, \quad \mathcal{S}_4^{(-3)} = 0$$
$$\mathcal{P}_4^{(-3),*}(x) = -\frac{4}{x^4}\;, \quad \mathcal{Q}_4^{(-3),*}(x) = \frac{x^2 + 8}{x^5}\;, \quad \mathcal{R}_4^{(-3),*} = 0\;, \quad \mathcal{S}_4^{(-3),*} = 0$$
$$\mathcal{P}_4^{(-4)}(x) = \frac{x^4 - 3\,x^2 + 360}{105\,x^5}\;, \quad \mathcal{Q}_4^{(-4)}(x) = -\frac{x^6 - x^4 - 96\,x^2 + 720}{105x^6}\;,$$
$$\mathcal{R}_4^{(-4)} = \frac{1}{105}\;, \quad \mathcal{S}_4^{(-4)} = 0$$
$$\mathcal{P}_4^{(-4),*}(x) = -\frac{x^4 + 3\,x^2 + 360}{105x^5}\;, \quad \mathcal{Q}_4^{(-4),*}(x) = -\frac{x^6 + x^4 - 96\,x^2 - 720}{105x^6}\;,$$
$$\mathcal{R}_4^{(-4),*} = \frac{1}{105}\;, \quad \mathcal{S}_4^{(-4),*} = 0$$
$$\mathcal{P}_4^{(-5)}(x) = \frac{x^4 - 8\,x^2 + 576}{192\,x^6}\;, \quad \mathcal{Q}_4^{(-5)}(x) = -\frac{x^6 - 4\,x^4 - 320\,x^2 + 2304}{384\,x^7}\;,$$
$$\mathcal{R}_4^{(-5)} = 0\;, \quad \mathcal{S}_4^{(-5)} = \frac{1}{384}$$
$$\mathcal{P}_4^{(-5),*}(x) = -\frac{x^4 + 8\,x^2 + 576}{192\,x^6}\;, \quad \mathcal{Q}_4^{(-5),*}(x) = -\frac{x^6 + 4\,x^4 - 320\,x^2 - 2304}{384\,x^7}\;,$$
$$\mathcal{R}_4^{(-5),*} = 0\;, \quad \mathcal{S}_4^{(-5),*} = \frac{1}{384}$$
$$\mathcal{P}_4^{(-6)}(x) = -\frac{x^6 - 3\,x^4 + 45\,x^2 - 2520}{945\,x^7}\;, \quad \mathcal{Q}_4^{(-6)}(x) = \frac{x^8 - x^6 + 9\,x^4 + 720\,x^2 - 5040}{945\,x^8}\;,$$
$$\mathcal{R}_4^{(-6)} = -\frac{1}{945}\;, \quad \mathcal{S}_4^{(-6)} = 0$$
$$\mathcal{P}_4^{(-6),*}(x) = -\frac{x^6 + 3\,x^4 + 45\,x^2 + 2520}{945\,x^7}\;, \quad \mathcal{Q}_4^{(-6),*}(x) = \frac{x^8 + x^6 + 9\,x^4 - 720\,x^2 - 5040}{945\,x^8}\;,$$
$$\mathcal{R}_4^{(-6),*} = \frac{1}{945}\;, \quad \mathcal{S}_4^{(-6),*} = 0$$

$\mathbf{Z}_5(\mathbf{x})$:

$$\int J_5(x)\,dx = -\frac{x^2 - 48}{x^2} J_0(x) + \frac{12\,x^2 - 96}{x^3} J_1(x)$$

$$\int I_5(x)\,dx = \frac{x^2+48}{x^2} I_0(x) - \frac{12\,x^2+96}{x^3} I_1(x)$$

$$\int x\,J_5(x)\,dx = -\frac{x^2-64}{x} J_0(x) + \frac{8\,x^2-128}{x^2} J_1(x) + 5\Lambda_0(x)$$

$$\int x\,I_5(x)\,dx = \frac{x^2+64}{x} I_0(x) - \frac{8\,x^2+128}{x^2} I_1(x) - 5\Lambda_0^*(x)$$

$$\int x^2\,J_5(x)\,dx = -(x^2-72) J_0(x) + \frac{14\,x^2-192}{x} J_1(x)$$

$$\int x^2\,I_5(x)\,dx = (x^2+72) I_0(x) - \frac{14\,x^2+192}{x} I_1(x)$$

$$\int x^3\,J_5(x)\,dx = -(x^3-87\,x) J_0(x) + (15\,x^2-384) J_1(x) + 105\Lambda_0(x)$$

$$\int x^3\,I_5(x)\,dx = (x^3+87\,x) I_0(x) - (15\,x^2+384) I_1(x) + 105\Lambda_0^*(x)$$

$$\int x^4\,J_5(x)\,dx = -(x^4-104\,x^2+384) J_0(x) + (16\,x^3-400\,x) J_1(x)$$

$$\int x^4\,I_5(x)\,dx = (x^4+104\,x^2+384) I_0(x) - (16\,x^3+400\,x) I_1(x)$$

$$\int \frac{J_5(x)\,dx}{x} = -\frac{4\,x^2-192}{5\,x^3} J_0(x) - \frac{x^4-56\,x^2+384}{5\,x^4} J_1(x) + \frac{1}{5}\,\Lambda_0(x)$$

$$\int \frac{I_5(x)\,dx}{x} = \frac{4\,x^2+192}{5\,x^3} I_0(x) - \frac{x^4+56\,x^2+384}{5\,x^4} I_1(x) + \frac{1}{5}\Lambda_0^*(x)$$

$$\int \frac{J_5(x)\,dx}{x^2} = -\frac{x^2-32}{x^4} J_0(x) + \frac{10\,x^2-64}{x^5} J_1(x)$$

$$\int \frac{I_5(x)\,dx}{x^2} = \frac{x^2+32}{x^4} I_0(x) - \frac{10\,x^2+64}{x^5} I_1(x)$$

$$\mathcal{P}_5^{(5)}(x) = -(x^5-123\,x^3+945\,x)\,, \quad \mathcal{Q}_5^{(5)}(x) = 17\,x^4-561\,x^2\,, \quad \mathcal{R}_5^{(5)} = 945\,, \quad \mathcal{S}_5^{(5)} = 0$$

$$\mathcal{P}_5^{(5),*}(x) = x^5+123\,x^3+945\,x\,, \quad \mathcal{Q}_5^{(5),*}(x) = -(17\,x^4+561\,x^2)\,, \quad \mathcal{R}_5^{(5),*} = -945\,, \quad \mathcal{S}_5^{(5),*} = 0$$

$$\mathcal{P}_5^{(6)}(x) = -(x^6-144\,x^4+1920\,x^2)\,, \quad \mathcal{Q}_5^{(6)}(x) = 18\,x^5-768\,x^3+3840\,x\,,$$

$$\mathcal{R}_5^{(6)} = 0\,, \quad \mathcal{S}_5^{(6)} = 0$$

$$\mathcal{P}_5^{(6),*}(x) = x^6+144\,x^4+1920\,x^2\,, \quad \mathcal{Q}_5^{(6),*}(x) = -(18\,x^5+768\,x^3+3840\,x)\,,$$

$$\mathcal{R}_5^{(6),*} = 0\,, \quad \mathcal{S}_5^{(6),*} = 0$$

$$\mathcal{P}_5^{(7)}(x) = -(x^7-167\,x^5+3465\,x^3-10395\,x)\,, \quad \mathcal{Q}_5^{(7)}(x) = 19\,x^6-1027\,x^4+10395\,x^2\,,$$

$$\mathcal{R}_5^{(7)} = -10395\,, \quad \mathcal{S}_5^{(7)} = 0$$

$$\mathcal{P}_5^{(7),*}(x) = x^7+167\,x^5+3465\,x^3+10395\,x\,, \quad \mathcal{Q}_5^{(7),*}(x) = -(19\,x^6+1027\,x^4+10395\,x^2)\,,$$

$$\mathcal{R}_5^{(7),*} = -10395\,, \quad \mathcal{S}_5^{(7),*} = 0$$

$$\mathcal{P}_5^{(8)}(x) = -(x^8-192\,x^6+5760\,x^4-46080\,x^2)\,, \quad \mathcal{Q}_5^{(8)}(x) = 20\,x^7-1344\,x^5+23040\,x^3-92160\,x\,,$$

$$\mathcal{R}_5^{(8)} = 0\,, \quad \mathcal{S}_5^{(8)} = 0$$

$$\mathcal{P}_5^{(8),*}(x) = x^8+192\,x^6+5760\,x^4+46080\,x^2\,, \quad \mathcal{Q}_5^{(8),*}(x) = -(20\,x^7+1344\,x^5+23040\,x^3+92160\,x)\,,$$

$$\mathcal{R}_5^{(8),*} = 0\,, \quad \mathcal{S}_5^{(8),*} = 0$$

$$\mathcal{P}_5^{(9)}(x) = -(x^9-219\,x^7+9009\,x^5-135135\,x^3+405405\,x)\,,$$

$$\mathcal{Q}_5^{(9)}(x) = 21\,x^8-1725\,x^6+45045\,x^4-405405\,x^2\,, \quad \mathcal{R}_5^{(9)} = 405405\,, \quad \mathcal{S}_5^{(9)} = 0$$

$$\mathcal{P}_5^{(9),*}(x) = x^9 + 219\,x^7 + 9009\,x^5 + 135135\,x^3 + 405405\,x\;,$$

$$\mathcal{Q}_5^{(9),*}(x) = -(21\,x^8 + 1725\,x^6 + 45045\,x^4 + 405405\,x^2)\;,\quad \mathcal{R}_5^{(9),*} = -405405\;,\quad \mathcal{S}_5^{(9),*} = 0$$

$$\mathcal{P}_5^{(10)}(x) = -(x^{10} - 248\,x^8 + 13440\,x^6 - 322560\,x^4 + 2580480\,x^2)\;,$$

$$\mathcal{Q}_5^{(10)}(x) = 22\,x^9 - 2176\,x^7 + 80640\,x^5 - 1290240\,x^3 + 5160960\,x\;,\quad \mathcal{R}_5^{(10)} = 0\;,\quad \mathcal{S}_5^{(10)} = 0$$

$$\mathcal{P}_5^{(10),*}(x) = x^{10} + 248\,x^8 + 13440\,x^6 + 322560\,x^4 + 2580480\,x^2\;,$$

$$\mathcal{Q}_5^{(10),*}(x) = -(22\,x^9 + 2176\,x^7 + 80640\,x^5 + 1290240\,x^3 + 5160960\,x)\;,\quad \mathcal{R}_5^{(10),*} = 0\;,\quad \mathcal{S}_5^{(10),*} = 0$$

$$\mathcal{P}_5^{(-3)}(x) = \frac{x^4 - 108\,x^2 + 2880}{105x^5}\;,\quad \mathcal{Q}_5^{(-3)}(x) = -\frac{x^6 - x^4 - 936\,x^2 + 5760}{105x^6}\;,\quad \mathcal{R}_5^{(-3)} = \frac{1}{105}\;,\quad \mathcal{S}_5^{(-3)} = 0$$

$$\mathcal{P}_5^{(-3),*}(x) = \frac{x^4 + 108\,x^2 + 2880}{105x^5}\;,\quad \mathcal{Q}_5^{(-3),*}(x) = \frac{x^6 + x^4 - 936\,x^2 - 5760}{105x^6}\;,\quad \mathcal{R}_5^{(-3),*} = -\frac{1}{105}\;,\quad \mathcal{S}_5^{(-3),*} = 0$$

$$\mathcal{P}_5^{(-4)}(x) = -\frac{x^2 - 24}{x^6}\;,\quad \mathcal{Q}_5^{(-4)}(x) = \frac{8\,x^2 - 48}{x^7}\;,\quad \mathcal{R}_5^{(-4)} = 0\;,\quad \mathcal{S}_5^{(-4)} = 0$$

$$\mathcal{P}_5^{(-4),*}(x) = \frac{x^2 + 24}{x^6}\;,\quad \mathcal{Q}_5^{(-4),*}(x) = -\frac{8\,x^2 + 48}{x^7}\;,\quad \mathcal{R}_5^{(-4),*} = 0\;,\quad \mathcal{S}_5^{(-4),*} = 0$$

$$\mathcal{P}_5^{(-5)}(x) = \frac{x^6 - 3\,x^4 - 900\,x^2 + 20160}{945\,x^7}\;,\quad \mathcal{Q}_5^{(-5)}(x) = -\frac{x^8 - x^6 + 9\,x^4 - 6840\,x^2 + 40320}{945\,x^8}\;,$$

$$\mathcal{R}_5^{(-5)} = \frac{1}{945}\;,\quad \mathcal{S}_5^{(-5)} = 0$$

$$\mathcal{P}_5^{(-5),*}(x) = -\frac{x^6 + 3\,x^4 - 900\,x^2 - 20160}{945\,x^7}\;,\quad \mathcal{Q}_5^{(-5),*}(x) = -\frac{x^8 + x^6 + 9\,x^4 + 6840\,x^2 + 40320}{945\,x^8}\;,$$

$$\mathcal{R}_5^{(-5),*} = \frac{1}{945}\;,\quad \mathcal{S}_5^{(-5),*} = 0$$

$$\mathcal{P}_5^{(-6)}(x) = \frac{x^6 - 8\,x^4 - 1728\,x^2 + 36864}{1920\,x^8}\;,\quad \mathcal{Q}_5^{(-6)}(x) = -\frac{x^8 - 4\,x^6 + 64\,x^4 - 25344\,x^2 + 147456}{3840\,x^9}\;,$$

$$\mathcal{R}_5^{(-6)} = 0\;,\quad \mathcal{S}_5^{(-6)} = \frac{1}{3840}$$

$$\mathcal{P}_5^{(-6),*}(x) = -\frac{x^6 + 8\,x^4 - 1728\,x^2 - 36864}{1920\,x^8}\;,\quad \mathcal{Q}_5^{(-6),*}(x) = -\frac{x^8 + 4\,x^6 + 64\,x^4 + 25344\,x^2 + 147456}{3840\,x^9}\;,$$

$$\mathcal{R}_5^{(-6),*} = 0\;,\quad \mathcal{S}_5^{(-6),*} = \frac{1}{3840}$$

$\mathbf{Z_6(x)}$:

$$\int J_6(x)\,dx = -\frac{16\,x^2 - 384}{x^3}J_0(x) - \frac{2\,x^4 - 128\,x^2 + 768}{x^4}J_1(x) + \Lambda_0(x)$$

$$\int I_6(x)\,dx = -\frac{16\,x^2 + 384}{x^3}I_0(x) + \frac{2\,x^4 + 128\,x^2 + 768}{x^4}I_1(x) - \Lambda_0^*(x)$$

$$\int x\,J_6(x)\,dx = -\frac{18\,x^2 - 480}{x^2}J_0(x) - \frac{x^4 - 144\,x^2 + 960}{x^3}J_1(x)$$

$$\int x\,I_6(x)\,dx = -\frac{18\,x^2 + 480}{x^2}I_0(x) + \frac{x^4 + 144\,x^2 + 960}{x^3}I_1(x)$$

$$\int x^2\,J_6(x)\,dx = -\frac{19\,x^2 - 640}{x}J_0(x) - \frac{x^4 - 128\,x^2 + 1280}{x^2}J_1(x) + 35\Lambda_0(x)$$

$$\int x^2\,I_6(x)\,dx = -\frac{19\,x^2 + 640}{x}I_0(x) + \frac{x^4 + 128\,x^2 + 1280}{x^2}I_1(x) + 35\Lambda_0^*(x)$$

$$\int x^3\,J_6(x)\,dx = -(20\,x^2 - 768)J_0(x) - \frac{x^4 - 184\,x^2 + 1920}{x}J_1(x)$$

$$\int x^3\, I_6(x)\, dx = -(20\, x^2 + 768) I_0(x) + \frac{x^4 + 184\, x^2 + 1920}{x} I_1(x)$$

$$\int x^4\, J_6(x)\, dx = -(21\, x^3 - 975\, x) J_0(x) - (x^4 - 207\, x^2 + 3840) J_1(x) + 945 \Lambda_0(x)$$

$$\int x^4\, I_6(x)\, dx = -(21\, x^3 + 975\, x) I_0(x) + (x^4 + 207\, x^2 + 3840) I_1(x) - 945 \Lambda_0^*(x)$$

$$\int \frac{J_6(x)\, dx}{x} = -\frac{16\, x^2 - 320}{x^4} J_0(x) - \frac{x^4 - 112\, x^2 + 640}{x^5} J_1(x)$$

$$\int \frac{I_6(x)\, dx}{x} = -\frac{16\, x^2 + 320}{x^4} I_0(x) + \frac{x^4 + 112\, x^2 + 640}{x^5} I_1(x)$$

$$\int \frac{J_6(x)\, dx}{x^2} = \frac{x^4 - 528\, x^2 + 9600}{35\, x^5} J_0(x) - \frac{x^6 + 34\, x^4 - 3456\, x^2 + 19200}{35\, x^6} J_1(x) + \frac{1}{35} \Lambda_0(x)$$

$$\int \frac{I_6(x)\, dx}{x^2} = -\frac{x^4 + 528\, x^2 + 9600}{35\, x^5} I_0(x) - \frac{x^6 - 34\, x^4 - 3456\, x^2 - 19200}{35\, x^6} I_1(x) + \frac{1}{35} \Lambda_0^*(x)$$

$$\mathcal{P}_6^{(5)}(x) = -(22\, x^4 - 1232\, x^2 + 3840)\ , \quad \mathcal{Q}_6^{(5)}(x) = -(x^5 - 232\, x^3 + 4384\, x)\ ,$$

$$\mathcal{R}_6^{(5)} = 0\ , \quad \mathcal{S}_6^{(5)} = 0$$

$$\mathcal{P}_6^{(5),*}(x) = -(22\, x^4 + 1232\, x^2 + 3840)\ , \quad \mathcal{Q}_6^{(5),*}(x) = 22\, x^4 + 1232\, x^2 + 3840\ ,$$

$$\mathcal{R}_6^{(5),*} = 0\ , \quad \mathcal{S}_6^{(5),*} = 0$$

$$\mathcal{P}_6^{(6)}(x) = -(23\, x^5 - 1545\, x^3 + 10395\, x)\ , \quad \mathcal{Q}_6^{(6)}(x) = -(x^6 - 259\, x^4 + 6555\, x^2)\ ,$$

$$\mathcal{R}_6^{(6)} = 10395\ , \quad \mathcal{S}_6^{(6)} = 0$$

$$\mathcal{P}_6^{(6),*}(x) = -(23\, x^5 + 1545\, x^3 + 10395\, x)\ , \quad \mathcal{Q}_6^{(6),*}(x) = x^6 + 259\, x^4 + 6555\, x^2\ ,$$

$$\mathcal{R}_6^{(6),*} = 10395\ , \quad \mathcal{S}_6^{(6),*} = 0$$

$$\mathcal{P}_6^{(7)}(x) = -(24\, x^6 - 1920\, x^4 + 23040\, x^2)\ , \quad \mathcal{Q}_6^{(7)}(x) = -(x^7 - 288\, x^5 + 9600\, x^3 - 46080\, x)\ ,$$

$$\mathcal{R}_6^{(7)} = 0\ , \quad \mathcal{S}_6^{(7)} = 0$$

$$\mathcal{P}_6^{(7),*}(x) = -(24\, x^6 + 1920\, x^4 + 23040\, x^2)\ , \quad \mathcal{Q}_6^{(7),*}(x) = x^7 + 288\, x^5 + 9600\, x^3 + 46080\, x\ ,$$

$$\mathcal{R}_6^{(7),*} = 0\ , \quad \mathcal{S}_6^{(7),*} = 0$$

$$\mathcal{P}_6^{(8)}(x) = -(25\, x^7 - 2363\, x^5 + 45045\, x^3 - 135135\, x)\ , \quad \mathcal{Q}_6^{(8)}(x) = -(x^8 - 319\, x^6 + 13735\, x^4 - 135135\, x^2)\ ,$$

$$\mathcal{R}_6^{(8)} = -135135\ , \quad \mathcal{S}_6^{(8)} = 0$$

$$\mathcal{P}_6^{(8),*}(x) = -(25\, x^7 + 2363\, x^5 + 45045\, x^3 + 135135\, x)\ , \quad \mathcal{Q}_6^{(8),*}(x) = x^8 + 319\, x^6 + 13735\, x^4 + 135135\, x^2\ ,$$

$$\mathcal{R}_6^{(8),*} = 135135\ , \quad \mathcal{S}_6^{(8),*} = 0$$

$$\mathcal{P}_6^{(9)}(x) = -(26\, x^8 - 2880\, x^6 + 80640\, x^4 - 645120\, x^2)\ ,$$

$$\mathcal{Q}_6^{(9)}(x) = -(x^9 - 352\, x^7 + 19200\, x^5 - 322560\, x^3 + 1290240\, x)\ , \quad \mathcal{R}_6^{(9)} = 0\ , \quad \mathcal{S}_6^{(9)} = 0$$

$$\mathcal{P}_6^{(9),*}(x) = -(26\, x^8 + 2880\, x^6 + 80640\, x^4 + 645120\, x^2)\ ,$$

$$\mathcal{Q}_6^{(9),*}(x) = x^9 + 352\, x^7 + 19200\, x^5 + 322560\, x^3 + 1290240\, x\ , \quad \mathcal{R}_6^{(9),*} = 0\ , \quad \mathcal{S}_6^{(9),*} = 0$$

$$\mathcal{P}_6^{(10)}(x) = -(27\, x^9 - 3477\, x^7 + 135135\, x^5 - 2027025\, x^3 + 6081075\, x)\ ,$$

$$\mathcal{Q}_6^{(10)}(x) = -(x^{10} - 387\, x^8 + 26259\, x^6 - 675675\, x^4 + 6081075\, x^2)\ , \quad \mathcal{R}_6^{(10)} = 6081075\ , \quad \mathcal{S}_6^{(10)} = 0$$

$$\mathcal{P}_6^{(10),*}(x) = -(27\, x^9 + 3477\, x^7 + 135135\, x^5 + 2027025\, x^3 + 6081075\, x)\ ,$$

$$\mathcal{Q}_6^{(10),*}(x) = x^{10} + 387\, x^8 + 26259\, x^6 + 675675\, x^4 + 6081075\, x^2\ , \quad \mathcal{R}_6^{(10),*} = 6081075\ , \quad \mathcal{S}_6^{(10),*} = 0$$

$$\mathcal{P}_6^{(-3)}(x) = -\frac{14\,x^2 - 240}{x^6}\,, \quad \mathcal{Q}_6^{(-3)}(x) = -\frac{x^4 - 88\,x^2 + 480}{x^7}\,, \quad \mathcal{R}_6^{(-3)} = 0\,, \quad \mathcal{S}_6^{(-3)} = 0$$

$$\mathcal{P}_6^{(-3),*}(x) = -\frac{14\,x^2 + 240}{x^6}\,, \quad \mathcal{Q}_6^{(-3),*}(x) = \frac{x^4 + 88\,x^2 + 480}{x^7}\,, \quad \mathcal{R}_6^{(-3),*} = 0\,, \quad \mathcal{S}_6^{(-3),*} = 0$$

$$\mathcal{P}_6^{(-4)}(x) = \frac{x^6 - 3\,x^4 - 12240\,x^2 + 201600}{945\,x^7}\,, \quad \mathcal{Q}_6^{(-4)}(x) = -\frac{x^8 - x^6 + 954\,x^4 - 74880\,x^2 + 403200}{945\,x^8}\,,$$

$$\mathcal{R}_6^{(-4)} = \frac{1}{945}\,, \quad \mathcal{S}_6^{(-4)} = 0$$

$$\mathcal{P}_6^{(-4),*}(x) = \frac{x^6 + 3\,x^4 - 12240\,x^2 - 201600}{945\,x^7}\,, \quad \mathcal{Q}_6^{(-4),*}(x) = \frac{x^8 + x^6 + 954\,x^4 + 74880\,x^2 + 403200}{945\,x^8}\,,$$

$$\mathcal{R}_6^{(-4),*} = -\frac{1}{945}\,, \quad \mathcal{S}_6^{(-4),*} = 0$$

$$\mathcal{P}_6^{(-5)}(x) = -\frac{12\,x^2 - 192}{x^8}\,, \quad \mathcal{Q}_6^{(-5)}(x) = -\frac{x^4 - 72\,x^2 + 384}{x^9}\,, \quad \mathcal{R}_6^{(-5)} = 0\,, \quad \mathcal{S}_6^{(-5)} = 0$$

$$\mathcal{P}_6^{(-5),*}(x) = -\frac{12\,x^2 + 192}{x^8}\,, \quad \mathcal{Q}_6^{(-5),*}(x) = \frac{x^4 + 72\,x^2 + 384}{x^9}\,, \quad \mathcal{R}_6^{(-5),*} = 0\,, \quad \mathcal{S}_6^{(-5),*} = 0$$

$$\mathcal{P}_6^{(-6)}(x) = \frac{x^8 - 3\,x^6 + 45\,x^4 - 115920\,x^2 + 1814400}{10395\,x^9}\,,$$

$$\mathcal{Q}_6^{(-6)}(x) = -\frac{x^{10} - x^8 + 9\,x^6 + 10170\,x^4 - 685440\,x^2 + 3628800}{10395\,x^{10}}\,,$$

$$\mathcal{R}_6^{(-6)} = \frac{1}{10395}\,, \quad \mathcal{S}_6^{(-6)} = 0$$

$$\mathcal{P}_6^{(-6),*}(x) = -\frac{x^8 + 3\,x^6 + 45\,x^4 + 115920\,x^2 + 1814400}{10395\,x^9}\,,$$

$$\mathcal{Q}_6^{(-6),*}(x) = -\frac{x^{10} + x^8 + 9\,x^6 - 10170\,x^4 - 685440\,x^2 - 3628800}{10395\,x^{10}}\,,$$

$$\mathcal{R}_6^{(-6),*} = \frac{1}{10395}\,, \quad \mathcal{S}_6^{(-6),*} = 0$$

$\mathbf{Z_7(x)}$:

$$\int J_7(x)\,dx = \frac{x^4 - 240\,x^2 + 3840}{x^4} J_0(x) - \frac{24\,x^4 - 1440\,x^2 + 7680}{x^5} J_1(x)$$

$$\int I_7(x)\,dx = \frac{x^4 + 240\,x^2 + 3840}{x^4} I_0(x) - \frac{24\,x^4 + 1440\,x^2 + 7680}{x^5} I_1(x)$$

$$\int x\,J_7(x)\,dx = \frac{x^4 - 256\,x^2 + 4608}{x^3} J_0(x) - \frac{32\,x^4 - 1664\,x^2 + 9216}{x^4} J_1(x) + 7\Lambda_0(x)$$

$$\int x\,I_7(x)\,dx = \frac{x^4 + 256\,x^2 + 4608}{x^3} I_0(x) - \frac{32\,x^4 + 1664\,x^2 + 9216}{x^4} I_1(x) + 7\Lambda_0^*(x)$$

$$\int x^2\,J_7(x)\,dx = \frac{x^4 - 288\,x^2 + 5760}{x^2} J_0(x) - \frac{26\,x^4 - 1920\,x^2 + 11520}{x^3} J_1(x)$$

$$\int x^2\,I_7(x)\,dx = \frac{x^4 + 288\,x^2 + 5760}{x^2} I_0(x) - \frac{26\,x^4 + 1920\,x^2 + 11520}{x^3} I_1(x)$$

$$\int x^3\,J_7(x)\,dx = \frac{x^4 - 315\,x^2 + 7680}{x} J_0(x) - \frac{27\,x^4 - 1920\,x^2 + 15360}{x^2} J_1(x) + 315\Lambda_0(x)$$

$$\int x^3\,I_7(x)\,dx = \frac{x^4 + 315\,x^2 + 7680}{x} I_0(x) - \frac{27\,x^4 + 1920\,x^2 + 15360}{x^2} I_1(x) - 315\Lambda_0^*(x)$$

$$\int x^4\,J_7(x)\,dx = (x^4 - 344\,x^2 + 9600) J_0(x) - \frac{28\,x^4 - 2608\,x^2 + 23040}{x} J_1(x)$$

$$\int x^4\, I_7(x)\, dx = (x^4 + 344\, x^2 + 9600) I_0(x) - \frac{28\, x^4 + 2608\, x^2 + 23040}{x} I_1(x)$$

$$\int \frac{J_7(x)\, dx}{x} = \frac{8\, x^4 - 1536\, x^2 + 23040}{7\, x^5} J_0(x) - \frac{x^6 + 160\, x^4 - 8832\, x^2 + 46080}{7\, x^6} J_1(x) + \frac{1}{7}\Lambda_0(x)$$

$$\int \frac{I_7(x)\, dx}{x} = \frac{8\, x^4 + 1536\, x^2 + 23040}{7\, x^5} I_0(x) + \frac{x^6 - 160\, x^4 - 8832\, x^2 - 46080}{7\, x^6} I_1(x) - \frac{1}{7}\Lambda_0^*(x)$$

$$\int \frac{J_7(x)\, dx}{x^2} = \frac{x^4 - 200\, x^2 + 2880}{x^6} J_0(x) - \frac{22\, x^4 - 1120\, x^2 + 5760}{x^7} J_1(x)$$

$$\int \frac{I_7(x)\, dx}{x^2} = \frac{x^4 + 200\, x^2 + 2880}{x^6} I_0(x) - \frac{22\, x^4 + 1120\, x^2 + 5760}{x^7} I_1(x)$$

$$\mathcal{P}_7^{(5)}(x) = x^5 - 375\, x^3 + 12645\, x\,, \quad \mathcal{Q}_7^{(5)}(x) = -(29\, x^4 - 3045\, x^2 + 46080)\,,$$

$$\mathcal{R}_7^{(5)} = 10395\,, \quad \mathcal{S}_7^{(5)} = 0$$

$$\mathcal{P}_7^{(5),*}(x) = x^5 + 375\, x^3 + 12645\, x\,, \quad \mathcal{Q}_7^{(5),*}(x) = -(29\, x^4 + 3045\, x^2 + 46080)\,,$$

$$\mathcal{R}_7^{(5),*} = 10395\,, \quad \mathcal{S}_7^{(5),*} = 0$$

$$\mathcal{P}_7^{(6)}(x) = x^6 - 408\, x^4 + 16704\, x^2 - 46080\,, \quad \mathcal{Q}_7^{(6)}(x) = -(30\, x^5 - 3552\, x^3 + 56448\, x)\,,$$

$$\mathcal{R}_7^{(6)} = 0\,, \quad \mathcal{S}_7^{(6)} = 0$$

$$\mathcal{P}_7^{(6),*}(x) = x^6 + 408\, x^4 + 16704\, x^2 + 46080\,, \quad \mathcal{Q}_7^{(6),*}(x) = -(30\, x^5 + 3552\, x^3 + 56448\, x)\,,$$

$$\mathcal{R}_7^{(6),*} = 0\,, \quad \mathcal{S}_7^{(6),*} = 0$$

$$\mathcal{P}_7^{(7)}(x) = x^7 - 443\, x^5 + 22005\, x^3 - 135135\, x\,, \quad \mathcal{Q}_7^{(7)}(x) = -(31\, x^6 - 4135\, x^4 + 89055\, x^2)\,,$$

$$\mathcal{R}_7^{(7)} = 135135\,, \quad \mathcal{S}_7^{(7)} = 0$$

$$\mathcal{P}_7^{(7),*}(x) = x^7 + 443\, x^5 + 22005\, x^3 + 135135\, x\,, \quad \mathcal{Q}_7^{(7),*}(x) = -(31\, x^6 + 4135\, x^4 + 89055\, x^2)\,,$$

$$\mathcal{R}_7^{(7),*} = -135135\,, \quad \mathcal{S}_7^{(7),*} = 0$$

$$\mathcal{P}_7^{(8)}(x) = x^8 - 480\, x^6 + 28800\, x^4 - 322560\, x^2\,, \quad \mathcal{Q}_7^{(8)}(x) = -(32\, x^7 - 4800\, x^5 + 138240\, x^3 - 645120\, x)\,,$$

$$\mathcal{R}_7^{(8)} = 0\,, \quad \mathcal{S}_7^{(8)} = 0$$

$$\mathcal{P}_7^{(8),*}(x) = x^8 + 480\, x^6 + 28800\, x^4 + 322560\, x^2\,, \quad \mathcal{Q}_7^{(8),*}(x) = -(32\, x^7 + 4800\, x^5 + 138240\, x^3 + 645120\, x)\,,$$

$$\mathcal{R}_7^{(8),*} = 0\,, \quad \mathcal{S}_7^{(8),*} = 0$$

$$\mathcal{P}_7^{(9)}(x) = x^9 - 519\, x^7 + 37365\, x^5 - 675675\, x^3 + 2027025\, x\,,$$

$$\mathcal{Q}_7^{(9)}(x) = -(33\, x^8 - 5553\, x^6 + 209865\, x^4 - 2027025\, x^2)\,, \quad \mathcal{R}_7^{(9)} = -2027025\,, \quad \mathcal{S}_7^{(9)} = 0$$

$$\mathcal{P}_7^{(9),*}(x) = x^9 + 519\, x^7 + 37365\, x^5 + 675675\, x^3 + 2027025\, x\,,$$

$$\mathcal{Q}_7^{(9),*}(x) = -(33\, x^8 + 5553\, x^6 + 209865\, x^4 + 2027025\, x^2)\,, \quad \mathcal{R}_7^{(9),*} = -2027025\,, \quad \mathcal{S}_7^{(9),*} = 0$$

$$\mathcal{P}_7^{(10)}(x) = x^{10} - 560\, x^8 + 48000\, x^6 - 1290240\, x^4 + 10321920\, x^2\,,$$

$$\mathcal{Q}_7^{(10)}(x) = -(34\, x^9 - 6400\, x^7 + 311040\, x^5 - 5160960\, x^3 + 20643840\, x)\,, \quad \mathcal{R}_7^{(10)} = 0\,, \quad \mathcal{S}_7^{(10)} = 0$$

$$\mathcal{P}_7^{(10),*}(x) = x^{10} + 560\, x^8 + 48000\, x^6 + 1290240\, x^4 + 10321920\, x^2\,,$$

$$\mathcal{Q}_7^{(10),*}(x) = -(34\, x^9 + 6400\, x^7 + 311040\, x^5 + 5160960\, x^3 + 20643840\, x)\,, \quad \mathcal{R}_7^{(10),*} = 0\,, \quad \mathcal{S}_7^{(10),*} = 0$$

$$\mathcal{P}_7^{(-3)}(x) = \frac{x^6 + 312\, x^4 - 57600\, x^2 + 806400}{315\, x^7}\,,$$

$$\mathcal{Q}_7^{(-3)}(x) = -\frac{x^8 - x^6 + 6624\, x^4 - 316800\, x^2 + 1612800}{315\, x^8}\,, \quad \mathcal{R}_7^{(-3)} = \frac{1}{315}\,, \quad \mathcal{S}_7^{(-3)} = 0$$

$$\mathcal{P}_7^{(-3),*}(x) = \frac{x^6 - 312\,x^4 - 57600\,x^2 - 806400}{315\,x^7}\ ,$$

$$\mathcal{Q}_7^{(-3),*}(x) = -\frac{x^8 + x^6 + 6624\,x^4 + 316800\,x^2 + 1612800}{315\,x^8}\ ,\quad \mathcal{R}_7^{(-3),*} = \frac{1}{315}\ ,\quad \mathcal{S}_7^{(-3),*} = 0$$

$$\mathcal{P}_7^{(-4)}(x) = \frac{x^4 - 168\,x^2 + 2304}{x^8}\ ,\quad \mathcal{Q}_7^{(-4)}(x) = -\frac{20\,x^4 - 912\,x^2 + 4608}{x^9}\ ,\quad \mathcal{R}_7^{(-4)} = 0\ ,\quad \mathcal{S}_7^{(-4)} = 0$$

$$\mathcal{P}_7^{(-4),*}(x) = \frac{x^4 + 168\,x^2 + 2304}{x^8}\ ,\quad \mathcal{Q}_7^{(-4),*}(x) = -\frac{20\,x^4 + 912\,x^2 + 4608}{x^9}\ ,\quad \mathcal{R}_7^{(-4),*} = 0\ ,\quad \mathcal{S}_7^{(-4),*} = 0$$

$$\mathcal{P}_7^{(-5)}(x) = \frac{x^8 - 3\,x^6 + 10440\,x^4 - 1612800\,x^2 + 21772800}{10395\,x^9}\ ,$$

$$\mathcal{Q}_7^{(-5)}(x) = -\frac{x^{10} - x^8 + 9\,x^6 + 197280\,x^4 - 8668800\,x^2 + 43545600}{10395\,x^{10}}\ ,\quad \mathcal{R}_7^{(-5)} = \frac{1}{10395}\ ,\quad \mathcal{S}_7^{(-5)} = 0$$

$$\mathcal{P}_7^{(-5),*}(x) = \frac{x^8 + 3\,x^6 + 10440\,x^4 + 1612800\,x^2 + 21772800}{10395\,x^9}\ ,$$

$$\mathcal{Q}_7^{(-5),*}(x) = \frac{x^{10} + x^8 + 9\,x^6 - 197280\,x^4 - 8668800\,x^2 - 43545600}{10395\,x^{10}}\ ,\quad \mathcal{R}_7^{(-5),*} = -\frac{1}{10395}\ ,\quad \mathcal{S}_7^{(-5),*} = 0$$

$$\mathcal{P}_7^{(-6)}(x) = \frac{x^4 - 144\,x^2 + 1920}{x^{10}}\ ,\quad \mathcal{Q}_7^{(-6)}(x) = -\frac{18\,x^4 - 768\,x^2 + 3840}{x^{11}}\ ,\quad \mathcal{R}_7^{(-6)} = 0\ ,\quad \mathcal{S}_7^{(-6)} = 0$$

$$\mathcal{P}_7^{(-6),*}(x) = \frac{x^4 + 144\,x^2 + 1920}{x^{10}}\ ,\quad \mathcal{Q}_7^{(-6),*}(x) = -\frac{18\,x^4 + 768\,x^2 + 3840}{x^{11}}\ ,\quad \mathcal{R}_7^{(-6),*} = 0\ ,\quad \mathcal{S}_7^{(-6),*} = 0$$

$\mathbf{Z_8(x)}$:

$$\int J_8(x)\,dx = \frac{32\,x^4 - 3456\,x^2 + 46080}{x^5} J_0(x) - \frac{448\,x^4 - 18432\,x^2 + 92160}{x^6} J_1(x) + \Lambda_0(x)$$

$$\int I_8(x)\,dx = -\frac{32\,x^4 + 3456\,x^2 + 46080}{x^5} I_0(x) + \frac{448\,x^4 + 18432\,x^2 + 92160}{x^6} I_1(x) + \Lambda_0^*(x)$$

$$\int x\,J_8(x)\,dx = \frac{32\,x^4 - 3840\,x^2 + 53760}{x^4} J_0(x) + \frac{x^6 - 480\,x^4 + 21120\,x^2 - 107520}{x^5} J_1(x)$$

$$\int x\,I_8(x)\,dx = -\frac{32\,x^4 + 3840\,x^2 + 53760}{x^4} I_0(x) + \frac{x^6 + 480\,x^4 + 21120\,x^2 + 107520}{x^5} I_1(x)$$

$$\int x^2\,J_8(x)\,dx = \frac{33\,x^4 - 4224\,x^2 + 64512}{x^3} J_0(x) + \frac{x^6 - 576\,x^4 + 24576\,x^2 - 129024}{x^4} J_1(x) + 63\Lambda_0(x)$$

$$\int x^2\,I_8(x)\,dx = -\frac{33\,x^4 + 4224\,x^2 + 64512}{x^3} I_0(x) + \frac{x^6 + 576\,x^4 + 24576\,x^2 + 129024}{x^4} I_1(x) - 63\Lambda_0^*(x)$$

$$\int x^3\,J_8(x)\,dx = \frac{34\,x^4 - 4800\,x^2 + 80640}{x^2} J_0(x) + \frac{x^6 - 548\,x^4 + 28800\,x^2 - 161280}{x^3} J_1(x)$$

$$\int x^3\,I_8(x)\,dx = -\frac{34\,x^4 + 4800\,x^2 + 80640}{x^2} I_0(x) + \frac{x^6 + 548\,x^4 + 28800\,x^2 + 161280}{x^3} I_1(x)$$

$$\int x^4\,J_8(x)\,dx = \frac{35\,x^4 - 5385\,x^2 + 107520}{x} J_0(x) + \frac{x^6 - 585\,x^4 + 30720\,x^2 - 215040}{x^2} J_1(x) + 3465\Lambda_0(x)$$

$$\int x^4\,I_8(x)\,dx = -\frac{35\,x^4 + 5385\,x^2 + 107520}{x} I_0(x) + \frac{x^6 + 585\,x^4 + 30720\,x^2 + 215040}{x^2} I_1(x) + 3465\Lambda_0^*(x)$$

$$\int \frac{J_8(x)\,dx}{x} = \frac{30\,x^4 - 3120\,x^2 + 40320}{x^6} J_0(x) + \frac{x^6 - 420\,x^4 + 16320\,x^2 - 80640}{x^7} J_1(x)$$

$$\int \frac{I_8(x)\,dx}{x} = -\frac{30\,x^4 + 3120\,x^2 + 40320}{x^6} I_0(x) + \frac{x^6 + 420\,x^4 + 16320\,x^2 + 80640}{x^7} I_1(x)$$

$$\int \frac{J_8(x)\,dx}{x^2} = \frac{x^6 + 1824\,x^4 - 178560\,x^2 + 2257920}{63x^7} J_0(x) -$$

$$-\frac{x^8-64\,x^6+24768\,x^4-921600\,x^2+4515840}{63x^8}J_1(x)+\frac{1}{63}\Lambda_0(x)$$

$$\int\frac{I_8(x)\,dx}{x^2}=\frac{x^6-1824\,x^4-178560\,x^2-2257920}{63x^7}I_0(x)+$$

$$+\frac{x^8+64\,x^6+24768\,x^4+921600\,x^2+4515840}{63x^8}I_1(x)-\frac{1}{63}\Lambda_0^*(x)$$

$$\mathcal{P}_8^{(5)}(x)=36\,x^4-6048\,x^2+138240\ ,\quad \mathcal{Q}_8^{(5)}(x)=\frac{x^6-624\,x^4+40896\,x^2-322560}{x}\ ,$$

$$\mathcal{R}_8^{(5)}=0\ ,\quad \mathcal{S}_8^{(5)}=0$$

$$\mathcal{P}_8^{(5),*}(x)=-(36\,x^4+6048\,x^2+138240)\ ,\quad \mathcal{Q}_8^{(5),*}(x)=\frac{x^6+624\,x^4+40896\,x^2+322560}{x}\ ,$$

$$\mathcal{R}_8^{(5),*}=0\ ,\quad \mathcal{S}_8^{(5),*}=0$$

$$\mathcal{P}_8^{(6)}(x)=37\,x^5-6795\,x^3+187425\,x\ ,\quad \mathcal{Q}_8^{(6)}(x)=x^6-665\,x^4+49185\,x^2-645120\ ,$$

$$\mathcal{R}_8^{(6)}=135135\ ,\quad \mathcal{S}_8^{(6)}=0$$

$$\mathcal{P}_8^{(6),*}(x)=-(37\,x^5+6795\,x^3+187425\,x)\ ,\quad \mathcal{Q}_8^{(6),*}(x)=x^6+665\,x^4+49185\,x^2+645120\ ,$$

$$\mathcal{R}_8^{(6),*}=-135135\ ,\quad \mathcal{S}_8^{(6),*}=0$$

$$\mathcal{P}_8^{(7)}(x)=38\,x^6-7632\,x^4+256896\,x^2-645120\ ,\quad \mathcal{Q}_8^{(7)}(x)=x^7-708\,x^5+59328\,x^3-836352\,x\ ,$$

$$\mathcal{R}_8^{(7)}=0\ ,\quad \mathcal{S}_8^{(7)}=0$$

$$\mathcal{P}_8^{(7),*}(x)=-(38\,x^6+7632\,x^4+256896\,x^2+645120)\ ,\quad \mathcal{Q}_8^{(7),*}(x)=x^7+708\,x^5+59328\,x^3+836352\,x\ ,$$

$$\mathcal{R}_8^{(7),*}=0\ ,\quad \mathcal{S}_8^{(7),*}=0$$

$$\mathcal{P}_8^{(8)}(x)=39\,x^7-8565\,x^5+353115\,x^3-2027025\,x\ ,\quad \mathcal{Q}_8^{(8)}(x)=x^8-753\,x^6+71625\,x^4-1381905\,x^2\ ,$$

$$\mathcal{R}_8^{(8)}=2027025\ ,\quad \mathcal{S}_8^{(8)}=0$$

$$\mathcal{P}_8^{(8),*}(x)=-(39\,x^7+8565\,x^5+353115\,x^3+2027025\,x)\ ,\quad \mathcal{Q}_8^{(8),*}(x)=x^8+753\,x^6+71625\,x^4+1381905\,x^2\ ,$$

$$\mathcal{R}_8^{(8),*}=2027025\ ,\quad \mathcal{S}_8^{(8),*}=0$$

$$\mathcal{P}_8^{(9)}(x)=40\,x^8-9600\,x^6+483840\,x^4-5160960\,x^2\ ,$$

$$\mathcal{Q}_8^{(9)}(x)=x^9-800\,x^7+86400\,x^5-2257920\,x^3+10321920\,x\ ,\quad \mathcal{R}_8^{(9)}=0\ ,\quad \mathcal{S}_8^{(9)}=0$$

$$\mathcal{P}_8^{(9),*}(x)=-(40\,x^8+9600\,x^6+483840\,x^4+5160960\,x^2)\ ,$$

$$\mathcal{Q}_8^{(9),*}(x)=x^9+800\,x^7+86400\,x^5+2257920\,x^3+10321920\,x\ ,\quad \mathcal{R}_8^{(9),*}=0\ ,\quad \mathcal{S}_8^{(9),*}=0$$

$$\mathcal{P}_8^{(10)}(x)=41\,x^9-10743\,x^7+658245\,x^5-11486475\,x^3+34459425\,x\ ,$$

$$\mathcal{Q}_8^{(10)}(x)=x^{10}-849\,x^8+104001\,x^6-3613785\,x^4+34459425\,x^2\ ,\quad \mathcal{R}_8^{(10)}=-34459425\ ,\quad \mathcal{S}_8^{(10)}=0$$

$$\mathcal{P}_8^{(10),*}(x)=-(41\,x^9+10743\,x^7+658245\,x^5+11486475\,x^3+34459425\,x)\ ,$$

$$\mathcal{Q}_8^{(10),*}(x)=x^{10}+849\,x^8+104001\,x^6+3613785\,x^4+34459425\,x^2\ ,\quad \mathcal{R}_8^{(10),*}=34459425\ ,\quad \mathcal{S}_8^{(10),*}=0$$

$$\mathcal{P}_8^{(-3)}(x)=\frac{28\,x^4-2592\,x^2+32256}{x^8}\ ,\quad \mathcal{Q}_8^{(-3)}(x)=\frac{x^6-368\,x^4+13248\,x^2-64512}{x^9}\ ,$$

$$\mathcal{R}_8^{(-3)}=0\ ,\quad \mathcal{S}_8^{(-3)}=0$$

$$\mathcal{P}_8^{(-3),*}(x)=-\frac{28\,x^4+2592\,x^2+32256}{x^8}\ ,\quad \mathcal{Q}_8^{(-3),*}(x)=\frac{x^6+368\,x^4+13248\,x^2+64512}{x^9}\ ,$$

$$\mathcal{R}_8^{(-3),*}=0\ ,\quad \mathcal{S}_8^{(-3),*}=0$$

$$\mathcal{P}_8^{(-4)}(x)=\frac{x^8-3\,x^6+93600\,x^4-8265600\,x^2+101606400}{3465\,x^9}\ ,$$

$$\mathcal{Q}_8^{(-4)}(x) = -\frac{x^{10} - x^8 - 3456\,x^6 + 1195200\,x^4 - 41932800\,x^2 + 203212800}{3465\,x^{10}}\ ,$$

$$\mathcal{R}_8^{(-4)} = \frac{1}{3465}\ , \quad \mathcal{S}_8^{(-4)} = 0$$

$$\mathcal{P}_8^{(-4),*}(x) = -\frac{x^8 + 3\,x^6 + 93600\,x^4 + 8265600\,x^2 + 101606400}{3465\,x^9}\ ,$$

$$\mathcal{Q}_8^{(-4),*}(x) = -\frac{x^{10} + x^8 - 3456\,x^6 - 1195200\,x^4 - 41932800\,x^2 - 203212800}{3465\,x^{10}}\ ,$$

$$\mathcal{R}_8^{(-4),*} = \frac{1}{3465}\ , \quad \mathcal{S}_8^{(-4),*} = 0$$

$$\mathcal{P}_8^{(-5)}(x) = \frac{26\,x^4 - 2208\,x^2 + 26880}{x^{10}}\ , \quad \mathcal{Q}_8^{(-5)}(x) = \frac{x^6 - 324\,x^4 + 11136\,x^2 - 53760}{x^{11}}\ ,$$

$$\mathcal{R}_8^{(-5)} = 0\ , \quad \mathcal{S}_8^{(-5)} = 0$$

$$\mathcal{P}_8^{(-5),*}(x) = -\frac{26\,x^4 + 2208\,x^2 + 26880}{x^{10}}\ , \quad \mathcal{Q}_8^{(-5),*}(x) = \frac{x^6 + 324\,x^4 + 11136\,x^2 + 53760}{x^{11}}\ ,$$

$$\mathcal{R}_8^{(-5),*} = 0\ , \quad \mathcal{S}_8^{(-5),*} = 0$$

$$\mathcal{P}_8^{(-6)}(x) = \frac{x^{10} - 3\,x^8 + 45\,x^6 + 3376800\,x^4 - 277603200\,x^2 + 3353011200}{135135\,x^{11}}\ ,$$

$$\mathcal{Q}_8^{(-6)}(x) = -\frac{x^{12} - x^{10} + 9\,x^8 - 135360\,x^6 + 41227200\,x^4 - 1393459200\,x^2 + 6706022400}{135135\,x^{12}}\ ,$$

$$\mathcal{R}_8^{(-6)} = \frac{1}{135135}\ , \quad \mathcal{S}_8^{(-6)} = 0$$

$$\mathcal{P}_8^{(-6),*}(x) = \frac{x^{10} + 3\,x^8 + 45\,x^6 - 3376800\,x^4 - 277603200\,x^2 - 3353011200}{135135\,x^{11}}\ ,$$

$$\mathcal{Q}_8^{(-6),*}(x) = \frac{x^{12} + x^{10} + 9\,x^8 + 135360\,x^6 + 41227200\,x^4 + 1393459200\,x^2 + 6706022400}{135135\,x^{12}}\ ,$$

$$\mathcal{R}_8^{(-6),*} = -\frac{1}{135135}\ , \quad \mathcal{S}_8^{(-6),*} = 0$$

$\mathbf{Z}_9(\mathbf{x})$:

$$\int J_9(x)\,dx = -\frac{x^6 - 720\,x^4 + 53760\,x^2 - 645120}{x^6} J_0(x) + \frac{40\,x^6 - 8160\,x^4 + 268800\,x^2 - 1290240}{x^7} J_1(x)$$

$$\int I_9(x)\,dx = \frac{x^6 + 720\,x^4 + 53760\,x^2 + 645120}{x^6} I_0(x) - \frac{40\,x^6 + 8160\,x^4 + 268800\,x^2 + 1290240}{x^7} I_1(x)$$

$$\int x\,J_9(x)\,dx =$$

$$= -\frac{x^6 - 768\,x^4 + 59904\,x^2 - 737280}{x^5} J_0(x) + \frac{32\,x^6 - 8832\,x^4 + 304128\,x^2 - 1474560}{x^6} J_1(x) + 9\Lambda_0(x)$$

$$\int x\,I_9(x)\,dx =$$

$$= \frac{x^6 + 768\,x^4 + 59904\,x^2 + 737280}{x^5} I_0(x) - \frac{32\,x^6 + 8832\,x^4 + 304128\,x^2 + 1474560}{x^6} I_1(x) - 9\Lambda_0^*(x)$$

$$\int x^2\,J_9(x)\,dx = -\frac{x^6 - 800\,x^4 + 67200\,x^2 - 860160}{x^4} J_0(x) + \frac{42\,x^6 - 9600\,x^4 + 349440\,x^2 - 1720320}{x^5} J_1(x)$$

$$\int x^2\,I_9(x)\,dx = \frac{x^6 + 800\,x^4 + 67200\,x^2 + 860160}{x^4} I_0(x) - \frac{42\,x^6 + 9600\,x^4 + 349440\,x^2 + 1720320}{x^5} I_1(x)$$

$$\int x^3\,J_9(x)\,dx = -\frac{x^6 - 843\,x^4 + 75264\,x^2 - 1032192}{x^3} J_0(x) +$$

$$+\frac{43\,x^6-11136\,x^4+408576\,x^2-2064384}{x^4}J_1(x)+693\Lambda_0(x)$$

$$\int x^3\,I_9(x)\,dx=\frac{x^6+843\,x^4+75264\,x^2+1032192}{x^3}I_0(x)-$$

$$-\frac{43\,x^6+11136\,x^4+408576\,x^2+2064384}{x^4}I_1(x)+693\Lambda_0^*(x)$$

$$\int x^4\,J_9(x)\,dx=-\frac{x^6-888\,x^4+86400\,x^2-1290240}{x^2}J_0(x)+\frac{44\,x^6-11376\,x^4+483840\,x^2-2580480}{x^3}J_1(x)$$

$$\int x^4\,I_9(x)\,dx=\frac{x^6+888\,x^4+86400\,x^2+1290240}{x^2}I_0(x)-\frac{44\,x^6+11376\,x^4+483840\,x^2+2580480}{x^3}I_1(x)$$

$$\int\frac{J_9(x)\,dx}{x}=-\frac{8\,x^6-6144\,x^4+437760\,x^2-5160960}{9x^7}J_0(x)-$$

$$-\frac{x^8-352\,x^6+67968\,x^4-2165760\,x^2+10321920}{9x^8}J_1(x)+\frac{1}{9}\Lambda_0(x)$$

$$\int\frac{I_9(x)\,dx}{x}=\frac{8\,x^6+6144\,x^4+437760\,x^2+5160960}{9x^7}I_0(x)-$$

$$-\frac{x^8+352\,x^6+67968\,x^4+2165760\,x^2+10321920}{9x^8}I_1(x)+\frac{1}{9}\Lambda_0^*(x)$$

$$\int\frac{J_9(x)\,dx}{x^2}=-\frac{x^6-648\,x^4+44352\,x^2-516096}{x^8}J_0(x)+\frac{38\,x^6-7008\,x^4+217728\,x^2-1032192}{x^9}J_1(x)$$

$$\int\frac{I_9(x)\,dx}{x^2}=\frac{x^6+648\,x^4+44352\,x^2+516096}{x^8}I_0(x)-\frac{38\,x^6+7008\,x^4+217728\,x^2+1032192}{x^9}I_1(x)$$

$$\mathcal{P}_9^{(5)}(x)=-\frac{x^6-935\,x^4+98805\,x^2-1720320}{x}\;,\quad \mathcal{Q}_9^{(5)}(x)=\frac{45\,x^6-12405\,x^4+537600\,x^2-3440640}{x^2}\;,$$

$$\mathcal{R}_9^{(5)}=45045\;,\quad \mathcal{S}_9^{(5)}=0$$

$$\mathcal{P}_9^{(5),*}(x)=\frac{x^6+935\,x^4+98805\,x^2+1720320}{x}\;,\quad \mathcal{Q}_9^{(5),*}(x)=-\frac{45\,x^6+12405\,x^4+537600\,x^2+3440640}{x^2}\;,$$

$$\mathcal{R}_9^{(5),*}=-45045\;,\quad \mathcal{S}_9^{(5),*}=0$$

$$\mathcal{P}_9^{(6)}(x)=-(x^6-984\,x^4+113472\,x^2-2257920)\;,\quad \mathcal{Q}_9^{(6)}(x)=\frac{46\,x^6-13536\,x^4+710784\,x^2-5160960}{x}\;,$$

$$\mathcal{R}_9^{(6)}=0\;,\quad \mathcal{S}_9^{(6)}=0$$

$$\mathcal{P}_9^{(6),*}(x)=x^6+984\,x^4+113472\,x^2+2257920\;,\quad \mathcal{Q}_9^{(6),*}(x)=-\frac{46\,x^6+13536\,x^4+710784\,x^2+5160960}{x}\;,$$

$$\mathcal{R}_9^{(6),*}=0\;,\quad \mathcal{S}_9^{(6),*}=0$$

$$\mathcal{P}_9^{(7)}(x)=-(x^7-1035\,x^5+130725\,x^3-3133935\,x)\;,\quad \mathcal{Q}_9^{(7)}(x)=47\,x^6-14775\,x^4+876015\,x^2-10321920\;,$$

$$\mathcal{R}_9^{(7)}=2027025\;,\quad \mathcal{S}_9^{(7)}=0$$

$$\mathcal{P}_9^{(7),*}(x)=x^7+1035\,x^5+130725\,x^3+3133935\,x\;,\quad \mathcal{Q}_9^{(7),*}(x)=-(47\,x^6+14775\,x^4+876015\,x^2+10321920)\;,$$

$$\mathcal{R}_9^{(7),*}=2027025\;,\quad \mathcal{S}_9^{(7),*}=0$$

$$\mathcal{P}_9^{(8)}(x)=-(x^8-1088\,x^6+150912\,x^4-4432896\,x^2+10321920)\;,$$

$$\mathcal{Q}_9^{(8)}(x)=48\,x^7-16128\,x^5+1087488\,x^3-14026752\,x\;,\quad \mathcal{R}_9^{(8)}=0\;,\quad \mathcal{S}_9^{(8)}=0$$

$$\mathcal{P}_9^{(8),*}(x)=x^8+1088\,x^6+150912\,x^4+4432896\,x^2+10321920\;,$$

$$\mathcal{Q}_9^{(8),*}(x)=-(48\,x^7+16128\,x^5+1087488\,x^3+14026752\,x)\;,\quad \mathcal{R}_9^{(8),*}=0\;,\quad \mathcal{S}_9^{(8),*}=0$$

$$\mathcal{P}_9^{(9)}(x)=-(x^9-1143\,x^7+174405\,x^5-6325515\,x^3+34459425\,x)\;,$$

$$\mathcal{Q}_9^{(9)}(x) = 49\,x^8 - 17601\,x^6 + 1355865\,x^4 - 24137505\,x^2\ , \quad \mathcal{R}_9^{(9)} = 34459425\ , \quad \mathcal{S}_9^{(9)} = 0$$

$$\mathcal{P}_9^{(9),*}(x) = x^9 + 1143\,x^7 + 174405\,x^5 + 6325515\,x^3 + 34459425\,x\ ,$$

$$\mathcal{Q}_9^{(9),*}(x) = -(49\,x^8 + 17601\,x^6 + 1355865\,x^4 + 24137505\,x^2)\ ,$$

$$\mathcal{R}_9^{(9),*} = -34459425\ , \quad \mathcal{S}_9^{(9),*} = 0$$

$$\mathcal{P}_9^{(10)}(x) = -(x^{10} - 1200\,x^8 + 201600\,x^6 - 9031680\,x^4 + 92897280\,x^2)\ ,$$

$$\mathcal{Q}_9^{(10)}(x) = 50\,x^9 - 19200\,x^7 + 1693440\,x^5 - 41287680\,x^3 + 185794560\,x\ ,$$

$$\mathcal{R}_9^{(10)} = 0\ , \quad \mathcal{S}_9^{(10)} = 0$$

$$\mathcal{P}_9^{(10),*}(x) = x^{10} + 1200\,x^8 + 201600\,x^6 + 9031680\,x^4 + 92897280\,x^2\ ,$$

$$\mathcal{Q}_9^{(10),*}(x) = -(50\,x^9 + 19200\,x^7 + 1693440\,x^5 + 41287680\,x^3 + 185794560\,x)\ ,$$

$$\mathcal{R}_9^{(10),*} = 0\ , \quad \mathcal{S}_9^{(10),*} = 0$$

$$\mathcal{P}_9^{(-3)}(x) = \frac{x^8 - 696\,x^6 + 426240\,x^4 - 28224000\,x^2 + 325140480}{693\,x^9}\ ,$$

$$\mathcal{Q}_9^{(-3)}(x) = -\frac{x^{10} - x^8 - 25632\,x^6 + 4521600\,x^4 - 137733120\,x^2 + 650280960}{693\,x^{10}}\ ,$$

$$\mathcal{R}_9^{(-3)} = \frac{1}{693}\ , \quad \mathcal{S}_9^{(-3)} = 0$$

$$\mathcal{P}_9^{(-3),*}(x) = \frac{x^8 + 696\,x^6 + 426240\,x^4 + 28224000\,x^2 + 325140480}{693\,x^9}\ ,$$

$$\mathcal{Q}_9^{(-3),*}(x) = \frac{x^{10} + x^8 - 25632\,x^6 - 4521600\,x^4 - 137733120\,x^2 - 650280960}{693\,x^{10}}\ ,$$

$$\mathcal{R}_9^{(-3),*} = -\frac{1}{693}\ , \quad \mathcal{S}_9^{(-3),*} = 0$$

$$\mathcal{P}_9^{(-4)}(x) = -\frac{x^6 - 584\,x^4 + 37632\,x^2 - 430080}{x^{10}}\ , \quad \mathcal{Q}_9^{(-4)}(x) = \frac{36\,x^6 - 6096\,x^4 + 182784\,x^2 - 860160}{x^{11}}\ ,$$

$$\mathcal{R}_9^{(-4)} = 0\ , \quad \mathcal{S}_9^{(-4)} = 0$$

$$\mathcal{P}_9^{(-4),*}(x) = \frac{x^6 + 584\,x^4 + 37632\,x^2 + 430080}{x^{10}}\ , \quad \mathcal{Q}_9^{(-4),*}(x) = -\frac{36\,x^6 + 6096\,x^4 + 182784\,x^2 + 860160}{x^{11}}\ ,$$

$$\mathcal{R}_9^{(-4),*} = 0\ , \quad \mathcal{S}_9^{(-4),*} = 0$$

$$\mathcal{P}_9^{(-5)}(x) = \frac{x^{10} - 3\,x^8 - 45000\,x^6 + 24998400\,x^4 - 1574899200\,x^2 + 17882726400}{45045\,x^{11}}\ ,$$

$$\mathcal{Q}_9^{(-5)}(x) = -\frac{x^{12} - x^{10} + 9\,x^8 - 1576800\,x^6 + 257443200\,x^4 - 7620480000\,x^2 + 35765452800}{45045\,x^{12}}\ ,$$

$$\mathcal{R}_9^{(-5)} = \frac{1}{45045}\ , \quad \mathcal{S}_9^{(-5)} = 0$$

$$\mathcal{P}_9^{(-5),*}(x) = -\frac{x^{10} + 3\,x^8 - 45000\,x^6 - 24998400\,x^4 - 1574899200\,x^2 - 17882726400}{45045\,x^{11}}\ ,$$

$$\mathcal{Q}_9^{(-5),*}(x) = -\frac{x^{12} + x^{10} + 9\,x^8 + 1576800\,x^6 + 257443200\,x^4 + 7620480000\,x^2 + 35765452800}{45045\,x^{12}}\ ,$$

$$\mathcal{R}_9^{(-5),*} = \frac{1}{45045}\ , \quad \mathcal{S}_9^{(-5),*} = 0$$

$$\mathcal{P}_9^{(-6)}(x) = -\frac{x^6 - 528\,x^4 + 32640\,x^2 - 368640}{x^{12}}\ ,$$

$$\mathcal{Q}_9^{(-6)}(x) = \frac{34\,x^6 - 5376\,x^4 + 157440\,x^2 - 737280}{x^{13}}\ , \quad \mathcal{R}_9^{(-6)} = 0\ , \quad \mathcal{S}_9^{(-6)} = 0$$

$$\mathcal{P}_9^{(-6),*}(x) = \frac{x^6 + 528\,x^4 + 32640\,x^2 + 368640}{x^{12}}\ ,$$

$$\mathcal{Q}_9^{(-6),*}(x) = -\frac{34\,x^6 + 5376\,x^4 + 157440\,x^2 + 737280}{x^{13}}\ , \quad \mathcal{R}_9^{(-6),*} = 0\ , \quad \mathcal{S}_9^{(-6),*} = 0$$

$\mathbf{Z_{10}(x)}$:

$$\int J_{10}(x)\,dx = -\frac{48\,x^6 - 15744\,x^4 + 921600\,x^2 - 10321920}{x^7}J_0(x) -$$
$$-\frac{2\,x^8 - 1152\,x^6 + 154368\,x^4 - 4423680\,x^2 + 20643840}{x^8}J_1(x) + \Lambda_0(x)$$

$$\int I_{10}(x)\,dx = -\frac{48\,x^6 + 15744\,x^4 + 921600\,x^2 + 10321920}{x^7}I_0(x) +$$
$$+\frac{2\,x^8 + 1152\,x^6 + 154368\,x^4 + 4423680\,x^2 + 20643840}{x^8}I_1(x) - \Lambda_0^*(x)$$

$$\int x\,J_{10}(x)\,dx = -\frac{50\,x^6 - 16800\,x^4 + 1021440\,x^2 - 11612160}{x^6}J_0(x) -$$
$$-\frac{x^8 - 1200\,x^6 + 168000\,x^4 - 4945920\,x^2 + 23224320}{x^7}J_1(x)$$

$$\int x\,I_{10}(x)\,dx = -\frac{50\,x^6 + 16800\,x^4 + 1021440\,x^2 + 11612160}{x^6}I_0(x) +$$
$$+\frac{x^8 + 1200\,x^6 + 168000\,x^4 + 4945920\,x^2 + 23224320}{x^7}I_1(x)$$

$$\int x^2\,J_{10}(x)\,dx = -\frac{51\,x^6 - 18048\,x^4 + 1142784\,x^2 - 13271040}{x^5}J_0(x)$$
$$-\frac{x^8 - 1152\,x^6 + 183552\,x^4 - 5603328\,x^2 + 26542080}{x^6}J_1(x) + 99\Lambda_0(x)$$

$$\int x^2\,I_{10}(x)\,dx = -\frac{51\,x^6 + 18048\,x^4 + 1142784\,x^2 + 13271040}{x^5}I_0(x) +$$
$$+\frac{x^8 + 1152\,x^6 + 183552\,x^4 + 5603328\,x^2 + 26542080}{x^6}I_1(x) + 99\Lambda_0^*(x)$$

$$\int x^3\,J_{10}(x)\,dx = -\frac{52\,x^6 - 19200\,x^4 + 1290240\,x^2 - 15482880}{x^4}J_0(x) -$$
$$-\frac{x^8 - 1304\,x^6 + 201600\,x^4 - 6451200\,x^2 + 30965760}{x^5}J_1(x)$$

$$\int x^3\,I_{10}(x)\,dx = -\frac{52\,x^6 + 19200\,x^4 + 1290240\,x^2 + 15482880}{x^4}I_0(x) +$$
$$+\frac{x^8 + 1304\,x^6 + 201600\,x^4 + 6451200\,x^2 + 30965760}{x^5}I_1(x)$$

$$\int x^4\,J_{10}(x)\,dx = -\frac{53\,x^6 - 20559\,x^4 + 1462272\,x^2 - 18579456}{x^3}J_0(x) -$$
$$-\frac{x^8 - 1359\,x^6 + 231168\,x^4 - 7569408\,x^2 + 37158912}{x^4}J_1(x) + 9009\Lambda_0(x)$$

$$\int x^4\,I_{10}(x)\,dx = -\frac{53\,x^6 + 20559\,x^4 + 1462272\,x^2 + 18579456}{x^3}I_0(x) +$$
$$+\frac{x^8 + 1359\,x^6 + 231168\,x^4 + 7569408\,x^2 + 37158912}{x^4}I_1(x) - 9009\Lambda_0^*(x)$$

$$\int \frac{J_{10}(x)\,dx}{x} = -\frac{48\,x^6 - 14784\,x^4 + 838656\,x^2 - 9289728}{x^8}J_0(x) -$$

$$-\frac{x^8 - 1104\,x^6 + 142464\,x^4 - 3999744\,x^2 + 18579456}{x^9} J_1(x)$$

$$\int \frac{I_{10}(x)\,dx}{x} = -\frac{48\,x^6 + 14784\,x^4 + 838656\,x^2 + 9289728}{x^8} I_0(x) +$$

$$+\frac{x^8 + 1104\,x^6 + 142464\,x^4 + 3999744\,x^2 + 18579456}{x^9} I_1(x)$$

$$\int \frac{J_{10}(x)\,dx}{x^2} = \frac{x^8 - 4656\,x^6 + 1376640\,x^4 - 76124160\,x^2 + 836075520}{99x^9} J_0(x) -$$

$$-\frac{x^{10} + 98\,x^8 - 104832\,x^6 + 13075200\,x^4 - 361267200\,x^2 + 1672151040}{99x^{10}} J_1(x) + \frac{1}{99}\Lambda_0(x)$$

$$\int \frac{I_{10}(x)\,dx}{x^2} = -\frac{x^8 + 4656\,x^6 + 1376640\,x^4 + 76124160\,x^2 + 836075520}{99x^9} I_0(x) -$$

$$-\frac{x^{10} - 98\,x^8 - 104832\,x^6 - 13075200\,x^4 - 361267200\,x^2 - 1672151040}{99x^{10}} I_1(x) + \frac{1}{99}\Lambda_0^*(x)$$

$$\mathcal{P}_{10}^{(5)}(x) = -\frac{54\,x^6 - 22032\,x^4 + 1693440\,x^2 - 23224320}{x^2}\ ,$$

$$\mathcal{Q}_{10}^{(5)}(x) = -\frac{x^8 - 1416\,x^6 + 245664\,x^4 - 9031680\,x^2 + 46448640}{x^3}\ , \quad \mathcal{R}_{10}^{(5)} = 0\ , \quad \mathcal{S}_{10}^{(5)} = 0$$

$$\mathcal{P}_{10}^{(5),*}(x) = -\frac{54\,x^6 + 22032\,x^4 + 1693440\,x^2 + 23224320}{x^2}\ ,$$

$$\mathcal{Q}_{10}^{(5),*}(x) = \frac{x^8 + 1416\,x^6 + 245664\,x^4 + 9031680\,x^2 + 46448640}{x^3}\ , \quad \mathcal{R}_{10}^{(5),*} = 0\ , \quad \mathcal{S}_{10}^{(5),*} = 0$$

$$\mathcal{P}_{10}^{(6)}(x) = -\frac{55\,x^6 - 23625\,x^4 + 1965915\,x^2 - 30965760}{x}\ ,$$

$$\mathcal{Q}_{10}^{(6)}(x) = -\frac{x^8 - 1475\,x^6 + 272475\,x^4 - 10321920\,x^2 + 61931520}{x^2}\ ,$$

$$\mathcal{R}_{10}^{(6)} = 675675\ , \quad \mathcal{S}_{10}^{(6)} = 0$$

$$\mathcal{P}_{10}^{(6),*}(x) = -\frac{55\,x^6 + 23625\,x^4 + 1965915\,x^2 + 30965760}{x}\ ,$$

$$\mathcal{Q}_{10}^{(6),*}(x) = \frac{x^8 + 1475\,x^6 + 272475\,x^4 + 10321920\,x^2 + 61931520}{x^2}\ ,$$

$$\mathcal{R}_{10}^{(6),*} = 675675\ , \quad \mathcal{S}_{10}^{(6),*} = 0$$

$$\mathcal{P}_{10}^{(7)}(x) = -(56\,x^6 - 25344\,x^4 + 2299392\,x^2 - 41287680)\ ,$$

$$\mathcal{Q}_{10}^{(7)}(x) = -\frac{x^8 - 1536\,x^6 + 302976\,x^4 - 13630464\,x^2 + 92897280}{x}\ , \quad \mathcal{R}_{10}^{(7)} = 0\ , \quad \mathcal{S}_{10}^{(7)} = 0$$

$$\mathcal{P}_{10}^{(7),*}(x) = -(56\,x^6 + 25344\,x^4 + 2299392\,x^2 + 41287680)\ ,$$

$$\mathcal{Q}_{10}^{(7),*}(x) = \frac{x^8 + 1536\,x^6 + 302976\,x^4 + 13630464\,x^2 + 92897280}{x}\ , \quad \mathcal{R}_{10}^{(7),*} = 0\ , \quad \mathcal{S}_{10}^{(7),*} = 0$$

$$\mathcal{P}_{10}^{(8)}(x) = -(57\,x^7 - 27195\,x^5 + 2706165\,x^3 - 58437855\,x)\ ,$$

$$\mathcal{Q}_{10}^{(8)}(x) = -(x^8 - 1599\,x^6 + 337575\,x^4 - 17150175\,x^2 + 185794560)\ ,$$

$$\mathcal{R}_{10}^{(8)} = 34459425\ , \quad \mathcal{S}_{10}^{(8)} = 0$$

$$\mathcal{P}_{10}^{(8),*}(x) = -(57\,x^7 + 27195\,x^5 + 2706165\,x^3 + 58437855\,x)\ ,$$

$$\mathcal{Q}_{10}^{(8),*}(x) = x^8 + 1599\,x^6 + 337575\,x^4 + 17150175\,x^2 + 185794560\ ,$$

$$\mathcal{R}_{10}^{(8),*} = -34459425\ , \quad \mathcal{S}_{10}^{(8),*} = 0$$

$$\mathcal{P}_{10}^{(9)}(x) = -(58\,x^8 - 29184\,x^6 + 3200256\,x^4 - 84953088\,x^2 + 185794560)\ ,$$

$$\mathcal{Q}_{10}^{(9)}(x) = -(x^9 - 1664\,x^7 + 376704\,x^5 - 21832704\,x^3 + 262803456\,x)\ , \quad \mathcal{R}_{10}^{(9)} = 0\ , \quad \mathcal{S}_{10}^{(9)} = 0$$

$$\mathcal{P}_{10}^{(9),*}(x) = -(58\,x^8 + 29184\,x^6 + 3200256\,x^4 + 84953088\,x^2 + 185794560)\ ,$$

$$\mathcal{Q}_{10}^{(9),*}(x) = x^9 + 1664\,x^7 + 376704\,x^5 + 21832704\,x^3 + 262803456\,x\ , \quad \mathcal{R}_{10}^{(9),*} = 0\ , \quad \mathcal{S}_{10}^{(9),*} = 0$$

$$\mathcal{P}_{10}^{(10)}(x) = -(59\,x^9 - 31317\,x^7 + 3797535\,x^5 - 125345745\,x^3 + 654729075\,x)\ ,$$

$$\mathcal{Q}_{10}^{(10)}(x) = -(x^{10} - 1731\,x^8 + 420819\,x^6 - 28019355\,x^4 + 468934515\,x^2)\ ,$$

$$\mathcal{R}_{10}^{(10)} = 654729075\ , \quad \mathcal{S}_{10}^{(10)} = 0$$

$$\mathcal{P}_{10}^{(10),*}(x) = -(59\,x^9 + 31317\,x^7 + 3797535\,x^5 + 125345745\,x^3 + 654729075\,x)\ ,$$

$$\mathcal{Q}_{10}^{(10),*}(x) = x^{10} + 1731\,x^8 + 420819\,x^6 + 28019355\,x^4 + 468934515\,x^2\ ,$$

$$\mathcal{R}_{10}^{(10),*} = 654729075\ , \quad \mathcal{S}_{10}^{(10),*} = 0$$

$$\mathcal{P}_{10}^{(-3)}(x) = -\frac{46\,x^6 - 13104\,x^4 + 709632\,x^2 - 7741440}{x^{10}}\ ,$$

$$\mathcal{Q}_{10}^{(-3)}(x) = -\frac{x^8 - 1016\,x^6 + 122976\,x^4 - 3354624\,x^2 + 15482880}{x^{11}}\ , \quad \mathcal{R}_{10}^{(-3)} = 0\ , \quad \mathcal{S}_{10}^{(-3)} = 0$$

$$\mathcal{P}_{10}^{(-3),*}(x) = -\frac{46\,x^6 + 13104\,x^4 + 709632\,x^2 + 7741440}{x^{10}}\ ,$$

$$\mathcal{Q}_{10}^{(-3),*}(x) = \frac{x^8 + 1016\,x^6 + 122976\,x^4 + 3354624\,x^2 + 15482880}{x^{11}}\ , \quad \mathcal{R}_{10}^{(-3),*} = 0\ , \quad \mathcal{S}_{10}^{(-3),*} = 0$$

$$\mathcal{P}_{10}^{(-4)}(x) = \frac{x^{10} - 3\,x^8 - 405360\,x^6 + 111484800\,x^4 - 5933813760\,x^2 + 64377815040}{9009\,x^{11}}\ ,$$

$$\mathcal{Q}_{10}^{(-4)}(x) = -\frac{x^{12} - x^{10} + 9018\,x^8 - 8784000\,x^6 + 1035820800\,x^4 - 27962081280\,x^2 + 128755630080}{9009\,x^{12}}\ ,$$

$$\mathcal{R}_{10}^{(-4)} = \frac{1}{9009}\ , \quad \mathcal{S}_{10}^{(-4)} = 0$$

$$\mathcal{P}_{10}^{(-4),*}(x) = \frac{x^{10} + 3\,x^8 - 405360\,x^6 - 111484800\,x^4 - 5933813760\,x^2 - 64377815040}{9009\,x^{11}}\ ,$$

$$\mathcal{Q}_{10}^{(-4),*}(x) = \frac{x^{12} + x^{10} + 9018\,x^8 + 8784000\,x^6 + 1035820800\,x^4 + 27962081280\,x^2 + 128755630080}{9009\,x^{12}}\ ,$$

$$\mathcal{R}_{10}^{(-4),*} = -\frac{1}{9009}\ , \quad \mathcal{S}_{10}^{(-4),*} = 0$$

$$\mathcal{P}_{10}^{(-5)}(x) = -\frac{44\,x^6 - 11712\,x^4 + 614400\,x^2 - 6635520}{x^{12}}\ ,$$

$$\mathcal{Q}_{10}^{(-5)}(x) = -\frac{x^8 - 936\,x^6 + 107904\,x^4 - 2887680\,x^2 + 13271040}{x^{13}}\ ,$$

$$\mathcal{R}_{10}^{(-5)} = 0\ , \quad \mathcal{S}_{10}^{(-5)} = 0$$

$$\mathcal{P}_{10}^{(-5),*}(x) = -\frac{44\,x^6 + 11712\,x^4 + 614400\,x^2 + 6635520}{x^{12}}\ ,$$

$$\mathcal{Q}_{10}^{(-5),*}(x) = \frac{x^8 + 936\,x^6 + 107904\,x^4 + 2887680\,x^2 + 13271040}{x^{13}}\ ,$$

$$\mathcal{R}_{10}^{(-5),*} = 0\ , \quad \mathcal{S}_{10}^{(-5),*} = 0$$

$$\mathcal{P}_{10}^{(-6)}(x) = \frac{x^{12} - 3\,x^{10} + 45\,x^8 - 29055600\,x^6 + 7506172800\,x^4 - 388949299200\,x^2 + 4184557977600}{675675\,x^{13}}\ ,$$

$$\mathcal{Q}_{10}^{(-6)}(x) =$$

$$-\frac{x^{14}-x^{12}+9\,x^{10}+675450\,x^{8}-607420800\,x^{6}+68660524800\,x^{4}-1824038092800\,x^{2}+8369115955200}{675675\,x^{14}}\,,$$

$$\mathcal{R}_{10}^{(-6)}=\frac{1}{675675}\,,\quad \mathcal{S}_{10}^{(-6)}=0$$

$$\mathcal{P}_{10}^{(-6),*}(x)=-\frac{x^{12}+3\,x^{10}+45\,x^{8}+29055600\,x^{6}+7506172800\,x^{4}+388949299200\,x^{2}+4184557977600}{675675\,x^{13}}\,,$$

$$\mathcal{Q}_{10}^{(-6),*}(x)=$$

$$-\frac{x^{14}+x^{12}+9\,x^{10}-675450\,x^{8}-607420800\,x^{6}-68660524800\,x^{4}-1824038092800\,x^{2}-8369115955200}{675675\,x^{14}}\,,$$

$$\mathcal{R}_{10}^{(-6),*}=\frac{1}{675675}\,,\quad \mathcal{S}_{10}^{(-6),*}=0$$

1.1.8. Second Antiderivatives of $x^n\,Z_\nu(x)$:

$\Phi(x)$, $\Psi(x)$ and $\Psi_K(x)$ are the same as in I., page 9.

a) $x^{2n+1}\,Z_0(x)$:

With the functions $\Phi(x)$, $\Psi(x)$ and $\Psi_K(x)$ as defined on page ??? holds:

$$x\,J_0(x)=\frac{d^2\,\Phi(x)}{dx^2}\;,\quad x\,I_0(x)=-\frac{d^2\,\Psi(x)}{dx^2}\;,\quad x\,K_0(x)=-\frac{d^2\,\Psi_K(x)}{dx^2}$$

$$x^3\,J_0(x)=\frac{d^2}{dx^2}\left[-x^3\,J_0(x)+5x^2\,J_1(x)-9\,\Phi(x)\right]$$

$$x^3\,I_0(x)=\frac{d^2}{dx^2}\left[x^3\,I_0(x)-5x^2\,I_1(x)+9\,\Psi(x)\right]$$

$$x^3\,K_0(x)=\frac{d^2}{dx^2}\left[x^3\,K_0(x)+5x^2\,K_1(x)-9\,\Psi_K(x)\right]$$

$$x^5\,J_0(x)=\frac{d^2}{dx^2}\left[(-x^5+43x^3)\,J_0(x)+(9x^4-161x^2)\,J_1(x)+225\Phi(x)\right]$$

$$x^5\,I_0(x)=\frac{d^2}{dx^2}\left[(x^5+43\,x^3)\,I_0(x)-(9\,x^4+161\,x^2)\,I_1(x)+225\,\Psi(x)\right]$$

$$x^5\,K_0(x)=\frac{d^2}{dx^2}\left[(x^5+43\,x^3)\,K_0(x)+(9\,x^4+161\,x^2)\,K_1(x)-225\,\Psi_K(x)\right]$$

$$x^7\,J_0(x)=\frac{d^2}{dx^2}\left[(-x^7+101\,x^5-2523\,x^3)\,J_0(x)+(13\,x^6-649\,x^4+8721\,x^2)\,J_1(x)-11025\,\Phi(x)\right]$$

$$x^7\,I_0(x)=\frac{d^2}{dx^2}\left[(x^7+101\,x^5+2523\,x^3)\,I_0(x)-(13\,x^6+649\,x^4+8721\,x^2)\,I_1(x)+11025\,\Psi(x)\right]$$

$$x^7\,K_0(x)=\frac{d^2}{dx^2}\left[(x^7+101\,x^5+2523\,x^3)\,K_0(x)+(13\,x^6+649\,x^4+8721\,x^2)\,K_1(x)-11025\,\Psi_K(x)\right]$$

The formulas for $I_1(x)$ and $K_1(x)$ vary in two signs.

$$x^9\,J_0(x)=\frac{d^2}{dx^2}\left[(-x^9+183\,x^7-10629\,x^5+223947\,x^3)\,J_0(x)+\right.$$
$$\left.+(17\,x^8-1665\,x^6+62361\,x^4-745569\,x^2)\,J_1(x)+893025\,\Phi(x)\right]$$

$$x^9\,I_0(x)=\frac{d^2}{dx^2}\left[(x^9+183\,x^7+10629\,x^5+223947\,x^3)\,I_0(x)-\right.$$
$$\left.-(17\,x^8+1665\,x^6+62361\,x^4+745569\,x^2)\,I_1(x)+893025\,\Psi(x)\right]$$

$$x^{11}\,J_0(x)=\frac{d^2}{dx^2}\left[(-x^{11}+289\,x^9-30207\,x^7+1479645\,x^5-28645875\,x^3)\,J_0(x)+\right.$$
$$\left.+(21\,x^{10}-3401\,x^8+249849\,x^6-8319825\,x^4+93310425\,x^2)\,J_1(x)-108056025\,\Phi(x)\right]$$

$$x^{11}\,I_0(x)=\frac{d^2}{dx^2}\left[(x^{11}+289\,x^9+30207\,x^7+1479645\,x^5+28645875\,x^3)\,I_0(x)-\right.$$
$$\left.-(21\,x^{10}+3401\,x^8+249849\,x^6+8319825\,x^4+93310425\,x^2)\,I_1(x)+108056025\,\Psi(x)\right]$$

$$x^{13}\,J_0(x)=\frac{d^2}{dx^2}\left[(-x^{13}+419\,x^{11}-68841\,x^9+6064983\,x^7-273100005\,x^5+5025472875\,x^3)\,J_0(x)+\right.$$
$$\left.+(25\,x^{12}-6049\,x^{10}+734769\,x^8-47984481\,x^6+1498210425\,x^4-16138101825\,x^2)\,J_1(x)+18261468225\,\Phi(x)\right]$$

$$x^{13}\,I_0(x)=\frac{d^2}{dx^2}\left[(x^{13}+419\,x^{11}+68841\,x^9+6064983\,x^7+273100005\,x^5+5025472875\,x^3)\,I_0(x)-\right.$$
$$\left.-(25\,x^{12}+6049\,x^{10}+734769\,x^8+47984481\,x^6+1498210425\,x^4+16138101825\,x^2)\,I_1(x)+18261468225\,\Psi(x)\right]$$

$$x^{15}\,J_0(x)=\frac{d^2}{dx^2}\left[(-x^{15}+573\,x^{13}-136035\,x^{11}+18830025\,x^9-1524979575\,x^7+65296102725\,x^5-\right.$$

$$-1161520209675\,x^3)\,J_0(x)+(29\,x^{14}-9801\,x^{12}+1778625\,x^{10}-192049425\,x^8+11758658625\,x^6-$$
$$-352491752025\,x^4+3692650536225\,x^2)\,J_1(x)-4108830350625\,\Phi(x)\big]$$

$$x^{15}\,I_0(x)=\frac{d^2}{dx^2}\,\big[(x^{15}+573\,x^{13}+136035\,x^{11}+18830025\,x^9+1524979575\,x^7+65296102725\,x^5+$$
$$+1161520209675\,x^3\,I_0(x)-(29\,x^{14}+9801\,x^{12}+1778625\,x^{10}+192049425\,x^8+11758658625\,x^6+$$
$$+352491752025\,x^4+3692650536225\,x^2\,I_1(x)+4108830350625\,\Psi(x)\big]$$

Recurrence relations:

$$x^{2n+3}\,J_0(x)=\frac{d^2}{dx^2}\,\left\{x^{2n}\,\left[\frac{2n(2n+1)(4n+5)}{4n+1}\,x-x^3\right]\,J_0(x)+(4n+5)x^{2n+2}\,J_1(x)\right\}-$$
$$-\frac{x^{2n-1}[(4n+3)(8n^2+12n+3)x^2+4n^2(2n+1)^2(4n+5)]\,J_1(x)}{4n+1}$$

$$x^{2n+3}\,I_0(x)=\frac{d^2}{dx^2}\,\left\{x^{2n}\,\left[\frac{2n(2n+1)(4n+5)}{4n+1}\,x+x^3\right]\,I_0(x)-(4n+5)x^{2n+2}\,I_1(x)\right\}+$$
$$+\frac{x^{2n-1}[(4n+3)(8n^2+12n+3)x^2-4n^2(2n+1)^2(4n+5)]\,I_1(x)}{4n+1}$$

$$x^{2n+3}\,K_0(x)=\frac{d^2}{dx^2}\,\left\{x^{2n}\,\left[\frac{2n(2n+1)(4n+5)}{4n+1}\,x+x^3\right]\,K_0(x)+(4n+5)x^{2n+2}\,K_1(x)\right\}+$$
$$+\frac{x^{2n-1}[(4n+3)(8n^2+12n+3)x^2-4n^2(2n+1)^2(4n+5)]\,K_1(x)}{4n+1}$$

b) $x^{2n}\,Z_1(x)$:

With the functions $\Phi(x)$, $\Psi(x)$ and $\Psi_K(x)$ as defined on page 9 holds:

$$J_1(x)=\frac{d^2}{dx^2}\,[-x\,J_0(x)-\Phi(x)]\;,\quad I_1(x)=\frac{d^2}{dx^2}\,[x\,I_0(x)+\Psi(x)]\;,\quad K_1(x)=\frac{d^2}{dx^2}\,[-x\,K_0(x)-\;\Psi_K(x)]$$

$$x^2\,J_1(x)=\frac{d^2}{dx^2}\,\big[-x^2\,J_1(x)+3\,\Phi(x)\big]$$

$$x^2\,I_1(x)=\frac{d^2}{dx^2}\,\big[x^2\,I_1(x)+3\,\Psi(x)\big]$$

$$x^2\,K_1(x)=\frac{d^2}{dx^2}\,\big[x^2\,K_1(x)-3\,\Psi_K(x)\big]$$

$$x^4\,J_1(x)=\frac{d^2}{dx^2}\,\big[-7x^3\,J_0(x)+(-x^4+29\,x^2)\,J_1(x)-45\,\Phi(x)\big]$$

$$x^4\,I_1(x)=\frac{d^2}{dx^2}\,\big[-7\,I_0(x)+(x^4+29x^2)\,I_1(x)+45\,\Psi(x)\big]$$

$$x^4\,K_1(x)=\frac{d^2}{dx^2}\,\big[7x^3\,K_0(x)+(x^4+29x^2)\,K_1(x)-45\,\Psi_K(x)\big]$$

$$x^6\,J_1(x)=\frac{d^2}{dx^2}\,\big[(-11\,x^5+333\,x^3)\,J_0(x)+(-x^6+79\,x^4-1191\,x^2)\,J_1(x)+1575\,\Phi(x)\big]$$

$$x^6\,I_1(x)=\frac{d^2}{dx^2}\,\big[-(11\,x^5+333\,x^3)\,I_0(x)+(x^6+79\,x^4+1191\,x^2)\,I_1(x)+1575\,\Psi(x)\big]$$

$$x^6\,K_1(x)=\frac{d^2}{dx^2}\,\big[(11\,x^5+333\,x^3)\,K_0(x)+(x^6+79\,x^4+1191\,x^2)\,K_1(x)-1575\,\Psi_K(x)\big]$$

The formulas for $I_1(x)$ and $K_1(x)$ vary in two signs.

$$x^8\,J_1(x)=$$

$$= \frac{d^2}{dx^2}\,\left[(-15\,x^7+1053\,x^5-23859\,x^3)\,J_0(x)+(-x^8+153\,x^6-6417\,x^4+80793\,x^2)\,J_1(x)-99225\,\Phi(x)\right]$$

$$x^8\,I_1(x) =$$

$$= \frac{d^2}{dx^2}\,\left[-(15\,x^7+1053\,x^5+23859\,x^3)\,I_0(x)+(x^8+153\,x^6+6417\,x^4+80793\,x^2)\,I_1(x)-99225\,\Psi(x)\right]$$

$$x^{10}\,J_1(x) = \frac{d^2}{dx^2}\,\left[(-19\,x^9+2397\,x^7-126135\,x^5+2537145\,x^3)\,J_0(x)+\right.$$
$$\left.+(-x^{10}+251\,x^8-20619\,x^6+722835\,x^4-8348715\,x^2)\,J_1(x)+9823275\,\Phi(x)\right]$$

$$x^{10}\,I_1(x) = \frac{d^2}{dx^2}\,\left[-(19\,x^9+2397\,x^7+126135\,x^5+2537145\,x^3)\,I_0(x)+\right.$$
$$\left.+(x^{10}+251\,x^8+20619\,x^6+722835\,x^4+8348715\,x^2)\,I_1(x)+9823275\,\Psi(x)\right]$$

$$x^{12}\,J_1(x) = \frac{d^2}{dx^2}\,\left[(-23\,x^{11}+4557\,x^9-431091\,x^7+20156985\,x^5-379769175\,x^3)\,J_0(x)+\right.$$
$$+(-x^{12}+373\,x^{10}-50613\,x^8+3478437\,x^6-111844125\,x^4+1227781125\,x^2)\,J_1(x)-$$
$$\left.-1404728325\,\Phi(x)\right]$$

$$x^{12}\,I_1(x) = \frac{d^2}{dx^2}\,\left[-(23\,x^{11}+4557\,x^9+431091\,x^7+20156985\,x^5+379769175\,x^3)\,I_0(x)+\right.$$
$$+(x^{12}+373\,x^{10}+50613\,x^8+3478437\,x^6+111844125\,x^4+1227781125\,x^2)\,I_1(x)+$$
$$\left.+1404728325\,\Psi(x)\right]$$

$$x^{14}\,J_1(x) =$$

$$= \frac{d^2}{dx^2}\,\left[(-27\,x^{13}+7725\,x^{11}-1147815\,x^9+96504345\,x^7-4229210475\,x^5+76443776325\,x^3)\,J_0(x)+\right.$$
$$+(-x^{14}+519\,x^{12}-105135\,x^{10}+11943135\,x^8-752944815\,x^6+23003997975\,x^4-244194893775\,x^2)\,J_1(x)+$$
$$\left.+273922023375\,\Phi(x)\right]$$

$$x^{14}\,I_1(x) =$$

$$= \frac{d^2}{dx^2}\,\left[-(27\,x^{13}+7725\,x^{11}+1147815\,x^9+96504345\,x^7+4229210475\,x^5+76443776325\,x^3)\,I_0(x)+\right.$$
$$+(x^{14}+519\,x^{12}+105135\,x^{10}+11943135\,x^8+752944815\,x^6+23003997975\,x^4+244194893775\,x^2)\,I_1(x)+$$
$$\left.+273922023375\,\Psi(x)\right]$$

$$x^{16}\,J_1(x) = \frac{d^2}{dx^2}\,\left[(-31\,x^{15}+12093\,x^{13}-2594835\,x^{11}+342689625\,x^9-27008454375\,x^7+1136044984725\,x^5-\right.$$
$$-19953933471675\,x^3)\,J_0(x)+(-x^{16}+689\,x^{14}-194841\,x^{12}+33059025\,x^{10}-3445473825\,x^8+206400006225\,x^6-$$
$$\left.-6096404738025\,x^4+63191238930225\,x^2)\,J_1(x)-69850115960625\,\Phi(x)\right]$$

$$x^{16}\,I_1(x) = \frac{d^2}{dx^2}\,\left[-(31\,x^{15}+12093\,x^{13}+2594835\,x^{11}+342689625\,x^9+27008454375\,x^7+1136044984725\,x^5+\right.$$
$$+19953933471675\,x^3)\,I_0(x)+(x^{16}+689\,x^{14}+194841\,x^{12}+33059025\,x^{10}+3445473825\,x^8+206400006225\,x^6+$$
$$\left.+6096404738025\,x^4+63191238930225\,x^2)\,I_1(x)+69850115960625\,\Psi(x)\right]$$

<u>Recurrence relations:</u>

$$x^{2n+2}\,J_1(x) = \frac{d^2}{dx^2}\,\left\{\left[\frac{2n(4n+3)(2n+1)}{4n-1}-x^2\right]\,x^{2n}\,J_1(x)-(4n+3)\,x^{2n+1}\,J_0(x)\right\}-$$
$$-\frac{(4n+1)(8n^2+4n-3)x^2+4n(n-1)(4n+3)(2n+1)(2n-1)}{4n-1}\,x^{2n-2}\,J_1(x)$$

$$x^{2n+2}\, I_1(x) = \frac{d^2}{dx^2}\,\left\{\left[\frac{2n(4n+3)(2n+1)}{4n-1} + x^2\right] x^{2n}\, I_1(x) - (4n+3)\, x^{2n+1}\, I_0(x)\right\} +$$

$$+ \frac{(4n+1)(8n^2+4n-3)x^2 - 4n(n-1)(4n+3)(2n+1)(2n-1)}{4n-1}\, x^{2n-2}\, I_1(x)$$

$$x^{2n+2}\, K_1(x) = \frac{d^2}{dx^2}\,\left\{\left[\frac{2n(4n+3)(2n+1)}{4n-1} + x^2\right] x^{2n}\, K_1(x) + (4n+3)\, x^{2n+1}\, K_0(x)\right\} +$$

$$+ \frac{(4n+1)(8n^2+4n-3)x^2 - 4n(n-1)(4n+3)(2n+1)(2n-1)}{4n-1}\, x^{2n-2}\, K_1(x)$$

1.1.9. Higher antiderivatives:

$\Phi(x)$, $\Psi(x)$ and $\Psi_K(x)$ are the same as in I., page 9. See also [1], 11.2. .

$$\begin{aligned}
J_0(x) &= \frac{d^2}{dx^2}\left\{x^2 J_0(x) - xJ_1(x) + x\,\Phi(x)\right\}\\
&= \frac{d^3}{dx^3}\left\{\frac{x^3}{2}J_0(x) - \frac{x^2}{2}J_1(x) + \frac{x^2-1}{2}\,\Phi(x)\right\}\\
&= \frac{d^4}{dx^4}\left\{\frac{x^4-2x^2}{6}J_0(x) - \frac{x^3-4x}{6}J_1(x) + \frac{x^3-3x}{6}\,\Phi(x)\right\}\\
&= \frac{d^5}{dx^5}\left\{\frac{x^5-5x^3}{24}J_0(x) - \frac{x^4-7x^2}{24}J_1(x) + \frac{x^4-6x^2+9}{24}\,\Phi(x)\right\}\\
&= \frac{d^6}{dx^6}\left\{\frac{x^6-9x^4+32x^2}{120}J_0(x) - \frac{x^5-11x^3+64x}{120}J_1(x) + \frac{x^5-10x^3+45x}{120}\,\Phi(x)\right\}\\
&= \frac{d^7}{dx^7}\left\{\frac{x^7-14x^5+117x^3}{720}J_0(x) - \frac{x^6-16x^4+159x^2}{720}J_1(x) + \frac{x^6-15x^4+135x^2-225}{720}\,\Phi(x)\right\}\\
I_0(x) &= \frac{d^2}{dx^2}\left\{x^2 I_0(x) - xI_1(x) + x\,\Psi(x)\right\}\\
&= \frac{d^3}{dx^3}\left\{\frac{x^3}{2}I_0(x) - \frac{x^2}{2}I_1(x) + \frac{x^2+1}{2}\,\Psi(x)\right\}\\
&= \frac{d^4}{dx^4}\left\{\frac{x^4+2x^2}{6}I_0(x) - \frac{x^3+4x}{6}I_1(x) + \frac{x^3+3x}{6}\,\Psi(x)\right\}\\
&= \frac{d^5}{dx^5}\left\{\frac{x^5+5x^3}{24}I_0(x) - \frac{x^4+7x^2}{24}I_1(x) + \frac{x^4+6x^2+9}{24}\,\Psi(x)\right\}\\
&= \frac{d^6}{dx^6}\left\{\frac{x^6+9x^4+32x^2}{120}I_0(x) - \frac{x^5+11x^3+64x}{120}I_1(x) + \frac{x^5+10x^3+45x}{120}\,\Psi(x)\right\}\\
&= \frac{d^7}{dx^7}\left\{\frac{x^7+14x^5+117x^3}{720}I_0(x) - \frac{x^6+16x^4+159x^2}{720}I_1(x) + \frac{x^6+15x^4+135x^2+225}{720}\,\Psi(x)\right\}\\
K_0(x) &= \frac{d^2}{dx^2}\left\{x^2 K_0(x) - xK_1(x) + x\,\Psi_K(x)\right\}\\
&= \frac{d^3}{dx^3}\left\{\frac{x^3}{2}K_0(x) - \frac{x^2}{2}K_1(x) + \frac{x^2+1}{2}\,\Psi_K(x)\right\}\\
&= \frac{d^4}{dx^4}\left\{\frac{x^4+2x^2}{6}K_0(x) - \frac{x^3+4x}{6}K_1(x) + \frac{x^3+3x}{6}\,\Psi_K(x)\right\} \quad \ldots
\end{aligned}$$

The formulas for $K_0(x)$ are similar to such for $I_0(x)$.

Let

$$J_0(x) = \frac{d^n}{dx^n}\left\{\frac{A_n(x)\,J_0(x) - B_n(x)\,J_1(x) + C_n(x)\,\Phi(x)}{(n-1)!}\right\},$$

then holds

$$\begin{aligned}
A_8 &= x^8 - 20\,x^6 + 291\,x^4 - 1152\,x^2\\
B_8 &= x^7 - 22\,x^5 + 345\,x^3 - 2304\,x\\
C_8 &= x^7 - 21\,x^5 + 315\,x^3 - 1575\,x\\
\hline
A_9 &= x^9 - 27\,x^7 + 599\,x^5 - 5541\,x^3\\
B_9 &= x^8 - 29\,x^6 + 667\,x^4 - 7407\,x^2\\
C_9 &= x^8 - 28\,x^6 + 630\,x^4 - 6300\,x^2 + 11025\\
\hline
A_{10} &= x^{10} - 35\,x^8 + 1095\,x^6 - 17613\,x^4 + 73728\,x^2\\
B_{10} &= x^9 - 37\,x^7 + 1179\,x^5 - 20583\,x^3 + 147456\,x\\
C_{10} &= x^9 - 36\,x^7 + 1134\,x^5 - 18900\,x^3 + 99225\,x\\
\hline
A_{11} &= x^{11} - 44\,x^9 + 1842\,x^7 - 45180\,x^5 + 439605\,x^3\\
B_{11} &= x^{10} - 46\,x^8 + 1944\,x^6 - 49770\,x^4 + 581535\,x^2\\
C_{11} &= x^{10} - 45\,x^8 + 1890\,x^6 - 47250\,x^4 + 496125\,x^2 - 893025\\
\hline
A_{12} &= x^{12} - 54\,x^{10} + 2912\,x^8 - 100770\,x^6 + 1702215\,x^4 - 7372800\,x^2\\
B_{12} &= x^{11} - 56\,x^9 + 3034\,x^7 - 107640\,x^5 + 1973205\,x^3 - 14745600\,x\\
C_{12} &= x^{11} - 55\,x^9 + 2970\,x^7 - 103950\,x^5 + 1819125\,x^3 - 9823275\,x
\end{aligned}$$

$A_{13} = x^{13} - 65\,x^{11} + 4386\,x^9 - 203202\,x^7 + 5231565\,x^5 - 52454925\,x^3$
$B_{13} = x^{12} - 67\,x^{10} + 4530\,x^8 - 213174\,x^6 + 5731245\,x^4 - 68891175\,x^2$
$C_{13} = x^{12} - 66\,x^{10} + 4455\,x^8 - 207900\,x^6 + 5457375\,x^4 - 58939650\,x^2 + 108056025$

$A_{14} = x^{14} - 77\,x^{12} + 6354\,x^{10} - 379386\,x^8 + 13778685\,x^6 - 239546025\,x^4 + 1061683200\,x^2$
$B_{14} = x^{13} - 79\,x^{11} + 6522\,x^9 - 393462\,x^7 + 14661765\,x^5 - 276270075\,x^3 + 2123366400\,x$
$C_{14} = x^{13} - 78\,x^{11} + 6435\,x^9 - 386100\,x^7 + 14189175\,x^5 - 255405150\,x^3 + 1404728325\,x$

$A_{15} = x^{15} - 90\,x^{13} + 8915\,x^{11} - 666348\,x^9 + 32399703\,x^7 - 858134970\,x^5 + 8776408725\,x^3$
$B_{15} = x^{14} - 92\,x^{12} + 9109\,x^{10} - 685728\,x^8 + 33896331\,x^6 - 937030500\,x^4 + 11465661375\,x^2$
$C_{15} = x^{14} - 91\,x^{12} + 9009\,x^{10} - 675675\,x^8 + 33108075\,x^6 - 893918025\,x^4 + 9833098275\,x^2 -$
$- 18261468225$

$A_{16} = x^{16} - 104\,x^{14} + 12177\,x^{12} - 1113480\,x^{10} + 69776595\,x^8 - 2606612400\,x^6 + 46180633275\,x^4 -$
$- 208089907200\,x^2$
$B_{16} = x^{15} - 106\,x^{13} + 12399\,x^{11} - 1139580\,x^9 + 72217215\,x^7 - 2767649850\,x^5 + 53076402225\,x^3 -$
$- 416179814400\,x$
$C_{16} = x^{15} - 105\,x^{13} + 12285\,x^{11} - 1126125\,x^9 + 70945875\,x^7 - 2681754075\,x^5 + 49165491375\,x^3 -$
$- 273922023375\,x$

$A_{17} = x^{17} - 119\,x^{15} + 16257\,x^{13} - 1785015\,x^{11} + 140033835\,x^9 - 7003021725\,x^7 +$
$+ 188956336275\,x^5 - 1959828398325\,x^3$
$B_{17} = x^{16} - 121\,x^{14} + 16509\,x^{12} - 1819485\,x^{10} + 143878995\,x^8 - 7315353675\,x^6 +$
$+ 205891548975\,x^4 - 2550046679775\,x^2$
$C_{17} = x^{16} - 120\,x^{14} + 16380\,x^{12} - 1801800\,x^{10} + 141891750\,x^8 - 7151344200\,x^6 + 196661965500\,x^4 -$
$- 2191376187000\,x^2 + 4108830350625$

$A_{18} = x^{18} - 135\,x^{16} + 21281\,x^{14} - 2762727\,x^{12} + 265161195\,x^{10} - 17089901325\,x^8 +$
$+ 650296717875\,x^6 - 11675732422725\,x^4 + 53271016243200\,x^2$
$B_{18} = x^{17} - 137\,x^{15} + 21565\,x^{13} - 2807469\,x^{11} + 271036755\,x^9 - 17667841275\,x^7 +$
$+ 689522590575\,x^5 - 13385846919375\,x^3 + 106542032486400\,x$
$C_{18} = x^{17} - 136\,x^{15} + 21420\,x^{13} - 2784600\,x^{11} + 268017750\,x^9 - 17367550200\,x^7 +$
$+ 668650682700\,x^5 - 12417798393000\,x^3 + 69850115960625\,x$

$A_{19} = x^{19} - 152\,x^{17} + 27384\,x^{15} - 4148856\,x^{13} + 478163970\,x^{11} - 38580445800\,x^9 +$
$+ 1965171423600\,x^7 - 53722981355400\,x^5 + 563060968600725\,x^3$
$B_{19} = x^{18} - 154\,x^{16} + 27702\,x^{14} - 4206042\,x^{12} + 486902160\,x^{10} - 39605788350\,x^8 +$
$+ 2050859043450\,x^6 - 58451723167950\,x^4 + 730304613424575\,x^2$
$C_{19} = x^{18} - 153\,x^{16} + 27540\,x^{14} - 4176900\,x^{12} + 482431950\,x^{10} - 39076987950\,x^8 +$
$+ 2005952048100\,x^6 - 55880092768500\,x^4 + 628651043645625\,x^2 - 1187451971330625$

$A_{20} = x^{20} - 170\,x^{18} + 34710\,x^{16} - 6069258\,x^{14} + 827072928\,x^{12} - 81642319470\,x^{10} +$
$+ 5359374805050\,x^8 - 206598479046750\,x^6 + 3746290783676175\,x^4 - 17259809262796800\,x^2$
$B_{20} = x^{19} - 172\,x^{17} + 35064\,x^{15} - 6141348\,x^{13} + 839759706\,x^{11} - 83394724500\,x^9 +$
$+ 5536741392000\,x^7 - 218854228527900\,x^5 + 4287004731290925\,x^3 - 34519618525593600\,x$
$C_{20} = x^{19} - 171\,x^{17} + 34884\,x^{15} - 6104700\,x^{13} + 833291550\,x^{11} - 82495863450\,x^9 +$
$+ 5444726987700\,x^7 - 212344352520300\,x^5 + 3981456609755625\,x^3 - 22561587455281875\,x$

To get the functions for $I_0(x)$ or $K_0(x)$ one has to change all '-' in the fractions to '+' .
The higher antiderivatives of $J_1(x)$, $I_1(x)$ and $K_1(x)$ follow from the previous tables and the formulas $J_1(x) = -J_0'(x)$ and $I_1(x) = I_0'(x)$, $K_1(x) = -K_0'(x)$.

1.1.10. Some Integrals of the Type $x^{2n+1}\, Z_1(x^2+\alpha)/(x^2+\alpha)$

$$\int \frac{x^5\, J_1(x^2+i)\, dx}{x^2+i} = \frac{1}{2}\, \left[-(x^2-i)\, J_0(x^2+i) + J_1(x^2+i)\right]$$

$$\int \frac{x^5\, J_1(x^2-i)\, dx}{x^2-i} = -\frac{1}{2}\, \left[(x^2+i)\, J_0(x^2-i) - J_1(x^2-i)\right]$$

$$\int \frac{x^5\, I_1(x^2+1)\, dx}{x^2+1} = \frac{1}{2}\, \left[(x^2-1)\, I_0(x^2+1) - I_1(x^2+1)\right]$$

$$\int \frac{x^5\, I_1(x^2-1)\, dx}{x^2-1} = \frac{1}{2}\, \left[(x^2+1)\, I_0(x^2-1) - I_1(x^2-1)\right]$$

$$\int \frac{x^5\, K_1(x^2+1)\, dx}{x^2+1} = -\frac{1}{2}\, \left[(x^2-1)\, K_0(x^2+1) + K_1(x^2+1)\right]$$

$$\int \frac{x^5\, K_1(x^2-1)\, dx}{x^2-1} = -\frac{1}{2}\, \left[(x^2+1)\, K_0(x^2-1) + K_1(x^2-1)\right]$$

$$\int \frac{x^7\, J_1(x^2+\sqrt{3}\,i)\, dx}{x^2+\sqrt{3}\,i} = \frac{1}{2}\, \left[(-x^4+\sqrt{3}\,i\,x+3)\, J_0(x^2+\sqrt{3}\,i) + (2x^2-\sqrt{3}\,i) J_1(x^2+\sqrt{3}\,i)\right]$$

$$\int \frac{x^7\, J_1(x^2-\sqrt{3}\,i)\, dx}{x^2-\sqrt{3}\,i} = \frac{1}{2}\, \left[(-x^4-\sqrt{3}\,i\,x+3)\, J_0(x^2-\sqrt{3}\,i) + (2x^2+\sqrt{3}\,i) J_1(x^2-\sqrt{3}\,i)\right]$$

$$\int \frac{x^7\, I_1(x^2+\sqrt{3})\, dx}{x^2+\sqrt{3}} = \frac{1}{2}\, \left[(x^4-\sqrt{3}x^2+3)\, I_0(x^2+\sqrt{3}) + (-2\,x^2+\sqrt{3}) I_1(x^2+\sqrt{3})\right]$$

$$\int \frac{x^7\, I_1(x^2-\sqrt{3})\, dx}{x^2-\sqrt{3}} = \frac{1}{2}\, \left[(x^4+\sqrt{3}x^2+3)\, I_0(x^2-\sqrt{3}) - (2\,x^2+\sqrt{3}) I_1(x^2-\sqrt{3})\right]$$

$$\int \frac{x^7\, K_1(x^2+\sqrt{3})\, dx}{x^2+\sqrt{3}} = \frac{1}{2}\, \left[-(x^4-\sqrt{3}x^2+3)\, K_0(x^2+\sqrt{3}) + (-2\,x^2+\sqrt{3}) K_1(x^2+\sqrt{3})\right]$$

$$\int \frac{x^7\, K_1(x^2-\sqrt{3})\, dx}{x^2-\sqrt{3}} = -\frac{1}{2}\, \left[(x^4+\sqrt{3}x^2+3)\, K_0(x^2-\sqrt{3}) + (2\,x^2+\sqrt{3}) K_1(x^2-\sqrt{3})\right]$$

$\lambda = \sqrt{2\sqrt{3}-3} = 0.68125\ 00386$

$$\int \frac{x^9\, J_1(x^2+\lambda)\, dx}{x^2+\lambda} =$$

$$= \frac{1}{2}\, \left[\left(-x^6+\lambda\, x^4+6\,x^2-2\sqrt{3}x^2+2\,\lambda\sqrt{3}\right)\, J_0\left(x^2+\lambda\right) + \left(3\,x^4-2\,\lambda\, x^2-6+2\sqrt{3}\right)\, J_1\left(x^2+\lambda\right)\right]$$

$$\int \frac{x^9\, J_1(x^2-\lambda)\, dx}{x^2-\lambda} =$$

$$= \frac{1}{2}\, \left[-\left(x^6+\lambda\, x^4-6\,x^2+2\sqrt{3}x^2+2\,\lambda\sqrt{3}\right)\, J_0\left(x^2-\lambda\right) + \left(3\,x^4+2\,\lambda\, x^2-6+2\sqrt{3}\right)\, J_1\left(x^2-\lambda\right)\right]$$

$$\int \frac{x^9\, I_1(x^2+\lambda\, i)\, dx}{x^2+\lambda\, i} =$$

$$= \frac{1}{2}\, \left[\left(x^6-\lambda\, i\, x^4+6\,x^2-2\sqrt{3}x^2+2i\,\lambda\sqrt{3}\right)\, I_0\left(x^2+\lambda\, i\right) + \left(-3\,x^4+2i\,\lambda x^2-6+2\sqrt{3}\right)\, I_1\left(x^2+\lambda\, i\right)\right]$$

$$\int \frac{x^9\, I_1(x^2-\lambda\, i)\, dx}{x^2-\lambda\, i} =$$

$$= \frac{1}{2}\, \left[\left(x^6+\lambda\, i\, x^4+6\,x^2-2\sqrt{3}x^2-2i\,\lambda\sqrt{3}\right)\, I_0\left(x^2-\lambda\, i\right) + \left(-3\,x^4-2i\,\lambda x^2-6+2\sqrt{3}\right)\, I_1\left(x^2-\lambda\, i\right)\right]$$

$$\int \frac{x^9\, K_1(x^2+\lambda\, i)\, dx}{x^2+\lambda\, i} =$$

$$= \frac{1}{2}\left[\left(-x^6+\lambda\,i\,x^4-6\,x^2+2\,\sqrt{3}x^2-2\,\lambda\sqrt{3}\,i\right)\,K_0\left(x^2+\lambda\,i\right)+\left(-3\,x^4+2\,\lambda\,i\,x^2-6+2\,\sqrt{3}\right)K_1\left(x^2+\lambda\,i\right)\right]$$

$$\int\frac{x^9\,K_1(x^2-\lambda\,i)\,dx}{x^2-\lambda\,i}=$$

$$= \frac{1}{2}\left[\left(-x^6-\lambda\,i\,x^4-6\,x^2+2\,\sqrt{3}x^2+2\,\lambda\,\sqrt{3}\,i\right)\,K_0\left(x^2-\lambda\,i\right)+\left(-3\,x^4-2\,\lambda\,i\,x^2-6+2\,\sqrt{3}\right)K_1\left(x^2-\lambda\,i\right)\right]$$

$\mu := \sqrt{2\,\sqrt{3}+3} = 2.54245\ 97568$

$$\int\frac{x^9\,J_1(x^2+\mu\,i)\,dx}{x^2+\mu\,i}=$$

$$= \frac{1}{2}\left[\left(-x^6+\mu\,i\,x^4+6\,x^2+2\,\sqrt{3}x^2-2\,\mu\,\sqrt{3}\,i\right)\,J_0\left(x^2+\mu\,i\right)+\left(3\,x^4-2\mu\,i\,x^2-6-2\,\sqrt{3}\right)J_1\left(x^2+\mu\,i\right)\right]$$

$$\int\frac{x^9\,J_1(x^2-\mu\,i)\,dx}{x^2-\mu\,i}=$$

$$= \frac{1}{2}\left[\left(-x^6-\mu\,i\,x^4+6\,x^2+2\,\sqrt{3}x^2+2\,\mu\,\sqrt{3}\,i\right)\,J_0\left(x^2-\mu\,i\right)+\left(3\,x^4+2\mu\,i\,x^2-6-2\,\sqrt{3}\right)J_1\left(x^2-\mu\,i\right)\right]$$

$$\int\frac{x^9\,I_1(x^2+\mu\,)\,dx}{x^2+\mu}=$$

$$= \frac{1}{2}\left[\left(x^6-\mu\,x^4+6\,x^2+2\,\sqrt{3}x^2-2\,\mu\,\sqrt{3}\right)\,I_0\left(x^2+\mu\,\right)-\left(3\,x^4-2\mu\,x^2+6+2\,\sqrt{3}\right)I_1\left(x^2+\mu\,\right)\right]$$

$$\int\frac{x^9\,I_1(x^2-\mu\,)\,dx}{x^2-\mu}=$$

$$= \frac{1}{2}\left[\left(x^6+\mu\,x^4+6\,x^2+2\,\sqrt{3}x^2+2\,\mu\,\sqrt{3}\right)\,I_0\left(x^2-\mu\,\right)-\left(3\,x^4+2\mu\,x^2+6+2\,\sqrt{3}\right)I_1\left(x^2-\mu\,\right)\right]$$

$$\int\frac{x^9\,K_1(x^2+\mu\,)\,dx}{x^2+\mu}=$$

$$= -\frac{1}{2}\left[\left(x^6-\mu\,x^4+6\,x^2+2\,\sqrt{3}x^2-2\,\mu\,\sqrt{3}\right)\,K_0\left(x^2+\mu\,\right)+\left(3\,x^4-2\mu\,x^2+6+2\,\sqrt{3}\right)K_1\left(x^2+\mu\,\right)\right]$$

$$\int\frac{x^9\,K_1(x^2-\mu\,)\,dx}{x^2-\mu}=$$

$$= -\frac{1}{2}\left[\left(x^6+\mu\,x^4+6\,x^2+2\,\sqrt{3}x^2+2\,\mu\,\sqrt{3}\right)\,K_0\left(x^2-\mu\,\right)+\left(3\,x^4+2\mu\,x^2+6+2\,\sqrt{3}\right)K_1\left(x^2-\mu\,\right)\right]$$

$\eta := \sqrt{2\sqrt{10}-5} = 1.15089\ 32706$

$$\int\frac{x^{11}\,J_1(x^2+\eta)\,dx}{x^2+\eta}=\frac{1}{2}\left[\left(-x^8+\sqrt{\eta}x^6+13\,x^4-2\,\sqrt{10}x^4-4\,\sqrt{\eta}x^2+\right.\right.$$

$$\left.+\,2\,\sqrt{\eta}\sqrt{10}x^2+6\,\sqrt{10}-30\right)\,J_0\left(x^2+\eta\right)+$$

$$\left.+\left(4\,x^6-3\,\sqrt{\eta}x^4-26\,x^2+4\,\sqrt{10}x^2+4\,\sqrt{\eta}-2\,\sqrt{\eta}\sqrt{10}\right)\,J_1\left(x^2+\eta\right)\right]$$

$$\int\frac{x^{11}\,J_1(x^2-\eta)\,dx}{x^2-\eta}=\frac{1}{2}\left[\left(-x^8-\sqrt{\eta}x^6+13\,x^4-2\,\sqrt{10}x^4+4\,\sqrt{\eta}x^2-\right.\right.$$

$$\left.-\,2\,\sqrt{\eta}\sqrt{10}x^2+6\,\sqrt{10}-30\right)\,J_0\left(x^2-\eta\right)+$$

$$\left.+\left(4\,x^6+3\,\sqrt{\eta}x^4-26\,x^2+4\,\sqrt{10}x^2-(4-2\sqrt{10})\,\sqrt{\eta}\right)\,J_1\left(x^2-\eta\right)\right]$$

$$\int\frac{x^{11}\,I_1(x^2+\eta\,i)\,dx}{x^2+\eta\,i}=\frac{1}{2}\left[\left(x^8-\sqrt{\eta}\,i\,x^6+13\,x^4-2\,\sqrt{10}x^4-4\,\sqrt{\eta}\,i\,x^2+\right.\right.$$

$$\left.+\,2\,\sqrt{\eta}\sqrt{10}\,i\,x^2-6\,\sqrt{10}+30\right)\,I_0\left(x^2+\eta\,i\,\right)+$$

$$+\left(-4\,x^6+3\,\sqrt{\eta}\,i\,x^4-26\,x^2+4\,\sqrt{10}x^2+(4-2\sqrt{10})\,\sqrt{\eta}\,i\right)I_1\left(x^2+\eta\,i\right)\Big]$$

$$\int\frac{x^{11}\,I_1(x^2-\eta\,i)\,dx}{x^2-\eta\,i}=\frac{1}{2}\,\Big[\Big(x^8+\sqrt{\eta}\,i\,x^6+13\,x^4-2\,\sqrt{10}x^4+4\,\sqrt{\eta}\,i\,x^2+$$

$$-2\,\sqrt{\eta}\sqrt{10}\,i\,x^2-6\,\sqrt{10}+30\Big)\,I_0\left(x^2-\eta\,i\right)+$$

$$+\left(-4\,x^6-3\,\sqrt{\eta}\,i\,x^4-26\,x^2+4\,\sqrt{10}x^2-(4-2\sqrt{10})\,\sqrt{\eta}\,i\right)I_1\left(x^2-\eta\,i\right)\Big]$$

$$\int\frac{x^{11}\,K_1(x^2+\eta\,i)\,dx}{x^2+\eta\,i}=\frac{1}{2}\,\Big[\Big(-x^8+\sqrt{\eta}\,i\,x^6-13\,x^4+2\,\sqrt{10}x^4+4\,\sqrt{\eta}\,i\,x^2-$$

$$-2\,\sqrt{\eta}\sqrt{10}\,i\,x^2+6\,\sqrt{10}-30\Big)\,K_0\left(x^2+\eta\,i\right)+$$

$$+\left(-4\,x^6+3\,\sqrt{\eta}\,i\,x^4-26\,x^2+4\,\sqrt{10}x^2+(4-2\sqrt{10})\,\sqrt{\eta}\,i\right)K_1\left(x^2+\eta\,i\right)\Big]$$

$$\int\frac{x^{11}\,K_1(x^2-\eta\,i)\,dx}{x^2-\eta\,i}=\frac{1}{2}\,\Big[\Big(-x^8-\sqrt{\eta}\,i\,x^6-13\,x^4+2\,\sqrt{10}x^4-4\,\sqrt{\eta}\,i\,x^2+$$

$$+2\,\sqrt{\eta}\sqrt{10}\,i\,x^2+6\,\sqrt{10}-30\Big)\,K_0\left(x^2-\eta\,i\right)+$$

$$+\left(-4\,x^6-3\,\sqrt{\eta}\,i\,x^4-26\,x^2+4\,\sqrt{10}x^2+(2\sqrt{10}-4)\,\sqrt{\eta}\,i\right)K_1\left(x^2-\eta\,i\right)\Big]$$

$\varrho=\sqrt{2\sqrt[3]{25}+4\sqrt[3]{5}+5}=4.20570\;31830$

$$\int\frac{x^{13}\,J_1(x^2+\varrho i)\,dx}{x^2+\varrho i}=\frac{1}{2}\,\Big\{-x^{10}+\varrho i\,x^8+2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6-\varrho i\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4-$$

$$-[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2+12\varrho i\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\;J_0(x^2+\varrho i)+\frac{1}{2}\,\Big\{5x^8-4\varrho i\,x^6-6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4+$$

$$+4\varrho i\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2+36\sqrt[3]{5}+24\sqrt[3]{25}+120\Big\}\;J_1(x^2+\varrho i)$$

$$\int\frac{x^{13}\,J_1(x^2-\varrho i)\,dx}{x^2-\varrho i}=\frac{1}{2}\,\Big\{-x^{10}-\varrho i\,x^8+2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6+\varrho i\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4-$$

$$-[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2-12\varrho i\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\;J_0(x^2-\varrho i)+\frac{1}{2}\,\Big\{5x^8+4\varrho i\,x^6-6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4-$$

$$-4\varrho i\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2+36\sqrt[3]{5}+24\sqrt[3]{25}+120\Big\}\;J_1(x^2-\varrho i)$$

$$\int\frac{x^{13}\,I_1(x^2+\varrho)\,dx}{x^2+\varrho}=\frac{1}{2}\,\Big\{x^{10}-\varrho\,x^8+2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6-\varrho\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4+$$

$$+[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2-12\varrho\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\;J_0(x^2+\varrho)+\frac{1}{2}\,\Big\{-5x^8+4\varrho\,x^6-6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4+$$

$$+4\varrho\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2-36\sqrt[3]{5}-24\sqrt[3]{25}-120\Big\}\;I_1(x^2+\varrho)$$

$$\int\frac{x^{13}\,I_1(x^2-\varrho)\,dx}{x^2-\varrho}=\frac{1}{2}\,\Big\{x^{10}+\varrho\,x^8+2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6+\varrho\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4+$$

$$+[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2+12\varrho\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\;J_0(x^2-\varrho)-\frac{1}{2}\,\Big\{5x^8+4\varrho\,x^6 6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4+$$

$$+4\varrho\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2+36\sqrt[3]{5}+24\sqrt[3]{25}+120\Big\}\;I_1(x^2-\varrho)$$

$$\int\frac{x^{13}\,K_1(x^2+\varrho)\,dx}{x^2+\varrho}=\frac{1}{2}\,\Big\{-x^{10}+\varrho\,x^8-2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6+\varrho\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4-$$

$$-[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2+12\varrho\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\;K_0(x^2+\varrho)+\frac{1}{2}\,\Big\{-5x^8+4\varrho\,x^6-6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4+$$

$$+4\varrho\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2-36\sqrt[3]{5}-24\sqrt[3]{25}-120\Big\}\;K_1(x^2+\varrho)$$

$$\int\frac{x^{13}\,K_1(x^2-\varrho)\,dx}{x^2-\varrho}=-\frac{1}{2}\,\Big\{x^{10}+\varrho\,x^8+2(\sqrt[3]{25}+2\sqrt[3]{5}+10)\,x^6+\varrho\,[8+4\sqrt[3]{5}+2\sqrt[3]{25}]\,x^4+$$

$$+[120+36\sqrt[3]{5}+24\sqrt[3]{25}]\,x^2+12\varrho\,[\sqrt[3]{5}+\sqrt[3]{25}]\Big\}\;K_0(x^2-\varrho)-\frac{1}{2}\,\Big\{5x^8+4\varrho\,x^6 6\,[2\sqrt[3]{5}+\sqrt[3]{25}+10]\,x^4+$$

$$+4\varrho\,[(2\sqrt[3]{5}+\sqrt[3]{25}+4]\,x^2+36\sqrt[3]{5}+24\sqrt[3]{25}+120\Big\}\;K_1(x^2-\varrho)$$

$$\sigma=\sqrt{5-2\sqrt[3]{5}-\sqrt[3]{25}+\sqrt{3}\,(2\sqrt[3]{5}-\sqrt[3]{25})\,i}\;=0.35429\,92342+1.21222\,83526\,i$$

$$\int\frac{x^{13}\,J_1(x^2+\sigma i)\,dx}{x^2+\sigma i}=\frac{1}{2}\,\Big\{-x^{10}+\sigma i x^8\;+\Big[(2\sqrt{3}\,i-2)\sqrt[3]{5}-(1+\sqrt{3}\,i)\sqrt[3]{25}+20\Big]\;x^6+$$

$$+\Big[2(\sqrt{3}+i)\sqrt[3]{5}+(i-\sqrt{3})\sqrt[3]{25}-8i\Big]\;\sigma\,x^4\;+\Big[18(1-\sqrt{3}\,i)\sqrt[3]{5}+12(1+\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\;x^2-$$

$$-6\;\Big[(\sqrt{3}+i)\sqrt[3]{5}-(\sqrt{3}-i)\sqrt[3]{25}\Big]\;\sigma\Big\}\;J_0(x^2+\sigma i)+$$

$$+\frac{1}{2}\;\Big\{5x^8-4\sigma i x^6\;+3\;\Big[2(1-\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}-20\Big]\;x^4-$$

$$-2\;\Big[2(\sqrt{3}\;+i)\sqrt[3]{5}-(\sqrt{3}\;-i)\sqrt[3]{25}-8i\Big]\;\sigma\,x^2\;+\Big[18(\sqrt{3}\,i-1)\sqrt[3]{5}-12(1+\sqrt{3}\,i)\sqrt[3]{25}+120\Big]\;\Big\}\;J_1(x^2+\sigma i)$$

$$\int\frac{x^{13}\,J_1(x^2-\sigma i)\,dx}{x^2-\sigma i}=\frac{1}{2}\,\Big\{-x^{10}-\sigma i x^8\;+\Big[(2\sqrt{3}\,i-2)\sqrt[3]{5}-(1+\sqrt{3}\,i)\sqrt[3]{25}+20\Big]\;x^6-$$

$$-\Big[2(\sqrt{3}+i)\sqrt[3]{5}+(i-\sqrt{3})\sqrt[3]{5}-8\Big]\;\sigma\,x^4\;+\Big[18(1-\sqrt{3}\,i)\sqrt[3]{5}+12(1+\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\;x^2+$$

$$+6\;\Big[(1+\sqrt{3}\,i)\sqrt[3]{5}-(\sqrt{3}-i)\sqrt[3]{25}\Big]\;\sigma\Big\}\;J_0(x^2-\sigma i)+$$

$$+\frac{1}{2}\;\Big\{5x^8+4\sigma i x^6\;+\Big[6(1-\sqrt{3}\,i)\sqrt[3]{5}+3(1+\sqrt{3}\,i)\sqrt[3]{25}-60\Big]\;x^4+$$

$$+2\;\Big[2(\sqrt{3}+i)\sqrt[3]{5}-(\sqrt{3}-i)\sqrt[3]{25}-8i\Big]\;\sigma\,x^2\;+\Big[18(\sqrt{3}\,i-1)\sqrt[3]{5}-12(1+\sqrt{3}\,i)\sqrt[3]{25}+120\Big]\;\Big\}\;J_1(x^2-\sigma i)$$

$$\int\frac{x^{13}\,I_1(x^2+\sigma)\,dx}{x^2+\sigma}=\frac{1}{2}\,\Big\{x^{10}-\sigma\,x^8\;+\Big[(2\sqrt{3}\,i-2)\sqrt[3]{5}-(1+\sqrt{3}\,i)\sqrt[3]{25}+20\Big]\;x^6+$$

$$+\Big[(2-2\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}-8\Big]\;\sigma\,x^4\;+\Big[18(\sqrt{3}\,i-1)\sqrt[3]{5}-12(1+\sqrt{3}\,i)\sqrt[3]{25}+120\Big]\;x^2+$$

$$+6\,\Big[(1-\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}\Big]\;\sigma\Big\}\;I_0(x^2+\sigma)+$$

$$+\frac{1}{2}\;\Big\{-5x^8+4\sigma\,x^6\;+\Big[6(1-\sqrt{3}\,i)\sqrt[3]{5}+3(1+\sqrt{3}\,i)\sqrt[3]{25}-60\Big]\;x^4+$$

$$+2\;\Big[2(\sqrt{3}\,i-1)\sqrt[3]{5}-(1+\sqrt{3}\,i)\sqrt[3]{25}+8\Big]\;\sigma\;x^2\;+\Big[18(1-\sqrt{3}\,i)\sqrt[3]{5}+12(1+\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\;\Big\}\;I_1(x^2+\sigma)$$

$$\int\frac{x^{13}\,I_1(x^2-\sigma)\,dx}{x^2-\sigma}=\frac{1}{2}\,\Big\{x^{10}+\sigma\,x^8\;+\Big[(2\sqrt{3}\,i-2)\sqrt[3]{5}-(1+\sqrt{3}\,i)\sqrt[3]{25}+20\Big]\;x^6-$$

$$-\Big[(2-2\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}-8\Big]\;\sigma\,x^4\;+\Big[18(\sqrt{3}\,i-1)\sqrt[3]{5}-12(1+\sqrt{3}\,i)\sqrt[3]{25}+120\Big]\;x^2-$$

$$-6\,\Big[(1-\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}\Big]\;\sigma\Big\}\;I_0(x^2-\sigma)+$$

$$+\frac{1}{2}\;\Big\{-5x^8-4\sigma\,x^6\;+\Big[6(1-\sqrt{3}\,i)\sqrt[3]{5}+3(1+\sqrt{3}\,i)\sqrt[3]{25}-60\Big]\;x^4-$$

$$-2\;\Big[2(1-\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}-8\Big]\;\sigma\;x^2\;+\Big[18(1-\sqrt{3}\,i)\sqrt[3]{5}+12(1+\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\;\Big\}\;I_1(x^2-\sigma)$$

$$\int \frac{x^{13}\,K_1(x^2+\sigma)\,dx}{x^2+\sigma} = \frac{1}{2}\,\Big\{-x^{10}+\sigma\,x^8\,-\Big[(2\sqrt{3}\,i-2)\sqrt[3]{5}-(1+\sqrt{3}\,i)\sqrt[3]{25}+20\Big]\,x^6-$$

$$-\Big[(2-2\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}-8\Big]\,\sigma\,x^4\,-\Big[18(\sqrt{3}\,i-1)\sqrt[3]{5}-12(1+\sqrt{3}\,i)\sqrt[3]{25}+120\Big]\,x^2-$$

$$-6\Big[(1-\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}\Big]\,\sigma\Big\}\,K_0(x^2+\sigma)+$$

$$+\frac{1}{2}\,\Big\{-5x^8+4\sigma\,x^6\,+\Big[6(1-\sqrt{3}\,i)\sqrt[3]{5}+3(1+\sqrt{3}\,i)\sqrt[3]{25}-60\Big]\,x^4+$$

$$+2\,\Big[2(\sqrt{3}\,i-1)\sqrt[3]{5}-(1+\sqrt{3}\,i)\sqrt[3]{25}+8\Big]\,\sigma\,x^2\,+\Big[18(1-\sqrt{3}\,i)\sqrt[3]{5}+12(1+\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\,\Big\}\,K_1(x^2+\sigma)$$

$$\int \frac{x^{13}\,K_1(x^2-\sigma)\,dx}{x^2-\sigma} = \frac{1}{2}\,\Big\{-x^{10}-\sigma\,x^8\,+\Big[(2-2\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}-20\Big]\,x^6-$$

$$-\Big[(2-2\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}-8\Big]\,\sigma\,x^4\,+\Big[18(1-\sqrt{3}\,i)\sqrt[3]{5}+12(1+\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\,x^2+$$

$$+6\Big[(1-\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}\Big]\,\sigma\Big\}\,K_0(x^2-\sigma)+$$

$$+\frac{1}{2}\,\Big\{-5x^8-4\sigma\,x^6\,+\Big[6(1-\sqrt{3}\,i)\sqrt[3]{5}+3(1+\sqrt{3}\,i)\sqrt[3]{25}-60\Big]\,x^4-$$

$$-2\,\Big[2(1-\sqrt{3}\,i)\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}-8\Big]\,\sigma\,x^2\,+\Big[18(1-\sqrt{3}\,i)\sqrt[3]{5}+12(1+\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\,\Big\}\,K_1(x^2-\sigma)$$

$$\nu = \sqrt{5-2\sqrt[3]{5}-\sqrt[3]{25}-\sqrt{3}\,(2\sqrt[3]{5}-\sqrt[3]{25})\,i}\;=0.35429\,92342-1.21222\,83526\,i$$

$$\int \frac{x^{13}\,J_1(x^2+\nu i)\,dx}{x^2+\nu i} = \frac{1}{2}\,\Big\{-x^{10}+\nu i x^8\,-\Big[(2+2\sqrt{3}\,i)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}-20\Big]\,x^6+$$

$$+\Big[(2(i-\sqrt{3})\sqrt[3]{5}+(\sqrt{3}+i)\sqrt[3]{25}-8i\Big]\,\nu\,x^4\,+\Big[18(1+\sqrt{3}\,i)\sqrt[3]{5}+12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\,x^2-$$

$$-6\,\Big[(i-\sqrt{3})\sqrt[3]{5}+(1+\sqrt{3}\,i)\sqrt[3]{25}\Big]\,\nu\Big\}\,J_0(x^2+\nu i)+$$

$$+\frac{1}{2}\,\Big\{5x^8-4\nu i x^6\,+\Big[(6+6\sqrt{3}\,i)\sqrt[3]{5}+3(1-\sqrt{3}\,i)\sqrt[3]{25}-60\Big]\,x^4-$$

$$-2\,\Big[2(i-\sqrt{3}\,i)\sqrt[3]{5}+(\sqrt{3}+i)\sqrt[3]{25}-8i\Big]\,\nu\,x^2\,-\Big[18(\sqrt{3}\,i+1)\sqrt[3]{5}+12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\,\Big\}\,J_1(x^2+\nu i)$$

$$\int \frac{x^{13}\,J_1(x^2-\nu i)\,dx}{x^2-\nu i} = \frac{1}{2}\,\Big\{-x^{10}-\nu i x^8\,-\Big[(2+2\sqrt{3}\,i)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}-20\Big]\,x^6-$$

$$-\Big[2(i-\sqrt{3})\sqrt[3]{5}+(\sqrt{3}+i)\sqrt[3]{25}-8i\Big]\,\nu\,x^4\,+\Big[18(1+\sqrt{3}\,i)\sqrt[3]{5}+12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\,x^2+$$

$$+6\,\Big[(i-\sqrt{3})\sqrt[3]{5}+(\sqrt{3}+i)\sqrt[3]{25}\Big]\,\nu\Big\}\,J_0(x^2-\nu i)+$$

$$+\frac{1}{2}\,\Big\{5x^8+4\nu i x^6\,+\Big[6(1+\sqrt{3}\,i)\sqrt[3]{5}-3(1-\sqrt{3}\,i)\sqrt[3]{25}-60\Big]\,x^4+$$

$$+2\,\Big[2(i-\sqrt{3})\sqrt[3]{5}+(\sqrt{3}+i)\sqrt[3]{25}-8i\Big]\,\nu\,x^2\,-\Big[18(\sqrt{3}\,i+1)\sqrt[3]{5}-12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\,\Big\}\,J_1(x^2-\nu i)$$

$$\int \frac{x^{13}\,I_1(x^2+\nu)\,dx}{x^2+\nu} = \frac{1}{2}\,\Big\{x^{10}-\nu x^8\,-\Big[(2+2\sqrt{3}\,i)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}-20\Big]\,x^6+$$

$$+\Big[(2\sqrt{3}\,i+2)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}-8\Big]\,\nu\,x^4\,-\Big[18(1+\sqrt{3}\,i)\sqrt[3]{5}+12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\,x^2+$$

$$+6\,\Big[(\sqrt{3}\,i+1)\sqrt[3]{5}+(1-\sqrt{3}\,i)\sqrt[3]{25}\Big]\,\nu\Big\}\,I_0(x^2+\nu)+$$

$$+\frac{1}{2}\,\Big\{-5x^8+4\nu\,x^6\,+\Big[(6+6\sqrt{3}\,i)\sqrt[3]{5}+3(1-\sqrt{3}\,i)\sqrt[3]{25}-60\Big]\,x^4-$$

$$-2\,\Big[2(1+\sqrt{3}\,i)\sqrt[3]{5}-(\sqrt{3}\,i-1)\sqrt[3]{25}-8\Big]\,\nu\,x^2\,+\Big[18(\sqrt{3}\,i+1)\sqrt[3]{5}+12(1-\sqrt{3}\,i)\sqrt[3]{25}-120\Big]\,\Big\}\,I_1(x^2+\nu)$$

$$\int \frac{x^{13}\, I_1(x^2-\nu)\, dx}{x^2-\nu} = \frac{1}{2}\, \Big\{x^{10}+\nu x^8\, - \Big[(2+2\sqrt{3}\, i)\sqrt[3]{5}+(1-\sqrt{3}\, i)\sqrt[3]{25}-20\Big]\; x^6 -$$

$$-\Big[(2\sqrt{3}\, i+2)\sqrt[3]{5}+(1-\sqrt{3}\, i)\sqrt[3]{25}-8\Big]\; \nu\, x^4\, - \Big[18(1+\sqrt{3}\, i)\sqrt[3]{5}+12(1-\sqrt{3}\, i)\sqrt[3]{25}-120\Big]\; x^2 -$$

$$-6\, \Big[(\sqrt{3}\, i+1)\sqrt[3]{5}+(1-\sqrt{3}\, i)\sqrt[3]{25}\Big]\; \nu\Big\}\; I_0(x^2-\nu)+$$

$$+\frac{1}{2}\, \Big\{-5x^8-4\nu\, x^6\, + \Big[6(1+\sqrt{3}\, i)\sqrt[3]{5}+3(1-\sqrt{3}\, i)\sqrt[3]{25}-60\Big]\; x^4 +$$

$$+2\, \Big[2(1+\sqrt{3}\, i)\sqrt[3]{5}+(\sqrt{3}\, i-1)\sqrt[3]{25}-8\Big]\; \nu\, x^2\, + \Big[18(\sqrt{3}\, i+1)\sqrt[3]{5}+12(1-\sqrt{3}\, i)\sqrt[3]{25}-120\Big]\Big\}\; I_1(x^2-\nu)$$

$$\int \frac{x^{13}\, K_1(x^2+\nu)\, dx}{x^2+\nu} = \frac{1}{2}\, \Big\{-x^{10}+\nu x^8\, + \Big[(2+2\sqrt{3}\, i)\sqrt[3]{5}+(1-\sqrt{3}\, i)\sqrt[3]{25}-20\Big]\; x^6 -$$

$$-\Big[(2\sqrt{3}\, i+2)\sqrt[3]{5}+(1-\sqrt{3}\, i)\sqrt[3]{25}-8\Big]\; \nu\, x^4\, + \Big[18(1+\sqrt{3}\, i)\sqrt[3]{5}+12(1-\sqrt{3}\, i)\sqrt[3]{25}-120\Big]\; x^2 -$$

$$-6\, \Big[(\sqrt{3}\, i+1)\sqrt[3]{5}+(1-\sqrt{3}\, i)\sqrt[3]{25}\Big]\; \nu\Big\}\; K_0(x^2+\nu)+$$

$$+\frac{1}{2}\, \Big\{-5x^8+4\nu\, x^6\, + \Big[6(1+\sqrt{3}\, i)\sqrt[3]{5}+3(1-\sqrt{3}\, i)\sqrt[3]{25}-60\Big]\; x^4 -$$

$$-2\, \Big[2(1+\sqrt{3}\, i)\sqrt[3]{5}-(\sqrt{3}\, i-1)\sqrt[3]{25}-8\Big]\; \nu\, x^2\, + \Big[18(\sqrt{3}\, i+1)\sqrt[3]{5}+12(1-\sqrt{3}\, i)\sqrt[3]{25}-120\Big]\Big\}\; K_1(x^2+\nu)$$

$$\int \frac{x^{13}\, K_1(x^2-\nu)\, dx}{x^2-\nu} = \frac{1}{2}\, \Big\{-x^{10}-\nu x^8\, + \Big[(2+2\sqrt{3}\, i)\sqrt[3]{5}+(1-\sqrt{3}\, i)\sqrt[3]{25}-20\Big]\; x^6 +$$

$$+\Big[(2\sqrt{3}\, i+2)\sqrt[3]{5}+(1-\sqrt{3}\, i)\sqrt[3]{25}-8\Big]\; \nu\, x^4\, + \Big[18(1+\sqrt{3}\, i)\sqrt[3]{5}+12(1-\sqrt{3}\, i)\sqrt[3]{25}-120\Big]\; x^2 +$$

$$+6\, \Big[(\sqrt{3}\, i+1)\sqrt[3]{5}+(1-\sqrt{3}\, i)\sqrt[3]{25}\Big]\; \nu\Big\}\; K_0(x^2-\nu)+$$

$$+\frac{1}{2}\, \Big\{-5x^8-4\nu\, x^6\, + \Big[6(1+\sqrt{3}\, i)\sqrt[3]{5}+3(1-\sqrt{3}\, i)\sqrt[3]{25}-60\Big]\; x^4 +$$

$$+2\, \Big[2(1+\sqrt{3}\, i)\sqrt[3]{5}-(\sqrt{3}\, i-1)\sqrt[3]{25}-8\Big]\; \nu\, x^2\, + \Big[18(\sqrt{3}\, i+1)\sqrt[3]{5}+12(1-\sqrt{3}\, i)\sqrt[3]{25}-120\Big]\Big\}\; K_1(x^2-\nu)$$

1.2. Elementary Function and Bessel Function

1.2.1. Integrals of the type $\int x^{n+1/2} \cdot Z_\nu(x)\,dx$

With the Lommel functions $s_{\mu,\nu}$ (see [7], 8.57, or [8], 10 -7) holds:

$$\int \sqrt{x}\, J_0(x)\,dx = \sqrt{x}\, J_1(x) - \frac{x}{4}\,\left[2\, s_{-1/2,1}(x)\, J_0(x) + s_{-3/2,0}(x)\, J_1(x)\right] \;,$$

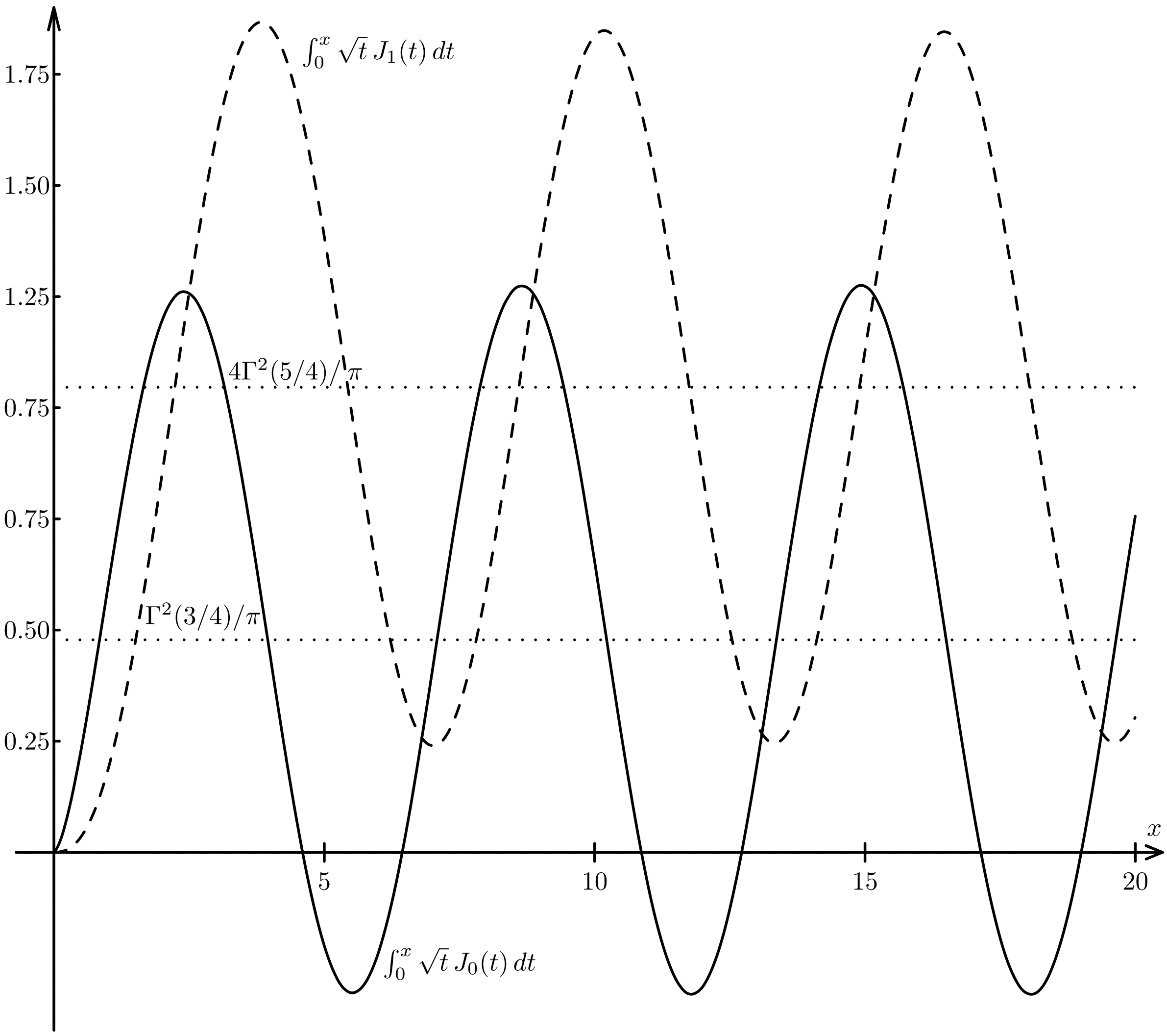
$$\int \sqrt{x}\, J_1(x)\,dx = \frac{x}{2}\,\left[s_{-1/2,0}(x)\, J_1(x) - 2\, s_{1/2,0}(x)\, J_0(x)\right].$$

$$x = t^2 \quad\Longrightarrow\quad \int x^{(2n-1)/2}\, J_\nu(x)\,dx = 2\int t^{2n}\, J_\nu(t^2)\,dt$$

Differential equations:

$$\int \sqrt{x}\, J_0(x)\,dx = y(x) \quad\Longrightarrow\quad x^2\, y''' + \left(x^2 + \frac{1}{4}\right) y' \;=\; 0$$

$$\int \sqrt{x}\, J_1(x)\,dx = z(x) \quad\Longrightarrow\quad x^2\, z''' + \left(x^2 - \frac{3}{4}\right) z' \;=\; 0$$



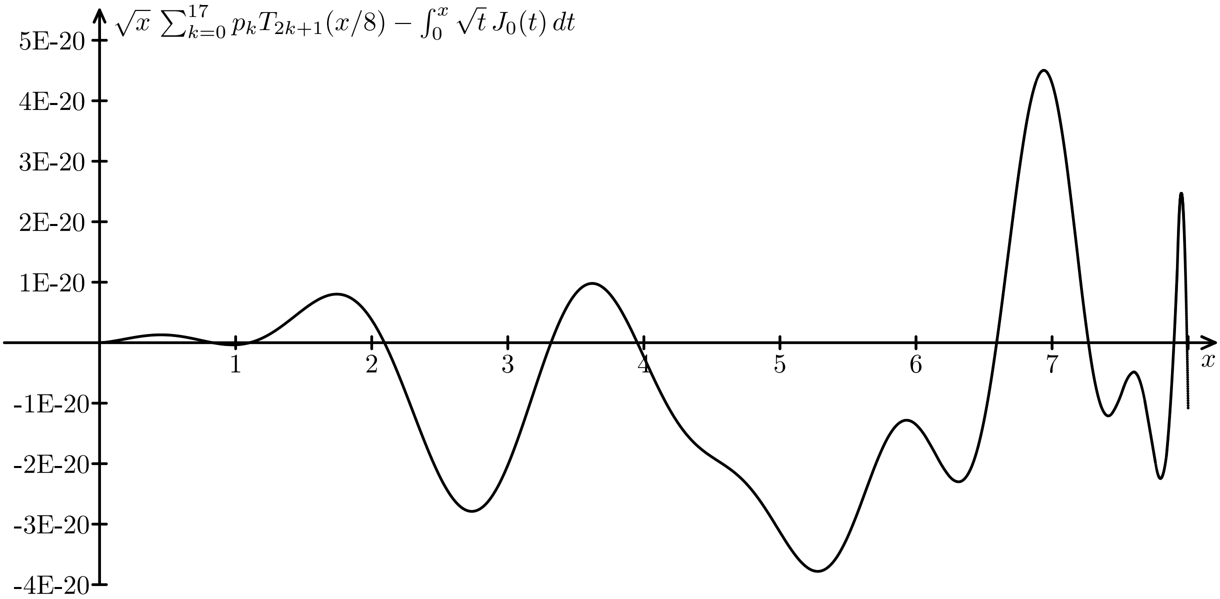
Function $J_0(x)$:
Approximation by Chebyshev polynomials, based on [2], 9.7.:
For $0 \le x \le 8$ holds

$$\int_0^x \sqrt{t}\, J_0(t)\, dt \quad \approx \quad \sqrt{x} \cdot \sum_{k=0}^{17} p_k\, T_{2k+1}\left(\frac{x}{8}\right)$$

with the following coefficients:

k	p_k	k	p_k
0	0.29396 17718 67412 06150	9	-0.00000 03762 56494 36038
1	-0.09593 33355 26137 75008	10	0.00000 00146 30781 61468
2	0.39583 39734 26816 07917	11	-0.00000 00004 68853 72762
3	-0.26902 16631 32696 96017	12	0.00000 00000 12605 05370
4	0.07963 03030 17678 07362	13	-0.00000 00000 00288 52942
5	-0.01366 63037 73087 91164	14	0.00000 00000 00005 69318
6	0.00155 43936 32776 04348	15	-0.00000 00000 00000 09787
7	-0.00012 67196 60682 08202	16	0.00000 00000 00000 00148
8	0.00000 77998 70507 77089	17	-0.00000 00000 00000 00002

This approximation differs from the true function as shown:



Asymptotic expansions for $x \to +\infty$:

$$\int_0^x \sqrt{t}\, J_0(t)\, dt \;\sim\; \frac{\Gamma^2(3/4)}{\pi} + \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{a_k}{x^k}\, \sin\left(x - \frac{2k+1}{4}\pi\right)$$

$$\frac{\Gamma^2(3/4)}{\pi} = 0.477\,988\,797\,486\,125$$

$$a_0 = 1\,, \quad a_1 = \frac{1}{8}\,, \quad a_2 = \frac{25}{128}\,, \quad a_3 = \frac{475}{1024}\,, \quad a_4 = \frac{49275}{32768}\,, \quad a_5 = \frac{1636335}{262144}\,, \quad a_6 = \frac{133308045}{4194304}\,,$$

$$a_7 = \frac{6456759075}{33554432}\,, \quad a_8 = \frac{2905671971475}{2147483648}\,, \quad a_9 = \frac{186381860485275}{17179869184}\,, \ldots$$

k	a_k	a_k/a_{k-1}	k	a_k	a_k/a_{k-1}
0	1.000 000 000	-	5	6.242 122 650	4.1510
1	0.125 000 000	0.1250	6	31.783 114 67	5.0917
2	0.195 312 500	1.5625	7	192.426 415 5	6.0544
3	0.463 867 188	2.3750	8	1 353.058 951	7.0316
4	1.503 753 662	3.2418	9	10 848.852 14	8.0180

Let

$$D_{0,n}(x) = \frac{\Gamma^2(3/4)}{\pi} + \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n} \frac{a_k}{x^k} \sin\left(x - \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t}\, J_0(t)\, dt \,,$$

then its first maximum and minimum values of interest are $D_{0,n}(x^*_{i,n})$.
In the case $x > x^*_{i,n}$ holds $|D_{0,n}(x)| < |D_{0,n}(x^*_{i,n})|$.

$n=0,\ i=$	1	2	3	4	5	6	7	8	9	10
$x^*_{i,0}$	1.143	4.058	7.146	10.264	13.394	16.527	19.664	22.801	25.940	29.079
$10^3 D_{0,0}(x^*_i)$	50.87	-21.784	13.293	-9.4707	7.3310	-5.9717	5.0342	-4.3496	3.8281	-3.4179

$n=1,\ i=$	1	2	3	4	5	6	7	8	9	10
$x^*_{i,1}$	2.473	5.561	8.681	11.812	14.947	18.085	21.223	24.363	27.503	30.643
$10^4 D_{0,1}(x^*_i)$	158.930	-43.0361	19.1585	-10.6855	6.7779	-4.6705	3.4092	-2.5962	2.0420	-1.6478

$n=2,\ i=$	3	4	5	6	7	8	9	10	11	12
$x^*_{i,2}$	7.100	10.233	13.369	16.508	19.647	22.787	25.927	29.068	32.209	35.350
$10^5 D_{0,2}(x^*_i)$	-85.8809	31.2037	-14.5367	7.8808	-4.7310	3.0556	-2.0851	1.4850	-1.0945	0.8296

$n=3,\ i=$	4	5	6	7	8	9	10	11	12	13
$x^*_{i,3}$	11.793	14.932	18.072	21.213	24.353	27.494	30.635	33.777	36.918	40.059
$10^7 D_{0,3}(x^*_i)$	548.133	-222.407	106.176	-56.7759	33.0053	-20.4558	13.3362	-9.0584	6.3648	-4.6012

$n=4,\ i=$	5	6	7	8	9	10	11	12	13	14
$x^*_{i,4}$	16.499	19.649	22.780	25.922	29.063	32.204	35.345	38.487	41.628	44.770
$10^8 D_{0,4}(x^*_i)$	-369.198	158.653	-76.8752	40.7761	-23.2093	13.9787	-8.8181	5.7815	-3.9164	2.7284

For $8 \le x \le 30$ the special approximation holds:

$$\int_0^x \sqrt{t}\, J_0(t)\, dt \;\approx\; 0.477\,988\,797\,935 + \sum_{k=0}^{9} \frac{c_k^{(0)}}{x^k} \sin\left(x - \frac{2k+1}{4}\pi\right)$$

with

$$-5.0 \cdot 10^{-9} < 0.477\,988\,797\,935 + \sum_{k=0}^{9} \frac{c_k^{(0)}}{x^k} \sin\left(x - \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t}\, J_0(t)\, dt < 6 \cdot 10^{-9} \,.$$

k	$c_k^{(0)}$	k	$c_k^{(0)}$
1	0.797 884 516 538	6	4.733 533 047 132
2	0.099 735 119 074	7	18.859 909 855 69
3	0.155 775 947 720	8	99.846 038 227 04
4	0.369 550 899 387	9	256.775 583 671 0
5	1.170 795 416 963	10	1 527.508 571 668

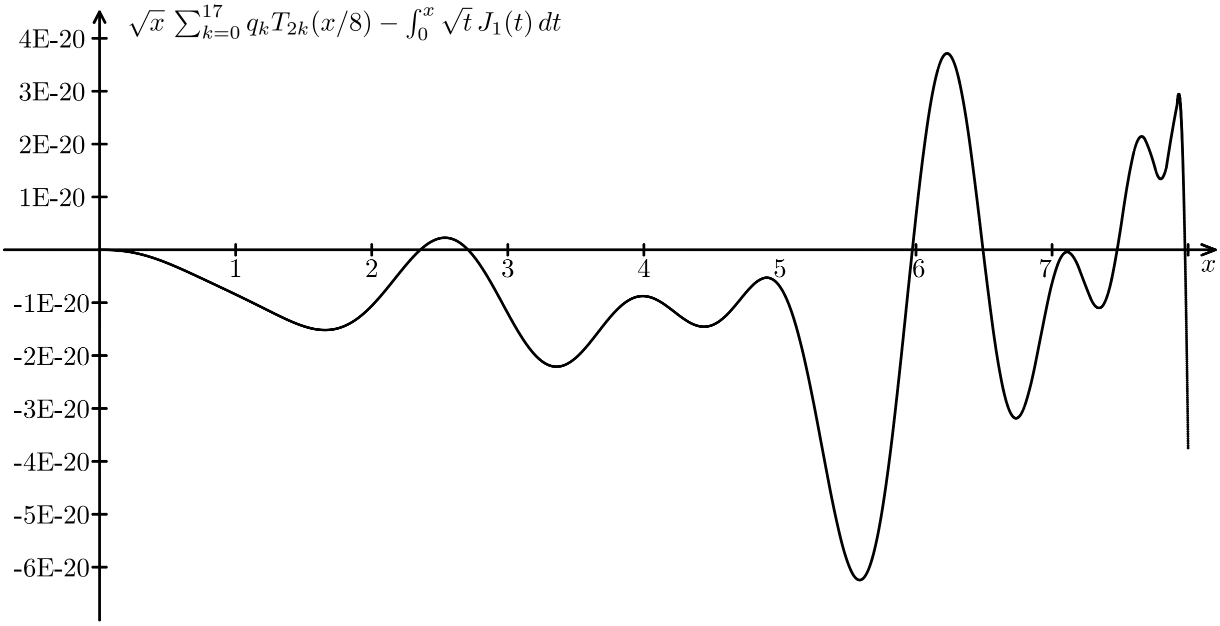
Function $J_1(x)$:
Approximation by Chebyshev polynomials, based on [2], 9.7.:

$$\int_0^x \sqrt{t}\, J_1(t)\, dt \quad \approx \quad \sqrt{x} \cdot \sum_{k=0}^{17} q_k\, T_{2k}\left(\frac{x}{8}\right) , \qquad 0 \le x \le 8$$

with the coefficients:

k	q_k	k	q_k
0	0.37975 25427 04720 47384	9	0.00000 17101 75937 74175
1	-0.24153 71053 32677 35417	10	-0.00000 00740 94524 46089
2	-0.12554 99442 21699 83184	11	0.00000 00026 16214 44822
3	0.31360 55017 12763 75964	12	-0.00000 00000 76805 72461
4	-0.14432 03488 73845 84716	13	0.00000 00000 01905 60154
5	0.03274 98779 87894 78550	14	-0.00000 00000 00040 50286
6	-0.00458 83639 64558 05653	15	0.00000 00000 00000 74601
7	0.00044 24559 29648 31876	16	-0.00000 00000 00000 01203
8	-0.00003 13683 81557 99050	17	0.00000 00000 00000 00017

Difference between approximation and true function:



Asymptotic expansion for $x \to \infty$:

$$\int_0^x \sqrt{t}\, J_1(t)\, dt \sim \frac{4\Gamma^2(5/4)}{\pi} - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{b_k}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right)$$

$$\frac{4\Gamma^2(5/4)}{\pi} = 1.046\,049\,620\,053\,102$$

$$b_0 = 1\,, \quad b_1 = \frac{3}{8}\,, \quad b_2 = -\frac{63}{128}\,, \quad b_3 = \frac{1113}{1024}\,, \quad b_4 = -\frac{111573}{32768}\,, \quad b_5 = \frac{3643101}{262144}\,, \quad b_6 = -\frac{294285915}{4194304}\,,$$

$$b_7 = \frac{14192615745}{33554432}\,, \quad b_8 = -\frac{6373074947085}{2147483648}\,, \quad b_9 = \frac{408344927902065}{17179869184}\,, \ \ldots$$

k	b_k	$\lvert b_k/b_{k-1}\rvert$	k	b_k	$\lvert b_k/b_{k-1}\rvert$
0	1.000 000 000	-	5	13.897 327 42	4.0815
1	0.375 000 000	0.3750	6	-70.163 229 70	5.0487
2	-0.492 187 500	1.3125	7	422.972 910 0	6.0284
3	1.086 914 063	2.2083	8	-2 967.694 284	7.0163
4	-3.404 937 744	3.1327	9	23 768.803 10	8.0092

Let

$$D_{1,n}(x) = \frac{4\Gamma^2(5/4)}{\pi} - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n} \frac{b_k}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t}\, J_1(t)\, dt\,,$$

then its first maximum and minimum values of interest are $D_{1,n}(x^*_{i,n})$.
In the case $x > x^*_{i,n}$ holds $|D_{1,n}(x)| < |D_{1,n}(x^*_{i,n})|$.

$n=0,\ i=$	1	2	3	4	5	6	7	8	9	10
$x^*_{i,0}$	2.470	6.470	8.675	11.807	14.943	18.081	21.220	24.360	27.500	30.641
$10^3 D_{1,0}(x^*_i)$	-98.4511	50.5537	-33.4821	24.9168	-19.8073	16.4242	-14.0227	12.2312	-10.8443	9.7390

$n=1,\ i=$	2	3	4	5	6	7	8	9	10	11
$x^*_{i,1}$	3.974	7.097	10.230	13.367	15.506	16.645	22.786	25.926	29.067	32.208
$10^4 D_{1,1}(x^*_i)$	-194.456	70.4454	-35.5457	21.2582	-14.0945	10.0128	-7.4731	5.7878	-4.6133	3.7625

$n=2,\ i=$	3	4	5	6	7	8	9	10	11	12
$x^*_{i,2}$	8.654	12.654	14.931	18.071	21.212	24.353	27.494	30.635	33.776	36.917
$10^5 D_{1,2}(x^*_i)$	117.286	-48.9485	24.7672	-14.1826	8.8517	-5.8853	4.1071	-2.9778	2.2269	-1.7083

$n=3,\ i=$	4	5	6	7	8	9	10	11	12	13
$x^*_{i,3}$	13.358	16.498	19.639	22.780	29.921	29.063	32.204	35.345	38.487	41.628
$10^7 D_{0,3}(x^*_i)$	-773.343	342.802	-173.926	97.2278	-58.4641	37.2099	-24.7842	17.1337	-12.2177	8.9436

$n=4,\ i=$	4	5	6	7	8	9	10	11	12	13
$x^*_{i,4}$	11.785	14.926	18.067	21.208	24.349	27.491	30.632	33.774	36.915	40.057
$10^7 D_{1,4}(x^*_i)$	409.325	-133.167	53.0102	-24.2927	12.3500	-6.7990	3.9863	-2.4597	1.5831	-1.0557

For $8 \le x \le 30$ holds

$$\int_0^x \sqrt{t}\, J_1(t)\, dt \;\approx\; 1.046\,049046\,618046\,299 + \sum_{k=0}^{9} \frac{c_k^{(1)}}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right)$$

with

$$-7.0 \cdot 10^{-9} < 1.046\,049\,046\,618 + \sum_{k=0}^{9} \frac{c_k^{(1)}}{x^k} \sin\left(x + \frac{2k+1}{4}\pi\right) - \int_0^x \sqrt{t}\, J_0(t)\, dt < 5 \cdot 10^{-9}\,.$$

k	$c_k^{(1)}$	k	$c_k^{(1)}$
1	-0.797 884 661 354	6	-10.914 764 692 58
2	-0.299 206 773 536	7	40.086 327 439 47
3	0.392 563 542 201	8	-264.350 611 778 9
4	-0.867 123 732 390	9	464.583 909 606 6
5	2.645 254 609 577	10	-5 043.243 969 567

Modified Bessel Functions:

$$\int_0^x \sqrt{t}\, I_0(t)\, dt \;=\; 2\sqrt{x} \sum_{k=0}^{\infty} \frac{x^{2n+1}}{4^k \cdot (k!)^2 \cdot (4k+3)} \;=$$

$$= 2\sqrt{x}\left(\frac{x}{3} + \frac{x^3}{28} + \frac{x^5}{704} + \frac{x^7}{34\,560} + \frac{x^9}{2\,801\,664} + \frac{x^{11}}{339\,148\,800} + \frac{x^{13}}{57\,330\,892\,800} + \frac{x^{15}}{12\,901\,574\,246\,400} + \ldots\right)$$

$$\int_0^x \sqrt{t}\, I_1(t)\, dt \;=\; \sqrt{x} \sum_{k=0}^{\infty} \frac{x^{2n+2}}{4^k \cdot (k!)^2 \cdot (4k+5) \cdot (k+1)} \;=$$

$$= \sqrt{x}\left(\frac{x^2}{5} + \frac{x^4}{72} + \frac{x^6}{2\,496} + \frac{x^8}{156\,672} + \frac{x^{10}}{15\,482\,880} + \frac{x^{12}}{2\,211\,840\,000} + \frac{x^{14}}{431\,043\,379\,200} + \ldots\right)$$

1.2.1. b) Integrals:

$$\int x^{3/2}\,J_0(x)\,dx = x^{3/2}J_1(x) - \frac{1}{2}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{3/2}\,J_1(x)\,dx = -x^{3/2}J_0(x) + \frac{3}{2}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{3/2}\,I_0(x)\,dx = x^{3/2}I_1(x) - \frac{1}{2}\int \sqrt{x}\,I_1(x)\,dx$$

$$\int x^{3/2}\,I_1(x)\,dx = x^{3/2}I_0(x) - \frac{3}{2}\int \sqrt{x}\,I_0(x)\,dx$$

$$\int x^{5/2}\,J_0(x)\,dx = \sqrt{x}\left[\frac{3x}{2}J_0(x) + x^2J_1(x)\right] - \frac{9}{4}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{5/2}\,J_1(x)\,dx = \sqrt{x}\left[-x^2J_0(x) + \frac{5}{2}xJ_1(x)\right] - \frac{5}{4}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{5/2}\,I_0(x)\,dx = \sqrt{x}\left[-\frac{3x}{2}I_0(x) + x^2I_1(x)\right] + \frac{9}{4}\int \sqrt{x}\,I_0(x)\,dx$$

$$\int x^{5/2}\,I_1(x)\,dx = \sqrt{x}\left[x^2J_0(x) - \frac{5}{2}xI_1(x)\right] + \frac{5}{4}\int \sqrt{x}\,I_1(x)\,dx$$

$$\int x^{7/2}\,J_0(x)\,dx = \sqrt{x}\left[\frac{5x^2}{2}J_0(x) + \left(x^3 - \frac{25}{4}\,x\right)J_1(x)\right] + \frac{25}{8}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{7/2}\,J_1(x)\,dx = \sqrt{x}\left[\left(-x^3 + \frac{21}{4}\,x\right)J_0(x) + \frac{7}{2}\,x^2J_1(x)\right] - \frac{63}{8}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{7/2}\,I_0(x)\,dx = \sqrt{x}\left[-\frac{5x^2}{2}I_0(x) + \left(x^3 + \frac{25}{4}\,x\right)I_1(x)\right] - \frac{25}{8}\int \sqrt{x}\,I_1(x)\,dx$$

$$\int x^{7/2}\,I_1(x)\,dx = \sqrt{x}\left[\left(x^3 + \frac{21}{4}\,x\right)I_0(x) - \frac{7}{2}\,x^2I_1(x)\right] - \frac{63}{8}\int \sqrt{x}\,I_0(x)\,dx$$

$$\int x^{9/2}\,J_0(x)\,dx = \sqrt{x}\left[\left(\frac{7}{2}\,x^3 - \frac{147}{8}\,x\right)J_0(x) + \left(x^4 - \frac{49}{4}\,x^2\right)J_1(x)\right] + \frac{441}{16}\int \sqrt{x}\,J_0(x)\,dx$$

$$\int x^{9/2}\,J_1(x)\,dx = \sqrt{x}\left[\left(-x^4 + \frac{45}{4}\,x^2\right)J_0(x) + \left(\frac{9}{2}\,x^3 - \frac{225}{8}\,x\right)J_1(x)\right] + \frac{225}{16}\int \sqrt{x}\,J_1(x)\,dx$$

$$\int x^{9/2}\,I_0(x)\,dx = \sqrt{x}\left[-\left(\frac{7}{2}\,x^3 + \frac{147}{8}\,x\right)I_0(x) + \left(x^4 + \frac{49}{4}\,x^2\right)I_1(x)\right] + \frac{441}{16}\int \sqrt{x}\,I_0(x)\,dx$$

$$\int x^{9/2}\,I_1(x)\,dx = \sqrt{x}\left[\left(x^4 + \frac{45}{4}\,x^2\right)I_0(x) - \left(\frac{9}{2}\,x^3 + \frac{225}{8}\,x\right)I_1(x)\right] + \frac{225}{16}\int \sqrt{x}\,I_1(x)\,dx$$

To find $\int x^{(2n+1)/2}\,Z_\nu(x)\,dx$ with $n > 4$ use the recurrence formulas (see page 64).

$$\int \frac{J_0(x)}{\sqrt{x}}\,dx = 2\sqrt{x}J_0(x) + 2\int \sqrt{x}J_1(x)\,dx$$

$$\int \frac{J_1(x)}{\sqrt{x}}\,dx = -2\sqrt{x}J_1(x) + 2\int \sqrt{x}J_0(x)\,dx$$

$$\int \frac{I_0(x)}{\sqrt{x}}\,dx = 2\sqrt{x}I_0(x) - 2\int \sqrt{x}J_1(x)\,dx$$

$$\int \frac{I_1(x)}{\sqrt{x}}\,dx = -2\sqrt{x}I_1(x) + 2\int \sqrt{x}I_0(x)\,dx$$

$$\int x^{-3/2}\cdot J_0(x)\,dx = \frac{\sqrt{x}}{x}[-2J_0(x) + 4xJ_1(x)] - 4\int \sqrt{x}J_0(x)\,dx$$

$$\int x^{-3/2} \cdot J_1(x)\,dx = \frac{\sqrt{x}}{3x}[4xJ_0(x) - 2J_1(x)] + \frac{4}{3}\int \sqrt{x}J_1(x)\,dx$$

$$\int x^{-3/2} \cdot I_0(x)\,dx = -\frac{\sqrt{x}}{x}[2I_0(x) + 4xI_1(x)] + 4\int \sqrt{x}I_0(x)\,dx$$

$$\int x^{-3/2} \cdot I_1(x)\,dx = \frac{\sqrt{x}}{3x}[4xI_0(x) - 2I_1(x)] - \frac{4}{3}\int \sqrt{x}I_1(x)\,dx$$

$$\int x^{-5/2} \cdot J_0(x)\,dx = \frac{\sqrt{x}}{9x^2}\left[\left(-8x^2 - 6\right)J_0(x) + 4xJ_1(x)\right] - \frac{8}{9}\int \sqrt{x}J_1(x)\,dx$$

$$\int x^{-5/2} \cdot J_1(x)\,dx = \frac{\sqrt{x}}{5x^2}[-4xJ_0(x) + (8x^2 - 2)J_1(x)] - \frac{8}{5}\int \sqrt{x}J_0(x)\,dx$$

$$\int x^{-5/2} \cdot I_0(x)\,dx = \frac{\sqrt{x}}{9x^2}\left[\left(8x^2 - 6\right)I_0(x) - 4xI_1(x)\right] - \frac{8}{9}\int \sqrt{x}I_1(x)\,dx$$

$$\int x^{-5/2} \cdot I_1(x)\,dx = \frac{\sqrt{x}}{5x^2}[-4xI_0(x) - (8x^2 + 2)I_1(x)] + \frac{8}{5}\int \sqrt{x}I_0(x)\,dx$$

$$\int x^{-7/2} \cdot J_0(x)\,dx = \frac{\sqrt{x}}{25\,x^3}[(8\,x^2 - 10)J_0(x) + (-16\,x^3 + 4\,x)J_1(x)] + \frac{16}{25}\int \sqrt{x}J_0(x)\,dx$$

$$\int x^{-7/2} \cdot J_1(x)\,dx = \frac{\sqrt{x}}{63\,x^3}[(-16\,x^3 - 12\,x)J_0(x) + (8\,x^2 - 18)J_1(x)] - \frac{16}{63}\int \sqrt{x}J_1(x)\,dx$$

$$\int x^{-7/2} \cdot I_0(x)\,dx = -\frac{\sqrt{x}}{25\,x^3}[(8\,x^2 + 10)I_0(x) + (16\,x^3 + 4\,x)I_1(x)] + \frac{16}{25}\int \sqrt{x}I_0(x)\,dx$$

$$\int x^{-7/2} \cdot I_1(x)\,dx = \frac{\sqrt{x}}{63\,x^3}[(16\,x^3 - 12\,x)I_0(x) - (8\,x^2 + 18)I_1(x)] - \frac{16}{63}\int \sqrt{x}I_1(x)\,dx$$

To find $\int x^{-(2n+1)/2}\,Z_\nu(x)\,dx$ with $n > 4$ use the recurrence formulas.

1.2.1. c) Recurrence Formulas:

$$\int x^{n+5/2}\,J_0(x)\,dx = x^{n+3/2}\left[\left(n + \frac{3}{2}\right)J_0(x) + x\,J_1(x)\right] - \left(n + \frac{3}{2}\right)^2 \int x^{n+1/2}\,J_0(x)\,dx$$

$$\int x^{n+5/2}\,J_1(x)\,dx = x^{n+3/2}\left[\left(n + \frac{5}{2}\right)J_1(x) - x\,J_0(x)\right] - \frac{(2n+1)(2n+5)}{4}\int x^{n+1/2}\,J_1(x)\,dx$$

$$\int x^{n+5/2}\,I_0(x)\,dx = x^{n+3/2}\left[x\,I_1(x) - \left(n + \frac{3}{2}\right)I_0(x)\right] + \left(n + \frac{3}{2}\right)^2 \int x^{n+1/2}\,I_0(x)\,dx$$

$$\int x^{n+5/2}\,I_1(x)\,dx = x^{n+3/2}\left[x\,I_0(x) - \left(n + \frac{5}{2}\right)I_1(x)\right] + \frac{(2n+1)(2n+5)}{4}\int x^{n+1/2}\,I_1(x)\,dx$$

1.2.2. Integrals of the type $\int x^n\, e^{\pm x} \cdot \left\{ \begin{array}{c} I_\nu(x) \\ K_\nu x \end{array} \right\} dx$

See also [1], 11.3.

a) Integrals with e^x :

$$\int e^x\, I_0(x)\, dx \;=\; xe^x\, [I_0(x) - I_1(x)] \;, \qquad \int \frac{e^x \cdot I_1(x)\, dx}{x} = e^x\, [I_0(x) - I_1(x)]$$

$$\int e^x\, K_0(x)\, dx = xe^x\, [K_0(x) + K_1(x)] \;, \qquad \int \frac{e^x\, K_1(x)\, dx}{x} = -e^x\, [K_0(x) + K_1(x)]$$

$$\int e^x\, I_1(x)\, dx \;=\; e^x\, [(1-x) I_0(x) + x I_1(x)]$$

$$\int e^x\, K_1(x)\, dx = -e^x\, [(1-x) K_0(x) - x K_1(x)]$$

$$\int xe^x\, I_0(x)\, dx \;=\; \frac{xe^x}{3}\, [x I_0(x) + (1-x) I_1(x)]$$

$$\int x\, e^x\, K_0(x)\, dx = \frac{xe^x}{3}\; [x\, K_0(x) + (x-1)\, K_1(x)]$$

$$\int x\, e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{3}\, [-x I_0(x) + (2+x) I_1(x)]$$

$$\int x\, e^x\, K_1(x)\, dx = \frac{xe^x}{3}\; [x\, K_0(x) + (x+2)\, K_1(x)]$$

$$\int x^2 e^x\, I_0(x)\, dx \;=\; \frac{xe^x}{15}\, \left[(2\,x + 3\,x^2) I_0(x) + (-4 + 4\,x - 3\,x^2) I_1(x)\right]$$

$$\int x^2\, e^x\, K_0(x)\, dx = \frac{xe^x}{15}\; \left[(3x^2 + 2x)\, K_0(x) + (3\,x^2 - 4\,x + 4)\, K_1(x)\right]$$

$$\int x^2 e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{5}\, \left[(x - x^2) I_0(x) + (-2 + 2x + x^2) I_1(x)\right]$$

$$\int x^2\, e^x\, K_1(x)\, dx = \frac{xe^x}{5}\; \left[(x^2 - x)\, K_0(x) + (x^2 + 2x - 2)\, K_1(x)\right]$$

$$\int x^3 e^x\, I_0(x)\, dx \;=\; \frac{xe^x}{35}\, \left[(-6\,x + 6\,x^2 + 5\,x^3) I_0(x) + (12 - 12\,x + 9\,x^2 - 5\,x^3) I_1(x)\right]$$

$$\int x^3\, e^x\, K_0(x)\, dx = \frac{xe^x}{35}\; \left[(5\,x^3 + 6\,x^2 - 6\,x)\, K_0(x) + (5\,x^3 - 9\,x^2 + 12\,x - 12)\, K_1(x)\right]$$

$$\int x^3 e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{35}\, \left[(-8\,x + 8\,x^2 - 5\,x^3) I_0(x) + (16 - 16\,x + 12\,x^2 + 5\,x^3) I_1(x)\right]$$

$$\int x^3\, e^x\, K_1(x)\, dx = \frac{xe^x}{35}\; \left[(5\,x^3 - 8\,x^2 + 8\,x)\, K_0(x) + (5\,x^3 + 12\,x^2 - 16\,x + 16)\, K_1(x)\right]$$

$$\int x^4 e^x\, I_0(x)\, dx \;=\; \frac{xe^x}{315}\, \left[(96\,x - 96\,x^2 + 60\,x^3 + 35\,x^4) I_0(x) + (-192 + 192\,x - 144\,x^2 + 80\,x^3 - 35\,x^4) I_1(x)\right]$$

$$\int x^4\, e^x\, K_0(x)\, dx = \frac{xe^x}{315}\; \left[(35\,x^4 + 60\,x^3 - 96\,x^2 + 96\,x)\, K_0(x) + (35\,x^4 - 80\,x^3 + 144\,x^2 - 192\,x + 192)\, K_1(x)\right]$$

$$\int x^4 e^x\, I_1(x)\, dx \;=\; \frac{xe^x}{63}\, \left[(24\,x - 24\,x^2 + 15\,x^3 - 7\,x^4) I_0(x) + (-48 + 48\,x - 36\,x^2 + 20\,x^3 + 7\,x^4) I_1(x)\right]$$

$$\int x^4\, e^x\, K_1(x)\, dx = \frac{xe^x}{63}\; \left[(7\,x^4 - 15\,x^3 + 24\,x^2 - 24\,x)\, K_0(x) + (7\,x^4 + 20\,x^3 - 36\,x^2 + 48\,x - 48)\, K_1(x)\right]$$

$$\int x^5 e^x\, I_0(x)\, dx \;=\; \frac{xe^x}{693}\, \left[(-480\,x + 480\,x^2 - 300\,x^3 + 140\,x^4 + 63\,x^5) I_0(x) + \right.$$
$$\left. +(960 - 960\,x + 720\,x^2 - 400\,x^3 + 175\,x^4 - 63\,x^5) I_1(x)\right]$$

$$\int x^5\, e^x\, K_0(x)\, dx = \frac{xe^x}{693}\, \big[(63\, x^5 + 140\, x^4 - 300\, x^3 + 480\, x^2 - 480\, x)\, K_0(x) +$$
$$+ (63\, x^5 - 175\, x^4 + 400\, x^3 - 720\, x^2 + 960\, x - 960)\, K_1(x)\big]$$

$$\int x^5 e^x\, I_1(x)\, dx \ = \ \frac{xe^x}{231}\, \big[(-192\, x + 192\, x^2 - 120\, x^3 + 56\, x^4 - 21\, x^5) I_0(x) +$$
$$+(384 - 384\, x + 288\, x^2 - 160\, x^3 + 70\, x^4 + 21\, x^5) I_1(x)\big]$$

$$\int x^5\, e^x\, K_1(x)\, dx = \frac{xe^x}{231}\, \big[(21\, x^5 - 56\, x^4 + 120\, x^3 - 192\, x^2 + 192\, x)\, K_0(x) +$$
$$+(21\, x^5 + 70\, x^4 - 160\, x^3 + 288\, x^2 - 384\, x + 384)\, K_1(x)\big]$$

$$\int x^6 e^x\, I_0(x)\, dx \ = \ \frac{xe^x}{1001}\, \big[(1920\, x - 1920\, x^2 + 1200\, x^3 - 560\, x^4 + 210 x^5 + 77 x^6) I_0(x) +$$
$$+ (-3840 + 3840\, x - 2880\, x^2 + 1600\, x^3 - 700\, x^4 + 252 x^5 - 77 x^6) I_1(x)\big]$$

$$\int x^6\, e^x\, K_0(x)\, dx = \frac{xe^x}{1001}\, \big[(77\, x^6 + 210\, x^5 - 560\, x^4 + 1200\, x^3 - 1920\, x^2 + 1920\, x)\, K_0(x) +$$
$$+ (77\, x^6 - 252\, x^5 + 700\, x^4 - 1600\, x^3 + 2880\, x^2 - 3840\, x + 3840)\, K_1(x)\big]$$

$$\int x^6 e^x\, I_1(x)\, dx \ = \ \frac{xe^x}{429}\, \big[(960\, x - 960\, x^2 + 600\, x^3 - 280\, x^4 + 105\, x^5 - 33\, x^6) I_0(x) +$$
$$+(-1920 + 1920\, x - 1440\, x^2 + 800\, x^3 - 350\, x^4 + 126\, x^5 + 33\, x^6) I_1(x)\big]$$

$$\int x^6\, e^x\, K_1(x)\, dx = \frac{xe^x}{429}\, \big[(33\, x^6 - 105\, x^5 + 280\, x^4 - 600\, x^3 + 960\, x^2 - 960\, x)\, K_0(x) +$$
$$+(33\, x^6 + 126\, x^5 - 350\, x^4 + 800\, x^3 - 1440\, x^2 + 1920\, x - 1920)\, K_1(x)\big]$$

$$\int x^7 e^x\, I_0(x)\, dx \ = \ \frac{xe^x}{2145}\, \big[(-13440\, x + 13440\, x^2 - 8400\, x^3 + 3920\, x^4 - 1470\, x^5 + 462\, x^6 + 143\, x^7) I_0(x) +$$
$$+(26880 - 26880\, x + 20160\, x^2 - 11200\, x^3 + 4900\, x^4 - 1764\, x^5 + 539\, x^6 - 143\, x^7) I_1(x)\big]$$

$$\int x^7\, e^x\, K_0(x)\, dx = \frac{xe^x}{2145}\, \big[(143\, x^7 + 462\, x^6 - 1470\, x^5 + 3920\, x^4 - 8400\, x^3 + 13440\, x^2 - 13440\, x)\, K_0(x) +$$
$$+(143\, x^7 - 539\, x^6 + 1764\, x^5 - 4900\, x^4 + 11200\, x^3 - 20160\, x^2 + 26880\, x - 26880)\, K_1(x)\big]$$

$$\int x^7 e^x\, I_1(x)\, dx \ = \ \frac{xe^x}{2145}\, \big[(-15360\, x + 15360\, x^2 - 9600\, x^3 + 4480\, x^4 - 1680\, x^5 + 528\, x^6 - 143\, x^7) I_0(x) +$$
$$+(30720 - 30720\, x + 23040\, x^2 - 12800\, x^3 + 5600\, x^4 - 2016\, x^5 + 616\, x^6 + 143\, x^7) I_1(x)\big]$$

$$\int x^7\, e^x\, K_1(x)\, dx = \frac{xe^x}{2145}\, \big[(143\, x^7 - 528\, x^6 + 1680\, x^5 - 4480\, x^4 + 9600\, x^3 - 15360\, x^2 + 15360\, x)\, K_0(x) +$$
$$+(143\, x^7 + 616\, x^6 - 2016\, x^5 + 5600\, x^4 - 12800\, x^3 + 23040\, x^2 - 30720\, x + 30720)\, K_1(x)\big]$$

Recurrence formulas:

$$\int x^n e^x I_0(x)\, dx = \frac{x^n e^x}{2n+1}[(n+x) I_0(x) - x I_1(x)] - \frac{n^2}{2n+1} \int x^{n-1} e^x I_0(x)\, dx \quad (*)$$

$$\int x^n e^x I_1(x)\, dx = \frac{x^n e^x}{2n+1}[(n+1-x) I_0(x) + x I_1(x)] - \frac{n(n+1)}{2n+1} \int x^{n-1} e^x I_0(x)\, dx \quad (*)$$

The last formula refers to $I_0(x)$ instead of $I_1(x)$.

$$\int x^n e^x K_0(x)\, dx = \frac{x^n e^x}{2n+1}[(n+x) K_0(x) + x K_1(x)] - \frac{n^2}{2n+1} \int x^{n-1} e^x K_0(x)\, dx$$

$$\int x^n e^x K_1(x)\, dx = \frac{x^n e^x}{2n+1}[(x-n-1) K_0(x) + x K_1(x)] + \frac{n(n+1)}{2n+1} \int x^{n-1} e^x K_0(x)\, dx$$

The last formula refers to $K_0(x)$ instead of $K_1(x)$.

b) Integrals with e^{-x} :

$$\int e^{-x}\, I_0(x)\, dx \;=\; xe^{-x}\,[I_0(x)+I_1(x)]\;,\qquad \int \frac{e^{-x}\cdot I_1(x)\,dx}{x} = e^{-x}\,[I_0(x)+I_1(x)]$$

$$\int e^{-x}\, K_0(x)\, dx = xe^{-x}\,[K_0(x)-K_1(x)]\;,\qquad \int \frac{e^{-x}\,K_1(x)\,dx}{x} = e^{-x}\,[K_0(x)-K_1(x)]$$

$$\int e^{-x}\, I_1(x)\, dx \;=\; e^{-x}\,[(1+x)I_0(x)+xI_1(x)]$$

$$\int e^{-x}\, K_1(x)\, dx = -e^{-x}\,[(1+x)\,K_0(x)-x\,K_1(x)]$$

$$\int xe^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{3}\,[xI_0(x)+(1+x)I_1(x)]$$

$$\int x\,e^{-x}\, K_0(x)\, dx = \frac{xe^{-x}}{3}\;[x\,K_0(x)-(x+1)\,K_1(x)]$$

$$\int xe^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{3}\,[xI_0(x)+(-2+x)I_1(x)]$$

$$\int x\,e^{-x}\, K_1(x)\, dx = \frac{xe^{-x}}{3}\;[-x\,K_0(x)+(x-2)\,K_1(x)]$$

$$\int x^2e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{15}\,\left[(-2\,x+3\,x^2)I_0(x)+(4+4\,x+3\,x^2)I_1(x)\right]$$

$$\int x^2\,e^{-x}\, K_0(x)\, dx = \frac{xe^{-x}}{15}\;\left[(3\,x^2-2\,x)\,K_0(x)-(3\,x^2+4\,x+4)\,K_1(x)\right]$$

$$\int x^2e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{5}\,\left[(x+x^2)I_0(x)+(-2-2x+x^2)I_1(x)\right]$$

$$\int x^2\,e^{-x}\, K_1(x)\, dx = \frac{xe^{-x}}{5}\;\left[-(x^2+x)\,K_0(x)+(x^2-2\,x-2)\,K_1(x)\right]$$

$$\int x^3e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{35}\,\left[(-6\,x-6\,x^2+5\,x^3)I_0(x)+(12+12\,x+9\,x^2+5\,x^3)I_1(x)\right]$$

$$\int x^3\,e^{-x}\, K_0(x)\, dx = \frac{xe^{-x}}{35}\;\left[(5\,x^3-6\,x^2-6\,x)\,K_0(x)-(5\,x^3+9\,x^2+12\,x+12)\,K_1(x)\right]$$

$$\int x^3e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{35}\,\left[(8\,x+8\,x^2+5\,x^3)I_0(x)+(-16-16\,x-12\,x^2+5\,x^3)I_1(x)\right]$$

$$\int x^3\,e^{-x}\, K_1(x)\, dx = \frac{xe^{-x}}{35}\;\left[-(5\,x^3+8\,x^2+8\,x)\,K_0(x)+(5\,x^3-12\,x^2-16\,x-16)\,K_1(x)\right]$$

$$\int x^4e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{315}\,\left[(-96\,x-96\,x^2-60\,x^3+35\,x^4)I_0(x)+(192+192\,x+144\,x^2+80\,x^3+35\,x^4)I_1(x)\right]$$

$$\int x^4\,e^{-x}\, K_0(x)\, dx =$$

$$= \frac{xe^{-x}}{315}\;\left[(35\,x^4-60\,x^3-96\,x^2-96\,x)\,K_0(x)-(35\,x^4+80\,x^3+144\,x^2+192\,x+192)\,K_1(x)\right]$$

$$\int x^4e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{63}\,\left[(24\,x+24\,x^2+15\,x^3+7\,x^4)I_0(x)+(-48-48\,x-36\,x^2-20\,x^3+7\,x^4)I_1(x)\right]$$

$$\int x^4\,e^{-x}\, K_1(x)\, dx =$$

$$= \frac{xe^{-x}}{63}\;\left[-(7\,x^4+15\,x^3+24\,x^2+24\,x)\,K_0(x)+(7\,x^4-20\,x^3-36\,x^2-48\,x-48)\,K_1(x)\right]$$

$$\int x^5 e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{693}\,\big[(-480\,x - 480\,x^2 - 300\,x^3 - 140\,x^4 + 63\,x^5)I_0(x)+$$
$$+(960 + 960\,x + 720\,x^2 + 400\,x^3 + 175\,x^4 + 63\,x^5)I_1(x)\big]$$

$$\int x^5\, e^{-x}\, K_0(x)\, dx = \frac{xe^{-x}}{693}\,\big[(63\,x^5 - 140\,x^4 - 300\,x^3 - 480\,x^2 - 480\,x)\, K_0(x)-$$
$$-(63\,x^5 + 175\,x^4 + 400\,x^3 + 720\,x^2 + 960\,x + 960)\, K_1(x)\big]$$

$$\int x^5 e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{231}\,\big[(192\,x + 192\,x^2 + 120\,x^3 + 56\,x^4 + 21\,x^5)I_0(x)+$$
$$+(-384 - 384\,x - 288\,x^2 - 160\,x^3 - 70\,x^4 + 21\,x^5)I_1(x)\big]$$

$$\int x^5\, e^{-x}\, K_1(x)\, dx = \frac{xe^{-x}}{231}\,\big[-(21\,x^5 + 56\,x^4 + 120\,x^3 + 192\,x^2 + 192\,x)\, K_0(x)+$$
$$+(21\,x^5 - 70\,x^4 - 160\,x^3 - 288\,x^2 - 384\,x - 384)\, K_1(x)\big]$$

$$\int x^6 e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{1001}\,\big[(-1920\,x - 1920\,x^2 - 1200\,x^3 - 560\,x^4 - 210x^5 + 77x^6)I_0(x)+$$
$$+\;(3840 + 3840\,x + 2880\,x^2 + 1600\,x^3 + 700\,x^4 + 252x^5 + 77x^6)I_1(x)\big]$$

$$\int x^6\, e^{-x}\, K_0(x)\, dx = \frac{xe^{-x}}{1001}\,\big[(77\,x^6 - 210\,x^5 - 560\,x^4 - 1200\,x^3 - 1920\,x^2 - 1920\,x)\, K_0(x)-$$
$$-(77\,x^6 + 252\,x^5 + 700\,x^4 + 1600\,x^3 + 2880\,x^2 + 3840\,x + 3840)\, K_1(x)\big]$$

$$\int x^6 e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{429}\,\big[(960\,x + 960\,x^2 + 600\,x^3 + 280\,x^4 + 105\,x^5 + 33\,x^6)I_0(x)+$$
$$+(-1920 - 1920\,x - 1440\,x^2 - 800\,x^3 - 350\,x^4 - 126\,x^5 + 33\,x^6)I_1(x)\big]$$

$$\int x^6\, e^{-x}\, K_1(x)\, dx = \frac{xe^{-x}}{429}\,\big[-(33\,x^6 + 105\,x^5 + 280\,x^4 + 600\,x^3 + 960\,x^2 + 960\,x)\, K_0(x)+$$
$$+(33\,x^6 - 126\,x^5 - 350\,x^4 - 800\,x^3 - 1440\,x^2 - 1920\,x - 1920)\, K_1(x)\big]$$

$$\int x^7 e^{-x}\, I_0(x)\, dx \;=\; \frac{xe^{-x}}{2145}\,\big[(-13440\,x - 13440\,x^2 - 8400\,x^3 - 3920\,x^4 - 1470\,x^5 - 462\,x^6 + 143\,x^7)I_0(x)+$$
$$+(26880 + 26880\,x + 20160\,x^2 + 11200\,x^3 + 4900\,x^4 + 1764\,x^5 + 539\,x^6 + 143\,x^7)I_1(x)\big]$$

$$\int x^7\, e^{-x}\, K_0(x)\, dx =$$
$$= \frac{xe^{-x}}{2145}\,\big[(143\,x^7 - 462\,x^6 - 1470\,x^5 - 3920\,x^4 - 8400\,x^3 - 13440\,x^2 - 13440\,x)\, K_0(x)-$$
$$-(143\,x^7 + 539\,x^6 + 1764\,x^5 + 4900\,x^4 + 11200\,x^3 + 20160\,x^2 + 26880\,x + 26880)\, K_1(x)\big]$$

$$\int x^7 e^{-x}\, I_1(x)\, dx \;=\; \frac{xe^{-x}}{2145}\,\big[(15360\,x + 15360\,x^2 + 9600\,x^3 + 4480\,x^4 + 1680\,x^5 + 528\,x^6 + 143\,x^7)I_0(x)+$$
$$+(-30720 - 30720\,x - 23040\,x^2 - 12800\,x^3 - 5600\,x^4 - 2016\,x^5 - 616\,x^6 + 143\,x^7)I_1(x)\big]$$

$$\int x^7\, e^{-x}\, K_1(x)\, dx =$$
$$= \frac{xe^{-x}}{2145}\,\big[-(143\,x^7 + 528\,x^6 + 1680\,x^5 + 4480\,x^4 + 9600\,x^3 + 15360\,x^2 + 15360\,x)\, K_0(x)+$$
$$+(143\,x^7 - 616\,x^6 - 2016\,x^5 - 5600\,x^4 - 12800\,x^3 - 23040\,x^2 - 30720\,x - 30720)\, K_1(x)\big]$$

Recurrence formulas:

$$\int x^n e^{-x} I_0(x)\,dx = \frac{x^n e^{-x}}{2n+1}[(x-n)I_0(x) + xI_1(x)] + \frac{n^2}{2n+1}\int x^{n-1}e^{-x}I_0(x)\,dx \quad (*)$$

$$\int x^n e^{-x} I_1(x)\,dx = \frac{x^n e^{-x}}{2n+1}[(n+1+x)I_0(x) + xI_1(x)] - \frac{n(n+1)}{2n+1}\int x^{n-1}e^{-x}I_0(x)\,dx \quad (*)$$

The last formula refers to $I_0(x)$ instead of $I_1(x)$.

$$\int x^n e^{-x} K_0(x)\,dx = \frac{x^n e^{-x}}{2n+1}[(x-n)K_0(x) - xK_1(x)] + \frac{n^2}{2n+1}\int x^{n-1}e^{-x}K_0(x)\,dx$$

$$\int x^n e^{-x} K_1(x)\,dx = \frac{x^n e^{-x}}{2n+1}[-(x+n+1)K_0(x) + xK_1(x)] + \frac{n(n+1)}{2n+1}\int x^{n-1}e^{-x}K_0(x)\,dx$$

The last formula refers to $K_0(x)$ instead of $K_1(x)$.

1.2.3. Integrals of the type $\int x^n \cdot \left\{ \begin{array}{c} \sinh \\ \cosh \end{array} \right\} x \cdot I_\nu(x)\,dx$

$$\int \frac{\sinh x\, I_1(x)\,dx}{x} = x\,\cosh x\, I_0(x) - \sinh x\, I_1(x)$$

$$\int \frac{\cosh x\, I_1(x)\,dx}{x} = x\,\sinh x\, I_0(x) - \cosh x\, I_1(x)$$

$$\int \sinh x\, I_0(x)\,dx = x\,\sinh x\, I_0(x) - x\,\cosh x\, I_1(x)$$

$$\int \cosh x\, I_0(x)\,dx = x\,\cosh x\, I_0(x) - x\,\sinh x\, I_1(x)$$

$$\int \sinh x\, I_1(x)\,dx = \sinh x\, I_0(x) - x\,\cosh x\, I_0(x) + x\,\sinh x\, I_1(x)$$

$$\int \cosh x\, I_1(x)\,dx = -x\,\sinh x\, I_0(x) + \cosh x\, I_0(x) + x\,\cosh x\, I_1(x)$$

$$\int x\,\sinh x\, I_0(x)\,dx = \frac{x^2}{3}\,\sinh x\, I_0(x) + \frac{x}{3}\,\sinh x\, I_1(x) - \frac{x^2}{3}\,\cosh x\, I_1(x)$$

$$\int x\,\cosh x\, I_0(x)\,dx = \frac{x^2}{3}\,\cosh x\, I_0(x) - \frac{x^2}{3}\,\sinh x\, I_1(x) + \frac{x}{3}\,\cosh x\, I_1(x)$$

$$\int x\,\sinh x\, I_1(x)\,dx = -\frac{x^2}{3}\,\cosh x\, I_0(x) + \frac{x^2}{3}\,\sinh x\, I_1(x) + \frac{2x}{3}\,\cosh x\, I_1(x)$$

$$\int x\,\cosh x\, I_1(x)\,dx = -\frac{x^2}{3}\,\sinh x\, I_0(x) + \frac{2x}{3}\,\sinh x\, I_1(x) + \frac{x^2}{3}\,\cosh x\, I_1(x)$$

$$\int x^2\,\sinh x\, I_0(x)\,dx = \frac{x^3}{5}\,\sinh x\, I_0(x) + \frac{2x^2}{15}\,\cosh x\, I_0(x) + \frac{4x^2}{15}\,\sinh x\, I_1(x) - \frac{3x^3+4x}{15}\,\cosh x\, I_1(x)$$

$$\int x^2\,\cosh x\, I_0(x)\,dx = \frac{2x^2}{15}\,\sinh x\, I_0(x) + \frac{x^3}{5}\,\cosh x\, I_0(x) - \frac{3\,x^3+4\,x}{15}\,\sinh x\, I_1(x) + \frac{4x^2}{15}\,\cosh x\, I_1(x)$$

$$\int x^2\,\sinh x\, I_1(x)\,dx = \frac{x^2}{5}\,\sinh x\, I_0(x) - \frac{x^3}{5}\,\cosh x\, I_0(x) + \frac{x^3-2\,x}{5}\,\sinh x\, I_1(x) + \frac{2x^2}{5}\,\cosh x\, I_1(x)$$

$$\int x^2\,\cosh x\, I_1(x)\,dx = -\frac{x^3}{5}\,\sinh x\, I_0(x) + \frac{x^2}{5}\,\cosh x\, I_0(x) + \frac{2x^2}{5}\,\sinh x\, I_1(x) + \frac{x^3-2\,x}{5}\,\cosh x\, I_1(x)$$

$$\int x^3\,\sinh x\, I_0(x)\,dx =$$

$$= \frac{5\,x^4-6\,x^2}{35}\,\sinh x\, I_0(x) + \frac{6x^3}{35}\,\cosh x\, I_0(x) + \frac{9\,x^3+12\,x}{35}\,\sinh x\, I_1(x) - \frac{5\,x^4+12\,x^2}{35}\,\cosh x\, I_1(x)$$

$$\int x^3\,\cosh x\, I_0(x)\,dx =$$

$$= \frac{6x^3}{35}\,\sinh x\, I_0(x) + \frac{5\,x^4-6\,x^2}{35}\,\cosh x\, I_0(x) - \frac{5\,x^4+12\,x^2}{35}\,\sinh x\, I_1(x) + \frac{9\,x^3+12\,x}{35}\,\cosh x\, I_1(x)$$

$$\int x^3\,\sinh x\, I_1(x)\,dx =$$

$$= \frac{8x^3}{35}\,\sinh x\, I_0(x) - \frac{5\,x^4+8\,x^2}{35}\,\cosh x\, I_0(x) + \frac{5\,x^4-16\,x^2}{35}\,\sinh x\, I_1(x) + \frac{12\,x^3+16\,x}{35}\,\cosh x\, I_1(x)$$

$$\int x^3\,\cosh x\, I_1(x)\,dx =$$

$$= -\frac{5\,x^4+8\,x^2}{35}\,\sinh x\, I_0(x) + \frac{8x^3}{35}\,\cosh x\, I_0(x) + \frac{12\,x^3+16\,x}{35}\,\sinh x\, I_1(x) + \frac{5\,x^4-16\,x^2}{35}\,\cosh x\, I_1(x)$$

$$\int x^4 \sinh x\, I_0(x)\, dx = \frac{35\, x^5 - 96\, x^3}{315} \sinh x\, I_0(x) + \frac{20\, x^4 + 32\, x^2}{105} \cosh x\, I_0(x) +$$
$$+ \frac{80\, x^4 + 192\, x^2}{315} \sinh x\, I_1(x) - \frac{35x^5 + 144x^3 + 192x}{315} \cosh x\, I_1(x)$$
$$\int x^4 \cosh x\, I_0(x)\, dx = \frac{20\, x^4 + 32\, x^2}{105} \sinh x\, I_0(x) + \frac{35\, x^5 - 96\, x^3}{315} \cosh x\, I_0(x) -$$
$$- \frac{35\, x^5 + 144\, x^3 + 192\, x}{315} \sinh x\, I_1(x) + \frac{80\, x^4 + 192\, x^2}{315} \cosh x\, I_1(x)$$
$$\int x^4 \sinh x\, I_1(x)\, dx = \frac{5\, x^4 + 8\, x^2}{21} \sinh x\, I_0(x) - \frac{7\, x^5 + 24\, x^3}{63} \cosh x\, I_0(x) +$$
$$+ \frac{7\, x^5 - 36\, x^3 - 48\, x}{63} \sinh x\, I_1(x) + \frac{20\, x^4 + 48\, x^2}{63} \cosh x\, I_1(x)$$
$$\int x^4 \cosh x\, I_1(x)\, dx = - \frac{7\, x^5 + 24\, x^3}{63} \sinh x\, I_0(x) + \frac{5\, x^4 + 8\, x^2}{21} \cosh x\, I_0(x) +$$
$$+ \frac{20\, x^4 + 48\, x^2}{63} \sinh x\, I_1(x) + \frac{7\, x^5 - 36\, x^3 - 48\, x}{63} \cosh x\, I_1(x)$$

$$\int x^5 \sinh x\, I_0(x)\, dx = \frac{21\, x^6 - 100\, x^4 - 160\, x^2}{231} \sinh x\, I_0(x) + \frac{140\, x^5 + 480\, x^3}{693} \cosh x\, I_0(x) +$$
$$+ \frac{175\, x^5 + 720\, x^3 + 960\, x}{693} \sinh x\, I_1(x) - \frac{63\, x^6 + 400\, x^4 + 960\, x^2}{693} \cosh x\, I_1(x)$$
$$\int x^5 \cosh x\, I_0(x)\, dx = \frac{140\, x^5 + 480\, x^3}{693} \sinh x\, I_0(x) + \frac{21\, x^6 - 100\, x^4 - 160\, x^2}{231} \cosh x\, I_0(x) -$$
$$- \frac{63\, x^6 + 400\, x^4 + 960\, x^2}{693} \sinh x\, I_1(x) + \frac{175\, x^5 + 720\, x^3 + 960\, x}{693} \cosh x\, I_1(x)$$
$$\int x^5 \sinh x\, I_1(x)\, dx = \frac{56\, x^5 + 192\, x^3}{231} \sinh x\, I_0(x) - \frac{7\, x^6 + 40\, x^4 + 64\, x^2}{77} \cosh x\, I_0(x) +$$
$$+ \frac{21\, x^6 - 160\, x^4 - 384\, x^2}{231} \sinh x\, I_1(x) + \frac{70\, x^5 + 288\, x^3 + 384\, x}{231} \cosh x\, I_1(x)$$
$$\int x^5 \cosh x\, I_1(x)\, dx = - \frac{7\, x^6 + 40\, x^4 + 64\, x^2}{77} \sinh x\, I_0(x) + \frac{56\, x^5 + 192\, x^3}{231} \cosh x\, I_0(x) +$$
$$+ \frac{70\, x^5 + 288\, x^3 + 384\, x}{231} \sinh x\, I_1(x) + \frac{21\, x^6 - 160\, x^4 - 384\, x^2}{231} \cosh x\, I_1(x)$$

$$\int x^6 \sinh x\, I_0(x)\, dx = \frac{77\, x^7 - 560\, x^5 - 1920\, x^3}{1001} \sinh x\, I_0(x) + \frac{210\, x^6 + 1200\, x^4 + 1920\, x^2}{1001} \cosh x\, I_0(x) +$$
$$+ \frac{252\, x^6 + 1600\, x^4 + 3840\, x^2}{1001} \sinh x\, I_1(x) - \frac{77\, x^7 + 700\, x^5 + 2880\, x^3 + 3840\, x}{1001} \cosh x\, I_1(x)$$
$$\int x^6 \cosh x\, I_0(x)\, dx = \frac{210\, x^6 + 1200\, x^4 + 1920\, x^2}{1001} \sinh x\, I_0(x) + \frac{77\, x^7 - 560\, x^5 - 1920\, x^3}{1001} \cosh x\, I_0(x) -$$
$$- \frac{77\, x^7 + 700\, x^5 + 2880\, x^3 + 3840\, x}{1001} \sinh x\, I_1(x) + \frac{252\, x^6 + 1600\, x^4 + 3840\, x^2}{1001} \cosh x\, I_1(x)$$
$$\int x^6 \sinh x\, I_1(x)\, dx = \frac{35\, x^6 + 200\, x^4 + 320\, x^2}{143} \sinh x\, I_0(x) - \frac{33\, x^7 + 280\, x^5 + 960\, x^3}{429} \cosh x\, I_0(x) +$$
$$+ \frac{33\, x^7 - 350\, x^5 - 1440\, x^3 - 1920\, x}{429} \sinh x\, I_1(x) + \frac{126\, x^6 + 800\, x^4 + 1920\, x^2}{429} \cosh x\, I_1(x)$$
$$\int x^6 \cosh x\, I_1(x)\, dx = - \frac{33\, x^7 + 280\, x^5 + 960\, x^3}{429} \sinh x\, I_0(x) + \frac{35\, x^6 + 200\, x^4 + 320\, x^2}{143} \cosh x\, I_0(x) +$$
$$+ \frac{126\, x^6 + 800\, x^4 + 1920\, x^2}{429} \sinh x\, I_1(x) + \frac{33\, x^7 - 350\, x^5 - 1440\, x^3 - 1920\, x}{429} \cosh x\, I_1(x)$$

Recurrence formulas:

$$\int x^{n+1}\,\sinh x\cdot I_0(x)\,dx =$$

$$= \frac{x^{n+1}}{2n+3}\,\left[x\,\sinh x\cdot I_0(x)+(n+1)\,\cosh x\cdot I_0(x)-x\,\cosh x\cdot I_1(x)\right]-\frac{(n+1)^2}{2n+3}\int x^n\,\cosh x\cdot I_0(x)\,dx$$

$$\int x^{n+1}\,\cosh x\cdot I_0(x)\,dx =$$

$$= \frac{x^{n+1}}{2n+3}\,\left[(n+1)\,\sinh x\cdot I_0(x)+x\,\cosh x\cdot I_0(x)-x\,\sinh x\cdot I_1(x)\right]-\frac{(n+1)^2}{2n+3}\int x^n\,\sinh x\cdot I_0(x)\,dx$$

$$\int x^{n+1}\,\cosh x\cdot I_0(x)\,dx =$$

$$= \frac{x^{n+1}}{2n+3}\,\left[(n+2)\,\sinh x\cdot I_0(x)-x\,\cosh x\cdot I_0(x)+x\,\sinh x\cdot I_1(x)\right]-\frac{(n+1)(n+2)}{2n+3}\int x^n\,\sinh x\cdot I_0(x)\,dx$$

$$\int x^{n+1}\,\cosh x\cdot I_0(x)\,dx =$$

$$= \frac{x^{n+1}}{2n+3}\,\left[-x\,\sinh x\cdot I_0(x)+(n+2)\,\cosh x\cdot I_0(x)+x\,\cosh x\cdot I_1(x)\right]-\frac{(n+1)}{2n+3}\int x^n\,\cosh x\cdot I_0(x)\,dx$$

1.2.4. Integrals of the type $\int x^n \cdot \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} x \cdot J_\nu(x)\,dx$

See also [1], 11.3.

$$\int \frac{\sin x \cdot J_0(x)\,dx}{x} = -\sin x\, J_1(x) - \cos x\, J_0(x)$$

$$\int \frac{\cos x \cdot J_0(x)\,dx}{x} = \sin x\, J_0(x) - \cos x\, J_1(x)$$

$$\int \sin x \cdot J_0(x)\,dx = x[\sin x \cdot J_0(x) - \cos x \cdot J_1(x)]$$

$$\int \cos x \cdot J_0(x)\,dx = x[\cos x \cdot J_0(x) + \sin x \cdot J_1(x)]$$

$$\int \sin x \cdot J_1(x)\,dx = (x\cos x - \sin x)J_0(x) + x \sin x \cdot J_1(x)$$

$$\int \cos x \cdot J_1(x)\,dx = -(x\sin x + \cos x)J_0(x) + x \cos x \cdot J_1(x)$$

$$\int x \sin x \cdot J_0(x)\,dx = \frac{x^2}{3} \sin x \cdot J_0(x) + \frac{x\sin x - x^2 \cos x}{3} \cdot J_1(x)$$

$$\int x \cos x \cdot J_0(x)\,dx = \frac{x^2}{3} \cos x \cdot J_0(x) + \frac{x^2\sin x - x \cos x}{3} \cdot J_1(x)$$

$$\int x \sin x \cdot J_1(x)\,dx = \frac{x^2}{3} \cdot \cos x \cdot J_0(x) + \frac{x^2\sin x - 2x \cos x}{3} \cdot J_1(x)$$

$$\int x \cos x \cdot J_1(x)\,dx = -\frac{x^2}{3} \sin x \cdot J_0(x) + \frac{2x\sin x + x^2 \cos x}{3} \cdot J_1(x)$$

$$\int x^2 \sin x \cdot J_0(x)\,dx = \frac{1}{15}\left\{\left[3\,x^3 \sin x - 2\,x^2 \cos x\right] \cdot J_0(x) + \left[4\,x^2 \sin x + (4\,x - 3\,x^3)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^2 \cos x \cdot J_0(x)\,dx = \frac{1}{15}\left\{\left[2\,x^2 \sin x + 3\,x^3 \cos x\right] \cdot J_0(x) + \left[(-4\,x + 3\,x^3)\sin x + 4\,x^2 \cos x\right] \cdot J_1(x)\right\}$$

$$\int x^2 \sin x \cdot J_1(x)\,dx = \frac{1}{5}\left\{\left[-x^2 \sin x + x^3 \cos x\right] \cdot J_0(x) + \left[(2\,x + x^3)\sin x - 2\,x^2 \cos x\right] \cdot J_1(x)\right\}$$

$$\int x^2 \cos x \cdot J_1(x)\,dx = \frac{1}{5}\left\{\left[-x^3 \sin x - x^2 \cos x\right] \cdot J_0(x) + \left[2\,x^2 \sin x + (2\,x + x^3)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^3 \sin x \cdot J_0(x)\,dx = \frac{1}{35}\left\{\left[(6\,x^2 + 5\,x^4)\sin x - 6\,x^3 \cos x\right] \cdot J_0(x) + \right.$$
$$\left. + \left[(-12\,x + 9\,x^3)\sin x + (12\,x^2 - 5\,x^4)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^3 \cos x \cdot J_0(x)\,dx = \frac{1}{35}\left\{\left[6\,x^3 \sin x + (6\,x^2 + 5\,x^4)\cos x\right] \cdot J_0(x) + \right.$$
$$\left. + \left[(-12\,x^2 + 5\,x^4)\sin x + (-12\,x + 9\,x^3)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^3 \sin x \cdot J_1(x)\,dx = \frac{1}{35}\left\{\left[-8x^3 \sin x + (-8\,x^2 + 5\,x^4)\cos x\right] \cdot J_0(x) + \right.$$
$$\left. + \left[(16\,x^2 + 5\,x^4)\sin x + (16\,x - 12\,x^3)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^3 \cos x \cdot J_1(x)\,dx = \frac{1}{35}\left\{\left[(8\,x^2 - 5\,x^4)\sin x - 8\,x^3 \cos x\right] \cdot J_0(x) + \right.$$
$$\left. + \left[(-16\,x + 12\,x^3)\sin x + (16\,x^2 + 5\,x^4)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^4 \sin x \cdot J_0(x)\,dx = \frac{1}{315}\left\{\left[(96\,x^3 + 35\,x^5)\sin x + (96\,x^2 - 60\,x^4)\cos x\right] \cdot J_0(x) + \right.$$
$$\left. + \left[(-192\,x^2 + 80\,x^4)\sin x + (-192\,x + 144\,x^3 - 35\,x^5)\cos x\right] \cdot J_1(x)\right\}$$

$$\int x^4 \cos x \cdot J_0(x)\,dx \;=\; \frac{1}{315}\left\{\left[(-96\,x^2+60\,x^4)\sin x+(96\,x^3+35\,x^5)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(192\,x-144\,x^3+35\,x^5)\sin x+(-192\,x^2+80\,x^4)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^4 \sin x \cdot J_1(x)\,dx \;=\; \frac{1}{315}\left\{\left[(120x^2-75x^4)\sin x+(-120\,x^3+35\,x^5)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(-240\,x+180\,x^3+35\,x^5)\sin x+(240\,x^2-100\,x^4)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^4 \cos x \cdot J_1(x)\,dx \;=\; \frac{1}{315}\left\{\left[(120\,x^3-35\,x^5)\sin x+(120\,x^2-75\,x^4)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(-240\,x^2+100\,x^4)\sin x+(-240\,x+180\,x^3+35\,x^5)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^5 \sin x \cdot J_0(x)\,dx \;=\; \frac{1}{693}\left\{\left[(-480\,x^2+300\,x^4+63\,x^6)\sin x+(480\,x^3-140\,x^5)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(960\,x-720\,x^3+175\,x^5)\sin x+(-960\,x^2+400\,x^4-63\,x^6)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^5 \cos x \cdot J_0(x)\,dx \;=\; \frac{1}{693}\left\{\left[(-480\,x^3+140\,x^5)\sin x+(-480\,x^2+300\,x^4+63\,x^6)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(960\,x^2-400\,x^4+63\,x^6)\sin x+(960\,x-720\,x^3+175\,x^5)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^5 \sin x \cdot J_1(x)\,dx \;=\; \frac{1}{231}\left\{\left[(192x^3-56x^5)\sin x+(192\,x^2-120\,x^4+21\,x^6)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(-384\,x^2+160\,x^4+21\,x^6)\sin x+(-384\,x+288\,x^3-70\,x^5)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^5 \cos x \cdot J_1(x)\,dx \;=\; \frac{1}{231}\left\{\left[(-192\,x^2+120\,x^4-21\,x^6)\sin x+(192\,x^3-56\,x^5)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(384\,x-288\,x^3+70\,x^5)\sin x+(-384\,x^2+160\,x^4+21\,x^6)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^6 \sin x \cdot J_0(x)\,dx \;=$$
$$=\;\frac{1}{1001}\left\{\left[(-1920\,x^3+560\,x^5+77\,x^7)\sin x+(-1920\,x^2+1200\,x^4-210\,x^6)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(3840\,x^2-1600\,x^4+252\,x^6)\sin x+(3840\,x-2880\,x^3+700\,x^5-77\,x^7)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^6 \cos x \cdot J_0(x)\,dx \;=$$
$$=\;\frac{1}{1001}\left\{\left[(1920\,x^2-1200\,x^4+210\,x^6)\sin x+(-1920\,x^3+560\,x^5+77\,x^7)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(-3840\,x+2880\,x^3-700\,x^5+77\,x^7)\sin x+(3840\,x^2-1600\,x^4+252\,x^6)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^6 \sin x \cdot J_1(x)\,dx \;=$$
$$=\;\frac{1}{429}\left\{\left[(-960x^2+600x^4-105x^6)\sin x+(960\,x^3-280\,x^5+33\,x^7)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(1920\,x-1440\,x^3+350\,x^5+33\,x^7)\sin x+(-1920\,x^2+800\,x^4-126\,x^6)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^6 \cos x \cdot J_1(x)\,dx \;=$$
$$=\;\frac{1}{429}\left\{\left[(-960\,x^3+280\,x^5-33\,x^7)\sin x+(-960\,x^2+600\,x^4-105\,x^6)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(1920\,x^2-800\,x^4+126\,x^6)\sin x+(1920\,x-1440\,x^3+350\,x^5+33\,x^7)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^7 \sin x \cdot J_0(x)\,dx \;=$$
$$=\;\frac{1}{2145}\left\{\left[(13440\,x^2-8400\,x^4+1470\,x^6+143\,x^8)\sin x+(-13440\,x^3+3920\,x^5-462\,x^7)\cos x\right]\cdot J_0(x)+\right.$$
$$\left.+\left[(-26880\,x+20160\,x^3-4900\,x^5+539\,x^7)\sin x+(26880\,x^2-11200\,x^4+1764\,x^6-143\,x^8)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^7 \cos x \cdot J_0(x)\,dx \;=$$

$$= \;\frac{1}{2145}\left\{\left[(13440\,x^3 - 3920\,x^5 + 462\,x^7)\sin x + (13440\,x^2 - 8400\,x^4 + 1470\,x^6 + 143\,x^8)\cos x\right]\cdot J_0(x) + \right.$$

$$\left. + \left[(-26880\,x^2 + 11200\,x^4 - 1764\,x^6 + 143\,x^8)\sin x + (-26880\,x + 20160\,x^3 - 4900\,x^5 + 539\,x^7)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^7 \sin x \cdot J_1(x)\,dx \;=$$

$$= \frac{1}{2145}\left\{\left[(-15360x^3 + 4480x^5 - 528x^7)\sin x + (-15360\,x^2 + 9600\,x^4 - 1680\,x^6 + 143\,x^8)\cos x\right]\cdot J_0(x) + \right.$$

$$\left. + \left[(30720\,x^2 - 12800\,x^4 + 2016\,x^6 + 143\,x^8)\sin x + (30720\,x - 23040\,x^3 + 5600\,x^5 - 616\,x^7)\cos x\right]\cdot J_1(x)\right\}$$

$$\int x^7 \cos x \cdot J_1(x)\,dx \;=$$

$$= \;\frac{1}{2145}\left\{\left[(15360\,x^2 - 9600\,x^4 + 1680\,x^6 - 143\,x^8)\sin x + (-15360\,x^3 + 4480\,x^5 - 528\,x^7)\cos x\right]\cdot J_0(x) + \right.$$

$$\left. + \left[(-30720\,x + 23040\,x^3 - 5600\,x^5 + 616\,x^7)\sin x + (30720\,x^2 - 12800\,x^4 + 2016\,x^6 + 143\,x^8)\cos x\right]\cdot J_1(x)\right\}$$

Recurrence formulas:

Let

$$S_n^{(\nu)} = \int x^n\,\sin x \cdot J_\nu(x)\,dx \quad , \qquad C_n^{(\nu)} = \int x^n\,\cos x \cdot J_\nu(x)\,dx$$

and

$$\sigma_n^{(\nu)} = x^n\,\sin x \cdot J_\nu(x) \quad , \qquad \gamma_n^{(\nu)} = x^n\,\cos x \cdot J_\nu(x)\;,$$

then holds

$$S_n^{(0)} \;=\; \frac{n^2 C_{n-1}^{(0)} - n\gamma_n^{(0)} + \sigma_{n+1}^{(0)} - \gamma_{n+1}^{(1)}}{2n+1}\;, \quad S_n^{(1)} \;=\; \frac{n(n+1)S_{n-1}^{(0)} - (n+1)\sigma_n^{(0)} + \gamma_{n+1}^{(0)} + \sigma_{n+1}^{(1)}}{2n+1}\;,$$

$$C_n^{(0)} \;=\; \frac{n\sigma_n^{(0)} - n^2 S_{n-1}^{(0)} + \gamma_{n+1}^{(0)} + \sigma_{n+1}^{(1)}}{2n+1}\;, \quad C_n^{(1)} \;=\; \frac{n(n+1)C_{n-1}^{(0)} - (n+1)\gamma_n^{(0)} - \sigma_{n+1}^{(0)} + \gamma_{n+1}^{(1)}}{2n+1}\;.$$

1.2.5. Integrals of the type $\int x^n \cdot e^{ax} \cdot Z_\nu(x)\,dx$

a) General facts:

Holds

$$\int e^{ax}\, J_0(x)\,dx \;=\; \int e^{-a\cdot(-x)}\, J_0(-x)\,dx\ ,$$

therefore the integral on the left hand side is discussed, assuming $x \geq 0$ and treating the cases $a > 0$ and $a < 0$ separately.

Let $\mathfrak{H}_\nu(x,a)$ denote the following functions:

$$\mathfrak{H}_1(x,a) \;=\; \sum_{k=1}^{\infty} b_k(a)\,x^k \quad , \qquad \mathfrak{H}_0(x,a) \;=\; \sum_{k=1}^{\infty} c_k(a)\,x^k$$

with

$$b_1(a) = 1\ , \quad b_2(a) = 0\ , \quad b_{k+2}(a) = -\frac{a(1+2k)b_{k+1}(a) + (1+a^2)b_k(a)}{k(k+2)}\ , \quad k \geq 1$$

and

$$c_k(a) = -(k+1)b_{k+1}(a) - a\,b_k(a)\ .$$

Then holds with $a \in \mathbb{R}$

$$\int_0^x e^{at}\, J_0(t)\,dt \;=\; e^{ax}\,\left[\,\mathfrak{H}_1(x,a)\,J_0(x) + \mathfrak{H}_0(x,a)\,J_1(x)\right]\ .$$

In the case $a = 0$ one has with the Struve functions

$$\mathfrak{H}_1(x,0) = x - \frac{\pi x}{2}\,\mathbf{H}_1(x)\ , \qquad \mathfrak{H}_0(x,0) = \frac{\pi x}{2}\,\mathbf{H}_0(x)\ .$$

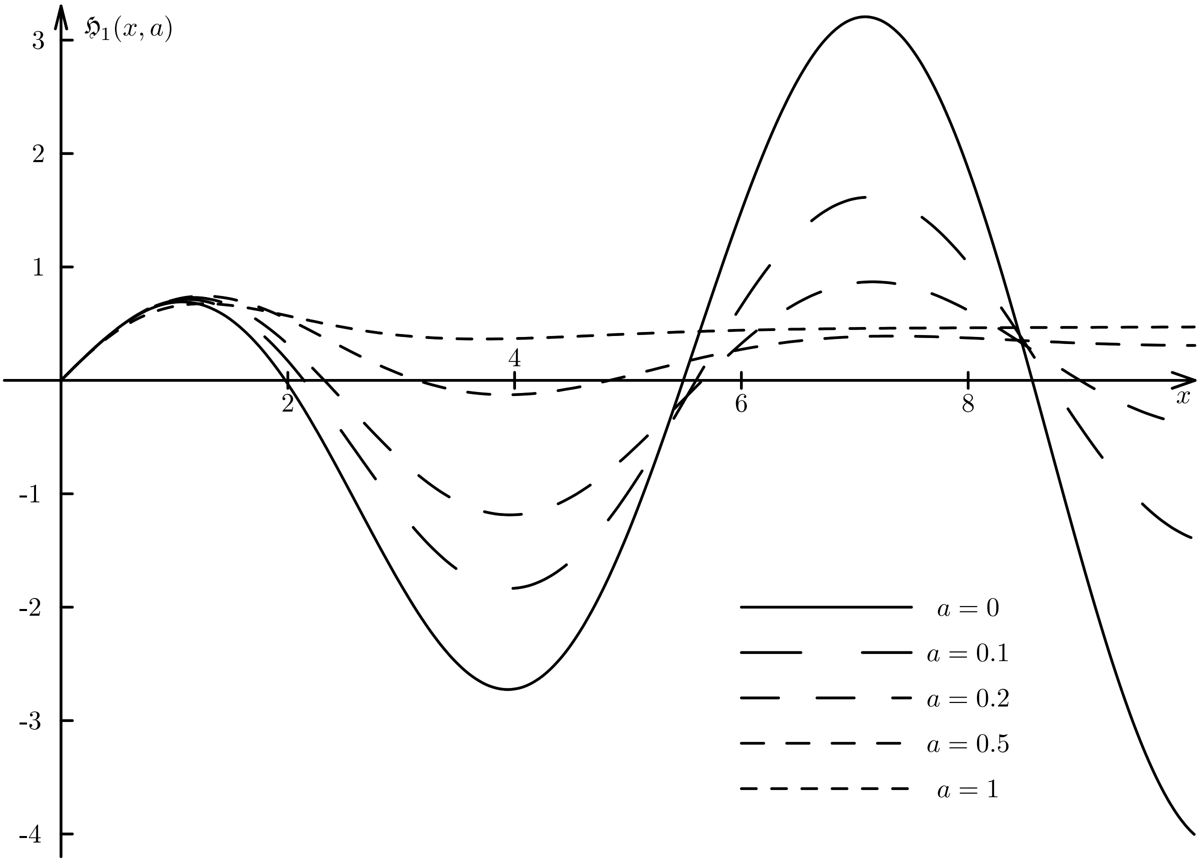
First terms of the power series:

$$\mathfrak{H}_1(x,a) = x - (a^2+1)\left[\frac{x^3}{3} - \frac{5a}{24}\,x^4 + \frac{27\,a^2-8}{360}\,x^5 - \frac{7a\,\left(8\,a^2-7\right)}{2\,880}\,x^6 + \frac{400\,a^4 - 691\,a^2 + 64}{100\,800}\,x^7 - \right.$$
$$- \frac{a\left(1\,080\,a^4 - 3076\,a^2 + 849\right)}{1\,612\,800}\,x^8 + \frac{9\,800\,a^6 - 41\,484\,a^4 + 22\,767\,a^2 - 1024}{101\,606\,400}\,x^9 -$$
$$- \frac{11a\,\left(1\,792\,a^6 - 10\,536\,a^4 + 9\,588\,a^2 - 1\,289\right)}{1\,625\,702\,400}\,x^{10} +$$
$$+ \frac{217\,728\,a^8 - 1\,695\,080\,a^6 + 2\,303\,364\,a^4 - 617\,289\,a^2 + 16\,384}{160\,944\,537\,600}\,x^{11} -$$
$$\left. - \frac{13\,\left(67\,200\,a^8 - 668\,576\,a^6 + 1\,266\,744\,a^4 - 564\,120\,a^2 + 44\,815\right)a}{6\,437\,781\,504\,000}\,x^{12} + \ldots\right]$$

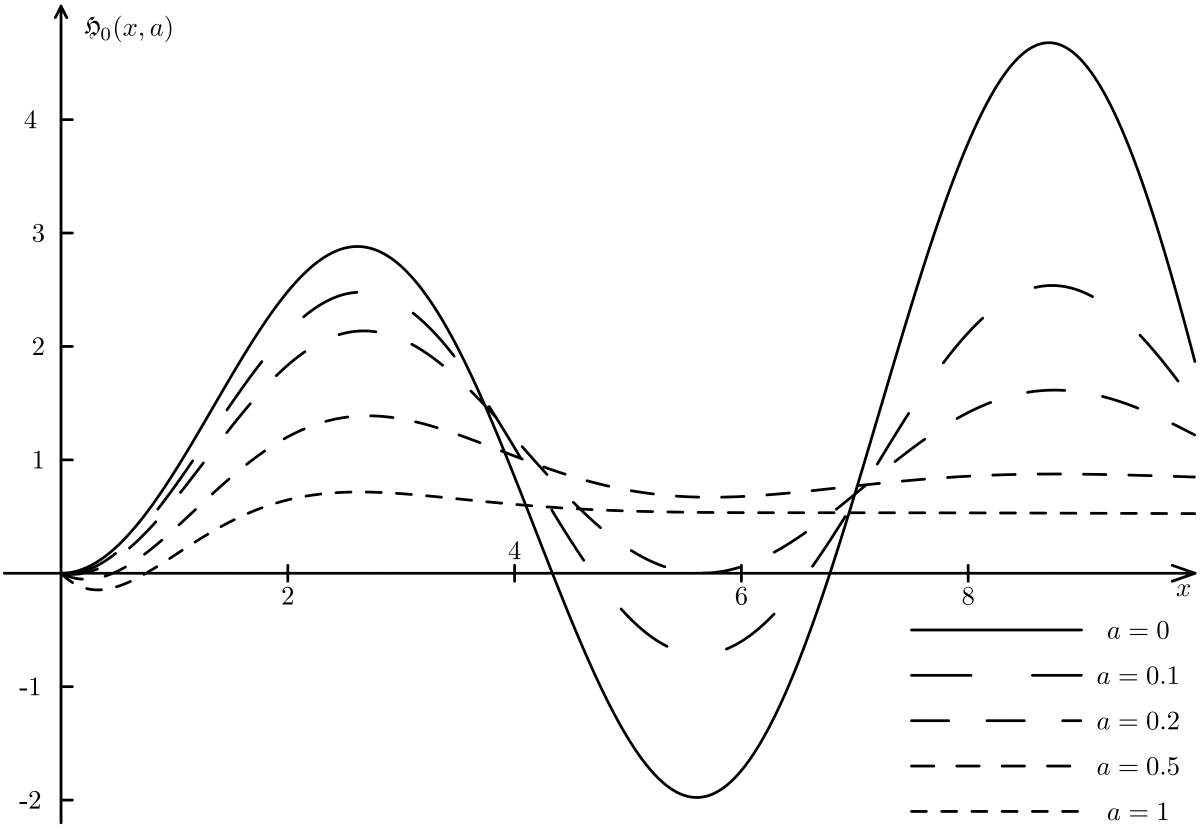
and

$$\mathfrak{H}_0(x,a) = -ax + (a^2+1)\left[x^2 - \frac{a}{2}\,x^3 + \frac{3\,a^2-2}{18}\,x^4 - \frac{a\left(12\,a^2-23\right)}{288}\,x^5 + \frac{60\,a^4 - 223\,a^2 + 32}{7\,200}\,x^6 - \right.$$
$$- \frac{a\left(40\,a^4 - 242\,a^2 + 103\right)}{28\,800}\,x^7 + \frac{280\,a^6 - 2\,494\,a^4 + 2\,103\,a^2 - 128}{1\,411\,200}\,x^8 -$$
$$- \frac{a\left(2240\,a^6 - 27512\,a^4 + 38356\,a^2 - 6967\right)}{90\,316\,800}\,x^9 + \frac{20160\,a^8 - 326008\,a^6 + 677076\,a^4 - 244839\,a^2 + 8192}{7\,315\,660\,800}\,x^{10} -$$
$$- \frac{\left(40\,320\,a^8 - 829\,424\,a^6 + 2\,397\,216\,a^4 - 1438\,890\,a^2 + 143\,995\right)}{146\,313\,216\,000}\,x^{11} +$$
$$\left. + \frac{443\,520\,a^{10} - 11\,300\,944\,a^8 + 43\,320\,176\,a^6 - 38\,861\,430\,a^4 + 7\,756\,835\,a^2 - 163\,840}{17\,703\,899\,136\,000}\,x^{12} - \ldots\right]\ .$$

b) The case $a > 0$:



$\mathfrak{H}_1(x,a)$
3
2
1
-1
-2
-3
-4
2
4
6
8
x
$a = 0$
$a = 0.1$
$a = 0.2$
$a = 0.5$
$a = 1$



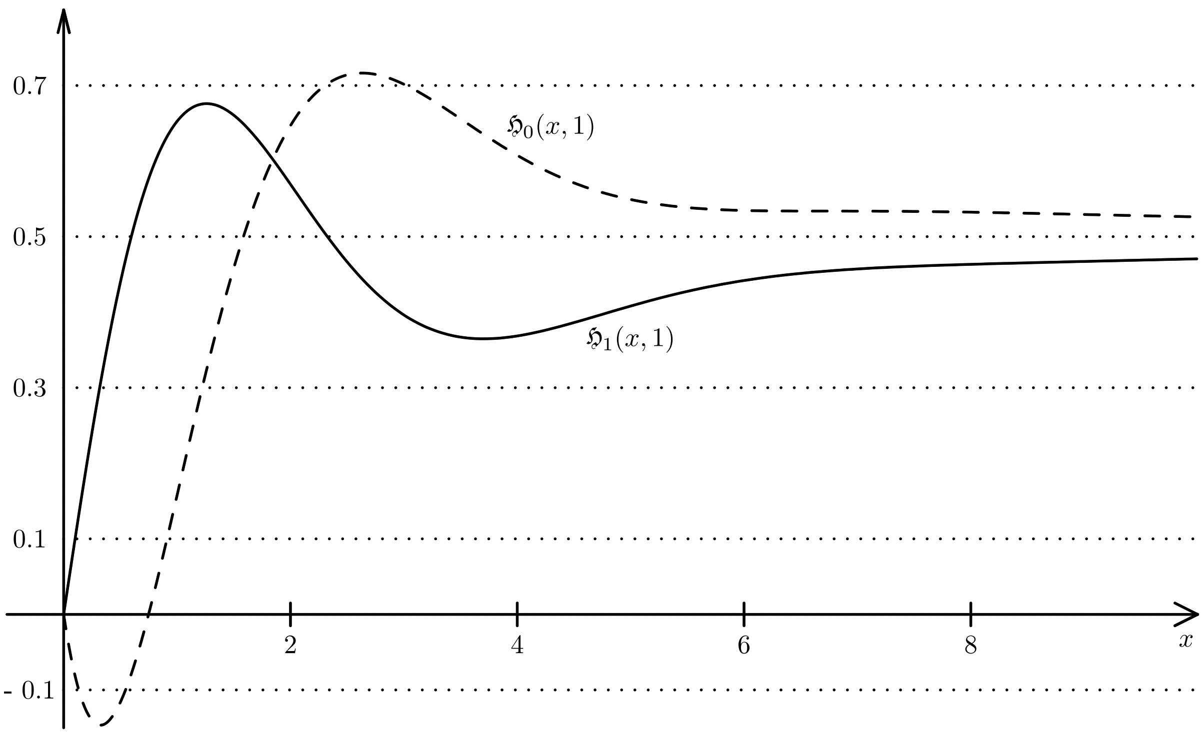
$\mathfrak{H}_0(x,a)$
4
3
2
1
-1
-2
2
4
6
8
x
$a = 0$
$a = 0.1$
$a = 0.2$
$a = 0.5$
$a = 1$

The special case $a = 1$:

$$\int_0^x e^t\, J_0(t)\, dt = e^x \left[\left(x - \frac{2x^3}{3} + \frac{5}{12}\, x^4 - \frac{19}{180}\, x^4 + \frac{7\, x^6}{1440}\, x^4 + \frac{227\, x^7}{50400} - \frac{1147\, x^8}{806400} + \frac{9941\, x^9}{50803200} + \dots\right) J_0(x) + \right.$$

$$\left. + \left(-x + 2x^2 - x^3 + \frac{x^4}{9} + \frac{11}{144}\, x^5 - \frac{131\, x^6}{3600} + \frac{11\, x^7}{1600} - \frac{239\, x^8}{705600} - \frac{2039\, x^9}{15052800} + \dots\right) J_1(x)\right]$$

k	$b_k(1)$	$b_k(1)$	$c_k(1)$	$c_k(1)$
1	1	1.000000000000000	-1	-1.000000000000000
2	0	0.000000000000000	2	2.000000000000000
3	$-2/3$	-0.666666666666667	-1	-1.000000000000000
4	$\frac{5}{12}$	0.416666666666667	$1/9$	0.111111111111111
5	$-\frac{19}{180}$	-0.105555555555556	$\frac{11}{144}$	0.076388888888889
6	$\frac{7}{1440}$	0.004861111111111	$-\frac{131}{3600}$	-0.036388888888889
7	$\frac{227}{50400}$	0.004503968253968	$\frac{11}{1600}$	0.006875000000000
8	$-\frac{1147}{806400}$	-0.001422371031746	$-\frac{239}{705600}$	-0.000338718820862
9	$\frac{9941}{50803200}$	0.000195676650290	$-\frac{2039}{15052800}$	-0.000135456526361
10	$-\frac{979}{162570240}$	-0.000006022012393	$\frac{134581}{3657830400}$	0.000036792575183
11	$-\frac{225107}{80472268800}$	-0.000002797323890	$-\frac{313217}{73156608000}$	-0.000004281458758
12	$\frac{1898819}{3218890752000}$	0.000000589898554	$\frac{1194317}{8851949568000}$	0.000000134921352
13	$-\frac{8554759}{153433792512000}$	-0.000000055755377	$\frac{16109741}{424893579264000}$	0.000000037914767
14	$\frac{14077813}{11047233060864000}$	0.000000001274329	$-\frac{517397957}{71807014895616000}$	-0.000000007205396
15	$\frac{18928541}{47871343263744000}$	0.000000000395404	$\frac{1217807483}{2010596417077248000}$	0.000000000605695
16	$-\frac{402561241}{6433908534647193600}$	-0.000000000062569	$-\frac{422808761}{30158946256158720000}$	-0.000000000014019
17	$\frac{36957033251}{8203233381675171840000}$	0.000000000004505	$-\frac{23427152899}{7720690241576632320000}$	-0.000000000003034
18	$-\frac{21450103637}{262503468213605498880000}$	-0.000000000000082	$\frac{76121087023}{171636883062742056960000}$	0.000000000000444
19	$-\frac{1614496500769}{84788620232994576138240000}$	-0.000000000000019	$-\frac{781266674809}{26775353757787760885760000}$	-0.000000000000029
20	$\frac{4906209165197}{2034926885591869827317760000}$	0.000000000000002	$\frac{5157087816757}{9665902706561381679759360000}$	0.000000000000001



Asymptotic formulas for $x \to +\infty$ in the case $a > 0$:

$$\mathfrak{H}_1(x,a) \sim \frac{a}{1+a^2} - \frac{1}{(1+a^2)^2\,x} - \frac{3a}{(1+a^2)^3\,x^2} - \frac{3\,(4\,a^2-1)}{(1+a^2)^4\,x^3} - \frac{15a\,\left(4\,a^2-3\right)}{(1+a^2)^5\,x^4} -$$

$$-\frac{360\,a^4-540\,a^2+45}{(1+a^2)^6\,x^5} - \frac{315a\,\left(8\,a^4-20\,a^2+5\right)}{(1+a^2)^7\,x^6} - \frac{20160\,a^6-75600\,a^4+37800\,a^2-1575}{(1+a^2)^8\,x^7} - \ldots$$

$$\mathfrak{H}_0(x,a) \sim \frac{1}{1+a^2} + \frac{a}{(1+a^2)^2\,x} + \frac{2\,a^2-1}{(1+a^2)^3\,x^2} + \frac{3\,a\,\left(2\,a^2-3\right)}{(1+a^2)^4\,x^3} + \frac{24\,a^4-72\,a^2+9}{(1+a^2)^5\,x^4} +$$

$$+\frac{15\,a\,\left(8\,a^4-40\,a^2+15\right)}{(1+a^2)^6\,x^5} + \frac{720\,a^6-5400\,a^4+4050\,a^2-225}{(1+a^2)^7\,x^6} + \frac{315\,a\,\left(16\,a^6-168\,a^4+210\,a^2-35\right)}{(1+a^2)^8\,x^7} + \ldots$$

The greater a the better these formulas. They cannot be used with $a = 0$.
The following tables show some relative errors. x_k denotes consecutive maxima or minima of this difference.

$$D_0(x) = \frac{aJ_0(x)+J_1(x)}{1+a^2} - \int_0^x e^{a(t-x)}\,J_0(t)\,dt \quad :$$

$a=0.1$		$a=0.3$		$a=1$		$a=3$	
x_k	$D_0(x_k)$	x_k	$D_0(x_k)$	x_k	$D_0(x_k)$	x_k	$D_0(x_k)$
60.226	-4.072E-3	13.386	-3.089E-3	2.399	-1.085E-1	1.757	-1.101E-2
63.622	-1.710E-4	15.961	-1.866E-2	6.200	1.879E-2	5.495	2.073E-3
66.565	-2.720E-3	19.421	5.504E-3	9.308	-1.068E-2	8.759	-1.015E-3
69.863	4.271E-4	22.413	-7.812E-3	12.481	6.748E-3	11.956	6.306E-4
72.883	-1.942E-3	25.632	5.035E-3	15.639	-4.767E-3	15.129	-4.404E-4
76.120	6.932E-4	28.741	-4.770E-3	18.791	3.595E-3	18.290	3.299E-4
79.188	-1.478E-3	31.900	3.866E-3	21.941	-2.835E-3	21.446	-2.590E-4
82.387	7.898E-4	35.036	-3.439E-3	25.089	2.310E-3	24.598	2.104E-4
85.485	-1.188E-3	38.182	2.988E-3	28.235	-1.929E-3	27.748	-1.753E-4
88.661	8.025E-4	41.323	-2.666E-3	31.381	1.642E-3	30.896	1.490E-4
91.776	-9.978E-4	44.466	2.382E-3	34.525	-1.420E-3	34.043	-1.286E-4
91.776	-9.978E-4	47.608	-2.151E-3	37.670	1.244E-3	37.189	1.126E-4

$$D_1(x) = \left(\frac{a}{1+a^2} - \frac{1}{(1+a^2)^2\,x}\right) J_0(x) + \left(\frac{1}{1+a^2} + \frac{a}{(1+a^2)^2\,x}\right) J_1(x) - \int_0^x e^{a(t-x)}\,J_0(t)\,dt \quad :$$

$a=0.1$		$a=0.3$		$a=1$		$a=3$	
x_k	$D_1(x_k)$	x_k	$D_1(x_k)$	x_k	$D_1(x_k)$	x_k	$D_1(x_k)$
111.782	-1.972E-5	29.538	-2.864E-4	2.061	-5.622E-2	1.816	-3.195E-3
115.333	-4.255E-6	33.053	7.320E-5	6.836	2.161E-3	5.526	2.229E-4
118.158	-1.252E-5	36.018	-1.159E-4	9.729	-1.174E-3	8.831	-6.917E-5
121.548	-3.783E-7	39.248	7.100E-5	12.930	5.554E-4	12.050	3.163E-5
124.497	-8.467E-6	42.348	-6.756E-5	16.086	-3.201E-4	15.237	-1.750E-5
127.789	1.493E-6	45.509	5.293E-5	19.240	2.033E-4	18.408	1.087E-5
130.814	-6.122E-6	48.642	-4.618E-5	22.391	-1.384E-4	21.571	-7.286E-6
134.046	2.317E-6	51.788	3.894E-5	25.540	9.919E-5	24.728	5.164E-6
137.118	-4.708E-6	54.928	-3.381E-5	28.687	-7.393E-5	27.882	-3.817E-6
140.313	2.604E-6	58.070	2.933E-5	31.833	5.683E-5	31.033	2.915E-6

If $x \to +\infty$, then the follwing direct asymptotic formula holds in the case $a > 0$:

$$\int_0^x e^{at} \cdot J_0(t)\, dt \;\sim\; \frac{e^{ax}}{\sqrt{\pi x}} \left[\sum_{k=0}^{\infty} \frac{\lambda_k}{x^k} \sin x + \sum_{k=0}^{\infty} \frac{\mu_k}{x^k} \cos x \right]$$

with

$$\lambda_0 = \frac{a+1}{a^2+1} \;, \quad \mu_0 = \frac{a-1}{a^2+1}$$

$$\lambda_1 = \frac{a^3+3\,a^2+9\,a-5}{8\,\left(a^2+1\right)^2} \;, \quad \mu_1 = \frac{-a^3+3\,a^2-9\,a-5}{8\,\left(a^2+1\right)^2}$$

$$\lambda_2 = \frac{-9\,a^5+15\,a^4+30\,a^3+270\,a^2-345\,a-129}{128\,\left(a^2+1\right)^3}$$

$$\mu_2 = \frac{-9\,a^5-15\,a^4+30\,a^3-270\,a^2-345\,a+129}{128\,\left(a^2+1\right)^3}$$

$$\lambda_3 = \frac{-75\,a^7-105\,a^6-105\,a^5+525\,a^4+5775\,a^3-12075\,a^2-9555\,a+2655}{1024\,\left(a^2+1\right)^4}$$

$$\mu_3 = \frac{75\,a^7-105\,a^6+105\,a^5+525\,a^4-5775\,a^3-12075\,a^2+9555\,a+2655}{1024\,\left(a^2+1\right)^4}$$

$$\lambda_4 = \left[32768\,\left(a^2+1\right)^5\right]^{-1} \cdot [3675\,a^9-4725\,a^8+11340\,a^7-8820\,a^6+92610\,a^5+727650\,a^4- \\ -1984500\,a^3-2407860\,a^2+1371195\,a+301035\,]$$

$$\mu_4 = \left[32768\,\left(a^2+1\right)^5\right]^{-1} \cdot [3675\,a^9+4725\,a^8+11340\,a^7+8820\,a^6+92610\,a^5-727650\,a^4- \\ -1984500\,a^3+2407860\,a^2+1371195\,a-301035\,]$$

$$\lambda_5 = \left[262144\,\left(a^2+1\right)^6\right]^{-1} \cdot [59535\,a^{11}+72765\,a^{10}+259875\,a^9+280665\,a^8+686070\,a^7+3056130\,a^6+ \\ +30124710\,a^5-98232750\,a^4-157827285\,a^3+135748305\,a^2+60259815\,a-10896795\,]$$

$$\mu_5 = \left[262144\,\left(a^2+1\right)^6\right]^{-1} \cdot [-59535\,a^{11}+72765\,a^{10}-259875\,a^9+280665\,a^8-686070\,a^7+3056130\,a^6- \\ -30124710\,a^5-98232750\,a^4+157827285\,a^3+135748305\,a^2-60259815\,a-10896795\,]$$

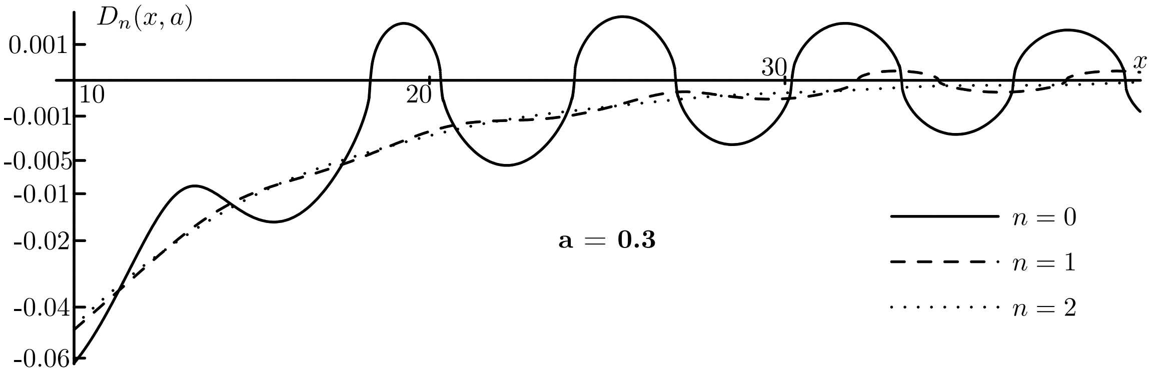
$$\lambda_6 = \left[4194304\,\left(a^2+1\right)^7\right]^{-1} \cdot [-2401245\,a^{13}+2837835\,a^{12}-13243230\,a^{11}+14864850\,a^{10}-34189155\,a^9+ \\ +49054005\,a^8+160540380\,a^7+2871889020\,a^6-11331475155\,a^5-22569301755\,a^4+25820244450\,a^3+ \\ -17234307090\,a^2-6264182925\,a-961319205\,]$$

$$\mu_6 = \left[4194304\,\left(a^2+1\right)^7\right]^{-1} \cdot [-2401245\,a^{13}-2837835\,a^{12}-13243230\,a^{11}-14864850\,a^{10}-34189155\,a^9- \\ -49054005\,a^8+160540380\,a^7-2871889020\,a^6-11331475155\,a^5+22569301755\,a^4+25820244450\,a^3- \\ -17234307090\,a^2-6264182925\,a+961319205\,]$$

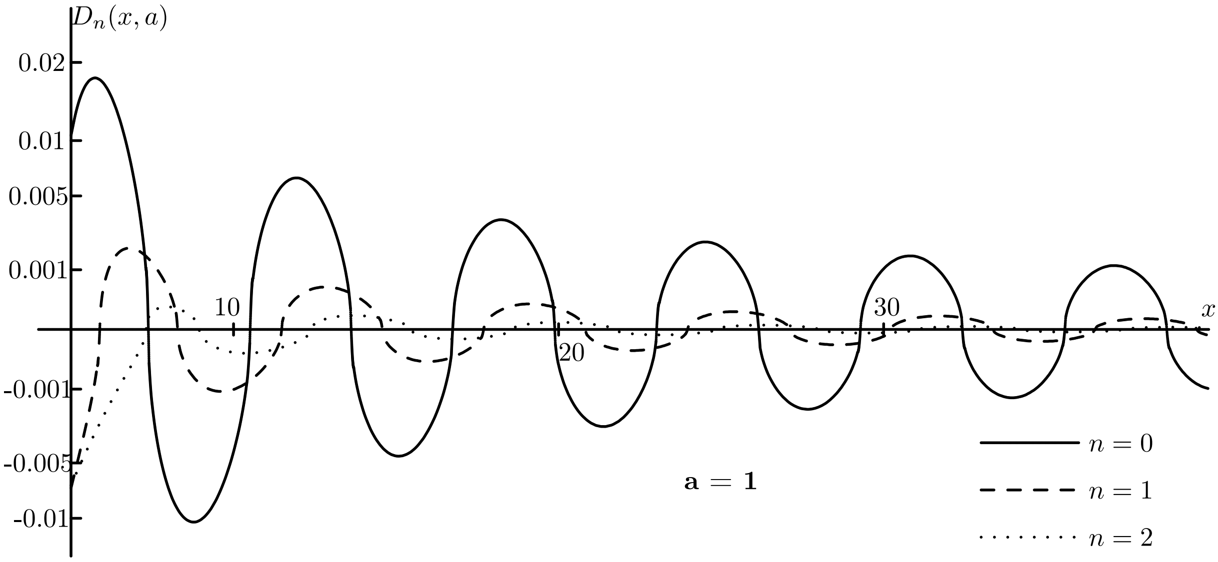
Let

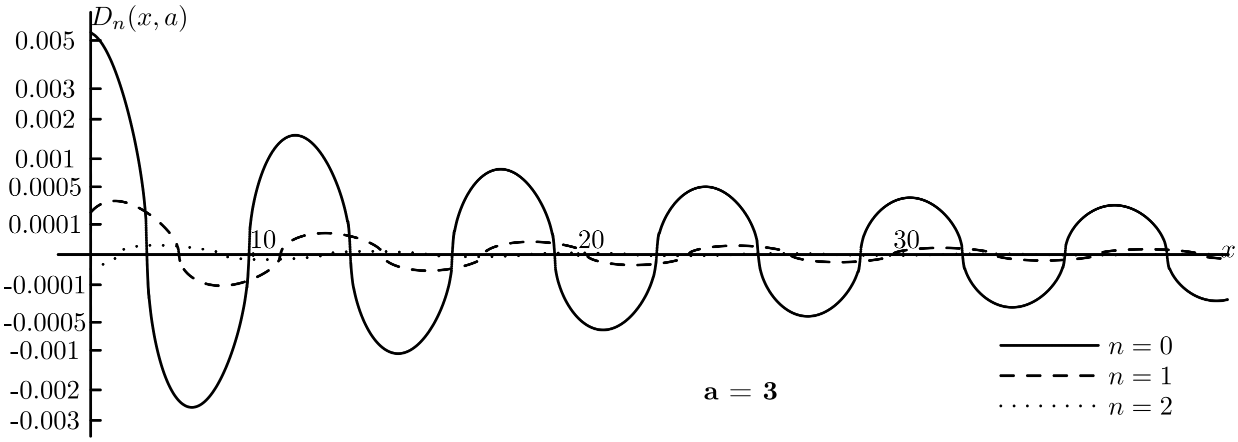
$$D_n(x,a) = \frac{1}{\sqrt{\pi x}} \left[\sum_{k=0}^{n} \frac{\lambda_k}{x^k} \sin x + \sum_{k=0}^{n} \frac{\mu_k}{x^k} \cos x \right] - e^{-ax} \int_0^x e^{at} \cdot J_0(t)\, dt$$

describe the 'relative difference' between the asymptotic approximation and the true function. With $a = 0.3$, $a = 1$ and $a = 3$ one has the following behaviour at $10 \le x \le 40$:



Note that there is a quadratic scale on the D_n-axis.
The first zero of $D_2(x, 0.3)$ is near $x = 48$.





Furthermore, let $\mathfrak{H}_\nu^*(x,a)$ denote the following functions:

$$\mathfrak{H}_1^*(x,a) = \sum_{k=1}^{\infty} b_k^*(a)\, x^k \quad , \qquad \mathfrak{H}_0^*(x,a) = \sum_{k=1}^{\infty} c_k^*(a)\, x^k$$

with

$$b_1^*(a) = 1 \;, \quad b_2^*(a) = 0 \;, \quad b_{k+2}^*(a) = -\frac{a(1+2k)b_{k+1}^*(a) + (1-a^2)b_k^*(a)}{k(k+2)} \quad , \quad k \ge 1$$

and

$$c_k^*(a) = -(k+1)b_{k+1}^*(a) - a\, b_k^*(a) \;.$$

Then holds with $a \in \mathbb{R}$

$$\int_0^x e^{at}\, I_0(t)\, dt = e^{ax}\, [\, \mathfrak{H}_1^*(x,a)\, I_0(x) + \mathfrak{H}_0^*(x,a)\, I_1(x)] \;.$$

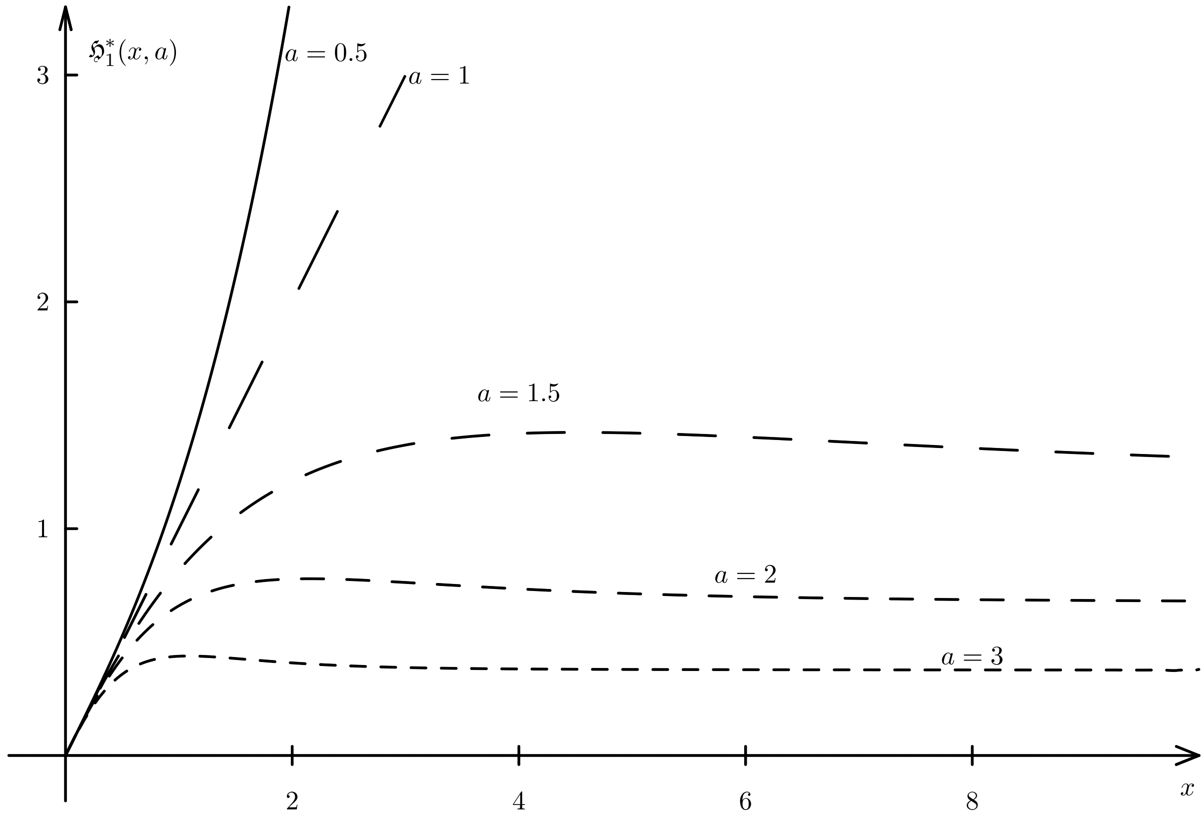
In the case $a = 0$ one has with the Struve functions

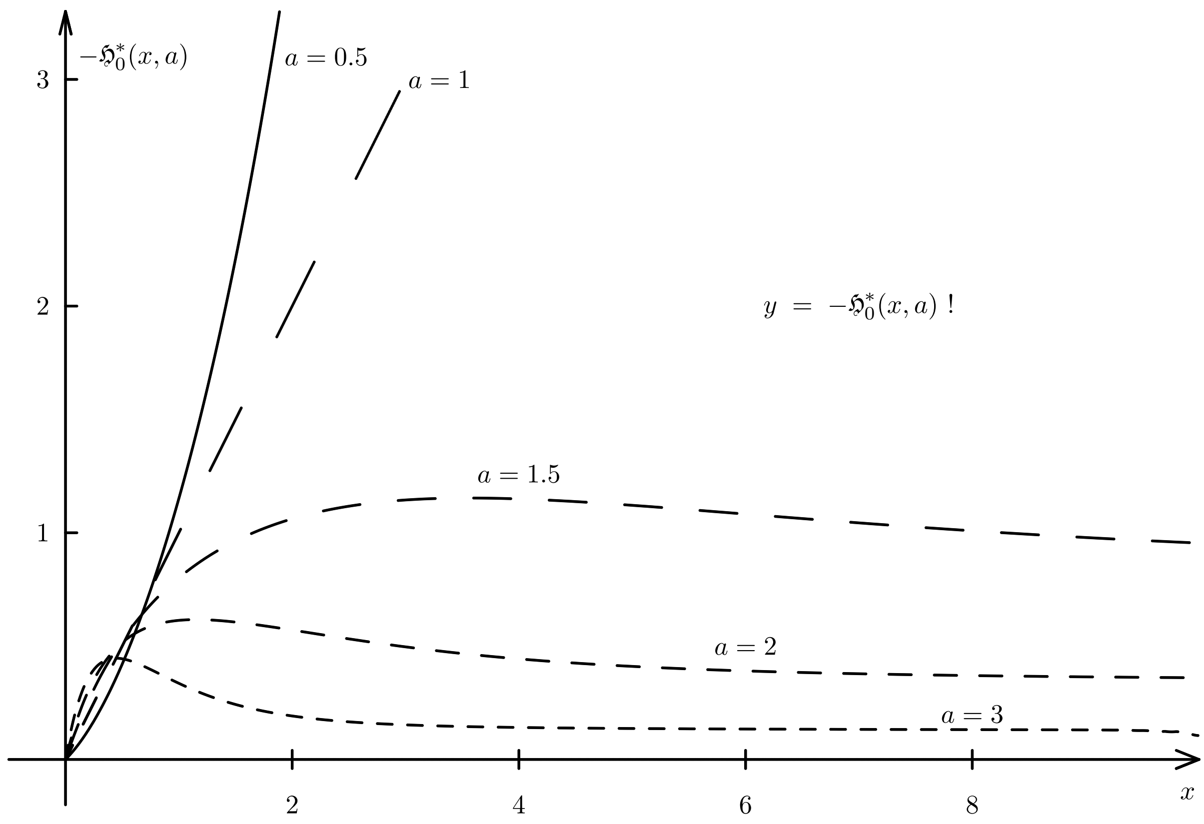
$$\mathfrak{H}_1^*(x,0) = x - \frac{\pi x}{2}\, \mathbf{L}_1(x) \;, \qquad \mathfrak{H}_0^*(x,0) = \frac{\pi x}{2}\, \mathbf{L}_0(x) \;.$$

With $a = 1$ holds $\mathfrak{H}_1^*(x,1) = -\mathfrak{H}_0^*(x,1) = x$ (see page 65).
First terms of the power series:

$$\mathfrak{H}_1^*(x,a) = x - \frac{a^2-1}{3}x^3 + \frac{5a^3-5a}{24}x^4 - \frac{27a^4-19a^2-8}{360}x^5 + \frac{56a^5-7a^3-49a}{2880}x^6 -$$
$$-\frac{400a^6+291a^4-627a^2-64}{100800}x^7 + \frac{1080a^7+1996a^5-2227a^3-849a}{1612800}x^8 -$$
$$-\frac{9800a^8+31684a^6-18717a^4-21743a^2-1024}{101606400}x^9 +$$
$$+\frac{19712a^9+96184a^7-10428a^5-91289a^3-14179a}{1625702400}x^{10} -$$
$$-\frac{217728a^{10}+1477352a^8+608284a^6-1686075a^4-600905a^2-16384}{160944537600}x^{11} +$$
$$+\frac{873600a^{11}+7817888a^9+7776184a^7-9134112a^5-6750965a^3-582595a}{6437781504000}x^{12} - \dots$$

$$\mathfrak{H}_0^*(x,a) = -ax + (a^2-1)x^2 - \frac{a^3-a}{2}x^3 + \frac{3a^4-a^2-2}{18}x^4 - \frac{12a^5+11a^3-23a}{288}x^5 +$$
$$+\frac{60a^6+163a^4-191a^2-32}{7200}x^6 - \frac{40a^7+202a^5-139a^3-103a}{28800}x^7 +$$
$$+\frac{280a^8+2214a^6-391a^4-1975a^2-128}{1411200}x^8 -$$
$$-\frac{2240a^9+25272a^7+10844a^5-31389a^3-6967a}{90316800}x^9 +$$
$$+\frac{20160a^{10}+305848a^8+351068a^6-432237a^4-236647a^2-8192}{7315660800}x^{10} -$$
$$-\frac{40320a^{11}+789104a^9+1567792a^7-958326a^5-1294895a^3-143995a}{146313216000}x^{11} +$$
$$+\frac{443520a^{12}+10857424a^{10}+32019232a^8-4458746a^6-31104595a^4-7592995a^2-163840}{17703899136000}x^{12} - \dots$$





If $0 < a < 1$, then $\mathfrak{H}_1^*(x,a)$ and $-\mathfrak{H}_0^*(x,a)$ are growing rapidly with $x \to +\infty$.

Asymptotic behaviour for $x \to +\infty$ and $a > 1$:

$$\mathfrak{H}_1^*(x,a) \sim \frac{a}{a^2-1} + \frac{1}{(a^2-1)^2\,x} + \frac{3a}{(a^2-1)^3\,x^2} + \frac{12\,a^2+3}{(a^2-1)^4\,x^3} + \frac{15a\,\left(4\,a^2+3\right)}{(a^2-1)^5\,x^4} + \frac{360\,a^4+540\,a^2+45}{(a^2-1)^6\,x^5} + \frac{315a\,\left(8\,a^4+20\,a^2+5\right)}{(a^2-1)^7\,x^6} + \frac{20160\,a^6+75600\,a^4+37800\,a^2+1575}{(a^2-1)^8\,x^7} + \ldots$$

$$\mathfrak{H}_0^*(x,a) \sim -\frac{1}{a^2-1} - \frac{a}{(a^2-1)^2\,x} - \frac{2\,a^2+1}{(a^2-1)^3\,x^2} - \frac{3\,a\left(2\,a^2+3\right)}{(a^2-1)^4\,x^3} - \frac{24\,a^4+72\,a^2+9}{(a^2-1)^5\,x^4} - \frac{15a\,\left(8\,a^4+40\,a^2+15\right)}{(a^2-1)^6\,x^5} - \frac{720\,a^6+5400\,a^4+4050\,a^2+225}{(a^2-1)^7\,x^6} - \frac{315\,\left(16\,a^6+168\,a^4+210\,a^2+35\right)a}{(a^2-1)^8\,x^7} - \ldots$$

Direct asymptotic formula for the case $x \to +\infty,\ a > 0$:

$$\int_0^x e^{at}\cdot I_0(t)\,dt \sim \frac{e^{(1+a)x}}{\sqrt{2\pi x}\,(1+a)} \sum_{k=0}^{\infty} \frac{\beta_k(a)}{x^k} = \frac{e^{(1+a)x}}{\sqrt{2\pi x}\,(1+a)} \left[1 + \frac{a+5}{8(1+a)\,x} + \frac{9\,a^2+42\,a+129}{128\,(1+a)^2\,x^2} + \frac{75\,a^3+405\,a^2+1065\,a+2655}{1024\,(1+a)^3\,x^3} + \frac{3675\,a^4+23100\,a^3+67410\,a^2+133980\,a+301035}{32768\,(1+a)^4\,x^4} + \frac{59535\,a^5+429975\,a^4+1426950\,a^3+3022110\,a^2+5120955\,a+10896795}{262144\,(1+a)^5\,x^5} + \frac{2401245\,a^6+19646550\,a^5+73856475\,a^4+173596500\,a^3+301964355\,a^2+465051510\,a+961319205}{4194304\,(1+a)^6\,x^6} + \frac{135135\,(370345+181955\,a+125205\,a^2+81815\,a^3+43435\,a^4+16569\,a^5+3927\,a^6+429\,a^7)}{33554432\,(1+a)^7\,x^7} + \ldots\right]$$

Some coefficients:

k	$\beta_k(0.1)$	$\beta_k(0.3)$	$\beta_k(1)$	$\beta_k(3)$	$\beta_k(10)$
1	0.579545	0.509615	0.375000	0.250000	0.170455
2	0.860602	0.658330	0.351563	0.164063	0.093556
3	2.029155	1.339262	0.512695	0.175781	0.094505
4	6.568555	3.717857	1.009369	0.265961	0.142222
5	27.098470	13.096614	2.498188	0.526314	0.285290
6	136.064853	55.981252	7.442518	1.296183	0.715146
7	805.747313	281.633987	25.915913	3.834025	2.150314

With fixed $x > 0$ and $a \to +\infty$ holds

$$\int_0^x e^{a\,t}\,J_0(t)\,dt \sim e^{ax}\left[\left(\frac{1}{a} - \frac{1}{a^3} - \frac{1}{a^4x} + \frac{x^2-3}{a^5x^2} + \frac{2\,x^2-12}{a^6x^3} - \frac{x^4-9\,x^2+60}{a^7x^4} - \frac{3\,x^4-51\,x^2+360}{a^8x^5} + \frac{x^6-18\,x^4+345\,x^2-2520}{a^9x^6} + \frac{4\,x^6-132\,x^4+2700\,x^2-20160}{a^{10}x^7} + \ldots\right) J_0(x) + \left(\frac{1}{a^2} + \frac{1}{a^3x} - \frac{x^2-2}{a^4x^2} - \frac{2\,x^2-6}{a^5x^3} + \frac{x^4-7\,x^2+24}{a^6x^4} + \frac{3\,x^4-33\,x^2+120}{a^7x^5} - \frac{x^6-15\,x^4+192\,x^2-720}{a^8x^6} - \frac{4\,x^6-96\,x^4+1320\,x^2-5040}{a^9x^7} + \frac{x^8-26\,x^6+729\,x^4-10440\,x^2+40320}{a^{10}x^8} + \ldots\right) J_1(x)\right]$$

and

$$\int_0^x e^{a\,t}\, I_0(t)\,dt \;\sim\; e^{ax}\, \left[\left(\frac{1}{a} + \frac{1}{a^3} + \frac{1}{a^4x} + \frac{x^2+3}{a^5x^2} + \frac{2\,x^2+12}{a^6x^3} + \frac{x^4+9\,x^2+60}{a^7x^4} + \frac{3\,x^4+51\,x^2+360}{a^8x^5} + \right.\right.$$

$$\left. \frac{x^6+18\,x^4+345\,x^2+2520}{a^9x^6} + \frac{4\,x^6+132\,x^4+2700\,x^2+20160}{a^{10}x^7} + \ldots\right) I_0(x) -$$

$$-\left(\frac{1}{a^2} + \frac{1}{a^3x} + \frac{x^2+2}{a^4x^2} + \frac{2\,x^2+6}{a^5x^3} + \frac{x^4+7\,x^2+24}{a^6x^4} + \frac{3\,x^4+33\,x^2+120}{a^7x^5} + \frac{x^6+15\,x^4+192\,x^2+720}{a^8x^6} + \right.$$

$$\left.\left. \frac{4\,x^6+96\,x^4+1320\,x^2+5040}{a^9x^7} + \frac{x^8+26\,x^6+729\,x^4+10440\,x^2+40320}{a^{10}x^8}\right) I_1(x)\right] .$$

c) The case $a < 0$:

To express this fact clearly it is written $a = -\alpha$ with $\alpha > 0$.
One has the Lipschitz integral (see [11], part I, §21, or the tables of Laplace transforms)

$$\int_0^\infty e^{a\,x}\, J_0(x)\,dx = \int_0^\infty e^{-\alpha\,x}\, J_0(x)\,dx = \frac{1}{\sqrt{1+\alpha^2}} = \frac{1}{\sqrt{1+a^2}} .$$

The representation

$$\int_0^x e^{a\,t}\, J_0(t)\,dt \;=\; e^{ax}\;[\;\mathfrak{H}_1(x,a)\, J_0(x) + \;\mathfrak{H}_0(x,a)\, J_1(x)]$$

keeps being true, but $\mathfrak{H}_0(x,a)$ and $\mathfrak{H}_1(x,a)$ are rapidly growing with x. For that reason other formulas are more applicable.

$$\int_0^x e^{-\alpha\,t}\, J_0(t)\,dt \;=\; \frac{1}{\sqrt{1+\alpha^2}} - e^{-\alpha\,x}\,\left[\frac{1}{\sqrt{1+\alpha^2}} - \sum_{k=1}^{\infty} \varphi_k(\alpha)\cdot x^{2k-1}\,\cdot\left(1+\frac{\alpha\,x}{2k}\right)\right]$$

with

$$\varphi_k(\alpha) = \frac{(-1)^{k-1}}{(2k-1)!\cdot a^2}\,\sum_{i=0}^{\infty}\frac{(-1)^i\cdot(2i+2k)!}{2^{2i+2k}\cdot[(i+k)!]^2\cdot a^{2i}}$$

Some first functions:

$$\varphi_1(\alpha) = \frac{\sqrt{1+\alpha^2}-\alpha}{\sqrt{1+\alpha^2}}$$

$$\varphi_2(\alpha) = \frac{1}{6}\cdot\frac{(2\,\alpha^2-1)\,\sqrt{1+\alpha^2}-2\,\alpha^3}{2\,\sqrt{1+\alpha^2}}$$

$$\varphi_3(\alpha) = \frac{1}{120}\cdot\frac{(8\,\alpha^4-4\,\alpha^2+3)\,\sqrt{1+\alpha^2}-8\,\alpha^5}{8\,\sqrt{1+\alpha^2}}$$

$$\varphi_4(\alpha) = \frac{1}{5040}\cdot\frac{(16\,\alpha^6-8\,\alpha^4+6\,\alpha^2-5)\,\sqrt{1+\alpha^2}-16\,\alpha^7}{16\,\sqrt{1+\alpha^2}}$$

$$\varphi_5(\alpha) = \frac{1}{362\,880}\cdot\frac{(128\,\alpha^8-64\,\alpha^6+48\,\alpha^4-40\,\alpha^2+35)\,\sqrt{1+\alpha^2}-128\,\alpha^9}{128\,\sqrt{1+\alpha^2}}$$

$$\varphi_6(\alpha) = \frac{1}{29\,916\,800}\cdot\frac{(256\,\alpha^{10}-128\,\alpha^8+96\,\alpha^6-80\,\alpha^4+70\,\alpha^2-63)\,\sqrt{1+\alpha^2}-256\,\alpha^{11}}{256\,\sqrt{1+\alpha^2}}$$

$$\varphi_7(\alpha) = \frac{1}{1\,307\,674\,368\,000}\cdot$$

$$\cdot\frac{+(1024\,\alpha^{12}-512\,\alpha^{10}+384\,\alpha^8-320\,\alpha^6+280\,\alpha^4-252\,\alpha^2+231)\,\sqrt{1+\alpha^2}-1024\,\alpha^{13}}{1024\,\sqrt{1+\alpha^2}}$$

Special values for the case $\alpha = 1$:

$$\varphi_1(1) = \frac{\sqrt{2}}{2+2\sqrt{2}} = 0.292893218813452\;, \quad \varphi_2(1) = \frac{1-\sqrt{2}}{12} = -0.034517796864425\;,$$

$$\varphi_3(1) = -\frac{1}{240}\sqrt{2} + \frac{7}{960} = 0.001399110156779\ , \quad \varphi_4(1) = -\frac{1}{10080}\sqrt{2} + \frac{1}{8960} = -0.000028691821664\ ,$$

$$\varphi_5(1) = -\frac{1}{725760}\sqrt{2} + \frac{107}{46448640} = 0.000000355022924\ ,$$

$$\varphi_6(1) = -\frac{1}{79833600}\sqrt{2} + \frac{151}{10218700800} = -0.000000002937686\ ,$$

$$\varphi_7(1) = -\frac{1}{12454041600}\sqrt{2} + \frac{167}{1275293859840} = 0.000000000017396\ ,$$

$$\varphi_8(1) = -\frac{1}{2615348736000}\sqrt{2} + \frac{1241}{2678117105664000} = -0.000000000000077\ ,$$

Asymptotic behaviour of $\varphi_k(\alpha)$ for $\alpha \to +\infty$:

$$\frac{1}{\sqrt{1+\alpha^2}} \sim \alpha^{-1} - \frac{1}{2}\alpha^{-3} + \frac{3}{8}\alpha^{-5} - \frac{5}{16}\alpha^{-7} + \frac{35}{128}\alpha^{-9} + \ldots$$

$$\varphi_1(\alpha) \sim \frac{1}{2}\alpha^{-2} - \frac{3}{8}\alpha^{-4} + \frac{5}{16}\alpha^{-6} - \frac{35}{128}\alpha^{-8} + \ldots$$

$$3!\cdot\varphi_2(\alpha) \sim -\frac{3}{8}\alpha^{-2} + \frac{5}{16}\alpha^{-4} - \frac{35}{128}\alpha^{-6} + \frac{63}{256}\alpha^{-8} + \ldots$$

$$5!\cdot\varphi_3(\alpha) \sim \frac{5}{16}\alpha^{-2} - \frac{35}{128}\alpha^{-4} + \frac{63}{256}\alpha^{-6} - \frac{231}{1024}\alpha^{-8}\ldots$$

$$7!\cdot\varphi_4(\alpha) \sim -\frac{35}{128}\alpha^{-2} + \frac{63}{256}\alpha^{-4} - \frac{231}{1024}\alpha^{-6} + \frac{429}{2048}\alpha^{-8} + \ldots$$

$$9!\cdot\varphi_5(\alpha) \sim \frac{63}{256}\alpha^{-2} - \frac{231}{1024}\alpha^{-4} + \frac{429}{2048}\alpha^{-6} - \frac{6435}{32768}\alpha^{-8} + \ldots$$

$$11!\cdot\varphi_6(\alpha) \sim -\frac{231}{1024}\alpha^{-2} + \frac{429}{2048}\alpha^{-4} - \frac{6435}{32768}\alpha^{-6} + \frac{12155}{65536}\alpha^{-8} + \ldots$$

$$13!\cdot\varphi_7(\alpha) \sim \frac{429}{2048}\alpha^{-2} - \frac{6435}{32768}\alpha^{-4} + \frac{12155}{65536}\alpha^{-6} - \frac{46189}{262144}\alpha^{-8} + \ldots$$

Direct asymptotic formula for $x \to \infty$:

$$\int_0^x e^{-\alpha t} J_0(t)\,dt \sim \frac{1}{\sqrt{1+\alpha^2}} + \frac{e^{-\alpha x}}{\sqrt{\pi x}}\left\{\left[-\frac{\alpha-1}{1+\alpha^2} - \frac{\alpha^3-3\alpha^2+9\alpha+5}{8(1+\alpha^2)^2 x} + \right.\right.$$

$$+\frac{3(3\alpha^5+5\alpha^4-10\alpha^3+90\alpha^2+115\alpha-43)}{128(1+\alpha^2)^3 x^2} +$$

$$+\frac{15(5\alpha^7-7\alpha^6+7\alpha^5+35\alpha^4-385\alpha^3-805\alpha^2+637\alpha+177)}{1024(1+\alpha^2)^4 x^3} -$$

$$-\frac{105(35\alpha^9+45\alpha^8+108\alpha^7+84\alpha^6+882\alpha^5-6930\alpha^4-18900\alpha^3+22932\alpha^2+13059\alpha-2867)}{32768(1+\alpha^2)^5 x^4} +$$

$$-\frac{945\,s_5}{262144(1+\alpha^2)^6 x^5} - \frac{34459425\,s_6}{17179869184(1+\alpha^2)^7 x^6} - \frac{135135\,s_7}{33554432(1+\alpha^2)^8 x^7} - \frac{2027025\,s_8}{2147483648(1+\alpha^2)^9 x^8} -$$

$$\left. -\frac{34459425\,s_9}{17179869184(1+\alpha^2)^{10} x^9} + \frac{654729075\,s_{10}}{274877906944(1+\alpha^2)^{11} x^{10}} + \ldots\right]\sin x +$$

$$+\left[-\frac{1+\alpha}{1+\alpha^2} + \frac{\alpha^3+3\alpha^2+9\alpha-5}{8(1+\alpha^2)^2 x} + \frac{3(3\alpha^5-5\alpha^4-10\alpha^3-90\alpha^2+115\alpha+43)}{128(1+\alpha^2)^3 x^2} - \right.$$

$$-\frac{15(5\alpha^7+7\alpha^6+7\alpha^5-35\alpha^4-385\alpha^3+805\alpha^2+637\alpha-177)}{(1+\alpha^2)^4 x^3} -$$

$$-\frac{105(35\alpha^9-45\alpha^8+108\alpha^7-84\alpha^6+882\alpha^5+6930\alpha^4-18900\alpha^3-22932\alpha^2+13059\alpha+2867)}{32768(1+\alpha^2)^5 x^4} +$$

$$+\frac{945\, c_5}{262144\,(1+\alpha^2)^6\, x^5}+\frac{10395\, c_6}{4194304\,(1+\alpha^2)^7\, x^6}-\frac{135135\, c_7}{33554432\,(1+\alpha^2)^8\, x^7}-\frac{2027025\, c_8}{2147483648\,(1+\alpha^2)^9\, x^8}+$$

$$+\frac{34459425\, c_9}{17179869184\,(1+\alpha^2)^{10}\, x^9}+\frac{654729075\,274877906944 c_{10}}{274877906944\,(1+\alpha^2)^{11}\, x^{10}}+\ldots]\ \cos x\ \Big\}$$

$$s_5 = 63\,\alpha^{11} - 77\,\alpha^{10} + 275\,\alpha^9 - 297\,\alpha^8 + 726\,\alpha^7 - 3234\,\alpha^6 + 31878\,\alpha^5 + 103950\,\alpha^4 - 167013\,\alpha^3 -$$
$$-143649\,\alpha^2 + 63767\,\alpha + 11531$$

$$c_5 = 63\,\alpha^{11} + 77\,\alpha^{10} + 275\,\alpha^9 + 297\,\alpha^8 + 726\,\alpha^7 + 3234\,\alpha^6 + 31878\,\alpha^5 - 103950\,\alpha^4 -$$
$$-167013\,\alpha^3 + 143649\,\alpha^2 + 63767\,\alpha - 11531$$

$$s_6 = 231\,\alpha^{13} + 273\,\alpha^{12} + 1274\,\alpha^{11} + 1430\,\alpha^{10} + 3289\,\alpha^9 + 4719\,\alpha^8 - 15444\,\alpha^7 + 276276\,\alpha^6 + 1090089\,\alpha^5 -$$
$$-2171169\,\alpha^4 - 2483910\,\alpha^3 + 1657942\,\alpha^2 + 602615\,\alpha - 92479$$

$$c_6 = 231\,\alpha^{13} - 273\,\alpha^{12} + 1274\,\alpha^{11} - 1430\,\alpha^{10} + 3289\,\alpha^9 - 4719\,\alpha^8 - 15444\,\alpha^7 - 276276\,\alpha^6 + 1090089\,\alpha^5 +$$
$$+2171169\,\alpha^4 - 2483910\,\alpha^3 - 1657942\,\alpha^2 + 602615\,\alpha + 92479$$

$$s_7 = 429\,\alpha^{15} - 495\,\alpha^{14} + 2835\,\alpha^{13} - 3185\,\alpha^{12} + 8385\,\alpha^{11} - 9867\,\alpha^{10} + 9295\,\alpha^9 + 57915\,\alpha^8 - 1151865\,\alpha^7 -$$
$$-5450445\,\alpha^6 + 13054041\,\alpha^5 + 18629325\,\alpha^4 - 16564405\,\alpha^3 - 9039225\,\alpha^2 + 2780805\,\alpha + 370345$$

$$c_7 = 429\,\alpha^{15} + 495\,\alpha^{14} + 2835\,\alpha^{13} + 3185\,\alpha^{12} + 8385\,\alpha^{11} + 9867\,\alpha^{10} + 9295\,\alpha^9 - 57915\,\alpha^8 - 1151865\,\alpha^7 +$$
$$+5450445\,\alpha^6 + 13054041\,\alpha^5 - 18629325\,\alpha^4 - 16564405\,\alpha^3 + 9039225\,\alpha^2 + 2780805\,\alpha - 370345$$

$$s_8 = 6435\,\alpha^{17} + 7293\,\alpha^{16} + 49368\,\alpha^{15} + 55080\,\alpha^{14} + 168980\,\alpha^{13} + 190060\,\alpha^{12} + 312936\,\alpha^{11} + 252824\,\alpha^{10} +$$
$$+2601170\,\alpha^9 - 39163410\,\alpha^8 - 210913560\,\alpha^7 + 591783192\,\alpha^6 + 1014047892\,\alpha^5 - 1126379540\,\alpha^4 -$$
$$-819264680\,\alpha^3 + 378189480\,\alpha^2 + 100843235\,\alpha - 11857475$$

$$c_8 = 6435\,\alpha^{17} - 7293\,\alpha^{16} + 49368\,\alpha^{15} - 55080\,\alpha^{14} + 168980\,\alpha^{13} - 190060\,\alpha^{12} + 312936\,\alpha^{11} - 252824\,\alpha^{10} +$$
$$+2601170\,\alpha^9 + 39163410\,\alpha^8 - 210913560\,\alpha^7 - 591783192\,\alpha^6 + 1014047892\,\alpha^5 + 1126379540\,\alpha^4 -$$
$$-819264680\,\alpha^3 - 378189480\,\alpha^2 + 100843235\,\alpha + 11857475$$

$$s_9 = 12155\,\alpha^{19} - 13585\,\alpha^{18} + 105963\,\alpha^{17} - 117249\,\alpha^{16} + 414732\,\alpha^{15} - 458660\,\alpha^{14} + 936700\,\alpha^{13} - 990964\,\alpha^{12} +$$
$$+1771978\,\alpha^{11} - 9884446\,\alpha^{10} + 168589850\,\alpha^9 + 1001839410\,\alpha^8 - 3209765988\,\alpha^7 - 6422303316\,\alpha^6 +$$
$$+8562147308\,\alpha^5 + 7783014460\,\alpha^4 - 4789707245\,\alpha^3 - 1916021465\,\alpha^2 + 450814995\,\alpha + 47442055$$

$$c_9 = 12155\,\alpha^{19} + 13585\,\alpha^{18} + 105963\,\alpha^{17} + 117249\,\alpha^{16} + 414732\,\alpha^{15} + 458660\,\alpha^{14} + 936700\,\alpha^{13} + 990964\,\alpha^{12} +$$
$$+1771978\,\alpha^{11} + 9884446\,\alpha^{10} + 168589850\,\alpha^9 - 1001839410\,\alpha^8 - 3209765988\,\alpha^7 + 6422303316\,\alpha^6 +$$

$$+8562147308\,\alpha^5 - 7783014460\,\alpha^4 - 4789707245\,\alpha^3 + 1916021465\,\alpha^2 + 450814995\,\alpha - 47442055$$

$$s_{10} = 46189\,\alpha^{21} + 51051\,\alpha^{20} + 450450\,\alpha^{19} + 494494\,\alpha^{18} + 1988217\,\alpha^{17} + 2177343\,\alpha^{16} + 5191256\,\alpha^{15} +$$
$$+5620200\,\alpha^{14} + 9265578\,\alpha^{13} + 12403846\,\alpha^{12} - 53260116\,\alpha^{11} + 1416154740\,\alpha^{10} + 9373133770\,\alpha^9 -$$
$$-33702542874\,\alpha^8 - 77051011752\,\alpha^7 + 119870062312\,\alpha^6 + 130763372649\,\alpha^5 - 100583852145\,\alpha^4 -$$
$$-53645367790\,\alpha^3 + 18934229790\,\alpha^2 + 3986102589\,\alpha - 379582629$$

$$c_{10} = 46189\,\alpha^{21} - 51051\,\alpha^{20} + 450450\,\alpha^{19} - 494494\,\alpha^{18} + 1988217\,\alpha^{17} - 2177343\,\alpha^{16} + 5191256\,\alpha^{15} -$$
$$-5620200\,\alpha^{14} + 9265578\,\alpha^{13} - 12403846\,\alpha^{12} - 53260116\,\alpha^{11} - 1416154740\,\alpha^{10} + 9373133770\,\alpha^9 +$$
$$+33702542874\,\alpha^8 - 77051011752\,\alpha^7 - 119870062312\,\alpha^6 + 130763372649\,\alpha^5 + 100583852145\,\alpha^4 -$$
$$-53645367790\,\alpha^3 - 18934229790\,\alpha^2 + 3986102589\,\alpha + 379582629$$

In the special case $\alpha = 1$ holds

$$\int_0^x e^{-t}\,J_0(t)\,dt \;\sim\; \frac{1}{\sqrt{2}} + \frac{e^{-x}}{\sqrt{\pi x}}\;\left[\left(\sum_{k=0}^{\infty}\frac{s_k^*}{x^k}\right)\sin x + \left(\sum_{k=0}^{\infty}\frac{c_k^*}{x^k}\right)\cos x\right]$$

with

k	s_k^*	s_k^*	c_k^*	c_k^*
0	0	0	-1	-1
1	$-\frac{3}{8}$	-0.375000000000000	$\frac{1}{4}$	0.250000000000000
2	$\frac{15}{32}$	0.468750000000000	$\frac{21}{128}$	0.164062500000000
3	$-\frac{315}{1024}$	-0.307617187500000	$-\frac{405}{512}$	-0.791015625000000
4	$-\frac{3465}{4096}$	-0.845947265625000	$\frac{59325}{32768}$	1.810455322265625
5	$\frac{1507275}{262144}$	5.749797821044922	$-\frac{284445}{131072}$	-2.170143127441406
6	$-\frac{22837815}{1048576}$	-21.77983760833740	$-\frac{38887695}{4194304}$	-9.271548986434937
7	$\frac{1422025605}{33554432}$	42.37966552376747	$\frac{1693106415}{16777216}$	100.9170064330101
8	$\frac{29462808375}{134217728}$	219.5150284096599	$-\frac{1167021130275}{2147483648}$	-543.4365618391894
9	$-\frac{56125340496225}{17179869184}$	-3266.924788257165	$\frac{11825475336675}{8589934592}$	1376.666517075500
10	$\frac{1515749532221925}{68719476736}$	22057.05870033016	$\frac{2498294907783675}{274877906944}$	9088.743928382155

The Laplace transform of $I_0(x)$ exists only in the case $\alpha > 1$:

$$\int_0^{\infty} e^{a\,x}\,I_0(x)\,dx = \int_0^{\infty} e^{-\alpha\,x}\,I_0(x)\,dx = \frac{1}{\sqrt{\alpha^2-1}} = \frac{1}{\sqrt{a^2-1}}\;.$$

If $\alpha > 1$ one has

$$\int_0^x e^{-\alpha\,t}\,I_0(t)\,dt \;=\; \frac{1}{\sqrt{\alpha^2-1}} - e^{-\alpha\,x}\;\left[\frac{1}{\sqrt{\alpha^2-1}} + \sum_{k=1}^{\infty}\varphi_k^*(\alpha)\cdot x^{2k-1}\cdot\left(1+\frac{\alpha\,x}{2k}\right)\right]$$

with

$$\varphi_k^*(\alpha) = \frac{1}{(2k-1)!\cdot a^2}\;\sum_{i=0}^{\infty}\frac{(2i+2k)!}{2^{2i+2k}\cdot[(i+k)!]^2\cdot a^{2i}}$$

This series fails to converge in the case $0 < \alpha \le 1$.
If $\alpha = 1$, then see page 65 for solutions with elementary functions.

Some first functions $\varphi_k^*(\alpha)$:

$$\varphi_1^*(\alpha) = \frac{\alpha - \sqrt{\alpha^2-1}}{\sqrt{\alpha}^2 - 1}$$

$$\varphi_2^*(\alpha) = \frac{1}{6} \cdot \frac{2\alpha^3 - (2\,\alpha^2+1)\sqrt{\alpha^2-1}}{2\sqrt{\alpha^2-1}}$$

$$\varphi_3^*(\alpha) = \frac{1}{120} \cdot \frac{8\alpha^5 - (8\,\alpha^4+4\,\alpha^2+3)\sqrt{\alpha^2-1}}{8\sqrt{\alpha^2-1}}$$

$$\varphi_4^*(\alpha) = \frac{1}{5\,040} \cdot \frac{16\alpha^7 - (16\,\alpha^6+8\,\alpha^4+6\,\alpha^2+5)\sqrt{\alpha^2-1}}{16\sqrt{\alpha^2-1}}$$

$$\varphi_5^*(\alpha) = \frac{1}{362\,880} \cdot \frac{128\alpha^9 - (128\,\alpha^8+64\,\alpha^6+48\,\alpha^4+40\,\alpha^2+35)\sqrt{\alpha^2-1}}{128\sqrt{\alpha^2-1}}$$

$$\varphi_6^*(\alpha) = \frac{1}{39\,916\,800} \cdot \frac{(256\,\alpha^{10}+128\,\alpha^8+96\,\alpha^6+80\,\alpha^4+70\,\alpha^2+63)\sqrt{\alpha^2-1}}{256\sqrt{\alpha^2-1}}$$

Asymptotic behaviour of $\varphi_k^*(\alpha)$ for $\alpha \to +\infty$:

$$\frac{1}{\sqrt{\alpha^2-1}} \sim \alpha^{-1} + \frac{1}{2}\alpha^{-3} + \frac{3}{8}\alpha^{-5} + \frac{5}{16}\alpha^{-7} + \frac{35}{128}\alpha^{-9} + \ldots$$

$$\varphi_1^*(\alpha) \sim \frac{1}{2}\alpha^{-2} + \frac{3}{8}\alpha^{-4} + \frac{5}{16}\alpha^{-6} + \frac{35}{128}\alpha^{-8} + \ldots$$

$$6\,\varphi_2^*(\alpha) \sim \frac{3}{8}\alpha^{-2} + \frac{5}{16}\alpha^{-4} + \frac{35}{128}\alpha^{-6} + \frac{63}{256}\alpha^{-8} + \ldots$$

$$120\,\varphi_3^*(\alpha) \sim \frac{5}{16}\alpha^{-2} + \frac{35}{128}\alpha^{-4} + \frac{63}{256}\alpha^{-6} + \frac{231}{1024}\alpha^{-8} + \ldots$$

$$5\,040\,\varphi_4^*(\alpha) \sim \frac{35}{128}\alpha^{-2} + \frac{63}{256}\alpha^{-4} + \frac{231}{1024}\alpha^{-6} + \frac{429}{2048}\alpha^{-8} + \ldots$$

$$362\,880\,\varphi_5^*(\alpha) \sim \frac{63}{256}\alpha^{-2} + \frac{231}{1024}\alpha^{-4} + \frac{429}{2048}\alpha^{-6} + \frac{6435}{32768}\alpha^{-8} + \ldots$$

$$39\,916\,800\,\varphi_6^*(\alpha) \sim \frac{231}{1024}\alpha^{-2} + \frac{429}{2048}\alpha^{-4} + \frac{6435}{32768}\alpha^{-6} + \frac{12155}{65536}\alpha^{-8} + \ldots$$

$$6\,227\,020\,800\,\varphi_7^*(\alpha) \sim \frac{429}{2048}\alpha^{-2} + \frac{6435}{32768}\alpha^{-4} + \frac{12155}{65536}\alpha^{-6} + \frac{46189}{262144}\alpha^{-8} + \ldots$$

If $0 < \alpha \le 1 \iff -1 \le a < 0$ and x is not very large, then $\mathfrak{H}_\nu^*(x,a)$ may be used.
With $0 < \alpha < 1$, $x >> 1$ one has the direct asymptotic formula

$$\int_0^x e^{-\alpha\,t}\,I_0(t)\,dt \sim \frac{e^{(1-\alpha)x}}{\sqrt{2\pi x}} \sum_{k=0}^{\infty} \frac{w_k(\alpha)}{x^k} = \frac{e^{(1-\alpha)x}}{\sqrt{2\pi x}} \left[\frac{1}{1-\alpha} - \frac{\alpha-5}{8\,(1-\alpha)^2\,x} - \frac{9\,\alpha^2-42\,\alpha+129}{128\,(1-\alpha)^3\,x^2} - \right.$$

$$- \frac{75\,\alpha^3-405\,\alpha^2+1065\,\alpha-2655}{1024\,(1-\alpha)^4\,x^3} - \frac{3675\,\alpha^4-23100\,\alpha^3+67410\,\alpha^2-133980\,\alpha+301035}{32768\,(1-\alpha)^5\,x^4} -$$

$$- \frac{59535\,\alpha^5-429975\,\alpha^4+1426950\,\alpha^3-3022110\,\alpha^2+5120955\,\alpha-10896795}{262144\,(1-\alpha)^6\,x^5} -$$

$$- \frac{10395}{4194304} \cdot \frac{z_6}{(1-\alpha)^7\,x^6} - \frac{135135}{33554432} \cdot \frac{z_7}{(1-\alpha)^8\,x^7} - \frac{2027025}{2147483648} \cdot \frac{z_8}{(1-\alpha)^9\,x^8} -$$

$$\left. - \frac{34459425}{17179869184} \cdot \frac{z_9}{(1-\alpha)^{10}\,x^9} - \frac{654729075}{274877906944} \cdot \frac{z_{10}}{(1-\alpha)^{11}x^{10}} + \ldots \right]$$

$$z_6 = 231\,\alpha^6 - 1890\,\alpha^5 + 7105\,\alpha^4 - 16700\,\alpha^3 + 29049\,\alpha^2 - 44738\,\alpha + 92479$$

$$z_7 = 429\,\alpha^7 - 3927\,\alpha^6 + 16569\,\alpha^5 - 43435\,\alpha^4 + 81815\,\alpha^3 - 125205\,\alpha^2 + 181955\,\alpha - 370345$$

$$z_8 = 6435\,\alpha^8 - 65208\,\alpha^7 + 305844\,\alpha^6 - 890568\,\alpha^5 + 1840370\,\alpha^4 - 2978440\,\alpha^3 + 4186740\,\alpha^2 - 5874040\,\alpha + 11857475$$

$$z_9 = 12155\,\alpha^9 - 135135\,\alpha^8 + 698412\,\alpha^7 - 2244396\,\alpha^6 + 5093802\,\alpha^5 - 8893010\,\alpha^4 + 12934780\,\alpha^3 - 17184540\,\alpha^2 + $$
$$+ 23605555\,\alpha - 47442055$$
$$z_{10} = 46189\,\alpha^{10} - 559130\,\alpha^9 + 3159585\,\alpha^8 - 11129976\,\alpha^7 + 27654858\,\alpha^6 - 52390044\,\alpha^5 + 80843770\,\alpha^4 - $$
$$- 109020920\,\alpha^3 + 139554825\,\alpha^2 - 189306330\,\alpha + 379582629$$

The first functions $w_k(\alpha)$:

x	$w_0(\alpha)$	$w_1(\alpha)$	$w_2(\alpha)$	$w_3(\alpha)$	$w_4(\alpha)$	$w_5(\alpha)$	$w_6(\alpha)$	$w_7(\alpha)$
0.1	1.1111E+00	7.5617E-01	1.3384E+00	3.7992E+00	1.4899E+01	7.4749E+01	4.5743E+02	3.3056E+03
0.2	1.2500E+00	9.3750E-01	1.8457E+00	5.8594E+00	2.5775E+01	1.4527E+02	9.9943E+02	8.1226E+03
0.3	1.4286E+00	1.1990E+00	2.6697E+00	9.6392E+00	4.8356E+01	3.1119E+02	2.4459E+03	2.2714E+04
0.4	1.6667E+00	1.5972E+00	4.1102E+00	1.7248E+01	1.0080E+02	7.5638E+02	6.9345E+03	7.5126E+04
0.5	2.0000E+00	2.2500E+00	6.8906E+00	3.4600E+01	2.4242E+02	2.1822E+03	2.4006E+04	3.1208E+05
0.6	2.5000E+00	3.4375E+00	1.3066E+01	8.1848E+01	7.1645E+02	8.0606E+03	1.1084E+05	1.8011E+06
0.7	3.3333E+00	5.9722E+00	3.0095E+01	2.5104E+02	2.9292E+03	4.3938E+04	8.0554E+05	1.7453E+07
0.8	5.0000E+00	1.3125E+01	9.8789E+01	1.2352E+03	2.1617E+04	4.8639E+05	1.3376E+07	4.3471E+08
0.9	1.0000E+01	5.1250E+01	7.6945E+02	1.9237E+04	6.7330E+05	3.0298E+07	1.6664E+09	1.0832E+11

With fixed $x >$ and $a = -\alpha \to -\infty$ holds

$$\int_0^x e^{-\alpha\,t}\,J_0(t)\,dt \;\sim\; \frac{1}{a} - \frac{1}{2a^3} + \frac{3}{8a^5} - \frac{5}{16a^7} + \frac{35}{128a^9} - \frac{63}{256a^{11}} + \ldots =$$
$$= \frac{1}{a} - \frac{0.5}{a^3} + \frac{0.375}{a^5} - \frac{0.3125}{a^7} + \frac{0.2734375}{a^9} - \frac{0.24609375}{a^{11}} + \ldots$$

and

$$\int_0^x e^{-\alpha\,t}\,I_0(t)\,dt \;\sim\; \frac{1}{a} + \frac{1}{2a^3} + \frac{3}{8a^5} + \frac{5}{16a^7} + \frac{35}{128a^9} + \frac{63}{256a^{11}} + \ldots\,.$$

d) Integrals:

Concerning the case '$a = \pm 1$ *and modified Bessel function*' see page 65.

$$\int e^{ax}\,J_1(x)\,dx = -e^{ax}\,J_0(x) + a\int e^{ax}\,J_0(x)\,dx$$

$$\int e^{ax}\,I_1(x)\,dx = e^{ax}\,I_0(x) - a\int e^{ax}\,I_0(x)\,dx$$

$$\int xe^{ax}J_0(x)\,dx = e^{ax}\left[\frac{ax}{a^2+1}\,J_0(x) + \frac{x}{a^2+1}\,J_1(x)\right] - \frac{a}{a^2+1}\int e^{ax}\,J_0(x)\,dx$$

$$\int xe^{ax}I_0(x)\,dx = e^{ax}\left[\frac{ax}{a^2-1}\,I_0(x) - \frac{x}{a^2-1}\,I_1(x)\right] - \frac{a}{a^2-1}\int e^{ax}I_0(x)\,dx$$

$$\int xe^{ax}J_1(x)\,dx = e^{ax}\left[-\frac{x}{a^2+1}\,J_0(x) + \frac{ax}{1+a^2}\,J_1(x)\right] + \frac{1}{a^2+1}\int e^{ax}J_0(x)\,dx$$

$$\int xe^{ax}I_1(x)\,dx = e^{ax}\left[-\frac{x}{a^2-1}\,I_0(x) + \frac{ax}{a^2-1}\,I_1(x)\right] + \frac{1}{a^2-1}\int e^{ax}I_0(x)\,dx$$

$$\int x^2e^{ax}J_0(x)\,dx = e^{ax}\left[\frac{a(a^2+1)x^2 + (-2a^2+1)x}{(a^2+1)^2}J_0(x) + \frac{(a^2+1)x^2 - 3ax}{(a^2+1)^2}J_1(x)\right] + \frac{2a^2-1}{(a^2+1)^2}\int e^{ax}J_0(x)\,dx$$

$$\int x^2e^{ax}I_0(x)\,dx = e^{ax}\left[\frac{a(a^2-1)x^2 - (2a^2+1)x}{(a^2-1)^2}\,I_0(x) + \frac{-(a^2-1)x^2 + 3ax}{(a^2-1)^2}\,I_1(x)\right] + \frac{2\,a^2+1}{(a^2-1)^2}\int e^{ax}I_0(x)\,dx$$

$$\int x^2e^{ax}J_1(x)\,dx = e^{ax}\left[\frac{-(a^2+1)x^2 + 3\,ax}{(a^2+1)^2}J_0(x) + \frac{a(a^2+1)x^2 + (2-a^2)x}{(a^2+1)^2}J_1(x)\right] - \frac{3a}{(a^2+1)^2}\int e^{ax}J_0(x)\,dx$$

$$\int x^2e^{ax}I_1(x)\,dx = e^{ax}\left[\frac{-(a^2-1)x^2 + 3\,ax}{(a^2-1)^2}\,I_0(x) + \frac{a(a^2-1)x^2 - (a^2+2)x}{(a^2-1)^2}I_1(x)\right] - \frac{3a}{(a^2-1)^2}\int e^{ax}I_0(x)\,dx$$

$$\int x^3 e^{ax} J_0(x)\,dx = e^{ax}\left[\frac{a\left(1+a^2\right)^2 x^3-\left(3\,a^2-2\right)\left(1+a^2\right)x^2+3\,a\left(2\,a^2-3\right)x}{(a^2+1)^3}J_0(x)+\right.$$

$$\left.+\frac{\left(1+a^2\right)^2 x^3-5\,a\left(1+a^2\right)x^2+(11\,a^2-4)x}{(a^2+1)^3}J_1(x)\right]-\frac{3a(2a^2-3)}{(a^2+1)^3}\int e^{ax}J_0(x)\,dx$$

$$\int x^3 e^{ax} I_0(x)\,dx = e^{ax}\left[\frac{a\left(a^2-1\right)^2 x^3+-\left(3\,a^2+2\right)\left(a^2-1\right)x^2+3\,a\left(2\,a^2+3\right)x}{(a^2-1)^3}I_0(x)+\right.$$

$$\left.+\frac{-\left(a^2-1\right)^2 x^3+5\,a\left(a^2-1\right)x^2-(11\,a^2+4)x}{(a^2-1)^3}I_1(x)\right]-\frac{3a\,(2a^2+3)}{(a^2-1)^3}\int e^{ax}I_0(x)\,dx$$

$$\int x^3 e^{ax} J_1(x)\,dx = e^{ax}\left[\frac{-\left(1+a^2\right)^2 x^3+5\,a\left(1+a^2\right)x^2-3\,\left(4\,a^2-1\right)x}{(a^2+1)^3}J_0(x)+\right.$$

$$\left.+\frac{a\left(1+a^2\right)^2 x^3-\left(2\,a^2-3\right)\left(1+a^2\right)x^2+a\left(2\,a^2-13\right)x}{(a^2+1)^3}J_1(x)\right]+\frac{12a^2-3}{(a^2+1)^3}\int e^{ax}J_0(x)\,dx$$

$$\int x^3 e^{ax} I_1(x)\,dx = e^{ax}\left[\frac{-\left(a^2-1\right)^2 x^3+5\,a\left(a^2-1\right)x^2-3(4a^2+1)x}{(a^2-1)^3}I_0(x)+\right.$$

$$\left.+\frac{a\left(a^2-1\right)^2 x^3-(2a^2+3)(a^2-1)x^2+a(2a^2+13)x}{(a^2-1)^3}I_1(x)\right]-\frac{3(4a^2+1)}{(a^2-1)^3}\int e^{ax}I_0(x)\,dx$$

Let

$$\int x^n e^{ax} J_\nu(x)\,dx = e^{ax}\left[\frac{P_n^{(\nu)}}{(a^2+1)^n}J_0(x)+\frac{Q_n^{(\nu)}}{(a^2+1)^n}J_1(x)\right]+\frac{R_n^{(\nu)}}{(a^2+1)^n}\int e^{ax}J_0(x)\,dx$$

and

$$\int x^n e^{ax} I_\nu(x)\,dx = e^{ax}\left[\frac{\mathfrak{P}_n^{(\nu)}}{(a^2-1)^n}I_0(x)+\frac{\mathfrak{Q}_n^{(\nu)}}{(a^2-1)^n}I_1(x)\right]+\frac{\mathfrak{R}_n^{(\nu)}}{(a^2-1)^n}\int e^{ax}I_0(x)\,dx\ ,$$

then holds

$$P_4^{(0)} = a\left(1+a^2\right)^3 x^4-\left(4\,a^2-3\right)\left(1+a^2\right)^2 x^3+a\left(12\,a^2-23\right)\left(1+a^2\right)x^2+(-24\,a^4+72\,a^2-9)x$$

$$Q_4^{(0)} = \left(1+a^2\right)^3 x^4-7\,a\left(1+a^2\right)^2 x^3+\left(26\,a^2-9\right)\left(1+a^2\right)x^2-5\,a\left(10\,a^2-11\right)x$$

$$R_4^{(0)} = 24\,a^4-72\,a^2+9$$

$$\mathfrak{P}_4^{(0)} = a\left(a^2-1\right)^3 x^4-\left(4\,a^2+3\right)\left(a^2-1\right)^2 x^3+a\left(12\,a^2+23\right)\left(a^2-1\right)x^2-(24\,a^4+72\,a^2+9)x$$

$$\mathfrak{Q}_4^{(0)} = -\left(a^2-1\right)^3 x^4+7a\,\left(a^2-1\right)^2 x^3-\left(26\,a^2+9\right)\left(a^2-1\right)x^2+5\,a\left(10\,a^2+11\right)x$$

$$\mathfrak{R}_4^{(0)} = 24\,a^4+72\,a^2+9$$

$$P_4^{(1)} = -\left(1+a^2\right)^3 x^4+7\,a\left(1+a^2\right)^2 x^3-\left(27\,a^2-8\right)\left(1+a^2\right)x^2+15\,a\left(4\,a^2-3\right)x$$

$$Q_4^{(1)} = a\left(1+a^2\right)^3 x^4-\left(3\,a^2-4\right)\left(1+a^2\right)^2 x^3+a\left(6\,a^2-29\right)\left(1+a^2\right)x^2+(-6\,a^4+83\,a^2-16)x$$

$$R_4^{(1)} = -15a\,\left(4\,a^2-3\right)$$

$$\mathfrak{P}_4^{(1)} = -\left(a^2-1\right)^3 x^4+7\,a\left(a^2-1\right)^2 x^3-\left(27\,a^2+8\right)\left(a^2-1\right)x^2+15\,a\left(4\,a^2+3\right)x$$

$$\mathfrak{Q}_4^{(1)} = a\left(a^2-1\right)^3 x^4-\left(4+3\,a^2\right)\left(a^2-1\right)^2 x^3+a\left(6\,a^2+29\right)\left(a^2-1\right)x^2-(6\,a^4+83\,a^2+16)x$$

$$\mathfrak{R}_4^{(1)} = -15a\ \left(4\,a^2+3\right)$$

$$P_5^{(0)} = a\left(1+a^2\right)^4 x^5 - \left(5\,a^2-4\right)\left(a^2+1\right)^3 x^4 + a\left(20\,a^2-43\right)\left(a^2+1\right)^2 x^3 -$$
$$-\left(a^2+1\right)\left(60\,a^4-223\,a^2+32\right)x^2 + 15\ \left(8\,a^4-40\,a^2+15\right)ax$$
$$Q_5^{(0)} = \left(a^2+1\right)^4 x^5 - 9\,a\left(a^2+1\right)^3 x^4 + \left(47\,a^2-16\right)\left(a^2+1\right)^2 x^3 -$$
$$-7\,a\left(22\,a^2-23\right)\left(a^2+1\right)x^2 + (274\,a^4-607\,a^2+64)x$$
$$R_5^{(0)} = -15a\ \left(8\,a^4-40\,a^2+15\right)$$

$$\mathfrak{P}_5^{(0)} = a\left(a^2-1\right)^4 x^5 - \left(4+5\,a^2\right)\left(a^2-1\right)^3 x^4 + a\left(20\,a^2+43\right)\left(a^2-1\right)^2 x^3 -$$
$$-\left(60\,a^4+223\,a^2+32\right)\left(a^2-1\right)x^2 + 15\,a\left(8\,a^4+40\,a^2+15\right)x$$
$$\mathfrak{Q}_5^{(0)} = -\left(a^2-1\right)^4 x^5 + 9\,a\left(a^2-1\right)^3 x^4 - \left(16+47\,a^2\right)\left(a^2-1\right)^2 x^3 + 7\,a\left(22\,a^2+23\right)\left(a^2-1\right)x^2 -$$
$$-(274\,a^4+607\,a^2+64)x$$
$$\mathfrak{R}_5^{(0)} = -15a\ \left(8\,a^4+40\,a^2+15\right)$$

$$P_5^{(1)} = -\left(a^2+1\right)^4 x^5 + 9\ \left(a^2+1\right)^3 ax^4 - 3\ \left(16\,a^2-5\right)\left(a^2+1\right)^2 x^3 + 21\,a\left(8\,a^2-7\right)\left(a^2+1\right)x^2 +$$
$$+(-360\,a^4+540\,a^2-45)x$$
$$Q_5^{(1)} = a\left(a^2+1\right)^4 x^5 - \left(-5+4\,a^2\right)\left(a^2+1\right)^3 x^4 + 3\,a\left(4\,a^2-17\right)\left(a^2+1\right)^2 x^3 -$$
$$-3\ \left(a^2+1\right)\left(8\,a^4-82\,a^2+15\right)x^2 + 3\,a\left(8\,a^4-194\,a^2+113\right)x$$
$$R_5^{(1)} = 360\,a^4-540\,a^2+45$$

$$\mathfrak{P}_5^{(1)} = -\left(a^2-1\right)^4 x^5 + 9\,a\left(a^2-1\right)^3 x^4 - 3\ \left(16\,a^2+5\right)\left(a^2-1\right)^2 x^3 + 21\,a\left(8\,a^2+7\right)\left(a^2-1\right)x^2 -$$
$$-(360\,a^4+540\,a^2+45)x$$
$$\mathfrak{Q}_5^{(1)} = a\left(a^2-1\right)^4 x^5 - \left(4\,a^2+5\right)\left(a^2-1\right)^3 x^4 + 3a\ \left(4\,a^2+17\right)\left(a^2-1\right)^2 x^3 -$$
$$-3\ \left(a^2-1\right)\left(8\,a^4+82\,a^2+15\right)x^2 + 3\,a\left(8\,a^4+194\,a^2+113\right)x$$
$$\mathfrak{R}_5^{(1)} = 360\,a^4+540\,a^2+45$$

$$P_6^{(0)} = a\left(a^2+1\right)^5 x^6 - \left(6\,a^2-5\right)\left(a^2+1\right)^4 x^5 + 3\,a\left(10\,a^2-23\right)\left(a^2+1\right)^3 x^4 -$$
$$-3\ \left(40\,a^4-166\,a^2+25\right)\left(a^2+1\right)^2 x^3 + 9\,a\left(a^2+1\right)\left(40\,a^4-242\,a^2+103\right)x^2 +$$
$$+(-720\,a^6+5400\,a^4-4050\,a^2+225)x$$
$$Q_6^{(0)} = \left(a^2+1\right)^5 x^6 - 11\,a\left(a^2+1\right)^4 x^5 + \left(74\,a^2-25\right)\left(a^2+1\right)^3 x^4 - 9\,a\left(38\,a^2-39\right)\left(a^2+1\right)^2 x^3 +$$
$$+9\ \left(a^2+1\right)\left(116\,a^4-244\,a^2+25\right)x^2 - 63\ \left(28\,a^4-104\,a^2+33\right)ax$$
$$R_6^{(0)} = 720\,a^6-5400\,a^4+4050\,a^2-225$$

$$\mathfrak{P}_6^{(0)} = a\left(a^2-1\right)^5 x^6 - \left(6\,a^2+5\right)\left(a^2-1\right)^4 x^5 + 3\,a\left(10\,a^2+23\right)\left(a^2-1\right)^3 x^4 -$$
$$-3\ \left(40\,a^4+166\,a^2+25\right)\left(a^2-1\right)^2 x^3 + 9\,a\left(40\,a^4+242\,a^2+103\right)\left(a^2-1\right)x^2 - (720\,a^6+5400\,a^4+4050\,a^2+225)x$$
$$\mathfrak{Q}_6^{(0)} = -\left(a^2-1\right)^5 x^6 + 11a\ \left(a^2-1\right)^4 x^5 - \left(74\,a^2+25\right)\left(a^2-1\right)^3 x^4 + 9\,a\left(38\,a^2+39\right)\left(a^2-1\right)^2 x^3 -$$
$$-9\ \left(116\,a^4+244\,a^2+25\right)\left(a^2-1\right)x^2 + 63\,a\left(28\,a^4+104\,a^2+33\right)x$$

$$\mathfrak{R}_6^{(0)} = 720\,a^6 + 5400\,a^4 + 4050\,a^2 + 225$$

$$P_6^{(1)} = -\left(a^2+1\right)^5 x^6 + 11\,a\left(a^2+1\right)^4 x^5 - 3\,\left(25\,a^2-8\right)\left(a^2+1\right)^3 x^4 + 9\,a\left(40\,a^2-37\right)\left(a^2+1\right)^2 x^3 -$$
$$-3\,\left(a^2+1\right)\left(400\,a^4 - 691\,a^2 + 64\right) x^2 + 315\,\left(8\,a^4 - 20\,a^2 + 5\right) ax$$
$$Q_6^{(1)} = a\left(a^2+1\right)^5 x^6 - \left(5\,a^2-6\right)\left(a^2+1\right)^4 x^5 + a\left(20\,a^2-79\right)\left(a^2+1\right)^3 x^4 -$$
$$-3\,\left(20\,a^4 - 179\,a^2 + 32\right)\left(a^2+1\right)^2 x^3 + 3\,a\left(a^2+1\right)\left(40\,a^4 - 718\,a^2 + 397\right) x^2 +$$
$$+(-120\,a^6 + 4554\,a^4 - 5337\,a^2 + 384)x$$
$$R_6^{(1)} = -315\,\left(8\,a^4 - 20\,a^2 + 5\right) a$$

$$\mathfrak{P}_6^{(1)} = -\left(a^2-1\right)^5 x^6 + 11\,a\left(a^2-1\right)^4 x^5 - 3\,\left(25\,a^2+8\right)\left(a^2-1\right)^3 x^4 + 9\,a\left(40\,a^2+37\right)\left(a^2-1\right)^2 x^3 -$$
$$-3\,\left(400\,a^4 + 691\,a^2 + 64\right)\left(a^2-1\right) x^2 + 315\,a\left(8\,a^4 + 20\,a^2 + 5\right) x$$
$$\mathfrak{Q}_6^{(1)} = a\left(a^2-1\right)^5 x^6 - \left(5\,a^2+6\right)\left(a^2-1\right)^4 x^5 + a\left(20\,a^2+79\right)\left(a^2-1\right)^3 x^4 -$$
$$-3\,\left(20\,a^4 + 179\,a^2 + 32\right)\left(a^2-1\right)^2 x^3 + 3\,a\left(a^2-1\right)\left(40\,a^4 + 718\,a^2 + 397\right) x^2 -$$
$$-(120\,a^6 + 4554\,a^4 + 5337\,a^2 + 384)x$$
$$\mathfrak{R}_6^{(1)} = -315\,\left(8\,a^4 + 20\,a^2 + 5\right) a$$

$$P_7^{(0)} = a\left(1+a^2\right)^6 x^7 - \left(7\,a^2-6\right)\left(a^2+1\right)^5 x^6 + a\left(42\,a^2-101\right)\left(a^2+1\right)^4 x^5 -$$
$$-3\,\left(70\,a^4 - 311\,a^2 + 48\right)\left(a^2+1\right)^3 x^4 + 3\,a\left(280\,a^4 - 1882\,a^2 + 841\right)\left(a^2+1\right)^2 x^3 +$$
$$-9\,\left(a^2+1\right)\left(280\,a^6 - 2494\,a^4 + 2103\,a^2 - 128\right) x^2 + 315\,\left(16\,a^6 - 168\,a^4 + 210\,a^2 - 35\right) ax$$
$$Q_7^{(0)} = \left(a^2+1\right)^6 x^7 - 13\,a\left(a^2+1\right)^5 x^6 + \left(107\,a^2-36\right)\left(a^2+1\right)^4 x^5 - 11\,\left(58\,a^2-59\right)\left(a^2+1\right)^3 ax^4 +$$
$$+9\,\left(306\,a^4 - 631\,a^2 + 64\right)\left(a^2+1\right)^2 x^3 - 9\,\left(a^2+1\right)\left(892\,a^4 - 3144\,a^2 + 969\right) ax^2 +$$
$$+(13068\,a^6 - 73188\,a^4 + 46575\,a^2 - 2304)x$$
$$R_7^{(0)} = -315\,\left(16\,a^6 - 168\,a^4 + 210\,a^2 - 35\right) a$$

$$\mathfrak{P}_7^{(0)} = a\left(a^2-1\right)^6 x^7 - \left(7\,a^2+6\right)\left(a^2-1\right)^5 x^6 + a\left(101 + 42\,a^2\right)\left(a^2-1\right)^4 x^5 -$$
$$-3\,\left(70\,a^4 + 311\,a^2 + 48\right)\left(a^2-1\right)^3 x^4 + 3\,a\left(280\,a^4 + 1882\,a^2 + 841\right)\left(a^2-1\right)^2 x^3 -$$
$$-9\,\left(280\,a^6 + 2494\,a^4 + 2103\,a^2 + 128\right)\left(a^2-1\right) x^2 + 315\,a\left(16\,a^6 + 168\,a^4 + 210\,a^2 + 35\right) x$$
$$\mathfrak{Q}_7^{(0)} = -\left(a^2-1\right)^6 x^7 + 13\,a\left(a^2-1\right)^5 x^6 - \left(107\,a^2+36\right)\left(a^2-1\right)^4 x^5 + 11\,a\left(59 + 58\,a^2\right)\left(a^2-1\right)^3 x^4 -$$
$$-9\,\left(306\,a^4 + 631\,a^2 + 64\right)\left(a^2-1\right)^2 x^3 + 9\,a\left(892\,a^4 + 3144\,a^2 + 969\right)\left(a^2-1\right) x^2 -$$
$$-(13068\,a^6 + 73188\,a^4 + 46575\,a^2 + 2304)x$$
$$\mathfrak{R}_7^{(0)} = -315a\,\left(16\,a^6 + 168\,a^4 + 210\,a^2 + 35\right)$$

$$P_7^{(1)} = -\left(1+a^2\right)^6 x^7 + 13\,a\left(a^2+1\right)^5 x^6 - \left(108\,a^2-35\right)\left(a^2+1\right)^4 x^5 + 33\,a\left(20\,a^2-19\right)\left(a^2+1\right)^3 x^4 -$$
$$-3\,\left(1000\,a^4 - 1828\,a^2 + 175\right)\left(a^2+1\right)^2 x^3 + 9\,a\left(a^2+1\right)\left(1080\,a^4 - 3076\,a^2 + 849\right) x^2 +$$
$$+(-20160\,a^6 + 75600\,a^4 - 37800\,a^2 + 1575)x$$
$$Q_7^{(1)} = a\left(a^2+1\right)^6 x^7 - \left(6\,a^2-7\right)\left(a^2+1\right)^5 x^6 + a\left(30\,a^2-113\right)\left(a^2+1\right)^4 x^5 -$$

$$-\left(120\,a^4-992\,a^2+175\right)\left(a^2+1\right)^3x^4+9\,a\left(40\,a^4-624\,a^2+337\right)\left(a^2+1\right)^2x^3-$$

$$-9\,\left(a^2+1\right)\left(80\,a^6-2248\,a^4+2502\,a^2-175\right)x^2+9\,a\left(80\,a^6-4408\,a^4+8654\,a^2-1873\right)x$$

$$R_7^{(1)}=20160\,a^6-75600\,a^4+37800\,a^2-1575$$

$$\mathfrak{P}_7^{(1)}=-\left(a^2-1\right)^6x^7+13\,a\left(a^2-1\right)^5x^6-\left(108\,a^2+35\right)\left(a^2-1\right)^4x^5+$$

$$+33\,a\left(20\,a^2+19\right)\left(a^2-1\right)^3x^4-3\,\left(1000\,a^4+1828\,a^2+175\right)\left(a^2-1\right)^2x^3+$$

$$+9\,a\left(1080\,a^4+3076\,a^2+849\right)\left(a^2-1\right)-(20160\,a^6+75600\,a^4+37800\,a^2+1575)x$$

$$\mathfrak{Q}_7^{(1)}=a\left(a^2-1\right)^6x^7-\left(6\,a^2+7\right)\left(a^2-1\right)^5x^6+a\left(30\,a^2+113\right)\left(a^2-1\right)^4x^5-$$

$$-\left(120\,a^4+992\,a^2+175\right)\left(a^2-1\right)^3x^4+9\,a\left(40\,a^4+624\,a^2+337\right)\left(a^2-1\right)^2x^3-$$

$$-9\,\left(80\,a^6+2248\,a^4+2502\,a^2+175\right)\left(a^2-1\right)x^2+9\,a\left(80\,a^6+4408\,a^4+8654\,a^2+1873\right)x$$

$$\mathfrak{R}_7^{(1)}=20160\,a^6+75600\,a^4+37800\,a^2+1575$$

$$P_8^{(0)}=a\left(a^2+1\right)^7x^8-\left(8\,a^2-7\right)\left(a^2+1\right)^6x^7+a\left(56\,a^2-139\right)\left(a^2+1\right)^5x^6-$$

$$-\left(336\,a^4-1564\,a^2+245\right)\left(a^2+1\right)^4x^5+3\,a\left(560\,a^4-4028\,a^2+1847\right)\left(a^2+1\right)^3x^4-$$

$$-3\,\left(2240\,a^6-22056\,a^4+19524\,a^2-1225\right)\left(a^2+1\right)^2x^3+$$

$$+9\,a\left(a^2+1\right)\left(2240\,a^6-27512\,a^4+38356\,a^2-6967\right)x^2+$$

$$+(-40320\,a^8+564480\,a^6-1058400\,a^4+352800\,a^2-11025)x$$

$$Q_8^{(0)}=\left(a^2+1\right)^7x^8-15\,a\left(a^2+1\right)^6x^7+\left(146\,a^2-49\right)\left(a^2+1\right)^5x^6-13\,a\left(82\,a^2-83\right)\left(a^2+1\right)^4x^5+$$

$$+\left(5944\,a^4-12136\,a^2+1225\right)\left(a^2+1\right)^3x^4-99\,a\left(248\,a^4-856\,a^2+261\right)\left(a^2+1\right)^2x^3+$$

$$+9\,\left(a^2+1\right)\left(7696\,a^6-40888\,a^4+25266\,a^2-1225\right)x^2-9\,a\left(12176\,a^6-95912\,a^4+101978\,a^2-15159\right)x$$

$$R_8^{(0)}=40320\,a^8-564480\,a^6+1058400\,a^4-352800\,a^2+11025$$

$$\mathfrak{P}_8^{(0)}=a\left(a^2-1\right)^7x^8-\left(8\,a^2+7\right)\left(a^2-1\right)^6x^7+a\left(56\,a^2+139\right)\left(a^2-1\right)^5x^6-$$

$$-\left(336\,a^4+1564\,a^2+245\right)\left(a^2-1\right)^4x^5+3\,a\left(560\,a^4+4028\,a^2+1847\right)\left(a^2-1\right)^3x^4-$$

$$-3\,\left(2240\,a^6+22056\,a^4+19524\,a^2+1225\right)\left(a^2-1\right)^2x^3+$$

$$+9\,a\left(2240\,a^6+27512\,a^4+38356\,a^2+6967\right)\left(a^2-1\right)x^2-$$

$$-(40320\,a^8+564480\,a^6+1058400\,a^4+352800\,a^2+11025)x$$

$$\mathfrak{Q}_8^{(0)}=-\left(a^2-1\right)^7x^8+15\,a\left(a^2-1\right)^6x^7-\left(146\,a^2+49\right)\left(a^2-1\right)^5x^6+13\,a\left(82\,a^2+83\right)\left(a^2-1\right)^4x^5-$$

$$-\left(5944\,a^4+12136\,a^2+1225\right)\left(a^2-1\right)^3x^4+99\,a\left(248\,a^4+856\,a^2+261\right)\left(a^2-1\right)^2x^3-$$

$$-9\,\left(7696\,a^6+40888\,a^4+25266\,a^2+1225\right)\left(a^2-1\right)x^2+$$

$$+9a\,\left(12176\,a^6+95912\,a^4+101978\,a^2+15159\right)x$$

$$\mathfrak{R}_8^{(0)}=40320\,a^8+564480\,a^6+1058400\,a^4+352800\,a^2+11025$$

$$P_8^{(1)}=-\left(a^2+1\right)^7x^8+15\,a\left(a^2+1\right)^6x^7-3\,\left(49\,a^2-16\right)\left(a^2+1\right)^5x^6+39\,a\left(28\,a^2-27\right)\left(a^2+1\right)^4x^5-$$

$$-9\,\left(700\,a^4-1317\,a^2+128\right)\left(a^2+1\right)^3x^4+99\,a\left(280\,a^4-844\,a^2+241\right)\left(a^2+1\right)^2x^3-$$

$$-9\,\left(a^2+1\right)\left(9800\,a^6-41484\,a^4+22767\,a^2-1024\right)x^2+2835\,a\left(64\,a^6-336\,a^4+280\,a^2-35\right)x$$

$$Q_8^{(1)} = a\left(a^2+1\right)^7 x^8 - \left(7\,a^2-8\right)\left(a^2+1\right)^6 x^7 + 3\,a\left(14\,a^2-51\right)\left(a^2+1\right)^5 x^6 -$$
$$-3\,\left(70\,a^4-549\,a^2+96\right)\left(a^2+1\right)^4 x^5 + 3\,a\left(280\,a^4-4016\,a^2+2139\right)\left(a^2+1\right)^3 x^4 -$$
$$-9\,\left(280\,a^6-6816\,a^4+7407\,a^2-512\right)\left(a^2+1\right)^2 x^3 + 9\,a\left(a^2+1\right)\left(560\,a^6-22872\,a^4+42666\,a^2-8977\right)x^2 +$$
$$+(-5040\,a^8+382248\,a^6-1130706\,a^4+490599\,a^2-18432)x$$
$$R_8^{(1)} = -2835a\,\left(64\,a^6-336\,a^4+280\,a^2-35\right)$$

$$\mathfrak{P}_8^{(1)} = -\left(a^2-1\right)^7 x^8 + 15\,a\left(a^2-1\right)^6 x^7 - 3\,\left(49\,a^2+16\right)\left(a^2-1\right)^5 x^6 + 39\,a\left(28\,a^2+27\right)\left(a^2-1\right)^4 x^5 -$$
$$-9\,\left(700\,a^4+1317\,a^2+128\right)\left(a^2-1\right)^3 x^4 + 99\,a\left(280\,a^4+844\,a^2+241\right)\left(a^2-1\right)^2 x^3 -$$
$$-9\,\left(9800\,a^6+41484\,a^4+22767\,a^2+1024\right)\left(a^2-1\right)x^2 + 2835\,a\left(64\,a^6+336\,a^4+280\,a^2+35\right)x$$
$$\mathfrak{Q}_8^{(1)} = a\left(a^2-1\right)^7 x^8 - \left(7\,a^2+8\right)\left(a^2-1\right)^6 x^7 + 3\,a\left(14\,a^2+51\right)\left(a^2-1\right)^5 x^6 -$$
$$-3\,\left(70\,a^4+549\,a^2+96\right)\left(a^2-1\right)^4 x^5 + 3\,a\left(280\,a^4+4016\,a^2+2139\right)\left(a^2-1\right)^3 x^4 -$$
$$-9\,\left(280\,a^6+6816\,a^4+7407\,a^2+512\right)\left(a^2-1\right)^2 x^3 + 9\,a\left(560\,a^6+22872\,a^4+42666\,a^2+8977\right)\left(a^2-1\right)x^2 -$$
$$-(5040\,a^8+382248\,a^6+1130706\,a^4+490599\,a^2+18432)x$$
$$\mathfrak{R}_8^{(1)} = -2835a\,\left(64\,a^6+336\,a^4+280\,a^2+35\right)$$

Recurrence formulas: Let

$$\mathbf{J}_n^{(\nu)} = \int x^n e^{ax} J_\nu(x)\,dx\,, \qquad \mathbf{I}_n^{(\nu)} = \int x^n e^{ax} I_\nu(x)\,dx\,,$$

then holds

$$\mathbf{J}_{n+1}^{(0)} = \frac{x^{n+1}e^{ax}[aJ_0(x)+J_1(x)] - (n+1)\,a\,\mathbf{J}_n^{(0)} - n\mathbf{J}_n^{(1)}}{a^2+1}\,,$$
$$\mathbf{J}_{n+1}^{(1)} = \frac{x^{n+1}e^{ax}[aJ_1(x)-J_0(x)] + (n+1)\,\mathbf{J}_n^{(0)} - a\,n\,\mathbf{J}_n^{(1)}}{a^2+1}$$

and

$$\mathbf{I}_{n+1}^{(0)} = \frac{x^{n+1}e^{ax}[aI_0(x)-I_1(x)] - (n+1)\,a\,\mathbf{I}_n^{(0)} + n\mathbf{I}_n^{(1)}}{a^2-1}\,,$$
$$\mathbf{I}_{n+1}^{(1)} = \frac{x^{n+1}e^{ax}[aI_1(x)-I_0(x)] + (n+1)\,\mathbf{I}_n^{(0)} - a\,n\,\mathbf{I}_n^{(1)}}{a^2-1}\,.$$

e) Special Cases:

$$\int x^2 \exp\left(\frac{x}{\sqrt{2}}\right) J_0(x)\,dx = \frac{x}{3}\left[\sqrt{2}x\,J_0(x) + 2(x-\sqrt{2})\,J_1(x)\right]\exp\left(\frac{x}{\sqrt{2}}\right)$$
$$\int x^2 \exp\left(-\frac{x}{\sqrt{2}}\right) J_0(x)\,dx = \frac{x}{3}\left[-\sqrt{2}x\,J_0(x) + 2(x+\sqrt{2})\,J_1(x)\right]\exp\left(-\frac{x}{\sqrt{2}}\right)$$
$$\int x^3 \exp\left(\sqrt{\frac{3}{2}}\,x\right) J_0(x)\,dx = \frac{x}{5}\left[(\sqrt{6}x^2-2\,x)\,J_0(x) + (x^2-\sqrt{6}x+2)\,J_1(x)\right]\exp\left(\sqrt{\frac{3}{2}}\,x\right)$$
$$\int x^3 \exp\left(-\sqrt{\frac{3}{2}}\,x\right) J_0(x)\,dx = \frac{x}{5}\left[-(\sqrt{6}x^2+2\,x)\,J_0(x) + (x^2+\sqrt{6}x+2)\,J_1(x)\right]\exp\left(\sqrt{\frac{3}{2}}\,x\right)$$
$$\int x^3 \exp\left(\frac{x}{2}\right) J_0(x)\,dx = \frac{2x}{5}\left[-(2x^2-4x)\,J_0(x) + (x^2+4x-8)\,J_1(x)\right]\exp\left(\frac{x}{2}\right)$$

$$\int x^3 \exp\left(-\frac{x}{2}\right)\, J_0(x)\, dx = -\frac{2x}{5}\left[(2x^2+4x)\, J_0(x) + (x^2-4x-8)\, J_1(x)\right] \exp\left(-\frac{x}{2}\right)$$

$$w \quad = \quad \pm\frac{\sqrt{6 \pm \sqrt{30}}}{2} \quad = \quad \pm\{0.361\,516;\ 1.693\,903\}$$

$$\int x^4\, e^{wx}\, J_0(x)\, dx = \frac{x}{5w(4w^2-3)} \big\{\big[(12\,w^2-3)x^3 - 24wx^2 + 24x\big]\ J_0(x) +$$
$$+ \big[(8\,w^3 - 12\,w)x^3 + (-24\,w^2+12)x^2 + 48wx - 48\big]\ J_1(x)\big\}\ e^{wx}$$

$$w \quad = \quad \pm\frac{\sqrt{3}}{2} \quad = \quad \pm\, 0.866\;025$$

$$\int x^4\, e^{wx}\, J_1(x)\, dx = \frac{x}{8\,w^4 - 24\,w^2 + 3} \big\{\big[(12\,w^2-3)x^3 - 24wx^2 + 24x\big]\ J_0(x) +$$
$$+ \big[(8\,w^3 - 12\,w)x^3 + (-24\,w^2+12)x^2 + 48wx - 48\big]\ J_1(x)\big\}\ e^{wx}$$

$$w \quad = \quad \pm\frac{\sqrt{10 \pm \sqrt{70}}}{2} \quad = \quad \pm\{0.639\;023;\ 2.142\;814\}$$

$$\int x^5\, e^{wx}\, J_0(x)\, dx = \frac{x}{3(8\,w^4 - 12\,w^2 + 1)} \big\{\big[(16\,w^3 - 12\,w)x^4 + (-48\,w^2+12)x^3 + 96wx - 96x\big]\ J_0(x) +$$
$$+ \big[(8\,w^4 - 24\,w^2 + 3)x^4 + (-32\,w^3 + 48\,w)x^3 + (96\,w^2 - 48)x^2 - 192wx + 192\big]\ J_1(x)\big\}\ e^{wx}$$

$$w \quad = \quad \pm\frac{\sqrt{3 \pm \sqrt{7}}}{2} \quad = \quad \pm\{0.297\;594;\ 1.188\;039\}$$

$$\int x^5\, e^{wx}\, J_1(x)\, dx = \frac{x}{w(8\,w^4 - 40\,w^2 + 15)} \big\{\big[(16\,w^3 - 12\,w)x^4 + (-48\,w^2+12)x^3 + 96wx^2 - 96x\big]\ J_0(x) +$$
$$+ \big[(8\,w^4 - 24\,w^2 + 3)x^4 + (-32\,w^3 + 48\,w)x^3 + (96\,w^2 - 48)x^2 - 192wx + 192\big]\ J_1(x)\big\}\ e^{wx}$$

$$16w^6 - 120w^4 + 90w^2 - 5 = 0 \quad \Longrightarrow \quad w \in \{\pm 0.245\;717\;164,\ \pm 0.881\;375\;831,\ \pm 2.581\;239\;958\}$$

$$\int x^6\, e^{wx}\, J_0(x)\, dx = \frac{x}{7w(8w^4 - 20w^2 + 5)}\cdot$$
$$\cdot\big\{\big[(40\,w^4 - 60\,w^2 + 5)x^5 + (-160\,w^3 + 120\,w)x^4 + (480\,w^2 - 120)x^3 - 960wx^2 + 960x\big]\ J_0(x) +$$
$$+ \big[(16\,w^5 - 80\,w^3 + 30\,w)x^5 + (-80\,w^4 + 240\,w^2 - 30)x^4 + (320\,w^3 - 480\,w)x^3 +$$
$$+ (-960\,w^2 + 480)x^2 + 1920wx - 1920\big]\ J_1(x)\big\}\ e^{wx}$$

$$w \quad = \quad \pm\frac{\sqrt{5 \pm \sqrt{15}}}{2} \quad = \quad \pm\{0.530\;805,\ 1.489\;378\}$$

$$\int x^6\, e^{wx}\, J_1(x)\, dx = \frac{x}{16w^6 - 120w^4 + 90w^2 - 5} \big\{\big[(40\,w^4 - 60\,w^2 + 5)x^5 + (-160\,w^3 + 120\,w)x^4 +$$
$$+\ (480\,w^2 - 120)x^3 - 960wx^2 + 960w\big]\ J_0(x) +$$
$$\big[(16\,w^5 - 80\,w^3 + 30\,w)x^5 + (-80\,w^4 + 240\,w^2 - 30)x^4 + (320\,w^3 - 480\,w)x^3$$
$$+\ (-960\,w^2 + 480)x^2 + 1920x - 1920\big]\ J_1(x)\big\}\ e^{wx}$$

$$16w^6 - 168w^4 + 210w^2 - 35 = 0 \quad \Rightarrow \quad w \in \{\pm 0.444\,060\,144,\ \pm 1.105\,247\,947,\ \pm 3.013\,509\,178\}$$

$$\int x^7\, e^{wx}\, J_0(x)\, dx = \frac{x}{64w^6 - 240w^4 + 120w^2 - 5} \Big\{ \big[(48\,w^5 - 120\,w^3 + 30\,w)x^6 + (-240\,w^4 + 360\,w^2 - 30)x^5 +$$

$$+(960\,w^3 - 720\,w)x^4 + (-2880\,w^2 + 720)x^3 + 5760wx^2 - 5760x\big]\ J_0(x) +$$

$$+ \big[(16\,w^6 - 120\,w^4 + 90\,w^2 - 5)x^6 + (-96\,w^5 + 480\,w^3 - 180\,w)x^5 + (480\,w^4 - 1440\,w^2 + 180)x^4 +$$

$$+\ (-1920\,w^3 + 2880\,w)x^3 + (5760\,w^2 - 2880)x^2 - 11520wx + 11520\big]\ J_1(x)\Big\}\ e^{wx}$$

$$64w^6 - 240w^4 + 120w^2 - 5 = 0 \quad \Rightarrow \quad w \in \{\pm 0.214\,039\,849,\ \pm 0.733\,975\,352,\ \pm 1.779\,175\,968\}$$

$$\int x^7\, e^{wx}\, J_1(x)\, dx = \frac{x}{w(16w^6 - 168w^4 + 210w^2 - 35)} \cdot$$

$$\cdot \Big\{ \big[(48\,w^5 - 120\,w^3 + 30\,w)x^6 + (-240\,w^4 + 360\,w^2 - 30)x^5 + (960\,w^3 - 720\,w)x^4$$

$$+ (-2880\,w^2 + 720)x^3 + 5760wx^2 - 5760\big]\ J_0(x) +$$

$$+ \big[(16\,w^6 - 120\,w^4 + 90\,w^2 - 5)x^6 + (-96\,w^5 + 480\,w^3 - 180\,w)x^5 + (480\,w^4 - 1440\,w^2 + 180)x^4 +$$

$$+\ (-1920\,w^3 + 2880\,w)x^3 + (5760\,w^2 - 2880)x^2 - 11520wx + 11520\big]\ J_1(x)\Big\}\ e^{wx}$$

1.2.6. Integrals of the type $\int x^{-n-1/2}\,\sin x\,J_\nu(x)\,dx$ and $\int x^{-n-1/2}\,\cos x\,J_\nu(x)\,dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x),\ p = 1, 2.$
The next four integrals are special cases of the general integral 1.8.2.7 from [4].

$$\int \frac{\sin x\,J_0(x)\,dx}{x^{3/2}} = \frac{(4x\cos x - 2\sin x)J_0(x) + 4x\sin x\,J_1(x)}{\sqrt{x}}$$

$$\int \frac{\cos x\,J_0(x)\,dx}{x^{3/2}} = \frac{(-4x\sin x - 2\cos x)J_0(x) + 4x\cos x\,J_1(x)}{\sqrt{x}}$$

$$\int \frac{\sin x\,J_1(x)\,dx}{x^{3/2}} = \frac{4x\sin x\,J_0(x) - (4x\cos x + 2\sin x)\,J_1(x)}{3\sqrt{x}}$$

$$\int \frac{\cos x\,J_1(x)\,dx}{x^{3/2}} = \frac{4x\cos\,xJ_0(x) + (4x\sin x - 2\cos x)J_1(x)}{3\sqrt{x}}$$

$$\int \frac{\sin x\,J_0(x)\,dx}{x^{5/2}} = \frac{[(-32\,x^2 - 6)\sin x - 12\,x\cos x]\,J_0(x) + [4\,x\sin x + 32\,x^2\cos x]\,J_1(x)}{9\,x^{3/2}}$$

$$\int \frac{\cos x\,J_0(x)\,dx}{x^{5/2}} = \frac{[12\,x\sin x - (32\,x^2 + 6)\cos x]\,J_0(x) + [-32\,x^2\sin x + 4\,x\cos x]\,J_1(x)}{9\,x^{3/2}}$$

$$\int \frac{\sin x\,J_1(x)\,dx}{x^{5/2}} = \frac{[-12\,x\sin x + 32\,x^2\cos x]\,J_0(x) + [(32\,x^2 - 6)\sin x - 4\,x\cos x]\,J_1(x)}{15\,x^{3/2}}$$

$$\int \frac{\cos x\,J_1(x)\,dx}{x^{5/2}} = \frac{[-32\,x^2\sin x - 12\,x\cos x]\,J_0(x) + [4\,x\sin x + (32\,x^2 - 6)\cos x]\,J_1(x)}{15\,x^{3/2}}$$

$$\int \frac{\sin x\,J_0(x)\,dx}{x^{7/2}} =$$

$$= \frac{[(192\,x^2 - 90)\sin x + (-512\,x^3 - 60\,x)\cos x]\,J_0(x) + [(-512\,x^3 + 36\,x)\sin x + 64\,x^2\cos x]\,J_1(x)}{225\,x^{5/2}}$$

$$\int \frac{\cos x\,J_0(x)\,dx}{x^{7/2}} =$$

$$= \frac{[(512\,x^3 + 60\,x)\sin x + (192\,x^2 - 90)\cos x]\,J_0(x) + [-64\,x^2\sin x + (-512\,x^3 + 36\,x)\cos x]\,J_1(x)}{225\,x^{5/2}}$$

$$\int \frac{\sin x\,J_1(x)\,dx}{x^{7/2}} =$$

$$= \frac{[(-512\,x^3 - 60\,x)\sin x - 192\,x^2\cos x]\,J_0(x) + [(64\,x^2 - 90)\sin x + (512\,x^3 - 36\,x)\cos x]\,J_1(x)}{315\,x^{5/2}}$$

$$\int \frac{\cos x\,J_1(x)\,dx}{x^{7/2}} =$$

$$= \frac{[192\,x^2\sin x + (-512\,x^3 - 60\,x)\cos x]\,J_0(x) + [(-512\,x^3 + 36\,x)\sin x + (64\,x^2 - 90)\cos x]\,J_1(x)}{315\,x^{5/2}}$$

Let

$$\int \frac{\sin x\,J_\nu(x)\,dx}{x^{n+1/2}} = \frac{[P_n^{(s,\nu)}(x)\,\sin x + Q_n^{(s,\nu)}(x)\cos x]\,J_0(x) + [R_n^{(s,\nu)}(x)\,\sin x + S_n^{(s,\nu)}(x)\,\cos x]\,J_1(x)}{N_n^{(s,\nu)}\,x^{n-1/2}}$$

and

$$\int \frac{\cos x\,J_\nu(x)\,dx}{x^{n+1/2}} = \frac{[P_n^{(c,\nu)}(x)\,\sin x + Q_n^{(c,\nu)}(x)\cos x]\,J_0(x) + [R_n^{(c,\nu)}(x)\,\sin x + S_n^{(c,\nu)}(x)\,\cos x]\,J_1(x)}{N_n^{(c,\nu)}\,x^{n-1/2}}\,,$$

then holds

$$P_4^{(s,0)}(x) = 4096\,x^4 + 480\,x^2 - 1050\;, \quad Q_4^{(s,0)}(x) = 1536\,x^3 - 420\,x\;, \quad R_4^{(s,0)}(x) = -512\,x^3 + 300\,x\;,$$
$$S_4^{(s,0)}(x) = -4096\,x^4 + 288\,x^2\;, \quad N_4^{(s,0)} = 3675$$
$$P_4^{(c,0)}(x) = -1536\,x^3 + 420\,x\;, \quad Q_4^{(c,0)}(x) = 4096\,x^4 + 480\,x^2 - 1050\;, \quad R_4^{(c,0)}(x) = 4096\,x^4 - 288\,x^2\;,$$
$$S_4^{(c,0)}(x) = -512\,x^3 + 300\,x\;, \quad N_4^{(c,0)} = 3675$$
$$P_4^{(s,1)}(x) = 1536\,x^3 - 420\,x\;, \quad Q_4^{(s,1)}(x) = -4096\,x^4 - 480\,x^2\;, \quad R_4^{(s,1)}(x) = -4096\,x^4 + 288\,x^2 - 1050\;,$$
$$S_4^{(s,1)}(x) = 512\,x^3 - 300\,x\;, \quad N_4^{(s,1)} = 4725$$
$$P_4^{(c,1)}(x) = 4096\,x^4 + 480\,x^2\;, \quad Q_4^{(c,1)}(x) = 1536\,x^3 - 420\,x\;, \quad R_4^{(c,1)}(x) = -512\,x^3 + 300\,x\;,$$
$$S_4^{(c,1)}(x) = -4096\,x^4 + 288\,x^2 - 1050\;, \quad N_4^{(c,1)} = 4725$$

$$P_5^{(s,0)}(x) = -49152\,x^4 + 13440\,x^2 - 66150\;, \quad Q_5^{(s,0)}(x) = 131072\,x^5 + 15360\,x^3 - 18900\,x\;,$$
$$R_5^{(s,0)}(x) = 131072\,x^5 - 9216\,x^3 + 14700\,x\;, \quad S_5^{(s,0)}(x) = -16384\,x^4 + 9600\,x^2\;, \quad N_5^{(s,0)} = 297675$$
$$P_5^{(c,0)}(x) = -131072\,x^5 - 15360\,x^3 + 18900\,x\;, \quad Q_5^{(c,0)}(x) = -49152\,x^4 + 13440\,x^2 - 66150\;,$$
$$R_5^{(c,0)}(x) = 16384\,x^4 - 9600\,x^2\;, \quad S_5^{(c,0)}(x) = 131072\,x^5 - 9216\,x^3 + 14700\,x\;, \quad N_5^{(c,0)} = 297675$$
$$P_5^{(s,1)}(x) = 131072\,x^5 + 15360\,x^3 - 18900\,x\;, \quad Q_5^{(s,1)}(x) = 49152\,x^4 - 13440\,x^2\;,$$
$$R_5^{(s,1)}(x) = -16384\,x^4 + 9600\,x^2 - 66150\;, \quad S_5^{(s,1)}(x) = -131072\,x^5 + 9216\,x^3 - 14700\,x\;, \quad N_5^{(s,1)} = 363825$$
$$P_5^{(c,1)}(x) = -49152\,x^4 + 13440\,x^2\;, \quad Q_5^{(c,1)}(x) = 131072\,x^5 + 15360\,x^3 - 18900\,x\;,$$
$$R_5^{(c,1)}(x) = 131072\,x^5 - 9216\,x^3 + 14700\,x\;, \quad S_5^{(c,1)}(x) = -16384\,x^4 + 9600\,x^2 - 66150\;, \quad N_5^{(c,1)} = 363825$$

$$P_6^{(s,0)}(x) = -1048576\,x^6 - 122880\,x^4 + 151200\,x^2 - 1309770\;, \quad Q_6^{(s,0)}(x) = -393216\,x^5 + 107520\,x^3 - 291060\,x\;,$$
$$R_6^{(s,0)}(x) = 131072\,x^5 - 76800\,x^3 + 238140\,x\;, \quad S_6^{(s,0)}(x) = 1048576\,x^6 - 73728\,x^4 + 117600\,x^2\;, \quad N_6^{(s,0)} = 7203735$$
$$P_6^{(c,0)}(x) = 393216\,x^5 - 107520\,x^3 + 291060\,x\;, \quad Q_6^{(c,0)}(x) = -1048576\,x^6 - 122880\,x^4 + 151200\,x^2 - 1309770\;,$$
$$R_6^{(c,0)}(x) = -1048576\,x^6 + 73728\,x^4 - 117600\,x^2\;, \quad S_6^{(c,0)}(x) = 131072\,x^5 - 76800\,x^3 + 238140\,x\;,$$
$$N_6^{(c,0)} = 7203735$$
$$P_6^{(s,1)}(x) = -393216\,x^5 + 107520\,x^3 - 291060\,x\;, \quad Q_6^{(s,1)}(x) = 1048576\,x^6 + 122880\,x^4 - 151200\,x^2\;,$$
$$R_6^{(s,1)}(x) = 1048576\,x^6 - 73728\,x^4 + 117600\,x^2 - 1309770\;, \quad S_6^{(s,1)}(x) = -131072\,x^5 + 76800\,x^3 - 238140\,x\;,$$
$$N_6^{(s,1)} = 8513505$$
$$P_6^{(c,1)}(x) = -1048576\,x^6 - 122880\,x^4 + 151200\,x^2\;, \quad Q_6^{(c,1)}(x) = -393216\,x^5 + 107520\,x^3 - 291060\,x\;,$$
$$R_6^{(c,1)}(x) = 131072\,x^5 - 76800\,x^3 + 238140\,x\;, \quad S_6^{(c,1)}(x) = 1048576\,x^6 - 73728\,x^4 + 117600\,x^2 - 1309770\;,$$
$$N_6^{(c,1)} = 8513505$$
$$P_7^{(s,0)}(x) = 6291456\,x^6 - 1720320\,x^4 + 4656960\,x^2 - 62432370\;,$$
$$Q_7^{(s,0)}(x) = -16777216\,x^7 - 1966080\,x^5 + 2419200\,x^3 - 11351340\,x\;,$$
$$R_7^{(s,0)}(x) = -16777216\,x^7 + 1179648\,x^5 - 1881600\,x^3 + 9604980\,x\;,$$
$$S_7^{(s,0)}(x) = 2097152\,x^6 - 1228800\,x^4 + 3810240\,x^2\;, \quad N_7^{(s,0)} = 405810405$$

$$P_7^{(c,0)}(x) = 16777216\,x^7 + 1966080\,x^5 - 2419200\,x^3 + 11351340\,x\ ,$$
$$Q_7^{(c,0)}(x) = 6291456\,x^6 - 1720320\,x^4 + 4656960\,x^2 - 62432370\ ,$$
$$R_7^{(c,0)}(x) = -2097152\,x^6 + 1228800\,x^4 - 3810240\,x^2\ ,$$
$$S_7^{(c,0)}(x) = -16777216\,x^7 + 1179648\,x^5 - 1881600\,x^3 + 9604980\,x\ , \quad N_7^{(c,0)} = 405810405$$
$$P_7^{(s,1)}(x) = -16777216\,x^7 - 1966080\,x^5 + 2419200\,x^3 - 11351340\,x\ ,$$
$$Q_7^{(s,1)}(x) = -6291456\,x^6 + 1720320\,x^4 - 4656960\,x^2\ ,$$
$$R_7^{(s,1)}(x) = 2097152\,x^6 - 1228800\,x^4 + 3810240\,x^2 - 62432370\ ,$$
$$S_7^{(s,1)}(x) = 16777216\,x^7 - 1179648\,x^5 + 1881600\,x^3 - 9604980\,x\ , \quad N_7^{(s,1)} = 468242775$$
$$P_7^{(c,1)}(x) = 6291456\,x^6 - 1720320\,x^4 + 4656960\,x^2\ ,$$
$$Q_7^{(c,1)}(x) = -16777216\,x^7 - 1966080\,x^5 + 2419200\,x^3 - 11351340\,x\ ,$$
$$R_7^{(c,1)}(x) = -16777216\,x^7 + 1179648\,x^5 - 1881600\,x^3 + 9604980\,x\ ,$$
$$S_7^{(c,1)}(x) = 2097152\,x^6 - 1228800\,x^4 + 3810240\,x^2 - 62432370\ , \quad N_7^{(c,1)} = 468242775$$

$$P_8^{(s,0)}(x) = 134217728\,x^8 + 15728640\,x^6 - 19353600\,x^4 + 90810720\,x^2 - 1739187450\ ,$$
$$Q_8^{(s,0)}(x) = 50331648\,x^7 - 13762560\,x^5 + 37255680\,x^3 - 267567300\,x\ ,$$
$$R_8^{(s,0)}(x) = -16777216\,x^7 + 9830400\,x^5 - 30481920\,x^3 + 231891660\,x\ ,$$
$$S_8^{(s,0)}(x) = -134217728\,x^8 + 9437184\,x^6 - 15052800\,x^4 + 76839840\,x^2\ , \quad N_8^{(s,0)} = 13043905875$$
$$P_8^{(c,0)}(x) = -50331648\,x^7 + 13762560\,x^5 - 37255680\,x^3 + 267567300\,x\ ,$$
$$Q_8^{(c,0)}(x) = 134217728\,x^8 + 15728640\,x^6 - 19353600\,x^4 + 90810720\,x^2 - 1739187450\ ,$$
$$R_8^{(c,0)}(x) = 134217728\,x^8 - 9437184\,x^6 + 15052800\,x^4 - 76839840\,x^2\ ,$$
$$S_8^{(c,0)}(x) = -16777216\,x^7 + 9830400\,x^5 - 30481920\,x^3 + 231891660\,x\ , \quad N_8^{(c,0)} = 13043905875$$
$$P_8^{(s,1)}(x) = 50331648\,x^7 - 13762560\,x^5 + 37255680\,x^3 - 267567300\,x\ ,$$
$$Q_8^{(s,1)}(x) = -134217728\,x^8 - 15728640\,x^6 + 19353600\,x^4 - 90810720\,x^2\ ,$$
$$R_8^{(s,1)}(x) = -134217728\,x^8 + 9437184\,x^6 - 15052800\,x^4 + 76839840\,x^2 - 1739187450\ ,$$
$$S_8^{(s,1)}(x) = 16777216\,x^7 - 9830400\,x^5 + 30481920\,x^3 - 231891660\,x\ , \quad N_8^{(s,1)} = 14783093325$$
$$P_8^{(c,1)}(x) = 134217728\,x^8 + 15728640\,x^6 - 19353600\,x^4 + 90810720\,x^2\ ,$$
$$Q_8^{(c,1)}(x) = 50331648\,x^7 - 13762560\,x^5 + 37255680\,x^3 - 267567300\,x\ ,$$
$$R_8^{(c,1)}(x) = -16777216\,x^7 + 9830400\,x^5 - 30481920\,x^3 + 231891660\,x\ ,$$
$$S_8^{(c,1)}(x) = -134217728\,x^8 + 9437184\,x^6 - 15052800\,x^4 + 76839840\,x^2 - 1739187450\ ,$$
$$N_8^{(c,1)} = 14783093325$$

<u>Recurrence relations:</u>

$$\text{Let} \qquad I_n^{(s,\nu)} = \int \frac{\sin x J_\nu(x)\,dx}{x^{n+1/2}} \qquad \text{and} \qquad I_n^{(c,\nu)} = \int \frac{\cos x J_\nu(x)\,dx}{x^{n+1/2}}\ , \qquad \text{then holds}$$

$$I_{n+1}^{(s,0)} = \frac{2}{2n+1}\left[I_n^{(c,0)} - I_n^{(s,1)} - \frac{\sin x \cdot J_0(x)}{x^{n+1/2}}\right]\ , \qquad I_{n+1}^{(s,1)} = \frac{2}{2n+3}\left[I_n^{(s,0)} + I_n^{(c,1)} - \frac{\sin x \cdot J_1(x)}{x^{n+1/2}}\right]$$
$$I_{n+1}^{(c,0)} = \frac{2}{2n+1}\left[-I_n^{(s,0)} - I_n^{(c,1)} - \frac{\cos x \cdot J_0(x)}{x^{n+1/2}}\right]\ , \qquad I_{n+1}^{(c,1)} = \frac{2}{2n+3}\left[I_n^{(c,0)} - I_n^{(s,1)} - \frac{\cos x \cdot J_1(x)}{x^{n+1/2}}\right]$$

1.2.7. Integrals of the Type $\int x^{-n-1/2}\, e^{\pm x} \left\{ \begin{array}{c} I_\nu(x) \\ K_\nu(x) \end{array} \right\} dx$

a) Integrals $\int x^{-n-1/2}\, e^x\, I_\nu(x)\, dx$:

See also [4], 1.11.2. and 1.11.? .

$$\int \frac{e^x\, I_0(x)\, dx}{x^{3/2}} = \frac{(4x-2)\, I_0(x) - 4x\, I_1(x)}{\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{3/2}} = \frac{4x\, I_0(x) - (2+4x)\, I_1(x)}{3\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, K_0(x)\, dx}{x^{3/2}} = \frac{(4x-2)\, K_0(x) + 4x\, K_1(x)}{\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{3/2}} = -\frac{4x\, K_0(x) + (2+4x)\, K_1(x)}{3\sqrt{x}}\, e^x$$

$$\int \frac{e^x\, I_0(x)\, dx}{x^{5/2}} = \frac{(-6-12x+32x^2)\, I_0(x) - (4x+32x^2)\, I_1(x)}{9\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{5/2}} = \frac{(-12x+32x^2)\, I_0(x) - (6+4x+32x^2)\, I_1(x)}{15\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, K_0(x)\, dx}{x^{5/2}} = \frac{(-6-12x+32x^2)\, K_0(x) + (4x+32x^2)\, K_1(x)}{9\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{5/2}} = \frac{(12x-32x^2)\, K_0(x) - (6+4x+32x^2)\, K_1(x)}{15\, x^{3/2}}\, e^x$$

$$\int \frac{e^x\, I_0(x)\, dx}{x^{7/2}} = \frac{(512\, x^3 - 192\, x^2 - 60\, x - 90)\, I_0(x) - (512\, x^3 + 64\, x^2 + 36\, x)\, I_1(x)}{225\, x^{5/2}}\, e^x$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{7/2}} = \frac{(512\, x^3 - 192\, x^2 - 60\, x)\, I_0(x) - (512\, x^3 + 64\, x^2 + 36\, x + 90)\, I_1(x)}{315\, x^{5/2}}\, e^x$$

$$\int \frac{e^x\, K_0(x)\, dx}{x^{7/2}} = \frac{(512\, x^3 - 192\, x^2 - 60\, x - 90)\, K_0(x) + (512\, x^3 + 64\, x^2 + 36\, x)\, K_1(x)}{225\, x^{5/2}}\, e^x$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{7/2}} = \frac{(-512\, x^3 + 192\, x^2 + 60\, x)\, K_0(x) - (512\, x^3 + 64\, x^2 + 36\, x + 90)\, K_1(x)}{315\, x^{5/2}}\, e^x$$

Let

$$\int \frac{e^x\, I_0(x)\, dx}{x^{n+1/2}} = \frac{P_n^{(0,+)}(x)\, I_0(x) - Q_n^{(0,+)}(x)\, I_1(x)}{N_n^{(0,+)}\, x^{n-1/2}}\, e^x\ ,$$

$$\int \frac{e^x\, I_1(x)\, dx}{x^{n+1/2}} = \frac{P_n^{(1,+)}(x)\, I_0(x) - Q_n^{(1,+)}(x)\, I_1(x)}{N_n^{(1,+)}\, x^{n-1/2}}\, e^x\ ,$$

then holds[1] with the same coefficients P_n and Q_n

$$\int \frac{e^x\, K_0(x)\, dx}{x^{n+1/2}} = \frac{P_n^{(0,+)}(x)\, K_0(x) + Q_n^{(0,+)}(x)\, K_1(x)}{N_n^{(0,+)}\, x^{n-1/2}}\, e^x\ ,$$

$$\int \frac{e^x\, K_1(x)\, dx}{x^{n+1/2}} = \frac{-P_n^{(1,+)}(x)\, K_0(x) - Q_n^{(1,+)}(x)\, K_1(x)}{N_n^{(1,+)}\, x^{n-1/2}}\, e^x\ .$$

$$P_4^{(0,+)}(x) = 4096\, x^4 - 1536\, x^3 - 480\, x^2 - 420\, x - 1050\ ,$$

[1] Note that there are signs '-' in the numerators!

$$Q_4^{(0,+)}(x) = 4096\,x^4 + 512\,x^3 + 288\,x^2 + 300\,x\ , \quad N_4^{(0,+)} = 3675$$
$$P_4^{(1,+)}(x) = 4096\,x^4 - 1536\,x^3 - 480\,x^2 - 420\,x\ ,$$
$$Q_4^{(1,+)}(x) = 4096\,x^4 + 512\,x^3 + 288\,x^2 + 300\,x + 1050\ , \quad N_4^{(1,+)} = 4725$$

$$P_5^{(0,+)}(x) = 131072\,x^5 - 49152\,x^4 - 15360\,x^3 - 13440\,x^2 - 18900\,x - 66150\ ,$$
$$Q_5^{(0,+)}(x) = 131072\,x^5 + 16384\,x^4 + 9216\,x^3 + 9600\,x^2 + 14700\,x\ , \quad N_5^{(0,+)} = 297675$$
$$P_5^{(1,+)}(x) = 131072\,x^5 - 49152\,x^4 - 15360\,x^3 - 13440\,x^2 - 18900\,x\ ,$$
$$Q_5^{(1,+)}(x) = 131072\,x^5 + 16384\,x^4 + 9216\,x^3 + 9600\,x^2 + 14700\,x + 66150\ , \quad N_5^{(1,+)} = 363825$$

$$P_6^{(0,+)}(x) = 1048576\,x^6 - 393216\,x^5 - 122880\,x^4 - 107520\,x^3 - 151200\,x^2 - 291060\,x - 1309770$$
$$Q_6^{(0,+)}(x) = 1048576\,x^6 + 131072\,x^5 + 73728\,x^4 + 76800\,x^3 + 117600\,x^2 + 238140\,x\ ,$$
$$N_6^{(0,+)} = 7203735$$
$$P_6^{(1,+)}(x) = 1048576\,x^6 - 393216\,x^5 - 122880\,x^4 - 107520\,x^3 - 151200\,x^2 - 291060\,x\ ,$$
$$Q_6^{(1,+)}(x) = 1048576\,x^6 + 131072\,x^5 + 73728\,x^4 + 76800\,x^3 + 117600\,x^2 + 238140\,x + 1309770\ ,$$
$$N_6^{(1,+)} = 8513505$$

$$P_7^{(0,+)}(x) =$$
$$= 16777216\,x^7 - 6291456\,x^6 - 1966080\,x^5 - 1720320\,x^4 - 2419200\,x^3 - 4656960\,x^2 - 11351340\,x - 62432370\ ,$$
$$Q_7^{(0,+)}(x) =$$
$$= 16777216\,x^7 + 2097152\,x^6 + 1179648\,x^5 + 1228800\,x^4 + 1881600\,x^3 + 3810240\,x^2 + 9604980\,x\ ,$$
$$N_7^{(0,+)} = 405810405$$
$$P_7^{(1,+)}(x) =$$
$$= 16777216\,x^7 - 6291456\,x^6 - 1966080\,x^5 - 1720320\,x^4 - 2419200\,x^3 - 4656960\,x^2 - 11351340\,x\ ,$$
$$Q_7^{(1,+)}(x) =$$
$$= 16777216\,x^7 + 2097152\,x^6 + 1179648\,x^5 + 1228800\,x^4 + 1881600\,x^3 + 3810240\,x^2 + 9604980\,x + 62432370\ ,$$
$$N_7^{(1,+)} = 468242775$$

Recurrence relations:

$$\text{With} \qquad \mathbf{I}_n^{(\nu,+)} = \int \frac{e^x\, I_\nu(x)\, dx}{x^{n+1/2}} \qquad \text{and} \qquad \mathbf{K}_n^{(\nu,+)} = \int \frac{e^x\, K_\nu(x)\, dx}{x^{n+1/2}} \qquad \text{holds}$$
$$\mathbf{I}_n^{(0,+)} = \frac{2}{2n-1}\left[\mathbf{I}_{n-1}^{(0,+)} + \mathbf{I}_{n-1}^{(1,+)} - \frac{e^x\, I_0(x)}{x^{n-1/2}}\right]\ , \qquad \mathbf{I}_n^{(1,+)} = \frac{2}{2n+1}\left[\mathbf{I}_{n-1}^{(0,+)} + \mathbf{I}_{n-1}^{(1,+)} - \frac{e^x\, I_1(x)}{x^{n-1/2}}\right]\ ,$$
$$\mathbf{K}_n^{(0,+)} = \frac{2}{2n-1}\left[\mathbf{K}_{n-1}^{(0,+)} - \mathbf{K}_{n-1}^{(1,+)} - \frac{e^x\, K_0(x)}{x^{n-1/2}}\right]\ , \qquad \mathbf{K}_n^{(1,+)} = \frac{2}{2n+1}\left[-\mathbf{K}_{n-1}^{(0,+)} + \mathbf{K}_{n-1}^{(1,+)} - \frac{e^x\, K_1(x)}{x^{n-1/2}}\right]$$

b) Integrals $\int x^{-n-1/2}\, e^{-x}\, I_\nu(x)\, dx$:

See also [4], 1.11.2.1 and 1.11.?. .

$$\int \frac{e^{-x}\, I_0(x)\, dx}{x^{3/2}} = -\,\frac{(4x+2)\, I_0(x) + 4x\, I_1(x)}{\sqrt{x}}\, e^{-x}$$
$$\int \frac{e^{-x}\, I_1(x)\, dx}{x^{3/2}} = \frac{4x\, I_0(x) + (4x-2)\, I_1(x)}{3\sqrt{x}}\, e^{-x}$$

$$\int \frac{e^{-x}\,K_0(x)\,dx}{x^{3/2}} = \frac{-(4x+2)\,K_0(x)+4x\,K_1(x)}{\sqrt{x}}\,e^{-x}$$
$$\int \frac{e^{-x}\,K_1(x)\,dx}{x^{3/2}} = \frac{-4x\,K_0(x)+(4x-2)\,K_1(x)}{3\sqrt{x}}\,e^{-x}$$

$$\int \frac{e^{-x}\,I_0(x)\,dx}{x^{5/2}} = \frac{(32\,x^2+12\,x-6)\,I_0(x)+(32\,x^2-4\,x)\,I_1(x)}{9\,x^{3/2}}\,e^{-x}$$
$$\int \frac{e^{-x}\,I_1(x)\,dx}{x^{5/2}} = \frac{(-32\,x^2-12\,x)\,I_0(x)+(-32\,x^2+4\,x-6)\,I_1(x)}{15\,x^{3/2}}\,e^{-x}$$
$$\int \frac{e^{-x}\,K_0(x)\,dx}{x^{5/2}} = \frac{(32\,x^2+12\,x-6)\,K_0(x)-(32\,x^2-4\,x)\,K_1(x)}{9\,x^{3/2}}\,e^{-x}$$
$$\int \frac{e^{-x}\,K_1(x)\,dx}{x^{5/2}} = \frac{(32\,x^2+12\,x)\,K_0(x)+(-32\,x^2+4\,x-6)\,K_1(x)}{15\,x^{3/2}}\,e^{-x}$$

$$\int \frac{e^{-x}\,I_0(x)\,dx}{x^{7/2}} = \frac{(-512\,x^3-192\,x^2+60\,x-90)\,I_0(x)+(-512\,x^3+64\,x^2-36\,x)\,I_1(x)}{225\,x^{5/2}}\,e^{-x}$$
$$\int \frac{e^{-x}\,I_1(x)\,dx}{x^{7/2}} = \frac{(512\,x^3+192\,x^2-60\,x)\,I_0(x)+(512\,x^3-64\,x^2+36\,x-90)\,I_1(x)}{315\,x^{5/2}}\,e^{-x}$$

$$\int \frac{e^{-x}\,K_0(x)\,dx}{x^{7/2}} = \frac{(-512\,x^3-192\,x^2+60\,x-90)\,K_0(x)-(-512\,x^3+64\,x^2-36\,x)\,K_1(x)}{225\,x^{5/2}}\,e^{-x}$$
$$\int \frac{e^{-x}\,K_1(x)\,dx}{x^{7/2}} = \frac{-(512\,x^3+192\,x^2-60\,x)\,K_0(x)+(512\,x^3-64\,x^2+36\,x-90)\,K_1(x)}{315\,x^{5/2}}\,e^{-x}$$

Let

$$\int \frac{e^{-x}\,I_0(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(0,-)}(x)\,I_0(x)+Q_n^{(0,-)}(x)\,I_1(x)}{N_n^{(0,-)}\,x^{n-1/2}}\,e^{x}\;,$$
$$\int \frac{e^{-x}\,I_1(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(1,-)}(x)\,I_0(x)+Q_n^{(1,-)}(x)\,I_1(x)}{N_n^{(1,-)}\,x^{n-1/2}}\,e^{x}\;,$$

then holds

$$\int \frac{e^{-x}\,K_0(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(0,-)}(x)\,K_0(x)-Q_n^{(0,-)}(x)\,K_1(x)}{N_n^{(0,-)}\,x^{n-1/2}}\,e^{x}\;,$$
$$\int \frac{e^{-x}\,K_1(x)\,dx}{x^{n+1/2}} = \frac{-P_n^{(1,-)}(x)\,K_0(x)+Q_n^{(1,-)}(x)\,K_1(x)}{N_n^{(1,-)}\,x^{n-1/2}}\,e^{x}\;.$$

$$P_4^{(0,-)}(x) = 4096\,x^4+1536\,x^3-480\,x^2+420\,x-1050\;,$$
$$Q_4^{(0,-)}(x) = 4096\,x^4-512\,x^3+288\,x^2-300\,x\;,\quad N_4^{(0,-)} = 3675$$
$$P_4^{(1,-)}(x) = -4096\,x^4-1536\,x^3+480\,x^2-420\,x\;,$$
$$Q_4^{(1,-)}(x) = -4096\,x^4+512\,x^3-288\,x^2+300\,x-1050\;,\quad N_4^{(1,-)} = 4725$$

$$P_5^{(0,-)}(x) = -131072\,x^5-49152\,x^4+15360\,x^3-13440\,x^2+18900\,x-66150\;,$$
$$Q_5^{(0,-)}(x) = -131072\,x^5+16384\,x^4-9216\,x^3+9600\,x^2-14700\,x\;,\quad N_5^{(0,-)} = 297675$$
$$P_5^{(1,-)}(x) = 131072\,x^5+49152\,x^4-15360\,x^3+13440\,x^2-18900\,x\;,$$
$$Q_5^{(1,-)}(x) = 131072\,x^5-16384\,x^4+9216\,x^3-9600\,x^2+14700\,x-66150\;,\quad N_5^{(1,-)} = 363825$$

$$P_6^{(0,-)}(x) = 1048576\,x^6 + 393216\,x^5 - 122880\,x^4 + 107520\,x^3 - 151200\,x^2 + 291060\,x - 1309770\ ,$$
$$Q_6^{(0,-)}(x) = 1048576\,x^6 - 131072\,x^5 + 73728\,x^4 - 76800\,x^3 + 117600\,x^2 - 238140\,x\ ,$$
$$N_6^{(0,-)} = 7203735$$
$$P_6^{(1,-)}(x) = -1048576\,x^6 - 393216\,x^5 + 122880\,x^4 - 107520\,x^3 + 151200\,x^2 - 291060\,x\ ,$$
$$Q_6^{(1,-)}(x) = -1048576\,x^6 + 131072\,x^5 - 73728\,x^4 + 76800\,x^3 - 117600\,x^2 + 238140\,x - 1309770\ ,$$
$$N_6^{(1,-)} = 8513505$$

$$P_7^{(0,-)}(x) =$$
$$= -16777216\,x^7 - 6291456\,x^6 + 1966080\,x^5 - 1720320\,x^4 + 2419200\,x^3 - 4656960\,x^2 + 11351340\,x - 62432370\ ,$$
$$Q_7^{(0,-)}(x) = -16777216\,x^7 + 2097152\,x^6 - 1179648\,x^5 + 1228800\,x^4 - 1881600\,x^3 + 3810240\,x^2 - 9604980\,x\ ,$$
$$N_7^{(0,-)} = 405810405$$
$$P_7^{(1,-)}(x) = 16777216\,x^7 + 6291456\,x^6 - 1966080\,x^5 + 1720320\,x^4 - 2419200\,x^3 + 4656960\,x^2 - 11351340\,x\ ,$$
$$Q_7^{(1,-)}(x) =$$
$$= 16777216\,x^7 - 2097152\,x^6 + 1179648\,x^5 - 1228800\,x^4 + 1881600\,x^3 - 3810240\,x^2 + 9604980\,x - 62432370\ ,$$
$$N_7^{(1,-)} = 468242775$$

Recurrence relations:

$$\text{With} \qquad \mathbf{I}_n^{(\nu,-)} = \int \frac{e^{-x}\,I_\nu(x)\,dx}{x^{n+1/2}} \qquad \text{and} \qquad \mathbf{K}_n^{(\nu,-)} = \int \frac{e^{-x}\,K_\nu(x)\,dx}{x^{n+1/2}} \qquad \text{holds}$$

$$\mathbf{I}_n^{(0,-)} = \frac{2}{2n-1}\left[-\mathbf{I}_{n-1}^{(0,-)} + \mathbf{I}_{n-1}^{(1,-)} - \frac{e^{-x}\,I_0(x)}{x^{n-1/2}}\right]\ , \qquad \mathbf{I}_n^{(1,-)} = \frac{2}{2n+1}\left[\mathbf{I}_{n-1}^{(0,-)} - \mathbf{I}_{n-1}^{(1,-)} - \frac{e^{-x}\,I_1(x)}{x^{n-1/2}}\right]\ ,$$
$$\mathbf{K}_n^{(0,-)} = -\frac{2}{2n-1}\left[\mathbf{K}_{n-1}^{(0,-)} + \mathbf{K}_{n-1}^{(1,-)} + \frac{e^{-x}\,K_0(x)}{x^{n-1/2}}\right]\ , \qquad \mathbf{K}_n^{(1,-)} = -\frac{2}{2n+1}\left[\mathbf{K}_{n-1}^{(0,-)} + \mathbf{K}_{n-1}^{(1,-)} + \frac{e^{-x}\,K_1(x)}{x^{n-1/2}}\right]$$

c) General formulas

Let
$$P_n^{(\nu,\pm)}(x) = \sum_{k=0}^{n} \vartheta_k^{(n,\nu,\pm)}\,x^k \qquad \text{and} \quad Q_n^{(\nu,\pm)}(x) = \sum_{k=0}^{n} \eta_k^{(n,\nu,\pm)}\,x^k\ ,$$
then holds
$$\int \frac{e^{\pm x}\,I_\nu(x)\,dx}{x^{n+1/2}} = \frac{P_n^{(\nu,\pm)}(x)\,I_0(x)\ +\ Q_n^{(\nu,\pm)}(x)\,I_1(x)}{x^{n-1/2}}\ .$$
The integrals with $K_\nu(x)$ may be expressed as described before.

I. $\nu = 0,\ e^x$:
$$\vartheta_0^{(n,0,+)} = -\frac{2}{2n-1}\ , \quad \vartheta_1^{(n,0,+)} = -\frac{2^2}{(2n-1)(2n-3)}\ ,$$
$$\vartheta_2^{(n,0,+)} = -\frac{2^5(n-1)}{(2n-1)^2(2n-3)(2n-5)} = \frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,0,+)}\ ,$$
$$\vartheta_3^{(n,0,+)} = -\frac{2^8(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)(2n-7)} = \frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,0,+)}\ ,$$

$$\vartheta_4^{(n,0,+)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n-1)^2(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = \frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,0,+)}\ , \quad \ldots\ ,$$

which gives $\vartheta_0^{(n,0,+)} = -\,2/(2n-1)$ and

$$\vartheta_k^{(n,0,+)} = -\,\frac{2^{k-1}\cdot\Gamma(n-k-\frac{1}{2})\cdot\Gamma(n-k+\frac{3}{2})\cdot(n-1)!}{\Gamma^2(n+\frac{1}{2})\cdot(n-k)!}\ ,\ k>0\ .$$

Furthermore

$$\eta_1^{(n,0,+)} = -\frac{2^2}{(2n-1)^2}\ , \quad \eta_2^{(n,0,+)} = -\frac{2^5\,(n-1)}{(2n-1)^2(2n-3)^2} = \frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,0,+)}\ ,$$

$$\eta_3^{(n,0,+)} = -\frac{2^8\,(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)^2} = \frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_2^{(n,0,+)}\ ,\ \ldots\ ,$$

which gives $\eta_0^{(n,0,+)} = 0$ and

$$\eta_k^{(n,0,+)} = -\frac{2^{k-1}\cdot\Gamma^2(n-k+\frac{1}{2})\cdot(n-1)!}{\Gamma^2(n+\frac{1}{2})\cdot(n-k)!}\ , \quad k>0\ .$$

II. $\nu=1,\ e^x$:

$$\vartheta_0^{(n,1,+)} = 0\ , \quad \vartheta_1^{(n,1,+)} = -\frac{2^2}{(2n+1)(2n-3)}\ ,$$

$$\vartheta_2^{(n,1,+)} = -\frac{2^5(n-1)}{(2n+1)(2n-1)(2n-3)(2n-5)} = \frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,1,+)}\ ,$$

$$\vartheta_3^{(n,1,+)} = -\frac{2^8(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)(2n-7)} = \frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,1,+)}\ ,$$

$$\vartheta_4^{(n,1,+)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = \frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,1,+)}\ , \quad \ldots\ ,$$

which gives $\vartheta_0^{(n,1,+)} = 0$ and

$$\vartheta_k^{(n,1,+)} = -\,\frac{2^{k-1}\cdot\Gamma(n-k-\frac{1}{2})\cdot\Gamma(n-k+\frac{3}{2})\cdot(n-1)!}{\Gamma(n+\frac{3}{2})\cdot\Gamma(n-\frac{1}{2})\cdot(n-k)!}\ ,\ k>0\ .$$

Furthermore

$$\eta_0^{(n,1,+)} = -\frac{2}{2n+1}\ , \quad \eta_1^{(n,1,+)} = -\frac{2^2}{(2n+1)(2n-1)}\ ,$$

$$\eta_2^{(n,1,+)} = -\frac{2^5\,(n-1)}{(2n+1)(2n-1)(2n-3)^2} = \frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,1,+)}\ ,$$

$$\eta_3^{(n,1,+)} = -\frac{2^8\,(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2} = \frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_1^{(n,2,+)}\ ,\ \ldots\ ,$$

which gives $\eta_0^{(n,1,+)} = -2/(2n+1)$ and

$$\eta_k^{(n,0,+)} = -\frac{2^{k-1}\cdot\Gamma^2(n-k+\frac{1}{2})\cdot(n-1)!}{\Gamma(n+\frac{3}{2})\cdot\Gamma(n-\frac{1}{2})\cdot(n-k)!}\ , \quad k>1\ .$$

III. $\nu=0,\ e^{-x}$:

$$\vartheta_0^{(n,0,-)} = -2/(2n-1)\ , \quad \vartheta_1^{(n,0,-)} = \frac{2^2}{(2n-1)(2n-3)}\ ,$$

$$\vartheta_2^{(n,0,-)} = -\frac{2^5(n-1)}{(2n-1)^2(2n-3)(2n-5)} = -\,\frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,0,-)}\ ,$$

$$\vartheta_3^{(n,0,-)} = \frac{2^8(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)(2n-7)} = \frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,0,-)}\ ,$$

$$\vartheta_4^{(n,0,-)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n-1)^2(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = \frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,0,-)}\ , \quad \ldots\ ,$$

which gives $\vartheta_0^{(n,0,-)} = -2/(2n-1)$ and

$$\vartheta_k^{(n,0,-)} = \frac{(-2)^{k-1}\cdot\Gamma(n-k-\frac{1}{2})\cdot\Gamma(n-k+\frac{3}{2})\cdot(n-1)!}{\Gamma^2(n+\frac{1}{2})\cdot(n-k)!}\ ,\ k>0\ .$$

Furthermore

$$\eta_1^{(n,0,-)} = -\,\frac{2^2}{(2n-1)^2}\ ,\quad \eta_2^{(n,0,-)} = \frac{2^5\,(n-1)}{(2n-1)^2(2n-3)^2} = -\frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,0,-)}\ ,$$

$$\eta_3^{(n,0,-)} = -\frac{2^8\,(n-1)(n-2)}{(2n-1)^2(2n-3)^2(2n-5)^2} = -\frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_2^{(n,0,-)}\ ,\ \ldots\ ,$$

which gives $\eta_0^{(n,0,-)} = 0$ and

$$\eta_k^{(n,0,-)} = -\frac{(-2)^{k-1}\cdot\Gamma^2(n-k+\frac{1}{2})\cdot(n-1)!}{\Gamma^2(n+\frac{1}{2})\cdot(n-k)!}\ ,\quad k>0\ .$$

IV. $\nu = 1,\ e^{-x}$:

$$\vartheta_0^{(n,1,-)} = 0\ ,\quad \vartheta_1^{(n,1,-)} = -\frac{2^2}{(2n+1)(2n-3)}\ ,$$

$$\vartheta_2^{(n,1,-)} = \frac{2^5(n-1)}{(2n+1)(2n-1)(2n-3)(2n-5)} = -\frac{2^3\,(n-1)}{(2n-1)(2n-5)}\,\vartheta_1^{(n,1,-)}\ ,$$

$$\vartheta_3^{(n,1,-)} = \frac{2^8(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)(2n-7)} = -\frac{2^3\,(n-2)}{(2n-3)(2n-7)}\,\vartheta_2^{(n,1,-)}\ ,$$

$$\vartheta_4^{(n,1,-)} = -\frac{2^{11}(n-1)(n-2)(n-3)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2(2n-7)(2n-9)} = -\frac{2^3\,(n-3)}{(2n-5)(2n-9)}\,\vartheta_3^{(n,1,-)}\ , \quad \ldots\ ,$$

which gives $\vartheta_0^{(n,1,-)} = 0$ and

$$\vartheta_k^{(n,1,-)} = -\,\frac{(-2)^{k-1}\cdot\Gamma(n-k-\frac{1}{2})\cdot\Gamma(n-k+\frac{3}{2})\cdot(n-1)!}{\Gamma(n+\frac{3}{2})\cdot\Gamma(n-\frac{1}{2})\cdot(n-k)!}\ ,\ k>0\ .$$

Furthermore

$$\eta_0^{(n,1,-)} = -\frac{2}{2n+1}\ ,\quad \eta_1^{(n,1,-)} = \frac{2^2}{(2n+1)(2n-1)}\ ,$$

$$q\eta_2^{(n,1,-)} = -\frac{2^5\,(n-1)}{(2n+1)(2n-1)(2n-3)^2} = \frac{2^3\,(n-1)}{(2n-3)^2}\,\eta_1^{(n,1,-)}\ ,$$

$$\eta_3^{(n,1,-)} = \frac{2^8\,(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)^2(2n-5)^2} = \frac{2^3\,(n-2)}{(2n-5)^2}\,\eta_2^{(n,1,-)}\ ,\ \ldots\ ,$$

which gives $\eta_0^{(n,1,-)} = -2/(2n+1)$ and

$$\eta_k^{(n,1,-)} = \frac{(-2)^{k-1}\cdot\Gamma^2(n-k+\frac{1}{2})\cdot(n-1)!}{\Gamma(n+\frac{3}{2})\cdot\Gamma(n-\frac{1}{2})\cdot(n-k)!}\ .$$

When $n > 0$, then the functions $P_n^{(\nu,\pm)}(x)$ and $Q_n^{(\nu,\pm)}(x)$ are polynomials. In the case $n \leq 0$ they are power series with the radius of convergence $R = +\infty$. Their coefficients may be found by the given recurrence relations.

$$P_0^{(0,+)} = 2-\frac{4}{3}\,x+\frac{32}{15}\,x^2-\frac{512}{315}\,x^3+\frac{4096}{4725}\,x^4-\frac{131072}{363825}\,x^5+\frac{1048576}{8513505}\,x^6-\frac{16777216}{468242775}\,x^7+\frac{134217728}{14783093325}\,x^8- -\frac{8589934592}{4213181597625}\,x^9+\frac{68719476736}{167122870039125}\,x^{10}+\dots$$

$$Q_0^{(0,+)} = -4\,x+\frac{32}{9}\,x^2-\frac{512}{225}\,x^3+\frac{4096}{3675}\,x^4-\frac{131072}{297675}\,x^5+\frac{1048576}{7203735}\,x^6-\frac{16777216}{405810405}\,x^7+\frac{134217728}{13043905875}\,x^8- -\frac{8589934592}{3769688797875}\,x^9+\frac{68719476736}{151206406225875}\,x^{10}+\dots$$

$$P_{-1}^{(0,+)} = \frac{2}{3}-\frac{4}{15}\,x+\frac{64}{315}\,x^2-\frac{512}{4725}\,x^3+\frac{16384}{363825}\,x^4-\frac{131072}{8513505}\,x^5+\frac{2097152}{468242775}\,x^6-\frac{16777216}{14783093325}\,x^7+ +\frac{1073741824}{4213181597625}\,x^8-\frac{8589934592}{167122870039125}\,x^9+\frac{137438953472}{14606538841419525}\,x^{10}+\dots$$

$$Q_{-1}^{(0,+)} = -\frac{4}{9}\,x+\frac{64}{225}\,x^2-\frac{512}{3675}\,x^3+\frac{16384}{297675}\,x^4-\frac{131072}{7203735}\,x^5+\frac{2097152}{405810405}\,x^6-\frac{16777216}{13043905875}\,x^7+ +\frac{1073741824}{3769688797875}\,x^8-\frac{8589934592}{151206406225875}\,x^9+\frac{137438953472}{13336405029122175}\,x^{10}+\dots$$

$$P_{-2}^{(0,+)} = \frac{2}{5}-\frac{4}{35}\,x+\frac{32}{525}\,x^2-\frac{1024}{40425}\,x^3+\frac{8192}{945945}\,x^4-\frac{131072}{52026975}\,x^5+\frac{1048576}{1642565925}\,x^6-\frac{67108864}{468131288625}\,x^7+ +\frac{536870912}{18569207782125}\,x^8-\frac{8589934592}{1622948760157725}\,x^9+\frac{68719476736}{77458918098436875}\,x^{10}+\dots$$

$$Q_{-2}^{(0,+)} = -\frac{4}{25}\,x+\frac{96}{1225}\,x^2-\frac{1024}{33075}\,x^3+\frac{8192}{800415}\,x^4-\frac{131072}{45090045}\,x^5+\frac{1048576}{1449322875}\,x^6-\frac{67108864}{418854310875}\,x^7+ +\frac{536870912}{16800711802875}\,x^8-\frac{8589934592}{1481822781013575}\,x^9+\frac{68719476736}{71262204650561925}\,x^{10}+\dots$$

$$P_0^{(1,+)} = \frac{4}{3}\,x-\frac{32}{15}\,x^2+\frac{512}{315}\,x^3-\frac{4096}{4725}\,x^4+\frac{131072}{363825}\,x^5-\frac{1048576}{8513505}\,x^6+\frac{16777216}{468242775}\,x^7-\frac{134217728}{14783093325}\,x^8+ +\frac{8589934592}{4213181597625}\,x^9-\frac{68719476736}{167122870039125}\,x^{10}+\dots$$

$$Q_0^{(1,+)} = \frac{2}{3}-\frac{4}{15}\,x+\frac{32}{245}\,x^2-\frac{1024}{19845}\,x^3+\frac{8192}{480249}\,x^4-\frac{131072}{27054027}\,x^5+\frac{1048576}{869593725}\,x^6-\frac{67108864}{251312586525}\,x^7+ +\frac{536870912}{10080427081725}\,x^8--\frac{8589934592}{889093668608145}\,x^9+\frac{68719476736}{42757322790337155}\,x^{10}+\dots$$

$$P_{-1}^{(1,+)} = -\frac{4}{5}\,x+\frac{64}{105}\,x^2-\frac{512}{1575}\,x^3+\frac{16384}{121275}\,x^4-\frac{131072}{2837835}\,x^5+\frac{2097152}{156080925}\,x^6-\frac{16777216}{4927697775}\,x^7+ +\frac{1073741824}{1404393865875}\,x^8-\frac{8589934592}{55707623346375}\,x^9+\frac{137438953472}{4868846280473175}\,x^{10}+\dots$$

$$Q_{-1}^{(1,+)} = 2-\frac{4}{3}\,x+\frac{64}{75}\,x^2-\frac{512}{1225}\,x^3+\frac{16384}{99225}\,x^4-\frac{131072}{2401245}\,x^5+\frac{2097152}{135270135}\,x^6-\frac{16777216}{4347968625}\,x^7+ +\frac{1073741824}{1256562932625}\,x^8-\frac{8589934592}{50402135408625}\,x^9+\frac{137438953472}{4445468343040725}\,x^{10}+\dots$$

$$P_{-2}^{(1,+)} = -\frac{4}{21}\,x+\frac{32}{315}\,x^2-\frac{1024}{24255}\,x^3+\frac{8192}{567567}\,x^4-\frac{131072}{31216185}\,x^5+\frac{1048576}{985539555}\,x^6-\frac{67108864}{280878773175}\,x^7+ +\frac{536870912}{11141524669275}\,x^8-\frac{8589934592}{973769256094635}\,x^9+\frac{68719476736}{46475350859062125}\,x^{10}+\dots$$

$$Q_{-2}^{(1,+)} = \frac{2}{3}-\frac{4}{15}\,x+\frac{32}{245}\,x^2-\frac{1024}{19845}\,x^3+\frac{8192}{480249}\,x^4-\frac{131072}{27054027}\,x^5+\frac{1048576}{869593725}\,x^6-\frac{67108864}{251312586525}\,x^7+$$

$$+\frac{536870912}{10080427081725}\,x^8-\frac{8589934592}{889093668608145}\,x^9+\frac{68719476736}{42757322790337155}\,x^{10}+\dots$$

$$P_0^{(0,-)}=2+\frac{4}{3}\,x+\frac{32}{15}\,x^2+\frac{512}{315}\,x^3+\frac{4096}{4725}\,x^4+\frac{131072}{363825}\,x^5+\frac{1048576}{8513505}\,x^6+\frac{16777216}{468242775}\,x^7+\frac{134217728}{14783093325}\,x^8+$$
$$+\frac{8589934592}{4213181597625}\,x^9+\frac{68719476736}{167122870039125}\,x^{10}+\dots$$

$$Q_0^{(0,-)}=-4\,x-\frac{32}{9}\,x^2-\frac{512}{225}\,x^3-\frac{4096}{3675}\,x^4-\frac{131072}{297675}\,x^5-\frac{1048576}{7203735}\,x^6-\frac{16777216}{405810405}\,x^7-\frac{134217728}{13043905875}\,x^8-$$
$$-\frac{8589934592}{3769688797875}\,x^9-\frac{68719476736}{151206406225875}\,x^{10}+\dots$$

$$P_{-1}^{(0,-)}=\frac{2}{3}+\frac{4}{15}\,x+\frac{64}{315}\,x^2+\frac{512}{4725}\,x^3+\frac{16384}{363825}\,x^4+\frac{131072}{8513505}\,x^5+\frac{2097152}{468242775}\,x^6+\frac{16777216}{14783093325}\,x^7+$$
$$+\frac{1073741824}{4213181597625}\,x^8+\frac{8589934592}{167122870039125}\,x^9+\frac{137438953472}{14606538841419525}\,x^{10}+\dots$$

$$Q_{-1}^{(0,-)}=-\frac{4}{9}\,x-\frac{64}{225}\,x^2-\frac{512}{3675}\,x^3-\frac{16384}{297675}\,x^4-\frac{131072}{7203735}\,x^5-\frac{2097152}{405810405}\,x^6-\frac{16777216}{13043905875}\,x^7-$$
$$-\frac{1073741824}{3769688797875}\,x^8-\frac{8589934592}{151206406225875}\,x^9-\frac{137438953472}{13336405029122175}\,x^{10}+\dots$$

$$P_{-2}^{(0,-)}=\frac{2}{5}+\frac{4}{35}\,x+\frac{32}{525}\,x^2+\frac{1024}{40425}\,x^3+\frac{8192}{945945}\,x^4+\frac{131072}{52026975}\,x^5+\frac{1048576}{1642565925}\,x^6+\frac{67108864}{468131288625}\,x^7+$$
$$+\frac{536870912}{18569207782125}\,x^8+\frac{8589934592}{1622948760157725}\,x^9+\frac{68719476736}{77458918098436875}\,x^{10}+\dots$$

$$Q_{-2}^{(0,-)}=-\frac{4}{25}\,x-\frac{96}{1225}\,x^2-\frac{1024}{33075}\,x^3-\frac{8192}{800415}\,x^4-\frac{131072}{45090045}\,x^5-\frac{1048576}{1449322875}\,x^6-\frac{67108864}{418854310875}\,x^7-$$
$$-\frac{536870912}{16800711802875}\,x^8-\frac{8589934592}{1481822781013575}\,x^9-\frac{68719476736}{71262204650561925}\,x^{10}+\dots$$

$$P_0^{(1,-)}=\frac{4}{3}\,x+\frac{32}{15}\,x^2+\frac{512}{315}\,x^3+\frac{4096}{4725}\,x^4+\frac{131072}{363825}\,x^5+\frac{1048576}{8513505}\,x^6+\frac{16777216}{468242775}\,x^7+\frac{134217728}{14783093325}\,x^8+$$
$$+\frac{8589934592}{4213181597625}\,x^9+\frac{68719476736}{167122870039125}\,x^{10}+\dots$$

$$Q_0^{(1,-)}=-2-4\,x-\frac{32}{9}\,x^2-\frac{512}{225}\,x^3-\frac{4096}{3675}\,x^4-\frac{131072}{297675}\,x^5-\frac{1048576}{7203735}\,x^6-\frac{16777216}{405810405}\,x^7-\frac{134217728}{13043905875}\,x^8-$$
$$-\frac{8589934592}{3769688797875}\,x^9-\frac{68719476736}{151206406225875}\,x^{10}+\dots$$

$$P_{-1}^{(1,-)}=-\frac{4}{5}\,x-\frac{64}{105}\,x^2-\frac{512}{1575}\,x^3-\frac{16384}{121275}\,x^4-\frac{131072}{2837835}\,x^5-\frac{2097152}{156080925}\,x^6-\frac{16777216}{4927697775}\,x^7-$$
$$-\frac{1073741824}{1404393865875}\,x^8-\frac{8589934592}{55707623346375}\,x^9-\frac{137438953472}{4868846280473175}\,x^{10}+\dots$$

$$Q_{-1}^{(1,-)}=2+\frac{4}{3}\,x+\frac{64}{75}\,x^2+\frac{512}{1225}\,x^3+\frac{16384}{99225}\,x^4+\frac{131072}{2401245}\,x^5+\frac{2097152}{135270135}\,x^6+\frac{16777216}{4347968625}\,x^7+$$
$$-\frac{1073741824}{1256562932625}\,x^8+\frac{8589934592}{50402135408625}\,x^9+\frac{137438953472}{4445468343040725}\,x^{10}+\dots$$

$$P_{-2}^{(1,-)}=-\frac{4}{21}\,x-\frac{32}{315}\,x^2-\frac{1024}{24255}\,x^3-\frac{8192}{567567}\,x^4-\frac{131072}{31216185}\,x^5-\frac{1048576}{985539555}\,x^6-\frac{67108864}{280878773175}\,x^7-$$
$$-\frac{536870912}{11141524669275}\,x^8-\frac{8589934592}{973769256094635}\,x^9-\frac{68719476736}{46475350859062125}\,x^{10}+\dots$$

$$Q_{-2}^{(1,-)}=\frac{2}{3}+\frac{4}{15}\,x+\frac{32}{245}\,x^2+\frac{1024}{19845}\,x^3+\frac{8192}{480249}\,x^4+\frac{131072}{27054027}\,x^5+\frac{1048576}{869593725}\,x^6+\frac{67108864}{251312586525}\,x^7+$$
$$+\frac{536870912}{10080427081725}\,x^8+\frac{8589934592}{889093668608145}\,x^9+\frac{68719476736}{42757322790337155}\,x^{10}+\dots$$

1.2.8. Integrals of the Type $\int x^{-n-1/2} \left\{ \begin{matrix} \sinh x \\ \cosh x \end{matrix} \right\} \left\{ \begin{matrix} I_\nu(x) \\ K_\nu(x) \end{matrix} \right\} dx$

$$\int x^{-3/2} \sinh x \cdot I_0(x)\, dx = \frac{2}{\sqrt{x}} \left[-\sinh x\, I_0(x) + 2x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) \right]$$

$$\int x^{-3/2} \sinh x \cdot I_1(x)\, dx = \frac{2}{3\sqrt{x}} \left[2x \sinh x\, I_0(x) - \sinh x\, I_1(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-3/2} \sinh x \cdot K_0(x)\, dx = \frac{2}{\sqrt{x}} \left[-\sinh x\, K_0(x) + 2x \cosh x\, K_0(x) + 2x \sinh x\, K_1(x) \right]$$

$$\int x^{-3/2} \sinh x \cdot K_1(x)\, dx = -\frac{2}{3\sqrt{x}} \left[2x \sinh x\, K_0(x) + \sinh x\, K_1(x) + 2x \cosh x\, K_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot I_0(x)\, dx = \frac{2}{\sqrt{x}} \left[2x \sinh x\, I_0(x) - \cosh x\, I_0(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot I_1(x)\, dx = \frac{2}{3\sqrt{x}} \left[2x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) - \cosh x\, I_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot K_0(x)\, dx = \frac{2}{\sqrt{x}} \left[2x \sinh x\, K_0(x) - \cosh x\, K_0(x) + 2x \cosh x\, K_1(x) \right]$$

$$\int x^{-3/2} \cosh x \cdot K_1(x)\, dx = -\frac{2}{3\sqrt{x}} \left[2x \cosh x\, K_0(x) + 2x \sinh x\, K_1(x) + \cosh x\, K_1(x) \right]$$

$$\int x^{-5/2} \sinh x\, I_0(x)\, dx =$$

$$= \frac{2}{9x^{3/2}} \left[(16x^2 - 3) \sinh x\, I_0(x) - 6x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) - 16x^2 \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2} \sinh x \cdot I_1(x)\, dx =$$

$$= \frac{2}{15x^{/2}} \left[-6x \sinh x\, I_0(x) + 16x^2 \cosh x\, I_0(x) - (16x^2 + 3) \sinh x\, I_1(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2} \sinh x \cdot K_0(x)\, dx =$$

$$= \frac{2}{9x^{3/2}} \left[(16x^2 - 3) \sinh x\, K_0(x) - 6x \cosh x\, K_0(x) + 2x \sinh x\, K_1(x) + 16x^2 \cosh x\, K_1(x) \right]$$

$$\int x^{-5/2} \sinh x \cdot K_1(x)\, dx =$$

$$= \frac{2}{15x^{3/2}} \left[6x \sinh x\, K_0(x) - 16x^2 \cosh x\, K_0(x) - (16x^2 + 3) \sinh x\, K_1(x) - 2x \cosh x\, K_1(x) \right]$$

$$\int x^{-5/2} \cosh x \cdot I_0(x)\, dx =$$

$$= \frac{2}{9x^{3/2}} \left[-6x \sinh x\, I_0(x) + (16x^2 - 3) \cosh x\, I_0(x) - 16x^2 \sinh x\, I_1(x) - 2x \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2} \cosh x \cdot I_1(x)\, dx =$$

$$= \frac{2}{15x^{3/2}} \left[16x^2 \sinh x\, I_0(x) - 6x \cosh x\, I_0(x) - 2x \sinh x\, I_1(x) - (16x^2 + 3) \cosh x\, I_1(x) \right]$$

$$\int x^{-5/2}\cosh x\cdot K_0(x)\,dx =$$

$$= \frac{2}{9x^{3/2}}\,\left[-6x\sinh x\,K_0(x)+(16x^2-3)\cosh x\,K_0(x)+16x^2\sinh x\,K_1(x)+2x\cosh x\,K_1(x)\right]$$

$$\int x^{-5/2}\cosh x\cdot K_1(x)\,dx =$$

$$= \frac{2}{15x^{3/2}}\,\left[-16\,x^2\sinh x\,K_0(x)+6\,x\cosh x\,K_0(x)-2\,x\sinh x\,K_1(x)-\left(3+16\,x^2\right)\,\cosh x\,K_1(x)\right]$$

Let

$$\int x^{-(2n+1)/2}\sinh x\cdot I_0(x)\,dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}}\,\left[P_n^{(s0)}(x)\sinh x\,I_0(x)+Q_n^{(s0)}(x)\cosh x\,I_0(x)+R_n^{(s0)}(x)\sinh x\,I_1(x)+S_n^{(s0)}(x)\cosh x\,I_1(x)\right]\,,$$

$$\int x^{-(2n+1)/2}\sinh x\cdot I_1(x)\,dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}}\,\left[P_n^{(s1)}(x)\sinh x\,I_0(x)+Q_n^{(s1)}(x)\cosh x\,I_0(x)+R_n^{(s1)}(x)\sinh x\,I_1(x)+S_n^{(s1)}(x)\cosh x\,I_1(x)\right]\,,$$

$$\int x^{-(2n+1)/2}\cosh x\cdot I_0(x)\,dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}}\,\left[P_n^{(c0)}(x)\sinh x\,I_0(x)+Q_n^{(c0)}(x)\cosh x\,I_0(x)+R_n^{(c0)}(x)\sinh x\,I_1(x)+S_n^{(c0)}(x)\cosh x\,I_1(x)\right]\,,$$

$$\int x^{-(2n+1)/2}\cosh x\cdot I_1(x)\,dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}}\,\left[P_n^{(c1)}(x)\sinh x\,I_0(x)+Q_n^{(c1)}(x)\cosh x\,I_0(x)+R_n^{(c1)}(x)\sinh x\,I_1(x)+S_n^{(c1)}(x)\cosh x\,I_1(x)\right]\,,$$

then holds

$$\int x^{-(2n+1)/2}\sinh x\cdot K_0(x)\,dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}}\,\left[P_n^{(s0)}(x)\sinh x\,K_0(x)+Q_n^{(s0)}(x)\cosh x\,K_0(x)-R_n^{(s0)}(x)\sinh x\,K_1(x)-S_n^{(s0)}(x)\cosh x\,K_1(x)\right]\,,$$

$$\int x^{-(2n+1)/2}\sinh x\cdot K_1(x)\,dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}}\,\left[-P_n^{(s1)}(x)\sinh x\,K_0(x)-Q_n^{(s1)}(x)\cosh x\,K_0(x)+R_n^{(s1)}(x)\sinh x\,K_1(x)+S_n^{(s1)}(x)\cosh x\,K_1(x)\right]\,,$$

$$\int x^{-(2n+1)/2}\cosh x\cdot K_0(x)\,dx =$$

$$= \frac{1}{M_n x^{(2n-1)/2}}\,\left[P_n^{(c0)}(x)\sinh x\,K_0(x)+Q_n^{(c0)}(x)\cosh x\,K_0(x)-R_n^{(c0)}(x)\sinh x\,K_1(x)-S_n^{(c0)}(x)\cosh x\,K_1(x)\right]\,,$$

$$\int x^{-(2n+1)/2}\cosh x\cdot K_1(x)\,dx =$$

$$= \frac{1}{N_n x^{(2n-1)/2}}\,\left[-P_n^{(c1)}(x)\sinh x\,K_0(x)-Q_n^{(c1)}(x)\cosh x\,K_0(x)+R_n^{(c1)}(x)\sinh x\,K_1(x)+S_n^{(c1)}(x)\cosh x\,K_1(x)\right]\,.$$

$M_3 = 225\,,\ P_3^{(s0)}(x) = -192\,x^2-90\,,\ Q_3^{(s0)}(x) = 512\,x^3-60\,x\,,\ R_3^{(s0)}(x) = -512\,x^3-36\,x\,,\ S_3^{(s0)}(x) = -64\,x^2$

$P_3^{(c0)}(x) = 512\,x^3-60\,x\,,\ Q_3^{(c0)}(x) = -192\,x^2-90\,,\ R_3^{(c0)}(x) = -64\,x^2\,,\ S_3^{(c0)}(x) = -512\,x^3-36\,x$

$N_3 = 315\,,\ P_3^{(s1)}(x) = 512\,x^3-60\,x\,,\ Q_3^{(s1)}(x) = -192\,x^2\,,\ R_3^{(s1)}(x) = -64\,x^2-90\,,\ S_3^{(s1)}(x) = -512\,x^3-36\,x$

$$P_3^{(c1)}(x) = -192\,x^2\,,\; Q_3^{(c1)}(x) = 512\,x^3 - 60\,x\,,\; R_3^{(c1)}(x) = -512\,x^3 - 36\,x\,,\; S_3^{(c1)}(x) = -64\,x^2 - 90$$

$$M_4 = 3675\,,\; P_4^{(s0)}(x) = 4096\,x^4 - 480\,x^2 - 1050\,,\; Q_4^{(s0)}(x) = -1536\,x^3 - 420\,x$$
$$R_4^{(s0)}(x) = -512\,x^3 - 300\,x\,,\; S_4^{(s0)}(x) = -4096\,x^4 - 288\,x^2$$
$$P_4^{(c0)}(x) = -1536\,x^3 - 420\,x\,,\; Q_4^{(c0)}(x) = 4096\,x^4 - 480\,x^2 - 1050\,,$$
$$R_4^{(c0)}(x) = -4096\,x^4 - 288\,x^2\,,\; S_4^{(c0)}(x) = -512\,x^3 - 300\,x$$

$$N_4 = 4725\,,\; P_4^{(s1)}(x) = -1536\,x^3 - 420\,x\,,\; Q_4^{(s1)}(x) = 4096\,x^4 - 480\,x^2\,,$$
$$R_4^{(s1)}(x) = -4096\,x^4 - 288\,x^2 - 1050\,,\; S_4^{(s1)}(x) = -512\,x^3 - 300\,x$$
$$P_4^{(c1)}(x) = 4096\,x^4 - 480\,x^2\,,\; Q_4^{(c1)}(x) = -1536\,x^3 - 420\,x\,,$$
$$R_4^{(c1)}(x) = -512\,x^3 - 300\,x\,,\; S_4^{(c1)}(x) = -4096\,x^4 - 288\,x^2 - 1050$$

$$M_5 = 297675\,,\; P_5^{(s0)}(x) = -49152\,x^4 - 13440\,x^2 - 66150\,,\; Q_5^{(s0)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,$$
$$R_5^{(s0)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x\,,\; S_5^{(s0)}(x) = -16384\,x^4 - 9600\,x^2$$
$$P_5^{(c0)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,\; Q_5^{(c0)}(x) = -49152\,x^4 - 13440\,x^2 - 66150\,,$$
$$R_5^{(c0)}(x) = -16384\,x^4 - 9600\,x^2\,,\; S_5^{(c0)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x$$

$$N_5 = 363825\,,\; P_5^{(s1)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,\; Q_5^{(s1)}(x) = -49152\,x^4 - 13440\,x^2\,,$$
$$R_5^{(s1)}(x) = -16384\,x^4 - 9600\,x^2 - 66150\,,\; S_5^{(s1)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x$$
$$P_5^{(c1)}(x) = -49152\,x^4 - 13440\,x^2\,,\; Q_5^{(c1)}(x) = 131072\,x^5 - 15360\,x^3 - 18900\,x\,,$$
$$R_5^{(c1)}(x) = -131072\,x^5 - 9216\,x^3 - 14700\,x\,,\; S_5^{(c1)}(x) = -16384\,x^4 - 9600\,x^2 - 66150$$

$$M_6 = 7203735\,,$$
$$P_6^{(s0)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2 - 1309770\,,\; Q_6^{(s0)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,$$
$$R_6^{(s0)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x\,,\; S_6^{(s0)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2$$
$$P_6^{(c0)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,\; Q_6^{(c0)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2 - 1309770\,,$$
$$R_6^{(c0)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2\,,\; S_6^{(c0)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x$$

$$N_6 = 8513505\,,\; P_6^{(s1)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,\; Q_6^{(s1)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2\,,$$
$$R_6^{(s1)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2 - 1309770\,,\; S_6^{(s1)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x$$
$$P_6^{(c1)}(x) = 1048576\,x^6 - 122880\,x^4 - 151200\,x^2\,,\; Q_6^{(c1)}(x) = -393216\,x^5 - 107520\,x^3 - 291060\,x\,,$$
$$R_6^{(c1)}(x) = -131072\,x^5 - 76800\,x^3 - 238140\,x\,,\; S_6^{(c1)}(x) = -1048576\,x^6 - 73728\,x^4 - 117600\,x^2 - 1309770$$

$$M_7 = 405810405\,,\; P_7^{(s0)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2 - 62432370\,,$$
$$Q_7^{(s0)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$
$$R_7^{(s0)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x\,,$$
$$S_7^{(s0)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2$$

$$P_7^{(c0)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$
$$Q_7^{(c0)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2 - 62432370\,,$$
$$R_7^{(c0)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2\,,$$
$$S_7^{(c0)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x$$

$$N_7 = 468242775\,,\; P_7^{(s1)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$
$$Q_7^{(s1)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2\,,$$
$$R_7^{(s1)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2 - 62432370\,,$$
$$S_7^{(s1)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x$$
$$P_7^{(c1)}(x) = -6291456\,x^6 - 1720320\,x^4 - 4656960\,x^2\,,$$
$$Q_7^{(c1)}(x) = 16777216\,x^7 - 1966080\,x^5 - 2419200\,x^3 - 11351340\,x\,,$$
$$R_7^{(c1)}(x) = -16777216\,x^7 - 1179648\,x^5 - 1881600\,x^3 - 9604980\,x\,,$$
$$S_7^{(c1)}(x) = -2097152\,x^6 - 1228800\,x^4 - 3810240\,x^2 - 62432370$$

$$M_8 = 13043905875\,,\; P_8^{(s0)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2 - 1739187450\,,$$
$$Q_8^{(s0)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$
$$R_8^{(s0)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x\,,$$
$$S_8^{(s0)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2$$
$$P_8^{(c0)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$
$$Q_8^{(c0)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2 - 1739187450\,,$$
$$R_8^{(c0)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2\,,$$
$$S_8^{(c0)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x$$

$$N_8 = 14783093325\,,\; P_8^{(s1)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$
$$Q_8^{(s1)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2\,,$$
$$R_8^{(s1)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2 - 1739187450\,,$$
$$S_8^{(s1)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x$$
$$P_8^{(c1)}(x) = 134217728\,x^8 - 15728640\,x^6 - 19353600\,x^4 - 90810720\,x^2\,,$$
$$Q_8^{(c1)}(x) = -50331648\,x^7 - 13762560\,x^5 - 37255680\,x^3 - 267567300\,x\,,$$
$$R_8^{(c1)}(x) = -16777216\,x^7 - 9830400\,x^5 - 30481920\,x^3 - 231891660\,x\,,$$
$$S_8^{(c1)}(x) = -134217728\,x^8 - 9437184\,x^6 - 15052800\,x^4 - 76839840\,x^2 - 1739187450$$

Recurrence Relations:

$$\int x^{-(n+1/2)} \sinh x \cdot I_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \cosh x \cdot I_0(x)\, dx -$$

$$- \frac{2x^{-n+1/2}}{(2n-1)^2} \left[(2n-1) \sinh x \cdot I_0(x) - 2x \cosh x \cdot I_0(x) + 2x \sinh x \cdot I_1(x) \right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot I_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \sinh x \cdot I_0(x)\, dx -$$

$$- \frac{2x^{-n+1/2}}{(2n-1)^2} \left[(2n-1) \cosh x \cdot I_0(x) + 2x \cosh x \cdot I_1(x) - 2x \sinh x \cdot I_0(x) \right]$$

$$\int x^{-(n+1/2)} \sinh x \cdot I_1(x)\, dx = \frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \sinh x \cdot I_0(x)\, dx +$$

$$+ \frac{2x^{-n+1/2}}{4n^2-1} \left[2x \sinh x \cdot I_0(x) - 2x \cosh x \cdot I_1(x) - (2n-1) \sinh x \cdot I_1(x) \right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot I_1(x)\, dx = \frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \cosh x \cdot I_0(x)\, dx +$$

$$+ \frac{2x^{-n+1/2}}{4n^2-1} \left[2x \cosh x \cdot I_0(x) - 2x \sinh x \cdot I_1(x) - (2n-1) \cosh x \cdot I_1(x) \right]$$

$$\int x^{-(n+1/2)} \sinh x \cdot K_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \cosh x \cdot K_0(x)\, dx +$$

$$+ \frac{2x^{-n+1/2}}{(2n-1)^2} \left[-(2n-1) \sinh x \cdot K_0(x) + 2x \cosh x \cdot K_0(x) + 2x \sinh x \cdot K_1(x) \right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot K_0(x)\, dx = \frac{8(n-1)}{(2n-1)^2} \int x^{-(n-1/2)} \sinh x \cdot K_0(x)\, dx +$$

$$+ \frac{2x^{-n+1/2}}{(2n-1)^2} \left[-(2n-1) \cosh x \cdot K_0(x) + 2x \cosh x \cdot K_1(x) + 2x \sinh x \cdot K_0(x) \right]$$

$$\int x^{-(n+1/2)} \sinh x \cdot K_1(x)\, dx = -\frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \sinh x \cdot K_0(x)\, dx -$$

$$- \frac{2x^{-n+1/2}}{4n^2-1} \left[2x \sinh x \cdot K_0(x) + 2x \cosh x \cdot K_1(x) + (2n-1) \sinh x \cdot K_1(x) \right]$$

$$\int x^{-(n+1/2)} \cosh x \cdot K_1(x)\, dx = -\frac{8(n-1)}{4n^2-1} \int x^{-(n-1/2)} \cosh x \cdot K_0(x)\, dx -$$

$$- \frac{2x^{-n+1/2}}{4n^2-1} \left[2x \cosh x \cdot K_0(x) + 2x \sinh x \cdot K_1(x) + (2n-1) \cosh x \cdot K_1(x) \right]$$

1.2.9. Integrals of the type $\int x^{2n+1}\ \ln x \cdot Z_0(x)\,dx$

$$\int x\ \ln x \cdot J_0(x)\,dx = J_0(x) + x\ln x \cdot J_1(x)$$

$$\int x\ \ln x \cdot I_0(x)\,dx = -I_0(x) + x\ln x \cdot I_1(x)$$

$$\int x\ \ln x \cdot K_0(x)\,dx = -K_0(x) - x\ln x \cdot K_1(x)$$

$$\int x^3\ \ln x \cdot J_0(x)\,dx = \left(x^2 - 4 + 2x^2\ln x\right)\ J_0(x) + \left[-4x + \left(x^3 - 4x\right)\ln x\right]\ J_1(x)$$

$$\int x^3\ \ln x \cdot I_0(x)\,dx = \left(-x^2 - 4 - 2x^2\ln x\right)\ I_0(x) + \left[4x + \left(x^3 + 4x\right)\ln x\right]\ I_1(x)$$

$$\int x^3\ \ln x \cdot K_0(x)\,dx = \left(-x^2 - 4 - 2x^2\ln x\right)\ K_0(x) - \left[4x + \left(x^3 + 4x\right)\ln x\right]\ K_1(x)$$

$$\int x^5\ \ln x \cdot J_0(x)\,dx =$$
$$= \left[x^4 - 32\,x^2 + 64 + \left(4\,x^4 - 32\,x^2\right)\ln x\right]\ J_0(x) + \left[-8\,x^3 + 96\,x + \left(x^5 - 16\,x^3 + 64\,x\right)\ln x\right]\ J_1(x)$$

$$\int x^5\ \ln x \cdot I_0(x)\,dx =$$
$$= \left[-x^4 - 32\,x^2 - 64 + \left(-4\,x^4 - 32\,x^2\right)\ln x\right]\ I_0(x) + \left[8\,x^3 + 96\,x + \left(x^5 + 16\,x^3 + 64\,x\right)\ln x\right]\ I_1(x)$$

$$\int x^5\ \ln x \cdot K_0(x)\,dx =$$
$$= \left[-x^4 - 32\,x^2 - 64 + \left(-4\,x^4 - 32\,x^2\right)\ln x\right]\ K_0(x) - \left[8\,x^3 + 96\,x + \left(x^5 + 16\,x^3 + 64\,x\right)\ln x\right]\ K_1(x)$$

The integrals with $K_0(x)$ can be found by comparison with the integrals for $I_0(x)$.

$$\int x^7\ \ln x \cdot J_0(x)\,dx = \left[x^6 - 84\,x^4 + 1536\,x^2 - 2304 + \left(6\,x^6 - 144\,x^4 + 1152\,x^2\right)\ln x\right]\ J_0(x) +$$
$$+ \left[-12\,x^5 + 480\,x^3 - 4224\,x + \left(x^7 - 36\,x^5 + 576\,x^3 - 2304\,x\right)\ln x\right]\ J_1(x)$$

$$\int x^7\ \ln x \cdot I_0(x)\,dx = \left[-x^6 - 84\,x^4 - 1536\,x^2 - 2304 + \left(-6\,x^6 - 144\,x^4 - 1152\,x^2\right)\ln x\right]\ I_0(x) +$$
$$+ \left[12\,x^5 + 480\,x^3 + 4224\,x + \left(x^7 + 36\,x^5 + 576\,x^3 + 2304\,x\right)\ln x\right]\ I_1(x)$$

$$\int x^9\ \ln x \cdot J_0(x)\,dx =$$
$$= \left[x^8 - 160\,x^6 + 7680\,x^4 - 116736\,x^2 + 147456 + \left(8\,x^8 - 384\,x^6 + 9216\,x^4 - 73728\,x^2\right)\ln x\right]\ J_0(x) +$$
$$+ \left[-16\,x^7 + 1344\,x^5 - 39936\,x^3 + 307200\,x + \left(x^9 - 64\,x^7 + 2304\,x^5 - 36864\,x^3 + 147456\,x\right)\ln x\right]\ J_1(x)$$

$$\int x^9\ \ln x \cdot I_0(x)\,dx =$$
$$= \left[-x^8 - 160\,x^6 - 7680\,x^4 - 116736\,x^2 - 147456 + \left(-8\,x^8 - 384\,x^6 - 9216\,x^4 - 73728\,x^2\right)\ln x\right]\ I_0(x) +$$
$$+ \left[16\,x^7 + 1344\,x^5 + 39936\,x^3 + 307200\,x + \left(x^9 + 64\,x^7 + 2304\,x^5 + 36864\,x^3 + 147456\,x\right)\ln x\right]\ I_1(x)$$

Let

$$\int x^n\ \ln x \cdot J_0(x)\,dx = (P_n(x) + Q_n(x)\ln x)J_0(x) + (R_n(x) + S_n(x)\ln x)J_1(x)\ ,$$

$$\int x^n\ \ln x \cdot I_0(x)\,dx = (P_n^*(x) + Q_n^*(x)\ln x)I_0(x) + (R_n^*(x) + S_n^*(x)\ln x)I_1(x)\ ,$$

$$\int x^n\ \ln x \cdot K_0(x)\,dx = (P_n^*(x) + Q_n^*(x)\ln x)K_0(x) - (R_n^*(x) + S_n^*(x)\ln x)K_1(x)\ ,$$

then holds

$$P_{11} = x^{10} - 260\,x^8 + 23680\,x^6 - 952320\,x^4 + 13148160\,x^2 - 14745600$$
$$Q_{11} = 10\,x^{10} - 800\,x^8 + 38400\,x^6 - 921600\,x^4 + 7372800\,x^2$$
$$R_{11} = -20\,x^9 + 2880\,x^7 - 180480\,x^5 + 4730880\,x^3 - 33669120\,x$$
$$S_{11} = x^{11} - 100\,x^9 + 6400\,x^7 - 230400\,x^5 + 3686400\,x^3 - 14745600\,x$$
$$P^*_{11} = -x^{10} - 260\,x^8 - 23680\,x^6 - 952320\,x^4 - 13148160\,x^2 - 14745600$$
$$Q^*_{11} = -10\,x^{10} - 800\,x^8 - 38400\,x^6 - 921600\,x^4 - 7372800\,x^2$$
$$R^*_{11} = 20\,x^9 + 2880\,x^7 + 180480\,x^5 + 4730880\,x^3 + 33669120\,x$$
$$S^*_{11} = x^{11} + 100\,x^9 + 6400\,x^7 + 230400\,x^5 + 3686400\,x^3 + 14745600\,x$$

$$P_{13} = x^{12} - 384\,x^{10} + 56640\,x^8 - 4331520\,x^6 + 159252480\,x^4 - 2070282240\,x^2 + 2123366400$$
$$Q_{13} = 12\,x^{12} - 1440\,x^{10} + 115200\,x^8 - 5529600\,x^6 + 132710400\,x^4 - 1061683200\,x^2$$
$$R_{13} = -24\,x^{11} + 5280\,x^9 - 568320\,x^7 + 31518720\,x^5 - 769720320\,x^3 + 5202247680\,x$$
$$S_{13} = x^{13} - 144\,x^{11} + 14400\,x^9 - 921600\,x^7 + 33177600\,x^5 - 530841600\,x^3 + 2123366400\,x$$
$$P^*_{13} = -x^{12} - 384\,x^{10} - 56640\,x^8 - 4331520\,x^6 - 159252480\,x^4 - 2070282240\,x^2 - 2123366400$$
$$Q^*_{13} = -12\,x^{12} - 1440\,x^{10} - 115200\,x^8 - 5529600\,x^6 - 132710400\,x^4 - 1061683200\,x^2$$
$$R^*_{13} = 24\,x^{11} + 5280\,x^9 + 568320\,x^7 + 31518720\,x^5 + 769720320\,x^3 + 5202247680\,x$$
$$S^*_{13} = x^{13} + 144\,x^{11} + 14400\,x^9 + 921600\,x^7 + 33177600\,x^5 + 530841600\,x^3 + 2123366400\,x$$

$$P_{15} = x^{14} - 532\,x^{12} + 115584\,x^{10} - 14327040\,x^8 + 1003806720\,x^6 -$$
$$-34929377280\,x^4 + 435502448640\,x^2 - 416179814400$$
$$Q_{15} = 14\,x^{14} - 2352\,x^{12} + 282240\,x^{10} - 22579200\,x^8 +$$
$$+1083801600\,x^6 - 26011238400\,x^4 + 208089907200\,x^2$$
$$R_{15} = -28\,x^{13} + 8736\,x^{11} - 1438080\,x^9 + 137195520\,x^7 -$$
$$-7106641920\,x^5 + 165728747520\,x^3 - 1079094804480\,x$$
$$S_{15} = x^{15} - 196\,x^{13} + 28224\,x^{11} - 2822400\,x^9 + 180633600\,x^7 -$$
$$-6502809600\,x^5 + 104044953600\,x^3 - 416179814400\,x$$
$$P^*_{15} = -x^{14} - 532\,x^{12} - 115584\,x^{10} - 14327040\,x^8 - 1003806720\,x^6 -$$
$$-34929377280\,x^4 - 435502448640\,x^2 - 416179814400$$
$$Q^*_{15} = -14\,x^{14} - 2352\,x^{12} - 282240\,x^{10} - 22579200\,x^8 -$$
$$-1083801600\,x^6 - 26011238400\,x^4 - 208089907200\,x^2$$
$$R^*_{15} = 28\,x^{13} + 8736\,x^{11} + 1438080\,x^9 + 137195520\,x^7 +$$
$$+7106641920\,x^5 + 165728747520\,x^3 + 1079094804480\,x$$
$$S^*_{15} = x^{15} + 196\,x^{13} + 28224\,x^{11} + 2822400\,x^9 + 180633600\,x^7 +$$
$$+6502809600\,x^5 + 104044953600\,x^3 + 416179814400\,x$$

Recurrence formulas:

$$\int x^{2n+1} \cdot \ln x \cdot J_0(x)\, dx =$$

$$= x^{2n} \ln x [2nJ_0(x) + xJ_1(x)] - 2n \int x^{2n-1} J_0(x)\, dx - \int x^{2n} J_1(x)\, dx - 4n^2 \int x^{2n-1} \cdot \ln x \cdot J_0(x)\, dx$$

$$\int x^{2n+1} \cdot \ln x \cdot I_0(x)\, dx =$$

$$= x^{2n} \ln x [xI_1(x) - 2nI_0(x)] + 2n \int x^{2n-1} I_0(x)\, dx - \int x^{2n} I_1(x)\, dx + 4n^2 \int x^{2n-1} \cdot \ln x \cdot I_0(x)\, dx$$

$$\int x^{2n+1} \cdot \ln x \cdot K_0(x)\, dx =$$

$$= -\, x^{2n} \ln x [2nJ_0(x) + xJ_1(x)] + 2n \int x^{2n-1} J_0(x)\, dx + \int x^{2n} J_1(x)\, dx + 4n^2 \int x^{2n-1} \cdot \ln x \cdot J_0(x)\, dx$$

The integrals of the type $\int x^m Z_\nu(x)\, dx$ are described before.

1.2.10. Integrals of the type $\int x^{2n}\,\ln x\cdot Z_1(x)\,dx$

$$\int \ln x\cdot J_1(x)\,dx = -\ln x\cdot J_0(x) + \int \frac{J_0(x)}{x}\,dx$$

$$\int \ln x\cdot I_1(x)\,dx = \ln x\cdot I_0(x) - \int \frac{I_0(x)}{x}\,dx$$

Concerning the integrals on the right hand side see 1.1.3, page 15.

$$\int x^2\,\ln x\cdot J_1(x)\,dx = \left(2 - x^2\ln x\right) J_0(x) + x\,(1+2\ln x)\,J_1(x)$$

$$\int x^2\,\ln x\cdot I_1(x)\,dx = \left(2 + x^2\ln x\right) I_0(x) - x\,(1+2\ln x)\,I_1(x)$$

$$\int x^4\,\ln x\cdot J_1(x)\,dx = \left[6\,x^2 - 16 + \left(-x^4 + 8\,x^2\right)\ln x\right]\,J_0(x) + \left[x^3 - 20\,x + \left(4\,x^3 - 16\,x\right)\ln x\right]\,J_1(x)$$

$$\int x^4\,\ln x\cdot I_1(x)\,dx = \left[6\,x^2 + 16 + \left(x^4 + 8\,x^2\right)\ln x\right]\,I_0(x) + \left[-x^3 - 20\,x + \left(-4\,x^3 - 16\,x\right)\ln x\right]\,I_1(x)$$

$$\int x^6\,\ln x\cdot J_1(x)\,dx = \left[10\,x^4 - 224\,x^2 + 384 + \left(-x^6 + 24\,x^4 - 192\,x^2\right)\ln x\right]\,J_0(x) +$$
$$+\left[x^5 - 64\,x^3 + 640\,x + \left(6\,x^5 - 96\,x^3 + 384\,x\right)\ln x\right]\,J_1(x)$$

$$\int x^6\,\ln x\cdot I_1(x)\,dx = \left[10\,x^4 + 224\,x^2 + 384 + \left(x^6 + 24\,x^4 + 192\,x^2\right)\ln x\right]\,I_0(x) +$$
$$+\left[-x^5 - 64\,x^3 - 640\,x + \left(-6\,x^5 - 96\,x^3 - 384\,x\right)\ln x\right]\,I_1(x)$$

$$\int x^8\,\ln x\cdot J_1(x)\,dx = \left[14\,x^6 - 816\,x^4 + 13440\,x^2 - 18432 + \left(-x^8 + 48\,x^6 - 1152\,x^4 + 9216\,x^2\right)\ln x\right]\,J_0(x) +$$
$$+\left[x^7 - 132\,x^5 + 4416\,x^3 - 36096\,x + \left(8\,x^7 - 288\,x^5 + 4608\,x^3 - 18432\,x\right)\ln x\right]\,J_1(x)$$

$$\int x^8\,\ln x\cdot I_1(x)\,dx = \left[14\,x^6 + 816\,x^4 + 13440\,x^2 + 18432 + \left(x^8 + 48\,x^6 + 1152\,x^4 + 9216\,x^2\right)\ln x\right]\,I_0(x) +$$
$$+\left[-x^7 - 132\,x^5 - 4416\,x^3 - 36096\,x + \left(-8\,x^7 - 288\,x^5 - 4608\,x^3 - 18432\,x\right)\ln x\right]\,I_1(x)$$

Let

$$\int x^n\,\ln x\cdot J_1(x)\,dx = [P_n(x) + Q_n(x)\ln x]J_0(x) + [R_n(x) + S_n(x)\ln x]J_1(x)\ ,$$

$$\int x^n\,\ln x\cdot I_1(x)\,dx = [P_n^*(x) + Q_n^*(x)\ln x]I_0(x) + [R_n^*(x) + S_n^*(x)\ln x]I_1(x)\ ,$$

then holds:

$$P_{10}(x) = 18\,x^8 - 1984\,x^6 + 86016\,x^4 - 1241088\,x^2 + 1474560$$
$$Q_{10}(x) = -x^{10} + 80\,x^8 - 3840\,x^6 + 92160\,x^4 - 737280\,x^2$$
$$R_{10}(x) = x^9 - 224\,x^7 + 15744\,x^5 - 436224\,x^3 + 3219456\,x$$
$$S_{10}(x) = 10\,x^9 - 640\,x^7 + 23040\,x^5 - 368640\,x^3 + 1474560\,x$$
$$P_{10}^*(x) = 18\,x^8 + 1984\,x^6 + 86016\,x^4 + 1241088\,x^2 + 1474560$$
$$Q_{10}^*(x) = x^{10} + 80\,x^8 + 3840\,x^6 + 92160\,x^4 + 737280\,x^2$$
$$R_{10}^*(x) = -x^9 - 224\,x^7 - 15744\,x^5 - 436224\,x^3 - 3219456\,x$$
$$S_{10}^*(x) = -10\,x^9 - 640\,x^7 - 23040\,x^5 - 368640\,x^3 - 1474560\,x$$

$$P_{12}(x) = 22\,x^{10} - 3920\,x^8 + 322560\,x^6 - 12349440\,x^4 + 165150720\,x^2 - 176947200$$
$$Q_{12}(x) = -x^{12} + 120\,x^{10} - 9600\,x^8 + 460800\,x^6 - 11059200\,x^4 + 88473600\,x^2$$

$$R_{12}(x) = x^{11} - 340\,x^9 + 40960\,x^7 - 2396160\,x^5 + 60456960\,x^3 - 418775040\,x$$
$$S_{12}(x) = 12\,x^{11} - 1200\,x^9 + 76800\,x^7 - 2764800\,x^5 + 44236800\,x^3 - 176947200\,x$$
$$P_{12}^*(x) = 22\,x^{10} + 3920\,x^8 + 322560\,x^6 + 12349440\,x^4 + 165150720\,x^2 + 176947200$$
$$Q_{12}^*(x) = x^{12} + 120\,x^{10} + 9600\,x^8 + 460800\,x^6 + 11059200\,x^4 + 88473600\,x^2$$
$$R_{12}^*(x) = -x^{11} - 340\,x^9 - 40960\,x^7 - 2396160\,x^5 - 60456960\,x^3 - 418775040\,x$$
$$S_{12}^*(x) = -12\,x^{11} - 1200\,x^9 - 76800\,x^7 - 2764800\,x^5 - 44236800\,x^3 - 176947200\,x$$

$$P_{14}(x) = 26\,x^{12} - 6816\,x^{10} + 908160\,x^8 - 66170880\,x^6 +$$
$$+2362245120\,x^4 - 30045634560\,x^2 + 29727129600$$
$$Q_{14}(x) = -x^{14} + 168\,x^{12} - 20160\,x^{10} + 1612800\,x^8 -$$
$$-77414400\,x^6 + 1857945600\,x^4 - 14863564800\,x^2$$
$$R_{14}(x) = x^{13} - 480\,x^{11} + 88320\,x^9 - 8878080\,x^7 +$$
$$+474439680\,x^5 - 11306926080\,x^3 + 74954833920\,x$$
$$S_{14}(x) = 14\,x^{13} - 2016\,x^{11} + 201600\,x^9 - 12902400\,x^7 +$$
$$+464486400\,x^5 - 7431782400\,x^3 + 29727129600\,x$$
$$P_{14}^*(x) = 26\,x^{12} + 6816\,x^{10} + 908160\,x^8 + 66170880\,x^6 +$$
$$+2362245120\,x^4 + 30045634560\,x^2 + 29727129600$$
$$Q_{14}^*(x) = x^{14} + 168\,x^{12} + 20160\,x^{10} + 1612800\,x^8 +$$
$$+77414400\,x^6 + 1857945600\,x^4 + 14863564800\,x^2$$
$$R_{14}^*(x) = -x^{13} - 480\,x^{11} - 88320\,x^9 - 8878080\,x^7 -$$
$$-474439680\,x^5 - 11306926080\,x^3 - 74954833920\,x$$
$$S_{14}^*(x) = -14\,x^{13} - 2016\,x^{11} - 201600\,x^9 - 12902400\,x^7 -$$
$$-464486400\,x^5 - 7431782400\,x^3 - 29727129600\,x$$

Recurrence formulas:

$$\int x^{2n+2} \cdot \ln x \cdot J_1(x)\,dx = x^{2n+1} \ln x[(2n+2)J_1(x) - xJ_0(x)] +$$
$$+ \int x^{2n+1} J_0(x)\,dx - (2n+2) \int x^{2n} J_1(x)\,dx - 4n(n+1) \int x^{2n} \cdot \ln x \cdot J_1(x)\,dx$$
$$\int x^{2n+2} \cdot \ln x \cdot I_1(x)\,dx = x^{2n+1} \ln x[xI_0(x) - (2n+2)I_1(x)] -$$
$$- \int x^{2n+1} I_0(x)\,dx + (2n+2) \int x^{2n} I_1(x)\,dx + 4n(n+1) \int x^{2n} \cdot \ln x \cdot I_1(x)\,dx$$

The integrals of the type $\int x^m Z_\nu(x)\,dx$ are described before.

1.2.11. Integrals of the type $\int x^{2n+\nu} \ln x \cdot Z_\nu(x)\, dx$

a) The Functions Λ_k and Λ_k^*, $k = 0, 1$:

Let

$$\Lambda_0(x) \; = \; \sum_{k=0}^{\infty} \alpha_k \, x^{2k+1} \; = \; \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} \cdot (k!)^2 \cdot (2k+1)} \; x^{2k+1} = \int_0^x J_0(t)\, dt = x\, J_0(x) + \Phi(x) \; .$$

($\Phi(x)$ and further on $\Psi(x)$ defined as on page 9)

$$\Lambda_0(x) = x - \frac{x^3}{12} + \frac{x^5}{320} - \frac{x^7}{16\,128} + \frac{x^9}{1\,327\,104} - \frac{x^{11}}{162\,201\,600} + \frac{x^{13}}{27\,603\,763\,200} - \frac{x^{15}}{6\,242\,697\,216\,000} + \dots$$

k	α_k	$1/\alpha_k$
0	1.0000000000000000E+00	1
1	-8.3333333333333329E-02	-12
2	3.1250000000000002E-03	320
3	-6.2003968253968251E-05	-16128
4	7.5352044753086416E-07	1327104
5	-6.1651672979797980E-09	-162201600
6	3.6226944592830012E-11	27603763200
7	-1.6018716996829596E-13	-6242697216000
8	5.5211570531352012E-16	1811214552268800
9	-1.5246859958300586E-18	-655872751986278400
10	3.4486945143775135E-21	289964795614986240000
11	-6.5058017249306315E-24	-153708957370763182080000
12	1.0391211088430869E-26	96235173310390861824000000
13	-1.4232975959389203E-29	-70259375330450160400465920000
14	1.6902284962328838E-32	59163598426411660994999746560000
15	-1.7568683294176930E-35	-56919461934375356612430790656000000
16	1.6117104111016952E-38	-56919461934375356612430790656000000
17	-1.3145438350557573E-41	-76072016263922024994318857577431040000000
18	9.5948102742224525E-45	104223009253931112643645081672942092288000000
19	-6.3038564554696844E-48	-158633053760659041611878822148470455926784000000
20	3.7477195390749646E-51	266828931453826490506134634177940048943513600000000

Values of this function may be found in [1], Table 11.1 .

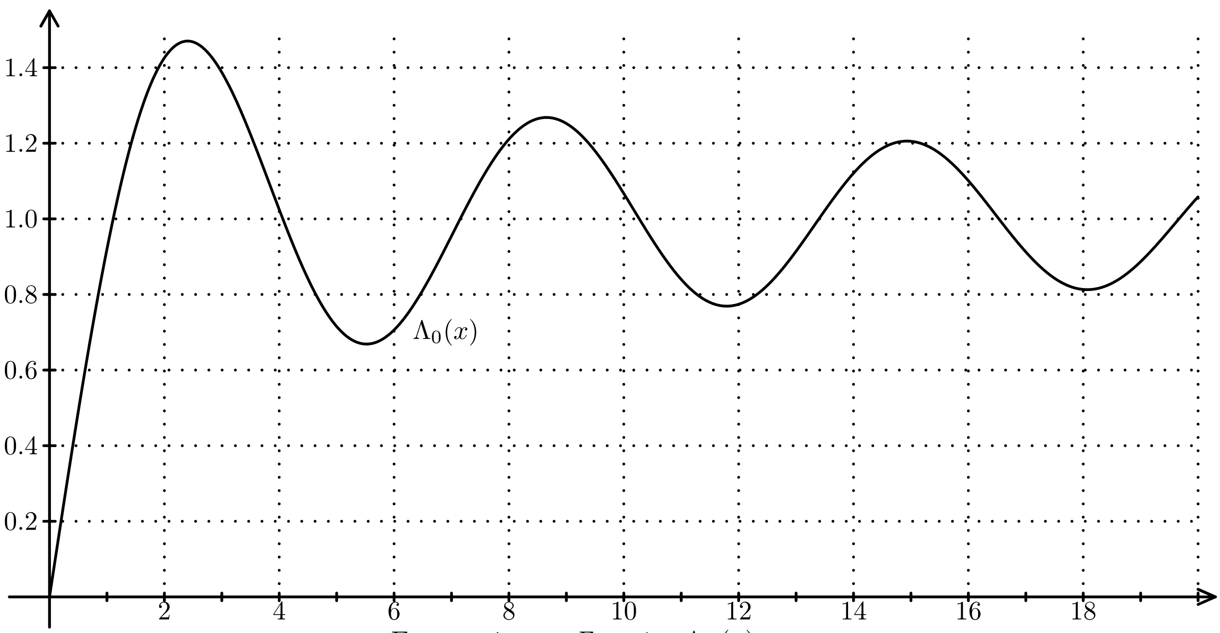


FIGURE 1 : *Function* $\Lambda_0\,(x)$

Approximations with Chebyshev polynomials are given in [1], table 9.3 .

The maxima and minima of $\Lambda_0(x)$ are situated in the zeros of $J_0(x)$:

k	1	2	3	4	5	6	7	8
max	1.470300	1.268168	1.205654	1.172888	1.151982	1.137178	1.125991	1.117157
min	0.668846	0.769119	0.812831	0.838567	0.855986	0.868771	0.878666	0.886617

Asymptotic expansion:

$$\Lambda_0(x) \sim 1 + \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{\lambda_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right)$$

Recurrence relation:

$$\lambda_{k+1} = -\frac{2k+1}{16(k+1)} \left[(12k+10)\lambda_k + (4k^2-1)\lambda_{k-1}\right]$$

If $k > 1$, then up to $k \approx 30$ holds

$$\lambda_k \approx (-1)^k \, \Gamma(s_k) \quad \text{with} \quad s_k = k + \frac{1}{2} - \frac{1}{3\sqrt{k}} \, .$$

Coefficients of the asymptotic formula:

k	λ_k	λ_k	$q_k = \lvert\lambda_k/\lambda_{k-1}\rvert$	$\lvert\lambda_k\rvert/\Gamma(s_k)$
0	1	1	-	-
1	$-\frac{5}{8}$	-0.625	0.625	-
2	$\frac{129}{128}$	1.007812500	1.612500000	0.882203509
3	$-\frac{2655}{1024}$	-2.592773438	2.572674419	0.958684418
4	$\frac{301035}{32768}$	9.186859131	3.543255650	0.992044824
5	$-\frac{10896795}{262144}$	-41.56797409	4.524720963	1.007474317
6	$\frac{961319205}{4194304}$	229.1963589	5.513772656	1.014554883
7	$-\frac{50046571575}{33554432}$	-1491.504060	6.507538198	1.017456390
8	$\frac{24035398261875}{2147483648}$	11192.35450	7.504072427	1.018151550
9	$-\frac{1634825936118375}{17179869184}$	-95159.39374	8.502178320	1.017633944
10	$\frac{248523783571238175}{274877906944}$	904124.2577	9.501156135	1.016430975
11	$-\frac{20877210220441199625}{2199023255552}$	-9493856.042	10.50060980	1.014835786
12	$\frac{7683027147736313147775}{70368744177664}$	109182382.6	11.50032001	1.014835786

Roughly spoken, the item

$$\frac{\lambda_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right)$$

in the asymptotic series should not be used if $|x| < q_k$.

Let

$$d_n(x) = 1 + \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{n} \frac{\lambda_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right) - \Lambda_0(x) \, .$$

The following table gives some consecutive maxima and minima of interest of this functions:

$n=0$		$n=1$		$n=2$		$n=3$	
x	$d_n(x)$	x	$d_n(x)$	x	$d_n(x)$	x	$d_n(x)$
3.953	-5.390E-2	5.510	-9.219-3	3.936	1.020E-2	5.501	2.128E-3
7.084	2.479E-2	8.647	3.315E-3	7.074	-1.751-3	8.642	-3.501E-4
10.221	-1.475E-2	11.787	-1.592E-3	10.214	5.360E-4	11.783	9.561E-5
13.360	1.000E-2	14.927	9.002E-4	13.355	-2.197E-4	14.924	-3.469E-5
16.500	-7.337E-3	18.068	-5.647E-4	16.496	1.076E-4	18.065	1.511E-5

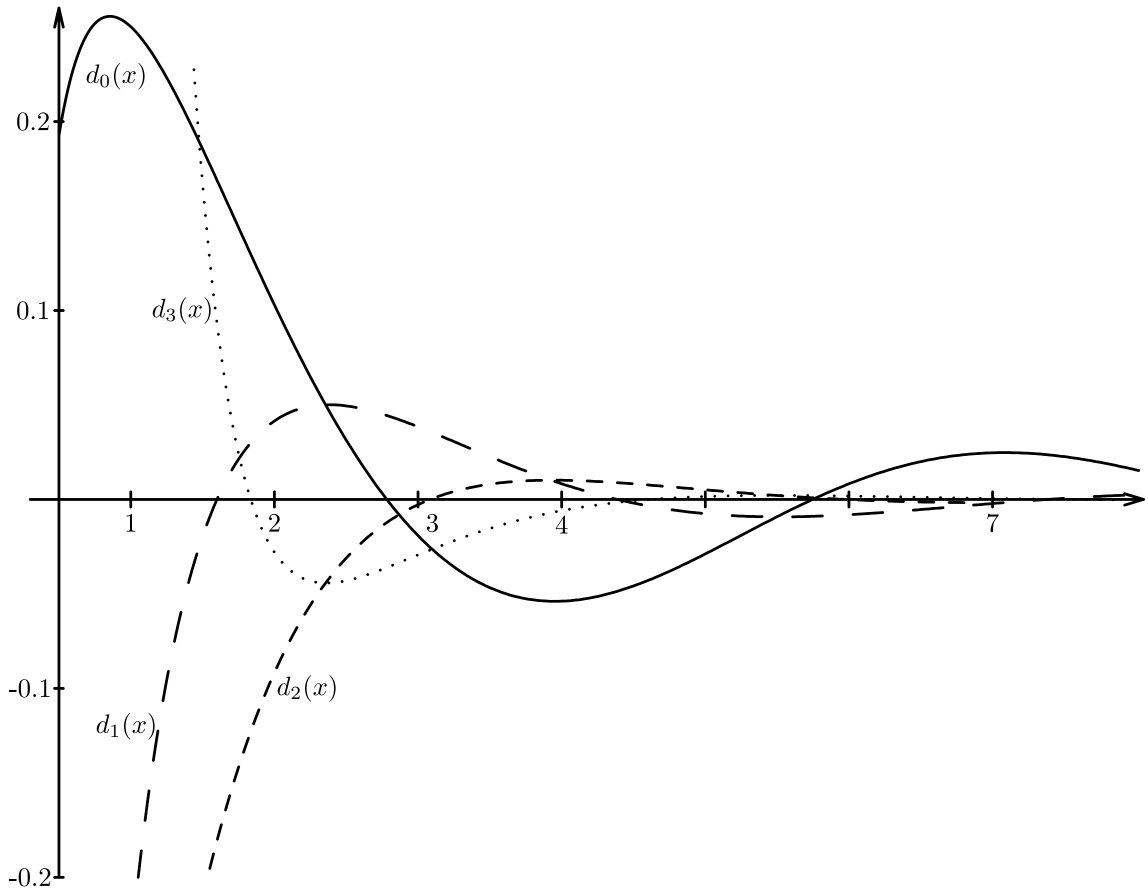


FIGURE 1 : *Differences* $d_0(x)$ ... $d_3(x)$

Let

$$\Lambda_0^*(x) = \sum_{k=0}^{\infty} |\alpha_k|\, x^{2k+1} = \sum_{k=0}^{\infty} \frac{1}{2^{2k}\cdot(k!)^2\cdot(2k+1)}\, x^{2k+1} = \int_0^x I_0(t)\,dt = x\, I_0(x) + \Psi(x)\ .$$

Asymptotic expansion (see $\Lambda_0(x)$):

$$\Lambda_0^*(x) = \frac{e^x}{\sqrt{2\pi x}}\left[1 + \frac{5}{8x} + \frac{129}{128x^2} + \frac{2655}{1024x^3} + \ldots\right]$$

Furthermore, let

$$\Lambda_1(x) = \sum_{k=0}^{\infty} \beta_k\, x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\cdot(k!)^2\cdot(2k+1)^2}\, x^{2k+1} = \sum_{k=0}^{\infty} \frac{\alpha_k}{2k+1}\, x^{2k+1}\ .$$

$\Lambda_1(x)$ can be written as a hypergeometric function.
One has

$$\Lambda_0(x) = x\,\Lambda_1'(x)\,,\ \Lambda_1(0) = 0 \quad\Longleftrightarrow\quad \Lambda_1(x) = \int_0^x \frac{\Lambda_0(t)\,dt}{t}\ .$$

$$\Lambda_1(x) = x - \frac{x^3}{36} + \frac{x^5}{1\,600} - \frac{x^7}{112\,896} + \frac{x^9}{11\,943\,936} - \frac{x^{11}}{1\,784\,217\,600} + \frac{x^{13}}{358\,848\,921\,600} - \frac{x^{15}}{93\,640\,458\,240\,000} + \ldots$$

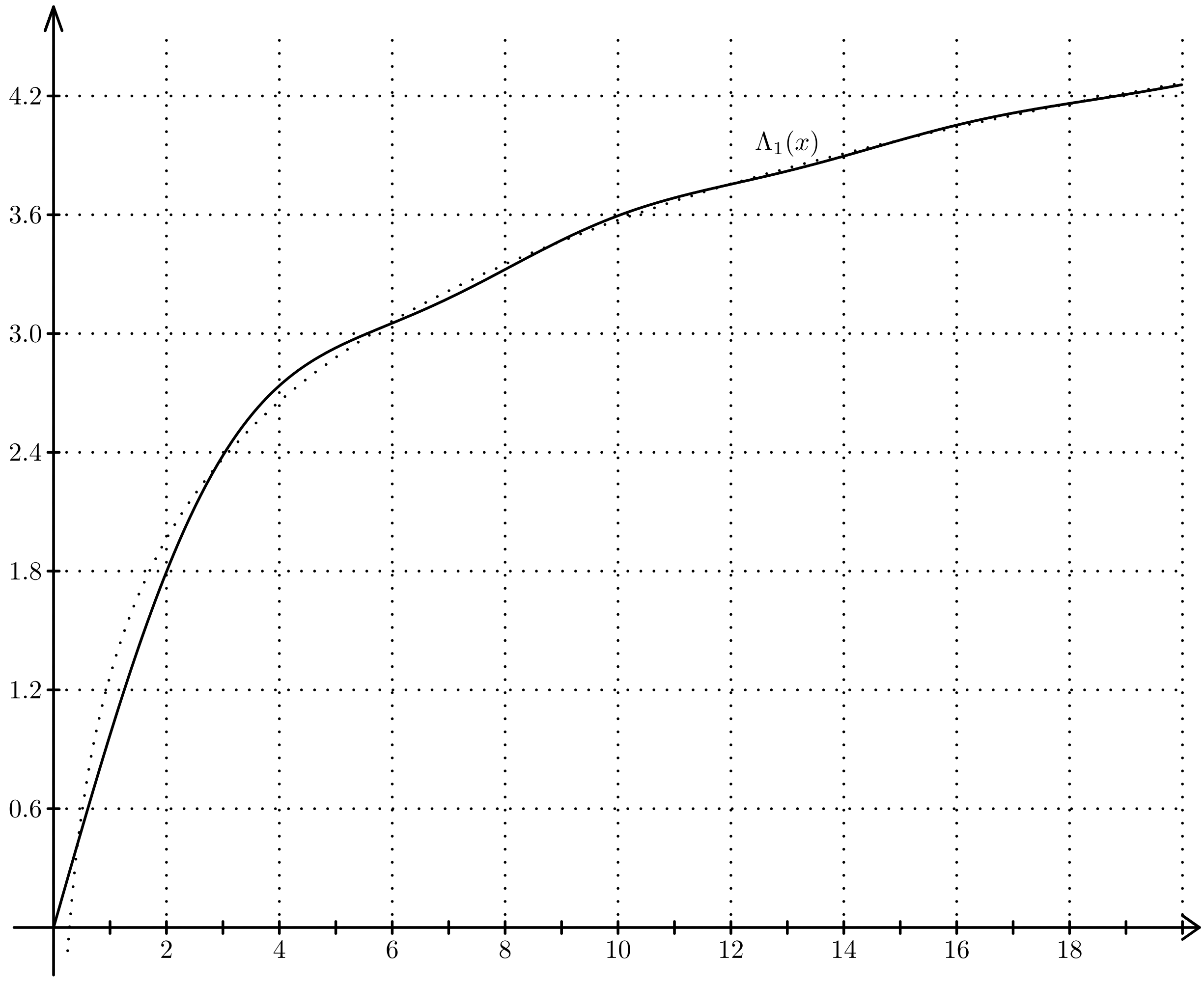


FIGURE 2 : *Function* $\Lambda_1(x)$, *dots:* $C + \ln 2x$

Asymptotic series with Euler's constant $\mathbf{C} = 0.577\ 215\ 664\ 901\ 533$:

$$\Lambda_1(x) \sim C + \ln 2x - \sqrt{\frac{2}{\pi x}} \sum_{k=1}^{\infty} \frac{\mu_k}{x^k} \sin\left(x + \frac{2k-1}{4}\pi\right)$$

k	μ_k	μ_k	$\lvert\mu_k/\mu_{k-1}\rvert$
1	1	1	1
2	$-\frac{17}{8}$	-2.125	2.125
3	$\frac{809}{128}$	6.320 312 500	2.974
4	$-\frac{25307}{1024}$	-24.713 867 187 500	3.910
5	$\frac{3945243}{32768}$	120.399 261 474 609	4.871
6	$-\frac{184487487}{262144}$	-703.763 912 200 928	5.845
7	$\frac{20148017853}{4194304}$	4 803.661 788 225 174	6.826
8	$-\frac{1258927642755}{33554432}$	-37 518.967 472 166	7.810
9	$\frac{708892035920595}{2147483648}$	330 103.578 008 988	8.798
10	$-\frac{55510620666083595}{17179869184}$	-3 231 143.384 827 510	9.788
11	$\frac{9574308055473282135}{274877906944}$	34 831 129.798 379	10.780
12	$-\frac{901713551323983156045}{2199023255552}$	-410 051 848.723 007	11.773

Some consecutive maxima and minima of the differences

$$\delta_n(x) \;=\; C + \ln 2x - \sqrt{\frac{2}{\pi x}}\,\sum_{k=1}^{n} \frac{\mu_k}{x^k}\,\sin\left(x + \frac{3-2k}{4}\,\pi\right) - \Lambda_1\,(x)$$

$n=1$	x	2.4704	5.569	8.689	11.819	14.953	18.090
	$\delta_n(x)$	9.374E-2	-1.855E-2	6.824E-3	-3.309E-3	1.880E-3	-1.182E-3
$n=2$	x	3.991	7.115	10.246	13.380	16.517	19.655
	$\delta_n(x)$	2.312E-2	-4.117E-3	1.279E-3	-5.284E-4	2.598E-4	-1.436E-4
$n=3$	x	5.541	8.673	11.808	14.945	18.083	21.222
	$\delta_n(x)$	5.439E-3	-9.148E-4	2.525E-4	-9.214E-5	4.028E-5	-1.997E-5
$n=4$	x	10.237	13.374	16.512	19.651	22.791	25.931
	$\delta_n(x)$	-2.043E-4	5.163E-5	-1.707E-5	6.771E-6	-3.061E-6	1.527E-6

Let

$$\Lambda_1^*(x) \;=\; \sum_{k=0}^{\infty} \lvert\beta_k\rvert\, x^{2k+1} \;=\; \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^{2k}\cdot(k!)^2\cdot(2k+1)^2}\;.$$

b) Basic Integrals:

$$\int \ln x\cdot J_0(x)\,dx \;=\; \Lambda_0\,(x)\cdot\ln x - \Lambda_1\,(x) + c$$

$$\int \ln x\cdot I_0(x)\,dx \;=\; \Lambda_0^*\,(x)\cdot\ln x - \Lambda_1^*\,(x) + c$$

In particular, let

$$\int_0^x \ln t\cdot J_0(t)\,dt \;=\; F\,(x) \quad\text{and}\quad \int_0^x \ln t\cdot I_0(t)\,dt \;=\; F^*\,(x)\;.$$

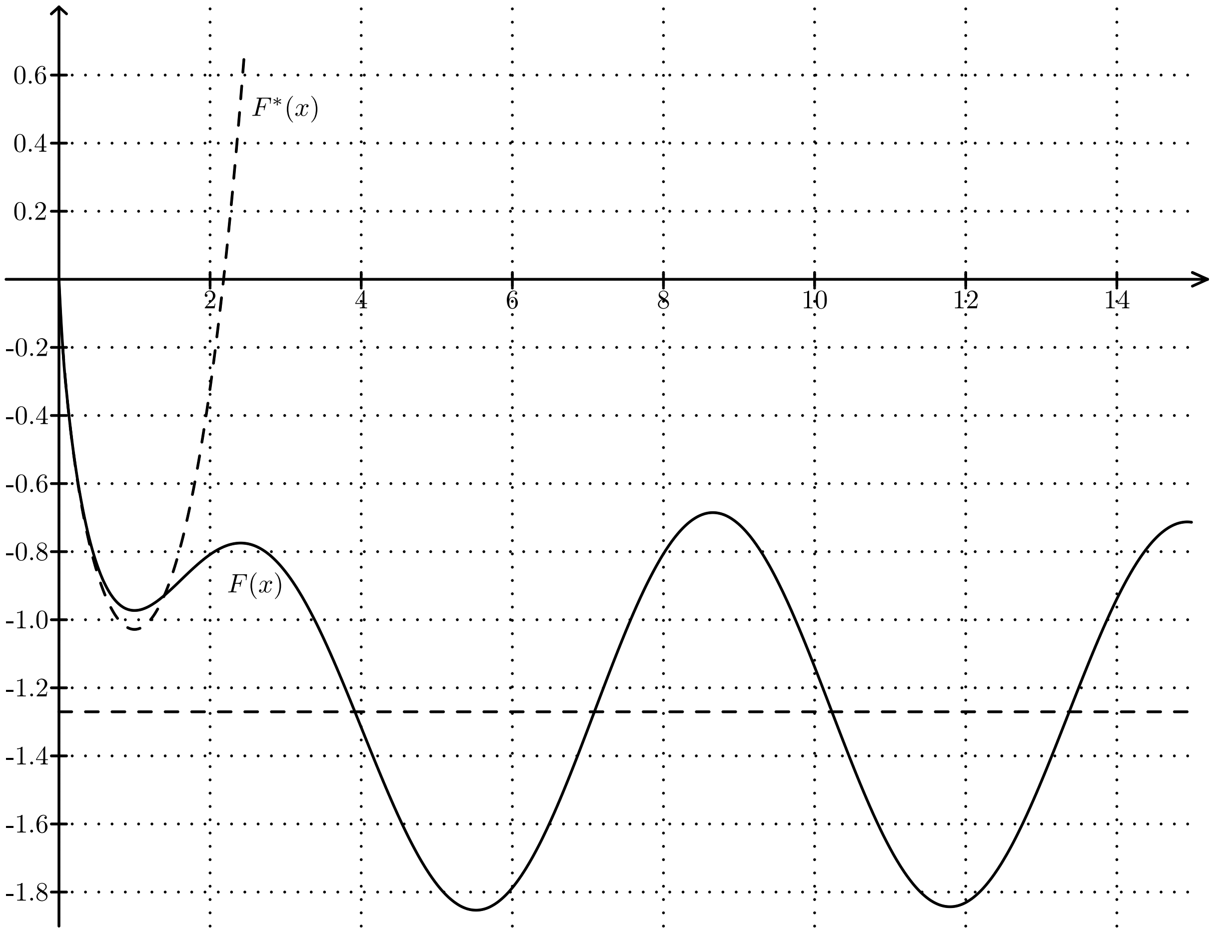


FIGURE 3 : *Functions* $F(x)$ and $F^*(x)$

Holds ([7], 6.772) with Euler's constant $\mathbf{C} = 0.577...$

$$\lim_{x\to\infty} F(x) \;=\; -\ln 2 - \mathbf{C} \;=\; -1.270\,362\,845\,461\,478\,170\;.$$

Asymptotic expansion:

$$F(x) \;\sim\; -\ln 2 - \mathbf{C} + \sqrt{\frac{2}{\pi x}}\left[\ln x \cdot \sin\left(x - \frac{\pi}{4}\right) + \sum_{k=1}^{\infty} \frac{\lambda_k \ln x + \mu_k}{x^k} \cdot \sin\left(x + \frac{(2k-1)\pi}{4}\right)\right] =$$

$$= -\ln 2 - \mathbf{C} + \sqrt{\frac{2}{\pi x}}\left[\ln x \cdot \sin\left(x - \frac{\pi}{4}\right) - \frac{5\ln x + 8}{8x}\,\sin\left(x + \frac{\pi}{4}\right) + \frac{129\ln x + 272}{128x^2}\,\sin\left(x + \frac{3\pi}{4}\right) - \right.$$

$$\left. - \frac{2\,655\ln x + 6\,472}{1\,024x^3}\,\sin\left(x + \frac{5\pi}{4}\right) + \frac{301\,035\ln x + 809\,824}{32\,768x^4}\,\sin\left(x + \frac{7\pi}{4}\right) + \,\ldots\right]$$

Let

$$\Delta_n(x) = -\ln 2 - \mathbf{C} + \sqrt{\frac{2}{\pi x}}\left[\ln x \cdot \sin\left(x - \frac{\pi}{4}\right) + \sum_{k=1}^{n} \frac{\lambda_k \ln x + \mu_k}{x^k} \cdot \sin\left(x + \frac{(2k-1)\pi}{4}\right)\right] - F(x)$$

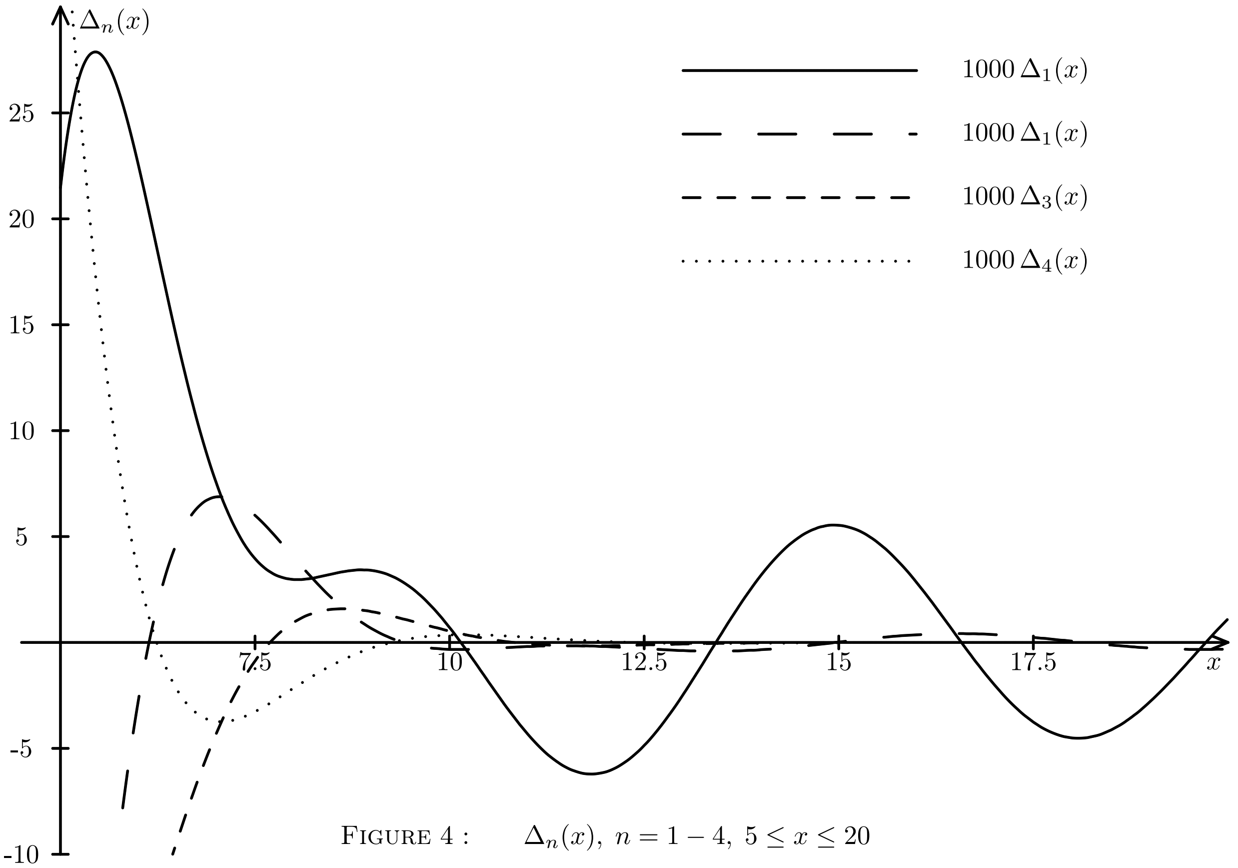


Figure 4 : $\Delta_n(x),\ n = 1 - 4,\ 5 \le x \le 20$

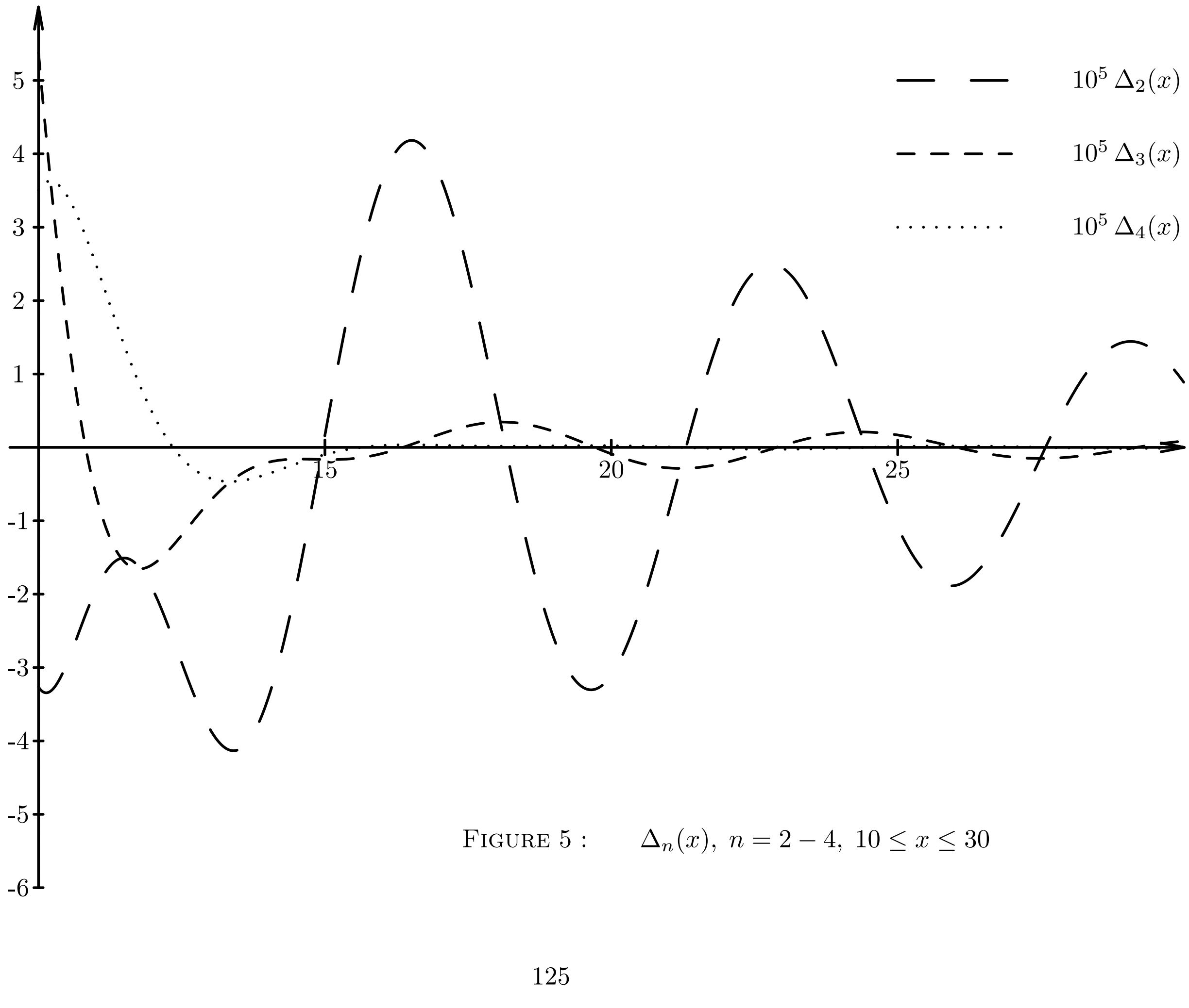


Figure 5 : $\Delta_n(x),\ n = 2 - 4,\ 10 \le x \le 30$

Some consecutive maxima and minima of the differences $\Delta_n(x)$:

$n=1$	x	11.822	14.944	18.079	21.217	24.356	27.496
	$\Delta_n(x)$	-6.213E-4	5.541E-4	-4.520E-4	3.644E-4	-2.959E-4	2.431E-4
$n=2$	x	7.046	10.134	13.404	16.513	19.647	22.785
	$\Delta_n(x)$	6.878E-4	-3.344E-5	-4.133E-5	4.183E-5	-3.303E-5	2.498E-5
$n=3$	x	18.083	21.215	24.354	27.494	30.634	33.775
	$\Delta_n(x)$	3.458-6	-2.863E-6	2.100E-6	-1.507E-6	1.087E-6	-7.949E-7
$n=4$	x	22.785	25.923	29.063	32.204	35.345	38.486
	$\Delta_n(x)$	-2.648E-7	1.982E-7	-1.385E-7	9.570E-8	-6.666E-8	4.709E-8

c) Integrals of $x^{2n} \ln x \cdot Z_0(x)$:

$$\int \ln x \cdot J_0(x)\,dx = [x\,J_0(x) + \Phi(x)] \cdot \ln x - \Lambda_1(x)$$

$$\int \ln x \cdot I_0(x)\,dx = [x\,I_0(x) + \Psi(x)] \cdot \ln x - \Lambda_1^*(x)$$

$$\int x^2 \ln x \cdot J_0(x)\,dx = [x^2\,J_1(x) - \Phi(x)] \cdot \ln x - xJ_0(x) + \Lambda_1(x) - 2\Phi(x)$$

$$\int x^2 \ln x \cdot I_0(x)\,dx = [x^2\,I_1(x) + \Psi(x)] \cdot \ln x + xI_0(x) - \Lambda_1^*(x) + 2\Psi(x)$$

$$\int x^4 \ln x \cdot J_0(x)\,dx =$$

$$= [3\,x^3\,J_0(x) + (x^4 - 9\,x^2)\,J_1(x) + 9\Phi(x)]\;\cdot \ln x + (x^3 + 9\,x)\,J_0(x) - 6\,x^2\,J_1(x) - 9\Lambda_1(x) + 24\Phi(x)$$

$$\int x^4 \ln x \cdot I_0(x)\,dx =$$

$$= [-3\,x^3\,I_0(x) + (x^4 + 9\,x^2)\,I_1(x) + 9\Psi(x)]\;\cdot \ln x + (-x^3 + 9\,x)\,I_0(x) + 6\,x^2\,I_1(x) - 9\Lambda_1^*(x) + 24\Psi(x)$$

Let

$$\int x^n \ln x \cdot J_0(x)\,dx =$$

$$= [P_n(x)\,J_0(x) + Q_n(x)\,J_1(x) + p_n\,\Phi(x)] \cdot \ln x + R_n(x)\,J_0(x) + S_n(x)\,J_1(x) - p_n\,\Lambda_1(x) + q_n\,\Phi(x)$$

and

$$\int x^n \ln x \cdot I_0(x)\,dx =$$

$$= [P_n^*(x)\,I_0(x) + Q_n^*(x)\,I_1(x) + p_n^*\,\Psi(x)] \cdot \ln x + R_n^*(x)\,I_0(x) + S_n^*(x)\,I_1(x) - p_n^*\,\Lambda_1^*(x) + q_n^*\,\Psi(x)\;,$$

then holds

$$P_6(x) = 5\,x^5 - 75\,x^3\;,\quad Q_6(x) = x^6 - 25\,x^4 + 225\,x^2\;,\quad R_6(x) = x^5 - 55\,x^3 - 225\,x$$

$$S_6(x) = -10\,x^4 + 240\,x^2\;,\quad p_6 = -225\;,\quad q_6 = -690$$

$$P_6^*(x) = -5\,x^5 - 75\,x^3\;,\quad Q_6^*(x) = x^6 + 25\,x^4 + 225\,x^2\;,\quad R_6^*(x) = -x^5 - 55\,x^3 + 225\,x$$

$$S_6^*(x) = 10\,x^4 + 240\,x^2\;,\quad p_6^* = 225\;,\quad q_6^* = 690$$

$$P_8(x) = 7\,x^7 - 245\,x^5 + 3675\,x^3\;,\quad Q_8(x) = x^8 - 49\,x^6 + 1225\,x^4 - 11025\,x^2$$

$$R_8(x) = x^7 - 119\,x^5 + 3745\,x^3 + 11025\,x\;,\quad S_8(x) = -14\,x^6 + 840\,x^4 - 14910\,x^2\;,\quad p_8 = 11025\;,\quad q_8 = 36960$$

$$P_8^*(x) = -7\,x^7 - 245\,x^5 - 3675\,x^3\;,\quad Q_8^*(x) = x^8 + 49\,x^6 + 1225\,x^4 + 11025\,x^2$$

$$R_8^*(x) = -x^7 - 119\,x^5 - 3745\,x^3 + 11025\,x\;,\quad S_8^*(x) = 14\,x^6 + 840\,x^4 + 14910\,x^2\;,\quad p_8^* = 11025\;,\quad q_8^* = 36960$$

$$P_{10}(x) = 9\,x^9 - 567\,x^7 + 19845\,x^5 - 297675\,x^3\ , \quad Q_{10}(x) = x^{10} - 81\,x^8 + 3969\,x^6 - 99225\,x^4 + 893025\,x^2$$
$$R_{10}(x) = x^9 - 207\,x^7 + 14049\,x^5 - 369495\,x^3 - 893025\,x\ , \quad S_{10}(x) = -18\,x^8 + 2016\,x^6 - 90090\,x^4 + 1406160\,x^2$$
$$p_{10} = -893025\ , \quad q_{10} = -3192210$$
$$P^*_{10}(x) = -9\,x^9 - 567\,x^7 - 19845\,x^5 - 297675\,x^3\ , \quad Q^*_{10}(x) = x^{10} + 81\,x^8 + 3969\,x^6 + 99225\,x^4 + 893025\,x^2$$
$$R^*_{10}(x) = -x^9 - 207\,x^7 - 14049\,x^5 - 369495\,x^3 + 893025\,x\ , \quad S^*_{10}(x) = 18\,x^8 + 2016\,x^6 + 90090\,x^4 + 1406160\,x^2$$
$$p^*_{10} = 893025\ , \quad q^*_{10} = 3192210$$

$$P_{12}(x) = 11\,x^{11} - 1089\,x^9 + 68607\,x^7 - 2401245\,x^5 + 36018675\,x^3$$
$$Q_{12}(x) = x^{12} - 121\,x^{10} + 9801\,x^8 - 480249\,x^6 + 12006225\,x^4 - 108056025\,x^2$$
$$R_{12}(x) = x^{11} - 319\,x^9 + 37521\,x^7 - 2136519\,x^5 + 51257745\,x^3 + 108056025\,x$$
$$S_{12}(x) = -22\,x^{10} + 3960\,x^8 - 331254\,x^6 + 13083840\,x^4 - 189791910\,x^2$$
$$p_{12} = 108056025\ , \quad q_{12} = 405903960$$
$$P^*_{12}(x) = -11\,x^{11} - 1089\,x^9 - 68607\,x^7 - 2401245\,x^5 - 36018675\,x^3$$
$$Q^*_{12}(x) = x^{12} + 121\,x^{10} + 9801\,x^8 + 480249\,x^6 + 12006225\,x^4 + 108056025\,x^2$$
$$R^*_{12}(x) = -x^{11} - 319\,x^9 - 37521\,x^7 - 2136519\,x^5 - 51257745\,x^3 + 108056025\,x$$
$$Q^*_{12}(x) = 22\,x^{10} + 3960\,x^8 + 331254\,x^6 + 13083840\,x^4 + 189791910\,x^2$$
$$p^*_{12} = 108056025\ , \quad q^*_{12} = 405903960$$

$$P_{14}(x) = 13\,x^{13} - 1859\,x^{11} + 184041\,x^9 - 11594583\,x^7 + 405810405\,x^5 - 6087156075\,x^3$$
$$Q_{14}(x) = x^{14} - 169\,x^{12} + 20449\,x^{10} - 1656369\,x^8 + 81162081\,x^6 - 2029052025\,x^4 + 18261468225\,x^2$$
$$R_{14}(x) = x^{13} - 455\,x^{11} + 82225\,x^9 - 8124831\,x^7 + 423504081\,x^5 - 9599044455\,x^3 - 18261468225\,x$$
$$S_{14}(x) = -26\,x^{12} + 6864\,x^{10} - 924066\,x^8 + 68468400\,x^6 - 2523330810\,x^4 + 34884289440\,x^2$$
$$p_{14} = -18261468225\ , \quad q_{14} = -71407225890$$
$$P^*_{14}(x) = -13\,x^{13} - 1859\,x^{11} - 184041\,x^9 - 11594583\,x^7 - 405810405\,x^5 - 6087156075\,x^3$$
$$Q^*_{14}(x) = x^{14} + 169\,x^{12} + 20449\,x^{10} + 1656369\,x^8 + 81162081\,x^6 + 2029052025\,x^4 + 18261468225\,x^2$$
$$R^*_{14}(x) = -x^{13} - 455\,x^{11} - 82225\,x^9 - 8124831\,x^7 - 423504081\,x^5 - 9599044455\,x^3 + 18261468225\,x$$
$$S^*_{14}(x) = 26\,x^{12} + 6864\,x^{10} + 924066\,x^8 + 68468400\,x^6 + 2523330810\,x^4 + 34884289440\,x^2$$
$$p^*_{14} = 18261468225\ , \quad q^*_{14} = 71407225890$$

$$P_{16}(x) = 15\,x^{15} - 2925\,x^{13} + 418275\,x^{11} - 41409225\,x^9 + 2608781175\,x^7 - 91307341125\,x^5 + 1369610116875\,x^3$$
$$Q_{16}(x) = x^{16} - 225\,x^{14} + 38025\,x^{12} - 4601025\,x^{10} + 372683025\,x^8 - 18261468225\,x^6 +$$
$$+ 456536705625\,x^4 - 4108830350625\,x^2$$
$$R_{16}(x) = x^{15} - 615\,x^{13} + 158145\,x^{11} - 24021855\,x^9 + 2175924465\,x^7 - 107462730375\,x^5 +$$
$$+ 2342399684625\,x^3 + 4108830350625\,x$$
$$S_{16}(x) = -30\,x^{14} + 10920\,x^{12} - 2157870\,x^{10} + 257605920\,x^8 - 17840252430\,x^6 +$$
$$+ 628620993000\,x^4 - 8396809170750\,x^2$$
$$p_{16} = 4108830350625\ , \quad q_{16} = 16614469872000$$

$$P_{16}^*(x) = -15\,x^{15} - 2925\,x^{13} - 418275\,x^{11} - 41409225\,x^9 - 2608781175\,x^7 - 91307341125\,x^5 - 1369610116875\,x^3$$

$$Q_{16}^*(x) = x^{16} + 225\,x^{14} + 38025\,x^{12} + 4601025\,x^{10} + 372683025\,x^8 + 18261468225\,x^6 +$$
$$+ 456536705625\,x^4 + 4108830350625\,x^2$$

$$R_{16}^*(x) = -x^{15} - 615\,x^{13} - 158145\,x^{11} - 24021855\,x^9 - 2175924465\,x^7 - 107462730375\,x^5 -$$
$$- 2342399684625\,x^3 + 4108830350625\,x$$

$$S_{16}^*(x) = 30\,x^{14} + 10920\,x^{12} + 2157870\,x^{10} + 257605920\,x^8 + 17840252430\,x^6 +$$
$$+ 628620993000\,x^4 + 8396809170750\,x^2$$

$$p_{16}^* = 4108830350625\,, \quad q_{16}^* = 16614469872000$$

$$P_{18}(x) = 17\,x^{17} - 4335\,x^{15} + 845325\,x^{13} - 120881475\,x^{11} + 11967266025\,x^9 - 753937759575\,x^7 +$$
$$+ 26387821585125\,x^5 - 395817323776875\,x^3$$

$$Q_{18}(x) = x^{18} - 289\,x^{16} + 65025\,x^{14} - 10989225\,x^{12} + 1329696225\,x^{10} - 107705394225\,x^8 +$$
$$+ 5277564317025\,x^6 - 131939107925625\,x^4 + 1187451971330625\,x^2$$

$$R_{18}(x) = x^{17} - 799\,x^{15} + 277185\,x^{13} - 59925255\,x^{11} + 8350229745\,x^9 - 717540730335\,x^7 +$$
$$+ 34161178676625\,x^5 - 723520252830375\,x^3 - 1187451971330625\,x$$

$$S_{18}(x) = -34\,x^{16} + 16320\,x^{14} - 4448730\,x^{12} + 780059280\,x^{10} - 87119333730\,x^8 + 5776722871920\,x^6 -$$
$$- 197193714968250\,x^4 + 2566378082268000\,x^2$$

$$p_{18} = -1187451971330625\,, \quad q_{18} = -4941282024929250$$

$$P_{18}^*(x) = -17\,x^{17} - 4335\,x^{15} - 845325\,x^{13} - 120881475\,x^{11} - 11967266025\,x^9 - 753937759575\,x^7 -$$
$$- 26387821585125\,x^5 - 395817323776875\,x^3$$

$$Q_{18}^*(x) = x^{18} + 289\,x^{16} + 65025\,x^{14} + 10989225\,x^{12} + 1329696225\,x^{10} + 107705394225\,x^8 +$$
$$+ 5277564317025\,x^6 + 131939107925625\,x^4 + 1187451971330625\,x^2$$

$$R_{18}^*(x) = -x^{17} - 799\,x^{15} - 277185\,x^{13} - 59925255\,x^{11} - 8350229745\,x^9 - 717540730335\,x^7 -$$
$$- 34161178676625\,x^5 - 723520252830375\,x^3 + 1187451971330625\,x$$

$$S_{18}^*(x) = 34\,x^{16} + 16320\,x^{14} + 4448730\,x^{12} + 780059280\,x^{10} + 87119333730\,x^8 + 5776722871920\,x^6 +$$
$$+ 197193714968250\,x^4 + 2566378082268000\,x^2$$

$$p_{18}^* = 1187451971330625\,, \quad q_{18}^* = 4941282024929250$$

Recurrence formulas:

$$\int x^{2n+2} \cdot \ln x \cdot J_0(x)\,dx = x^{2n+1} \ln x[(2n+1)J_0(x) + xJ_1(x)] -$$
$$-(2n+1)\int x^{2n} J_0(x)\,dx - \int x^{2n+1} J_1(x)\,dx - (2n+1)^2 \int x^{2n} \cdot \ln x \cdot J_0(x)\,dx$$

$$\int x^{2n+2} \cdot \ln x \cdot I_0(x)\,dx = -x^{2n+1} \ln x[(2n+1)I_0(x) - xI_1(x)] +$$
$$+(2n+1)\int x^{2n} I_0(x)\,dx - \int x^{2n+1} I_1(x)\,dx + (2n+1)^2 \int x^{2n} \cdot \ln x \cdot I_0(x)\,dx$$

The integrals of the type $\int x^m Z_\nu(x)\,dx$ are described before.

d) Integrals of $x^{2n+1}\,\ln x \cdot Z_1(x)$:

$$\int x\,\ln x \cdot J_1(x)\,dx = \Phi(x)\cdot \ln x + x\,J_0(x) - \Lambda_1(x) + \Phi(x)$$

$$\int x\,\ln x \cdot I_1(x)\,dx = -\Psi(x)\cdot \ln x - x\,I_0(x) + \Lambda_1^*(x) - \Psi(x)$$

$$\int x^3\,\ln x \cdot J_1(x)\,dx = [-x^3\,J_0(x) + 3x^2\,J_1(x) - 3\,\Phi(x)]\,\ln x - 3x\,J_0(x) + x^2\,J_1(x) + 3\,\Lambda_1(x) - 7\,\Phi(x)$$

$$\int x^3\,\ln x \cdot I_1(x)\,dx = [x^3\,I_0(x) - 3x^2\,I_1(x) - 3\,\Psi(x)]\,\ln x - 3x\,I_0(x) - x^2\,I_1(x) + 3\,\Lambda_1^*(x) - 7\,\Psi(x)$$

Let

$$\int x^n \ln x \cdot J_0(x)\,dx =$$

$$= [P_n(x)\,J_0(x) + Q_n(x)\,J_1(x) + p_n\,\Phi(x)]\cdot \ln x + R_n(x)\,J_0(x) + S_n(x)\,J_1(x) - p_n\,\Lambda_1(x) + q_n\,\Phi(x)$$

and

$$\int x^n \ln x \cdot I_0(x)\,dx =$$

$$= [P_n^*(x)\,I_0(x) + Q_n^*(x)\,I_1(x) + p_n^*\,\Psi(x)]\cdot \ln x + R_n^*(x)\,I_0(x) + S_n^*(x)\,I_1(x) - p_n^*\,\Lambda_1^*(x) + q_n^*\,\Psi(x)\ ,$$

then holds

$$P_5(x) = -x^5 + 15\,x^3\ ,\quad Q_5(x) = 5\,x^4 - 45\,x^2\ ,\quad R_5(x) = 8\,x^3 + 45\,x\ ,\quad S_5(x) = x^4 - 39\,x^2\ ,$$

$$p_5 = 45\ ,\quad q_5 = 129$$

$$P_5^*(x) = x^5 + 15\,x^3\ ,\quad Q_5^*(x) = -5\,x^4 - 45\,x^2\ ,\quad R_5^*(x) = 8\,x^3 - 45\,x\ ,\quad S_5^*(x) = -x^4 - 39\,x^2$$

$$p_5^* = -45\ ,\quad q_5^* = -129$$

$$P_7(x) = -x^7 + 35\,x^5 - 525\,x^3\ ,\quad Q_7(x) = 7\,x^6 - 175\,x^4 + 1575\,x^2$$

$$R_7(x) = 12\,x^5 - 460\,x^3 - 1575\,x\ ,\quad S_7(x) = x^6 - 95\,x^4 + 1905\,x^2\ ,\quad p_7 = -1575\ ,\quad q_7 = -5055$$

$$P_7^*(x) = x^7 + 35\,x^5 + 525\,x^3\ ,\quad Q_7^*(x) = -7\,x^6 - 175\,x^4 - 1575\,x^2$$

$$R_7^*(x) = 12\,x^5 + 460\,x^3 - 1575\,x\ ,\quad S_7^*(x) = -x^6 - 95\,x^4 - 1905\,x^2\ ,\quad p_7^* = -1575\ ,\quad q_7^* - 5055$$

$$P_9(x) = -x^9 + 63\,x^7 - 2205\,x^5 + 33075\,x^3\ ,\quad Q_9(x) = 9\,x^8 - 441\,x^6 + 11025\,x^4 - 99225\,x^2$$

$$R_9(x) = 16\,x^7 - 1316\,x^5 + 37380\,x^3 + 99225\,x\ ,\quad S_9(x) = x^8 - 175\,x^6 + 8785\,x^4 - 145215\,x^2$$

$$p_9 = 99225\ ,\quad q_9 = 343665$$

$$P_9^*(x) = x^9 + 63\,x^7 + 2205\,x^5 + 33075\,x^3\ ,\quad Q_9^*(x) = -9\,x^8 - 441\,x^6 - 11025\,x^4 - 99225\,x^2$$

$$R_9^*(x) = 16\,x^7 + 1316\,x^5 + 37380\,x^3 - 99225\,x\ ,\quad S_9^*(x) = -x^8 - 175\,x^6 - 8785\,x^4 - 145215\,x^2$$

$$p_9^* = -99225\ ,\quad q_9^* = -343665$$

$$P_{11}(x) = -x^{11} + 99\,x^9 - 6237\,x^7 + 218295\,x^5 - 3274425\,x^3$$

$$Q_{11}(x) = 11\,x^{10} - 891\,x^8 + 43659\,x^6 - 1091475\,x^4 + 9823275\,x^2$$

$$R_{11}(x) = 20\,x^9 - 2844\,x^7 + 174384\,x^5 - 4362120\,x^3 - 9823275\,x$$

$$S_{11}(x) = x^{10} - 279\,x^8 + 26145\,x^6 - 1090215\,x^4 + 16360785\,x^2$$

$$p_{11} = -9823275\ ,\quad q_{11} = -36007335$$

$$P^*_{11}(x) = x^{11} + 99\,x^9 + 6237\,x^7 + 218295\,x^5 + 3274425\,x^3$$
$$Q^*_{11}(x) = -11\,x^{10} - 891\,x^8 - 43659\,x^6 - 1091475\,x^4 - 9823275\,x^2$$
$$R^*_{11}(x) = 20\,x^9 + 2844\,x^7 + 174384\,x^5 + 4362120\,x^3 - 9823275\,x$$
$$R^*_{11}(x) = -x^{10} - 279\,x^8 - 26145\,x^6 - 1090215\,x^4 - 16360785\,x^2$$
$$p^*_{11} = -9823275\ , \quad q^*_{11} = -36007335$$

$$P_{13}(x) = -x^{13} + 143\,x^{11} - 14157\,x^9 + 891891\,x^7 - 31216185\,x^5 + 468242775\,x^3$$
$$Q_{13}(x) = 13\,x^{12} - 1573\,x^{10} + 127413\,x^8 - 6243237\,x^6 + 156080925\,x^4 - 1404728325\,x^2$$
$$R_{13}(x) = 24\,x^{11} - 5236\,x^9 + 556380\,x^7 - 30175992\,x^5 + 702369360\,x^3 + 1404728325\,x$$
$$S_{13}(x) = x^{12} - 407\,x^{10} + 61281\,x^8 - 4786551\,x^6 + 182096145\,x^4 - 2575350855\,x^2$$
$$p_{13} = 1404728325\ , \quad q_{13} = 5384807505$$
$$P^*_{13}(x) = x^{13} + 143\,x^{11} + 14157\,x^9 + 891891\,x^7 + 31216185\,x^5 + 468242775\,x^3$$
$$R^*_{13}(x) = -13\,x^{12} - 1573\,x^{10} - 127413\,x^8 - 6243237\,x^6 - 156080925\,x^4 - 1404728325\,x^2$$
$$R^*_{13}(x) = 24\,x^{11} + 5236\,x^9 + 556380\,x^7 + 30175992\,x^5 + 702369360\,x^3 - 1404728325\,x$$
$$S^*_{13}(x) = -x^{12} - 407\,x^{10} - 61281\,x^8 - 4786551\,x^6 - 182096145\,x^4 - 2575350855\,x^2$$
$$p^*_{13} = -1404728325\ , \quad q^*_{13} = -5384807505$$

$$P_{15}(x) = -x^{15} + 195\,x^{13} - 27885\,x^{11} + 2760615\,x^9 - 173918745\,x^7 +$$
$$+ 6087156075\,x^5 - 91307341125\,x^3$$
$$Q_{15}(x) = 15\,x^{14} - 2535\,x^{12} + 306735\,x^{10} - 24845535\,x^8 + 1217431215\,x^6 -$$
$$- 30435780375\,x^4 + 273922023375\,x^2$$
$$R_{15}(x) = 28\,x^{13} - 8684\,x^{11} + 1417416\,x^9 - 133467048\,x^7 + 6758371620\,x^5 -$$
$$- 150072822900\,x^3 - 273922023375\,x$$
$$S_{15}(x) = x^{14} - 559\,x^{12} + 123409\,x^{10} - 15517359\,x^8 + 1108188081\,x^6 -$$
$$- 39879014175\,x^4 + 541525809825\,x^2$$
$$p_{15} = -273922023375\ , \quad q_{15} = -1089369856575$$
$$P^*_{15}(x) = x^{15} + 195\,x^{13} + 27885\,x^{11} + 2760615\,x^9 + 173918745\,x^7 +$$
$$+ 6087156075\,x^5 + 91307341125\,x^3$$
$$Q^*_{15}(x) = -15\,x^{14} - 2535\,x^{12} - 306735\,x^{10} - 24845535\,x^8 - 1217431215\,x^6 -$$
$$- 30435780375\,x^4 - 273922023375\,x^2$$
$$R^*_{15}(x) = 28\,x^{13} + 8684\,x^{11} + 1417416\,x^9 + 133467048\,x^7 + 6758371620\,x^5 +$$
$$+ 150072822900\,x^3 - 273922023375\,x$$
$$S^*_{15}(x) = -x^{14} - 559\,x^{12} - 123409\,x^{10} - 15517359\,x^8 - 1108188081\,x^6 -$$
$$- 39879014175\,x^4 - 541525809825\,x^2$$
$$p^*_{15} = -273922023375\ , \quad q^*_{15} = -1089369856575$$

$$P_{17}(x) = -x^{17} + 255\,x^{15} - 49725\,x^{13} + 7110675\,x^{11} - 703956825\,x^9 + 44349279975\,x^7 -$$
$$- 1552224799125\,x^5 + 23283371986875\,x^3$$

$$Q_{17}(x) = 17\,x^{16} - 3825\,x^{14} + 646425\,x^{12} - 78217425\,x^{10} + 6335611425\,x^{8} - 310444959825\,x^{6} +$$
$$+ 7761123995625\,x^{4} - 69850115960625\,x^{2}$$
$$R_{17}(x) = 32\,x^{15} - 13380\,x^{13} + 3106740\,x^{11} - 449780760\,x^{9} + 39599497080\,x^{7} - 1918173757500\,x^{5} +$$
$$+ 41190404755500\,x^{3} + 69850115960625\,x$$
$$S_{17}(x) = x^{16} - 735\,x^{14} + 223665\,x^{12} - 41284815\,x^{10} + 4751983665\,x^{8} - 321545759535\,x^{6} +$$
$$+ 11143093586625\,x^{4} - 146854586253375\,x^{2}$$
$$p_{17} = 69850115960625\,, \quad q_{17} = 286554818174625$$
$$P^*_{17}(x) = x^{17} + 255\,x^{15} + 49725\,x^{13} + 7110675\,x^{11} + 703956825\,x^{9} + 44349279975\,x^{7} + 1552224799125\,x^{5} +$$
$$+ 23283371986875\,x^{3}$$
$$Q^*_{17}(x) = -17\,x^{16} - 3825\,x^{14} - 646425\,x^{12} - 78217425\,x^{10} - 6335611425\,x^{8} - 310444959825\,x^{6} -$$
$$- 7761123995625\,x^{4} - 69850115960625\,x^{2}$$
$$R^*_{17}(x) = 32\,x^{15} + 13380\,x^{13} + 3106740\,x^{11} + 449780760\,x^{9} + 39599497080\,x^{7} + 1918173757500\,x^{5} +$$
$$+ 41190404755500\,x^{3} - 69850115960625\,x$$
$$S^*_{17}(x) = -x^{16} - 735\,x^{14} - 223665\,x^{12} - 41284815\,x^{10} - 4751983665\,x^{8} - 321545759535\,x^{6} -$$
$$- 11143093586625\,x^{4} - 146854586253375\,x^{2}$$
$$p^*_{17} = -69850115960625\,, \quad q^*_{17} = -286554818174625$$

$$P_{19}(x) = -x^{19} + 323\,x^{17} - 82365\,x^{15} + 16061175\,x^{13} - 2296748025\,x^{11} + 227378054475\,x^{9} -$$
$$- 14324817431925\,x^{7} + 501368610117375\,x^{5} - 7520529151760625\,x^{3}$$
$$Q_{19}(x) = 19\,x^{18} - 5491\,x^{16} + 1235475\,x^{14} - 208795275\,x^{12} + 25264228275\,x^{10} - 2046402490275\,x^{8} +$$
$$+ 100273722023475\,x^{6} - 2506843050586875\,x^{4} + 22561587455281875\,x^{2}$$
$$R_{19}(x) = 36\,x^{17} - 19516\,x^{15} + 6111840\,x^{13} - 1259461320\,x^{11} + 170621631180\,x^{9} - 14387211635940\,x^{7} +$$
$$+ 675450216441000\,x^{5} - 14142702127554000\,x^{3} - 22561587455281875\,x$$
$$S_{19}(x) = x^{18} - 935\,x^{16} + 375105\,x^{14} - 95515095\,x^{12} + 16150822545\,x^{10} - 1762972735095\,x^{8} +$$
$$+ 115035298883505\,x^{6} - 3878619692322375\,x^{4} + 49948635534422625\,x^{2}$$
$$p_{19} = -22561587455281875\,, \quad q_{19} = -95071810444986375$$
$$P^*_{19}(x) = x^{19} + 323\,x^{17} + 82365\,x^{15} + 16061175\,x^{13} + 2296748025\,x^{11} + 227378054475\,x^{9} +$$
$$+ 14324817431925\,x^{7} + 501368610117375\,x^{5} + 7520529151760625\,x^{3}$$
$$Q^*_{19}(x) = -19\,x^{18} - 5491\,x^{16} - 1235475\,x^{14} - 208795275\,x^{12} - 25264228275\,x^{10} - 2046402490275\,x^{8} -$$
$$- 100273722023475\,x^{6} - 2506843050586875\,x^{4} - 22561587455281875\,x^{2}$$
$$R^*_{19}(x) = 36\,x^{17} + 19516\,x^{15} + 6111840\,x^{13} + 1259461320\,x^{11} + 170621631180\,x^{9} + 14387211635940\,x^{7} +$$
$$+ 675450216441000\,x^{5} + 14142702127554000\,x^{3} - 22561587455281875\,x$$
$$S^*_{19}(x) = -x^{18} - 935\,x^{16} - 375105\,x^{14} - 95515095\,x^{12} - 16150822545\,x^{10} - 1762972735095\,x^{8} -$$
$$- 115035298883505\,x^{6} - 3878619692322375\,x^{4} - 49948635534422625\,x^{2}$$
$$p^*_{19} = -22561587455281875\,, \quad q^*_{19} = -95071810444986375$$

Recurrence formulas:

$$\int x^{2n+1} \cdot \ln x \cdot J_1(x)\,dx = x^{2n} \ln x \,[(2n+1)J_1(x) - xJ_0(x)] +$$
$$+ \int x^{2n} J_0(x)\,dx - (2n+1)\int x^{2n-1} J_1(x)\,dx - (4n^2-1)\int x^{2n-1} \cdot \ln x \cdot J_1(x)\,dx$$
$$\int x^{2n+1} \cdot \ln x \cdot I_1(x)\,dx = -x^{2n} \ln x \,[xI_0(x) - (2n+1)I_1(x)] -$$
$$- \int x^{2n} I_0(x)\,dx + (2n+1)\int x^{2n-1} I_1(x)\,dx + (4n^2-1)\int x^{2n-1} \cdot \ln x \cdot I_1(x)\,dx$$

The integrals of the type $\int x^m Z_\nu x(x)\,dx$ are described before.

1.2.12. Integrals of the type $\int x^n\, e^{\pm x}\, \ln x \cdot Z_\nu(x)\, dx$

n = 0:

$$\int e^x\, \ln x\, I_0(x)\, dx = e^x\, \{(1-2x)\, I_0(x) + 2x\, I_1(x) + \ln x\, [x\, I_0(x) - x\, I_1(x)]\, \}$$

$$\int e^{-x}\, \ln x\, I_0(x)\, dx = e^{-x}\, \{-(1+2x)\, I_0(x) - 2x\, I_1(x) + \ln x\, [x\, I_0(x) + x\, I_1(x)]\, \}$$

$$\int e^x\, \ln x\, K_0(x)\, dx = e^x\, \{(1-2x)\, K_0(x) - 2x\, K_1(x) + \ln x\, [x\, K_0(x) + x\, K_1(x)]\, \}$$

$$\int e^{-x}\, \ln x\, K_0(x)\, dx = e^{-x}\, \{-(1+2x)\, K_0(x) + 2x\, K_1(x) + \ln x\, [x\, K_0(x) - x\, K_1(x)]\, \}$$

n = 1:

$$\int x\, e^x\, \ln x\, I_0(x)\, dx =$$

$$= \frac{e^x}{9}\, \{(-2\,x^2 + 3\,x - 3)\, I_0(x) + (2\,x^2 - 2\,x)\, I_1(x) + \ln x\, [3x^2\, I_0(x) - (3\,x^2 - 3\,x)\, I_1(x)]\, \}$$

$$\int x\, e^{-x}\, \ln x\, I_0(x)\, dx =$$

$$= \frac{e^{-x}}{9}\, \{-(2\,x^2 + 3\,x + 3)\, I_0(x) - (2\,x^2 + 2\,x)\, I_1(x) + \ln x\, [3x^2\, I_0(x) + (3\,x^2 + 3\,x)\, I_1(x)]\, \}$$

$$\int x\, e^x\, \ln x\, K_0(x)\, dx =$$

$$= \frac{e^x}{9}\, \{(-2\,x^2 + 3\,x - 3)\, K_0(x) + (-2\,x^2 + 2\,x)\, K_1(x) + \ln x\, [3\,x^2\, K_0(x) + (3\,x^2 - 3\,x)\, K_1(x)]\, \}$$

$$\int x\, e^{-x}\, \ln x\, K_0(x)\, dx =$$

$$= \frac{e^{-x}}{9}\, \{-(2\,x^2 + 3\,x + 3)\, K_0(x) + (2\,x^2 + 2\,x)\, K_1(x) + \ln x\, [3x^2\, K_0(x) - (3\,x^2 + 3\,x)\, K_1(x)]\, \}$$

$$\int x\, e^x\, \ln x\, I_1(x)\, dx =$$

$$= \frac{e^x}{9}\, \{(2\,x^2 + 6\,x - 6)\, I_0(x) - (2\,x^2 + 7\,x)\, I_1(x) + \ln x\, [-3x^2\, I_0(x) + (3\,x^2 + 6\,x)\, I_1(x)]\, \}$$

$$\int x\, e^{-x}\, \ln x\, I_1(x)\, dx =$$

$$= \frac{e^{-x}}{9}\, \{(-2x^2 + 6x + 6)\, I_0(x) - (2x^2 - 7x)\, I_1(x) + \ln x\, [3x^2\, I_0(x) + (3x^2 - 6x)\, I_1(x)]\, \}$$

$$\int x\, e^x\, \ln x\, K_1(x)\, dx =$$

$$= \frac{e^x}{9}\, \{-(2x^2 + 6x - 6)\, K_0(x) - (2x^2 + 7x)\, K_1(x) + \ln x\, [3x^2\, K_0(x) + (3x^2 + 6x)\, K_1(x)]\, \}$$

$$\int x\, e^{-x}\, \ln x\, K_1(x)\, dx =$$

$$= \frac{e^{-x}}{9}\, \{(2x^2 - 6x - 6)\, K_0(x) - (2x^2 - 7x)\, K_1(x) - \ln x\, [3x^2\, K_0(x) - (3x^2 - 3x)\, K_1(x)]\, \}$$

n = 2:

$$\int x^2\, e^x\, \ln x\, I_0(x)\, dx = \frac{e^x}{225}\, \{(-18\,x^3 + 13\,x^2 - 60\,x + 60)\, I_0(x) + (18\,x^3 - 4\,x^2 + 4\,x)\, I_1(x) +$$

$$+ \ln x\, [(45\,x^3 + 30\,x^2)\, I_0(x) + (-45\,x^3 + 60\,x^2 - 60\,x)\, I_1(x)]\, \}$$

$$\int x^2 e^{-x} \ln x\, I_0(x)\, dx = \frac{e^{-x}}{225} \{-(18x^3+13x^2+60x+60)\, I_0(x) - (18x^3+4x^2+4x)\, I_1(x)+$$
$$+\ln x\, [(45x^3-30x^2)\, I_0(x) + (45x^3+60x^2+60x)\, I_1(x)]\, \}$$
$$\int x^2 e^{x} \ln x\, K_0(x)\, dx = \frac{e^{x}}{225} \{(-18x^3+13x^2-60x+60)\, K_0(x) + (-18x^3+4x^2-4x)\, K_1(x)+$$
$$+\ln x\, [(45x^3+30x^2)\, K_0(x) + (45x^3-60x^2+60x)\, K_1(x)]\, \}$$
$$\int x^2 e^{-x} \ln x\, K_0(x)\, dx = \frac{e^{-x}}{225} \{-(18x^3+13x^2+60x+60)\, K_0(x) + (18x^3+4x^2+4x)\, K_1(x)+$$
$$+\ln x\, [(45x^3-30x^2)\, K_0(x) - (45x^3+60x^2+60x)\, K_1(x)]\, \}$$
$$\int x^2 e^{x} \ln x\, I_1(x)\, dx = \frac{e^{x}}{75} \{(6x^3+4x^2-30x+30)\, I_0(x) + (-6x^3-7x^2+7x)\, I_1(x)+$$
$$+\ln x\, [(-15x^3+15x^2)\, I_0(x) + (15x^3+30x^2-30x)\, I_1(x)]\, \}$$
$$\int x^2 e^{-x} \ln x\, I_1(x)\, dx = \frac{e^{-x}}{75} \{(-6x^3+4x^2+30x+30)\, I_0(x) + (-6x^3+7x^2+7x)\, I_1(x)+$$
$$+\ln x\, [(15x^3+15x^2)\, I_0(x) + (15x^3-30x^2-30x)\, I_1(x)]\, \}$$
$$\int x^2 e^{x} \ln x\, K_1(x)\, dx = \frac{e^{x}}{75} \{(-6x^3-4x^2+30x-30)\, K_0(x) + (-6x^3-7x^2+7x)\, K_1(x)+$$
$$+\ln x\, [(15x^3-15x^2)\, K_0(x) + (15x^3+30x^2-30x)\, K_1(x)]\, \}$$
$$\int x^2 e^{-x} \ln x\, K_1(x)\, dx = \frac{e^{-x}}{75} \{(6x^3-4x^2-30x-30)\, K_0(x) + (-6x^3+7x^2+7x)\, K_1(x)+$$
$$+\ln x\, [(-15x^3-15x^2)\, K_0(x) + (15x^3-30x^2-30x)\, K_1(x)]\, \}$$

n = 3:

$$\int x^3 e^{x} \ln x\, I_0(x)\, dx =$$
$$= \frac{e^{x}}{1225} \{(-50x^4+31x^3-171x^2+420x-420)\, I_0(x) + (50x^4-6x^3-132x^2+132x)\, I_1(x)+$$
$$+\ln x\, [(175x^4+210x^3-210x^2)\, I_0(x) + (-175x^4+315x^3-420x^2+420x)\, I_1(x)]\, \}$$
$$\int x^3 e^{-x} \ln x\, I_0(x)\, dx =$$
$$= \frac{e^{-x}}{1225} \{-(50x^4+31x^3+171x^2+420x+420)\, I_0(x) - (50x^4+6x^3-132x^2-132x)\, I_1(x)+$$
$$+\ln x\, [(175x^4-210x^3-210x^2)\, I_0(x) + (175x^4+315x^3+420x^2+420x)\, I_1(x)]\, \}$$
$$\int x^3 e^{x} \ln x\, K_0(x)\, dx =$$
$$= \frac{e^{x}}{1225} \{(-50x^4+31x^3-171x^2+420x-420)\, K_0(x) + (-50x^4+6x^3+132x^2-132x)\, K_1(x)+$$
$$+\ln x\, [(175x^4+210x^3-210x^2)\, K_0(x) + (175x^4-315x^3+420x^2-420x)\, K_1(x)]\, \}$$
$$\int x^3 e^{-x} \ln x\, K_0(x)\, dx =$$
$$= \frac{e^{-x}}{1225} \{-(50x^4+31x^3+171x^2+420x+420)\, K_0(x) + (50x^4+6x^3-132x^2-132x)\, K_1(x)+$$
$$+\ln x\, [(175x^4-210x^3-210x^2)\, K_0(x) - (175x^4+315x^3+420x^2+420x)\, K_1(x)]\, \}$$
$$\int x^3 e^{x} \ln x\, I_1(x)\, dx =$$
$$= \frac{e^{x}}{3675} \{(150x^4+54x^3-614x^2+1680x-1680)\, I_0(x) + (-150x^4-129x^3-388x^2+388x)\, I_1(x)+$$

$$+\ln x\,[(-525\,x^4+840\,x^3-840\,x^2)\,I_0(x)+(525\,x^4+1260\,x^3-1680\,x^2+1680\,x)\,I_1(x)]\,\}$$

$$\int x^3\,e^{-x}\,\ln x\,I_1(x)\,dx=$$

$$=\frac{e^{-x}}{3675}\,\{(-150\,x^4+54\,x^3+614\,x^2+1680\,x+1680)\,I_0(x)+(-150\,x^4+129\,x^3-388\,x^2-388\,x)\,I_1(x)+$$

$$+\ln x\,[(525\,x^4+840\,x^3+840\,x^2)\,I_0(x)+(525\,x^4-1260\,x^3-1680\,x^2-1680\,x)\,I_1(x)]\,\}$$

$$\int x^3\,e^{x}\,\ln x\,K_1(x)\,dx=$$

$$=\frac{e^{x}}{3675}\,\{(-150\,x^4-54\,x^3+614\,x^2-1680\,x+1680)\,K_0(x)+(-150\,x^4-129\,x^3-388\,x^2+388\,x)\,K_1(x)+$$

$$+\ln x\,[(525\,x^4-840\,x^3+840\,x^2)\,K_0(x)+(525\,x^4+1260\,x^3-1680\,x^2+1680\,x)\,K_1(x)]\,\}$$

$$\int x^3\,e^{-x}\,\ln x\,K_1(x)\,dx=$$

$$=\frac{e^{-x}}{3675}\,\{(150\,x^4-54\,x^3-614\,x^2-1680\,x-1680)\,K_0(x)+(-150\,x^4+129\,x^3-388\,x^2-388\,x)\,K_1(x)+$$

$$+\ln x\,[(-525\,x^4-840\,x^3-840\,x^2)\,K_0(x)+(525\,x^4-1260\,x^3-1680\,x^2-1680\,x)\,K_1(x)]\,\}$$

n = 4:

$$\int x^4\,e^{x}\,\ln x\,I_0(x)\,dx=\frac{e^{x}}{99225}\,\{(-2450\,x^5+1425\,x^4-12864\,x^3+33024\,x^2-60480\,x+60480)\,I_0(x)+$$

$$+(2450\,x^5-200\,x^4-11736\,x^3+35808\,x^2-35808\,x)\,I_1(x)+\ln x\,[(11025\,x^5+18900\,x^4-30240\,x^3+30240\,x^2)\,I_0(x)+$$

$$+(-11025\,x^5+25200\,x^4-45360\,x^3+60480\,x^2-60480\,x)\,I_1(x)]\,\}$$

$$\int x^4\,e^{-x}\,\ln x\,I_0(x)\,dx=\frac{e^{-x}}{99225}\,\{-(2450\,x^5+1425\,x^4+12864\,x^3+33024\,x^2+60480\,x+60480)\,I_0(x)-$$

$$-(2450\,x^5+200\,x^4-11736\,x^3-35808\,x^2-35808\,x)\,I_1(x)+\ln x\,[(11025\,x^5-18900\,x^4-30240\,x^3-30240\,x^2)\,I_0(x)+$$

$$+(11025\,x^5+25200\,x^4+45360\,x^3+60480\,x^2+60480\,x)\,I_1(x)]\,\}$$

$$\int x^4\,e^{x}\,\ln x\,K_0(x)\,dx=\frac{e^{x}}{99225}\,\{(-2450\,x^5+1425\,x^4-12864\,x^3+33024\,x^2-60480\,x+60480)\,K_0(x)+$$

$$+(-2450\,x^5+200\,x^4+11736\,x^3-35808\,x^2+35808\,x)\,K_1(x)+\ln x\,[(11025\,x^5+18900\,x^4-30240\,x^3+$$

$$+30240\,x^2)\,K_0(x)+(11025\,x^5-25200\,x^4+45360\,x^3-60480\,x^2+60480\,x)\,K_1(x)]\,\}$$

$$\int x^4\,e^{-x}\,\ln x\,K_0(x)\,dx=\frac{e^{-x}}{99225}\,\{-(2450\,x^5+1425\,x^4+12864\,x^3+33024\,x^2+60480\,x+60480)\,K_0(x)+$$

$$+(2450\,x^5+200\,x^4-11736\,x^3-35808\,x^2-35808\,x)\,K_1(x)+\ln x\,[(11025\,x^5-18900\,x^4-30240\,x^3-$$

$$-30240\,x^2)\,K_0(x)-(11025\,x^5+25200\,x^4+45360\,x^3+60480\,x^2+60480\,x)\,K_1(x)]\,\}$$

$$\int x^4\,e^{x}\,\ln x\,I_1(x)\,dx=\frac{e^{x}}{19845}\,\{(490\,x^5+120\,x^4-2838\,x^3+7878\,x^2-15120\,x+15120)\,I_0(x)+$$

$$+(-490\,x^5-365\,x^4-2367\,x^3+8196\,x^2-8196\,x)\,I_1(x)+\ln x\,[(-2205\,x^5+4725\,x^4-7560\,x^3+7560\,x^2)\,I_0(x)+$$

$$+(2205\,x^5+6300\,x^4-11340\,x^3+15120\,x^2-15120\,x)\,I_1(x)]\,\}$$

$$\int x^4\,e^{-x}\,\ln x\,I_1(x)\,dx=\frac{e^{-x}}{19845}\,\{(-490\,x^5+120\,x^4+2838\,x^3+7878\,x^2+15120\,x+15120)\,I_0(x)+$$

$$+(-490\,x^5+365\,x^4-2367\,x^3-8196\,x^2-8196\,x)\,I_1(x)+\ln x\,[(2205\,x^5+4725\,x^4+7560\,x^3+7560\,x^2)\,I_0(x)+$$

$$+(2205\,x^5-6300\,x^4-11340\,x^3-15120\,x^2-15120\,x)\,I_1(x)]\,\}$$

$$\int x^4\,e^{x}\,\ln x\,K_1(x)\,dx=\frac{e^{x}}{19845}\,\{(-490\,x^5-120\,x^4+2838\,x^3-7878\,x^2+15120\,x-15120)\,K_0(x)+$$

$$+(-490\,x^5-365\,x^4-2367\,x^3+8196\,x^2-8196\,x)\,K_1(x)+\ln x\,[(2205\,x^5-4725\,x^4+7560\,x^3-7560\,x^2)\,K_0(x)+$$
$$+(2205\,x^5+6300\,x^4-11340\,x^3+15120\,x^2-15120\,x)\,K_1(x)]\,\}$$

$$\int x^4\,e^{-x}\,\ln x\,K_1(x)\,dx=\frac{e^{-x}}{19845}\,\{(490\,x^5-120\,x^4-2838\,x^3-7878\,x^2-15120\,x-15120)\,K_0(x)+$$
$$+(-490\,x^5+365\,x^4-2367\,x^3-8196\,x^2-8196\,x)\,K_1(x)+\ln x\,[(-2205\,x^5-4725\,x^4-7560\,x^3-7560\,x^2)\,K_0(x)+$$
$$+(2205\,x^5-6300\,x^4-11340\,x^3-15120\,x^2-15120\,x)\,K_1(x)]\,\}$$

<u>n = 5:</u>

$$\int x^5\,e^{x}\,\ln x\,I_0(x)\,dx=$$
$$=\frac{e^{x}}{480249}\,\{(-7938\,x^6+4459\,x^5-61035\,x^4+214080\,x^3-435840\,x^2+665280\,x-665280)\,I_0(x)+$$
$$+(7938\,x^6-490\,x^5-58280\,x^4+237960\,x^3-539040\,x^2+539040\,x)\,I_1(x)+$$
$$+\ln x\,[(43659\,x^6+97020\,x^5-207900\,x^4+332640\,x^3-332640\,x^2)\,I_0(x)+$$
$$+(-43659\,x^6+121275\,x^5-277200\,x^4+498960\,x^3-665280\,x^2+665280\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,I_0(x)\,dx=$$
$$=\frac{e^{-x}}{480249}\,\{-(7938\,x^6+4459\,x^5+61035\,x^4+214080\,x^3+435840\,x^2+665280\,x+665280)\,I_0(x)-$$
$$-(7938\,x^6+490\,x^5-58280\,x^4-237960\,x^3-539040\,x^2-539040\,x)\,I_1(x)+$$
$$+\ln x\,[(43659\,x^6-97020\,x^5-207900\,x^4-332640\,x^3-332640\,x^2)\,I_0(x)+$$
$$-(43659\,x^6+121275\,x^5+277200\,x^4+498960\,x^3+665280\,x^2+665280\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^{x}\,\ln x\,K_0(x)\,dx=$$
$$=\frac{e^{x}}{480249}\,\{(-7938\,x^6+4459\,x^5-61035\,x^4+214080\,x^3-435840\,x^2+665280\,x-665280)\,K_0(x)+$$
$$+(-7938\,x^6+490\,x^5+58280\,x^4-237960\,x^3+539040\,x^2-539040\,x)\,K_1(x)+$$
$$+\ln x\,[(43659\,x^6+97020\,x^5-207900\,x^4+332640\,x^3-332640\,x^2)\,K_0(x)+$$
$$+(43659\,x^6-121275\,x^5+277200\,x^4-498960\,x^3+665280\,x^2-665280\,x)\,K_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,K_0(x)\,dx=$$
$$=\frac{e^{-x}}{480249}\,\{-(7938\,x^6+4459\,x^5+61035\,x^4+214080\,x^3+435840\,x^2+665280\,x+665280)\,K_0(x)+$$
$$+(7938\,x^6+490\,x^5-58280\,x^4-237960\,x^3-539040\,x^2-539040\,x)\,K_1(x)+$$
$$+\ln x\,[(43659\,x^6-97020\,x^5-207900\,x^4-332640\,x^3-332640\,x^2)\,K_0(x)-$$
$$-(43659\,x^6+121275\,x^5+277200\,x^4+498960\,x^3+665280\,x^2+665280\,x)\,K_1(x)]\,\}$$

$$\int x^5\,e^{x}\,\ln x\,I_1(x)\,dx=$$
$$=\frac{e^{x}}{800415}\,\{(13230\,x^6+2450\,x^5-108210\,x^4+405984\,x^3-849504\,x^2+1330560\,x-1330560)\,I_0(x)+$$
$$+(-13230\,x^6-9065\,x^5-98080\,x^4+442656\,x^3-1033728\,x^2+1033728\,x)\,I_1(x)+$$
$$+\ln x\,[(-72765\,x^6+194040\,x^5-415800\,x^4+665280\,x^3-665280\,x^2)\,I_0(x)+$$
$$+(72765\,x^6+242550\,x^5-554400\,x^4+997920\,x^3-1330560\,x^2+1330560\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,I_1(x)\,dx=$$

$$= \frac{e^{-x}}{800415} \{(-13230\,x^6 + 2450\,x^5 + 108210\,x^4 + 405984\,x^3 + 849504\,x^2 + 1330560\,x + 1330560)\,I_0(x) +$$
$$+(-13230\,x^6 + 9065\,x^5 - 98080\,x^4 - 442656\,x^3 - 1033728\,x^2 - 1033728\,x)\,I_1(x) +$$
$$+\ln x\,[(72765\,x^6 + 194040\,x^5 + 415800\,x^4 + 665280\,x^3 + 665280\,x^2)\,I_0(x) +$$
$$+(72765\,x^6 - 242550\,x^5 - 554400\,x^4 - 997920\,x^3 - 1330560\,x^2 - 1330560\,x)\,I_1(x)]\,\}$$

$$\int x^5\,e^x\,\ln x\,K_1(x)\,dx =$$
$$= \frac{e^{x}}{800415} \{(-13230\,x^6 - 2450\,x^5 + 108210\,x^4 - 405984\,x^3 + 849504\,x^2 - 1330560\,x + 1330560)\,K_0(x) +$$
$$+(-13230\,x^6 - 9065\,x^5 - 98080\,x^4 + 442656\,x^3 - 1033728\,x^2 + 1033728\,x)\,K_1(x) +$$
$$+\ln x\,[(72765\,x^6 - 194040\,x^5 + 415800\,x^4 - 665280\,x^3 + 665280\,x^2)\,K_0(x) +$$
$$+(72765\,x^6 + 242550\,x^5 - 554400\,x^4 + 997920\,x^3 - 1330560\,x^2 + 1330560\,x)\,K_1(x)]\,\}$$

$$\int x^5\,e^{-x}\,\ln x\,K_1(x)\,dx =$$
$$= \frac{e^{-x}}{800415} \{(13230\,x^6 - 2450\,x^5 - 108210\,x^4 - 405984\,x^3 - 849504\,x^2 - 1330560\,x - 1330560)\,K_0(x) +$$
$$+(-13230\,x^6 + 9065\,x^5 - 98080\,x^4 - 442656\,x^3 - 1033728\,x^2 - 1033728\,x)\,K_1(x) +$$
$$+\ln x\,[(-72765\,x^6 - 194040\,x^5 - 415800\,x^4 - 665280\,x^3 - 665280\,x^2)\,K_0(x) +$$
$$+(72765\,x^6 - 242550\,x^5 - 554400\,x^4 - 997920\,x^3 - 1330560\,x^2 - 1330560\,x)\,K_1(x)]\,\}$$

n = 6:

$$\int x^6\,e^x\,\ln x\,I_0(x)\,dx =$$
$$= \frac{e^{x}}{9018009} \{(-106722\,x^7 + 58653\,x^6 - 1137388\,x^5 + 5114220\,x^4 - 14236800\,x^3 + 25768320\,x^2 - 34594560\,x +$$
$$+34594560)\,I_0(x) + (106722\,x^7 - 5292\,x^6 - 1106420\,x^5 + 5617760\,x^4 - 17030880\,x^3 + 34239360\,x^2 -$$
$$-34239360\,x)\,I_1(x) + \ln x\,[(693693\,x^7 + 1891890\,x^6 - 5045040\,x^5 + 10810800\,x^4 - 17297280\,x^3 + 17297280\,x^2)\,I_0(x) +$$
$$+(-693693\,x^7 + 2270268\,x^6 - 6306300\,x^5 + 14414400\,x^4 - 25945920\,x^3 + 34594560\,x^2 - 34594560\,x)\,I_1(x)]\,\}$$

$$\int x^6\,e^{-x}\,\ln x\,I_0(x)\,dx =$$
$$= \frac{e^{-x}}{9018009} \{-(106722\,x^7 + 58653\,x^6 + 1137388\,x^5 + 5114220\,x^4 + 14236800\,x^3 + 25768320\,x^2 + 34594560\,x +$$
$$+34594560)\,I_0(x) - (106722\,x^7 + 5292\,x^6 - 1106420\,x^5 - 5617760\,x^4 - 17030880\,x^3 - 34239360\,x^2 -$$
$$-34239360\,x)\,I_1(x) + \ln x\,[(693693\,x^7 - 1891890\,x^6 - 5045040\,x^5 - 10810800\,x^4 - 17297280\,x^3 - 17297280\,x^2)\,I_0(x) +$$
$$+(693693\,x^7 + 2270268\,x^6 + 6306300\,x^5 + 14414400\,x^4 + 25945920\,x^3 + 34594560\,x^2 + 34594560\,x)\,I_1(x)]\,\}$$

$$\int x^6\,e^x\,\ln x\,K_0(x)\,dx =$$
$$= \frac{e^{x}}{9018009} \{(-106722\,x^7 + 58653\,x^6 - 1137388\,x^5 + 5114220\,x^4 - 14236800\,x^3 + 25768320\,x^2 - 34594560\,x +$$
$$+34594560)\,K_0(x) + (-106722\,x^7 + 5292\,x^6 + 1106420\,x^5 - 5617760\,x^4 + 17030880\,x^3 - 34239360\,x^2 +$$
$$+34239360\,x)\,K_1(x) + \ln x\,[(693693\,x^7 + 1891890\,x^6 - 5045040\,x^5 + 10810800\,x^4 - 17297280\,x^3 +$$
$$+17297280\,x^2)\,K_0(x) + (693693\,x^7 - 2270268\,x^6 + 6306300\,x^5 - 14414400\,x^4 + 25945920\,x^3 - 34594560\,x^2 +$$
$$+34594560\,x)\,K_1(x)]\,\}$$

$$\int x^6\,e^{-x}\,\ln x\,K_0(x)\,dx =$$

$$= \frac{e^{-x}}{9018009} \{-(106722\,x^7 + 58653\,x^6 + 1137388\,x^5 + 5114220\,x^4 + 14236800\,x^3 + 25768320\,x^2 + 34594560\,x +$$
$$+34594560)\,K_0(x) + (106722\,x^7 + 5292\,x^6 - 1106420\,x^5 - 5617760\,x^4 - 17030880\,x^3 - 34239360\,x^2 -$$
$$-34239360\,x)\,K_1(x) + \ln x\,[(693693\,x^7 - 1891890\,x^6 - 5045040\,x^5 - 10810800\,x^4 - 17297280\,x^3 -$$
$$-17297280\,x^2)\,K_0(x) - (693693\,x^7 + 2270268\,x^6 + 6306300\,x^5 + 14414400\,x^4 + 25945920\,x^3 + 34594560\,x^2 +$$
$$+34594560\,x)\,K_1(x)]\,\}$$

$$\int x^6\,e^x\,\ln x\,I_1(x)\,dx = \frac{e^x}{3864861} \{(45738\,x^7 + 6804\,x^6 - 508634\,x^5 + 2428410\,x^4 - 6912480\,x^3 + 12678240\,x^2 -$$
$$-17297280\,x + 17297280)\,I_0(x) + (-45738\,x^7 - 29673\,x^6 - 478135\,x^5 + 2637280\,x^4 - 8206560\,x^3 + 16707840\,x^2 -$$
$$-16707840\,x)\,I_1(x) + \ln x\,[(-297297\,x^7 + 945945\,x^6 - 2522520\,x^5 + 5405400\,x^4 - 8648640\,x^3 + 8648640\,x^2)\,I_0(x) +$$
$$+(297297\,x^7 + 1135134\,x^6 - 3153150\,x^5 + 7207200\,x^4 - 12972960\,x^3 + 17297280\,x^2 - 17297280\,x)\,I_1(x)]\,\}$$

$$\int x^6\,e^{-x}\,\ln x\,I_1(x)\,dx = \frac{e^{-x}}{3864861} \{(-45738\,x^7 + 6804\,x^6 + 508634\,x^5 + 2428410\,x^4 + 6912480\,x^3 + 12678240\,x^2 +$$
$$+17297280\,x + 17297280)\,I_0(x) + (-45738\,x^7 + 29673\,x^6 - 478135\,x^5 - 2637280\,x^4 - 8206560\,x^3 - 16707840\,x^2 -$$
$$-16707840\,x)\,I_1(x) + \ln x\,[(297297\,x^7 + 945945\,x^6 + 2522520\,x^5 + 5405400\,x^4 + 8648640\,x^3 + 8648640\,x^2)\,I_0(x) +$$
$$+(297297\,x^7 - 1135134\,x^6 - 3153150\,x^5 - 7207200\,x^4 - 12972960\,x^3 - 17297280\,x^2 - 17297280\,x)\,I_1(x)]\,\}$$

$$\int x^6\,e^x\,\ln x\,K_1(x)\,dx = \frac{e^x}{3864861} \{(-45738\,x^7 - 6804\,x^6 + 508634\,x^5 - 2428410\,x^4 + 6912480\,x^3 - 12678240\,x^2 +$$
$$+17297280\,x - 17297280)\,K_0(x) + (-45738\,x^7 - 29673\,x^6 - 478135\,x^5 + 2637280\,x^4 - 8206560\,x^3 + 16707840\,x^2 -$$
$$-16707840\,x)\,K_1(x) + \ln x\,[(297297\,x^7 - 945945\,x^6 + 2522520\,x^5 - 5405400\,x^4 + 8648640\,x^3 - 8648640\,x^2)\,K_0(x) +$$
$$+(297297\,x^7 + 1135134\,x^6 - 3153150\,x^5 + 7207200\,x^4 - 12972960\,x^3 + 17297280\,x^2 - 17297280\,x)\,K_1(x)]\,\}$$

$$\int x^6\,e^{-x}\,\ln x\,K_1(x)\,dx = \frac{e^{-x}}{3864861} \{(45738\,x^7 - 6804\,x^6 - 508634\,x^5 - 2428410\,x^4 - 6912480\,x^3 - 12678240\,x^2 -$$
$$-17297280\,x - 17297280)\,K_0(x) + (-45738\,x^7 + 29673\,x^6 - 478135\,x^5 - 2637280\,x^4 - 8206560\,x^3 - 16707840\,x^2 -$$
$$-16707840\,x)\,K_1(x) + \ln x\,[(-297297\,x^7 - 945945\,x^6 - 2522520\,x^5 - 5405400\,x^4 - 8648640\,x^3 - 8648640\,x^2)\,K_0(x) +$$
$$+(297297\,x^7 - 1135134\,x^6 - 3153150\,x^5 - 7207200\,x^4 - 12972960\,x^3 - 17297280\,x^2 - 17297280\,x)\,K_1(x)]\,\}$$

Recurrence relations:

About the recurrence relations for the integrals $\int x^n\,e^{\pm x}\,I_\nu(x)\,dx$ and $\int x^n\,e^{\pm x}\,K_\nu(x)\,dx$ see the pages 66 and 69.

$$\int x^{n+1}\,e^x\,\ln(x)\,I_0(x)\,dx = \frac{x^{n+1}\,e^x}{2n+3} \{[(n+1+x)\,I_0(x) - xI_1(x)]\,\ln(x) - I_0(x)\} -$$
$$-\frac{(n+1)^2}{2n+3} \int x^n\,\ln x\,e^x\,I_0(x)\,dx + \frac{2}{2n+3} \int x^{n+1}\,e^x\,I_1(x)\,dx$$

$$\int x^{n+1}\,e^x\,\ln(x)\,K_0(x)\,dx = \frac{x^{n+1}\,e^x}{2n+3} \{[(n+1+x)\,K_0(x) + xK_1(x)]\,\ln(x) - K_0(x)\} -$$
$$-\frac{(n+1)^2}{2n+3} \int x^n\,\ln x\,e^x\,K_0(x)\,dx - \frac{2}{2n+3} \int x^{n+1}\,e^x\,K_1(x)\,dx$$

$$\int x^{n+1}\,e^{-x}\,\ln(x)\,I_0(x)\,dx = \frac{x^{n+1}\,e^{-x}}{2n+3} \{[-(n+1-x)\,I_0(x) + xI_1(x)]\,\ln(x) + I_0(x)\} +$$
$$+\frac{(n+1)^2}{2n+3} \int x^n\,\ln x\,e^x\,I_0(x)\,dx - \frac{2}{2n+3} \int x^{n+1}\,e^x\,I_1(x)\,dx$$

$$\int x^{n+1}\, e^{-x}\, \ln(x)\, K_0(x)\, dx = \frac{x^{n+1}\, e^{-x}}{2n+3} \left\{ [(x-n-1)\, K_0(x) - xK_1(x)]\, \ln(x) + K_0(x) \right\} +$$

$$+ \frac{(n+1)^2}{2n+3} \int x^n\, \ln x\, e^x\, K_0(x)\, dx + \frac{2}{2n+3} \int x^{n+1}\, e^x\, K_1(x)\, dx$$

The following recurrence relations for the integrals $\int x^{n+1}\, e^{\pm x}\, Z_1(x)\, dx$ refer to $\int x^n\, e^{\pm x}\, Z_0(x)\, dx$ instead of $\int x^n\, e^{\pm x}\, Z_1(x)\, dx$.

$$\int x^{n+1}\, e^x\, \ln(x)\, I_1(x)\, dx = \frac{x^{n+1}\, e^x}{2n+3} \left\{ [(n+2-x)\, I_0(x) + xI_1(x)]\, \ln(x) - \frac{n+2}{n+1}\, I_0(x) \right\} -$$

$$- \frac{(n+1)(n+2)}{2n+3} \int x^n\, \ln x\, e^x\, I_0(x)\, dx + \frac{1}{n+1} \int x^{n+1}\, e^x\, I_0(x)\, dx + \frac{1}{(n+1)(2n+3)} \int x^{n+1}\, e^x\, I_1(x)\, dx$$

$$\int x^{n+1}\, e^{-x}\, \ln(x)\, I_1(x)\, dx = \frac{x^{n+1}\, e^{-x}}{2n+3} \left\{ [(n+2+x)\, I_0(x) + xI_1(x)]\, \ln(x) - \frac{n+2}{n+1}\, I_0(x) \right\} -$$

$$- \frac{(n+1)(n+2)}{2n+3} \int x^n\, \ln x\, e^x\, I_0(x)\, dx - \frac{1}{n+1} \int x^{n+1}\, e^x\, I_0(x)\, dx - \frac{1}{(n+1)(2n+3)} \int x^{n+1}\, e^x\, I_1(x)\, dx$$

$$\int x^{n+1}\, e^x\, \ln(x)\, K_1(x)\, dx = \frac{x^{n+1}\, e^x}{2n+3} \left\{ [(x-n-2)\, K_0(x) + xK_1(x)]\, \ln(x) + \frac{n+2}{n+1}\, K_0(x) \right\} +$$

$$+ \frac{(n+1)(n+2)}{2n+3} \int x^n\, \ln x\, e^x\, K_0(x)\, dx - \frac{1}{n+1} \int x^{n+1}\, e^x\, K_0(x)\, dx + \frac{1}{(n+1)(2n+3)} \int x^{n+1}\, e^x\, K_1(x)\, dx$$

$$\int x^{n+1}\, e^{-x}\, \ln(x)\, K_1(x)\, dx = \frac{x^{n+1}\, e^{-x}}{2n+3} \left\{ [-(n+2+x)\, K_0(x) + xK_1(x)]\, \ln(x) + \frac{n+2}{n+1}\, K_0(x) \right\} +$$

$$+ \frac{(n+1)(n+2)}{2n+3} \int x^n\, \ln x\, e^x\, K_0(x)\, dx + \frac{1}{n+1} \int x^{n+1}\, e^x\, K_0(x)\, dx + \frac{1}{(n+1)(2n+3)} \int x^{n+1}\, e^x\, K_1(x)\, dx$$

1.2.13. $\int x^n\, e^{-x^2}\, J_\nu(\alpha x)\, dx$

a) The Case $\alpha = 1$, Basic Integrals:

Some improper integrals: From [13], 4.14. (34) and (35) one has (or [14], 8.2.(21); see also [7], 6.643 and 9.235; or [4], 2.12.9.1.-3.)

$$\int_0^\infty e^{-x^2}\, J_0(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{2}\, I_0\left(\frac{1}{8}\right) = 0.78515\,05503$$

$$\int_0^\infty x\, e^{-x^2}\, J_0(x)\, dx = \frac{e^{-1/4}}{2} = 0.38940\,03915$$

$$\int_0^\infty x^2\, e^{-x^2}\, J_0(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{16}\left[3I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] = 0.30055\,34957$$

$$\int_0^\infty x^3\, e^{-x^2}\, J_0(x)\, dx = \frac{3e^{-1/4}}{8} = 0.29205\,02937$$

$$\int_0^\infty x^4\, e^{-x^2}\, J_0(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{64}\left[13I_0\left(\frac{1}{8}\right) + 7I_1\left(\frac{1}{8}\right)\right] = 0.32968\,09799$$

$$\int_0^\infty x^5\, e^{-x^2}\, J_0(x)\, dx = \frac{17e^{-1/4}}{32} = 0.41373\,79160$$

$$\int_0^\infty x^6\, e^{-x^2}\, J_0(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{256}\left[87I_0\left(\frac{1}{8}\right) + 69I_1\left(\frac{1}{8}\right)\right] = 0.56005\,83093$$

$$\int_0^\infty \frac{e^{-x^2}\, J_1(x)\, dx}{x} = \frac{\sqrt{\pi}\, e^{-1/8}}{4}\left[I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] = 0.41706\,34325$$

$$\int_0^\infty e^{-x^2}\, J_1(x)\, dx = 1 - e^{-1/4} = 0.22119\,92169$$

$$\int_0^\infty x\, e^{-x^2}\, J_1(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{8}\left[I_0\left(\frac{1}{8}\right) - I_1\left(\frac{1}{8}\right)\right] = 0.18404\,35589$$

$$\int_0^\infty x^2\, e^{-x^2}\, J_1(x)\, dx = \frac{e^{-1/4}}{4} = 0.19470\,01958$$

$$\int_0^\infty x^3\, e^{-x^2}\, J_1(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{32}\left[5I_0\left(\frac{1}{8}\right) - I_1\left(\frac{1}{8}\right)\right] = 0.24229\,85273$$

$$\int_0^\infty x^4\, e^{-x^2}\, J_1(x)\, dx = \frac{7e^{-1/4}}{16} = 0.34072\,53426$$

$$\int_0^\infty x^5\, e^{-x^2}\, J_1(x)\, dx = \frac{\sqrt{\pi}\, e^{-1/8}}{128}\left[43I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] = 0.52828\,82809$$

$$\int_0^\infty x^6\, e^{-x^2}\, J_1(x)\, dx = \frac{73e^{-1/4}}{64} = 0.88831\,96432$$

Let

$$F_\nu(x) = \int_0^x e^{-t^2}\, J_\nu(t)\, dt = \nu + e^{-x^2}\, \left[P_\nu(x)\, J_0(x) + Q_\nu(x)\, J_1(x)\right]\,, \quad \nu = 0,\ 1,$$

with

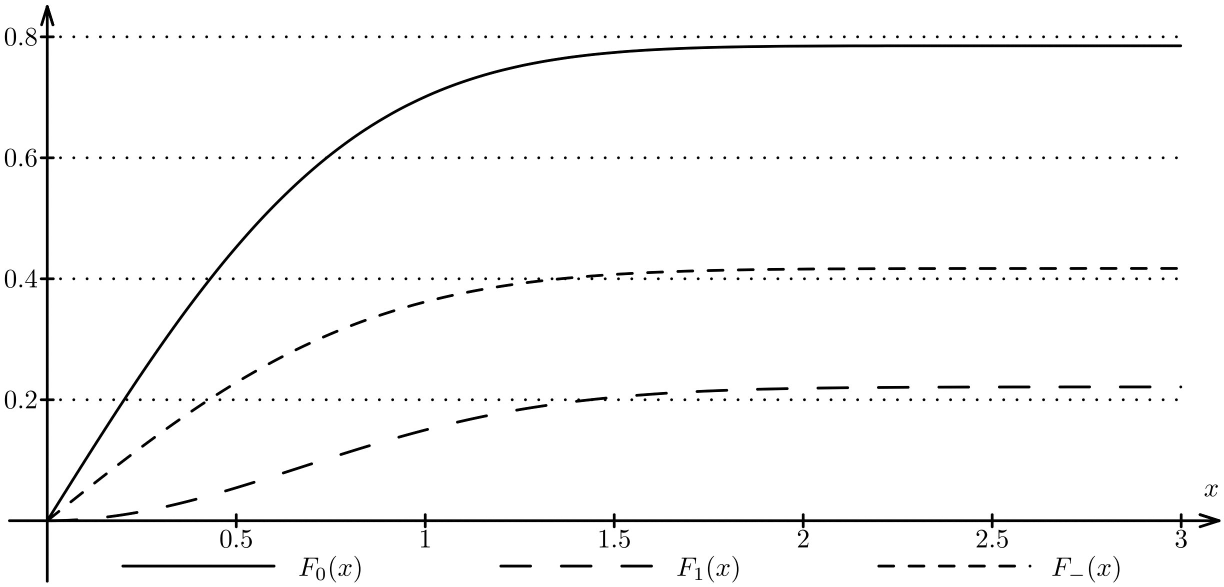
$$P_\nu(x) = \sum_{k=0}^\infty a_k^{(\nu)}\, x^{2k+1-\nu} = \sum_{k=0}^\infty \frac{\alpha_k^{(\nu)}}{\gamma_k^{(\nu)}}\, x^{2k+1-\nu} \quad \text{and} \quad Q_\nu(x) = \sum_{k=0}^\infty b_k^{(\nu)}\, x^{2k+\nu} = \sum_{k=0}^\infty \frac{\beta_k^{(\nu)}}{\delta_k^{(\nu)}}\, x^{2k+\nu}\,.$$

Furthermore, let

$$F_-(x) = \int_0^x \frac{e^{-t^2}\, J_1(t)\, dt}{t} = e^{-x^2}\, \left[P_-(x)\, J_0(x) + Q_-(x)\, J_1(x)\right]$$

with

$$P_-(x) = \sum_{k=0}^{\infty} a_k^{(-)}\, x^{2k+1} = \sum_{k=0}^{\infty} \frac{\alpha_k^{(-)}}{\gamma_k^{(-)}}\, x^{2k+1} \quad \text{and} \quad Q_-(x) = \sum_{k=0}^{\infty} b_k^{(-)}\, x^{2k+\nu} = \sum_{k=0}^{\infty} \frac{\beta_k^{(-)}}{\delta_k^{(-)}}\, x^{2k} \ .$$



The minimum and maximum values of $F_s(x)$ are located in the zeros $x_k^{(\nu)}$ of $J_\nu(x)$, $0 < x_k^{(\nu)} < x_{k+1}^{(\nu)}$.
Let $\Delta_k^{(s)} = F_s(x_k^{(\nu)}) - \lim_{x\to\infty} F_s(x)$.
The following table shows, that $F_s(x)$ must not be be computed for large values of x.

s	Value	$k=1$	$k=2$	$k=3$	$k=4$
0	x_k	2.4048	5.5201	8.6537	11.7915
	$\Delta_k^{(0)}$	$5.1225 \cdot 10^{-5}$	$-1.5215 \cdot 10^{-16}$	$2.6395 \cdot 10^{-36}$	$-1.6974 \cdot 10^{-64}$
1	x_k	3.8317	7.0156	10.1735	13.3237
	$\Delta_k^{(1)}$	$2.5200 \cdot 10^{-9}$	$-6.1486 \cdot 10^{-25}$	$6.6365 \cdot 10^{-49}$	$-2.4330 \cdot 10^{-81}$
-	$\Delta_k^{(-)}$	$6.2084 \cdot 10^{-10}$	$-8.5976 \cdot 10^{-26}$	$6.4624 \cdot 10^{-50}$	$-1.8160 \cdot 10^{-82}$

Remark: In any case, if $F_s(x)$ is written as

$$F_s(x) = e^{-x^2}\, \psi_s(x) + \lim_{x\to\infty} F_s(x) \ ,$$

then one has

x	1.52	2.15	2.63	3.03	3.39	3.72	4.01	4.29	4.55	4.80
$\exp(-x^2)$	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}

.

The influence of $\psi_s(x)$ vanishes soon.
Another estimation: From

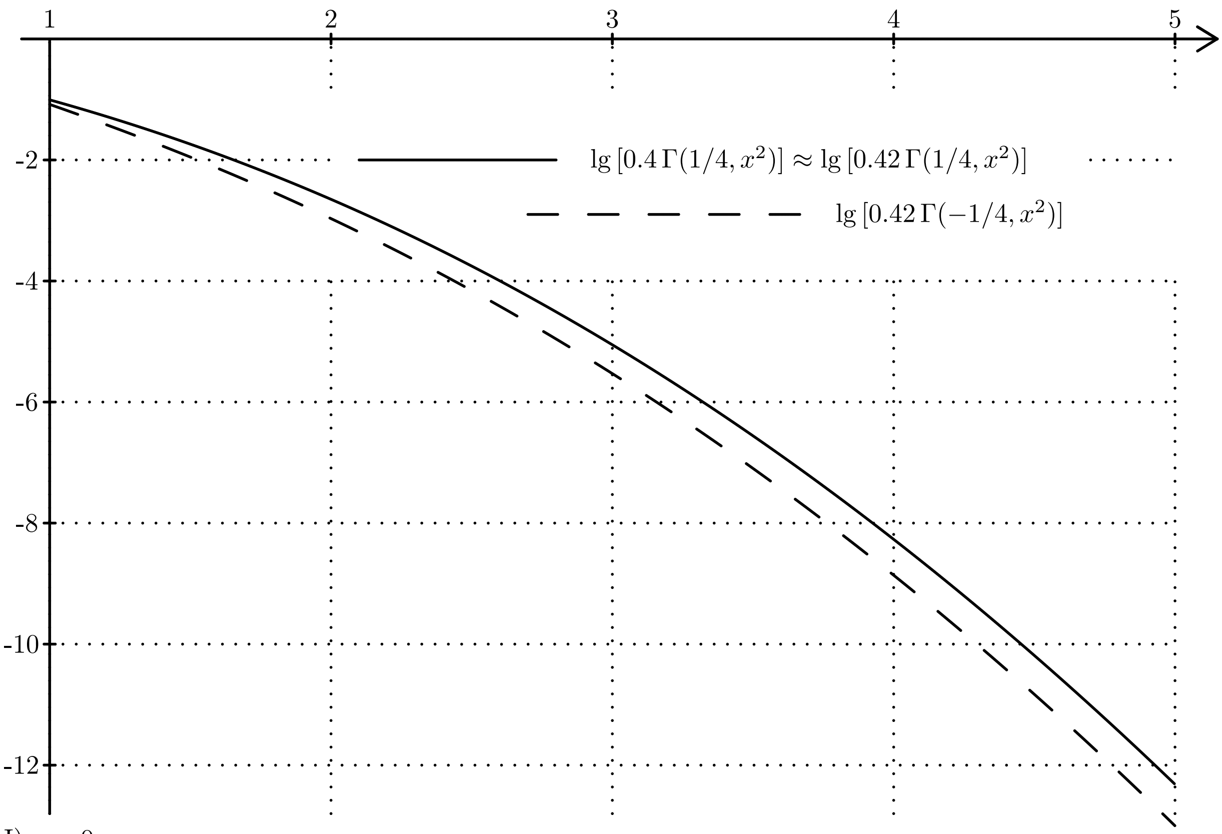
$$F_0(x) = \int_0^x e^{-t^2}\, J_0(t)\, dt = \lim_{x\to\infty} F_0(x) - \int_x^\infty e^{-t^2}\, J_0(t)\, dt$$

one has (substituting $t^2 = s$)

$$\left|\int_x^\infty e^{-t^2}\, J_0(t)\, dt\right| < \int_x^\infty e^{-t^2}\, |J_0(t)|\, dt < \int_x^\infty \frac{8\, e^{-t^2}\, dt}{\sqrt{t}} = \int_{x^2}^\infty \frac{8\, e^{-s}\, ds}{2 s^{3/4}} = 0.4\, \Gamma\left(\frac{1}{4}, x^2\right)$$

with the (upper) incomplete Gamma function. By the same way follows

$$\left|\int_x^\infty e^{-t^2}\, J_1(t)\, dt\right| < 0.42\, \Gamma\left(\frac{1}{4}, x^2\right) \quad \text{and} \quad \left|\int_x^\infty \frac{e^{-t^2}\, J_1(t)\, dt}{t}\right| < 0.42\, \Gamma\left(-\frac{1}{4}, x^2\right)$$



I) $s = 0$:

With $k \geq 1$ holds ($P_0(x)$ and $Q_0(x)$ as defined before)

$$a_{k+1}^{(0)} = \frac{(8k+3)a_k^{(0)} - 4a_{k-1}^{(0)}}{(2k+1)(2k+3)} \, , \quad b_{k+1}^{(0)} = \frac{(8k-1)b_k^{(0)} - 4b_{k-1}^{(0)}}{(2k+1)^2} \, .$$

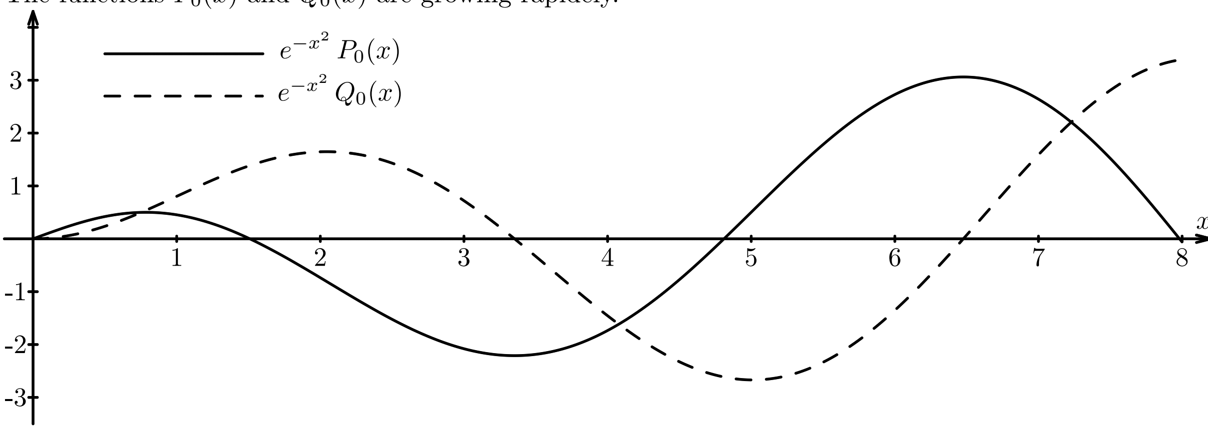
$$\int_0^x e^{-t^2} J_0(t)\,dt =$$

$$= e^{-x^2} \left[\left(x + \frac{x^3}{3} - \frac{x^5}{45} - \frac{79}{1575}\,x^7 - \ldots \right) J_0(x) + \left(x^2 + \frac{7}{9}\,x^4 + \frac{23}{75}\,x^6 + \ldots \right) J_1(x) \right]$$

k	$\alpha_k^{(0)}$	$\gamma_k^{(0)}$	$a_k^{(0)}$
0	1	1	1.00000 00000
1	1	3	0.33333 33333
2	-1	45	-0.02222 22222
3	-79	1575	-0.05015 87302
4	-1993	99225	-0.02008 56639
5	-7121	1403325	-0.00507 43769
6	-1354193	1404728325	-0.00096 40248
7	-40551359	273922023375	-0.00014 80398
8	-1336259641	69850115960625	-0.00001 91304
9	-48167009767	22561587455281875	-0.00000 21349
10	-1886078276353	9002073394657468125	-0.00000 02095
11	-79669949349167	4348001449619557104375	-0.00000 00183
12	-515151737265743	3571572619330350478593755	-0.00000 00014
13	-9145224759056621	88819371717557400059765625	-0.00000 00001

k	$\beta_k^{(0)}$	$\delta_k^{(0)}$	$b_k^{(0)}$
0	0	1	0.00000 00000
1	1	1	1.00000 00000
2	7	9	0.77777 77778
3	23	75	0.3066666667
4	887	11025	0.0804535147
5	13973	893025	0.0156468184
6	85853	36018675	0.0023835691
7	5342341	18261468225	0.0002925472
8	119718871	4108830350625	0.0000291370
9	33755333	14659900880625	0.0000023026
10	5066536837	38970014695486875	0.0000001300
11	26744808373	11120208311047460625	0.0000000024
12	-19585169733827	33334677780416604466875	-0.0000000006
13	-594894329175841	5682047348934648488671875	-0.0000000001

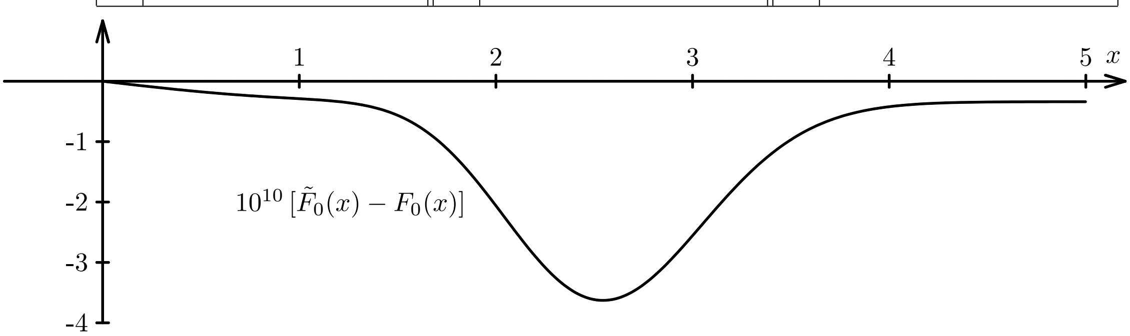
One has $a_0^{(0)}, a_1^{(0)} > 0$, $a_2^{(0)}, \ldots, a_{22}^{(0)} < 0$, but $a_{23}^{(0)} > 0$. Holds $b_{11}^{(0)} \cdot b_{12}^{(0)} < 0$ and $b_{41}^{(0)} \cdot b_{42}^{(0)} < 0$.
First positive zeros of $P_0(x)$: 1.5091, 4.8104, 7.9822 .
Maxima: $P_0(1.1915) = 1.3870$, $P_0(7.9191) = 3.6414 \cdot 10^{26}$, mimimum: $P_0(4.7057) = -1.1187 \cdot 10^9$.
First positive zeros of $Q_0(x)$: 3.3521, 6.4769, 9.6124.
Maxima: $Q_0(3.2008) = 9162.0$, $Q_0(9.5602) = 9.5630 \cdot 10^{38}$, minimum: $Q_0(6.3994) = -1.4373 \cdot 10^{17}$
The functions $P_0(x)$ and $Q_0(x)$ are growing rapidely.



Approximation:

$$F_0(x) \approx \tilde{F}_0(x) = 0.7851505503\,\mathrm{erf}(x) + e^{-x^2} \sum_{k=0}^{5} c_k^{(0)} x^{2k+1}$$

k	$c_k^{(0)}$	k	$c_k^{(0)}$	k	$c_k^{(0)}$
0	$1.14052\,47597 \cdot 10^{-1}$	1	$-7.29834\,93536 \cdot 10^{-3}$	2	$2.05660\,25858 \cdot 10^{-4}$
3	$-3.24389\,43744 \cdot 10^{-6}$	4	$3.26550\,30992 \cdot 10^{-8}$	5	$-2.27888\,93576 \cdot 10^{-10}$



Asymptotic expansion:

$$\int_0^x e^{-t^2}\,J_0(t)\,dx \quad \sim \quad \frac{\sqrt{\pi}\,e^{-1/8}}{2}\,I_0\left(\frac{1}{8}\right) +$$

$$+\frac{\sqrt{2}e^{-x^2}}{\sqrt{\pi x}}\left[\left(-\frac{1}{2x}+\frac{129}{256x^3}-\frac{76203}{65536x^5}+\ldots\right)\sin\left(x+\frac{\pi}{4}\right)+\left(-\frac{3}{16x^2}+\frac{921}{2048x^4}-\frac{775773}{524288x^6}+\ldots\right)\cos\left(x+\frac{\pi}{4}\right)\right]$$

See the remark on page 140.
Let

$$\varphi_1(x)=e^{x^2}\left[\int_0^x e^{-t^2}\,J_0(t)\,dt-\frac{\sqrt{\pi}\,e^{-1/8}}{2}\,I_0\left(\frac{1}{8}\right)\right]\ ,\quad \varphi_2(x)=-\frac{\sqrt{2}}{\sqrt{\pi x}}\cdot\frac{1}{2x}\sin\left(x+\frac{\pi}{4}\right)\ ,$$

$$\varphi_3(x)=-\frac{\sqrt{2}}{\sqrt{\pi x}}\left[\frac{1}{2x}\sin\left(x+\frac{\pi}{4}\right)+\frac{3}{16x^2}\cos\left(x+\frac{\pi}{4}\right)\right]\ .$$

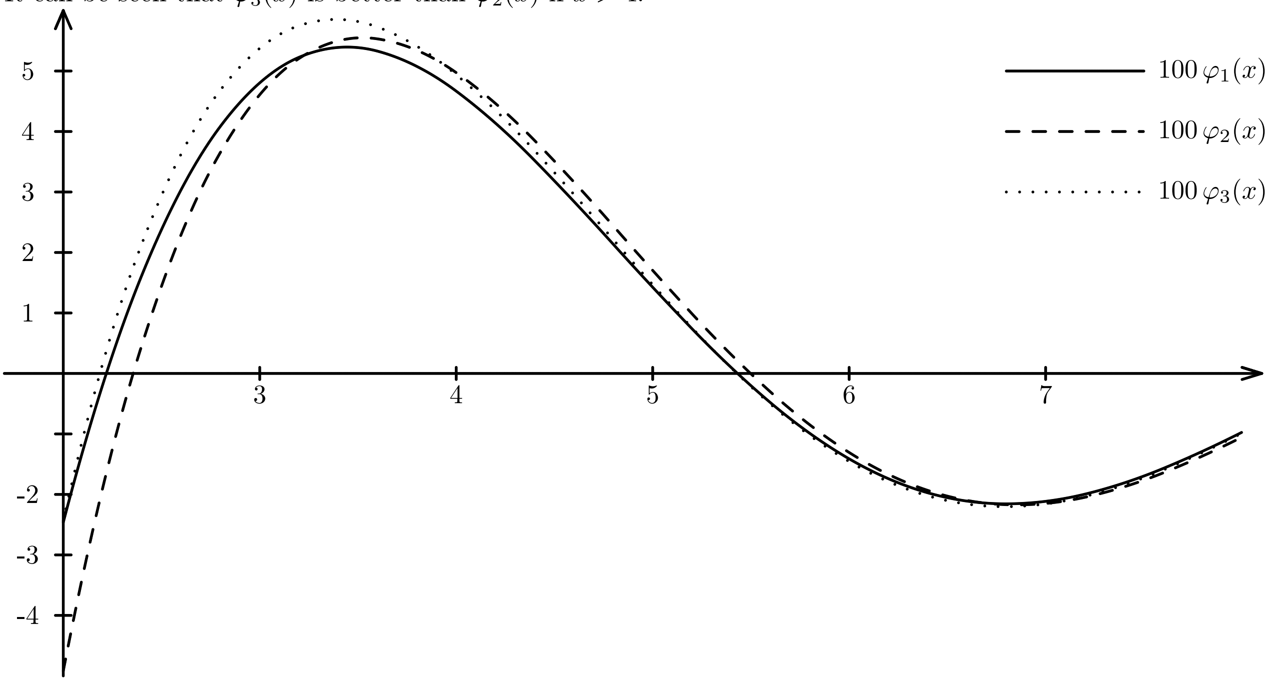
The following figure shows that $\varphi_1(x)\approx\varphi_2(x)$ if $x>2$. From this

$$e^{x^2}\left[\int_0^x e^{-t^2}\,J_0(t)\,dt-\frac{\sqrt{\pi}\,e^{-1/8}}{2}\,I_0\left(\frac{1}{8}\right)\right]\approx-\frac{\sqrt{2}}{\sqrt{\pi x}}\cdot\frac{1}{2x}\sin\left(x+\frac{\pi}{4}\right)$$

and therefore holds

$$\int_0^x e^{-t^2}\,J_0(t)\,dt\approx\frac{\sqrt{\pi}\,e^{-1/8}}{2}\,I_0\left(\frac{1}{8}\right)-\frac{\sqrt{2}}{\sqrt{\pi x}}\cdot\frac{e^{-x^2}}{2x}\sin\left(x+\frac{\pi}{4}\right)\ .$$

It can be seen that $\varphi_3(x)$ is better than $\varphi_2(x)$ if $x>4$.



II) $s=1$:
With $k\geq 1$ holds

$$a_{k+1}^{(1)}=\frac{(8k-1)a_k^{(1)}-4a_{k-1}^{(1)}}{4k(k+1)}\ ,\quad b_{k+1}^{(1)}=\frac{(8k+3)b_k^{(1)}-4b_{k-1}^{(1)}}{4(k+1)^2}\ .$$

$$\int_0^x e^{-t^2}\,J_1(t)\,dt=$$

$$=1+e^{-x^2}\left[\left(-1-x^2-\frac{3}{8}x^3-\frac{13}{192}x^5-\ldots\right)J_0(x)+\left(-\frac{x^3}{2}-\frac{11}{32}x^5-\frac{145}{1152}x^7-\ldots\right)J_1(x)\right]$$

k	$\alpha_k^{(1)}$	$\gamma_k^{(1)}$	$a_k^{(1)}$
0	-1	1	-1.00000 00000
1	-1	1	-1.00000 00000
2	-3	8	-0.37500 00000
3	-13	192	-0.06770 83333
4	-11	9216	-0.00119 35764
5	431	147456	0.00292 29058
6	17513	17694720	0.00098 97303
7	88033	424673280	0.00020 72958
8	3160567	95126814720	0.00003 32248
9	17176879	3913788948480	0.00000 43888
10	4895935679	9862748150169600	0.00000 04964
11	213635978321	4339609186074624000	0.00000 00492
12	9969483318887	229131365024740147200	0.00000 00044
13	495901729080313	14297797177543785185280000	0.00000 00003

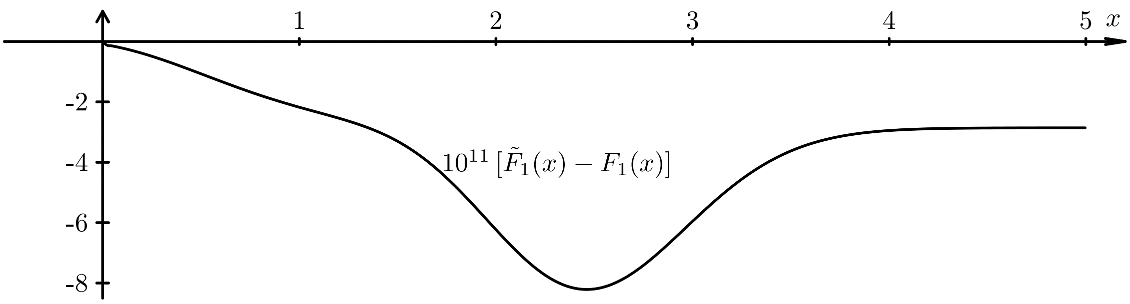
k	$\beta_k^{(1)}$	$\delta_k^{(1)}$	$b_k^{(1)}$
0	0	1	0.00000 00000
1	-1	2	-0.50000 00000
2	-11	32	-0.34375 00000
3	-145	1152	-0.12586 80556
4	-259	8192	-0.03161 62109
5	-8893	1474560	-0.00603 09516
6	-195919	212336640	-0.00092 26811
7	-231881	1981808640	-0.00011 70047
8	-19100009	1522029035520	-0.00001 25490
9	-567362171	493137407508480	-0.00000 11505
10	-5932850387	657516543344640000	-0.00000 00902
11	-569500272763	954714020936417280000	-0.00000 00060
12	-17366529773737	5499152760593763532800000	-0.00000 00003

One has $a_4^{(1)} \cdot a_5^{(1)} < 0,\ a_{27}^{(1)} \cdot a_{28}^{(1)} < 0$ and $b_{13}^{(1)} \cdot b_{14}^{(1)} < 0,\ b_{47}^{(1)} \cdot b_{48}^{(1)} < 0$.
First positive zeros of $P_1(x)$: 2.0140, 5.2565 and 8.4241.
Minima: $P_1(1.7541) = -7.0905$ and $P_1(8.3646) = -4.1348 \cdot 10^{29}$, maximum: $P_1(5.1608) = 7.8803 \cdot 10^{10}$
First positive zeros of $Q_1(x)$: 3.7880, 6.9155 and 10.0515.
Minima: $Q_1(3.6545) = -1.5968 \cdot 10^5$ and $Q_1(10.0017) = -4.3429 \cdot 10^{42}$,
maximum: $Q_1(6.8429) = 4.0769 \cdot 10^{19}$
$P_1(x)$ and $Q_1(x)$ are growing like $P_0(x)$ and $Q_0(x)$.

Approximation:

$$F_1(x) \approx \tilde{F}_1(x) = 0.22119\,92169 + e^{-x^2}\left[-0.22119\,92169 + \sum_{k=1}^{6} c_k^{(1)} x^{2k}\right]$$

k	$c_k^{(1)}$	k	$c_k^{(1)}$	k	$c_k^{(1)}$
1	$2.88007\,83071 \cdot 10^{-2}$	2	$-1.22460\,84642 \cdot 10^{-3}$	3	$2.58249\,56345 \cdot 10^{-5}$
4	$-3.25444\,94146 \cdot 10^{-7}$	5	$2.72785\,19850 \cdot 10^{-9}$	6	$-1.63082\,82205 \cdot 10^{-11}$



Asymptotic expansion:

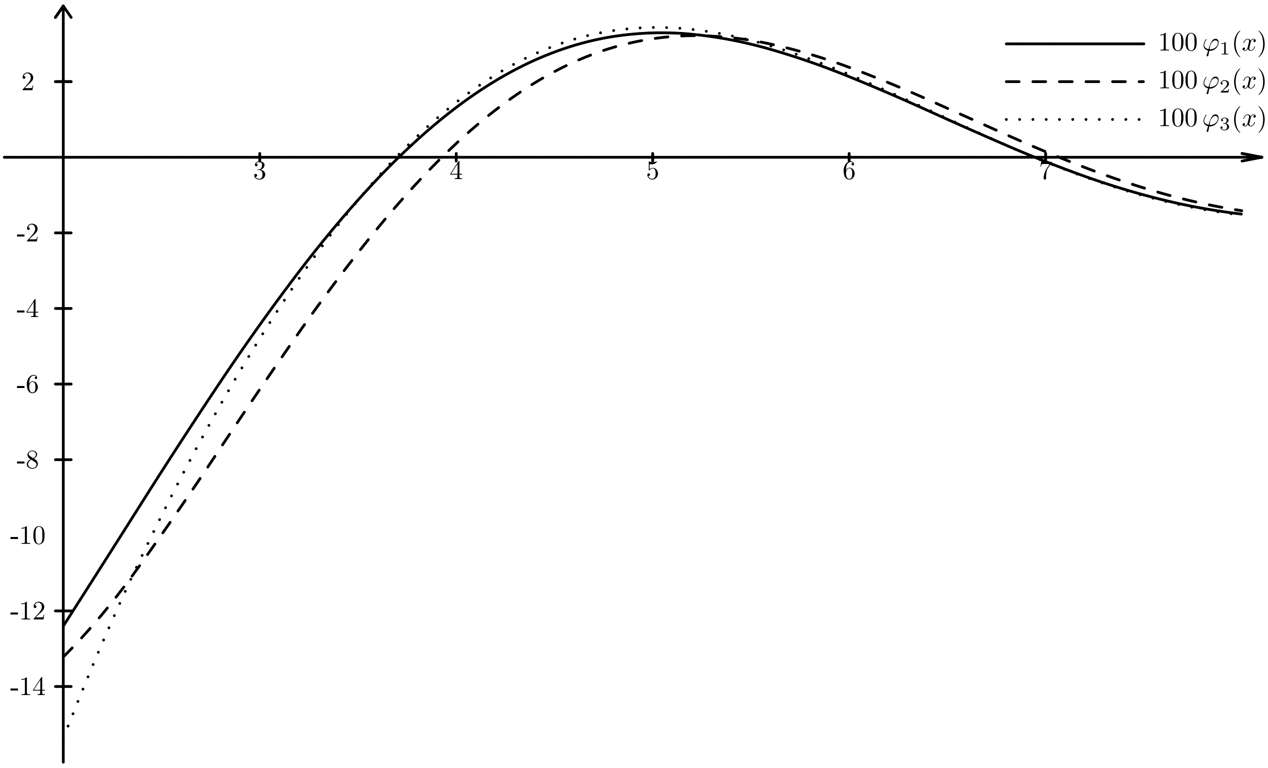
$$\int_0^x e^{-t^2}\, J_1(t)\, dt \quad \sim \quad 1 - e^{-1/4} +$$

$$+\frac{\sqrt{2}e^{-x^2}}{\sqrt{\pi x}} \left[\left(\frac{1}{2x} - \frac{137}{256x^3} + \frac{85019}{65536x^5} + \ldots \right) \cos\left(x+\frac{\pi}{4}\right) + \left(-\frac{7}{16x^2} + \frac{1773}{2048x^4} - \frac{1434089}{524288x^6} + \ldots \right) \sin\left(x+\frac{\pi}{4}\right) \right]$$

See the remark on page 140.
Let

$$\varphi_1(x) = e^{x^2} \left[\int_0^x e^{-t^2}\, J_1(t)\, dt - 1 + e^{-1/4} \right] , \quad \varphi_2(x) = \frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{1}{2x} \cos\left(x + \frac{\pi}{4}\right) ,$$

$$\varphi_3(x) = \frac{\sqrt{2}}{\sqrt{\pi x}} \left[\frac{1}{2x} \cos\left(x+\frac{\pi}{4}\right) - \frac{7}{16x^2} \sin\left(x+\frac{\pi}{4}\right) \right] .$$



III) $s = -$:

With $k \geq 1$ holds

$$a_{k+1}^{(-)} = \frac{(8k+3)a_k^{(-)} - 4a_{k-1}^{(-)}}{(2k+1)(2k+3)} , \quad b_{k+1}^{(-)} = \frac{(8k-1)b_k^{(-)} - 4b_{k-1}^{(-)}}{(2k+1)^2} .$$

$$\int_0^x \frac{e^{-t^2}\, J_0(t)\, dt}{t} =$$

$$= e^{-x^2} \left[\left(x + x^3 + \frac{7x^5}{15} + \frac{73}{525}\, x^7 - \ldots \right) J_0(x) + \left(-1 - x^2 - \frac{x^4}{3} - \frac{x^6}{25} + \ldots \right) J_1(x) \right]$$

k	$\alpha_k^{(0)}$	$\gamma_k^{(0)}$	$a_k^{(0)}$
0	1	1	1.00000 00000
1	1	1	1.00000 00000
2	7	15	0.46666 66667
3	73	525	0.13904 76190
4	991	33075	0.02996 22071
5	2327	467775	0.00497 46139
6	307991	468242775	0.00065 77592
7	6390233	91307341125	0.00006 99860
8	136790767	23283371986875	0.00000 58750
9	2646943729	7520529151760625	0.00000 03520
10	21787108711	3000691131552489375	0.00000 00726
11	-2416192168471	1449333816539852368125	-0.00000 00017
12	-37423740194359	119052420644345015953125	-0.00000 00003

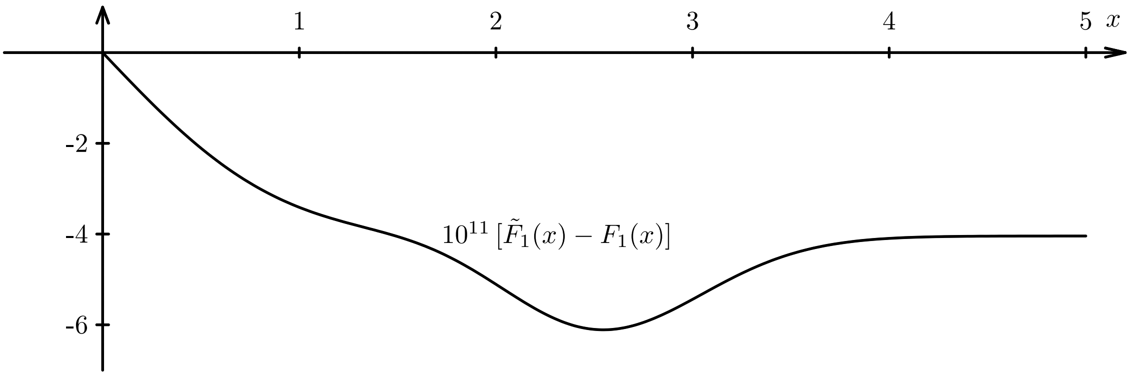
k	$\beta_k^{(-)}$	$\delta_k^{(-)}$	$b_k^{(-)}$
0	-1	1	-1.00000 00000
1	-1	1	-1.00000 00000
2	-1	3	-0.33333 33333
3	-1	25	-0.0400 00000
4	31	3675	0.00843 53741
5	1549	297675	0.00520 36617
6	16789	12006225	0.00139 83579
7	1617533	6087156075	0.00026 57289
8	54916223	1369610116875	0.00004 00962
9	24740029	4886633626875	0.00000 50628
10	7163341181	12990004898495625	0.00000 05145
11	195955925549	3706736103682486875	0.00000 00529
12	50273780536949	11111559260138868155625	0.00000 00045
13	661736445380167	1894015782978216162890625	0.00000 00003

One has $a_{10}^{(-)} \cdot a_{11}^{(-)} < 0,\ a_{41}^{(-)} \cdot a_{42}^{(-)} < 0$ and $b_3^{(-)} \cdot b_4^{(-)} < 0,\ b_{25}^{(-)} \cdot b_{26}^{(-)} < 0,\ b_{65}^{(-)} \cdot b_{66}^{(-)} < 0$.
First positive zeros of $P_-(x)$: 3.2793, 6.4739 and 9.6341.
Maxima: $P_-(3.1244) = 4757.3$ and $P_-(9.5821) = 1.2302 \cdot 10^{39}$, minimum: $P_-(6.3963) = -1.1684 \cdot 10^{17}$
First positive zeros of $Q_-(x)$: 1.8812, 4.9850 and 8.1173.
Minima: $Q_-(1.6008) = -4.7895$ and $Q_-(8.0555) = -2.7233 \cdot 10^{27}$, maximum: $Q_-(4.8841) = 5.2197 \cdot 10^9$
$P_1(x)$ and $Q_1(x)$ are growing like $P_0(x)$ and $Q_0(x)$.

Approximation:

$$F_-(x) \approx \tilde{F}_-(x) = 0.41706\,34325\,\mathrm{erf}(x) + e^{-x^2} \sum_{k=0}^{5} c_k^{(-)} x^{2k+1}$$

k	$c_k^{(-)}$	k	$c_k^{(-)}$	k	$c_k^{(-)}$
0	$2.93943\,11364 \cdot 10^{-2}$	1	$-1.23712\,57573 \cdot 10^{-3}$	2	$2.59830\,30409 \cdot 10^{-5}$
3	$-3.26773\,05781 \cdot 10^{-7}$	4	$2.73580\,96839 \cdot 10^{-9}$	5	$-1.63439\,98979 \cdot 10^{-11}$



Asymptotic expansion:

$$\int_0^x \frac{e^{-t^2}\, J_1(t)\, dt}{t} \sim \frac{\sqrt{\pi}\, e^{-1/8}}{4} \left[I_0\left(\frac{1}{8}\right) + I_1\left(\frac{1}{8}\right)\right] +$$

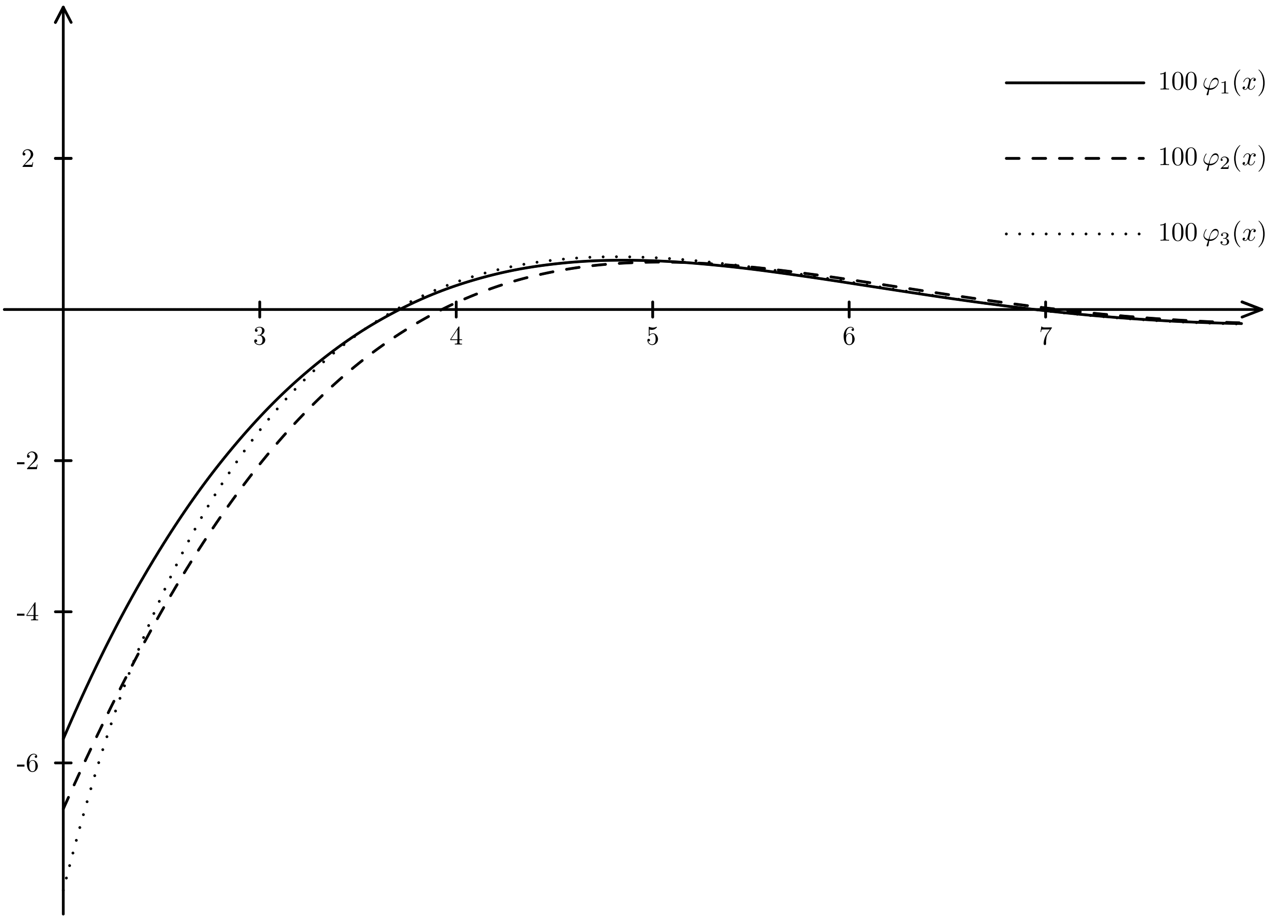
$$+\sqrt{\frac{2}{\pi x}}\, e^{-x^2} \left[\left(\frac{1}{2x^2} - \frac{201}{256x^4} + \frac{150683}{65536x^6} - \dots \right) \cos\left(x + \frac{\pi}{4}\right) + \left(-\frac{7}{16x^3} + \frac{2477}{2048x^5} - \frac{2419305}{524288x^7} + \dots \right) \sin\left(x + \frac{\pi}{4}\right) \right]$$

See the remark on page 140.

Let

$$\varphi 1-(x) = e^{x^2} \left[\int_0^x \frac{e^{-t^2}\, J_1(t)\, dt}{t} - \frac{\sqrt{\pi}\, e^{-1/8}}{4} \right] , \quad \varphi_2(x) = \frac{\sqrt{2}}{\sqrt{\pi x}} \cdot \frac{1}{2x^2} \cos\left(x + \frac{\pi}{4}\right) ,$$

$$\varphi_3(x) = \frac{\sqrt{2}}{\sqrt{\pi x}} \left[\frac{1}{2x^2} \cos\left(x + \frac{\pi}{4}\right) - \frac{7}{16x^3} \sin\left(x + \frac{\pi}{4}\right) \right] .$$



b) Integrals ($\alpha = 1$)

Let

$$\mathcal{J}_\nu \quad = \quad \int e^{-x^2}\, J_\nu(x)\, dx , \qquad \nu = 0, 1 \quad \text{and} \quad \mathcal{J}_- \quad = \quad \int \frac{e^{-x^2}\, J_\nu(x)\, dx}{x} .$$

$$\int x e^{-x^2} J_0(x)\, dx = -\frac{1}{2}\, e^{-x^2} J_0(x) - \frac{1}{2}\, \mathcal{J}_1$$

$$\int x e^{-x^2} J_1(x)\, dx = -\frac{1}{2}\, e^{-x^2} J_1(x) + \frac{1}{2}\, \mathcal{J}_0 - \frac{1}{2}\, \mathcal{J}_-$$

$$\int x^2 e^{-x^2} J_0(x)\, dx = \frac{e^{-x^2}}{4} \left[-2x J_0(x) + J_1(x) \right] + \frac{1}{4}\, \mathcal{J}_0 + \frac{1}{4}\, \mathcal{J}_-$$

$$\int x^2 e^{-x^2} J_1(x)\, dx = -\frac{e^{-x^2}}{4} \left[J_0(x) + 2x J_1(x) \right] - \frac{1}{4}\, \mathcal{J}_1$$

$$\int x^3 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{8}\,\left[-(4x^2+3)J_0(x)+2xJ_1(x)\right]-\frac{3}{8}\,\mathcal{J}_1$$

$$\int x^3 e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{8}\,\left[2xJ_0(x)+(4x^2+1)J_1(x)\right]+\frac{3}{8}\,\mathcal{J}_0-\frac{1}{8}\,\mathcal{J}_-$$

$$\int x^4 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{16}\,\left[-(8x^3+10x)J_0(x)+(4x^2+7)J_1(x)\right]+\frac{3}{16}\,\mathcal{J}_0+\frac{7}{16}\,\mathcal{J}_-$$

$$\int x^4 e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{16}\,\left[(4x^2+7)J_0(x)+(8x^3+6x)J_1(x)\right]-\frac{7}{16}\,\mathcal{J}_1$$

$$\int x^5 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{32}\,\left[-(16x^4+28x^2+17)J_0(x)+(8x^3+22x)J_1(x)\right]-\frac{17}{32}\,\mathcal{J}_1$$

$$\int x^5 e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{32}\,\left[(8x^3+22x)J_0(x)+(16x^4+20x^2-1)J_1(x)\right]+\frac{21}{32}\,\mathcal{J}_0+\frac{1}{32}\,\mathcal{J}_-$$

$$\int x^6 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{64}\,\left[-(32x^5+72x^3+78x)J_0(x)+(16x^4+60x^2+69)J_1(x)\right]+\frac{9}{64}\,\mathcal{J}_0+\frac{69}{64}\,\mathcal{J}_-$$

$$\int x^6 e^{-x^2} J_1(x)\,dx = -\frac{e^{-x^2}}{64}\,\left[(16x^4+60x^2+73)J_0(x)+(32x^5+56x^3+26x)J_1(x)\right]-\frac{73}{64}\,\mathcal{J}_1$$

$$\int x^7 e^{-x^2} J_0(x)\,dx = \frac{e^{-x^2}}{128}\,\left[-(64x^6+176x^4+276x^2+131)J_0(x)+(32x^5+152x^3+290x)J_1(x)\right]-\frac{131}{128}\,\mathcal{J}_1$$

$$\int x^7 e^{-x^2} J_1(x)\,dx =$$

$$= -\frac{e^{-x^2}}{128}\,\left[(32xu+152x^3+298x)J_0(x)+(64x^6+144x^4+140x^2-79)J_1(x)\right]+\frac{219}{128}\,\mathcal{J}_0+\frac{79}{128}\,\mathcal{J}_1$$

$$\int x^8 e^{-x^2} J_0(x)\,dx =$$

$$= \frac{e^{-x^2}}{256}\,\left[-(128\,x^7+416\,x^5+856\,x^3+794\,x)J_0(x)+(64\,x^6+368\,x^4+980\,x^2+887)J_1(x)\right]-\frac{93}{256}\,\mathcal{J}_0+\frac{887}{256}\,\mathcal{J}_-$$

$$\int x^8 e^{-x^2} J_1(x)\,dx =$$

$$= -\frac{e^{-x^2}}{256}\,\left[(64\,x^6+368\,x^4+996\,x^2+1007)J_0(x)+(128\,x^7+352\,x^5+520\,x^3+22\,x)J_1(x)\right]-\frac{1007}{256}\,\mathcal{J}_1$$

Recurrence relations:

$$\int x^{2n+2}\,e^{-x^2}\,J_0(x)\,dx =$$

$$= \frac{x^{2n}\,e^{-x^2}}{4}\,[J_1(x)-2xJ_0(x)]+\frac{4n+1}{4}\int x^{2n}\,e^{-x^2}\,J_0(x)\,dx-\frac{2n-1}{4}\int x^{2n-1}\,e^{-x^2}\,J_1(x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_0(x)\,dx =$$

$$= \frac{x^{2n-1}\,e^{-x^2}}{4}\,[J_1(x)-2\,x\,J_0(x)]+\frac{4n-1}{4}\int x^{2n-1}\,e^{-x^2}\,J_0(x)\,dx-\frac{n-1}{2}\int x^{2n-2}\,e^{-x^2}\,J_1(x)\,dx$$

$$\int x^{2n+2}\,e^{-x^2}\,J_1(x)\,dx = -\frac{x^{2n+1}\,e^{-x^2}}{2}\,J_1(x)+\frac{1}{2}\int x^{2n+1}\,e^{-x^2}\,J_0(x)\,dx+n\int x^{2n}\,e^{-x^2}\,J_1(x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_1(x)\,dx = -\frac{x^{2n}\,e^{-x^2}}{2}\,J_1(x)+\frac{1}{2}\int x^{2n}\,e^{-x^2}\,J_0(x)\,dx+\frac{2n-1}{2}\int x^{2n-1}\,e^{-x^2}\,J_1(x)\,dx$$

Otherwise:

$$\int x^{2n+2}\,e^{-x^2}\,J_0(x)\,dx = \frac{x^{2n-1}\,e^{-x^2}}{4}\,\left[x\,J_1(x)-(2\,x^2-2n+1)\,J_0(x)\right]+$$

$$+\frac{8n-1}{4}\int x^{2n}\,e^{-x^2}\,J_0(x)\,dx-\frac{(2n-1)^2}{4}\int x^{2n-2}\,e^{-x^2}\,J_0(x)\,dx$$

$$\int x^{2n+2}\,e^{-x^2}\,J_1(x)\,dx=-\frac{x^{2n-1}\,e^{-x^2}}{4}\,\left[x\,J_0(x)+(2\,x^2-2n)\,J_1(x)\right]+$$

$$+\frac{8n-1}{4}\int x^{2n}\,e^{-x^2}\,J_1(x)\,dx-n(n-1)\int x^{2n-2}\,e^{-x^2}\,J_1(x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_0(x)\,dx=-\frac{x^{2n-2}\,e^{-x^2}}{4}\,\left[2\,(x^2-n+1)\,J_0(x)-x\,J_1(x)\right]+$$

$$+\frac{8n-5}{4}\int x^{2n-1}\,e^{-x^2}\,J_0(x)\,dx-(n-1)^2\int x^{2n-3}\,e^{-x^2}\,J_0(x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_1(x)\,dx=-\frac{x^{2n-2}\,e^{-x^2}}{4}\,\left[x\,J_0(x)+(2\,x^2-2n+1)\,J_1(x)\right]+$$

$$+\frac{8n-5}{4}\int x^{2n-1}\,e^{-x^2}\,J_1(x)\,dx-\frac{(2n-1)(2n-3)}{4}\int x^{2n-3}\,e^{-x^2}\,J_1(x)\,dx$$

c) General Case $\alpha \neq 1$, Basic Integrals

Let

$$F_\nu(x;\alpha)=\int_0^x e^{-t^2}\,J_\nu(\alpha t)\,dt=\frac{\nu}{\alpha}+e^{-x^2}\,[P_\nu(x;\alpha)\,J_0(\alpha x)+Q_\nu(x;\alpha)\,J_1(\alpha x)]\,,\quad \nu=0,\,1,$$

with

$$P_\nu(x;\alpha)=\sum_{k=0}^{\infty} a_k^{(\nu;\alpha)}\,x^{2k+1-\nu}\quad\text{and}\quad Q_\nu(x;\alpha)=\sum_{k=0}^{\infty} b_k^{(\nu;\alpha)}\,x^{2k+\nu}\,.$$

Furthermore, let

$$F_-(x;\alpha)=\int_0^x \frac{e^{-t^2}\,J_1(\alpha t)\,dt}{t}=e^{-x^2}\,[P_-(x;\alpha)\,J_0(\alpha x)+Q_-(x;\alpha)\,J_1(\alpha x)]$$

with

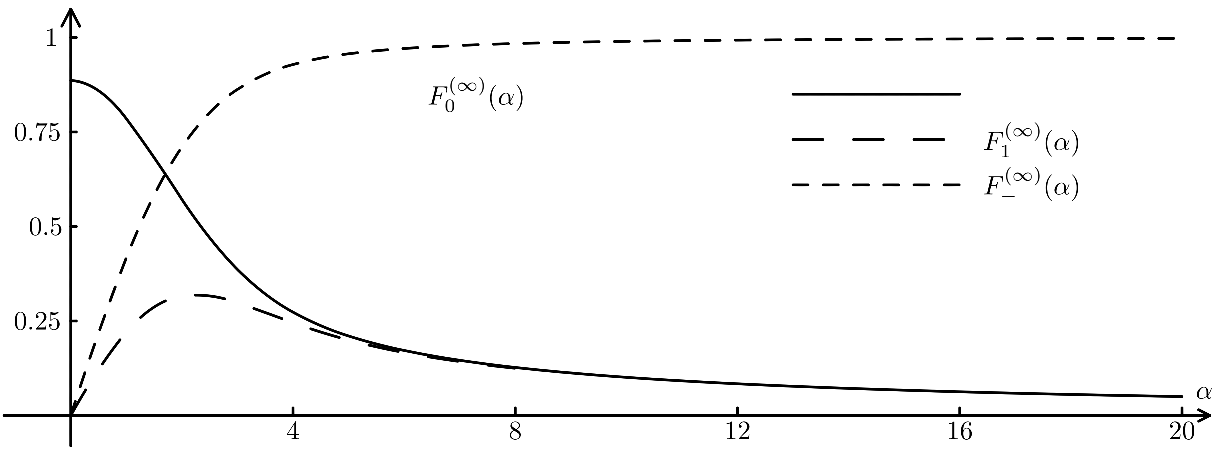
$$P_-(x;\alpha)=\sum_{k=0}^{\infty} a_k^{(-;\alpha)}\,x^{2k+1-\nu}\quad\text{and}\quad Q_\nu(x;\alpha)=\sum_{k=0}^{\infty} b_k^{(-;\alpha)}\,x^{2k+\nu}\,.$$

Some improper integrals:

$$F_0^{(\infty)}(\alpha)\;=\;\int_0^\infty e^{-x^2}\,J_0(\alpha x)\,dx=\frac{\sqrt{\pi}}{2}\,e^{-\alpha^2/8}\,I_0\left(\frac{\alpha^2}{8}\right)$$

$$F_1^{(\infty)}(\alpha)\;=\;\int_0^\infty e^{-x^2}\,J_1(\alpha x)\,dx=\frac{1-e^{-\alpha^2/4}}{\alpha}$$

$$F_-^{(\infty)}(\alpha)\;=\;\int_0^\infty \frac{e^{-x^2}\,J_1(\alpha x)\,dx}{x}\;=\;\frac{\sqrt{\pi}}{4}\,e^{-\alpha^2/8}\,\left[I_0\left(\frac{\alpha^2}{8}\right)+I_1\left(\frac{\alpha^2}{8}\right)\right]\quad\xrightarrow[\alpha\to\infty]{}\quad 1$$



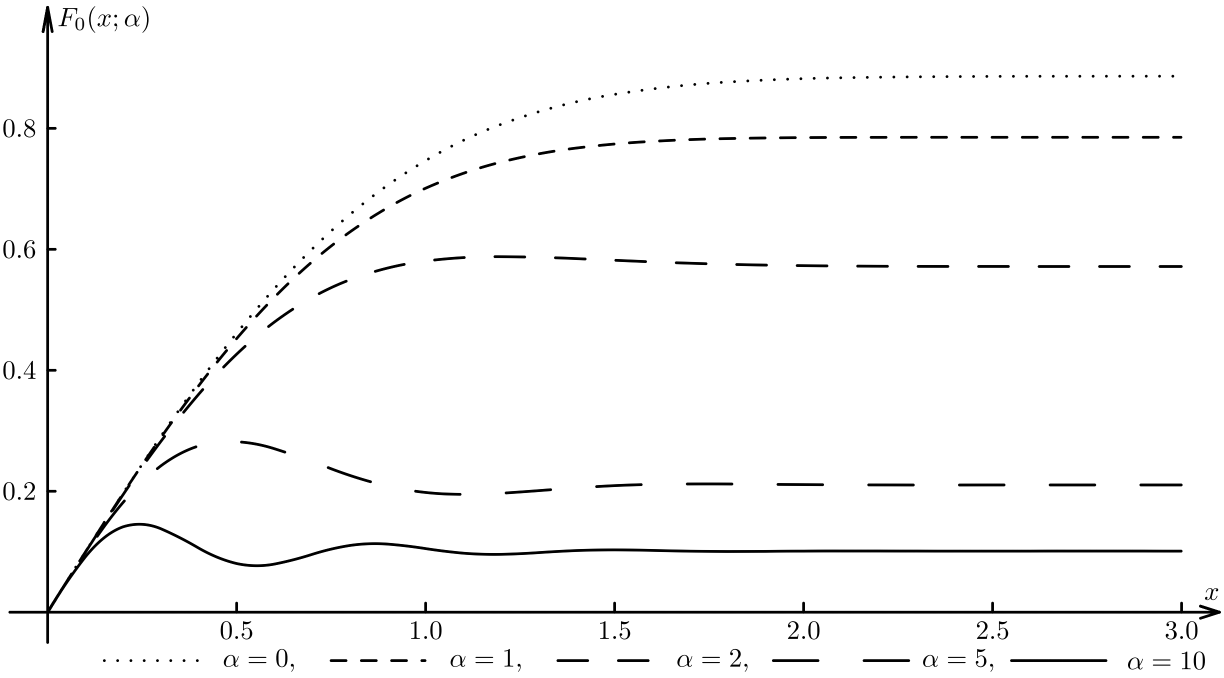
$F_1^{(\infty)}(2.2418) = 0.3191$ is the maximal value of this function.

The formulas from page 140 can be generalized:

$$\left|\int_x^\infty e^{-t^2}\, J_0(\alpha t)\, dt\right| < \frac{0.4}{\sqrt{\alpha}}\,\Gamma\left(\frac{1}{4}, x^2\right)\ , \quad \left|\int_x^\infty e^{-t^2}\, J_1(\alpha t)\, dt\right| < \frac{0.42}{\sqrt{\alpha}}\,\Gamma\left(\frac{1}{4}, x^2\right)\ ,$$

$$\left|\int_x^\infty \frac{e^{-t^2}\, J_1(\alpha t)\, dt}{t}\right| < \frac{0.42}{\sqrt{\alpha}}\,\Gamma\left(-\frac{1}{4}, x^2\right)\ .$$

I) $F_0(x;\alpha)$:



$$F_0(x;\alpha) = x - \frac{(\alpha^2+4)\,x^3}{12} - \frac{(\alpha^4+16\,\alpha^2+32)\,x^5}{320} - \frac{(\alpha^6+36\,\alpha^4+288\,\alpha^2+384)\,x^7}{}\frac{()\,x^7}{16128} +$$

$$+\frac{(\alpha^8+64\,\alpha^6+1152\,\alpha^4+6144\,\alpha^2+6144)\,x^9}{1327104} -$$

$$-\frac{(\alpha^{10}+100\,\alpha^8+3200\,\alpha^6+38400\,\alpha^4+153600\,\alpha^2+122880)\,x^{11}}{162201600} +$$

$$+\frac{(\alpha^{12}+144\,\alpha^{10}+7200\,\alpha^8+153600\,\alpha^6+1382400\,\alpha^4+4423680\,\alpha^2+2949120)\,x^{13}}{27603763200} - \dots$$

With the formulas defined on page 149 holds

$$F_0(x;\alpha) =$$

$$= e^{-x^2}\left[\left(x + \frac{2-\alpha^2}{3}\,x^3 + \frac{\alpha^4-14\alpha^2+12}{45}\,x^5 + \frac{-\alpha^6+34\alpha^4-232\alpha^2+120}{1575}\,x^7 + \dots\right) J_0(\alpha x) + \right.$$

$$\left. + \left(\alpha x^2 + \frac{8\alpha-\alpha^3}{9}\,x^4 + \frac{\alpha^5-24\alpha^3+92\alpha}{225}\,x^6 \dots\right) J_1(\alpha x)\right]$$

Recurrence relations:

$$a_{k+1}^{(0;\alpha)} = \frac{(4k+2-\alpha^2)a_k^{(0;\alpha)} - 2\alpha b_k^{(0;\alpha)}}{(2k+1)(2k+3)}\ , \quad b_{k+1}^{(0;\alpha)} = \frac{2b_k^{(0;\alpha)} + \alpha a_k^{(0;\alpha)}}{2k+1}$$

or

$$a_{k+1}^{(0;\alpha)} = \frac{(8k+4-\alpha^2)a_k^{(0;\alpha)} - 4\,a_{k-1}^{(0;\alpha)}}{(2k+1)(2k+3)}\ , \quad b_{k+1}^{(0;\alpha)} = \frac{(8k-\alpha^2)b_k^{(0;\alpha)} - 4b_{k-1}^{(0;\alpha)}}{2k+1}$$

This sums may be used for computation, for instance, by this way: One should define some ε and calculate

$$P_0(x;\alpha) \approx P_{(0;N)} = \sum_{k=0}^{N} a_k^{(0;\alpha)}\, x^{2k+1}$$

witht $N = N(x;\alpha)$ until

$$\left|a_{N-2}^{(0;\alpha)}\, x^{2N-3}\right| + \left|a_{N-1}^{(0;\alpha)}\, x^{2N-1}\right| + \left|a_N^{(0;\alpha)}\, x^{2N+1}\right| \ < \varepsilon\,(1 + |P_{(0;N)}|)\ .$$

The use of the simple condition $|a_N^{(0;\alpha)}\, x^{2N+1}| < \varepsilon$ could cause an error because $a_N^{(0;\alpha)} = 0$ (or $a_N^{(0;\alpha)} \approx 0$) is possible while $|a_{N+1}^{(0;\alpha)}\, x^{2N+3}| > \varepsilon$ holds.
Approximation (α not too large):

$$\hat{F}_0(x;\alpha) = \frac{\sqrt{\pi}}{2}\, e^{-\alpha^2/8}\, I_0\left(\frac{\alpha^2}{8}\right)\, \mathrm{erf}(x) + e^{-x^2} \sum_{k=0}^{5} c_k^{(0;\alpha)}\, x^{2k+1}$$

$$c_0^{(0;\alpha)} = -\frac{77\,\alpha^{12}}{1006632960} + \frac{21\,\alpha^{10}}{10485760} - \frac{35\,\alpha^8}{786432} + \frac{5\,\alpha^6}{6144} - \frac{3\,\alpha^4}{256} + \frac{\alpha^2}{8} =$$
$$= -0.00000\,00765\,\alpha^{12} + 0.00000\,20027\,\alpha^{10} - 0.00004\,45048\,\alpha^8 + 0.00081\,38021\,\alpha^6 -$$
$$-0.01171\,87500\,\alpha^4 + 0.12500\,00000\,\alpha^2$$

$$c_1^{(0;\alpha)} = -\frac{77\,\alpha^{12}}{1509949440} + \frac{7\,\alpha^{10}}{5242880} - \frac{35\,\alpha^8}{1179648} + \frac{5\,\alpha^6}{9216} - \frac{\alpha^4}{128} =$$
$$= -0.00000\,00510\,\alpha^{12} + 0.00000\,13351\,\alpha^{10} - 0.00002\,96699\,\alpha^8 + 0.00054\,25347\,\alpha^6 - 0.00781\,25000\,\alpha^4$$

$$c_2^{(0;\alpha)} = -\frac{77\,\alpha^{12}}{3774873600} + \frac{7\,\alpha^{10}}{13107200} - \frac{7\,\alpha^8}{589824} + \frac{\alpha^6}{4608} =$$
$$= -0.00000\,00204\,\alpha^{12} + 0.00000\,05341\,\alpha^{10} - 0.00001\,18679\,\alpha^8 + 0.00021\,70139\,\alpha^6$$

$$c_3^{(0;\alpha)} = -\frac{11\,\alpha^{12}}{1887436800} + \frac{\alpha^{10}}{6553600} - \frac{\alpha^8}{294912} = -0.00000\,00058\,\alpha^{12} + 0.00000\,01526\,\alpha^{10} - 0.00000\,33908\,\alpha^8$$

$$c_4^{(0;\alpha)} = -\frac{11\,\alpha^{12}}{8493465600} + \frac{\alpha^{10}}{29491200} = -0.00000\,00013\,\alpha^{12} + 0.00000\,00339\,\alpha^{10}$$

$$c_5^{(0;\alpha)} = -\frac{\alpha^{12}}{4246732800} = -0.00000\,00002\,\alpha^{12}$$

In any case holds $\hat{F}_0(0;\alpha) = F_0(0;\alpha) = 0$ and $\lim_{x\to\infty} \hat{F}_0(x;\alpha) = \lim_{x\to\infty} F_0(x;\alpha)$. $M(\alpha)$ denotes the maximal value of the difference between these two fundctions:

$$M(\alpha) \ = \ \max\left\{ |\hat{F}_0(x;\alpha) - F_0(x;\alpha)|\,;0 \le x < \infty \right\} = |\hat{F}_0(x^*;\alpha) - F_0(x^*;\alpha)|\ .$$

α	0.5	1	1.5	2	2.5	3
x^*	1.644	1.642	1.639	1.636	1.632	1.627
$-\log_{10} M(\alpha)$	12.88	8.67	6.23	4.50	3.17	2.09

In the case of small values of $|x|$ and $\alpha >> 1$ the following approximation may be used ($\Phi(x)$ defined as on page 9):

$$F_0(x;\alpha) \approx \hat{F}_0(x;\alpha) = \sigma^{(0)}(\alpha)\,\Phi(\alpha x) + \left[\sum_{k=0}^{7} \varphi_k^{(0)}(\alpha)\, x^{2k+1}\right] J_0(\alpha x) + \left[\sum_{k=0}^{7} \psi_k^{(0)}(\alpha)\, x^{2k+2}\right] J_1(\alpha x)$$

$$\sigma^{(0)}(\alpha) = 1 + \frac{1}{\alpha^2} + \frac{9}{2\,\alpha^4} + \frac{75}{2\,\alpha^6} + \frac{3675}{8\,\alpha^8} + \frac{59535}{8\,\alpha^{10}} + \frac{2401245}{16\,\alpha^{12}} + \frac{57972915}{16\,\alpha^{14}} + \frac{13043905875}{128\,\alpha^{16}} + \frac{418854310875}{128\,\alpha^{18}} +$$
$$+ \frac{30241281245175}{256\,\alpha^{20}} + \frac{1212400457192925}{256\,\alpha^{22}} + \frac{213786613951685775}{1024\,\alpha^{24}} + \frac{10278202593831046875}{1024\,\alpha^{26}} +$$
$$+ \frac{1070401384414690453125}{2048\,\alpha^{28}} + \frac{60013837619516978071875}{2048\,\alpha^{30}} =$$

$$= 1+0.04\left(\frac{5}{\alpha}\right)^{2}+0.0072\left(\frac{5}{\alpha}\right)^{4}+0.0024\left(\frac{5}{\alpha}\right)^{6}+0.00117\,6\left(\frac{5}{\alpha}\right)^{8}+0.00076\,2048\left(\frac{5}{\alpha}\right)^{10}+0.00061\,47187\left(\frac{5}{\alpha}\right)^{12}+$$

$$+0.00059\,36426\left(\frac{5}{\alpha}\right)^{14}+0.00066\,78479808\left(\frac{5}{\alpha}\right)^{16}+0.00085\,78136287\left(\frac{5}{\alpha}\right)^{-18}+0.00123\,86829\left(\frac{5}{\alpha}\right)^{20}+$$

$$+0.00198\,63969\left(\frac{5}{\alpha}\right)^{22}+0.00350\,26799\left(\frac{5}{\alpha}\right)^{24}+0.00673\,5922852\left(\frac{5}{\alpha}\right)^{26}+0.01402\,996503\left(\frac{5}{\alpha}\right)^{28}+$$

$$+0.03146\,45349\left(\frac{5}{\alpha}\right)^{30}$$

Let

$$\varphi_k^{(0)}(\alpha)=\sum_{j=0}^{7} s_j^{(k,0)}\left(\frac{5}{\alpha}\right)^{2k} \quad \text{and} \quad \psi_k^{(0)}(\alpha)=\sum_{j=0}^{7} t_j^{(k,0)}\left(\frac{5}{\alpha}\right)^{2k+1}$$

$s_j^{(k,0)}$:

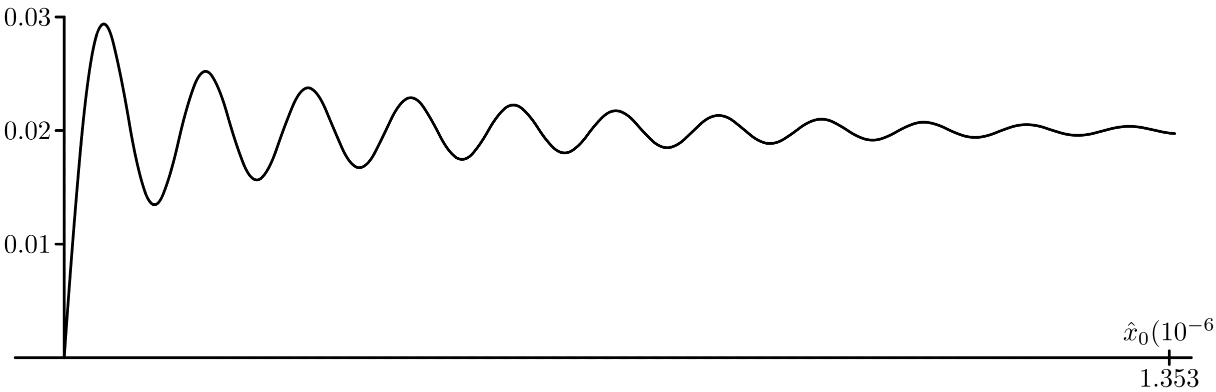
j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	1	0	0	0	0	0	0	0
1	0	0.0600000000	-0.0333333333	0.0116666667	-0.0030000000	0.0006111111	-0.0001031746	0.0000148810
2	0	0.0200000000	-0.0163333333	0.0075600000	-0.0024200000	0.0005901587	-0.0001160714	0.0000191138
3	0	0.0098000000	-0.0105840000	0.0060984000	-0.0023370286	0.0006639286	-0.0001490873	0.0000276003
4	0	0.0063504000	-0.0085377600	0.0058893120	-0.0026291571	0.0008527794	-0.0002152821	0.0000442608
5	0	0.0051226560	-0.0082450368	0.0066254760	-0.0033770063	0.0012314134	-0.0003452341	0.0000780465
6	0	0.0049470221	-0.0092756664	0.0085100558	-0.0048763971	0.0019747393	-0.0006087629	0.0001500895
7	0	0.0055653998	-0.0119140782	0.0122885206	-0.0078199677	0.0034821237	-0.0011706978	0.0003126149

$t_j^{(k,0)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	-0.2000000000	0.1000000000	-0.0333333333	0.0083333333	-0.0016666667	0.0002777778	-0.0000396825	0.0000049603
1	-0.0360000000	0.0333333333	-0.0163333333	0.0054000000	-0.0013444444	0.0002682540	-0.0000446429	0.0000063713
2	-0.0120000000	0.0163333333	-0.0105840000	0.0043560000	-0.0012983492	0.0003017857	-0.0000573413	0.0000092001
3	-0.0058800000	0.0105840000	-0.0085377600	0.0042066514	-0.0014606429	0.0003876270	-0.0000828008	0.0000147536
4	-0.0038102400	0.0085377600	-0.0082450368	0.0047324829	-0.0018761146	0.0005597334	-0.0001327824	0.0000260155
5	-0.0030735936	0.0082450368	-0.0092756664	0.0060786113	-0.0027091095	0.0008976088	-0.0002341396	0.0000500298
6	-0.0029682132	0.0092756664	-0.0119140782	0.0087775147	-0.0043444265	0.0015827835	-0.0004502684	0.0001042050
7	-0.0033392399	0.0119140782	-0.0172039289	0.0140759418	-0.0076606720	0.0030438144	-0.0009378448	0.0002336970

With $\hat{x}(\varepsilon,\alpha)$ defined by the condition: Holds $|\hat{F}_0(x;\alpha)-F_0(x;\alpha)|\le\varepsilon$ if $0\le x\le\hat{x}(\varepsilon,\alpha)$. Some values of $\hat{x}(\varepsilon,\alpha)$ are shown in the following table:

ε	$\alpha=5$	$\alpha=6$	$\alpha=8$	$\alpha=10$	$\alpha=15$	$\alpha=20$	$\alpha=30$	$\alpha=50$	$\alpha=75$	$\alpha=100$
10^{-3}	0.161	1.532	1.635	1.683	1.784	1.929	1.938	2.043	2.194	2.156
10^{-6}	0.0159	0.0737	1.555	1.254	1.193	1.289	1.299	1.353	1.406	1.434
10^{-9}	0.00159	0.00734	0.945	0.796	1.044	0.827	0.871	0.909	0.943	0.962



This function is approximated by $\hat{F}_0(x;50)$ with an accuracy of 10^{-6}.

Asymptotic expansion:

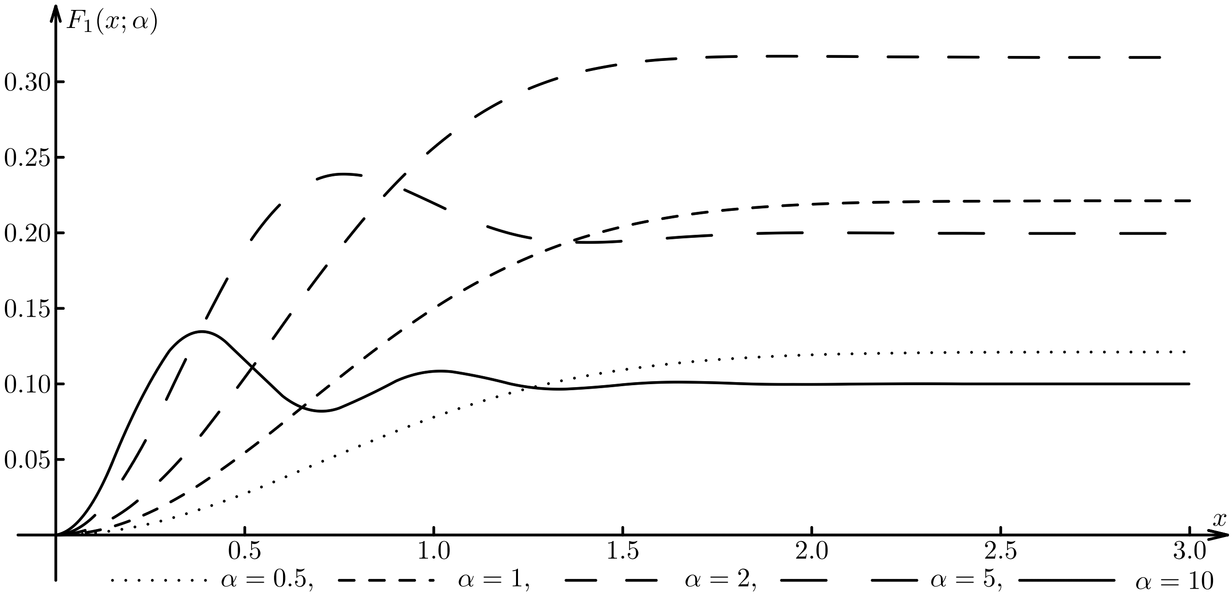
$$F_0(x;\alpha) \quad \sim \quad \frac{\sqrt{\pi}}{2}\,e^{-\alpha^2/8}\,I_0\left(\frac{\alpha^2}{8}\right)+$$

$$+\frac{2\,e^{-x^2}}{\sqrt{\pi\alpha x}}\left[\left(\frac{1}{2x}-\frac{32\alpha^4+120\alpha^2-15}{256x^3}-\frac{2048\alpha^8+32256\alpha^6+60480\alpha^4-5040\alpha^2-4725}{65536x^5}+\ldots\right)\cos\left(x+\frac{\pi}{4}\right)+\right.$$

$$+\left(-\frac{4\alpha^2+3}{16x^2}+\frac{128\alpha^6+1120\alpha^4+420\alpha^2+105}{2048x^4}-\right.$$

$$\left.\left.\frac{8192\alpha^{10}+202752\alpha^8+887040\alpha^6+221760\alpha^4+41580\alpha^2+72765}{524288x^6}+\ldots\right)\sin\left(x+\frac{\pi}{4}\right)\right]$$

II) $s=1$:



$$F_1(x;\alpha)=\frac{\alpha x^2}{4}-\frac{(\alpha^2+8)\alpha x^4}{64}+\frac{(\alpha^4+24\,\alpha^2+96)\alpha x^6}{2304}-\frac{(\alpha^6+48\,\alpha^4+576\,\alpha^2+1536)\alpha x^8}{147456}+$$

$$+\frac{(\alpha^8+80\,\alpha^6+1920\,\alpha^4+15360\,\alpha^2+30720)\alpha x^{10}}{14745600}-$$

$$-\frac{(\alpha^{10}+120\,\alpha^8+4800\,\alpha^6+76800\,\alpha^4+460800\,\alpha^2+737280)\alpha x^{12}}{2123366400}+$$

$$+\frac{(\alpha^{12}+168\,\alpha^{10}+10080\,\alpha^8+268800\,\alpha^6+3225600\,\alpha^4+15482880\,\alpha^2+20643840)\alpha x^{14}}{416179814400}-\ldots$$

Otherwise (see page 149):

$$F_1(x;\alpha)=\int_0^x e^{-t^2}\,J_1(\alpha t)\,dt=$$

$$=\frac{1}{\alpha}+\frac{e^{-x^2}}{\alpha}\left[\left(-1-x^2+\frac{\alpha^2-4}{8}\,x^4-\frac{\alpha^4-20\,\alpha^2+32}{192}\,x^6+\frac{\alpha^6-44\,\alpha^4+416\,\alpha^2-384}{9216}\,x^8+\ldots\right)J_0(\alpha x)+\right.$$

$$\left.+\left(-\frac{\alpha}{2}\,x^3+\frac{\alpha^3-12\,\alpha}{32}\,x^5-\frac{\alpha^5-32\,\alpha^3+176\,\alpha}{1152}\,x^7+\frac{\alpha^7-60\,\alpha^5+928\,\alpha^3-3200\,\alpha}{73728}\,x^9+\ldots\right)J_1(\alpha x)\right]$$

Recurrence relations:

$$a_{k+1}^{(1;\alpha)}=\frac{2a_k^{(1;\alpha)}-\alpha b_k^{(1;\alpha)}}{2k+2}\;,\quad b_{k+1}^{(1;\alpha)}=\frac{2\alpha a_k^{(1;\alpha)}+(4k+4-\alpha^2)b_k^{(1;\alpha)}}{(2k+2)^2}$$

or

$$a_{k+1}^{(1;\alpha)}=\frac{(8k-\alpha^2)a_k^{(1;\alpha)}-4a_{k-1}^{(1;\alpha)}}{4k(k+1)}\;,\quad b_{k+1}^{(1;\alpha)}=\frac{(8k+4-\alpha^2)b_k^{(1;\alpha)}-4b_{k-1}k^{(1;\alpha)}}{(2k+2)^2}$$

About using this formula see page 151.

Approximation:

$$F_1(x;\alpha) = c_0^{(1;\alpha)} \left(1 - e^{-x^2}\right) + e^{-x^2} \sum_{k=1}^{5} c_k^{(1;\alpha)}\, x^{2k}$$

$$c_0^{(1;\alpha)} = -0.00000\,03391\,\alpha^{11} + 0.00000\,81380\,\alpha^9 - 0.00016\,27604\,\alpha^7 + 0.00260\,41667\,\alpha^5 - 0.03125\,\alpha^3 + 0.25\,\alpha$$

$$c_1^{(1;\alpha)} = 0.00000\,03391\,\alpha^{11} - 0.00000\,81380\,\alpha^9 + 0.00016\,27604\,\alpha^7 - 0.00260\,41667\,\alpha^5 + 0.03125\,\alpha^3$$

$$c_2^{(1;\alpha)} = 0.00000\,01695\,\alpha^{11} - 0.00000\,40690\,\alpha^9 + 0.00008\,13802\,\alpha^7 - 0.00130\,20833\,\alpha^5$$

$$c_3^{(1;\alpha)} = 0.00000\,00565\,\alpha^{11} - 0.00000\,13563\,\alpha^9 + 0.00002\,71267\,\alpha^7$$

$$c_4^{(1;\alpha)} = 0.00000\,00141\,\alpha^{11} - 0.00000\,03391\,\alpha^9\ , \quad c_5^{(1;\alpha)} = 0.00000\,00028\,\alpha^{11}$$

In the case of small values of $|x|$ and $\alpha >> 1$ the following approximation may be used:

$$F_1(x;\alpha) = \left[\sum_{k=1}^{7} \varphi_k^{(1)}(\alpha)\, x^{2k}\right] J_0(\alpha x) + \left[\sum_{k=0}^{7} \psi_k^{(1)}(\alpha)\, x^{2k+1}\right] J_1(\alpha x)$$

Let

$$\varphi_k^{(1)}(\alpha) = \sum_{j=0}^{7} s_j^{(k,1)} \left(\frac{5}{\alpha}\right)^{2k+1} \quad \text{and} \quad \psi_k^{(1)}(\alpha) = \sum_{j=0}^{7} t_j^{(k,1)} \left(\frac{5}{\alpha}\right)^{2k+2}$$

$s_j^{(k,1)}$:

j	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	0.2000000000	-0.1000000000	0.0333333333	-0.0083333333	0.0016666667	-.0002777778	0.0000396825
1	0.0320000000	-0.0320000000	0.0160000000	-0.0053333333	0.0013333333	-0.0002666667	0.0000444444
2	0.0102400000	-0.0153600000	0.0102400000	-0.0042666667	0.0012800000	-0.0002986667	0.0000568889
3	0.0049152000	-0.0098304000	0.0081920000	-0.0040960000	0.0014336000	-0.0003822933	0.0000819200
4	0.0031457280	-0.0078643200	0.0078643200	-0.0045875200	0.0018350080	-0.0005505024	0.0001310720
5	0.0025165824	-0.0075497472	0.0088080384	-0.0058720256	0.0026424115	-0.0008808038	0.0002306867
6	0.0024159191	-0.0084557169	0.0112742892	-0.0084557169	0.0042278584	-0.0015502148	0.0004429185
7	0.0027058294	-0.0108233176	0.0162349764	-0.0135291470	0.0074410308	-0.0029764123	0.0009212705

$t_j^{(k,1)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	-0.0800000000	0.0800000000	-0.0400000000	0.0133333333	-0.0033333333	0.0006666667	-0.0001111111	0.0000158730
1	-0.0128000000	0.0256000000	-0.0192000000	0.0085333333	-0.0026666667	0.0006400000	-0.0001244444	0.0000203175
2	-0.0040960000	0.0122880000	-0.0122880000	0.0068266667	-0.0025600000	0.0007168000	-0.0001592889	0.0000292571
3	-0.0019660800	0.0078643200	-0.0098304000	0.0065536000	-0.0028672000	0.0009175040	-0.0002293760	0.0000468114
4	-0.0012582912	0.0062914560	-0.0094371840	0.0073400320	-0.0036700160	0.0013212058	-0.0003670016	0.0000823881
5	-0.0010066330	0.0060397978	-0.0105696461	0.0093952410	-0.0052848230	0.0021139292	-0.0006459228	0.0001581852
6	-0.0009663676	0.0067645735	-0.0135291470	0.0135291470	-0.0084557169	0.0037205154	-0.0012401718	0.0003290252
7	-0.0010823318	0.0086586541	-0.0194819717	0.0216466352	-0.0148820617	0.0071433896	-0.0025795574	0.0007370164

Asymptotic expansion:

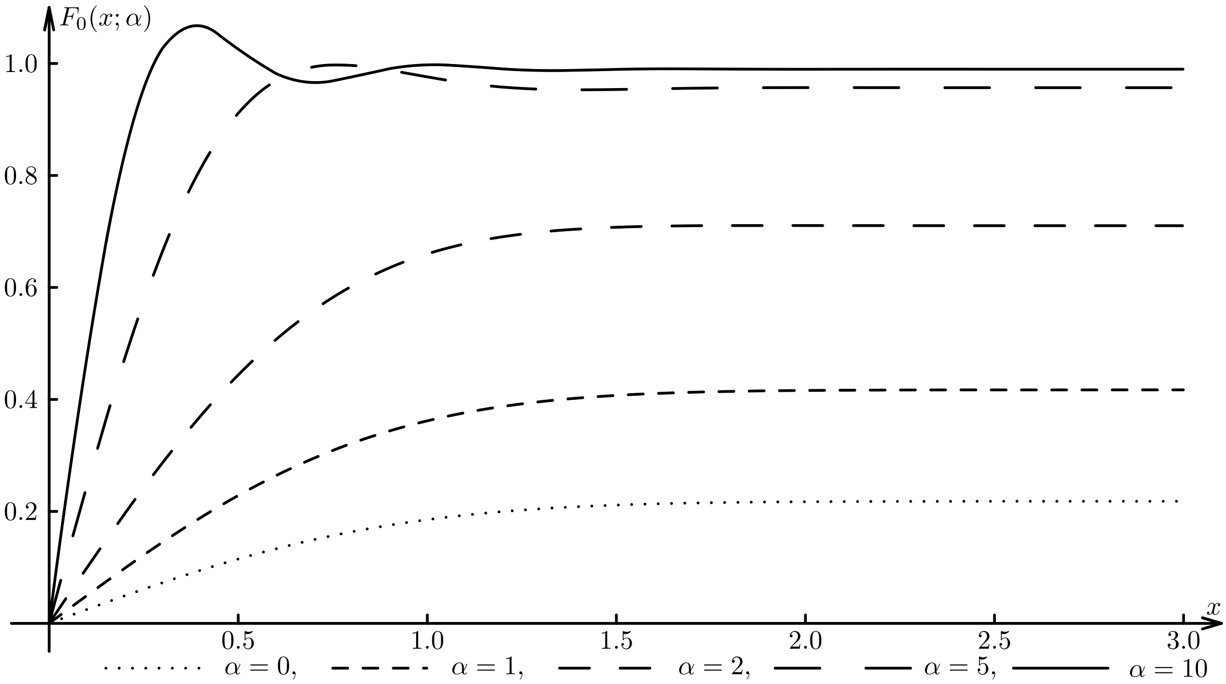
$$F_1(x;\alpha) \sim \frac{1 - e^{-\alpha^2/4}}{\alpha} + \sqrt{\frac{2}{\pi\alpha x}}\, e^{-x^2} \cdot$$

$$\cdot \left[\left(\frac{1}{2} - \frac{32\,\alpha^4 + 88\,\alpha^2 - 15}{256\,\alpha^2 x^2} + \frac{2048\,\alpha^8 + 26112\,\alpha^6 + 38976\,\alpha^4 - 4080\,\alpha^2 - 4725}{65536\,\alpha^4 x^4} + \ldots \right) \cos\left(x + \frac{\pi}{4}\right) - \right.$$

$$- \left(\frac{4\,\alpha^2 + 3}{16\,\alpha x} - \frac{128\,\alpha^6 + 864\,\alpha^4 + 324\,\alpha^2 + 105}{2048\,\alpha^3 x^3} + \right.$$

$$\left. \left. + \frac{8192\,\alpha^{10} + 169984\,\alpha^8 + 598272\,\alpha^6 + 149568\,\alpha^4 + 34860\,\alpha^2 + 72765}{524288\,\alpha^5 x^5} + \ldots \right) \sin\left(x + \frac{\pi}{4}\right) \right]$$

III) $s = -$:



$$F_-(x;\alpha) = \frac{\alpha x}{2} - \frac{(\alpha^2+8)\alpha x^3}{48} + \frac{(\alpha^4+24\,\alpha^2+96)\alpha x^5}{1920} - \frac{(\alpha^6+48\,\alpha^4+576\,\alpha^2+1536)\alpha x^7}{129024} +$$
$$+\frac{(\alpha^8+80\,\alpha^6+1920\,\alpha^4+15360\,\alpha^2+30720)\alpha x^9}{13271040} -$$
$$-\frac{(\alpha^{10}+120\,\alpha^8+4800\,\alpha^6+76800\,\alpha^4+460800\,\alpha^2+737280)\alpha x^{11}}{1946419200} +$$
$$+\frac{(\alpha^{12}+168\,\alpha^{10}+10080\,\alpha^8+268800\,\alpha^6+3225600\,\alpha^4+15482880\,\alpha^2+20643840)\alpha x^{13}}{386452684800} - \ldots$$

The formulas for $F_-(x;\alpha)$ and $F1-(x;\alpha)$ have the same α - factors in the numerators of their coefficients. Otherwise (see page 149):

$$F_-(x;\alpha) = \int_0^x \frac{e^{-t^2}\,J_1(t)\,dt}{t} =$$
$$= e^{-x^2}\left[\left(\alpha x + \frac{4\alpha-\alpha^3}{3}\,x^3 + \frac{\alpha^5-16\alpha^3+36\alpha}{45}\,x^5 + \frac{-\alpha^7+36\alpha^4-296\alpha^3+480}{1575}\,x^7+\ldots\right) J_0(\alpha x) + \right.$$
$$\left. + \left(-1+(\alpha^2-2)x^2 + \frac{-\alpha^4+10\alpha^2-12}{9}\,x^4 + \frac{\alpha^6-26\alpha^4+136\alpha^2-120}{225}\,x^6\ldots\right) J_1(\alpha x)\right]$$

Recurrence relations:

$$a_{k+1}^{(-;\alpha)} = \frac{(4k+2-\alpha^2)a_k^{(-;\alpha)} - 2\alpha\, b_k^{(-;\alpha)}}{(2k+1)(2k+3)}\;, \quad b_{k+1}^{(-;\alpha)} = \frac{2b_k^{(-;\alpha)} + \alpha\, a_k^{(-;\alpha)}}{2k+1}$$

or

$$a_{k+1}^{(-;\alpha)} = \frac{(8k+4-\alpha^2)a_k^{(-;\alpha)} - 4a_{k-1}^{(-;\alpha)}}{(2k+1)(2k+3)}\;, \quad b_{k+1}^{(-;\alpha)} = \frac{(8k-\alpha^2)b_k^{(-;\alpha)} - 4b_k^{(-;\alpha)\,2}}{2k+1}$$

About using this formula see page 151.

Approximation:

$$F_-(x;\alpha) = c_0^{(-;\alpha)}\operatorname{erf}(x) + e^{-x^2}\sum_{k=1}^{5} c_k^{(-;\alpha)}\,x^{2k-1}$$

$$c_0^{(-;\alpha)} = -0.00000\,01479\,\alpha^{11} + 0.00000\,39441\,\alpha^9 - 0.00009\,01517\,\alpha^7 + 0.00173\,09120\,\alpha^5 -$$

$$-0.02769\,45914\,\alpha^3 + 0.44311\,34627\,\alpha$$

$$c_1^{(-;\alpha)} = 0.00000\,016689\,\alpha^{11} - 0.00000\,44505\,\alpha^9 + 0.00010\,17253\,\alpha^7 - 0.00195\,3125\,\alpha^5 + 0.03125\,\alpha^3$$

$$c_2^{(-;\alpha)} = 0.00000\,01113\,\alpha^{11} - 0.00000\,29670\,\alpha^9 + 0.00006\,78168\,\alpha^7 - 0.00130\,20833\,\alpha^5$$

$$c_3^{(-;\alpha)} = 0.00000\,00445\,\alpha^{11} - 0.00000\,11868\,\alpha^9 + 0.00002\,71267\,\alpha^7$$

$$c_4^{(-;\alpha)} = 0.00000\,00127\,\alpha^{11} - 0.00000\,03391\,\alpha^9\,, \quad c_5^{(-;\alpha)} = 0.00000\,00028\,\alpha^{11}$$

In the case of small values of $|x|$ and $\alpha >> 1$ the following approximation may be used ($\Phi(x)$ defined as on page 9):

$$F_-(x;\alpha) = \sigma^{(-)}(\alpha)\,\Phi(\alpha x) + \left[\sum_{k=0}^{7} \varphi_k^{(-)}(\alpha)\,x^{2k+1}\right] J_0(\alpha x) + \left[\sum_{k=0}^{7} \psi_k^{(-)}(\alpha)\,x^{2k}\right] J_1(\alpha x)$$

Let

$$\varphi_k^{(0)}(\alpha) = \sum_{j=0}^{7} s_j^{(k,0)} \left(\frac{5}{\alpha}\right)^{2k-1} \quad \text{and} \quad \psi_k^{(-)}(\alpha) = \sum_{j=0}^{7} t_j^{(k,-)} \left(\frac{5}{\alpha}\right)^{2k}$$

$s_j^{(k,-)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	5	0	0	0	0	0	0	0
1	0	-0.1000000000	0.0333333333	-0.0083333333	0.0016666667	-0.0002777778	0.0000396825	-0.0000049603
2	0	-0.0200000000	0.0116666667	-0.0042000000	0.0011000000	-0.0002269841	0.0000386905	-0.0000056217
3	0	-0.0070000000	0.0058800000	-0.0027720000	0.0008988571	-0.0002213095	0.0000438492	-0.0000072632
4	0	-0.0035280000	0.0038808000	-0.0022651200	0.0008763857	-0.0002508175	0.0000566532	-0.0000105383
5	0	-0.0023284800	0.0031711680	-0.0022084920	0.0009932371	-0.0003240562	0.0000821986	-0.0000169666
6	0	-0.0019027008	0.0030918888	-0.0025029576	0.0012832624	-0.0004701760	0.0001323398	-0.0000300179
7	0	-0.0018551333	0.0035041406	-0.0032338212	0.0018618971	-0.0007569834	0.0002341396	-0.0000578917

$t_j^{(k,-)}$:

j	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0	1	0	0	0	0	0	0	0
1	0	0.0600000000	-0.0333333333	0.0116666667	-0.0030000000	0.0006111111	-0.0001031746	0.0000148810
2	0	0.0120000000	-0.0116666667	0.0058800000	-0.0019800000	0.0004993651	-0.0001005952	0.0000168651
3	0	0.0042000000	-0.0058800000	0.0038808000	-0.0016179429	0.0004868810	-0.0001140079	0.0000217897
4	0	0.0021168000	-0.0038808000	0.0031711680	-0.0015774943	0.0005517984	-0.0001472983	0.0000316148
5	0	0.0013970880	-0.0031711680	0.0030918888	-0.0017878269	0.0007129235	-0.0002137164	0.0000508999
6	0	0.0011416205	-0.0030918888	0.0035041406	-0.0023098723	0.0010343873	-0.0003440834	0.0000900537
7	0	0.0011130800	-0.0035041406	0.0045273497	-0.0033514147	0.0016653635	-0.0006087629	0.0001736750

Asymptotic expansion:

$$F_-(x;\alpha) \sim \frac{\sqrt{\pi}}{4}\,e^{-\alpha^2/8}\left[I_0\left(\frac{\alpha^2}{8}\right) + I_1\left(\frac{\alpha^2}{8}\right)\right] + \sqrt{\frac{2}{\pi\alpha x}}\,e^{-x^2}\cdot$$

$$\cdot\left[\left(\frac{1}{2x} - \frac{-32\,\alpha^4 - 152\,\alpha^2 + 15}{256\,\alpha^2 x^3} + \frac{2048\,\alpha^8 + 38400\,\alpha^6 + 86080\,\alpha^4 - 6000\,\alpha^2 - 4725}{65536\,\alpha^4 x^5} + \ldots\right)\cos\left(x + \frac{\pi}{4}\right) - \right.$$

$$-\left(\frac{4\,\alpha^2 + 3}{16\,\alpha x^2} - \frac{128\,\alpha^6 + 1376\,\alpha^4 + 516\,\alpha^2 + 105}{2048\,\alpha^3 x^4} + \right.$$

$$\left.\left. + \frac{8192\,\alpha^{10} + 235520\,\alpha^8 + 1224960\,\alpha^6 + 306240\,\alpha^4 + 48300\,\alpha^2 + 72765}{524288\,\alpha^5 x^6} + \ldots\right)\sin\left(x + \frac{\pi}{4}\right)\right]$$

d) Integrals ($\alpha \neq 1$)

$$\int x\, e^{-x^2}\, J_0(\alpha x)\, dx = -\frac{e^{-x^2}}{2}\, J_0(\alpha x) - \frac{\alpha}{2} \int e^{-x^2}\, J_1(\alpha x)\, dx$$

$$\int x\, e^{-x^2}\, J_1(\alpha x)\, dx = -\frac{e^{-x^2}}{2}\, J_1(\alpha x) - \frac{\alpha}{2} \int e^{-x^2}\, J_0(\alpha x)\, dx - \frac{1}{2} \int \frac{e^{-x^2}\, J_1(\alpha x)}{x}\, dx$$

Let

$$\int x^{2n+\nu}\, e^{-x^2}\, J_\nu(\alpha x)\, dx =$$

$$= e^{-x^2} \left[A_{n;\,\nu}(x;\alpha)\, J_0(\alpha x) + B_{n;\,\nu}(x;\alpha)\, J_1(\alpha x)\right] + P_{n;\,\nu} \int e^{-x^2}\, J_0(\alpha x)\, dx + Q_{n;\,\nu} \int \frac{e^{-x^2}\, J_1(\alpha x)\, dx}{x}$$

and

$$\int x^{2n+1-\nu}\, e^{-x^2}\, J_\nu(\alpha x)\, dx = e^{-x^2} \left[C_{n;\,\nu}(x;\alpha)\, J_0(\alpha x) + D_{n;\,\nu}(x;\alpha)\, J_1(\alpha x)\right] + R_{n;\,\nu} \int e^{-x^2}\, J_1(\alpha x)\, dx\,.$$

$P_{n;\,\nu},\ Q_{n;\,\nu},\ R_{n;\,\nu} = 0$ are omitted.

$$A_{1;\,0}(x;\alpha) = -\frac{x}{2}\,, \quad B_{1;\,0}(x;\alpha) = \frac{\alpha}{4}\,, \quad P_{1;\,0}(\alpha) = \frac{-\alpha^2+2}{4}\,, \quad Q_{1;\,0}(x;\alpha) = \frac{\alpha}{4}$$

$$A_{1;1}(x;\alpha) = -\frac{\alpha\, x}{8}\,, \quad B_{1;1}(x;\alpha) = \frac{-4\,x^2+\alpha^2-2}{8}\,, \quad P_{1;1}(\alpha) = \frac{-\alpha^3+4\,\alpha}{8}\,, \quad Q_{1;1}(x;\alpha) = \frac{\alpha^2-2}{8}$$

$$C_{1;0}(x;\alpha) = \frac{-4\,x^2+\alpha^2-4}{8}\,, \quad D_{1;0}(x;\alpha) = \frac{\alpha\, x}{4}\,, \quad R_{1;0}(\alpha) = \frac{\alpha^3-4\,\alpha}{8}$$

$$C_{1;1}(x;\alpha) = -\frac{\alpha}{4}\,, \quad D_{1;1}(x;\alpha) = -\frac{x}{2}\,, \quad R_{1;1}(\alpha) = -\frac{\alpha^2}{4}$$

$$A_{2;0}(x;\alpha) = \frac{-4\,x^3+(\alpha^2-6)\,x}{8}\,, \quad B_{2;0}(x;\alpha) = \frac{4\,\alpha\,x^2-\alpha^3+8\,\alpha}{16}\,,$$

$$P_{2;0}(\alpha) = \frac{\alpha^4-10\,\alpha^2+12}{16}\,, \quad Q_{2;0}(x;\alpha) = \frac{-\alpha^3+8\,\alpha}{16}$$

$$A_{2;1}(x;\alpha) = \frac{-4\,\alpha\,x^3+\left(\alpha^3-12\,\alpha\right)x}{16}\,, \quad B_{2;1}(x;\alpha) = \frac{-12-16\,x^4+\left(4\,\alpha^2-24\right)x^2-\alpha^4+14\,\alpha^2}{32}\,,$$

$$P_{2;1}(\alpha) = \frac{\alpha^5-16\,\alpha^3+36\,\alpha}{32}\,, \quad Q_{2;1}(x;\alpha) = \frac{-\alpha^4+14\,\alpha^2-12}{32}$$

$$C_{2;0}(x;\alpha) = \frac{-16\,x^4+\left(4\,\alpha^2-32\right)x^2-\alpha^4+16\,\alpha^2-32}{32}\,, \quad D_{2;0}(x;\alpha) = \frac{4\,\alpha\,x^3+\left(-\alpha^3+12\,\alpha\right)x}{16}\,,$$

$$R_{2;0}(\alpha) = \frac{-\alpha^5+16\,\alpha^3-32\,\alpha}{32}$$

$$C_{2;1}(x;\alpha) = \frac{-4\,\alpha\,x^2+\alpha^3-8\,\alpha}{16}\,, \quad D_{2;1}(x;\alpha) = \frac{-4\,x^3+\left(\alpha^2-4\right)x}{8}\,, \quad R_{2;1}(\alpha) = \frac{\alpha^4-8\,\alpha^2}{16}$$

$$A_{3;0}(x;\alpha) = \frac{-16\,x^5+(4\,\alpha^2-40)x^3-(\alpha^4-22\,\alpha^2+60)x}{32}\,,$$

$$B_{3;0}(x;\alpha) = \frac{16\,\alpha\,x^4+(64\,\alpha-4\,\alpha^3)x^2+\alpha^5-24\,\alpha^3+92\,\alpha}{64}\,,$$

$$P_{3;0}(\alpha) = \frac{-\alpha^6+26\,\alpha^4-136\,\alpha^2+120}{64}\,, \quad Q_{3;0}(x;\alpha) = \frac{\alpha^5-24\,\alpha^3+92\,\alpha}{64}$$

$$A_{3;1}(x;\alpha) = \frac{-16\,\alpha\,x^5+\left(4\,\alpha^3-80\,\alpha\right)x^3+\left(-\alpha^5+32\,\alpha^3-180\,\alpha\right)x}{64}\,,$$

$$B_{3;1}(x;\alpha) = \frac{-64\,x^6+\left(16\,\alpha^2-160\right)x^4+\left(-4\,\alpha^4+104\,\alpha^2-240\right)x^2+\alpha^6-34\,\alpha^4+232\,\alpha^2-120}{128}\,,$$

$$P_{3;1}(\alpha) = \frac{-\alpha^7 + 36\,\alpha^5 - 296\,\alpha^3 + 480\,\alpha}{128}\,, \quad Q_{3;1}(x;\alpha) = \frac{\alpha^6 - 34\,\alpha^4 + 232\,\alpha^2 - 120}{128}$$

$$C_{3;0}(x;\alpha) = \frac{-64\,x^6 + \left(16\,\alpha^2 - 192\right)x^4 + \left(-4\,\alpha^4 + 112\,\alpha^2 - 384\right)x^2 + \alpha^6 - 36\,\alpha^4 + 288\,\alpha^2 - 384}{128}\,,$$

$$D_{3;0}(x;\alpha) = \frac{16\,\alpha\,x^5 + \left(-4\,\alpha^3 + 80\,\alpha\right)x^3 + \left(\alpha^5 - 32\,\alpha^3 + 176\,\alpha\right)x}{64}\,,$$

$$R_{3;0}(\alpha) = \frac{\alpha^7 - 36\,\alpha^5 + 288\,\alpha^3 - 384\,\alpha}{128}$$

$$C_{3;1}(x;\alpha) = \frac{-16\,\alpha\,x^4 + \left(4\,\alpha^3 - 64\,\alpha\right)x^2 - \alpha^5 + 24\,\alpha^3 - 96\,\alpha}{64}\,,$$

$$D_{3;1}(x;\alpha) = \frac{-16\,x^5 + \left(4\,\alpha^2 - 32\right)x^3 + \left(-\alpha^4 + 20\,\alpha^2 - 32\right)x}{32}\,,$$

$$R_{3;1}(\alpha) = \frac{-\alpha^6 + 24\,\alpha^4 - 96\,\alpha^2}{64}$$

Recurrence relations:

$$\int x^{2n+2}\,e^{-x^2}\,J_0(\alpha x)\,dx =$$

$$= \frac{x^{2n}\,e^{-x^2}}{4}\,[\alpha J_1(\alpha x) - 2xJ_0(\alpha x)] + \frac{4n+2-\alpha}{4}\int x^{2n}\,e^{-x^2}\,J_0(\alpha x)\,dx - \frac{(2n-1)\,\alpha}{4}\int x^{2n-1}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_0(\alpha x)\,dx = -\frac{x^{2n}\,e^{-x^2}}{2}\,J_0(\alpha x) + n\int x^{2n-1}\,e^{-x^2}\,J_0(\alpha x)\,dx - \frac{\alpha}{2}\int x^{2n}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

$$\int x^{2n+2}\,e^{-x^2}\,J_1(\alpha x)\,dx =$$

$$= -\frac{x^{2n}\,e^{-x^2}}{4}\,[\alpha J_0(\alpha x) + 2xJ_1(\alpha x)] + \frac{n\,\alpha}{2}\int x^{2n-1}\,e^{-x^2}\,J_0(\alpha x)\,dx + \frac{4n-\alpha^2}{4}\int x^{2n}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

$$\int x^{2n+1}\,e^{-x^2}\,J_1(\alpha x)\,dx = -\frac{x^{2n}\,e^{-x^2}}{2}\,J_1(\alpha x) + \frac{\alpha}{2}\int x^{2n}\,e^{-x^2}\,J_0(\alpha x)\,dx + \frac{2n-1}{2}\int x^{2n-1}\,e^{-x^2}\,J_1(\alpha x)\,dx$$

e) Special Cases: $\nu = 0$

$$\int x^3\,e^{-x^2}\,J_0(2x)\,dx = \frac{x}{2}\,e^{-x^2}\,[J_1(2x) - xJ_0(2x)]$$

$$\int x^3\,e^{x^2}\,I_0(2x)\,dx = \frac{x}{2}\,e^{x^2}\,[xI_0(2x) - I_1(2x)]$$

$$\int x^3\,e^{x^2}\,K_0(2x)\,dx = \frac{x}{2}\,e^{x^2}\,[xK_0(2x) + K_1(2x)]$$

$$2\,\lambda = 2\sqrt{2+\sqrt{2}} = 3.69551\,81300\,, \qquad 2\,\mu = 2\sqrt{2-\sqrt{2}} = 1.53073\,37295$$

$$\int x^5\,e^{-x^2}\,J_0\,(2\lambda\,x)\;dx = \frac{x}{2}\,e^{-x^2}\;\left[x(\sqrt{2}-x^2)\,J_0\,((2\lambda\,x) + \sqrt{2+\sqrt{2}}\,(x^2+1-\sqrt{2})\,J_1\,(2\lambda\,x)\right]$$

$$\int x^5\,e^{x^2}\,I_0\,(2\lambda\,x)\;dx = \frac{x}{2}\,e^{x^2}\;\left[x(\sqrt{2}+x^2)\,I_0\,((2\lambda\,x) - \lambda\,(x^2-1+\sqrt{2})\,I_1\,(2\lambda\,x)\right]$$

$$\int x^5\,e^{x^2}\,K_0\,(2\lambda\,x)\;dx = \frac{x}{2}\,e^{x^2}\;\left[x(\sqrt{2}+x^2)\,K_0\,((2\lambda\,x) + \lambda\,(x^2-1+\sqrt{2})\,K_1\,(2\lambda\,x)\right]$$

$$\int x^5\,e^{-x^2}\,J_0\,(2\mu\,x)\;dx = \frac{x}{2}\,e^{-x^2}\;\left[-x(\sqrt{2}+x^2)\,J_0\,((2\mu\,x) + \mu\,(x^2+1+\sqrt{2})\,J_1\,(2\mu\,x)\right]$$

$$\int x^5\, e^{x^2}\, I_0\,(2\mu\, x)\; dx = \frac{x}{2}\, e^{x^2}\; \left[x(x^2-\sqrt{2})\, I_0\,((2\mu\, x) - \mu\,(x^2-1-\sqrt{2})\, I_1\,(2\mu\, x)\right]$$

$$\int x^5\, e^{x^2}\, K_0\,(2\mu\, x)\; dx = \frac{x}{2}\, e^{x^2}\; \left[x(x^2-\sqrt{2})\, K_0\,((2\mu\, x) + \mu\,(x^2-1-\sqrt{2})\, K_1\,(2\mu\, x)\right]$$

f) Special Cases: $\nu = 1$

$$\int x^4\, e^{-x^2}\, J_1(2\sqrt{2}x)\; dx = \frac{x}{\sqrt{2}}\, e^{-x^2}\, [(1-x^2)J_1(2\sqrt{2}x) - x\sqrt{2}\, J_0(2\sqrt{2}x)]$$

$$\int x^4\, e^{x^2}\, I_1(2\sqrt{2}x)\; dx = \frac{x}{\sqrt{2}}\, e^{x^2}\, [(1+x^2)I_1(2\sqrt{2}x) - x\sqrt{2}\, I_0(2\sqrt{2}x)]$$

$$\int x^4\, e^{x^2}\, K_1(2\sqrt{2}x)\; dx = \frac{x}{\sqrt{2}}\, e^{x^2}\, [(1+x^2)K_1(2\sqrt{2}x) + x\sqrt{2}\, K_0(2\sqrt{2}x)]$$

$$2\,\eta = 2\sqrt{3+\sqrt{3}} = 4.35065\ 54943\,, \qquad 2\,\varrho = 2\sqrt{3-\sqrt{3}} = 2.25206\ 50012$$

$$\int x^6\, e^{-x^2}\, J_1\,(2\eta\, x)\; dx =$$

$$= \frac{x}{2}\, e^{-x^2}\; \left[(-x^4+(1+\sqrt{3})x^2-\sqrt{3}+1)\, J_1\,((2\eta\, x) - \eta\, x\,(x^2+1-\sqrt{3})\, J_0\,(2\eta\, x)\right]$$

$$\int x^6\, e^{x^2}\, I_1\,(2\eta\, x)\; dx =$$

$$= \frac{x}{2}\, e^{x^2}\; \left[(x^4+(1+\sqrt{3})x^2+\sqrt{3}-1)\, I_1\,((2\eta\, x) - \eta\, x\,(x^2-1+\sqrt{3})\, I_0\,(2\eta\, x)\right]$$

$$\int x^6\, e^{x^2}\, K_1\,(2\eta\, x)\; dx =$$

$$= \frac{x}{2}\, e^{x^2}\; \left[(x^4+(1+\sqrt{3})x^2+\sqrt{3}-1)\, K_1\,((2\eta\, x) + \eta\, x\,(x^2-1+\sqrt{3})\, K_0\,(2\eta\, x)\right]$$

$$\int x^6\, e^{-x^2}\, J_1\,(2\varrho\, x)\; dx =$$

$$= \frac{x}{2}\, e^{-x^2}\; \left[(-x^4+(1-\sqrt{3})x^2+\sqrt{3}+1)\, J_1\,((2\varrho\, x) - \varrho\, x\,(x^2+1+\sqrt{3})\, J_0\,(2\varrho\, x)\right]$$

$$\int x^6\, e^{x^2}\, I_1\,(2\varrho\, x)\; dx =$$

$$= \frac{x}{2}\, e^{x^2}\; \left[(x^4+(1-\sqrt{3})x^2-\sqrt{3}-1)\, I_1\,((2\varrho\, x) - \varrho\, x\,(x^2-1-\sqrt{3})\, I_0\,(2\varrho\, x)\right]$$

$$\int x^6\, e^{x^2}\, K_1\,(2\varrho\, x)\; dx =$$

$$= \frac{x}{2}\, e^{x^2}\; \left[(x^4+(1-\sqrt{3})x^2-\sqrt{3}-1)\, K_1\,((2\varrho\, x) + \varrho\, x\,(x^2-1-\sqrt{3})\, K_0\,(27\, x)\right]$$

g) Integrals of $x^n\, e^{-(x+\beta)^2}\, J_\nu(\alpha x)$ or $x^n\, e^{-x^2}\, J_\nu(\alpha x+\gamma)$

With $z = x+\beta$ follows

$$\int x^n\, e^{-(x+\beta)^2\, J_\nu(\alpha x)}\, dx = \int (z-\beta)^n\, e^{-z^2}\, J_\nu(\alpha z - \alpha\beta)\, dx\,.$$

About the integrals on the right side see page 166. To describe the left integrals three basic integrals must be defined:

$$\int e^{-(x+\beta)^2\, J_0(\alpha x)}\, dx = F_0(x;\alpha,\beta) = e^{-(x+\beta)^2}\; \left[\left(\sum_{k=0}^{\infty} p_k^{(0)}(\alpha,\beta)\, x^k\right)\; J_0(\alpha x) + \left(\sum_{k=0}^{\infty} q_k^{(0)}(\alpha,\beta)\, x^k\right)\; J_1(\alpha x)\right]$$

$$\int e^{-(x+\beta)^2\, J_1(\alpha x)}\, dx = F_1(x;\alpha,\beta) =$$

$$= e^{-(x+\beta)^2}\left[\left(\sum_{k=0}^{\infty} p_k^{(1)}(\alpha,\beta)\, x^k\right) J_0(\alpha x) + \left(\sum_{k=0}^{\infty} q_k^{(1)}(\alpha,\beta)\, x^k\right) J_1(\alpha x)\right] - p_0\, e^{-\beta^2}$$

$$\int \frac{e^{-(x+\beta)^2\, J_1(\alpha x)}\, dx}{x} = F_-(x;\alpha,\beta) = e^{-(x+\beta)^2}\left[\left(\sum_{k=0}^{\infty} p_k^{(-)}(\alpha,\beta)\, x^k\right) J_0(\alpha x) + \left(\sum_{k=0}^{\infty} q_k^{(-)}(\alpha,\beta)\, x^k\right) J_1(\alpha x)\right]$$

I) $F_0(x;\alpha,\beta)$:

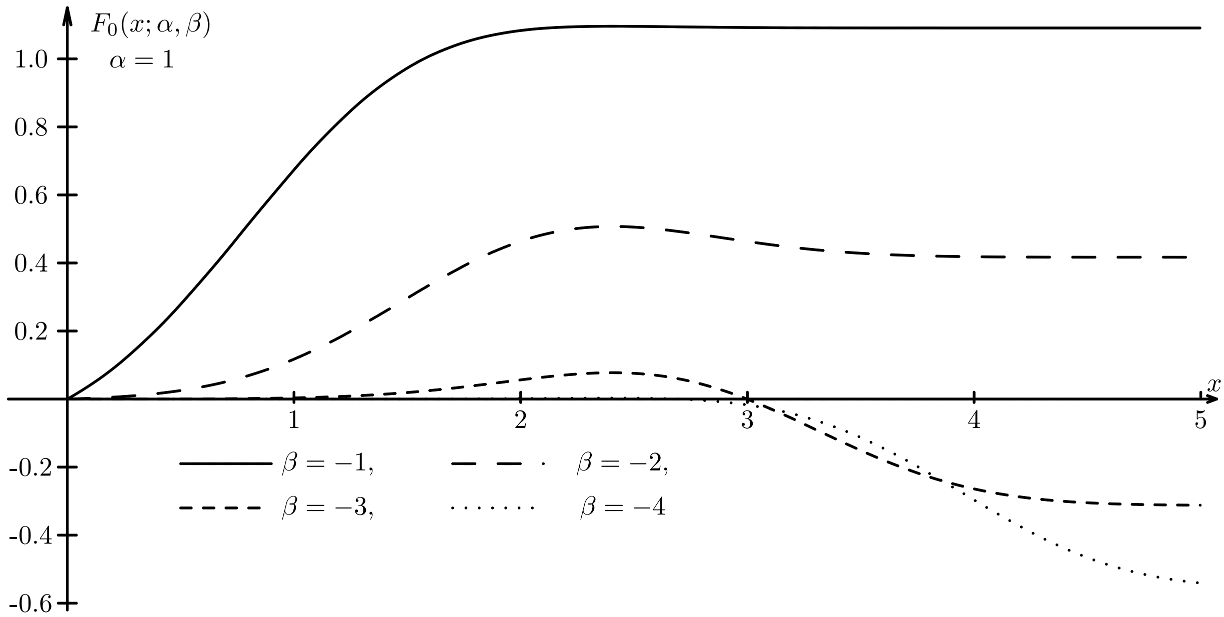
$$p_0^{(0)} = 0\,,\ p_1^{(0)} = 1\,,\ q_0^{(0)} = 0\,,\ q_1^{(0)} = 0\,,$$

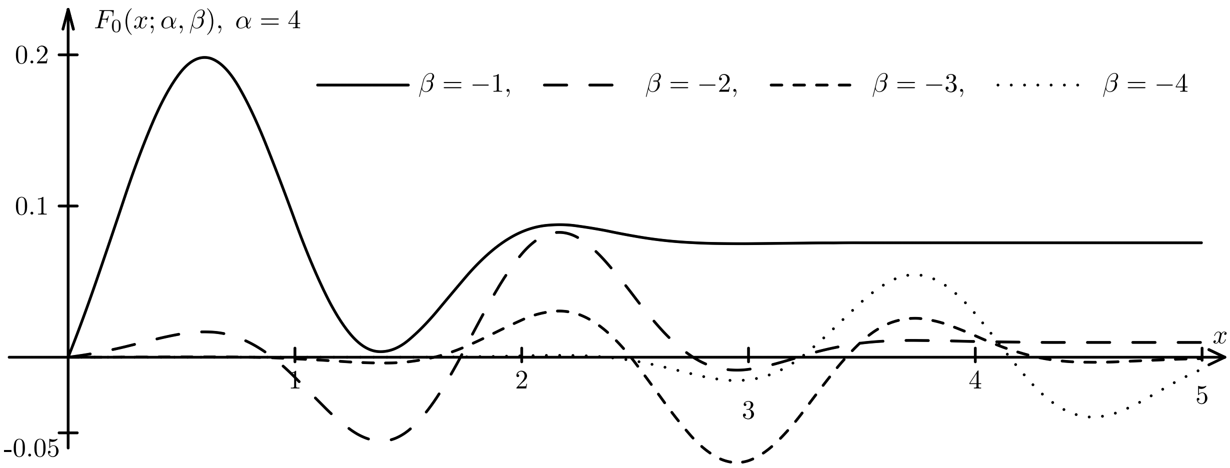
$$p_{n+1}^{(0)} = \frac{2p_{n-1}^{(0)}(\alpha,\beta) + 2\beta\, p_n^{(0)}(\alpha,\beta) - \alpha\, q_n^{(0)}(\alpha,\beta)}{n+1}\ , \quad q_{n+1}^{(0)}(\alpha,\beta) = \frac{2q_{n-1}^{(0)}(\alpha,\beta) + 2\beta\, q_n^{(0)}(\alpha,\beta) + \alpha\, p_n^{(0)}(\alpha,\beta)}{n}$$

$$F_0(x;\alpha,\beta) = e^{-(x+\beta)^2}\left[\left(x + \beta x^2 + \frac{2\beta^2-\alpha^2+2}{3}\, x^3 + \frac{\beta}{24}(8\beta^2 - 13\alpha^2 + 20)\, x^4 + \ldots\right) J_0(\alpha x) + \right.$$

$$\left. + \left(\alpha\, x^2 + \frac{3\alpha\beta}{2}\, x^3 + \frac{\alpha}{9}(11\beta^2 - \alpha^2 + 8)\, x^4 + \frac{\alpha\beta\,(200\beta^2 - 55\alpha^2 + 404)}{288}\, x^5 + \ldots\right) J_1(\alpha x)\right]$$

About the using of this formula see page 172, concerning $a_k^{(0;\alpha)}$ and $b_k^{(0;\alpha)}$.

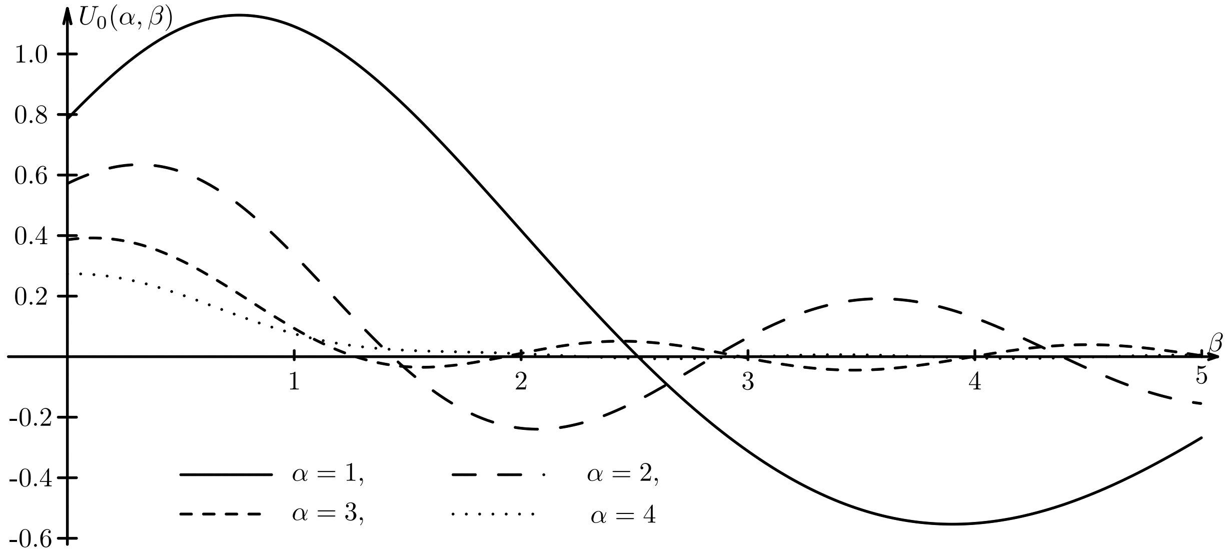


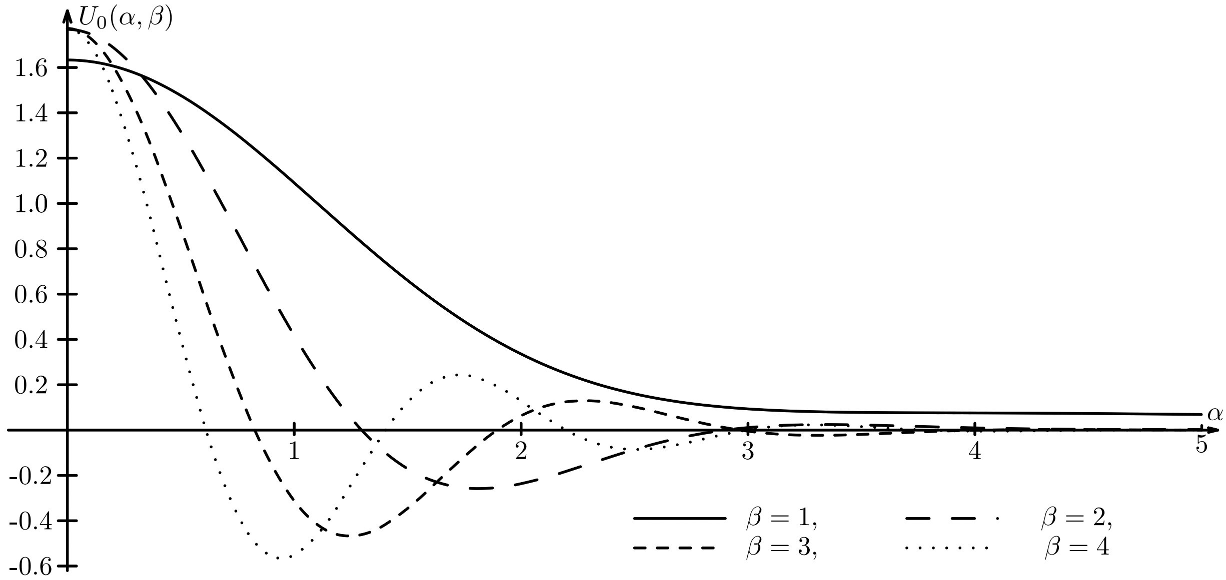


Let

$$U_0(\alpha,\beta) \quad = \quad \int_0^\infty e^{-(x-\beta)^2}\, J_0(\alpha x)\, dx \; .$$

The following pictures show some of this functions:





<u>II) $F_1(x;\alpha,\beta)$:</u>

$$p_0^{(1)} = -\,\frac{1}{\alpha}\;,\; p_1^{(1)} = -\,\frac{2\beta}{\alpha}\;,\; q_0^{(1)} = 0\;,\; q_1^{(1)} = 0\;,$$

$$p_{n+1}^{(1)} = \frac{2p_{n-1}^{(1)}(\alpha,\beta) + 2\beta\, p_n^{(1)}(\alpha,\beta) - \alpha\, q_n^{(1)}(\alpha,\beta)}{n+1}\;,$$

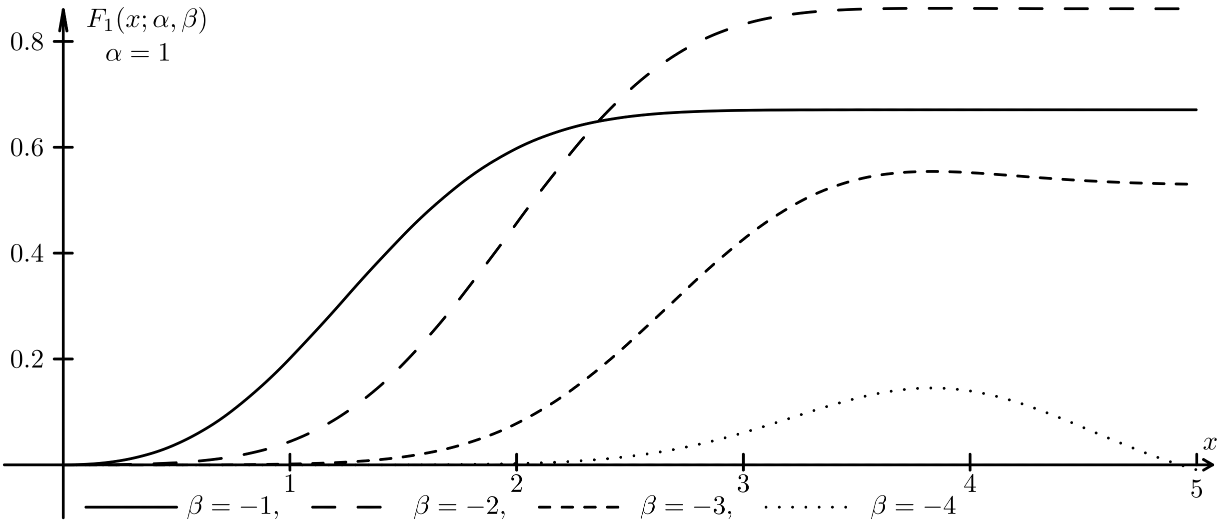
$$q_{n+1}^{(1)}(\alpha,\beta) = \frac{2q_{n-1}^{(1)}(\alpha,\beta) + 2\beta\, q_n^{(1)}(\alpha,\beta) + \alpha\, p_n^{(1)}(\alpha,\beta)}{n}$$

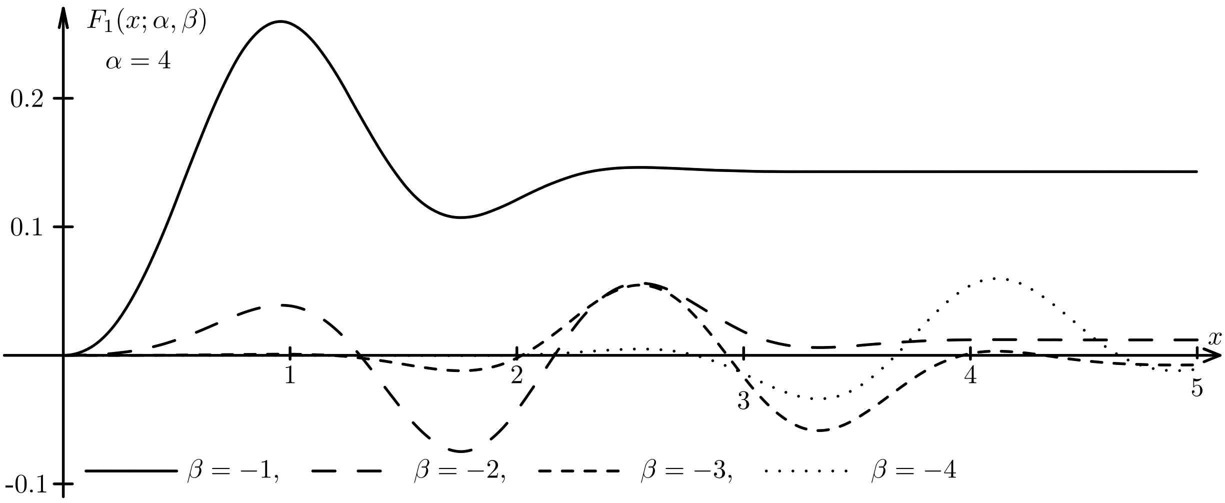
$$F_1(x;\alpha,\beta) = \frac{e^{-\beta^2}}{\alpha} -$$

$$-e^{-(x+\beta)^2}\left[\left(\frac{1}{\alpha} + \frac{2\beta x}{\alpha} + \frac{2\beta^2+1}{\alpha}x^2 + \frac{4\beta^3 + 2\beta(3-2\alpha^2)}{3\alpha}x^3 + \frac{16\beta^4 + \beta^2(48-26\alpha^2) + 12 - 3\alpha^2}{24\alpha}x^4 + \dots\right) J_0(\alpha x) + \right.$$

$$\left. + \left(2\beta\, x^2 + \frac{6\beta^2+1}{2}\,x^3 + \frac{22\beta^3 + \beta(21-2\alpha^2)}{9}\,x^4 + \frac{400\beta^4 + \beta^2\,(912-110\alpha^2) + 108 - 9\alpha^2}{288}\,x^5 + \dots\right) J_1(\alpha x)\right]$$

About the using of this formula see page 172, concerning $a_k^{(0;\alpha)}$ and $b_k^{(0;\alpha)}$.

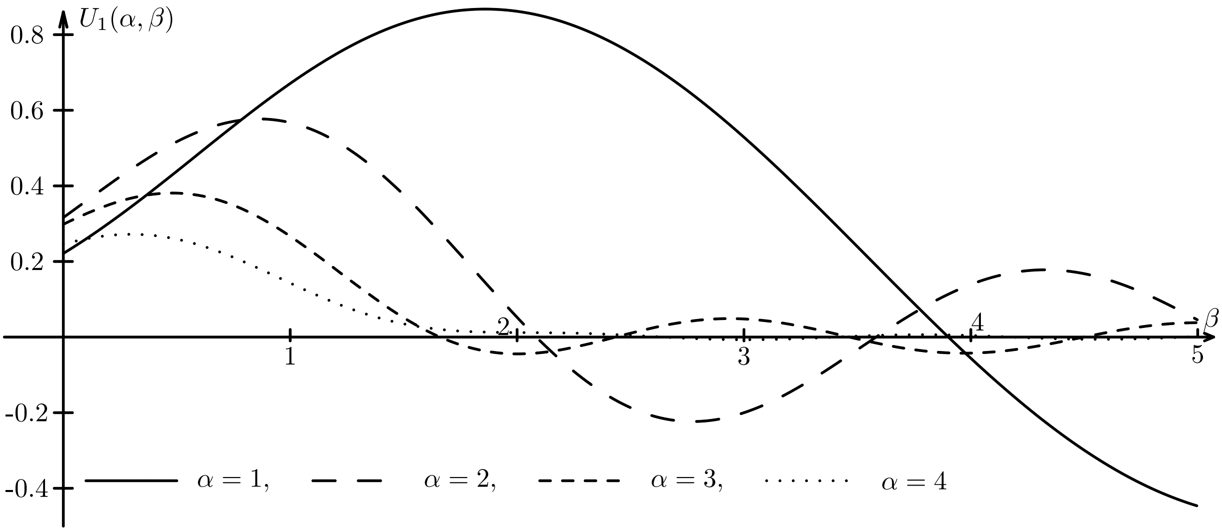


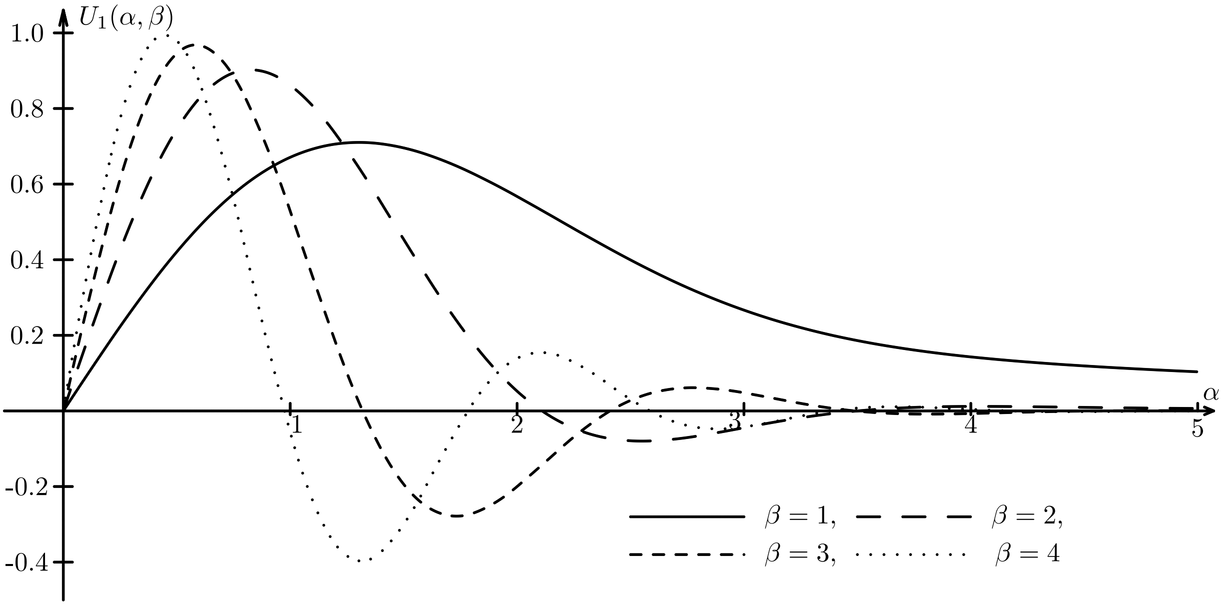


Let

$$U_1(\alpha,\beta) \quad = \quad \int_0^\infty e^{-(x-\beta)^2}\, J_1(\alpha x)\, dx\ .$$

The following pictures show some of this functions:





III) $F_{-}(x;\alpha,\beta)$:

$$p_0^{(-)} = \frac{2\beta}{\alpha}\,,\; p_1^{(-)} = \frac{\alpha^2+4\beta^2}{\alpha}\,,\; q_0^{(-)} = -1\,,\; q_1^{(-)} = 0\,,$$

$$p_{n+1}^{(-)} = \frac{2p_{n-1}^{(-)}(\alpha,\beta) + 2\beta\, p_n^{(-)}(\alpha,\beta) - \alpha\, q_n^{(-)}(\alpha,\beta)}{n+1}\;,$$

$$q_{n+1}^{(-)}(\alpha,\beta) = \frac{2q_{n-1}^{(-)}(\alpha,\beta) + 2\beta\, q_n^{(-)}(\alpha,\beta) + \alpha\, p_n^{(-)}(\alpha,\beta)}{n}$$

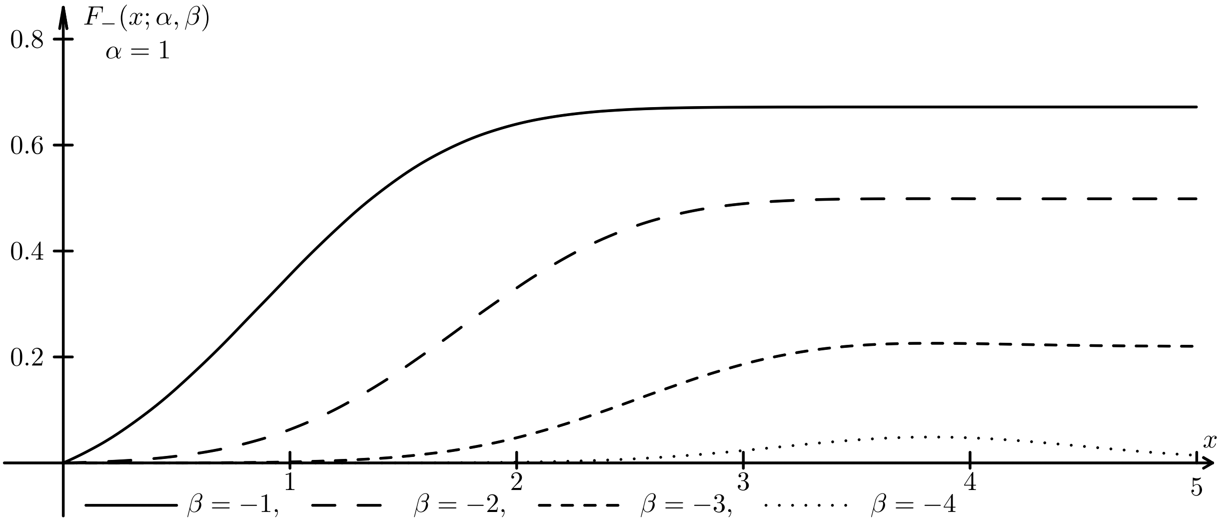
$$F_{-}(x;\alpha,\beta) =$$

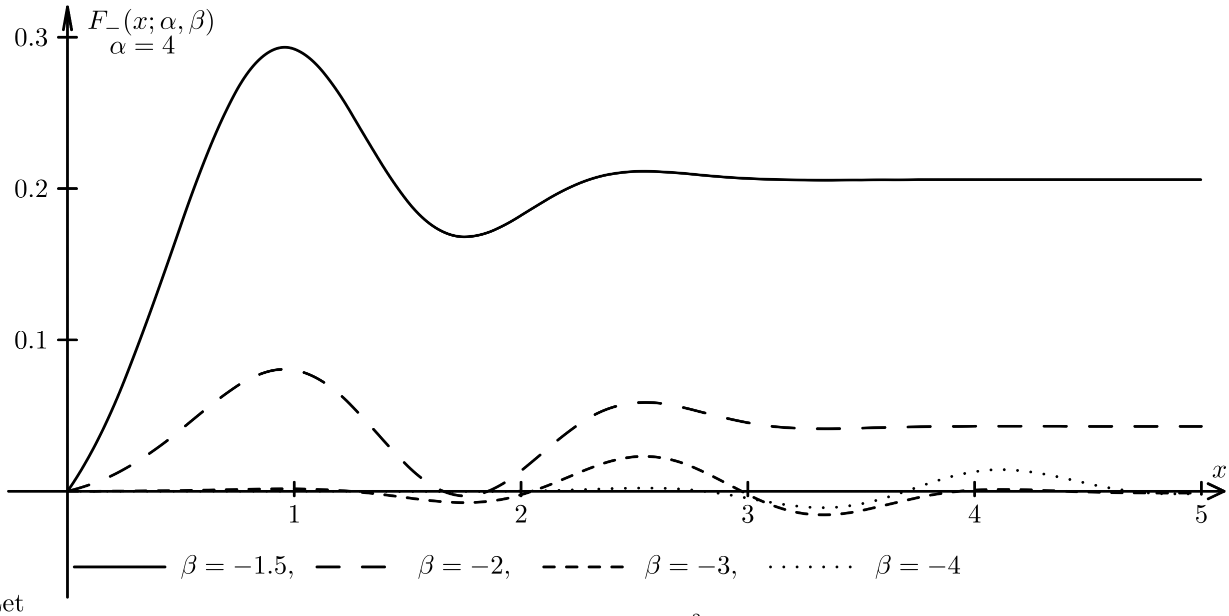
$$= e^{-(x+\beta)^2}\left[\left(\frac{2\beta}{\alpha} + \frac{\alpha^2+4\beta^2}{\alpha}\,x + \frac{\beta(4\beta^2+\alpha^2+2)}{\alpha}\,x^2 + \frac{8\beta^3+\beta^2(12-2\alpha^2)+\alpha^2(4-\alpha^2)}{\alpha}\,x^3 + \dots\right) J_0(\alpha x) + \right.$$

$$\left.\left(-x + (4\beta^2+\alpha^2-2)\,x^2 + \frac{\beta(12\beta^2+3\alpha^2-2)}{2}\,x^3 + \frac{44\beta^4+\beta^2(30+7\alpha^2)-\alpha^4+10\alpha^2-12}{9}\,x^4 + \dots\right) J_1(\alpha x)\right] -$$

$$-\frac{2\beta\, e^{-\beta^2}}{\alpha}$$

About the using of this formula see page 172, concerning $a_k^{(0;\alpha)}$ and $b_k^{(0;\alpha)}$.

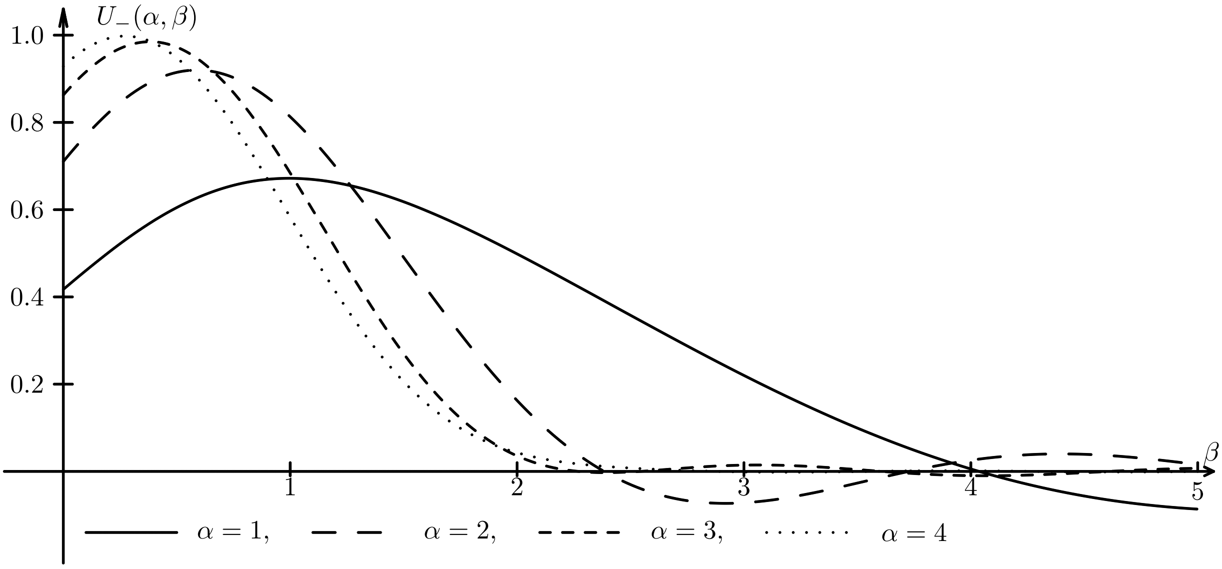


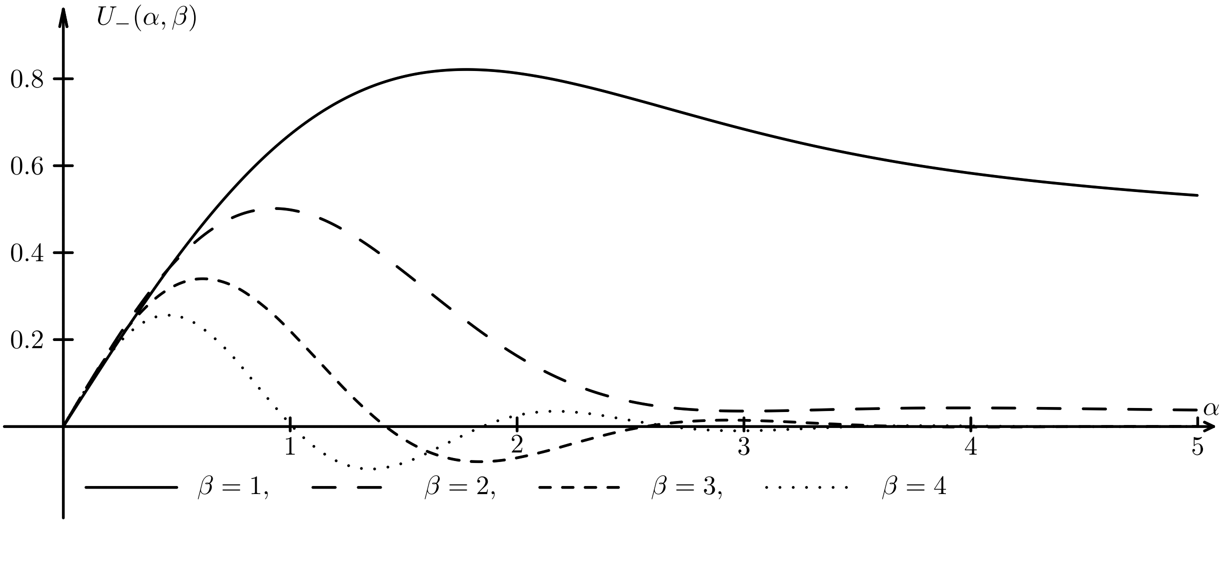


Let

$$U_-(\alpha,\beta) \quad = \quad \int_0^\infty \frac{e^{-(x-\beta)^2}\,J_1(\alpha x)\,dx}{x}\,.$$

The following pictures show some of this functions:





IV) Integrals:

$$\int x\, e^{-(x+\beta)^2}\, J_0(\alpha x)\, dx = -\,\frac{e^{-(x+\beta)^2}\, J_0(\alpha x)}{2} - \beta\, F_0(x;\alpha,\beta) - \frac{\alpha}{2}\, F_1(x;\alpha,\beta)$$

$$\int x\, e^{-(x+\beta)^2}\, J_1(\alpha x)\, dx = -\,\frac{e^{-(x+\beta)^2}\, J_1(\alpha x)}{2} + \frac{\alpha}{2}\, F_0(x;\alpha,\beta) - \beta\, F_1(x;\alpha,\beta) - \frac{1}{2}\, F_-(x;\alpha,\beta)$$

$$\int x^2\, e^{-(x+\beta)^2}\, J_0(\alpha x)\, dx =$$

$$= e^{-(x+\beta)^2}\, \left[\frac{\beta - x}{2}\, J_0(\alpha x) + \frac{\alpha}{4}\, J_1(\alpha x)\right] + \frac{2 + 4\beta^2 - \alpha^2}{4}\, F_0(x;\alpha,\beta) + \alpha\beta\, F_1(x;\alpha,\beta) + \frac{\alpha}{4}\, F_-(x;\alpha,\beta)$$

$$\int x^2\, e^{-(x+\beta)^2}\, J_1(\alpha x)\, dx = e^{-(x+\beta)^2}\, \left[-\frac{\alpha}{4}\, J_0(\alpha x) + \frac{\beta - x}{2}\, J_1(\alpha x)\right] -$$

$$-\alpha\beta\, F_0(x;\alpha,\beta) + \frac{4\beta^2 - \alpha^2}{4}\, F_1(x;\alpha,\beta) + \frac{\beta}{2}\, F_-(x;\alpha,\beta)$$

$$\int x^3\, e^{-(x+\beta)^2}\, J_0(\alpha x)\, dx = e^{-(x+\beta)^2}\, \left[\frac{\alpha^2 - 4\beta^2 - 4 + 4\beta x - 4x^2}{8}\, J_0(\alpha x) + \frac{\alpha(x - 2\beta)}{4}\, J_1(\alpha x)\right] -$$

$$-\frac{\beta(4\beta^2 + 6 - 3\alpha^2)}{4}\, F_0(x;\alpha,\beta) - \frac{\alpha(12\beta^2 + 4 - \alpha^2)}{8}\, F_1(x;\alpha,\beta) - \frac{\alpha\beta}{2}\, F_-(x;\alpha,\beta)$$

$$\int x^3\, e^{-(x+\beta)^2}\, J_1(\alpha x)\, dx = e^{-(x+\beta)^2}\, \left[\frac{\alpha(2\beta - x)}{4}\, J_0(\alpha x) - \frac{4\beta^2 + 2 - \alpha^2 - 4\beta x + 4x^2}{8}\, J_1(\alpha x)\right] +$$

$$+\frac{\alpha(12\beta^2 + 4 - \alpha^2)}{8}\, F_0(x;\alpha,\beta) - \frac{\beta(4\beta^2 + 2 - 3\alpha^2)}{4}\, F_1(x;\alpha,\beta) - \frac{4\beta^2 + 2 - \alpha^2}{8}\, F_-(x;\alpha,\beta)$$

$$\int x^4\, e^{-(x+\beta)^2}\, J_0(\alpha x)\, dx = e^{-(x+\beta)^2}\, \left[\frac{-4x^3 + 4\beta x^2 + x(\alpha^2 - 4\beta^2 - 6) + \beta(4\beta^2 - 3\alpha^2 + 10)}{8}\, J_0(\alpha x) + \right.$$

$$\left. + \frac{\alpha[4x(x^2 - 2\beta) + 12\beta^2 + 8 - \alpha^2)}{16}\, J_1(\alpha x)\right] + \frac{16\beta^4 + 24\beta^2(2 - \alpha^2) + \alpha^4 - 10\alpha^2 + 12}{16}\, F_0(x;\alpha,\beta) +$$

$$+\frac{\alpha\beta(8\beta^2 - 2\alpha^2 + 9)}{4}\, F_1(x;\alpha,\beta) + \frac{\alpha(12\beta^2 + 8 - \alpha^2)}{16}\, F_-(x;\alpha,\beta)$$

$$\int x^4\, e^{-(x+\beta)^2}\, J_1(\alpha x)\, dx = e^{-(x+\beta)^2}\, \left[\frac{\alpha[-4x^2 + 8\beta x - 12\beta^2 - 8 + \alpha^2]}{16}\, J_0(\alpha x) + \right.$$

$$\left. + \frac{-4x^3 + 4\beta x^2 + x(\alpha^2 - 4\beta^2 - 4) + \beta(4\beta^2 + 6 - 3\alpha^2)}{8}\, J_1(\alpha x)\right] + \frac{\alpha\beta(2\alpha^2 - 8\beta^2 - 9)}{4}\, F_0(x;\alpha,\beta) +$$

$$+\frac{\alpha^4 - 24\alpha^2\beta^2 + 16\beta^4 - 8\alpha^2 + 24\beta^2}{16}\, F_1(x;\alpha,\beta) + \frac{\beta(4\beta^2 + 6 - 3\alpha^2)}{8}\, F_-(x;\alpha,\beta)$$

Recurrence relations:

$$\text{Let} \qquad I_m^{(\nu)} \quad = \quad \int x^m\, e^{-(x+\beta)^2}\, J_\nu(\alpha x)\, dx\ .$$

$$I_{2n+1}^{(0)} = -\,\frac{x^{2n}\, e^{-(x+\beta)^2}\, J_0(\alpha x)}{2} - \beta I_{2n}^{(0)} + n\, I_{2n-1}^{(0)} - \frac{\alpha}{2}\, I_{2n}^{(1)}$$

$$I_{2n+1}^{(1)} = -\,\frac{x^{2n}\, e^{-(x+\beta)^2}\, J_1(\alpha x)}{2} + \frac{\alpha}{2}\, I_{2n}^{(0)} - \beta I_{2n}^{(1)} + \frac{2n - 1}{2}\, I_{2n-1}^{(1)}$$

$$I_{2n+2}^{(0)} = -\,\frac{x^{2n+1}\, e^{-(x+\beta)^2}\, J_0(\alpha x)}{2} + \frac{2n + 1}{2}\, I_{2n}^{(0)} - \frac{\alpha}{2}\, I_{2n+1}^{(1)} - \beta\, I_{2n+1}^{(0)}$$

$$I_{2n+2}^{(1)} = -\,\frac{x^{2n+1}\, e^{-(x+\beta)^2}\, J_1(\alpha x)}{2} + n\, I_{2n}^{(1)} - \beta I_{2n+1}^{(1)} + \frac{\alpha}{2}\, I_{2n+1}^{(0)}$$

Otherwise:

$$I_{n+1}^{(0)} = -\,\frac{x^{n-2}\, e^{-(x+\beta)^2}}{4}\, \left[(2\,x^2 + 2\,\beta x - n + 2)\, J_0(\alpha x) - \alpha\, x\, J_1(\alpha x)\right] -$$

$$-2\,\beta\,I_n^{(0)} - \left(\frac{\alpha^2}{4} + \beta^2 - n + 1\right) I_{n-1}^{(0)} + \frac{(2\,n-3)\,\beta}{2}\,I_{n-2}^{(0)} - \frac{(n-2)^2}{4}\,I_{n-3}^{(0)}$$

$$I_{n+1}^{(1)} = -\,\frac{x^{n-2}\,e^{-(x+\beta)^2}}{4}\,\left[\alpha\,x\,J_0(\alpha x) + (2\,x^2 + 2\,\beta x - n + 1)\,J_1(\alpha x)\right] -$$

$$-2\,\beta\,I_n^{(1)} - \left(\frac{\alpha^2}{4} + \beta^2 - n + 1\right) I_{n-1}^{(1)} + \frac{(2\,n-3)\,\beta}{2}\,I_{n-2}^{(1)} - \frac{(n-3)(n-1)}{4}\,I_{n-3}^{(1)}$$

<u>Integrals of the type</u> $\int x^n\,e^{-x^2}\,J_\nu(\alpha x+\gamma)\,dx$:

$$\int e^{-x^2}\,J_0(\alpha x+\gamma)\,dx = F_0\left(x - \frac{\gamma}{\alpha};\alpha,-\frac{\gamma}{\alpha}\right)\;,\quad \int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx = F_1\left(x - \frac{\gamma}{\alpha};\alpha,-\frac{\gamma}{\alpha}\right)$$

The integral $\int \frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{x}$ cannot be expressed by F_- .

$\int_0^y \frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{x}$ is not defined if $J_1(\gamma) \neq 0$.

$$\int \frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma} = \frac{1}{\alpha}\int \frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{x+\gamma/\alpha} = \frac{1}{\alpha}\,F_-\left(x - \frac{\gamma}{\alpha};\alpha,-\frac{\gamma}{\alpha}\right)$$

Integrals:

$$\int x\,e^{-x^2}\,J_0(\alpha x+\gamma)\,dx = -\,\frac{e^{-x^2}\,J_0(\alpha x+\gamma)}{2} - \frac{\alpha}{2}\int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx$$

$$\int x\,e^{-x^2}\,J_1(\alpha x+\gamma)\,dx =$$

$$= -\,\frac{e^{-x^2}\,J_1(\alpha x+\gamma)}{2} + \frac{\alpha}{2}\int e^{-x^2}\,J_0(\alpha x+\gamma)\,dx - \frac{\alpha}{2}\int \frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

$$\int \frac{x\,e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma} = \frac{1}{\alpha}\int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx - \frac{\gamma}{\alpha}\int \frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

$$\int x^2\,e^{-x^2}\,J_0(\alpha x+\gamma)\,dx =$$

$$= \frac{e^{-x^2}}{4}\,[a\,J_1(\alpha x+\gamma) - 2x\,J_0(\alpha x+\gamma)] + \frac{2-\alpha^2}{4}\int e^{-x^2}\,J_0(\alpha x+\gamma)\,dx + \frac{\alpha^2}{4}\int \frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

$$\int x^2\,e^{-x^2}\,J_1(\alpha x+\gamma)\,dx =$$

$$= -\,\frac{e^{-x^2}}{4}\,[\alpha J_0(\alpha x+\gamma) + 2x\,J_1(\alpha x+\gamma)] - \frac{\alpha^2}{4}\int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx + \frac{\gamma}{2}\int \frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

$$\int \frac{x^2\,e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma} =$$

$$= -\,\frac{e^{-x^2}\,J_1(\alpha x+\gamma)}{2\alpha} + \frac{1}{2}\int e^{-x^2}\,J_0(\alpha x+\gamma)\,dx - \frac{\gamma}{\alpha^2}\int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx + \frac{2\gamma^2-\alpha^2}{2\alpha^2}\int \frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

$$\int x^3\,e^{-x^2}\,J_0(\alpha x+\gamma)\,dx = \frac{e^{-x^2}}{8}\,\left[(\alpha^2-4-4x^2)\,J_0(\alpha x+\gamma) + 2\alpha x\,J_1(\alpha x+\gamma)\right] +$$

$$+\frac{\alpha(\alpha^2-4)}{8}\int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx - \frac{\alpha\gamma}{4}\int \frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

$$\int x^3\,e^{-x^2}\,J_1(\alpha x+\gamma)\,dx = \frac{e^{-x^2}}{8}\,\left[(\alpha^2-2-4x^2)\,J_1(\alpha x+\gamma) - 2\alpha x\,J_0(\alpha x+\gamma)\right] +$$

$$+\frac{\alpha(4-\alpha^2)}{8}\int e^{-x^2}\,J_0(\alpha x+\gamma)\,dx+\frac{\gamma}{2\alpha}\int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx+\frac{\alpha^4-2\alpha^2-4\gamma^2}{8\alpha}\int\frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

$$\int\frac{x^3\,e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}=-\frac{e^{-x^2}}{4\alpha^2}\left[\alpha^2\,J_0(\alpha x+\gamma)+(2\alpha x-2\gamma)\,J_1(\alpha x+\gamma)\right]-$$

$$-\frac{\gamma}{2\alpha}\int e^{-x^2}\,J_0(\alpha x+\gamma)\,dx+\frac{4\gamma^2-\alpha^4}{4\alpha^3}\int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx-\frac{\gamma(\gamma^2-\alpha^2)}{\alpha^3}\int\frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

$$\int x^4\,e^{-x^2}\,J_0(\alpha x+\gamma)\,dx=\frac{e^{-x^2}}{16}\left[2(\alpha^2-6-4x^2)x\,J_0(\alpha x+\gamma)+\alpha(4x^2-\alpha^2+8)\,J_1(\alpha x+\gamma)\right]+$$

$$+\frac{\alpha^4-10\alpha^2+12}{16}\int e^{-x^2}\,J_0(\alpha x+\gamma)\,dx-\frac{\gamma}{4}\int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx+\frac{\alpha^2(8-\alpha^2)+4\gamma^2}{16}\int\frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

$$\int x^4\,e^{-x^2}\,J_1(\alpha x+\gamma)\,dx=\frac{e^{-x^2}}{16\alpha}\left\{[\alpha^4-4\alpha^2(2+x^2)]\,J_0(\alpha x+\gamma)-[8\alpha x^3+2\alpha(4-\alpha^2)x+4\gamma]\,J_1(\alpha x+\gamma)\right\}+$$

$$+\frac{\gamma}{4}\int e^{-x^2}\,J_0(\alpha x+\gamma)\,dx-\frac{\alpha^4(8-\alpha^2)+8\gamma^2}{16\alpha^2}\int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx+\frac{\gamma(2\alpha^2-\alpha^4+4\gamma^2)}{8\alpha^2}\int\frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

$$\int\frac{x^4\,e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}=\frac{e^{-x^2}}{8\alpha^3}\left[2\alpha^2(\gamma-\alpha x)\,J_0(\alpha x+\gamma)-(2\alpha^2-\alpha^4+4\gamma^2-4\alpha\gamma x+4\alpha^2x^2)\,J_1(\alpha x+\gamma)\right]+$$

$$+\frac{4\alpha^2-\alpha^4+4\gamma^2}{8\alpha^2}\int e^{-x^2}\,J_0(\alpha x+\gamma)\,dx+\frac{\gamma(\alpha^4+2\alpha^2-4\gamma^2)}{4\alpha^4}\int e^{-x^2}\,J_1(\alpha x+\gamma)\,dx+$$

$$+\frac{\alpha^6-2\alpha^4-12\alpha^2\gamma^2+8\gamma^4}{8\alpha^4}\int\frac{e^{-x^2}\,J_1(\alpha x+\gamma)\,dx}{\alpha x+\gamma}$$

Recurrence relations:

$$\int x^{n+1}\,e^{-x^2}\,J_0(x)\,dx=-\frac{x^n\,e^{-x^2}\,J_0(x)}{2}+\frac{n}{2}\int x^{n-1}\,e^{-x^2}\,J_0(x)\,dx-\frac{\alpha}{2}\int x^n\,e^{-x^2}\,J_1(x)\,dx$$

$$\int x^{n+1}\,e^{-x^2}\,J_1(x)\,dx=-\frac{(\alpha\,x+\gamma)\,x^{n-1}\,e^{-x^2}\,J_1(x)}{2\,\alpha}-\frac{\gamma}{\alpha}\int x^n\,e^{-x^2}\,J_1(x)\,dx+$$

$$+\frac{n-1}{2}\int x^{n-1}\,e^{-x^2}\,J_1(x)\,dx+\frac{(n-1)\,\gamma}{2\,\alpha}\int x^{n-2}\,e^{-x^2}\,J_1(x)\,dx+$$

$$+\frac{\alpha}{2}\int x^n\,e^{-x^2}\,J_0(x)\,dx+\frac{\gamma}{2}\int x^{n-1}\,e^{-x^2}\,J_0(x)\,dx$$

h) $I_\nu(\alpha x)$; Special Case: $\alpha=1$

About the following improper integrals see also [4], 2.15.5, or [7], 6.618.4 .

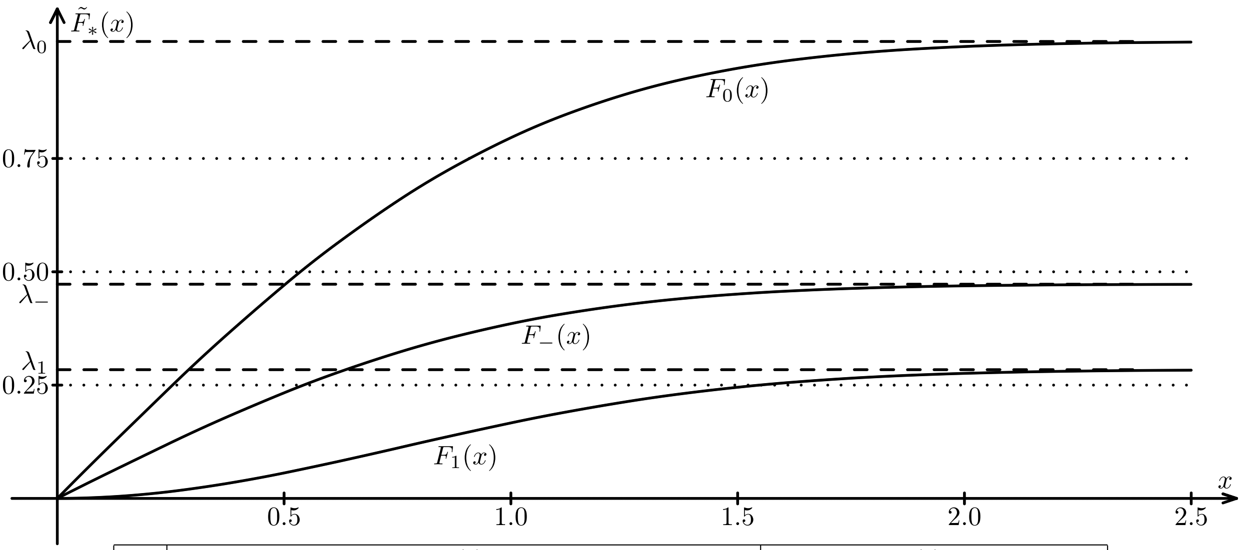
$$\lambda_0=\int_0^\infty\exp(-x^2)\,I_0(x)\,dx=\frac{\sqrt{\pi}}{2}\,e^{1/8}\,I_0\left(\frac{1}{8}\right)=1.00815\;32626$$

$$\lambda_1=\int_0^\infty\exp(-x^2)\,I_1(x)\,dx=e^{1/4}-1=0.28402\;54167$$

$$\lambda_-=\int_0^\infty\frac{\exp(-x^2)\,I_1(x)\,dx}{x}=\frac{\sqrt{\pi}}{4}\,e^{1/8}\left[I_0\left(\frac{1}{8}\right)-I_1\left(\frac{1}{8}\right)\right]=0.47263\;32148$$

Let

$$\tilde{F}_\nu(x)=\int_0^x e^{-t^2}\,I_\nu(t)\,dt=\sum_{k=0}^\infty a_k^{(\nu)}\,x^{2k+1+\nu}\quad\text{and}\quad\tilde{F}_-(x)=\int_0^x\frac{e^{-t^2}\,I_\nu(t)\,dt}{t}=\sum_{k=0}^\infty a_k^{(-)}\,x^{2k+1}\;.$$



k	$a_k^{(0)}$	$a_k^{(0)}$
0	1	1.00000 00000 00000E+00
1	$-\frac{1}{4}$	−2.50000 00000 00000E−01
2	$\frac{17}{320}$	5.31250 00000 00000E−02
3	$-\frac{131}{16128}$	−8.12251 98412 69841E−03
4	$\frac{121}{147456}$	8.20583 76736 11111E−04
5	$-\frac{4579}{162201600}$	−2.82303 01057 44949E−05
6	$\frac{-238703}{27603763200}$	−8.64748 03551 42302E−06
7	$\frac{4937119}{2080899072000}$	2.37258 93612 20110E−06
8	$-\frac{686726911}{1811214552268800}$	−3.79152 712824 5399E−07
9	$\frac{30843873469}{655872751986278400}$	4.70272 21935 33859E−08
10	$-\frac{474430811173}{96654931871662080000}$	−4.90850 08078 31997E−09
11	$\frac{68783074431677}{15370895737076318208000}$	4.47489 04428 36362E−10
12	$-\frac{3503765621989823}{962351733103908618240000000}$	−3.64083 68182 48353E−11
13	$\frac{20932954787378741}{78065972589389067111628800000}$	2.68144 41802 29685E−12
14	$-\frac{10684253218385246191}{5916359842641166099499974656000000}$	−1.80588 29250 68265E−13
15	$\frac{57988854926847793271}{51744965394886687829482536960000000}$	1.12066 66094 81936E−14
16	$-\frac{1333186970994605933602 1}{2068196191060658118975549761126400000000}$	−6.44613 39632 91420E−16
17	$\frac{262667894842474714543807 7}{76072016263922024994318857577431040000000000}$	3.45288 46183 22491E−17
18	$-\frac{16378970655878919827873581}{947481902308464660396773469754019020800000000}$	−1.72868 42752 32366E−18
19	$\frac{2523155270965161599945213 59}{311045203452272630611527102251902854758400000000}$	8.11186 04079 43308E−20
20	$-\frac{953899962106037194572915500479}{266828931453826490506134634177940048943513600000000}$	−3.57494 95263 07664E−21

k	$a_k^{(1)}$	$a_k^{(1)}$
0	$\frac{1}{4}$	2.50000 00000 00000E−01
1	$-\frac{7}{64}$	−1.09375 00000 00000E−01
2	$\frac{73}{2304}$	3.16840 27777 77778E−02
3	$-\frac{1007}{147456}$	−6.82915 58159 72222E−03
4	$\frac{17201}{14745600}$	1.16651 74696 18056E−03
5	$-\frac{348599}{2123366400}$	−1.64172 79655 55073E−04
6	$\frac{8127673}{416179814400}$	1.95292 34044 46595E−05
7	$-\frac{212763487}{106542032486400}$	−1.99699 10657 29472E−06
8	$\frac{874529239}{4931374075084800}$	1.77339 87032 50800E−07
9	$-\frac{189537494759}{13807847410237440000}$	−1.37267 95287 32862E−08
10	$\frac{6158207356841}{668299814655492096000 0}$	9.21473 74902 61837E−10
11	$-\frac{202178049269327}{3849406932415634472960000}$	−5.25218 69685 11883E−11
12	$\frac{6200780742937873}{2602199086312968903720960000}$	2.38290 02075 79313E−12
13	$-\frac{134044005546360727}{2040124083669367620517232640000}$	−6.57038 49397 86072E−14
14	$-\frac{3177788907789045479}{1836111675302430858465509376000000}$	−1.73071 65737 97519E−15
15	$\frac{2415582328462590247}{5481569549590930609529683968000000}$	4.40673 47985 08315E−16
16	$-\frac{930224724042489591102 07}{217348617896920071412339593078374400000 0}$	−4.27987 41167 22742E−17
17	$\frac{8952207071913676364932153}{2816838087944084125503921126295732224000000}$	3.17810 49504 50877E−18
18	$-\frac{824727586392452816381724407}{4067514198991257477227662106371037331456000000}$	−2.02759 60845 98758E−19
19	$\frac{755473884477487382047316442 41}{6508022718386011963564259370193659730329600000000}$	1.16083 47376 28028E−20
20	$-\frac{69976910379259362509514470397 59}{1148015207523292510372735352902161576430141440000000 0}$	−6.09546 89381 01116E−22

k	$a_k^{(-)}$	$a_k^{(-)}$
0	$\frac{1}{2}$	5.00000 00000 00000E−01
1	$-\frac{7}{48}$	−1.45833 33333 33333E−01
2	$\frac{73}{1920}$	3.80208 33333 33333E−02
3	$-\frac{1007}{129024}$	−7.80474 95039 68254E−03
4	$\frac{17201}{13271040}$	1.29613 05217 97840E−03
5	$-\frac{348599}{1946419200}$	−1.79097 59624 23716E−04
6	$\frac{8127673}{386452684800}$	2.10314 82817 11718E−05
7	$-\frac{212763487}{99883155456000}$	−2.13012 38034 44770E−06
8	$\frac{874529239}{4657408848691200}$	1.87771 62740 30259E−07
9	$-\frac{189537494759}{13117455039725568000}$	−1.44492 58197 18802E−08
10	$\frac{6158207356841}{637922550352969728000 0}$	9.65353 45136 07639E−10
11	$-\frac{202178049269327}{3689014976898316369920000}$	−5.48054 29236 64574E−11
12	$\frac{6200780742937873}{2502114506070162407424000000}$	2.47821 62158 82486E−12
13	$-\frac{134044005546360727}{1967262509252604491213045760000}$	−6.81373 25301 48519E−14
14	$-\frac{3177788907789045479}{1774907952792349829849992396800000}$	−1.79039 64556 52606E−15
15	$\frac{2415582328462590247}{5310270501166214027981881344000000}$	4.54888 75339 44067E−16
16	$-\frac{930224724042489591102 07}{210956011488187128135506075634892800000 0}$	−4.40956 72717 74947E−17
17	$\frac{8952207071913676364932153}{2738592585501192899795478872787517440000000}$	3.26890 79490 35188E−18

k	$a_k^{(-)}$	$a_k^{(-)}$
18	$-\frac{824727586392452816381724407}{396047435164938228045851310357179950694400000 0}$	−2.08239 59787 77103E−19
19	$\frac{7554738844774873820473164424 1}{634532215042636166447515288593881823707136000000 0}$	1.19059 97309 00542E−20
20	$-\frac{1706753911689252744134499277 99}{2733369541722125024696988935481337086738432000000 000}$	−6.24413 89122 01143E−22

Asymptotic formulas:

$$\int_x^\infty e^{-t^2}\, I_0(t)\, dt \;\sim\; \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\left[1+\frac{5}{8x}-\frac{47}{128x^2}-\frac{913}{1024x^3}+\frac{10123}{32768x^4}+\frac{625915}{262144x^5}+\ldots\right]=\frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\sum_{k=0}^{\infty}\frac{c_k^{(0)}}{x^k}$$

Let

$$\varphi_n^{(0)}(x)=\frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\sum_{k=0}^{n}\frac{c_k^{(0)}}{x^k}\quad\text{and}\quad q_n^{(0)}(x)=\left[\varphi_n^{(0)}(x)\Bigg/\int_x^\infty e^{-t^2}\, I_0(t)\, dt\right]-1\,.$$

Value	$x=1.5$	$x=2.0$	$x=2.5$	$x=3.0$	$x=3.5$	$x=4.0$	$x=5.0$	$x=6.0$
Exactly	5.90956E-02	1.11275E-02	1.37491E-03	1.09079E-04	5.47423E-06	1.72020E-07	4.06147E-09	1.38485E-15
$\varphi_0^{(0)}(x)$	5.1289E-02	9.5443E-03	1.1868E-03 7	9.5155E-05	4.8273E-06	1.5320E-07	3.6774E-11	1.2700E-15
$q_0^{(0)}(x)$	-0.1321	-0.1423	-0.1368	-0.1277	-0.1182	-0.1094	-0.0946	-0.0829
$\varphi_1^{(0)}(x)$	7.2659E-02	1.2527E-02	1.4835E-03	1.1498E-04	5.6893E-06	1.7714E-07	4.1370E-11	1.4023E-15
$q_1^{(0)}(x)$	0.2295	0.1258	0.0790	0.0541	0.0393	0.0297	0.0186	0.0126
$\varphi_2^{(0)}(x)$	6.4289E-02	1.1651E-02	1.4137E-03	1.1110E-04	5.5446E-06	1.7362E-07	4.0830E-11	1.3894E-15
$q_2^{(0)}(x)$	0.08788	0.04703	0.02824	0.01850	0.01285	0.00930	0.00530	0.00328
$\varphi_3^{(0)}(x)$	5.0740E-02	1.0587E-02	1.3460E-03	1.0795E-04	5.4442E-06	1.7149E-07	4.0568E-11	1.38414E-15
$q_3^{(0)}(x)$	-0.14140	-0.04856	-0.02101	-0.01031	-0.00549	-0.00310	-0.00116	-0.000508
$\varphi_4^{(0)}(x)$	5.38695E-02	1.07714E-02	1.35540E-03	1.08317E-04	5.45414E-06	1.71671E-07	4.05860E-11	1.38445E-15
$q_4^{(0)}(x)$	-0.088435	-0.032002	-0.014184	-0.006984	-0.003670	-0.002028	-0.000708	-0.000289
$\varphi_5^{(0)}(x)$	6.99960E-02	1.14835E-02	1.38442E-03	1.09252E-04	5.47608E-06	1.72029E-07	4.06141E-11	1.38484E-15
$q_5^{(0)}(x)$	0.184454	0.031997	0.006920	0.001588	0.000339	0.000048	-0.000016	-0.000008

$$\int_x^\infty e^{-t^2}\, I_1(t)\, dt \;\sim\; \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\left[1+\frac{1}{8x}-\frac{103}{128x^2}-\frac{677}{1024x^3}+\frac{30587}{32768x^4}+\frac{439535}{262144x^5}+\ldots\right]=\frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\sum_{k=0}^{\infty}\frac{c_k^{(1)}}{x^k}$$

Let

$$\varphi_n^{(1)}(x)=\frac{e^{-x^2+x}}{\sqrt{8\pi x^3}}\sum_{k=0}^{n}\frac{c_k^{(1)}}{x^k}\quad\text{and}\quad q_n^{(1)}(x)=\left[\varphi_n^{(1)}(x)\Bigg/\int_x^\infty e^{-t^2}\, I_1(t)\, dt\right]-1\,.$$

Value	$x=1.5$	$x=2.0$	$x=2.5$	$x=3.0$	$x=3.5$	$x=4.0$	$x=5.0$	$x=6.0$
Exactly	3.87808E-02	8.13674E-03	1.07802E-03	8.95917E-05	4.64340E-06	1.49352E-07	3.63793E-11	1.26530E-15
$\varphi_0^{(1)}(x)$	5.12888E-02	9.54434E-03	1.18677E-03	9.51549E-05	4.82727E-06	1.53199E-07	3.67735E-11	1.27004E-15
$q_0^{(1)}(x)$	0.3225	0.1730	0.1009	0.0621	0.0396	0.0258	0.0108	0.00375
$\varphi_1^{(1)}(x)$	5.55628E-02	1.01409E-02	1.24611E-03	9.91197E-05	4.99967E-06	1.57987E-07	3.76929E-11	1.29650E-15
$q_1^{(1)}(x)$	0.4327	0.2463	0.1559	0.1063	0.0767	0.0578	0.0361	0.0247
$\varphi_2^{(1)}(x)$	3.72200E-02	8.22081E-03	1.09331E-03	9.06119E-05	4.68257E-06	1.50282E-07	3.65092E-11	1.26811E-15
$q_2^{(1)}(x)$	-0.0402	0.0103	0.0142	0.0114	0.00844	0.00622	0.00357	0.00222
$\varphi_3^{(1)}(x)$	2.71730E-02	7.43205E-03	1.04309E-03	8.82819E-05	4.60814E-06	1.48699E-07	3.63147E-11	1.26423E-15
$q_3^{(1)}(x)$	-0.2993	-0.08661	-0.03240	-0.0146	-0.00759	-0.00437	-0.00177	-0.000852
$\varphi_4^{(1)}(x)$	3.66298E-02	7.98887E-03	1.07145E-03	8.93785E-05	4.63816E-06	1.49258E-07	3.63697E-11	1.26514E-15
$q_4^{(1)}(x)$	-0.0555	-0.0182	-0.00609	-0.00238	-0.00113	-0.000632	-0.000264	-0.000129
$\varphi_5^{(1)}(x)$	4.79543E-02	8.48896E-03	1.09183E-03	9.00351E-05	4.65357E-06	1.49509E-07	3.63894E-11	1.26542E-15
$q_5^{(1)}(x)$	0.2365	0.0433	0.0128	0.00495	0.00219	0.00105	0.000278	0.000088

The values of $\varphi_1^{(1)}(x)$ seem to be worse than such of $\varphi_0^{(1)}(x)$, but for $x > x^* = 7.0135\ldots$ one has

$$\varphi_0^{(1)}(x) \ < \ \int_x^\infty e^{-t^2}\, I_1(t)\, dt$$

and therefore $c_1^{(1)} = 1/8 > 0$ is true. Holds $\varphi_0^{(1)}(x^*) = 5.18\,\text{E-}21$.

$$\int_x^\infty \frac{e^{-t^2}\, I_1(t)\, dt}{t} \ \sim \ \frac{e^{-x^2+x}}{\sqrt{8\pi x^5}} \left[1 + \frac{1}{8x} - \frac{167}{128x^2} - \frac{997}{1024x^3} + \frac{75515}{32768x^4} + \frac{931183}{262144x^5} + \ldots\right] = \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}} \sum_{k=0}^\infty \frac{c_k^{(-)}}{x^k}$$

Let

$$\varphi_n^{(-)}(x) = \frac{e^{-x^2+x}}{\sqrt{8\pi x^3}} \sum_{k=0}^n \frac{c_k^{(.)}}{x^k} \quad \text{and} \quad q_n^{(-)}(x) = \left[\varphi_n^{(1)}(x) \Bigg/ \int_x^\infty \frac{e^{-t^2}\, I_1(t)\, dt}{t}\right] - 1\,.$$

Value	$x = 1.5$	$x = 2.0$	$x = 2.5$	$x = 3.0$	$x = 3.5$	$x = 4.0$	$x = 5.0$	$x = 6.0$
Exactly	2.17978E-02	3.64967E-03	4.00293E-04	2.83007E-05	1.27407E-06	3.61817E-08	7.12851E-12	2.07890E-16
$\varphi_0^{(-)}(x)$	3.41925E-02	4.77217E-03	4.74707E-04	3.17183E-05	1.37922E-06	3.82998E-08	7.35471E-12	2.11674E-16
$q_0^{(-)}(x)$	0.5686	0.3076	0.1859	0.1208	0.0825	0.0585	0.0317	0.0182
$\varphi_1^{(-)}(x)$	3.70419E-02	5.07043E-03	4.98442E-04	3.30399E-05	1.42848E-06	3.94967E-08	7.53858E-12	2.16084E-16
$q_1^{(-)}(x)$	0.6993	0.3893	0.2452	0.1675	0.1212	0.0916	0.0575	0.0394
$\varphi_2^{(-)}(x)$	1.72150E-02	3.51388E-03	3.99347E-04	2.84419E-05	1.28158E-06	3.63736E-08	7.15475E-12	2.08413E-16
$q_2^{(-)}(x)$	-0.2102	-0.0372	-0.00236	0.00499	0.00590	0.00530	0.00368	0.00251
$\varphi_3^{(-)}(x)$	7.35100E-03	2.93309E-03	3.69767E-04	2.72981E-05	1.25026E-06	3.57909E-08	7.09747E-12	2.07458E-16
$q_3^{(-)}(x)$	-0.6628	-0.1963	-0.0763	-0.0354	-0.0187	-0.0108	-0.00436	-0.00208
$\varphi_4^{(-)}(x)$	2.29160E-02	3.62044E-03	3.97773E-04	2.82005E-05	1.27144E-06	3.61357E-08	7.12458E-12	2.07835E-16
$q_4^{(-)}(x)$	0.0513	-0.00801	-0.00629	-0.00354	-0.00206	-0.00127	-0.000551	-0.000266
$\varphi_5^{(-)}(x)$	3.89105E-02	4.15018E-03	4.15040E-04	2.86642E-05	1.28077E-06	3.62686E-08	7.13294E-12	2.07931E-16
$q_5^{(-)}(x)$	0.7851	0.1371	0.0368	0.0128	0.00526	0.00240	0.000622	0.000199

Integrals:

$$\int x\, e^{-x^2}\, I_0(x)\, dx = -\frac{e^{-x^2}\, I_0(x)}{2} + \frac{1}{2} \int e^{-x^2}\, I_1(x)\, dx$$

$$\int x\, e^{-x^2}\, I_1(x)\, dx = -\frac{e^{-x^2}\, I_1(x)}{2} + \frac{1}{2} \int e^{-x^2}\, I_0(x)\, dx - \frac{1}{2} \int \frac{e^{-x^2}\, I_1(x)\, dx}{x}$$

$$\int x^2\, e^{-x^2}\, I_0(x)\, dx =$$

$$= -\frac{e^{-x^2}}{4}\, [2x\, I_0(x) + I_1(x)] + \frac{3}{4} \int e^{-x^2}\, I_0(x)\, dx - \frac{1}{4} \int \frac{e^{-x^2}\, I_1(x)\, dx}{x}$$

$$\int x^2\, e^{-x^2}\, I_1(x)\, dx = -\frac{e^{-x^2}}{4}\, [2x\, I_1(x) + I_0(x)] + \frac{1}{4} \int e^{-x^2}\, I_1(x)\, dx$$

Recurrence relations:

$$\int x^{2n+1}\, e^{-x^2}\, I_0(x)\, dx = -\frac{x^{2n} e^{-x^2}}{2}\, I_0(x) + \frac{1}{2} \int x^{2n}\, e^{-x^2}\, I_1(x)\, dx + n \int x^{2n-1}\, e^{-x^2}\, I_0(x)\, dx$$

$$\int x^{2n+1}\, e^{-x^2}\, I_1(x)\, dx = -\frac{x^{2n} e^{-x^2}}{2}\, I_1(x) + \frac{1}{2} \int x^{2n}\, e^{-x^2}\, I_0(x)\, dx + \frac{2n-1}{2} \int x^{2n-1}\, e^{-x^2}\, I_1(x)\, dx$$

$$\int x^{2n+2}\, e^{-x^2}\, I_0(x)\, dx =$$

$$= -\frac{x^{2n}e^{-x^2}}{4}\,[I_1(x) + 2xI_0(x)] + \frac{4n+3}{4}\int x^{2n}\,e^{-x^2}\,I_0(x)\,dx + \frac{2n-1}{4}\int x^{2n-1}\,e^{-x^2}\,I_1(x)\,dx$$

$$\int x^{2n+2}\,e^{-x^2}\,I_1(x)\,dx =$$

$$= -\frac{x^{2n}e^{-x^2}}{4}\,[I_0(x) + 2xI_1(x)] + \frac{4n+1}{4}\int x^{2n}\,e^{-x^2}\,I_1(x)\,dx + \frac{n}{2}\int x^{2n-1}\,e^{-x^2}\,I_0(x)\,dx$$

i) General case

About the following improper integrals see also [4], 2.15.5, or [7], 6.618.4 .

$$\int_0^\infty \exp(-x^2)\,I_0(\alpha x)\,dx = \frac{\sqrt{\pi}}{2}\,e^{\alpha^2/8}\,I_0\left(\frac{\alpha^2}{8}\right)$$

$$\int_0^\infty \exp(-x^2)\,I_1(\alpha x)\,dx = \frac{1}{\alpha}\,\left[e^{\alpha^2/4} - 1\right]$$

$$\int_0^\infty \frac{\exp(-x^2)\,I_1(\alpha x)\,dx}{x} = \frac{\sqrt{\pi}\,\alpha^2}{4}\,e^{\alpha^2/8}\,\left[I_0\left(\frac{\alpha^2}{8}\right) - I_1\left(\frac{\alpha^2}{8}\right)\right]$$

Let

$$\tilde{F}_\nu(x;\alpha) = \int_0^x e^{-t^2}\,I_\nu(\alpha t)\,dt = \sum_{k=0}^\infty a_k^{(\nu)}(\alpha)\,x^{2k+1+\nu} \text{ and } \tilde{F}_-(x;\alpha) = \int_0^x \frac{e^{-t^2}\,I_\nu(\alpha t)\,dt}{t} = \sum_{k=0}^\infty a_k^{(-)}(\alpha)\,x^{2k+1}$$

and $a_k^{(*)}(\alpha) = b_k^{(*)}(\alpha)/n_k^{(*)}$.

k	$n_k^{(0)}$, $b_k^{(0)}(\alpha)$
0	1 1
1	12 $\alpha^2 - 4$
2	320 $\alpha^4 - 16\,\alpha^2 + 32$
3	16128 $\alpha^6 - 36\,\alpha^4 + 288\,\alpha^2 - 384$
4	13 27104 $\alpha^8 - 64\,\alpha^6 + 1152\,\alpha^4 - 6144\,\alpha^2 + 6144$
5	1622 01600 $\alpha^{10} - 100\,\alpha^8 + 3200\,\alpha^6 - 38400\,\alpha^4 + 153600\,\alpha^2 - 122880$
6	2 76037 63200 $\alpha^{12} - 144\,\alpha^{10} + 7200\,\alpha^8 - 153600\,\alpha^6 + 1382400\,\alpha^4 - 4423680\,\alpha^2 + 2949120$
7	624 26972 16000 $\alpha^{14} - 196\,\alpha^{12} + 14112\,\alpha^{10} - 470400\,\alpha^8 + 7526400\,\alpha^6 - 54190080\,\alpha^4 + 144506880\,\alpha^2 - 82575360$

k	$n_k^{(1)}$, $b_k^{(1)}(\alpha)$
0	4 α
1	64 $\alpha^3 - 8\,\alpha$
2	2304 $\alpha^5 - 24\,\alpha^3 + 96\,\alpha$
3	1 47456 $\alpha^7 - 48\,\alpha^5 + 576\,\alpha^3 - 1536\,\alpha$
4	147 45600 $\alpha^9 - 80\,\alpha^7 + 1920\,\alpha^5 - 15360\,\alpha^3 + 30720\,\alpha$

k	$n_k^{(1)}\,,\ b_k^{(1)}(\alpha)$
5	21233 66400 $\alpha^{11} - 120\,\alpha^9 + 4800\,\alpha^7 - 76800\,\alpha^5 + 460800\,\alpha^3 - 737280\,\alpha$
6	41 61798 14400 $\alpha^{13} - 168\,\alpha^{11} + 10080\,\alpha^9 - 268800\,\alpha^7 + 3225600\,\alpha^5 - 15482880\,\alpha^3 + 20643840\,\alpha$
7	10654 20324 86400 $\alpha^{15} - 224\,\alpha^{13} + 18816\,\alpha^{11} - 752640\,\alpha^9 + 15052800\,\alpha^7 - 144506880\,\alpha^5 + 578027520\,\alpha^3 - 660602880\,\alpha$

k	$n_k^{(-)}\,,\ b_k^{(-)}(\alpha)$
0	2 α
1	48 $\alpha^3 - 8\,\alpha$
2	1920 $\alpha^5 - 24\,\alpha^3 + 96\,\alpha$
3	1 29024 $\alpha^7 - 48\,\alpha^5 + 576\,\alpha^3 - 1536\,\alpha$
4	132 71040 $\alpha^9 - 80\,\alpha^7 + 1920\,\alpha^5 - 15360\,\alpha^3 + 30720\,\alpha$
5	19464 19200 $\alpha^{11} - 120\,\alpha^9 + 4800\,\alpha^7 - 76800\,\alpha^5 + 460800\,\alpha^3 - 737280\,\alpha$
6	38 64526 84800 $\alpha^{13} - 168\,\alpha^{11} + 10080\,\alpha^9 - 268800\,\alpha^7 + 3225600\,\alpha^5 - 15482880\,\alpha^3 + 20643840\,\alpha$
7	9988 31554 56000 $\alpha^{15} - 224\,\alpha^{13} + 18816\,\alpha^{11} - 752640\,\alpha^9 + 15052800\,\alpha^7 - 144506880\,\alpha^5 + 578027520\,\alpha^3 - 660602880\,\alpha$

Asymptotic formulas: They are based on the asymptotic formulas for $I_\nu(x)$. So they are inaccurate for small values of α. At least they should not be used for $x < 2/\alpha$, roughly spoken.

$$\int_x^\infty e^{-t^2}\, I_0(\alpha t)\, dt \;\sim\; \frac{e^{-x^2+\alpha x}}{\sqrt{8\pi x^3}} \left[1 + \frac{4\alpha^2+1}{8\alpha x} + \frac{32\,\alpha^4 - 88\,\alpha^2 + 9}{128\alpha^2 x^2} + \frac{128\,\alpha^6 - 992\,\alpha^4 - 124\,\alpha^2 + 75}{1024\alpha^3 x^3} + \right.$$
$$+ \frac{2048\,\alpha^8 - 30208\,\alpha^6 + 37440\,\alpha^4 - 2832\,\alpha^2 + 3675}{32768\alpha^4 x^4} +$$
$$\left. + \frac{8192\,\alpha^{10} - 194560\,\alpha^8 + 721152\,\alpha^6 + 60096\,\alpha^4 - 28500\,\alpha^2 + 59535}{262144\alpha^5 x^5} + \ldots \right]$$

$$\int_x^\infty e^{-t^2}\, I_1(\alpha t)\, dt \;\sim\; \frac{e^{-x^2+\alpha x}}{\sqrt{8\pi x^3}} \left[1 + \frac{4\alpha^2-3}{8\alpha x} + \frac{32\,\alpha^4 - 120\,\alpha^2 - 15}{128\alpha^2 x^2} + \frac{128\,\alpha^6 - 1120\,\alpha^4 + 420\,\alpha^2 - 105}{1024\alpha^3 x^3} + \right.$$
$$+ \frac{2048\,\alpha^8 - 32256\,\alpha^6 + 60480\,\alpha^4 + 5040\,\alpha^2 - 4725}{32768\alpha^4 x^4} +$$
$$\left. + \frac{8192\,\alpha^{10} - 202752\,\alpha^8 + 887040\,\alpha^6 - 221760\,\alpha^4 + 41580\,\alpha^2 - 72765}{262144\alpha^5 x^5} + \ldots \right]$$

$$\int_x^\infty \frac{e^{-t^2}\, I_1(\alpha t)\, dt}{t} \;\sim\; \frac{e^{-x^2+\alpha x}}{\sqrt{8\pi\alpha x^5}} \left[1 + \frac{4\alpha^2-3}{8\alpha x} + \frac{32\,\alpha^4 - 184\,\alpha^2 - 15}{128\alpha^2 x^2} + \frac{128\,\alpha^6 - 1632\,\alpha^4 + 612\,\alpha^2 - 105}{1024\alpha^3 x^3} + \right.$$
$$+ \frac{2048\,\alpha^8 - 44544\,\alpha^6 + 115776\,\alpha^4 + 6960\,\alpha^2 - 4725}{32768\alpha^4 x^4} +$$
$$\left. + \frac{8192\,\alpha^{10} - 268288\,\alpha^8 + 1612032\,\alpha^6 - 403008\,\alpha^4 + 55020\,\alpha^2 - 72765}{262144\alpha^5 x^5} + \ldots \right]$$

Approximations for $\alpha < 1$:

$$\tilde{F}_0(x;\alpha) \approx \frac{\sqrt{\pi}}{2} \left[1 + \frac{\alpha^2}{8} + \frac{3\,\alpha^4}{256} + \frac{5\,\alpha^6}{6144} + \frac{35\,\alpha^8}{786432} + \frac{21\,\alpha^{10}}{10485760} + \frac{77\,\alpha^{12}}{1006632960} + \frac{143\,\alpha^{14}}{56371445760} + \right.$$

$$+\frac{143}{1924145348608}\,\alpha^{16}+\frac{2431\,\alpha^{18}}{1246846185897984}+\frac{46189\,\alpha^{20}}{997476948718387200}\Bigg]\ \mathrm{erf}\,(x)-$$

$$-\frac{\alpha^2 x}{8}\ \Bigg[1+\frac{3\,\alpha^2}{32}+\frac{5\alpha^4}{768}+\frac{35\alpha^6}{98304}+\frac{21\alpha^8}{1310720}+\frac{77\alpha^{10}}{125829120}+\frac{143\alpha^{12}}{7046430720}+\frac{143\alpha^{14}}{240518168576}+$$

$$+\frac{2431\alpha^{16}}{155855773237248}+\frac{46189\alpha^{18}}{124684618589798400}\Bigg]-\frac{\alpha^4 x^3}{128}\ \Bigg[1+\frac{5\alpha^2}{72}+\frac{35\alpha^4}{9216}+\frac{7\alpha^6}{40960}+\frac{77\alpha^8}{11796480}+$$

$$+\frac{143\alpha^{10}}{660602880}+\frac{143\alpha^{12}}{22548578304}+\frac{2431\alpha^{14}}{14611478740992}+\frac{2431\alpha^{14}}{14611478740992}+\frac{4199\alpha^{16}}{48983243017420800}\Bigg]-$$

$$-\frac{\alpha^6 x^5}{4608}\ \Bigg[1+\frac{7\alpha^2}{128}+\frac{63\alpha^4}{25600}+\frac{77\alpha^6}{819200}+\frac{143\alpha^8}{45875200}+\frac{429\alpha^{10}}{4697620480}+\frac{2431\alpha^{12}}{1014686023680}+\frac{46189\alpha^{14}}{811748818944000}\Bigg]-$$

$$-\frac{\alpha^8 x^7}{294912}\ \Bigg[1+\frac{9\alpha^2}{200}+\frac{11\alpha^4}{6400}+\frac{143\alpha^6}{2508800}+\frac{429\alpha^8}{256901120}+\frac{2431\alpha^{10}}{55490641920}+\frac{46189\alpha^{12}}{44392513536000}\Bigg]-$$

$$-\frac{\alpha^{10} x^9}{294912}\ \Bigg[1+\frac{11\alpha^2}{288}+\frac{143\alpha^4}{112896}+\frac{715\alpha^6}{19267584}+\frac{12155\alpha^8}{12485394432}+\frac{46189\alpha^{10}}{1997663109120}\Bigg]-$$

$$-\frac{\alpha^{12} x^{11}}{4246732800}\ \Bigg[1+\frac{13\alpha^2}{392}+\frac{195\alpha^4}{200704}+\frac{1105\alpha^6}{43352064}+\frac{4199\alpha^8}{6936330240}\Bigg]-$$

$$-\frac{\alpha^{14} x^{13}}{832359628800}\ \Bigg[1+\frac{15\,\alpha^2}{512}+\frac{85\alpha^4}{110592}+\frac{323\alpha^6}{17694720}\Bigg]-\frac{\alpha^{16} x^{15}}{213084064972800}\ \Bigg[1+\frac{17\alpha^2}{648}+\frac{323\alpha^4}{518400}\Bigg]-$$

$$-\frac{\alpha^{18} x^{17}}{69039237051187200}\ \Bigg[1+\frac{19\alpha^2}{800}\Bigg]-\frac{\alpha^{20} x^{19}}{69039237051187200}$$

$$\tilde{F}_1(x;\alpha)\approx\frac{\alpha(1-e^{-x^2})}{4}\ \Bigg[1+\frac{\alpha^2}{8}+\frac{\alpha^4}{96}+\frac{\alpha^6}{1536}+\frac{1\alpha^8}{30720}+\frac{\alpha^{10}}{737280}+\frac{\alpha^{12}}{20643840}+\frac{\alpha^{14}}{660602880}+$$

$$+\frac{\alpha^{16}}{23781703680}+\frac{\alpha^{18}}{951268147200}\Bigg]-\frac{\alpha^3 x^2}{32}\ \Bigg[1+\frac{\alpha^2}{12}+\frac{\alpha^4}{192}+\frac{\alpha^6}{3840}+$$

$$+\frac{\alpha^8}{92160}+\frac{\alpha^{10}}{2580480}+\frac{\alpha^{12}}{82575360}++\frac{\alpha^{14}}{2972712960}+\frac{\alpha^{16}}{118908518400}+\frac{\alpha^{18}}{5231974809600}\Bigg]-$$

$$-\frac{\alpha^5 x^4}{768}\ \Bigg[1+\frac{\alpha^2}{16}+\frac{\alpha^4}{320}+\frac{\alpha^6}{7680}+\frac{\alpha^8}{215040}+\frac{\alpha^{10}}{6881280}+\frac{\alpha^{12}}{247726080}+\frac{\alpha^{14}}{9909043200}+\frac{\alpha^{16}}{435997900800}\Bigg]-$$

$$-\frac{\alpha^7 x^6}{36864}\ \Bigg[1+\frac{\alpha^2}{20}+\frac{\alpha^4}{480}+\frac{\alpha^6}{13440}+\frac{\alpha^8}{430080}+\frac{\alpha^{10}}{15482880}+\frac{\alpha^{12}}{619315200}+\frac{\alpha^{14}}{27249868800}\Bigg]-$$

$$-\frac{\alpha^9 x^8}{2949120}\ \Bigg[1+\frac{\alpha^2}{24}+\frac{\alpha^4}{672}+\frac{\alpha^6}{21504}+\frac{\alpha^8}{774144}+\frac{\alpha^{10}}{30965760}+\frac{\alpha^{12}}{1362493440}\Bigg]-$$

$$-\frac{\alpha^{11} x^{10}}{353894400}\ \Bigg[1+\frac{\alpha^2}{28}+\frac{\alpha^4}{896}+\frac{\alpha^6}{32256}+\frac{\alpha^8}{1290240}+\frac{\alpha^{10}}{56770560}\Bigg]-$$

$$-\frac{\alpha^{13} x^{12}}{59454259200}\ \Bigg[1+\frac{\alpha^2}{32}+\frac{\alpha^4}{1152}+\frac{\alpha^6}{46080}+\frac{\alpha^8}{2027520}\Bigg]-\frac{\alpha^{15} x^{14}}{13317754060800}\ \Bigg[1+\frac{\alpha^2}{36}+\frac{\alpha^4}{1440}+\frac{\alpha^6}{63360}\Bigg]-$$

$$-\frac{\alpha^{17} x^{16}}{3835513169510400}\ \Bigg[1+\frac{\alpha^2}{40}+\frac{\alpha^4}{1760}\Bigg]-\frac{\alpha^{19} x^{18}}{1380784741023744000}\ \Bigg[1+\frac{\alpha^2}{44}\Bigg]-\frac{\alpha^{21} x^{20}}{607545286050447360000}$$

$$\tilde{F}_-(x;\alpha)\approx\frac{\sqrt{\pi}\,\alpha}{4}\ \Bigg[1+\frac{\alpha^2}{16}+\frac{\alpha^4}{256}+\frac{5\alpha^6}{24576}+\frac{7\alpha^8}{786432}+\frac{7\alpha^{10}}{20971520}+\frac{11\alpha^{12}}{1006632960}+\frac{143\,\alpha^{14}}{450971566080}+$$

$$+\frac{143\,\alpha^{16}}{17317308137472}+\frac{2431\alpha^{18}}{12468461858979840}+\frac{4199\,\alpha^{20}}{997476948718387200}\Bigg]\ \mathrm{erf}\,(x)-\frac{\alpha^3 x}{32}\ \Bigg[1+\frac{\alpha^2}{16}+\frac{5\alpha^4}{1536}+$$

$$+\frac{7\alpha^6}{49152}+\frac{7\alpha^8}{1310720}+\frac{11\alpha^{10}}{62914560}+\frac{143\alpha^{12}}{28185722880}+\frac{143\alpha^{14}}{1082331758592}+\frac{2431\alpha^{16}}{779278866186240}\Bigg]-$$

$$-\frac{\alpha^5 x^3}{32}\left[1+\frac{5\alpha^2}{96}+\frac{7\alpha^4}{3072}+\frac{7\alpha^6}{81920}+\frac{11\alpha^8}{3932160}+\frac{143\alpha^{10}}{1761607680}+\frac{143\alpha^{12}}{67645734912}+\frac{2431\alpha^{14}}{48704929136640}\right]-$$

$$-\frac{\alpha^7 x^5}{36864}\left[1+\frac{7\alpha^2}{160}+\frac{21\alpha^4}{12800}+\frac{11\alpha^6}{204800}+\frac{143\alpha^8}{91750400}+\frac{143\alpha^8}{91750400}+\frac{143\alpha^{10}}{3523215360}+\frac{2431\alpha^{12}}{2536715059200}\right]-$$

$$-\frac{\alpha^9 x^7}{2949120}\left[1+\frac{3\alpha^2}{80}+\frac{11\alpha^4}{8960}+\frac{143\alpha^6}{4014080}+\frac{143\alpha^8}{154140672}+\frac{2431\alpha^{10}}{110981283840}\right]-$$

$$-\frac{\alpha^{11} x^9}{353894400}\left[1+\frac{11\alpha^2}{336}+\frac{143\alpha^4}{150528}+\frac{715\alpha^6}{28901376}+\frac{2431\alpha^8}{4161798144}\right]-$$

$$-\frac{\alpha^{13} x^{11}}{59454259200}\left[1+\frac{13\alpha^2}{448}+\frac{65\alpha^4}{86016}+\frac{221\alpha^6}{12386304}\right]-\frac{\alpha^{15} x^{13}}{13317754060800}\left[1+\frac{5\alpha^2}{192}+\frac{17\alpha^4}{27648}\right]-$$

$$-\frac{\alpha^{17} x^{15}}{3835513169510400}\left[1+\frac{17\alpha^2}{720}\right]-\frac{\alpha^{19} x^{17}}{1380784741023744000}$$

<u>Integrals:</u>

$$\int x\, e^{-x^2}\, I_0(\alpha x)\, dx = -\frac{e^{-x^2}\, I_0(\alpha x)}{2}+\frac{\alpha}{2}\int e^{-x^2}\, I_1(\alpha x)\, dx$$

$$\int x\, e^{-x^2}\, I_1(\alpha x)\, dx = -\frac{e^{-x^2}\, I_1(\alpha x)}{2}+\frac{\alpha}{2}\int e^{-x^2}\, I_0(\alpha x)\, dx-\frac{1}{2}\int \frac{e^{-x^2}\, I_1(\alpha x)\, dx}{x}$$

$$\int x^2\, e^{-x^2}\, I_0(\alpha x)\, dx =$$

$$= -\frac{e^{-x^2}}{4}\left[2x\, I_0(\alpha x)+\alpha\, I_1(\alpha x)\right]+\frac{\alpha^2+2}{4}\int e^{-x^2}\, I_0(\alpha x)\, dx-\frac{\alpha}{4}\int \frac{e^{-x^2}\, I_1(\alpha x)\, dx}{x}$$

$$\int x^2\, e^{-x^2}\, I_1(\alpha x)\, dx = -\frac{e^{-x^2}}{4}\left[2x\, I_1(\alpha x)+\alpha\, I_0(\alpha x)\right]+\frac{\alpha^2}{4}\int e^{-x^2}\, I_1(\alpha x)\, dx$$

Recurrence relations:

$$\int x^{2n+1}\, e^{-x^2}\, I_0(\alpha x)\, dx = -\frac{x^{2n}e^{-x^2}}{2}\, I_0(\alpha x)+\frac{\alpha}{2}\int x^{2n}\, e^{-x^2}\, I_1(\alpha x)\, dx+n\int x^{2n-1}\, e^{-x^2}\, I_0(\alpha x)\, dx$$

$$\int x^{2n+1}\, e^{-x^2}\, I_1(\alpha x)\, dx = -\frac{x^{2n}e^{-x^2}}{2}\, I_1(\alpha x)+\frac{\alpha}{2}\int x^{2n}\, e^{-x^2}\, I_0(\alpha x)\, dx+\frac{2n-1}{2}\int x^{2n-1}\, e^{-x^2}\, I_1(\alpha x)\, dx$$

$$\int x^{2n+2}\, e^{-x^2}\, I_0(\alpha x)\, dx =$$

$$= -\frac{x^{2n}e^{-x^2}}{4}\left[\alpha I_1(\alpha x)+2xI_0(\alpha x)\right]+\frac{4n+2+\alpha^2}{4}\int x^{2n}\, e^{-x^2}\, I_1(\alpha x)\, dx+\frac{(2n-1)\alpha}{2}\int x^{2n-1}\, e^{-x^2}\, I_1(\alpha x)\, dx$$

$$\int x^{2n+2}\, e^{-x^2}\, I_1(\alpha x)\, dx =$$

$$= -\frac{x^{2n}e^{-x^2}}{4}\left[\alpha I_0(\alpha x)+2xI_1(\alpha x)\right]+\frac{4n+\alpha^2}{4}\int x^{2n}\, e^{-x^2}\, I_0(\alpha x)\, dx+\frac{n\alpha}{2}\int x^{2n-1}\, e^{-x^2}\, I_0(\alpha x)\, dx$$

1.3.12. Integrals with Orthogonal Polynomials $F_n(x)$: $\int F_n(x) \cdot Z_\nu(x)\,dx$

No simple recurrence relations were found.
Let $p_n(x)$ denote some system of orthogonal polynomials with degree$[p_n(x)] = n$. Furthermore, let

$$\int p_n(x)\, Z_\nu(x)\, dx = \sum_{k=0}^{n+1} \left[\lambda_k\, p_k(x)\, Z_0(x) + \mu_k\, p_k(x)\, Z_1(x)\right] + \gamma\, \Xi(x)\ ,$$

where $\Xi(x)$ denotes one of the functions of the type $\Phi(x)$ or $\Psi(x)$, defined as on page 9 and depending from $Z_\nu(x)$.
In the following the non-zero values of λ_k, μ_k and γ are given for $2 \le n \le 10$, if $p_1(x) = x$, otherwise for $1 \le n \le 10$.

a) Legendre Polynomials $P_n(x)$:

$$\int_{-1}^{1} P_n^2(x)\, dx \quad = \quad \frac{2}{2n+1}$$

First polynomials:

$$P_0(x) = 1\,,\ P_1(x) = x\,,\ P_2(x) = \frac{3x^2-1}{2}\,,\ P_3(x) = \frac{5x^3-3x}{2}\,,\ P_4(x) = \frac{35x^4-30x^2+3}{8}$$

$Z_\nu(x) \ = \ J_0(x)$:

n	λ_k	μ_k	γ
2	$\lambda_1 = -1/2$	$\mu_0 = 1/2\,,\ \mu_2 = 1$	-2
3	$\lambda_0 = 5/3\,,\ \lambda_2 = 10/3$	$\mu_1 = -10\,,\ \mu_3 = 1$	0
4	$\lambda_1 = 33/4\,,\ \lambda_3 = 21/4$	$\mu_0 = -27/2\,,\ \mu_2 = -105/4\,,\ \mu_4 = 1$	$87/2$
5	$\lambda_0 = -1253/15\,,\ \lambda_2 = -485/3\,,\ \lambda_4 = 36/5$	$\mu_1 = 2317/5\,,\ \mu_3 = -252/5\,,\ \mu_5 = 1$	0
6	$\lambda_1 = -1309/2\,,\ \lambda_3 = -1274/3\,,\ \lambda_5 = 55/6$	$\mu_0 = 1070\,,\ \mu_2 = 4155/2\,,\ \mu_4 = -165/2\,,$ $\mu_6 = 1$	$-6865/2$
7	$\lambda_0 = 349554/35\,,\ \lambda_2 = 135195/7\,,$ $\lambda_4 = -30519/35\,,\ \lambda_6 = 78/7$	$\mu_1 = -1937538/35\,,\ \mu_3 = 30129/5\,,$ $\mu_5 = -858/7\,,\ \mu_7 = 1$	0
8	$\lambda_1 = 878489/8\,,\ \lambda_3 = 3419941/48\,,$ $\lambda_5 = -37235/24\,,\ \lambda_7 = 105/8\,,$	$\mu_0 = -2872345/16\,,\ \mu_2 = -5576835/16\,,$ $\mu_4 = 13845\,,\ \mu_6 = -1365/8\,,\ \mu_8 = 1$	$4607015/8$
9	$\lambda_0 = -704613008/315\,,$ $\lambda_2 = -272518010/63\,,$ $\lambda_4 = 6835407/35\,,$ $\lambda_6 = -158197/63\,,\ \lambda_8 = 136/9$	$\mu_1 = 1301857612/105\,,$ $\mu_3 = -20244146/15\,,$ $\mu_5 = 576565/21\,,$ $\mu_7 = -680/3\,,\ \mu_9 = 1$	0
10	$\lambda_1 = -253102035/8\,,$ $\lambda_3 = -20527528\,,$ $\lambda_5 = 8939843/20\,,$ $\lambda_7 = -15195/4\,,$ $\lambda_9 = 171/10\,,$	$\mu_0 = 413776575/8\,,$ $\mu_2 = 200843175/2\,,$ $\mu_4 = -7977789/2\,,$ $\mu_6 = 983229/20\,,$ $\mu_8 = -2907/10\,,\ \mu_{10} = 1$	-165916233

$Z_\nu(x) \ = \ I_0(x)$:

n	λ_k	μ_k	γ
2	$\lambda_1 = -1/2$	$\mu_0 = 1/2\,,\ \mu_2 = 1$	-1
3	$\lambda_0 = -5/3\,,\ \lambda_2 = -10/3$	$\mu_1 = 10\,,\ \mu_3 = 1$	0
4	$\lambda_1 = -15/2\,,\ \lambda_3 = -21/4$	$\mu_0 = 51/4\,,\ \mu_2 = 105/4\,,\ \mu_4 = 1$	36
5	$\lambda_0 = -1267/15\,,\ \lambda_2 = -523/3\,,\ \lambda_4 = -36/5$	$\mu_1 = 2723/5\,,\ \mu_3 = 252/5\,,\ \mu_5 = 1$	0

n	λ_k	μ_k	γ
6	$\lambda_1 = -1291/2\,,\ \lambda_3 = -5299/12\,,\ \lambda_5 = -55/6$	$\mu_0 = 4385/4\,,\ \mu_2 = 9015/4\,,\ \mu_4 = 165/2\,,$ $\mu_6 = 1$	$6155/2$
7	$\lambda_0 = -371268/35\,,\ \lambda_2 = -153147/7\,,$ $\lambda_4 = -31257/35\,,\ \lambda_6 = -78/7$	$\mu_1 = 2392146/35\,,\ \mu_3 = 31647/5\,,$ $\mu_5 = 858/7\,,\ \mu_7 = 1$	0
8	$\lambda_1 = -1790753/16\,,\ \lambda_3 = -918799/12\,,$ $\lambda_5 = -4730/3\,,\ \lambda_7 = -105/8\,,$	$\mu_0 = 3041255/16\,,\ \mu_2 = 3126255/8\,,$ $\mu_4 = 114465/8\,,\ \mu_6 = 1365/8\,,\ \mu_8 = 1$	$1067255/2$
9	$\lambda_0 = -766234942/315\,,$ $\lambda_2 = -316070848/63\,,$ $\lambda_4 = -7167687/35\,,$ $\lambda_6 = -160043/63\,,\ \lambda_8 = -136/9$	$\mu_1 = 1645668398/105\,,$ $\mu_3 = 21771386/15\,,$ $\mu_5 = 590315/21\,,$ $\mu_7 = 680/3\,,\ \mu_9 = 1$	0
10	$\lambda_1 = -525156393/16\,,$ $\lambda_3 = -179631375/8\,,$ $\lambda_5 = -4623729/10\,,$ $\lambda_7 = -30657/8\,,$ $\lambda_9 = -171/10\,,$	$\mu_0 = 891878697/16\,,$ $\mu_2 = 114600720\,,$ $\mu_4 = 33567993/8\,,$ $\mu_6 = 2001597/40\,,$ $\mu_8 = 2907/10\,,\ \mu_{10} = 1$	156491622

$Z_\nu(x)\ =\ K_0(x):$

n	λ_k	μ_k	γ
2	$\lambda_1 = -1/2$	$\mu_0 = -1/2\,,\ \mu_2 = -1$	1
3	$\lambda_0 = -5/3\,,\ \lambda_2 = -10/3$	$\mu_1 = -10\,,\ \mu_3 = -1$	0
4	$\lambda_1 = -15/2\,,\ \lambda_3 = -21/4$	$\mu_0 = -51/4\,,\ \mu_2 = -105/4\,,\ \mu_4 = -1$	36
5	$\lambda_0 = -1267/15\,,\ \lambda_2 = -523/3\,,\ \lambda_4 = -36/5$	$\mu_1 = -2723/5\,,\ \mu_3 = -252/5\,,\ \mu_5 = -1$	0
6	$\lambda_1 = -1291/2\,,\ \lambda_3 = -5299/12\,,$ $\lambda_5 = -55/6$	$\mu_0 = -4385/4\,,\ \mu_2 = -9015/4\,,$ $\mu_4 = -165/2\,,\mu_6 = -1$	$6155/2$
7	$\lambda_0 = -371268/35\,,\ \lambda_2 = -153147/7\,,$ $\lambda_4 = -31257/35\,,\ \lambda_6 = -78/7$	$\mu_1 = -2392146/35\,,\ \mu_3 = -31647/5\,,$ $\mu_5 = -858/7\,,\ \mu_7 = -1$	0
8	$\lambda_1 = -1790753/16\,,\ \lambda_3 = -918799/12\,,$ $\lambda_5 = -4730/3\,,\ \lambda_7 = -105/8\,,$	$\mu_0 = -3041255/16\,,\ \mu_2 = -3126255/8\,,$ $\mu_4 = -114465/8\,,\ \mu_6 = -1365/8\,,$ $\mu_8 = -1$	$1067255/2$
9	$\lambda_0 = -766234942/315\,,$ $\lambda_2 = -316070848/63\,,$ $\lambda_4 = -7167687/35\,,$ $\lambda_6 = -160043/63\,,\ \lambda_8 = -136/9$	$\mu_1 = -1645668398/105\,,$ $\mu_3 = -21771386/15\,,$ $\mu_5 = -590315/21\,,$ $\mu_7 = -680/3\,,\ \mu_9 = -1$	0
10	$\lambda_1 = -525156393/16\,,$ $\lambda_3 = -179631375/8\,,$ $\lambda_5 = -4623729/10\,,$ $\lambda_7 = -30657/8\,,$ $\lambda_9 = -171/10\,,$	$\mu_0 = -891878697/16\,,$ $\mu_2 = -114600720\,,$ $\mu_4 = -33567993/8\,,$ $\mu_6 = -2001597/40\,,$ $\mu_8 = -2907/10\,,\ \mu_{10} = -1$	156491622

$Z_\nu(x)\ =\ J_1(x):$

n	λ_k	μ_k	γ
2	$\lambda_2 = -1\,,$	$\mu_1 = 3\,,$	0
3	$\lambda_1 = -3/2\,,\ \lambda_3 = -1$	$\mu_0 = 5/2\,,\ \mu_2 = 5$	-9
4	$\lambda_0 = 35/3\,,\ \lambda_2 = 70/3\,,\ \lambda_4 = -1$	$\mu_1 = -67\,,\ \mu_3 = 7$	0
5	$\lambda_1 = 291/4\,,\ \lambda_3 = 189/4\,,\ \lambda_5 = -1$	$\mu_0 = -119\,,\ \mu_2 = -925/4\,,\ \mu_4 = 9$	$765/2$
6	$\lambda_0 = -4536/5\,,\ \lambda_2 = -1755\,,$ $\lambda_4 = 396/5\,,\ \lambda_6 = -1\,,$	$\mu_1 = 25152/5\,,\ \mu_3 = -2737/5\,,$ $\mu_5 = 11$	0
7	$\lambda_1 = -33743/4\,,\ \lambda_3 = -65681/12\,,$ $\lambda_5 = 715/6\,,\ \lambda_7 = -1$	$\mu_0 = 13791\,,\ \mu_2 = 107105/4\,,$ $\mu_4 = -2127/2\,,\ \mu_6 = 13$	-44240

n	λ_k	μ_k	γ
8	$\lambda_0 = 5211558/35\,,\ \lambda_2 = 2015640/7\,,$ $\lambda_4 = -455013/35\,,\ \lambda_6 = 1170/7\,, \lambda_8 = -1$	$\mu_1 = -28887006/35\,,\ \mu_3 = 449198/5\,,$ $\mu_5 = -12793/7\,,\ \mu_7 = 15\,,$	0
9	$\lambda_1 = 14866827/8\,,$ $\lambda_3 = 19292091/16\,,$ $\lambda_5 = -210045/8\,,$ $\lambda_7 = 1785/8\,,\ \lambda_9 = -1$	$\mu_0 = -48609209/16\,,$ $\mu_2 = -94377775/16\,,$ $\mu_4 = 468603/2\,,$ $\mu_6 = -23101/8\,,\ \mu_8 = 17$	77965335/8
10	$\lambda_0 = -2668148626/63\,,$ $\lambda_2 = -5159701430/63\,,$ $\lambda_4 = 25883544/7\,,$ $\lambda_6 = -2995213/63\,,$ $\lambda_8 = 2584/9\,,\ \lambda_{10} = -1$	$\mu_1 = 4929726722/21\,,$ $\mu_3 = -76658236/3\,,$ $\mu_5 = 10916356/21\,,$ $\mu_7 = -12875/3\,,$ $\mu_9 = 19\,,$	0

$Z_\nu(x) \ = \ I_1(x) :$

n	λ_k	μ_k	γ
2	$\lambda_2 = 1\,,$	$\mu_1 = -3\,,$	0
3	$\lambda_1 = 3/2\,,\ \lambda_3 = 1$	$\mu_0 = -5/2\,,\ \mu_2 = -5$	-6
4	$\lambda_0 = 35/3\,,\ \lambda_2 = 70/3\,,\ \lambda_4 = 1$	$\mu_1 = -73\,,\ \mu_3 = -7$	0
5	$\lambda_1 = 69\,,\ \lambda_3 = 189/4\,,\ \lambda_5 = 1$	$\mu_0 = -469/4\,,\ \mu_2 = -965/4\,,\ \mu_4 = -9$	-330
6	$\lambda_0 = 4704/5\,,\ \lambda_2 = 1941\,,$ $\lambda_4 = 396/5\,,\ \lambda_6 = 1\,,$	$\mu_1 = -30318/5\,,\ \mu_3 = -2807/5\,,$ $\mu_5 = -11$	0
7	$\lambda_1 = 16921/2\,,\ \lambda_3 = 34727/6\,,$ $\lambda_5 = 715/6\,,\ \lambda_7 = 1$	$\mu_0 = -28737/2\,,\ \mu_2 = -29540\,,$ $\mu_4 = -2163/2\,,\ \mu_6 = -13$	$-80675/2$
8	$\lambda_0 = 5601948/35\,,\ \lambda_2 = 5601948/35\,,$ $\lambda_4 = 471627/35\,,\ \lambda_6 = 1170/7\,, \lambda_8 = 1$	$\mu_1 = -36094416/35\,,\ \mu_3 = -477512/5\,,$ $\mu_5 = -12947/7\,,\ \mu_7 = -15\,,$	0
9	$\lambda_1 = 30578169/16\,,$ $\lambda_3 = 5229679/4\,,$ $\lambda_5 = 53845/2\,,$ $\lambda_7 = 1785/8\,,\ \lambda_9 = 1$	$\mu_0 = -51931231/16\,,$ $\mu_2 = -53382655/8\,,$ $\mu_4 = -1954557/8\,,$ $\mu_6 = -23309/8\,,\ \mu_8 = -17$	-9112005
10	$\lambda_0 = 2921776286/63\,,$ $\lambda_2 = 2921776286/63\,,$ $\lambda_4 = 2921776286/63\,,$ $\lambda_6 = 3051347/63\,,$ $\lambda_8 = 2584/9\,,\ \lambda_{10} = 1$	$\mu_1 = -6275196562/21\,,$ $\mu_3 = -83017774/3\,,$ $\mu_5 = -11254826/21\,,$ $\mu_7 = -12965/3\,,$ $\mu_9 = -19\,,$	0

$Z_\nu(x) \ = \ K_1(x) :$

n	λ_k	μ_k	γ
2	$\lambda_2 = -1\,,$	$\mu_1 = -3\,,$	0
3	$\lambda_1 = -3/2\,,\ \lambda_3 = -1$	$\mu_0 = -5/2\,,\ \mu_2 = -5$	6
4	$\lambda_0 = -35/3\,,\ \lambda_2 = -70/3\,,\ \lambda_4 = -1$	$\mu_1 = -73\,,\ \mu_3 = -7$	0
5	$\lambda_1 = -69\,,\ \lambda_3 = -189/4\,,\ \lambda_5 = -1$	$\mu_0 = -469/4\,,\ \mu_2 = -965/4\,,\ \mu_4 = -9$	330
6	$\lambda_0 = -4704/5\,,\ \lambda_2 = -1941\,,$ $\lambda_4 = -396/5\,,\ \lambda_6 = -1\,,$	$\mu_1 = -30318/5\,,\ \mu_3 = -2807/5\,,$ $\mu_5 = -11$	0
7	$\lambda_1 = -16921/2\,,\ \lambda_3 = -34727/6\,,$ $\lambda_5 = -715/6\,,\ \lambda_7 = -1$	$\mu_0 = -28737/2\,,\ \mu_2 = -29540\,,$ $\mu_4 = -2163/2\,,\ \mu_6 = -13$	80675/2
8	$\lambda_0 = -5601948/35\,,\ \lambda_2 = -2310792/7\,,$ $\lambda_4 = -471627/35\,,\ \lambda_6 = -1170/7\,, \lambda_8 = -1$	$\mu_1 = -36094416/35\,,\ \mu_3 = -477512/5\,,$ $\mu_5 = -477512/5\,,\ \mu_7 = -15\,,$	0

n	λ_k	μ_k	γ
9	$\lambda_1 = -30578169/16$, $\lambda_3 = -5229679/4$, $\lambda_5 = -5229679/4$, $\lambda_7 = -1785/8$, $\lambda_9 = -1$	$\mu_0 = -51931231/16$, $\mu_2 = -53382655/8$, $\mu_4 = -1954557/8$, $\mu_6 = -23309/8$, $\mu_8 = -17$	$-23309/8$
10	$\lambda_0 = -2921776286/63$, $\lambda_2 = -6026143240/63$, $\lambda_4 = -27331536/7$, $\lambda_6 = -3051347/63$, $\lambda_8 = -2584/9$, $\lambda_{10} = -1$	$\mu_1 = -6275196562/21$, $\mu_3 = -83017774/3$, $\mu_5 = -11254826/21$, $\mu_7 = -12965/3$, $\mu_9 = -19$,	0

b) Chebyshev Polynomials of the First Kind $T_n(x)$:

$$\int_{-1}^{1} \frac{T_n^2(x)\,dx}{\sqrt{1-x^2}} = \begin{cases} \pi & , \quad n=0, \\ \frac{\pi}{2} & , \quad n>0 \end{cases}$$

First polynomials:

$$T_0(x) = 1\,,\; T_1(x) = x\,,\; T_2(x) = 2x^2 - 1\,,\; T_3(x) = 4x^3 - 3x\,,\; T_4(x) = 8x^4 - 8x^2 + 1$$

$Z_\nu(x) \;=\; J_0(x)$:

n	λ_k	μ_k	γ
2	$\lambda_1 = -1$,	$\mu_0 = 1$, $\mu_2 = 1$	-3
3	$\lambda_0 = 4$, $\lambda_2 = 4$	$\mu_1 = -16$, $\mu_3 = 1$	0
4	$\lambda_1 = 19$, $\lambda_3 = 6$	$\mu_0 = -37$, $\mu_2 = -36$, $\mu_4 = 1$	81
5	$\lambda_0 = -252$, $\lambda_2 = -244$, $\lambda_4 = 8$	$\mu_1 = 912$, $\mu_3 = -64$, $\mu_5 = 1$	0
6	$\lambda_1 = -1809$, $\lambda_3 = -586$, $\lambda_5 = 10$	$\mu_0 = 3517$, $\mu_2 = 3416$, $\mu_4 = -100$, $\mu_6 = 1$	-7651
7	$\lambda_0 = 35208$, $\lambda_2 = 34060$, $\lambda_4 = -1136$, $\lambda_6 = 12$	$\mu_1 = -127296$, $\mu_3 = 8944$, $\mu_5 = -144$, $\mu_7 = 1$	0
8	$\lambda_1 = 347651$, $\lambda_3 = 112614$, $\lambda_5 = -1942$, $\lambda_7 = 14$	$\mu_0 = -675881$, $\mu_2 = -656460$, $\mu_4 = 19224$, $\mu_6 = -196$, $\mu_8 = 1$	1470273
9	$\lambda_0 = -8890744$, $\lambda_2 = -8600812$, $\lambda_4 = 286864$, $\lambda_6 = -3052$, $\lambda_8 = 16$	$\mu_1 = 32144704$, $\mu_3 = -2258544$, $\mu_5 = 36368$, $\mu_7 = -256$, $\mu_9 = 1$	0
10	$\lambda_1 = -111510593$, $\lambda_3 = -36121434$, $\lambda_5 = 622906$, $\lambda_7 = -4514$, $\lambda_9 = 18$	$\mu_0 = 216791801$, $\mu_2 = 210562416$, $\mu_4 = -6166188$, $\mu_6 = 62872$, $\mu_8 = -324$, $\mu_{10} = 1$	-471596451

$Z_\nu(x) \;=\; I_0(x)$:

n	λ_k	μ_k	γ
2	$\lambda_1 = -1$,	$\mu_0 = 1$, $\mu_2 = 1$	1
3	$\lambda_0 = -4$, $\lambda_2 = -4$	$\mu_1 = 16$, $\mu_3 = 1$	0
4	$\lambda_1 = -17$, $\lambda_3 = -6$	$\mu_0 = 35$, $\mu_2 = 36$, $\mu_4 = 1$	65
5	$\lambda_0 = -260$, $\lambda_2 = -268$, $\lambda_4 = -8$	$\mu_1 = 1136$, $\mu_3 = 64$, $\mu_5 = 1$	0
6	$\lambda_1 = -1793$, $\lambda_3 = -614$, $\lambda_5 = -10$	$\mu_0 = 3685$, $\mu_2 = 3784$, $\mu_4 = 100$, $\mu_6 = 1$	6785
7	$\lambda_0 = -38536$, $\lambda_2 = -39692$, $\lambda_4 = -1168$, $\lambda_6 = -12$	$\mu_1 = 168256$, $\mu_3 = 9488$, $\mu_5 = 144$, $\mu_7 = 1$	0

n	λ_k	μ_k	γ
8	$\lambda_1 = -358049\,,\ \lambda_3 = -122614\,,$ $\lambda_5 = -1978\,,\ \lambda_7 = -14$	$\mu_0 = -735879\,,\ \mu_2 = 755660\,,$ $\mu_4 = 19976\,,\ \mu_6 = 196\,,\ \mu_8 = 1$	1355009
9	$\lambda_0 = -9996680\,,\ \lambda_2 = -10296596\,,$ $\lambda_4 = -302992\,,\ \lambda_6 = -3052\,,$ $\lambda_8 = -16$	$\mu_1 = 43647680\,,\ \mu_3 = 43647680\,,$ $\mu_5 = 43647680\,,\ \mu_7 = 256\,,$ $\mu_9 = 1$	0
10	$\lambda_1 = -117155329\,,$ $\lambda_3 = -40119878\,,$ $\lambda_5 = -40119878\,,\ \lambda_7 = -4558\,,$ $\lambda_9 = -18$	$\mu_0 = 240783081\,,$ $\mu_2 = 247255504\,,$ $\mu_4 = 6536236\,,\ \mu_6 = 64136\,,$ $\mu_8 = 324\,,\ \mu_{10} = 1$	443365249

$Z_\nu(x) \ = \ K_0(x)$:

n	λ_k	μ_k	γ
2	$\lambda_1 = -1\,,$	$\mu_0 = -1\,,\ \mu_2 = -1$	1
3	$\lambda_0 = -4\,,\ \lambda_2 = -4$	$\mu_1 = -16\,,\ \mu_3 = -1$	0
4	$\lambda_1 = -17\,,\ \lambda_3 = -6$	$\mu_0 = -35\,,\ \mu_2 = -36\,,\ \mu_4 = -1$	65
5	$\lambda_0 = -260\,,\ \lambda_2 = -268\,,\ \lambda_4 = -8$	$\mu_1 = -1136\,,\ \mu_3 = -64\,,\ \mu_5 = -1$	0
6	$\lambda_1 = -1793\,,\ \lambda_3 = -614\,,$ $\lambda_5 = -10$	$\mu_0 = -3685\,,\ \mu_2 = -3784\,,$ $\mu_4 = -100\,,\ \mu_6 = -1$	6785
7	$\lambda_0 = -38536\,,\ \lambda_2 = -39692\,,$ $\lambda_4 = -1186\,,\ \lambda_6 = -12$	$\mu_1 = -168256\,,\ \mu_3 = -9488\,,$ $\mu_5 = -144\,,\ \mu_7 = -1$	0
8	$\lambda_1 = -358049\,,\ \lambda_3 = -122614\,,$ $\lambda_5 = -1978\,,\ \lambda_7 = -14$	$\mu_0 = -735879\,,\ \mu_2 = -755660\,,$ $\mu_4 = -19976\,,\ \mu_6 = -196\,,\ \mu_8 = -1$	1470273
9	$\lambda_0 = -9996680\,,\ \lambda_2 = -10296596\,,$ $\lambda_4 = -302992\,,\ \lambda_6 = -3092\,,$ $\lambda_8 = -16$	$\mu_1 = -43647680\,,\ \mu_3 = -2461296\,,$ $\mu_5 = -37360\,,\ \mu_7 = -256\,,$ $\mu_9 = -1$	0
10	$\lambda_1 = -117155329\,,$ $\lambda_3 = -40119878\,,$ $\lambda_5 = -647210\,,\ \lambda_7 = -4558\,,$ $\lambda_9 = -18$	$\mu_0 = -240783081\,,$ $\mu_2 = -247255504\,,$ $\mu_4 = -6536236\,,\ \mu_6 = -64136\,,$ $\mu_8 = -324\,,\ \mu_{10} = -1$	443365249

$Z_\nu(x) \ = \ J_1(x)$:

n	λ_k	μ_k	γ
2	$\lambda_2 = -1\,,$	$\mu_1 = 4\,,$	0
3	$\lambda_1 = -3\,,\ \lambda_3 = -1$	$\mu_0 = 6\,,\ \mu_2 = 6$	-15
4	$\lambda_0 = 32\,,\ \lambda_2 = 32\,,\ \lambda_4 = -1$	$\mu_1 = -120\,,\ \mu_3 = 8$	0
5	$\lambda_1 = 185\,,\ \lambda_3 = 60\,,\ \lambda_5 = -1$	$\mu_0 = -360\,,\ \mu_2 = -350\,,\ \mu_4 = 10$	785
6	$\lambda_0 = -2976\,,\ \lambda_2 = -2880\,,\ \lambda_4 = 96\,,\ \lambda_6 = -1$	$\mu_1 = 10764\,,\ \mu_3 = -756\,,\ \mu_5 = 12$	0
7	$\lambda_1 = -25067\,,\ \lambda_3 = -8120\,,\ \lambda_5 = 140\,,$ $\lambda_7 = -1\,,$	$\mu_0 = 48734\,,\ \mu_2 = 47334\,,$ $\mu_4 = -1386\,,\ \mu_6 = 14\,,$	-106015
8	$\lambda_0 = 559360\,,\ \lambda_2 = 541120\,,$ $\lambda_4 = -18048\,,\ \lambda_6 = 192\,,\ \lambda_8 = -1$	$\mu_1 = -2022384\,,\ \mu_3 = 142096\,,$ $\mu_5 = -2288\,,\ \mu_7 = 16$	0
9	$\lambda_1 = 6225489\,,\ \lambda_3 = 2016612\,,$ $\lambda_5 = -34776\,,\ \lambda_7 = 252\,,$ $\lambda_9 = -1$	$\mu_0 = -12103200\,,$ $\mu_2 = -11755422\,,\ \mu_4 = 344250\,,$ $\mu_6 = -3510\,,\ \mu_8 = 18$	26328609
10	$\lambda_0 = -177115680\,,\ \lambda_2 = -171339840\,,$ $\lambda_4 = 5714720\,,\ \lambda_6 = -60800\,,$ $\lambda_8 = 320\,,\ \lambda_{10} = -1\,,$	$\mu_1 = 640366100\,,$ $\mu_3 = -44993260\,,$ $\mu_5 = 724500\,,\ \mu_7 = -5100\,,$ $\mu_9 = 20$	0

$Z_\nu(x) \; = \; I_1(x)$:

n	λ_k	μ_k	γ
2	$\lambda_2 = 1\,,$	$\mu_1 = -4\,,$	0
3	$\lambda_1 = 3\,,\ \lambda_3 = 1$	$\mu_0 = -6\,,\ \mu_2 = -6$	-9
4	$\lambda_0 = 32\,,\ \lambda_2 = 32\,,\ \lambda_4 = 1$	$\mu_1 = -136\,,\ \mu_3 = -8$	0
5	$\lambda_1 = 175\,,\ \lambda_3 = 60\,,\ \lambda_5 = 1$	$\mu_0 = -360\,,\ \mu_2 = -370\,,\ \mu_4 = -10$	-665
6	$\lambda_0 = 3168\,,\ \lambda_2 = 3264\,,\ \lambda_4 = 96\,,\ \lambda_6 = 1$	$\mu_1 = -13836\,,\ \mu_3 = -780\,,\ \mu_5 = -12$	0
7	$\lambda_1 = 25347\,,\ \lambda_3 = 8680\,,\ \lambda_5 = 140\,,$ $\lambda_7 = 1\,,$	$\mu_0 = -52094\,,\ \mu_2 = -53494\,,$ $\mu_4 = -1414\,,\ \mu_6 = -14\,,$	-95921
8	$\lambda_0 = 620800\,,\ \lambda_2 = 639424\,,$ $\lambda_4 = 18816\,,\ \lambda_6 = 192\,,\ \lambda_8 = 1$	$\mu_1 = -2710544\,,\ \mu_3 = -152848\,,$ $\mu_5 = -2320\,,\ \mu_7 = -16$	0
9	$\lambda_1 = 6477471\,,\ \lambda_3 = 2218212\,,$ $\lambda_5 = 35784\,,\ \lambda_7 = 252\,,$ $\lambda_9 = 1$	$\mu_0 = -13312800\,,$ $\mu_2 = -13670658\,,\ \mu_4 = -361386\,,$ $\mu_6 = -3546\,,\ \mu_8 = -18$	-24513489
10	$\lambda_0 = 200709600\,,\ \lambda_2 = 206731200\,,$ $\lambda_4 = 6083360\,,\ \lambda_6 = 62080\,,$ $\lambda_8 = 320\,,\ \lambda_{10} = 1\,,$	$\mu_1 = 62080\,,$ $\mu_3 = -49416980\,,$ $\mu_5 = -750100\,,\ \mu_7 = -5140\,,$ $\mu_9 = -20$	0

$Z_\nu(x) \; = \; K_1(x)$:

n	λ_k	μ_k	γ
2	$\lambda_2 = -1\,,$	$\mu_1 = -4\,,$	0
3	$\lambda_1 = -3\,,\ \lambda_3 = -1$	$\mu_0 = -6\,,\ \mu_2 = -6$	9
4	$\lambda_0 = -32\,,\ \lambda_2 = -32\,,\ \lambda_4 = -1$	$\mu_1 = -136\,,\ \mu_3 = -8$	0
5	$\lambda_1 = -175\,,\ \lambda_3 = -60\,,\ \lambda_5 = -1$	$\mu_0 = -360\,,\ \mu_2 = -370\,,\ \mu_4 = -10$	665
6	$\lambda_0 = -3168\,,\ \lambda_2 = -3264\,,\ \lambda_4 = -96\,,\ \lambda_6 = -1$	$\mu_1 = -13836\,,\ \mu_3 = -780\,,\ \mu_5 = -12$	0
7	$\lambda_1 = -25347\,,\ \lambda_3 = -8680\,,\ \lambda_5 = -140\,,$ $\lambda_7 = -1\,,$	$\mu_0 = -52094\,,\ \mu_2 = -53494\,,$ $\mu_4 = -1414\,,\ \mu_6 = -14\,,$	95921
8	$\lambda_0 = -620800\,,\ \lambda_2 = -639424\,,$ $\lambda_4 = -18816\,,\ \lambda_6 = -192\,,\ \lambda_8 = -1$	$\mu_1 = -2710544\,,\ \mu_3 = -152848\,,$ $\mu_5 = -152848\,,\ \mu_7 = -16$	0
9	$\lambda_1 = -6477471\,,\ \lambda_3 = -2218212\,,$ $\lambda_5 = -35784\,,\ \lambda_7 = -252\,,$ $\lambda_9 = -1$	$\mu_0 = -13312800\,,$ $\mu_2 = -13670658\,,\ \mu_4 = -361386\,,$ $\mu_6 = --3546\,,\ \mu_8 = -18$	24513489
10	$\lambda_0 = -200709600\,,\ \lambda_2 = -206731200\,,$ $\lambda_4 = -6083360\,,\ \lambda_6 = -62080\,,$ $\lambda_8 = -320\,,\ \lambda_{10} = -1\,,$	$\mu_1 = -876341780\,,$ $\mu_3 = -49416980\,,$ $\mu_5 = -750100\,,\ \mu_7 = -5140\,,$ $\mu_9 = -20$	0

b) Chebyshev Polynomials of the Second Kind $U_n(x)$:

$$\int_{-1}^{1} \sqrt{1-x^2}\, U_n^2(x)\, dx \quad = \quad \frac{\pi}{2}$$

First polynomials:

$$U_0(x) = 1\,,\ U_1(x) = 2x\,,\ U_2(x) = 4x^2 - 1\,,\ U_3(x) = 8x^3 - 4x\,,\ U_4(x) = 16x^4 - 12x^2 + 1$$

$Z_\nu(x) \; = \; J_0(x):$

n	λ_k	μ_k	γ
1	–	$\mu_1 = 1\,,$	0
2	$\lambda_1 = -1/2\,,$	$\mu_0 = 1\,,\ \mu_2 = 1$	-5
3	$\lambda_0 = 4\,,\ \lambda_2 = 4$	$\mu_1 = -16\,,\ \mu_3 = 1$	0
4	$\lambda_1 = 25/2\,,\ \lambda_3 = 6$	$\mu_0 = -37\,,\ \mu_2 = -36\,,\ \mu_4 = 1$	157
5	$\lambda_0 = -256\,,\ \lambda_2 = -248\,,\ \lambda_4 = 8$	$\mu_1 = 960\,,\ \mu_3 = -64\,,\ \mu_5 = 1\,,$	0
6	$\lambda_1 = -2421/2\,,\ \lambda_3 = -590\,,\ \lambda_5 = 10$	$\mu_0 = 3581\,,\ \mu_2 = 3480\,,\ \mu_4 = -100$ $\mu_6 = 1\,,$	-15145
7	$\lambda_0 = 36100\,,\ \lambda_2 = 34948\,,$ $\lambda_4 = -1140\,,\ \lambda_6 = 12$	$\mu_1 = -135280\,,\ \mu_3 = 9024$ $\mu_5 = -144\,,\ \mu_7 = 1$	0
8	$\lambda_1 = 467653/2\,,\ \lambda_3 = 113966$ $\lambda_5 = -1946\,,\ \lambda_7 = 14$	$\mu_0 = -691721\,,\ \mu_2 = -672204\,,$ $\mu_4 = 19320\,,\ \mu_6 = -196\,,\ \mu_8 = 1$	2925401
9	$\lambda_0 = -9144576\,,\ \lambda_2 = -8852728\,,$ $\lambda_4 = 288776\,,\ \lambda_6 = -3056\,,\lambda_8 = 16$	$\mu_1 = 34267968\,,\ \mu_3 = -2285888\,,$ $\mu_5 = 36480\,,\ \mu_7 = -256\,,\mu_9 = 1$	0
10	$\lambda_1 = -150310665/2\,,\ \lambda_3 = -36630374\,,$ $\lambda_5 = 625474\,,\ \lambda_7 = -4518\,,$ $\lambda_9 = 18$	$\mu_0 = 222329465\,,\ \mu_2 = 216056400\,,$ $\mu_4 = -6209740\,,\ \mu_6 = 63000\,,$ $\mu_8 = -324\,,\ \mu_{10} = 1$	-940267501

$Z_\nu(x) \; = \; I_0(x):$

n	λ_k	μ_k	γ
1	–	$\mu_1 = 1\,,$	0
2	$\lambda_1 = -1/2\,,$	$\mu_0 = 1\,,\ \mu_2 = 1$	3
3	$\lambda_0 = -4\,,\ \lambda_2 = -4$	$\mu_1 = 16\,,\ \mu_3 = 1$	0
4	$\lambda_1 = -23/2\,,\ \lambda_3 = -6$	$\mu_0 = 35\,,\ \mu_2 = 36\,,\ \mu_4 = 1$	133
5	$\lambda_0 = -256\,,\ \lambda_2 = -264\,,\ \lambda_4 = -8$	$\mu_1 = 1880\,,\ \mu_3 = 64\,,\ \mu_5 = 1\,,$	0
6	$\lambda_1 = -2381/2\,,\ \lambda_3 = -610\,,\ \lambda_5 = -10$	$\mu_0 = 3621\,,\ \mu_2 = 3720\,,\ \mu_4 = 100$ $\mu_6 = 1\,,$	13703
7	$\lambda_0 = -37636\,,\ \lambda_2 = -38788\,,$ $\lambda_4 = -1164\,,\ \lambda_6 = -12$	$\mu_1 = 159856\,,\ \mu_3 = 9408$ $\mu_5 = 144\,,\ \mu_7 = 1$	0
8	$\lambda_1 = -473251/2\,,\ \lambda_3 = -121246$ $\lambda_5 = -1974\,,\ \lambda_7 = -14$	$\mu_0 = 719719\,,\ \mu_2 = 739404\,,$ $\mu_4 = 19880\,,\ \mu_6 = 196\,,\ \mu_8 = 1$	2723721
9	$\lambda_0 = -9734400\,,\ \lambda_2 = -10032392\,,$ $\lambda_4 = -10032392\,,\ \lambda_6 = -3088\,,\lambda_8 = -16$	$\mu_1 = 41346240\,,\ \mu_3 = 2433344\,,$ $\mu_5 = 37248\,,\ \mu_7 = 256\,,\mu_9 = 1$	0
10	$\lambda_1 = -154544153/2\,,\ \lambda_3 = -39593914\,,$ $\lambda_5 = -644626\,,\ \lambda_7 = -4554\,,$ $\lambda_9 = -18$	$\mu_0 = 235030377\,,\ \mu_2 = 241458672\,,$ $\mu_4 = 6491980\,,\ \mu_6 = 64008\,,$ $\mu_8 = 324\,,\ \mu_{10} = 1$	889454219

$Z_\nu(x) \; = \; K_0(x):$

n	λ_k	μ_k	γ
1	–	$\mu_1 = -1\,,$	0
2	$\lambda_1 = -1/2\,,$	$\mu_0 = -1\,,\ \mu_2 = -1$	3
3	$\lambda_0 = -4\,,\ \lambda_2 = -4$	$\mu_1 = -16\,,\ \mu_3 = -1$	0
4	$\lambda_1 = -23/2\,,\ \lambda_3 = -6$	$\mu_0 = -35\,,\ \mu_2 = -36\,,\ \mu_4 = -1$	133
5	$\lambda_0 = -256\,,\ \lambda_2 = -264\,,\ \lambda_4 = -8$	$\mu_1 = -1088\,,\ \mu_3 = -64\,,\ \mu_5 = -1\,,$	0
6	$\lambda_1 = -2381/2\,,\ \lambda_3 = -610\,,\ \lambda_5 = -10$	$\mu_0 = -3621\,,\ \mu_2 = -3720\,,\ \mu_4 = -100$ $\mu_6 = -1\,,$	13703

n	λ_k	μ_k	γ
7	$\lambda_0 = -37636\,,\ \lambda_2 = -38788\,,$ $\lambda_4 = -1164\,,\ \lambda_6 = -12$	$\mu_1 = -159856\,,\ \mu_3 = -9408$ $\mu_5 = -144\,,\ \mu_7 = -1$	0
8	$\lambda_1 = -473251/2\,,\ \lambda_3 = -121246$ $\lambda_5 = -1974\,,\ \lambda_7 = -14$	$\mu_0 = -719719\,,\ \mu_2 = -739404\,,$ $\mu_4 = -19880\,,\ \mu_6 = -196\,,\ \mu_8 = -1$	2723721
9	$\lambda_0 = 2723721\,,\ \lambda_2 = -10032392\,,$ $\lambda_4 = -301064\,,\ \lambda_6 = -3088\,, \lambda_8 = -16$	$\mu_1 = -41346240\,,\ \mu_3 = -2433344\,,$ $\mu_5 = -37248\,,\ \mu_7 = -256\,, \mu_9 = -1$	0
10	$\lambda_1 = -154544153/2\,,\ \lambda_3 = -39593914\,,$ $\lambda_5 = -644626\,,\ \lambda_7 = -4554\,,$ $\lambda_9 = -18$	$\mu_0 = -235030377\,,\ \mu_2 = -241458672\,,$ $\mu_4 = -6491980\,,\ \mu_6 = -64008\,,$ $\mu_8 = -324\,,\ \mu_{10} = -1$	889454219

$Z_\nu(x) \ = \ J_1(x)$:

n	λ_k	μ_k	γ
1	–	–	2
2	$\lambda_2 = -1$	$\mu_1 = 4$	0
3	$\lambda_1 = -2\,,\ \lambda_3 = -1$	$\mu_0 = 6\,,\ \mu_2 = 6$	-28
4	$\lambda_0 = 32\,,\ \lambda_2 = 32\,,\ \lambda_4 = -1$	$\mu_1 = -124\,,\ \mu_3 = 8$	0
5	$\lambda_1 = 123\,,\ \lambda_3 = 60\,,\ \lambda_5 = -1$	$\mu_0 = -364\,,\ \mu_2 = -354\,,\ \mu_4 = 10$	1542
6	$\lambda_0 = -3040\,,\ \lambda_2 = -2944\,,\ \lambda_4 = 96\,,\ \lambda_6 = -1$	$\mu_1 = 11396\,,\ \mu_3 = -760\,,\ \mu_5 = 12$	0
7	$\lambda_1 = -16824\,,\ \lambda_3 = -8200\,,\ \lambda_5 = 140\,,$ $\lambda_7 = -1$	$\mu_0 = 49770\,,\ \mu_2 = 48366\,,$ $\mu_4 = -1390\,,\ \mu_6 = 14$	-210488
8	$\lambda_0 = 574560\,,\ \lambda_2 = 574560\,,\ \lambda_4 = -18144\,,$ $\lambda_6 = 192\,,\ \lambda_8 = -1$	$\mu_1 = -2153084\,,\ \mu_3 = 143624\,,$ $\mu_5 = -2292\,,\ \mu_7 = 16$	0
9	$\lambda_1 = 4192053\,,\ \lambda_3 = 2043188\,,$ $\lambda_5 = -34888\,,\ \lambda_7 = 252\,,\ \lambda_9 = -1$	$\mu_0 = -12401208\,,\ \mu_2 = -12051306\,,$ $\mu_4 = 346370\,,\ \mu_6 = -3514\,,\ \mu_8 = 18$	52446730
10	$\lambda_0 = -182316960\,,\ \lambda_2 = -176498336\,,$ $\lambda_4 = 5757376\,,\ \lambda_6 = -60928\,,$ $\lambda_8 = 320\,,\ \lambda_{10} = -1$	$\mu_1 = 683206276\,,\ \mu_3 = -45574136\,,$ $\mu_5 = 727308\,,\ \mu_7 = -5104\,,\ \mu_9 = 20$	0

$Z_\nu(x) \ = \ I_1(x)$:

n	λ_k	μ_k	γ
1	–	–	-2
2	$\lambda_2 = 1$	$\mu_1 = -4$	0
3	$\lambda_1 = 2\,,\ \lambda_3 = 1$	$\mu_0 = -6\,,\ \mu_2 = -6$	-20
4	$\lambda_0 = 32\,,\ \lambda_2 = 32\,,\ \lambda_4 = 1$	$\mu_1 = -132\,,\ \mu_3 = -8$	0
5	$\lambda_1 = 117\,,\ \lambda_3 = 60\,,\ \lambda_5 = 1$	$\mu_0 = -356\,,\ \mu_2 = -366\,,\ \mu_4 = -10$	-1350
6	$\lambda_0 = 3104\,,\ \lambda_2 = 3200\,,\ \lambda_4 = 96\,,\ \lambda_6 = 1$	$\mu_1 = -13188\,,\ \mu_3 = -776\,,\ \mu_5 = -12$	0
7	$\lambda_1 = 16784\,,\ \lambda_3 = 8600\,,\ \lambda_5 = 140\,,$ $\lambda_7 = 1$	$\mu_0 = -51050\,,\ \mu_2 = -51050\,,$ $\mu_4 = -1410\,,\ \mu_6 = -14$	-193192
8	$\lambda_0 = 605280\,,\ \lambda_2 = 623808\,,\ \lambda_4 = 18720\,,$ $\lambda_6 = 192\,,\ \lambda_8 = 1$	$\mu_1 = -2570884\,,\ \mu_3 = -2570884\,,$ $\mu_5 = -2316\,,\ \mu_7 = -16$	0
9	$\lambda_1 = 4276043\,,\ \lambda_3 = 2191028\,,$ $\lambda_5 = 35672\,,\ \lambda_7 = 252\,,\ \lambda_9 = 1$	$\mu_0 = -13005992\,,\ \mu_2 = -13361718\,,$ $\mu_4 = -359250\,,\ \mu_6 = -359250\,,\ \mu_8 = -18$	-49220170
10	$\lambda_0 = 195293280\,,\ \lambda_2 = 201271648\,,$ $\lambda_4 = 6040000\,,\ \lambda_6 = 61952\,,$ $\lambda_8 = 320\,,\ \lambda_{10} = 1$	$\mu_1 = -829495684\,,\ \mu_3 = -48818184\,,$ $\mu_5 = -747276\,,\ \mu_7 = -5136\,,\ \mu_9 = -20$	0

$Z_\nu(x) \;=\; K_1(x)$:

n	λ_k	μ_k	γ
1	–	–	2
2	$\lambda_2 = -1$	$\mu_1 = -4$	0
3	$\lambda_1 = -2\,,\ \lambda_3 = -1$	$\mu_0 = -6\,,\ \mu_2 = -6$	20
4	$\lambda_0 = -32\,,\ \lambda_2 = -32\,,\ \lambda_4 = -1$	$\mu_1 = -132\,,\ \mu_3 = -8$	0
5	$\lambda_1 = -117\,,\ \lambda_3 = -60\,,\ \lambda_5 = -1$	$\mu_0 = -356\,,\ \mu_2 = -366\,,\ \mu_4 = -10$	1350
6	$\lambda_0 = -3104\,,\ \lambda_2 = -3200\,,\ \lambda_4 = -96\,,\ \lambda_6 = -1$	$\mu_1 = -13188\,,\ \mu_3 = -776\,,\ \mu_5 = -12$	0
7	$\lambda_1 = -16784\,,\ \lambda_3 = -8600\,,\ \lambda_5 = -140\,,$ $\lambda_7 = -1$	$\mu_0 = -51050\,,\ \mu_2 = -52446\,,$ $\mu_4 = -1410\,,\ \mu_6 = -14$	-52446
8	$\lambda_0 = -605280\,,\ \lambda_2 = -623808\,,\ \lambda_4 = -623808\,,$ $\lambda_6 = -192\,,\ \lambda_8 = -1$	$\mu_1 = -623808\,,\ \mu_3 = -151304\,,$ $\mu_5 = -2316\,,\ \mu_7 = -16$	0
9	$\lambda_1 = -4276043\,,\ \lambda_3 = -2191028\,,$ $\lambda_5 = -35672\,,\ \lambda_7 = -252\,,\ \lambda_9 = -1$	$\mu_0 = -13005992\,,\ \mu_2 = -13361718\,,$ $\mu_4 = -359250\,,\ \mu_6 = -3542\,,\ \mu_8 = -18$	49220170
10	$\lambda_0 = -195293280\,,\ \lambda_2 = -195293280\,,$ $\lambda_4 = -6040000\,,\ \lambda_6 = -61952\,,$ $\lambda_8 = -320\,,\ \lambda_{10} = -1$	$\mu_1 = -829495684\,,\ \mu_3 = -48818184\,,$ $\mu_5 = -48818184\,,\ \mu_7 = -5136\,,\ \mu_9 = -20$	0

d) Ultraspherical (Gegenbauer) Polynomials $C_n^{(\alpha)}(x)$:

$$\int_{-1}^{1} (1-x^2)^{\alpha-1/2}\,[C_n^{(\alpha)}(x)]^2\,dx \;=\; \frac{\pi\,2^{1-2\alpha}\,\Gamma(n+2\alpha)}{n!\,(n+\alpha)\,[\Gamma(\alpha)]^2}\,,\quad \alpha > -1/2,\ \alpha \neq 0$$

First polynomials:

$$C_0^{(\alpha)}(x) = 1\,,\ C_1^{(\alpha)}(x) = 2\alpha x\,,\ C_2^{(\alpha)}(x) = 2\alpha(\alpha+1)x^2 - \alpha\,,\ C_3^{(\alpha)}(x) = \frac{\alpha(\alpha+1)[4(\alpha+2)\,x^2 - 6]\,x}{3}\,,$$

$$C_4^{(\alpha)}(x) = \frac{\alpha(\alpha+1)[4(\alpha+2)(\alpha+3)\,x^4 - 12(\alpha+2)\,x^2 + 3]}{6}$$

$Z_\nu(x) \;=\; J_0(x)$:

n	λ_k	μ_k	γ
1	–	$\mu_1 = 1$	0
2	$\lambda_1 = -1/2\,,$	$\mu_0 = \alpha\,,\ \mu_2 = 1$	$-\alpha(2\alpha+3)$
3	$\lambda_0 = 4\alpha(\alpha+2)/3\,,\ \lambda_2 = (4\alpha+8)/3$	$\mu_1 = -8(\alpha+1)(\alpha+2)/3\,,\ \mu_3 = 1$	0
4	$\lambda_1 = (6\alpha+19)(\alpha+1)/4\,,$ $\lambda_3 = 3(\alpha+3)/2\,,$	$\mu_0 = -\alpha(6\alpha^2+31\alpha+37)/2\,,$ $\mu_2 = -3(\alpha+2)(\alpha+3)\,,\ \mu_4 = 1$	$\alpha(\alpha+1)\cdot$ $\cdot(12\alpha^2+64\alpha+81)/2$
5	$\lambda_0 = -8\alpha(\alpha+3)(8\alpha^2+49\alpha+63)/15\,,$ $\lambda_2 = -4(\alpha+2)(16\alpha^2+111\alpha+183)/15\,,$ $\lambda_4 = (8\alpha+32)/15$	$\mu_1 = 8(\alpha+1)(\alpha+3)\cdot$ $\cdot\,(16\alpha^2+95\alpha+114)/15\,,$ $\mu_3 = -16(\alpha+3)(\alpha+4)/5\,,$ $\mu_5 = 1$	0
6	$\lambda_1 = -(\alpha+1)(60\alpha^3+728\alpha^2+$ $+2857a+3618)/12\,,$ $\lambda_3 = -(\alpha+3)\cdot$ $\cdot(30\alpha^2+269\alpha+586)/6\,,$ $\lambda_5 = (5\alpha+25)/3\,,$	$\mu_0 = \alpha(60\alpha^4+848\alpha^3+4303\alpha^2+$ $+9241\alpha+7034)/6\,,$ $\mu_2 = (\alpha+2)(\alpha+4)\cdot$ $\cdot(30\alpha^2+239\alpha+427)/3\,,$ $\mu_4 = -10(\alpha+4)(\alpha+5)/3\,,\ \mu_6 = 1$	$-\alpha(\alpha+1)(\alpha+2)\cdot$ $\cdot(120\alpha^3+1476\alpha^2+$ $+5898\alpha+7651)/6$

n	$\lambda_k,\ \mu_k,\ \gamma$
7	$\lambda_0 = 4\,\alpha\,(\alpha+4)\left(384\,\alpha^4+6184\,\alpha^3+35060\,\alpha^2+81882\,\alpha+66015\right)/105$ $\lambda_2 = 4\,(\alpha+2)\left(384\,\alpha^4+6880\,\alpha^3+44878\,\alpha^2+125928\,\alpha+127725\right)/105$ $\lambda_4 = -4\,(\alpha+4)\left(48\,\alpha^2+527\,\alpha+1420\right)/35\,,\ \lambda_6 = 12(aaa+6)/7$ $\mu_1 = -8\,(\alpha+4)\,(\alpha+1)\left(384\,\alpha^4+6112\,\alpha^3+34046\,\alpha^2+77343\,\alpha+59670\right)/105$ $\mu_3 = 8\,(\alpha+3)\,(\alpha+5)\left(48\,\alpha^2+479\,\alpha+1118\right)/35$ $\mu_5 = -24\,(\alpha+5)\,(\alpha+6)/7\,,\ \mu_7 = 1$ $\gamma = 0$
8	$\lambda_1 = (\alpha+1)\left(840\,\alpha^5+21100\,\alpha^4+207050\,\alpha^3+990585\,\alpha^2+2306355\,\alpha+2085906\right)/48$ $\lambda_3 = (\alpha+3)\left(420\,\alpha^4+9220\,\alpha^3+74459\,\alpha^2+261855\,\alpha+337842\right)/24$ $\lambda_5 = -(\alpha+5)\left(70\,\alpha^2+909\,\alpha+2913\right)/12\,,\ \lambda_7 = 7(\alpha+7)/4$ $\mu_0 = -\alpha\,(22780\,\alpha^5+249110\,\alpha^4+1401593\,\alpha^3+4262368\,\alpha^2+840\,\alpha^6+6609327\,\alpha+$ $+\,4055286)/24$ $\mu_2 = -(\alpha+2)\,(\alpha+5)\left(420\,\alpha^4+8380\,\alpha^3+60079\,\alpha^2+182319\,\alpha+196938\right)/12$ $\mu_4 = (\alpha+4)\,(\alpha+6)\left(70\,\alpha^2+839\,\alpha+2403\right)/6\,,\ \mu_6 = -7\,(\alpha+6)\,(\alpha+7)/2$ $\gamma = \alpha\,(\alpha+3)\,(\alpha+2)\,(\alpha+1)\left(1680\,\alpha^4+37440\,\alpha^3+307992\,\alpha^2+1108016\,\alpha+1470273\right)/24$
9	$\lambda_0 = -16\,\alpha\,(\alpha+5)\,(3072\,\alpha^6+92448\,\alpha^5+1111896\,\alpha^4+6804152\,\alpha^3+22211724\,\alpha^2+$ $+\,36455425\,\alpha+23338203)/945$ $\lambda_2 = -4\,(\alpha+2)\,(12288\,\alpha^6+404352\,\alpha^5+5416248\,\alpha^4+37733036\,\alpha^3+143879274\,\alpha^2+$ $+\,283935817\,\alpha+225771315)/945$ $\lambda_4 = 8\,(\alpha+4)\left(768\,\alpha^4+19944\,\alpha^3+191615\,\alpha^2+806754\,\alpha+1255030\right)/315$ $\lambda_6 = -4\,(\alpha+6)\left(96\,\alpha^2+1439\,\alpha+5341\right)/63\,,\ \lambda_8 = 16(\alpha+8)/9$ $\mu_1 = 16\,(\alpha+1)\,(\alpha+5)\,(6144\,\alpha^6+183744\,\alpha^5+2191452\,\alpha^4+13256458\,\alpha^3+42576930\,\alpha^2+$ $+\,68258003\,\alpha+42189924)/945$ $\mu_3 = -8\,(\alpha+3)\,(\alpha+6)\left(1536\,\alpha^4+36816\,\alpha^3+321406\,\alpha^2+1207917\,\alpha+1646855\right)/315$ $\mu_5 = 8\,(\alpha+5)\,(\alpha+7)\left(96\,\alpha^2+1343\,\alpha+4546\right)/63$ $\mu_7 = -32\,(\alpha+7)\,(\alpha+8)/9\,,\ \mu_9 = 1$ $\gamma = 0$
10	$\lambda_1 = -(\alpha+1)\,(15120\,\alpha^7+636720\,\alpha^6+11259512\,\alpha^5+108253632\,\alpha^4+610299361\,\alpha^3+$ $+\,2014357569\,\alpha^2+3597563754\,\alpha+2676254232)/240$ $\lambda_3 = -(\alpha+3)\,(7560\,\alpha^6+294420\,\alpha^5+4699778\,\alpha^4+39326439\,\alpha^3+181752061\,\alpha^2+$ $+\,439373754\,\alpha+433457208)/120$ $\lambda_5 = (\alpha+5)\left(1260\,\alpha^4+37772\,\alpha^3+420377\,\alpha^2+2057895\,\alpha+3737436\right)/60$ $\lambda_7 = (\alpha+5)\left(1260\,\alpha^4+37772\,\alpha^3+420377\,\alpha^2+2057895\,\alpha+3737436\right)/20\,,$ $\lambda_9 = 9(\alpha+9)/5$ $\mu_0 = \alpha\,(15120\,\alpha^8+666960\,\alpha^7+12530432\,\alpha^6+130674180\,\alpha^5+825228251\,\alpha^4+$ $+\,3221684250\,\alpha^3+7564578353\,\alpha^2+9721155030\,\alpha+5203003224)/120$ $\mu_2 = (\alpha+2)\,(\alpha+6)(7560\,\alpha^6+271740\,\alpha^5+3950078\,\alpha^4+29637161\,\alpha^3+120690694\,\alpha^2+$ $+\,252184351\,\alpha+210562416)/60$ $\mu_4 = -(\alpha+4)\,(\alpha+7)\left(1260\,\alpha^4+35252\,\alpha^3+362095\,\alpha^2+1616046\,\alpha+2642652\right)/30$ $\mu_6 = (\alpha+6)\,(\alpha+8)\left(126\,\alpha^2+2015\,\alpha+7859\right)/10$ $\mu_8 = -18\,(\alpha+8)\,(\alpha+9)/5\,,\ \mu_{10} = 1$ $\gamma = -\alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\,(\alpha+4)\,(30240\,\alpha^5+1066800\,\alpha^4+14886000\,\alpha^3+$ $+\,102678120\,\alpha^2+350009890\,\alpha+471596451)/120$

$Z_\nu(x) \;=\; I_0(x)\,,\; K_0(x)\;:$ Let $s \;=\; \begin{cases} \;\;1 & , \quad Z_\nu(x) = I_\nu(x) \\ -1 & , \quad Z_\nu(x) = K_\nu(x) \end{cases}$.

n	λ_k	μ_k	γ
1	–	$\mu_1 = s$	0
2	$\lambda_1 = -1/2\,,$	$\mu_0 = s\alpha\,,\ \mu_2 = s$	$\alpha(2\alpha+1)$
3	$\lambda_0 = -4\alpha(\alpha+2)/3\,,\ \lambda_2 = -(4\alpha+8)/3$	$\mu_1 = 8s(\alpha+1)(\alpha+2)/3\,,\ \mu_3 = s$	0
4	$\lambda_1 = -(6\alpha+17)(\alpha+1)/4\,,$ $\lambda_3 = -3(\alpha+3)/2\,,$	$\mu_0 = s\alpha\,(2\,\alpha+5)\,(3\,\alpha+7)/2\,,$ $\mu_2 = 3s(\alpha+2)(\alpha+3)\,,\ \mu_4 = s$	$\alpha(\alpha+1)\,\cdot$ $\cdot(2\,\alpha+5)(6\,\alpha+13)/2$

n	$\lambda_k,\ \mu_k,\ \gamma$
5	$\lambda_0=-8\,\alpha\ (\alpha+3)\,(8\,\alpha^2+47\,\alpha+65)/15$ $\lambda_2=-4\,(\alpha+2)(16\,\alpha^2+113\,\alpha+201)/15\,,\ \lambda_4=-(8\alpha+32)/15$ $\mu_1=8s(\alpha+1)(\alpha+3)(16\alpha^2+97\alpha+142)/15$ $\mu_3=16s(\alpha+3)(\alpha+4)/5\,,\ \mu_5=s$ $\gamma=0$
6	$\lambda_1=-(\alpha+1)(60\alpha^3+712\alpha^2+2785\alpha+3586)/12$ $\lambda_3=-\,(\alpha+3)\,(30\,\alpha^2+271\,\alpha+614)/6$ $\lambda_5=-(5\alpha+25)/3$ $\mu_0=s\alpha(60\alpha^4+832\alpha^3+4219\alpha^2+9245\alpha+7370)/6$ $\mu_2=s(\alpha+2)(\alpha+4)(30\alpha^2+271\alpha+614)/6$ $\mu_4=10s(\alpha+4)(\alpha+5)/3\,,\ \mu_6=s$ $\gamma=\alpha(\alpha+1)(\alpha+2)(120\alpha^3+1404\alpha^2+5394\alpha+6785)/6$
7	$\lambda_0=-4\,\alpha\ (\alpha+4)\,(384\,\alpha^4+6104\,\alpha^3+34836\,\alpha^2+84010\,\alpha+72255)/105$ $\lambda_2=-4\ (\alpha+2)\,(384\,\alpha^4+6944\,\alpha^3+46510\,\alpha^2+136712\,\alpha+148845)/105$ $\lambda_4=-4\ (\alpha+4)\,(48\,\alpha^2+529\,\alpha+1460)/35\,,\ \lambda_6=-12(\alpha+6)/7$ $\mu_1=8s\ (\alpha+1)\,(\alpha+4)\,(2\alpha+5)\,(192\,\alpha^3+2608\,\alpha^2+11399\,\alpha+15774)/105$ $\mu_3=8s\ (\alpha+3)\,(\alpha+5)\,(48\,\alpha^2+481\,\alpha+1186)/35$ $\mu_5=24s\ (\alpha+5)\,(\alpha+6)/7\,,\ \mu_7=s$ $\gamma=0$
8	$\lambda_1=-\,(\alpha+1)\,(840\,\alpha^5+20900\,\alpha^4+204570\,\alpha^3+983583\,\alpha^2+2320825\,\alpha+2148294)/48$ $\lambda_3=-\,(\alpha+3)\,(420\,\alpha^4+9260\,\alpha^3+75899\,\alpha^2+274055\,\alpha+367842)/24$ $\lambda_5=-\,(\alpha+5)\,(70\,\alpha^2+911\,\alpha+2967)/12\,,\ \lambda_7=-7(\alpha+7)/4$ $\mu_0=s\alpha\,(22580\,\alpha^5+246510\,\alpha^4+1395791\,\alpha^3+840\,\alpha^6+4312956\,\alpha^2+6879305\,\alpha+$ $+\,4415274)/24$ $\mu_2=s\ (\alpha+2)\,(\alpha+5)\,(420\,\alpha^4+8420\,\alpha^3+61719\,\alpha^2+195679\,\alpha+226698)/12$ $\mu_4=s\ (\alpha+4)\,(\alpha+6)\,(70\,\alpha^2+841\,\alpha+2497)/6\,,\ \mu_6=7s\,(\alpha+6)(\alpha+7)/2\,,\ \mu_8=s$ $\gamma=\alpha\ (\alpha+1)\,(\alpha+2)\,(\alpha+3)\,(1680\,\alpha^4+36480\,\alpha^3+293592\,\alpha^2+1036960\,\alpha+1355009)/24$
9	$\lambda_0=-16\,\alpha\ (\alpha+5)\,(3072\,\alpha^6+91872\,\alpha^5+1106136\,\alpha^4+6835828\,\alpha^3+$ $+\,22787712\,\alpha^2+38757095\,\alpha+26241285)/945$ $\lambda_2=-4\ (2\,\alpha+7)\,(\alpha+2)\,(6144\,\alpha^5+181824\,\alpha^4+2123580\,\alpha^3+$ $+\,12223112\,\alpha^2+34636535\,\alpha+38612235)/945$ $\lambda_4=-8\ (\alpha+4)\left(768\,\alpha^4+19992\,\alpha^3+193919\,\alpha^2+830610\,\alpha+1325590\right)/315$ $\lambda_6=-4\ (\alpha+6)\,(96\,\alpha^2+1441\,\alpha+5411)/63\,,\ \lambda_8=-16(\alpha+8)/9$ $\mu_1=16s\ (2\,\alpha+5)\,(\alpha+5)\,(\alpha+1)(3072\,\alpha^5+84768\,\alpha^4+910302\,\alpha^3+$ $+\,4735516\,\alpha^2+11880401\,\alpha+11457516)/945$ $\mu_3=8s\ (\alpha+6)\,(\alpha+3)\left(1536\,\alpha^4+36912\,\alpha^3+326782\,\alpha^2+1261965\,\alpha+1794695\right)/315$ $\mu_5=8s\ (\alpha+7)\,(\alpha+5)\left(96\,\alpha^2+1345\,\alpha+4670\right)/63$ $\mu_7=32s\ (\alpha+8)\,(\alpha+7)\,/9\,,\ \mu_9=s$ $\gamma=0$
10	$\lambda_1=-\,(\alpha+1)\,(15120\,\alpha^7+633360\,\alpha^6+11178872\,\alpha^5+107665608\,\alpha^4+610500641\,\alpha^3+$ $+\,2036115369\,\alpha^2+3694812314\,\alpha+2811727896)/240$ $\lambda_3=-\,(\alpha+3)\,(7560\,\alpha^6+295260\,\alpha^5+4750178\,\alpha^4+40279841\,\alpha^3+189816079\,\alpha^2+$ $471229966\,\alpha+481438536)/120$ $\lambda_5=-\,(\alpha+5)\left(1260\,\alpha^4+37828\,\alpha^3+423821\,\alpha^2+2100091\,\alpha+3883260\right)/60$ $\lambda_7=-\,(\alpha+7)\left(126\,\alpha^2+2143\,\alpha+9116\right)/20\,,\ \lambda_9=-9(\alpha+9)/5$ $\mu_0=s\alpha\,(15120\,\alpha^8+663600\,\alpha^7+12448112\,\alpha^6+130121436\,\alpha^5+827403511\,\alpha^4+$ $+\,3270380570\,\alpha^3+7829197173\,\alpha^2+10354621774\,\alpha+5778793944)/120$ $\mu_2=s\ (\alpha+6)\,(\alpha+2)\,(7560\,\alpha^6+272580\,\alpha^5+4002998\,\alpha^4+30600643\,\alpha^3+$ $+\,128278422\,\alpha^2+279464213\,\alpha+247255504)/60$ $\mu_4=s\ (\alpha+7)\,(\alpha+4)\left(1260\,\alpha^4+35308\,\alpha^3+366183\,\alpha^2+1664990\,\alpha+2801244\right)/30$ $\mu_6=s\ (\alpha+8)\,(\alpha+6)\left(126\,\alpha^2+2017\,\alpha+8017\right)/10$ $\mu_8=18s\ (\alpha+9)\,(\alpha+8)\,/5\,,\ \mu_{10}=s$ $\gamma=\alpha\ (\alpha+4)\,(\alpha+3)\,(\alpha+2)\,(\alpha+1)\,(30240\,\alpha^5+1050000\,\alpha^4+14449200\,\alpha^3+$ $+\,98461080\,\alpha^2+332098450\,\alpha+443365249)/120$

$Z_\nu(x) \ = \ J_1(x)$:

n	λ_k	μ_k	γ
1	$-$	$-$	2α
2	$\lambda_2 = -1\,,$	$\mu_1 = 2(\alpha+1)\,,$	0
3	$\lambda_1 = -(\alpha+1)\,,\ \lambda_3 = -1$	$\mu_0 = 2\alpha(\alpha+2)\,,\ \mu_2 = 2(\alpha+2)$	$-2\alpha(\alpha+1)(2\alpha+5)$
4	$\lambda_0 = 8\alpha(\alpha+2)(\alpha+3)/3\,,$ $\lambda_2 = 8(\alpha+2)(\alpha+3)/3\,,\ \lambda_4 = -1$	$\mu_1 = -2(\alpha+1)(8\alpha^2+40\alpha+45)/3\,,$ $\mu_3 = 2(\alpha+3)$	0
5	$\lambda_1 = (\alpha+1)(6\alpha^2+43\alpha+74)/2\,,$ $\lambda_3 = 3(\alpha+3)(\alpha+4)\,,\ \lambda_5 = -1$	$\mu_0 = -\alpha(\alpha+3)(6\alpha^2+37\alpha+48)\,,$ $\mu_2 = -2(\alpha+2)(3\alpha^2+21\alpha+35)\,,$ $\mu_4 = 2(\alpha+4)$	$6\alpha(\alpha+1)(\alpha+2)\cdot$ $\cdot(12\alpha^2+88\alpha+157)$
6	$\lambda_0 = 8\alpha(\alpha+3)(\alpha+4)\cdot$ $\cdot(16\alpha^2+114\alpha+155)/15$ $\lambda_2 = -8(\alpha+2)(\alpha+4)\cdot$ $\cdot(16\alpha^2+127\alpha+225)/15\,,$ $\lambda_4 = 16(\alpha+4)(\alpha+5)/5\,, \lambda_6 = -1$ $\lambda_6 = -1$	$\mu_1 = 2(\alpha+1)(128\alpha^4+1784\alpha^3+$ $+8872\alpha^2+18496\alpha+13455)/15\,,$ $\mu_3 = -2(\alpha+3)(4\alpha+15)(4\alpha+21)\,,$ $\mu_5 = 2(\alpha+5)$	0
7	$\lambda_1 = -\left(\alpha+1\right)\left(60\,\alpha^4+1088\,\alpha^3+7207\,\alpha^2+20631\,\alpha+21486\right)/6$ $\lambda_3 = -\left(\alpha+3\right)\left(\alpha+5\right)\left(30\,\alpha^2+299\,\alpha+696\right)/3$ $\lambda_5 = 10\ \left(\alpha+5\right)\left(\alpha+6\right)/3\,,\ \lambda_7 = -1$ $\mu_0 = \alpha\ \left(\alpha+4\right)\left(60\,\alpha^4+968\,\alpha^3+5501\,\alpha^2+12890\,\alpha+10443\right)/3$ $\mu_2 = 2\ \left(\alpha+2\right)\left(30\,\alpha^4+539\,\alpha^3+3528\,\alpha^2+9943\,\alpha+10143\right)/3$ $\mu_4 = -2\ \left(\alpha+4\right)\left(10\,\alpha^2+110\,\alpha+297\right)/3\,,\ \mu_6 = 2(\alpha+6)$ $\gamma = -\alpha\ \left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)\left(120\,\alpha^3+1836\,\alpha^2+9210\,\alpha+15145\right)/3$		
8	$\lambda_0 = 16\,\alpha\ \left(\alpha+4\right)\left(\alpha+5\right)\left(192\,\alpha^4+3476\,\alpha^3+21738\,\alpha^2+54394\,\alpha+45885\right)/105$ $\lambda_2 = 8\ \left(\alpha+2\right)\left(\alpha+5\right)\left(384\,\alpha^4+7648\,\alpha^3+54686\,\alpha^2+165307\,\alpha+177555\right)/105$ $\lambda_4 = -8\ \left(\alpha+4\right)\left(\alpha+6\right)\left(48\,\alpha^2+575\,\alpha+1645\right)/35$ $\lambda_6 = 24\ \left(\alpha+6\right)\left(\alpha+7\right)/7\,,\ \lambda_8 = -1$ $\mu_1 = -2\ \left(\alpha+1\right)(3072\,\alpha^6+82688\,\alpha^5+895344\,\alpha^4+4971392\,\alpha^3+14847744\,\alpha^2+$ $+22446320\,\alpha+13271895)/105$ $\mu_3 = 2\ \left(\alpha+3\right)\left(384\,\alpha^4+8440\,\alpha^3+68256\,\alpha^2+240440\,\alpha+310835\right)/35$ $\mu_4 = -2\ \left(\alpha+5\right)\left(24\,\alpha^2+312\,\alpha+1001\right)/7\,,\ \mu_7 = 2(\alpha+7)$ $\gamma = 0$		
9	$\lambda_1 = \left(\alpha+1\right)(840\,\alpha^6+27820\,\alpha^5+375610\,\alpha^4+2642633\,\alpha^3+10202207\,\alpha^2+$ $+20454222\,\alpha+16601304)/24$ $\lambda_3 = \left(\alpha+3\right)\left(\alpha+6\right)\left(420\,\alpha^4+10060\,\alpha^3+87739\,\alpha^2+329297\,\alpha+448136\right)/12$ $\lambda_5 = -\left(\alpha+5\right)\left(\alpha+7\right)\left(70\,\alpha^2+979\,\alpha+3312\right)/6\,,\ \lambda_7 = 7\ \left(\alpha+7\right)\left(\alpha+8\right)/2$ $\mu_0 = -\alpha\left(\alpha+5\right)(840\,\alpha^6+25300\,\alpha^5+304610\,\alpha^4+1866591\,\alpha^3+6104665\,\alpha^2+$ $+10045370\,\alpha+6455040)/12$ $\mu_2 = -\left(\alpha+2\right)(420\,\alpha^6+13840\,\alpha^5+185699\,\alpha^4+1296390\,\alpha^3+4956133\,\alpha^2+$ $+9813182\,\alpha+7836948)/6$ $\mu_4 = \left(\alpha+4\right)\left(70\,\alpha^4+1819\,\alpha^3+17489\,\alpha^2+73694\,\alpha+114750\right)/3$ $\mu_6 = -\left(\alpha+6\right)\left(7\,\alpha^2+105\,\alpha+390\right)\,,\ \mu_8 = 2(\alpha+8)$ $\gamma = \alpha\ \left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)\left(\alpha+4\right)(1680\,\alpha^4+44160\,\alpha^3+430392\,\alpha^2+$ $+1843040\,\alpha+2925401)/12$		
10	$\lambda_0 = -16\,\alpha\ \left(\alpha+5\right)\left(\alpha+6\right)(6144\,\alpha^6+203328\,\alpha^5+2666160\,\alpha^4+17587276\,\alpha^3+61053750\,\alpha^2+$ $+105127268\,\alpha+69739299)/945$ $\lambda_2 = -16\ \left(\alpha+2\right)\left(\alpha+6\right)(6144\,\alpha^6+220608\,\alpha^5+3202332\,\alpha^4+23982586\,\alpha^3+97424616\,\alpha^2+$ $+202902629\,\alpha+168662655)/945$ $\lambda_4 = 8\ \left(\alpha+4\right)\left(\alpha+7\right)\left(1536\,\alpha^4+42960\,\alpha^3+441070\,\alpha^2+1967321\,\alpha+3214530\right)/315$ $\lambda_6 = -8\ \left(\alpha+6\right)\left(\alpha+8\right)\left(96\,\alpha^2+1535\,\alpha+5985\right)/63$ $\lambda_8 = 32\ \left(\alpha+9\right)\left(\alpha+8\right)/9\,,\ \lambda_{10} = -1$ $\mu_1 = 2\ \left(\alpha+1\right)(98304\,\alpha^8+4316160\,\alpha^7+80617920\,\alpha^6+834540064\,\alpha^5+5220464816\,\alpha^4+$ $+20129267600\,\alpha^3+46486591360\,\alpha^2+58394288256\,\alpha+30257298225)/945$		

n	λ_k	μ_k	γ
	$\mu_3 = -2\ (\alpha+3)\,(12288\,\alpha^6 + 478848\,\alpha^5 + 7649264\,\alpha^4 + 64060608\,\alpha^3 + 296357968\,\alpha^2 +$ $+717278784\,\alpha + 708643845)/315$ $\mu_5 = 2\ (\alpha+5)\left(768\,\alpha^4 + 23032\,\alpha^3 + 256440\,\alpha^2 + 1255952\,\alpha + 2282175\right)/63$ $\mu_7 = -2\ (\alpha+7)\left(32\,\alpha^2 + 544\,\alpha + 2295\right)/9\,,\ \mu_9 = 2(\alpha+9)$ $\gamma = 0$		

$Z_\nu(x) \ = \ I_1(x)\,,\ K_1(x)$ with s defined as on page 185:

n	λ_k	μ_k	γ
1	$-$	$-$	$-2s\alpha$
2	$\lambda_2 = s\,,$	$\mu_1 = -2(\alpha+1)\,,$	0
3	$\lambda_1 = s(\alpha+1)\,,\ \lambda_3 = s$	$\mu_0 = -2\alpha(\alpha+2)\,,\ \mu_2 = -2(\alpha+2)$	$-2s\alpha(\alpha+1)(2\alpha+3)$
4	$\lambda_0 = 8s\alpha(\alpha+2)(\alpha+3)/3\,,$ $\lambda_2 = 8s(\alpha+2)(\alpha+3)/3\,,\ \lambda_4 = s$	$\mu_1 = -2(\alpha+1)(8\alpha^2+40\alpha+45)/3\,,$ $\mu_3 = 2(\alpha+3)$	0
5	$\lambda_1 = s(\alpha+1)(2\,\alpha+7)(3\,\alpha+10)/2\,,$ $\lambda_3 = 3s(\alpha+3)(\alpha+4)\,,\ \lambda_5 = s$	$\mu_0 = -\alpha(\alpha+3)(6\alpha^2+35\alpha+48)\,,$ $\mu_2 = -2(\alpha+2)(3\alpha^2+21\alpha+37)\,,$ $\mu_4 = -2(\alpha+4)$	$-s\alpha(\alpha+1)(\alpha+2)\cdot$ $\cdot(2\,\alpha+7)(6\,\alpha+19)$
6	$\lambda_0 = 8s\alpha(\alpha+3)(\alpha+4)\cdot$ $\cdot(16\alpha^2+110\alpha+165)/15$ $\lambda_2 = 8s(\alpha+2)(\alpha+4)\cdot$ $\cdot(16\alpha^2+129\alpha+255)/15\,,$ $\lambda_4 = 16s(\alpha+4)(\alpha+5)/5\,,\lambda_6 = s$	$\mu_1 = -2(\alpha+1)(2\alpha+5)\cdot$ $\cdot(64\alpha^3+740\alpha^2+2802\alpha+3459)/15\,,$ $\mu_3 = -2(\alpha+3)\cdot$ $\cdot(16\alpha^2+144\alpha+325)/5\,,$ $\mu_5 = -2(\alpha+5)$	0
7	$\lambda_1 = s(\alpha+1)(60\,\alpha^4 + 1072\,\alpha^3 + 7075\,\alpha^2 + 20419\,\alpha + 21726)/6$ $\lambda_3 = s(\alpha+3)(\alpha+5)\left(30\,\alpha^2 + 301\,\alpha + 744\right)/3$ $\lambda_5 = 10s\,(\alpha+5)(\alpha+6)/3\,,\ \lambda_7 = s$ $\mu_0 = -\alpha\ (\alpha+4)\left(60\,\alpha^4 + 952\,\alpha^3 + 5421\,\alpha^2 + 13034\,\alpha + 11163\right)/3$ $\mu_2 = -2\ (\alpha+2)\left(30\,\alpha^4 + 541\,\alpha^3 + 3612\,\alpha^2 + 10577\,\alpha + 11463\right)/3$ $\mu_4 = -2\ (\alpha+4)\left(10\,\alpha^2 + 110\,\alpha + 303\right)/3\,,\ \mu_6 = -2(\alpha+6)$ $\gamma = -s\alpha\,(\alpha+1)(\alpha+2)\,(\alpha+3)\left(120\,\alpha^3 + 1764\,\alpha^2 + 8562\,\alpha + 13703\right)/3$		
8	$\lambda_0 = 16s\,\alpha\,(\alpha+4)\,(\alpha+5)\left(192\,\alpha^4 + 3436\,\alpha^3 + 21738\,\alpha^2 + 54394\,\alpha + 45885\right)/105$ $\lambda_2 = 8s\,(\alpha+5)(\alpha+2)(2\alpha+7)\left(192\,\alpha^3 + 3184\,\alpha^2 + 17191\,\alpha + 29973\right)/105$ $\lambda_4 = 8s\,(\alpha+4)(\alpha+6)(48\,\alpha^2 + 577\,\alpha + 1715)/35$ $\lambda_6 = 24s\,(\alpha+7)(\alpha+6)\,,\ \lambda_8 = s$ $\mu_1 = -2\ (\alpha+1)\,(2\,\alpha+5)\,(1536\,\alpha^5 + 37760\,\alpha^4 + 364152\,\alpha^3 + 1718676\,\alpha^2 +$ $+3961102\,\alpha + 3557589)/105$ $\mu_3 = -2\ (\alpha+3)\left(384\,\alpha^4 + 8456\,\alpha^3 + 69216\,\alpha^2 + 249544\,\alpha + 334355\right)/35$ $\mu_4 = -2\ (\alpha+5)\left(24\,\alpha^2 + 312\,\alpha + 1015\right)/7\,,\ \mu_7 = -2(\alpha+7)$ $\gamma = 0$		
9	$\lambda_1 = s(\alpha+1)(840\,\alpha^6 + 27620\,\alpha^5 + 372010\,\alpha^4 + 2624431\,\alpha^3 + 10217789\,\alpha^2 +$ $+20796570\,\alpha + 17273256)/24$ $\lambda_3 = s(\alpha+3)(\alpha+6)(420\,\alpha^4 + 10100\,\alpha^3 + 89499\,\alpha^2 + 346057\,\alpha + 492936)/12$ $\lambda_5 = s(\alpha+5)(\alpha+7)(70\,\alpha^2 + 981\,\alpha + 3408)/6$ $\lambda_7 = 7s\ (\alpha+7)\,(\alpha+8)/2\,,\ \lambda_9 = s$ $\mu_0 = -\alpha\,(\alpha+5)(840\,\alpha^6 + 25100\,\alpha^5 + 301890\,\alpha^4 + 1863189\,\alpha^3 + 6200255\,\alpha^2 +$ $+10520550\,\alpha + 7100160)/12$ $\mu_2 = -\,(\alpha+2)\,(420\,\alpha^6 + 13880\,\alpha^5 + 188099\,\alpha^4 + 1336990\,\alpha^3 + 5253733\,\alpha^2 +$ $+10816542\,\alpha + 9113772)/6$ $\mu_4 = -(\alpha+4)(70\,\alpha^4 + 1821\,\alpha^3 + 17651\,\alpha^2 + 75546\,\alpha + 120462)/3$ $\mu_6 = -(\alpha+6)(7\,\alpha^2 + 105\,\alpha + 394)\,,\ \mu_8 = -2(\alpha+8)$ $\gamma = -s\alpha\,(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(1680\,\alpha^4 +$ $+43200\,\alpha^3 + 413112\,\alpha^2 + 1740304\,\alpha + 2723721)/12$		

n	λ_k	μ_k	γ
10	$\lambda_0 = 16s\,\alpha\,(\alpha+5)(\alpha+6)(6144\,\alpha^6 + 202176\,\alpha^5 + 2654640\,\alpha^4 + 17692100\,\alpha^3 + 62786346\,\alpha^2 + {}$ $+112260364\,\alpha + 79029405)/945$ $\lambda_2 = 16s\,(\alpha+2)(\alpha+6)(2\alpha+7)(3072\,\alpha^5 + 100128\,\alpha^4 + 1280094\,\alpha^3 + 8005750\,\alpha^2 +$ $+24436771\,\alpha + 29071575)/945$ $\lambda_4 = 8s\,(\alpha+4)(\alpha+7)(1536\,\alpha^4 + 43056\,\alpha^3 + 446734\,\alpha^2 + 2032409\,\alpha + 3421890)/315$ $\lambda_6 = 8s\,(\alpha+8)\,(\alpha+6)\,\big(96\,\alpha^2 + 1537\,\alpha + 6111\big)\,/63$ $\lambda_8 = 32s\,(\alpha+9)(\alpha+8)/9\,,\ \lambda_{10} = s$ $\mu_1 = -2\,(\alpha+1)(2\alpha+5)(49152\,\alpha^7 + 2044416\,\alpha^6 + 35778528\,\alpha^5 + 341048704\,\alpha^4 +$ $1909536808\,\alpha^3 + +6269569684\,\alpha^2 + 11156022222\,\alpha + 8281429821)/915$ $\mu_3 = -2\,\,(\alpha+3)\,(12288\,\alpha^6 + 479616\,\alpha^5 + 7710704\,\alpha^4 + 65331648\,\alpha^3 + 307586128\,\alpha^2 +$ $+762778176\,\alpha + 778317435)/315$ $\mu_5 = -2\,(\alpha+5)(768\,\alpha^4 + 23048\,\alpha^3 + 258120\,\alpha^2 + 1278448\,\alpha + 2362815)/63$ $\mu_7 = -2\,\,(\alpha+7)\,\big(32\,\alpha^2 + 544\,\alpha + 231^3\big)\,/9\,,\ \mu_9 = -2(\alpha+9)$ $\gamma = 0$		

d) Jacobi Polynomials $C_n^{(\alpha,\beta)}(x)$:

$$\int_{-1}^{1} (1-x)^\alpha\,(1+x)^\beta\,[C_n^{(\alpha,\beta)}(x)]^2\,dx \quad = \quad \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}\,, \quad \alpha,\beta \;>\; -1$$

First polynomials:

$$C_0^{(\alpha,\beta)}(x) = 1\,,\ C_1^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta)\,x + \alpha - \beta}{2}\,,$$

$$C_2^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+3)(\alpha+\beta+4)}{8}\,x^2 + \frac{(\alpha+\beta+3)(\alpha-\beta)}{4}\,x + \frac{(\alpha-\beta-1)(\alpha-\beta)}{8}\,,$$

$$C_3^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+4)(\alpha+\beta+5)(\alpha+\beta+6)}{48}\,x^3 +$$

$$+\frac{(\alpha+\beta+4)(\alpha+\beta+5)(\alpha-\beta)}{16}\,x^2 + \frac{(\alpha+\beta+4)\,[(\alpha-\beta)^2-(\alpha+\beta)-6]}{16}\,x + \frac{(\alpha-\beta)\,[(\alpha-\beta)^2-3(\alpha+\beta)-16]}{48}\,,$$

$$C_4^{(\alpha,\beta)}(x) =$$

$$= \frac{(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)}{384}\,x^4 + \frac{(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha-\beta)}{96}\,x^3 +$$

$$+\frac{(\alpha+\beta+5)(\alpha+\beta+6)[(\alpha-\beta)(\alpha-\beta-1)-8]}{64}\,x^2 + \frac{(\alpha+\beta+5)(\alpha-\beta)[(\alpha-\beta)(\alpha-\beta-3)-22]}{96}\,x +$$

$$+\frac{(\alpha-\beta)^4 - 6(\alpha^3-\alpha^2\beta-\alpha\beta^2+\beta^3) - (37\alpha^2-86\alpha\beta+37\beta^2) + 42(\alpha+\beta) + 144}{384}$$

$Z_\nu(x) \;=\; J_0(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = -(\alpha-\beta)^2/[2(\alpha+\beta+2)]$, $\lambda_1 = (\alpha-\beta)/(\alpha+\beta+2)$	$\mu_0 = -(\alpha-\beta)/2$, $\mu_1 = 1$	$(\alpha-\beta)/2$
2	$\lambda_0 = -(\alpha-\beta)\,[(\alpha-\beta)^2-\alpha-\beta-4]/[8(\alpha+\beta+2)]\,,\ \lambda_1 = [(\alpha-\beta)^2-\alpha-\beta-4]/[4(\alpha+\beta+2)]$ $\mu_0 = -\,[(\alpha-\beta)^2-\alpha-\beta-44]/8\,,\ \mu_2 = 1$ $\gamma = -(\alpha+2)(\beta+2)/2$		
3	$\lambda_0 = [(\alpha+\beta)(\alpha-\beta)^2(\alpha^2+6\,\alpha\,\beta+\beta^2) + (32\,\alpha^4+20\,\alpha^3\,\beta-72\alpha^2\,\beta^2+20\alpha\,\beta^3+32\,\beta^4) +$ $+(\alpha+\beta)(207\,\alpha^2-278\,\alpha\,\beta+207\,\beta^2) + (496\,\alpha^2-160\,\alpha\,\beta+496\,\beta^2) + 536\,(\alpha+\beta) +$ $+480]/[48\,(\alpha+\beta+2)(\alpha+\beta+3)]$ $\lambda_1 = -(\alpha-\beta)[((3\alpha+\beta)(\alpha+3\beta)+47(\alpha+\beta)+136]\;/\;[24(\alpha+\beta+2)]$ $\lambda_2 = (\alpha+\beta+5)(\alpha+\beta+6)\;/\;[3(\alpha+\beta+3)]$ $\mu_0 = (\alpha-\beta)[(\alpha+\beta)(3\alpha+\beta)(\alpha+3\beta)+(61\alpha^2+130\alpha\beta+61\beta^2)+318(\alpha+\beta) +$ $+512]\;/\;[48(\alpha+\beta+2)]$ $\mu_1 = -(\alpha+\beta+4)(\alpha+\beta+5)(\alpha+\beta+6)\;/\;[6(\alpha+\beta+2)]\,,\ \mu_3 = 1$ $\gamma = -\,(\alpha-\beta)[(\alpha^2+4\alpha\beta+\beta^2)+15(\alpha+\beta)+38]\;/\;24$		

n	λ_k	μ_k	γ
4	$\lambda_0 = (\alpha-\beta)\,[(\alpha^2-\beta^2)^2(\alpha^2+3\,\beta\,\alpha+\beta^2)+2\,(\alpha+\beta)(12\,\alpha^4+3\,\alpha^3\,\beta-32\alpha^2\,\beta^2+3\,\alpha\beta^3+12\,\beta^4)+$ $+(170\,\alpha^4-57\,\alpha^3\,\beta-502\,\alpha^2\beta^2-57\,\alpha\beta^3+170\,\beta^4)+2\,(\alpha+\beta)(137\,\alpha^2-855\,\alpha\,\beta+137\,\beta^2)-$ $-(1491\,\alpha^2+5142\,\alpha\,\beta+1491\,\beta^2)-6322\,(\alpha+\beta)-7152]\;/\;[96(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_1 = -[(\alpha+\beta)(\alpha-\beta)^2(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2)+(46\,\alpha^4-10\,\alpha^3\,\beta-144\,\alpha^2\beta^2-10\,\alpha\beta^3+46\,\beta^4)+$ $+3\,(\alpha+\beta)(39\,\alpha^2-230\,\alpha\,\beta+39\,\beta^2)-(814\,\alpha^2\,\alpha+2620\,\beta+814\,\beta^2)-4296(\,\alpha+\beta)-6336]\;/$ $/\;[96(\alpha+\beta+2)(\alpha+\beta+4)]$ $\lambda_2 = -\,(\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+7)(\alpha+\beta+24)\;/\;[48(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_3 = 3(\alpha+\beta+7)(\alpha+\beta+8)\;/\;[8(\alpha+\beta+4)]$ $\mu_0 = [(\alpha^2-\beta^2)^2(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2)(\alpha^2-\beta^2)^2+(\alpha+\beta)(47\,\alpha^4-8\,\alpha^3\,\beta-150\,\alpha^2\beta^2-8\,\alpha\beta^3+47\,\beta^4)+$ $+(63\,\alpha^4-498\,\alpha^3\,\beta-1170\,\alpha^2\beta^2-498\,\alpha\beta^3+63\,\beta^4)-(\alpha+\beta)(1823\,\alpha^2+1922\,\alpha\,\beta+1823\,\beta^2)-$ $-(9642\,\alpha^2+9684\,\alpha\,\beta+9642\,\beta^2)-17640(\,\alpha+\beta)-15552]\;/\;[192(\alpha+\beta+2)(\alpha+\beta+3)]$ $\mu_1 = (\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+40)\;/\;[96(\alpha+\beta+2)(\alpha+\beta+4)]$ $\mu_2 = -3(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)\;/\;[16(\alpha+\beta+3)(\alpha+\beta+4)]\,,\;\mu_4=1$ $\gamma = [(\alpha^2+4\,\alpha\,\beta+\beta^2)^2+6\,(\alpha+\beta)(7\,\alpha^2+26\,\alpha\,\beta+7\,\beta^2)+(539\,\alpha^2+1298\,\alpha\,\beta+539\,\beta^2)+$ $+2586\,(\alpha+\beta)+4176]\;/\;96$		
5	$\lambda_0 = -\,[(\alpha+\beta)^3(\alpha-\beta)^2(6\,\alpha^4+31\,\alpha^3\,\beta+54\,\alpha^2\beta^2+31\,\alpha\beta^3+6\,\beta^4)+$ $+(\alpha+\beta)^2(347\,\alpha^6+947\,\alpha^5\,\beta-171\,\alpha^4\beta^2-1734\,\alpha^3\beta^3-171\,\alpha^2\beta^4+947\,\alpha\beta^5+347\,\beta^6)+$ $+(\alpha+\beta)(8108\,\alpha^6+20327\,\alpha^5\,\beta-1360\,\alpha^4\beta^2-27910\,\alpha^3\beta^3-1360\,\alpha^2\beta^4+20327\,\alpha\beta^5+8108\,\beta^6)+$ $+(101076\,\alpha^6+250891\,\alpha^5\,\beta+42780\,\alpha^4\beta^2-215030\,\alpha^3\beta^3+42780\,\alpha^2\beta^4+250891\,\alpha\beta^5+101076\,\beta^6)+$ $+6\,(\alpha+\beta)(123960\,\alpha^4+78295\,\alpha^3\,\beta-112814\,\alpha^2\beta^2+78295\,\alpha\beta^3+123960\,\beta^4)+$ $+(3347073\,\alpha^4+3270528\,\alpha^3\,\beta-286530\,\alpha^2\beta^2+3270528\,\alpha\beta^3+3347073\,\beta^4)+$ $+6\,(\alpha+\beta)(1533317\,\alpha^2-594074\,\alpha\,\beta+1533317\,\beta^2)+(15202864\,\alpha^2+10520288\,\alpha\,\beta+15202864\,\beta^2)+$ $+15425344(\,\alpha+\beta)+9623040]\;/\;[960(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\lambda_1 = (\alpha-\beta)\,[(\alpha+\beta)^2\,(27\,\alpha^4+126\,\alpha^3\,\beta+206\,\alpha^2\beta^2+126\,\alpha\beta^3+27\,\beta^4)+$ $+(\alpha+\beta)(1341\,\alpha^4+5816\,\alpha^3\,\beta+9022\,\alpha^2\beta^2+5816\,\alpha\beta^3+1341\,\beta^4)+$ $+(26279\,\alpha^4+109098\,\alpha^3\,\beta+165798\,\alpha^2\beta^2+109098\,\alpha\beta^3+26279\,\beta^4)+$ $+(\alpha+\beta)(264131\,\alpha^2+544130\,\alpha\,\beta+264131\,\beta^2)+(1452462\,\alpha^2+2931644\,\alpha\,\beta+1452462\,\beta^2)+$ $+4170640\,(\alpha+\beta)+4892320]\;/\;[960(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\lambda_2 = -(\alpha+\beta+8)\,[(\alpha+\beta)^2(33\,\alpha^2+62\,\alpha\,\beta+33\,\beta^2)+16\,(\alpha+\beta)\,(64\,\alpha^2+127\,\alpha\,\beta+64\,\beta^2)+$ $+(11823\,\alpha^2+24306\,\alpha\,\beta+11823\,\beta^2)+61520\,(\alpha+\beta)+116400]\;/\;[480(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_3 = -\,(\alpha-\beta)\,(\alpha+\beta+7)(\alpha+\beta+9)(\alpha+\beta+40)\;/\;[80(\alpha+\beta+4)(\alpha+\beta+5)]$ $\lambda_4 = 2(\alpha+\beta+9)(\alpha+\beta+10)\;/\;[5(\alpha+\beta+5)]$ $\mu_0 = -\,(\alpha-\beta)[(\alpha+\beta)^3(27\,\alpha^4+126\,\alpha^3\,\beta+206\,\alpha^2\beta^2+126\,\alpha\beta^3+27\,\beta^4)+$ $+3\,(\alpha+\beta)^2\,(469\,\alpha^4+2024\,\alpha^3\,\beta+3134\,\alpha^2\beta^2+2024\,\alpha\beta^3+469\,\beta^4)+$ $+(\alpha+\beta)(29203\,\alpha^4+120818\,\alpha^3\,\beta+183438\,\alpha^2\beta^2+120818\,\alpha\beta^3+29203\,\beta^4)+$ $+(318057\,\alpha^4+1293792\,\alpha^3\,\beta+1951662\,\alpha^2\beta^2+1293792\,\alpha\beta^3+318057\,\beta^4)+$ $+2\,(\alpha+\beta)(993929\,\alpha^2+2033642\,\alpha\,\beta+993929\,\beta^2)+(7184328\,\alpha^2+14574096\,\alpha\,\beta+7184328\,\beta^2)+$ $+13918672(\,\alpha+\beta)+11039808]\;/\;[1920(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_1 = (\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)[(\alpha+\beta)^2(33\,\alpha^2+62\,\alpha\,\beta+33\,\beta^2)+$ $+2\,(\alpha+\beta)(443\,\alpha^2+898\,\alpha\,\beta+443\,\beta^2)+(8285\,\alpha^2+18710\,\alpha\,\beta+8285\,\beta^2)+36296\,(\alpha+\beta)+$ $+52960]\,/\,[960(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\mu_2 = (\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+70)\;/\;[160(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_3 = -(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+10)\;/\;[5(\alpha+\beta+4)(\alpha+\beta+5)]\,,\;\mu_5=1$ $\gamma = (\alpha-\beta)[(9\,\alpha^4+44\,\alpha^3\,\beta+74\,\alpha^2\beta^2+44\,\alpha\beta^3+9\,\beta^4)+10\,(\alpha+\beta)(31\,\alpha^2+76\,\alpha\,\beta+31\,\beta^2)+$ $+(3815\,\alpha^2+8130\,\beta\,\alpha+3815\,\beta^2)+19850\,(\alpha+\beta)+37216]\;/\;960$		

$Z_\nu(x) \;=\; I_0(x)\,,\; K_0(x)$ with s defined as on page 185:

n	$\lambda_k\,,\;\mu_k\,,\;\gamma$
1	$\lambda_0 = -(\alpha-\beta)^2/[2(\alpha+\beta+2)]\,,\;\lambda_1 = (\alpha-\beta)/(\alpha+\beta+2)$ $\mu_0 = -s(\alpha-\beta)/2\,,\;\mu_1 = s$ $\gamma = (\alpha-\beta)/2$
2	$\lambda_0 = -(\alpha-\beta)[(\alpha-\beta)^2-(\alpha+\beta)-4]\,/\,[8(\alpha+\beta+2)]$ $\lambda_1 = [(\alpha-\beta)^2-(\alpha+\beta)-4]\,/\,[4(\alpha+\beta+2)]$ $\mu_0 = -\,s[(\alpha-\beta)^2-(\alpha+\beta)-4]\,/\,8\,,\;\mu_2 = s$ $\gamma = [(\alpha^2+\beta^2)+3\,(\alpha+3\beta)+4]\,/\,4$

n	$\lambda_k\,,\ \mu_k\,,\ \gamma$
3	$\lambda_0 = -[(\alpha+\beta)(\alpha-\beta)^2(3\alpha^2+2\alpha\,\beta+3\beta^2) + (32\alpha^4-4\alpha^3\beta-24\alpha^2\beta^2-4\alpha\beta^3+32\beta^4) +$ $+(\alpha+\beta)(157\,\alpha^2-178\,\alpha\,\beta+157\,\beta^2) + (400\,\alpha^2+32\,\alpha\,\beta+400\,\beta^2) + 536\,(\alpha+\beta) + 480]\,/$ $/\,[48\,(\alpha+\beta+2)(\alpha+\beta+3)]$ $\lambda_1 = (\alpha-\beta)[(5\,\alpha^2+6\,\alpha\,\beta+5\,\beta^2) + 41\,(\alpha+\beta) + 104]\,/\,[24\,(\alpha+\beta+2)]$ $\lambda_2 = -(\alpha+\beta+5)(\alpha+\beta+6)\,/\,[3(\alpha+\beta+3)]$ $\mu_0 = -\,s(\alpha-\beta)[(\alpha+\beta)(5\,\alpha^2+6\,\alpha\,\beta+5\,\beta^2) + (59\,\alpha^2+110\,\alpha\,\beta+59\,\beta^2) + 274\,(\alpha+\beta) + 448]\,/$ $/\,[48\,(\alpha+\beta+2)]$ $\mu_1 = s(\alpha+\beta+4)(\alpha+\beta+5)(\alpha+\beta+6)\,/\,[6\,(\alpha+\beta+2)]\,,\ \mu_3 = s$ $\gamma = (\alpha-\beta)[(\alpha^2+\alpha\beta+\beta^2) + 6\,(\alpha+\beta) + 11]/12$
4	$\lambda_0 = -(\alpha-\beta)[(\alpha^2-\beta^2)^2(3\,\alpha^2+4\,\alpha\,\beta+3\,\beta^2) + (\alpha+\beta)(49\,\alpha^4-16\,\alpha^3\,\beta-74\,\alpha^2\beta^2-16\,\alpha\beta^3+49\,\beta^4)+$ $+(273\,\alpha^4-150\,\alpha^3\,\beta-750\,\alpha^2\beta^2-150\,\alpha\beta^3+273\,\beta^4) + 259\,(\alpha+\beta)(\alpha^2-10\,\alpha\,\beta+\beta^2) -$ $-(2988\,\alpha^2+8376\,\alpha\,\beta+2988\,\beta^2) - 11132\,(\alpha+\beta) - 12576]\,/\,[192\,(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_1 = (\alpha+\beta)(\alpha-\beta)^2(2\,\alpha^2+3\,\alpha\,\beta+2\,\beta^2) + (22\,\alpha^4-13\,\alpha^3\,\beta-54\,\alpha^2\beta^2-13\,\alpha\beta^3+22\,\beta^4)+$ $+2\,(\alpha+\beta)(14\,\alpha^2-139\,\alpha\,\beta+14\,\beta^2) - (460\,\alpha^2+1096\,\alpha\,\beta+460\,\beta^2) - 1992\,(\alpha+\beta) - 2880]\,/$ $/\,[48\,(\alpha+\beta+2)(\alpha+\beta+4)]$ $\lambda_2 = (\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+7)(\alpha+\beta+24)\,/\,[48\,(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_3 = -3\,(\alpha+\beta+7)(\alpha+\beta+8)\,/\,[8(\alpha+\beta+4)]$ $\mu_0 = -\,s[(\alpha^2-\beta^2)^2(2\,\alpha^2+3\,\alpha\,\beta+2\,\beta^2) + (\alpha+\beta)(23\,\alpha^4-14\,\alpha^3\,\beta-54\,\alpha^2\beta^2-14\,\alpha\beta^3+23\,\beta^4)+$ $+(\alpha^4-255\,\alpha^3\,\beta-488\,\alpha^2\beta^2-255\,\alpha\beta^3+\beta^4) - (\alpha+\beta)(1001\,\alpha^2+668\,\alpha\,\beta+1001\,\beta^2)-$ $-(4755\,\alpha^2+4230\,\alpha\,\beta+4755\,\beta^2) - 8334\,(\alpha+\beta) - 7344]\,/\,[96\,(\alpha+\beta+2)(\alpha+\beta+3)]$ $\mu_1 = -s(\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+40)\,/\,[96\,(\alpha+\beta+2)(\alpha+\beta+4)]$ $\mu_2 = 3\,s(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)\,/\,[16\,(\alpha+\beta+3)(\alpha+\beta+4)]\,,\ \mu_4 = s$ $\gamma = [(\alpha^2+\alpha\beta+\beta^2)^2 + 3\,(\alpha+\beta)(6\,\alpha^2+7\,\alpha\,\beta+6\,\beta^2) + (143\,\alpha^2+251\,\alpha\,\beta+143\,\beta^2)+$ $+558\,(\alpha+\beta) + 864]\,/\,24$
5	$\lambda_0 = -\,[(\alpha-\beta)^2(\alpha+\beta)^3(21\,\alpha^4+62\,\alpha^3\,\beta+90\,\alpha^2\beta^2+62\,\alpha\beta^3+21\,\beta^4) +$ $+2\,(\alpha+\beta)^2(454\,\alpha^6+691\,\alpha^5\,\beta-278\,\alpha^4\beta^2-1222\,\alpha^3\beta^3-278\,\alpha^2\beta^4+691\,\alpha\beta^5+454\,\beta^6)+$ $+2\,(\alpha+\beta)(8879\,\alpha^6+17123\,\alpha^5\,\beta-2195\,\alpha^4\beta^2-21630\,\alpha^3\beta^3-2195\,\alpha^2\beta^4+17123\,\alpha\beta^5+8879\,\beta^6)+$ $+(201004\,\alpha^6+452234\,\alpha^5\,\beta+82100\,\alpha^4\beta^2-340180\,\alpha^3\beta^3+82100\,\alpha^2\beta^4+452234\,\alpha\beta^5+201004\,\beta^6)+$ $+(\alpha+\beta)(1417457\,\alpha^4+811660\,\alpha^3\,\beta-1023546\,\alpha^2\beta^2+811660\,\alpha\beta^3+1417457\,\beta^4)+$ $+(6324712\,\alpha^4+6310656\,\alpha^3\,\beta+161968\,\alpha^2\beta^2+6310656\,\alpha\beta^3+6324712\,\beta^4)+$ $+4\,(\alpha+\beta)(4400183\,\alpha^2-1485118\,\alpha\,\beta+4400183\,\beta^2)+$ $+(29794976\,\alpha^2+21700672\,\alpha\,\beta+29794976\,\beta^2) + 30911872\,(\alpha+\beta) + 19461120]\,/$ $/\,[1920\,(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\lambda_1 = (\alpha-\beta)\,[(\alpha+\beta)^2(19\,\alpha^4+63\,\alpha^3\,\beta+92\,\alpha^2\beta^2+63\,\alpha\beta^3+19\,\beta^4)+$ $+(\alpha+\beta)(769\,\alpha^4+2834\,\alpha^3\,\beta+4166\,\alpha^2\beta^2+2834\,\alpha\beta^3+769\,\beta^4)+$ $+(13513\,\alpha^4+52321\,\alpha^3\,\beta+77696\,\alpha^2\beta^2+52321\,\alpha\beta^3+13513\,\beta^4)+$ $+(\alpha+\beta)(129267\,\alpha^2+252440\,\alpha\,\beta+129267\,\beta^2) + (699524\,\alpha^2+1388888\,\alpha\,\beta+699524\,\beta^2)+$ $+2013308\,(\alpha+\beta) + 2402480)\,/\,[480\,(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\lambda_2 = -\,(\alpha+\beta+8)\,[(\alpha+\beta)^2(31\,\alpha^2+66\,\alpha\,\beta+31\,\beta^2) + 16\,(\alpha+\beta)(64\,\alpha^2+129\,\alpha\,\beta+64\,\beta^2)+$ $+(12433\,\alpha^2+24206\,\alpha\,\beta+12433\,\beta^2) + 64432\,(\alpha+\beta) + 125520]\,/\,[480\,(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_3 = (\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+9)(\alpha+\beta+40)\,/\,[80\,(\alpha+\beta+5)(\alpha+\beta+4)]$ $\lambda_4 = -2\,(\alpha+\beta+9)(\alpha+\beta+10)\,/\,[5\,(\alpha+\beta+5)]$ $\mu_0 = -\,s(\alpha-\beta)\,[(\alpha+\beta)^3(19\,\alpha^4+63\,\alpha^3\,\beta+92\,\alpha^2\beta^2+63\,\alpha\beta^3+19\,\beta^4)+$ $+2\,(\alpha+\beta)^2(400\,\alpha^4+1481\,\alpha^3\,\beta+2180\,\alpha^2\beta^2+1481\,\alpha\beta^3+400\,\beta^4)+$ $+(\alpha+\beta)(14998\,\alpha^4+58225\,\alpha^3\,\beta+86558\,\alpha^2\beta^2+58225\,\alpha\beta^3+14998\,\beta^4)+$ $+2\,(1147\,\alpha^2+2246\,\alpha\,\beta+1147\,\beta^2)(69\,\alpha^2+137\,\alpha\,\beta+69\,\beta^2)+$ $+(\alpha+\beta)(995351\,\alpha^2+1949558\,\alpha\,\beta+995351\,\beta^2) + (3675970\,\alpha^2+7253060\,\alpha\,\beta+3675970\,\beta^2)+$ $+7327952\,(\alpha+\beta) + 6104544]\,/\,[960\,(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_1 = s(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)\,[(\alpha+\beta)^2(31\,\alpha^2+66\,\alpha\,\beta+31\,\beta^2)+$ $+6\,(\alpha+\beta)(151\,\alpha^2+298\,\alpha\,\beta+151\,\beta^2) + (9699\,\alpha^2+17258\,\alpha\,\beta+9699\,\beta^2)+$ $+39864\,(\alpha+\beta) + 62240]\,/\,[960\,(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\mu_2 = -\,s(\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+70)\,/\,[160\,(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_3 = s(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+10)\,/\,[5(\alpha+\beta+4)(\alpha+\beta+5)]\,,\ \mu_5 = s$ $\gamma = (\alpha-\beta)\,[(7\,\alpha^4+22\,\alpha^3\,\beta+32\,\alpha^2\beta^2+22\,\alpha\beta^3+7\,\beta^4) + 60\,(\alpha+\beta)(3\,\alpha^2+5\,\alpha\,\beta+3\,\beta^2)+$ $+(1845\,\alpha^2+3520\,\alpha\,\beta+1845\,\beta^2) + 8700\,(\alpha+\beta) + 15668]\,/\,480$

$Z_\nu(x) \; = \; J_1(x) :$

<table>
<tr><th>n</th><th>λ_k</th><th>μ_k</th><th>γ</th></tr>
<tr><td>1</td><td>$\lambda_0 = -(\alpha-\beta)/2$</td><td>–</td><td>$(\alpha+\beta+2)/2$</td></tr>
<tr><td>2</td><td>$\lambda_0 = -(\alpha-\beta)^2(\alpha+\beta+3)/$
$/[4(\alpha+\beta+2)]$
$\lambda_1 = (\alpha-\beta)(\alpha+\beta+3)/$
$/[2(\alpha+\beta+2)]\,,\ \lambda_2 = -1$</td><td>$\mu_0 = -(\alpha-\beta)(\alpha+\beta+3)(\alpha+\beta+4)/$
$/[4(\alpha+\beta+2)]$
$\mu_1 = (\alpha+\beta+3)(\alpha+\beta+4)/[2(\alpha+\beta+2]$</td><td>$(\alpha-\beta)(\alpha+\beta+3)/4$</td></tr>
<tr><td>3</td><td colspan="3">$\lambda_0 = -(\alpha-\beta)(\alpha+\beta+4)\,[(\alpha-\beta)^2-\alpha-\beta-6]\;/\;[16(\alpha+\beta+2)]$
$\lambda_1 = (\alpha+\beta+4)[(\alpha-\beta)^2-\alpha-\beta-6]\;/\;[8(\alpha+\beta+2)]\,,\ \lambda_3 = -1$
$\mu_0 = -\,(\alpha+\beta+4)(\alpha+\beta+5)\,[(\alpha+\beta)(\alpha-\beta)^2-(\alpha+\beta)^2-8\,(\alpha+\beta)-12]\;/$
$/\;[16(\alpha+\beta+2)(\alpha+\beta+3)]$
$\mu_1 = -(\alpha-\beta)(\alpha+\beta+5)/[2(\alpha+\beta+2)]\,,\ \mu_2 = (\alpha+\beta+5)(\alpha+\beta+6)/[2(\alpha+\beta+3)]$
$\gamma = -(\alpha+3)(\beta+3)(\alpha+\beta+4)/4$</td></tr>
<tr><td>4</td><td colspan="3">$\lambda_0 = (\alpha+\beta+5)\,[(\alpha+\beta)(\alpha-\beta)^2(\alpha^2+6\,\alpha\,\beta+\beta^2)+(44\,\alpha^4+20\,\alpha^3\,\beta-96\,\alpha^2\beta^2+20\alpha\,\beta^3+44\,\beta^4)+$
$+(\alpha+\beta)(369\,\alpha^2-554\,\alpha\,\beta+369\,\beta^2)+(1114\,\alpha^2-724\,\alpha\,\beta+1114\,\beta^2)+1256\,(\alpha+\beta)+1344]\,/$
$/\,[96\,(\alpha+\beta+2)(\alpha+\beta+3)]$
$\lambda_1 = -\,(\alpha-\beta)(\alpha+\beta+5)[(\alpha+\beta)(3\,\alpha+\beta)(\alpha+3\,\beta))+(83\,\alpha^2+182\,\alpha\,\beta+83\,\beta^2)+618\,(\alpha+\beta)+$
$+1432]\,/\,[48\,(\alpha+\beta+2)(\alpha+\beta+4)]$
$\lambda_2 = (\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)\,/\,[6\,(\alpha+\beta+3)(\alpha+\beta+4)]\,,\ \lambda_4 = -1$
$\mu_0 = (\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+6)[(\alpha+\beta)^2(3\,\alpha+\beta)(\alpha+3\,\beta)+4\,(\alpha+\beta)(22\,\alpha^2+47\,\alpha\,\beta+22\,\beta^2)+$
$+(715\,\alpha^2+1462\,\alpha\,\beta+715\,\beta^2)+2354\,(\alpha+\beta)+2696]\,/\,[96(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$
$\mu_1 = -(\alpha+\beta+6)[(\alpha+\beta)^4+24\,(\alpha+\beta)^3+(205\,\alpha^2+428\,\alpha\,\beta+205\,\beta^2)+786\,(\alpha+\beta)+1072]\,/$
$/\,[12(\alpha+\beta+2)(\alpha+\beta+4)]$
$\mu_2 = -\,(\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+7)\,/\,[2(\alpha+\beta+3)(\alpha+\beta+4)]$
$\mu_3 = (\alpha+\beta+7)(\alpha+\beta+8)\,/\,[2(\alpha+\beta+4)$
$\gamma = -(\alpha-\beta)(\alpha+\beta+5)[(\alpha^2+4\,\alpha\,\beta+\beta^2)+21\,(\alpha+\beta)+74]\,/\,48$</td></tr>
<tr><td>5</td><td colspan="3">$\lambda_0 = (\alpha-\beta)(\alpha+\beta+6)\,[(\alpha^2-\beta^2)^2(\alpha^2+3\,\alpha\,\beta+\beta^2)+$
$+2\,(\alpha+\beta)(15\,\alpha^4+3\,\alpha^3\,\beta-38\,\alpha^2\beta^2+3\,\alpha\beta^3+15\,\beta^4)+$
$+(248\,\alpha^4-81\,\beta\,\alpha^3-706\,\beta^2\alpha^2-81\,\beta^3\alpha+248\,\beta^4)+2\,(\alpha+\beta)(164\,\alpha^2-1341\,\alpha\,\beta+164\,\beta^2)-$
$-(4089\,\alpha^2+9810\,\alpha\,\beta+4089\,\beta^2)-17302\,(\alpha+\beta)-22896]\,/$
$/\,[192(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$
$\lambda_1 = \,-\,(\alpha+\beta+6)[(\alpha^2-\beta^2)^2(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2)+(\alpha+\beta)(73\,\alpha^4-218\,\alpha^2\beta^2+73\,\beta^4)+$
$+(419\,\alpha^4-650\,\alpha^3\,\beta-2298\,\alpha^2\beta^2-650\,\alpha\beta^3+419\,\beta^4)-(\alpha+\beta)(1489\,\alpha^2+7390\,\alpha\,\beta+1489\,\beta^2)-$
$-(21014\,\alpha^2+34028\,\alpha\,\beta+21014\,\beta^2)-67752\,(\alpha+\beta)-93120]\,/$
$/\,[192(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$
$\lambda_2 = \,-\,(\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+42)\;/\;[96(\alpha+\beta+3)(\alpha+\beta+4)]$
$\lambda_3 = 3\,(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+10)\;/\;[16(\alpha+\beta+4)(\alpha+\beta+5)]\,,\ \lambda_5 = -1$
$\mu_0 = (\alpha+\beta+6)(\alpha+\beta+7)\,[(\alpha+\beta)^3(\alpha-\beta)^2(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2)+$
$+(\alpha+\beta)^2(65\,\alpha^4-4\,\alpha^3\,\beta-194\,\alpha^2\beta^2-4\,\alpha\beta^3+65\,\beta^4)+$
$+(\alpha+\beta)(175\,\alpha^4-646\,\alpha^3\,\beta-1818\,\alpha^2\beta^2-646\,\alpha\beta^3+175\,\beta^4)-$
$-(\alpha+\beta)^2(4073\,\alpha^2+2390\,\alpha\,\beta+4073\,\beta^2)-2\,(\alpha+\beta)(17681\,\alpha^2+5330\,\alpha\,\beta+17681\,\beta^2)-$
$-(112392\,\alpha^2+107664\,\beta\,\alpha+112392\,\beta^2)-168576\,(\alpha+\beta)-130560]\,/$
$/\,[384(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)(\alpha+\beta+5)]$
$\mu_1 = (\alpha-\beta)(\alpha+\beta+7)\,[(\alpha+\beta)^5+86(\alpha+\beta)^4+1913\,(\alpha+\beta)^3+$
$+(17968\,\alpha^2+36128\,\alpha\,\beta+17968\,\beta^2)+77004(\alpha+\beta)+$
$+123168]\,/\,[192(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$
$\mu_2 = -(\alpha+\beta+8)[3(\alpha+\beta)^4+96(\alpha+\beta)^3+(1121\alpha^2+2290\alpha\beta+1121\beta^2)+5840(\alpha+\beta)+11100]\,/$
$/\,[32(\alpha+\beta+3)(\alpha+\beta+4)]$
$\mu_3 = -\,(\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+9)/[2(\alpha+\beta+4)(\alpha+\beta+5)]$
$\mu_4 = (\alpha+\beta+9)(\alpha+\beta+10)/[2(\alpha+\beta+5)]$
$\gamma = (\alpha+\beta+6)\,[(\alpha^2+4\,\alpha\,\beta+\beta^2)^2+6\,(\alpha+\beta)(9\,\alpha^2+34\,\alpha\,\beta+9\,\beta^2)+$
$+(911\,\alpha^2+2210\,\alpha\,\beta+911\,\beta^2)+5754\,(\alpha+\beta)+12240]/192$</td></tr>
</table>

$Z_\nu(x) = I_1(x)\,,\; K_1(x)$ with s defined as on page 185:

n	$\lambda_k\,,\;\mu_k\,,\;\gamma$
1	$\lambda_0 = s(\alpha-\beta)/2$ $\gamma = -s(\alpha+\beta+2)/2$
2	$\lambda_0 = s(\alpha-\beta)^2(\alpha+\beta+3)/[4(\alpha+\beta+2)]$ $\lambda_1 = -\,s(\alpha-\beta)(\alpha+\beta+3)/[2(\alpha+\beta+2)]\,,\; \lambda_2 = s$ $\mu_0 = (\alpha-\beta)(\alpha+\beta+3)(\alpha+\beta+4)/[4(\alpha+\beta+2)]$ $\mu_1 = -\,(\alpha+\beta+3)(\alpha+\beta+4)/[2(\alpha+\beta+2)]$ $\gamma = -\,s(\alpha-\beta)(\alpha+\beta+3)\,/\,4$
3	$\lambda_0 = s(\alpha-\beta)(\alpha+\beta+4)\;[(\alpha-\beta)^2-(\alpha+\beta)-6]\,/\,[16(\alpha+\beta+2)]$ $\lambda_1 = -\,s(\alpha+\beta+4)\,[(\alpha-\beta)^2-(\alpha+\beta)-6]\,/\,[8(\alpha+\beta+2)]\,,\; \lambda_3 = s$ $\mu_0 = (\alpha+\beta+4)(\alpha+\beta+5)\,[(\alpha+\beta)(\alpha-\beta)^2-(\alpha+\beta)^2-8\,(\alpha+\beta)-12]\,/\,[16(\alpha+\beta+2)(\alpha+\beta+3)]$ $\mu_1 = (\alpha-\beta)(\alpha+\beta+5)/[2(\alpha+\beta+2)]\,,\; \mu_2 = -(\alpha+\beta+5)(\alpha+\beta+6)/[2(\alpha+\beta+3)]$ $\gamma = -\,s(\alpha+\beta+4)\,[(\alpha^2+\beta^2)+5\,(\alpha+\beta)+12\,]\,/\,8$
4	$\lambda_0 = s(\alpha+\beta+5)\,[(\alpha+\beta)(\alpha-\beta)^2(3\,\alpha^2+2\,\alpha\,\beta+3\,\beta^2)+(44\,\alpha^4-4\,\alpha^3\,\beta-48\,\alpha^2\beta^2-4\alpha\,\beta^3+44\,\beta^4)+$ $+(\alpha+\beta)(307\,\alpha^2-430\,\alpha\,\beta+307\,\beta^2)+(982\,\alpha^2-460\,\alpha\,\beta+982\,\beta^2)+1256\,(\alpha+\beta)+1344]\,/$ $/\,[96\,(\alpha+\beta+2)(\alpha+\beta+3)]$ $\lambda_1 = -\,s(\alpha-\beta)(\alpha+\beta+5)\,[(\alpha+\beta)(5\,\alpha^2+6\,\alpha\,\beta+5\,\beta^2)+(85\,\alpha^2+154\,\alpha\,\beta+85\,\beta^2)+550\,(\alpha+\beta)+$ $+1256]\,/\,[48\,(\alpha+\beta+2)(\alpha+\beta+4)]$ $\lambda_2 = s(\alpha+\beta+5)(\alpha+\beta+6)(\alpha+\beta+7)(\alpha+\beta+8)\,/\,[6\,(\alpha+\beta+3)(\alpha+\beta+4)]\,,\; \lambda_4 = s$ $\mu_0 = (\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+6)\,[(\alpha+\beta)^2(5\,\alpha^2+6\,\alpha\,\beta+5\,\beta^2)+$ $+4\,(\alpha+\beta)(22\,\alpha^2+41\,\alpha\,\beta+22\,\beta^2)+(669\,\alpha^2+1306\,\alpha\,\beta+669\,\beta^2)+2222\,(\alpha+\beta)+2680]\,/$ $/\,[96\,(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_1 = -\,(\alpha+\beta+6)\,[(\alpha+\beta)^4+24\,(\alpha+\beta)^3+(217\,\alpha^2+416\,\alpha\,\beta+217\,\beta^2)+822\,(\alpha+\beta)+1168]\,/$ $/\,[12\,(\alpha+\beta+2)(\alpha+\beta+4)]$ $\mu_2 = (\alpha-\beta)(\alpha+\beta+5)(\alpha+\beta+7)\,/\,[2(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_3 = -\,(\alpha+\beta+7)(\alpha+\beta+8)\,/\,[2(\alpha+\beta+4)]$ $\gamma = -\,s(\alpha-\beta)(\alpha+\beta+5)\,[(\alpha^2+\alpha\beta+\beta^2)+9\,(\alpha+\beta)+26]\,/\,24$
5	$\lambda_0 = s(\alpha-\beta)(\alpha+\beta+6)\,[(\alpha^2-\beta^2)^2(3\,\alpha^2+4\,\alpha\,\beta+3\,\beta^2)+$ $+(\alpha+\beta)(61\,\alpha^4-16\,\alpha^3\,\beta-98\,\alpha^2\beta^2-16\,\alpha\beta^3+61\,\beta^4)+$ $+(417\,\alpha^4-198\,\alpha^3\,\beta-1134\,\alpha^2\beta^2-198\,\alpha\beta^3+417\,\beta^4)+(\alpha+\beta)(295\,\alpha^2-4342\,\alpha\,\beta+295\,\beta^2)-$ $-(8148\,\alpha^2+17064\,\alpha\,\beta+8148\,\beta^2)-32276\,(\alpha+\beta)-42912\,]\,/$ $/\,[384\,(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_1 = -\,s(\alpha+\beta+6)\,[(\alpha^2-\beta^2)^2(2\,\alpha^2+3\,\alpha\,\beta+2\,\beta^2)+$ $+2\,(\alpha+\beta)(19\,\alpha^4-9\,\alpha^3\beta\,-38\,\alpha^2\beta^2-9\,\alpha\beta^3+19\,\beta^4)+$ $+(168\,\alpha^4-359\,\alpha^3\,\beta-974\,\alpha^2\beta^2-359\,\alpha\beta^3+168\,\beta^4)-2(\alpha+\beta)(499\,\alpha^2+1513\,\alpha\,\beta+499\,\beta^2)-$ $-(10634\,\alpha^2+15188\,\beta\,\alpha+10634\,\beta^2)-32256\,(\alpha+\beta)-44160\,]\,/$ $/\;[96\,(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\lambda_2 = -\,s(\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+42)\,/\,[96\,(\alpha+\beta+3)(\alpha+\beta+4)]$ $\lambda_3 = 3\,s(\alpha+\beta+7)(\alpha+\beta+8)(\alpha+\beta+9)(\alpha+\beta+10)\,/\,[16\,(\alpha+\beta+4)(\alpha+\beta+5)]\,,\; \lambda_5 = s$ $\mu_0 = (\alpha+\beta+6)(\alpha+\beta+7)\,[(\alpha-\beta)^2(\alpha+\beta)^3(2\,\alpha^2+3\,\alpha\,\beta+2\,\beta^2)+$ $+(\alpha+\beta)^2(33\,\alpha^4-16\,\alpha^3\,\beta-70\,\alpha^2\beta^2-16\,\alpha\beta^3+33\,\beta^4)+$ $+(\alpha+\beta)(53\,\alpha^4-361\,\alpha^3\,\beta-740\,\alpha^2\beta^2-361\,\alpha\beta^3+53\,\beta^4)-$ $-3\,(\alpha+\beta)^2(749\,\alpha^2+208\,\alpha\,\beta+749\,\beta^2)-(\alpha+\beta)(18043\,\alpha^2+3286\,\alpha\,\beta+18043\,\beta^2)-$ $-(55926\,\alpha^2+49452\,\alpha\,\beta+55926\,\beta^2)-82392\,(\alpha+\beta)-64320\,]\,/$ $/\,[192\,(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\mu_1 = (\alpha-\beta)(\alpha+\beta+7)\,[(\alpha+\beta)^5+86\,(\alpha+\beta)^4+1913\,(\alpha+\beta)^3+$ $+(18160\,\alpha^2+36128\,\alpha\,\beta+18160\,\beta^2)+78348\,(\alpha+\beta)+127392\,]\,/$ $/\,[192\,(\alpha+\beta+2)(\alpha+\beta+4)(\alpha+\beta+5)]$ $\mu_2 = -\,(\alpha+\beta+8)\,[3\,(\alpha+\beta)^4+96\,(\alpha+\beta)^3+(1153\,\alpha^2+2258\,\alpha\,\beta+1153\,\beta^2)+$ $+5968\,(\alpha+\beta)+11580\,]\,/\,[32\,(\alpha+\beta+3)(\alpha+\beta+4)]$ $\mu_3 = (\alpha-\beta)(\alpha+\beta+7)(\alpha+\beta+9)\,/\,[2\,(\alpha+\beta+4)(\alpha+\beta+5)]$ $\mu_4 = (\alpha+\beta+9)(\alpha+\beta+10)\,/\,[2\,(\alpha+\beta+5)]$ $\gamma = -\,s(\alpha+\beta+6)\,[(\alpha^2+\alpha\,\beta+\beta^2)^2+3\,(\alpha+\beta)(8\,\alpha^2+9\,\alpha\,\beta+8\,\beta^2)+$ $+(251\,\alpha^2+431\,\alpha\,\beta+251\,\beta^2)+1284\,(\alpha+\beta)+2640\,]\,/\,48$

f) Laguerre Polynomials $L_n(x)$:

$$\int_0^\infty e^{-x}\, L_n^2(x)\, dx \quad = \quad 1$$

First polynomials:

$$L_0(x) = 1\,,\; L_1(x) = -x+1\,,\; L_2(x) = \frac{x^2}{2} - 2x + 1\,,\; L_3(x) = -\frac{x^3}{6} + \frac{3x^2}{2} - 3x + 1\,,$$

$$L_4(x) = \frac{x^4}{24} - \frac{2x^3}{3} + 3x^2 - 4x + 1$$

$Z_\nu(x) \;=\; J_0(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = 1\,,\; \lambda_1 = -1$	$\mu_0 = -1\,,\; \mu_1 = 1$	1
2	$\lambda_0 = 1\,,\; \lambda_1 = -1$	$\mu_0 = -1\,,\; \mu_2 = 1$	$1/2$
3	$\lambda_0 = 1/3\,,\; \lambda_1 = 1/3\,,\; \lambda_2 = -2/3$	$\mu_0 = -1/3\,,\; \mu_1 = -2/3\,,\; \mu_3 = 1$	$-1/2$
4	$\lambda_0 = -11/12\,,\; \lambda_1 = 25/12\,,\; \lambda_2 = -5/12\,,$ $\lambda_3 = -3/4$	$\mu_0 = 11/12\,,\; \mu_1 = -7/6\,,\; \mu_2 = -3/4\,,$ $\mu_4 = 1$	$-13/8$
5	$\lambda_0 = -131/60\,,\; \lambda_1 = 193/60\,,\; \lambda_2 = 19/60\,,$ $\lambda_3 = -11/20\,,\; \lambda_4 = -4/5$	$\mu_0 = 131/60\; \mu_1 = -31/30\,,$ $\mu_2 = -27/20\,,\; \mu_3 = -4/5\,,\; \mu_5 = 1$	$-17/8$
6	$\lambda_0 = -99/40\,,\; \lambda_1 = 97/40\,,\; \lambda_2 = 51/40\,,$ $\lambda_3 = 29/120\,,\; \lambda_4 = -19/30\,,\; \lambda_5 = -5/6$	$\mu_0 = 99/40\,,\; \mu_1 = 1/20\,,\; \mu_2 = -49/40\,,$ $\mu_3 = -22/15\,,\; \mu_4 = -5/6\,,\; \mu_6 = 1$	$-19/16$
7	$\lambda_0 = -239/280\,,\; \lambda_1 = -283/280\,,\; \lambda_2 = 73/40\,,$ $\lambda_3 = 167/120\,,\; \lambda_4 = 41/210\,,\; \lambda_5 = -29/42\,,$ $\lambda_6 = -6/7$	$\mu_0 = 239/280\,,\; \mu_1 = 261/140\,,$ $\mu_2 = 11/280\,,\; \mu_3 = -142/105\,,$ $\mu_4 = -65/42\,,\; \mu_5 = -6/7\,,\; \mu_7 = 1$	$23/16$
8	$\lambda_0 = 6117/2240\,,\; \lambda_1 = -13751/2240\,,$ $\lambda_2 = 381/320\,,\; \lambda_3 = 2099/960\,,$ $\lambda_4 = 2477/1680\,,\; \lambda_5 = 55/336\,,$ $\lambda_6 = -41/56\,,\; \lambda_7 = -7/8$	$\mu_0 = -6117/2240\,,\; \mu_1 = 3817/1120\,,$ $\mu_2 = 4967/2240\,,\; \mu_3 = 13/420\,,$ $\mu_4 = -485/336\,,\; \mu_5 = -45/28\,,$ $\mu_6 = -7/8\,,\; \mu_8 = 1$	$611/128$
9	$\lambda_0 = 134581/20160\,,$ $\lambda_1 = -197543/20160\,,$ $\lambda_2 = -2867/2880\,,\; \lambda_3 = 4769/2880\,,$ $\lambda_4 = 12287/5040\,,\; \lambda_5 = 1549/1008\,,$ $\lambda_6 = 71/504\,,\; \lambda_7 = -55/72\,,$ $\lambda_8 = -8/9$	$\mu_0 = -134581/20160\,,$ $\mu_1 = 31481/10080\,,$ $\mu_2 = 27677/6720\,,\; \mu_3 = 3103/1260\,,$ $\mu_4 = 25/1008\,,\; \mu_5 = -127/84\,,$ $\mu_6 = -119/72\,,\; \mu_7 = -8/9\,,$ $\mu_9 = 1$	$827/128$
10	$\lambda_0 = 313217/40320\,,$ $\lambda_1 = -305131/40320\,,$ $\lambda_2 = -23239/5760\,,$ $\lambda_3 = -4547/5760\,,$ $\lambda_4 = 19939/10080\,,$ $\lambda_5 = 26461/10080\,,$ $\lambda_6 = 7991/5040\,,\; \lambda_7 = 89/720\,,$ $\lambda_8 = -71/90\,,\; \lambda_9 = -9/10$	$\mu_0 = -313217/40320\,,$ $\mu_1 = -4043/20160\,,$ $\mu_2 = 51529/13440\,,$ $\mu_3 = 11651/2520\,,$ $\mu_4 = 5333/2016\,,$ $\mu_5 = 17/840\,,$ $\mu_6 = -1127/720\,,\; \mu_7 = -76/45\,,$ $\mu_8 = -9/10\,,\; \mu_{10} = 1$	$943/256$

$Z_\nu(x) \;=\; I_0(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = 1\,,\; \lambda_1 = -1$	$\mu_0 = -1\,,\; \mu_1 = 1$	1
2	$\lambda_0 = 1\,,\; \lambda_1 = -1$	$\mu_0 = -1\,,\; \mu_2 = 1$	$3/2$
3	$\lambda_0 = 5/3\,,\; \lambda_1 = -7/3\,,\; \lambda_2 = 2/3$	$\mu_0 = -5/3\,,\; \mu_1 = 2/3\,,\; \mu_3 = 1$	$5/2$
4	$\lambda_0 = 35/12\,,\; \lambda_1 = -49/12\,,\; \lambda_2 = 5/12\,,$ $\lambda_3 = 3/4$	$\mu_0 = -35/12\,,\; \mu_1 = 7/6\,,\; \mu_2 = 3/4\,,$ $\mu_4 = 1$	$35/8$

n	λ_k	μ_k	γ
5	$\lambda_0 = 21/4\,,\ \lambda_1 = -147/20\,,\ \lambda_2 = 3/4\,,$ $\lambda_3 = 11/20\,,\ \lambda_4 = 4/5$	$\mu_0 = -21/4\ \mu_1 = 21/10\,,$ $\mu_2 = 27/20\,,\ \mu_3 = 4/5\,,\ \mu_5 = 1$	63/8
6	$\lambda_0 = 77/8\,,\ \lambda_1 = -539/40\,,\ \lambda_2 = 11/8\,,$ $\lambda_3 = 121/120\,,\ \lambda_4 = 19/30\,,\ \lambda_5 = 5/6$	$\mu_0 = -77/8\,,\ \mu_1 = 77/20\,,\ \mu_2 = 99/40\,,$ $\mu_3 = 22/15\,,\ \mu_4 = 5/6\,,\ \mu_6 = 1$	231/16
7	$\lambda_0 = 143/8\,,\ \lambda_1 = -1001/40\,,\ \lambda_2 = 143/56\,,$ $\lambda_3 = 1573/840\,,\ \lambda_4 = 247/210\,,\ \lambda_5 = 29/42\,,$ $\lambda_6 = 6/7$	$\mu_0 = -143/8\,,\ \mu_1 = 143/20\,,$ $\mu_2 = 1287/280\,,\ \mu_3 = 286/105\,,$ $\mu_4 = 65/42\,,\ \mu_5 = 6/7\,,\ \mu_7 = 1$	429/16
8	$\lambda_0 = 429/16\,,\ \lambda_1 = -3003/64\,,$ $\lambda_2 = 2145/448\,,\ \lambda_3 = 1573/448\,,$ $\lambda_4 = 247/112\,,\ \lambda_5 = 145/112\,,$ $\lambda_6 = 41/56\,,\ \lambda_7 = 7/8$	$\mu_0 = -2145/64\,,\ \mu_1 = 429/32\,,$ $\mu_2 = 3861/448\,,\ \mu_3 = 143/28\,,$ $\mu_4 = 325/112\,,\ \mu_5 = 45/28\,,$ $\mu_6 = 7/8\,,\ \mu_8 = 1$	6435/128
9	$\lambda_0 = 12155/192\,,$ $\lambda_1 = -17017/192\,,$ $\lambda_2 = 12155/1344\,,\ \lambda_3 = 26741/4032\,,$ $\lambda_4 = 4199/1008\,,\ \lambda_5 = 2465/1008\,,$ $\lambda_6 = 697/504\,,\ \lambda_7 = 55/72\,,\ \lambda_8 = 8/9$	$\mu_0 = -12155/192\,,$ $\mu_1 = 2431/96\,,$ $\mu_2 = 7293/448\,,\ \mu_3 = 2431/252\,,$ $\mu_4 = 5525/1008\,,\ \mu_5 = 85/28\,,$ $\mu_6 = 119/72\,,\ \mu_7 = 8/9\,,\ \mu_9 = 1$	12155/128
10	$\lambda_0 = 46189/384\,,$ $\lambda_1 = -323323/1920\,,$ $\lambda_2 = 46189/2688\,,$ $\lambda_3 = 508079/40320\,,$ $\lambda_4 = 79781/10080\,,\ \lambda_5 = 9367/2016\,,$ $\lambda_6 = 13243/5040\,,\ \lambda_7 = 209/144\,,$ $\lambda_8 = 71/90\,,\ \lambda_9 = 9/10$	$\mu_0 = -46189/384\,,$ $\mu_1 = 46189/960\,,$ $\mu_2 = 138567/4480\,,$ $\mu_3 = 46189/2520\,,$ $\mu_4 = 20995/2016\,,\ \mu_5 = 323/56\,,$ $\mu_6 = 2261/720\,,\ \mu_7 = 76/45\,,$ $\mu_8 = 9/10\,,\ \mu_{10} = 1$	943/256

$Z_\nu(x) \ = \ K_0(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = 1\,,\ \lambda_1 = -1$	$\mu_0 = 1\,,\ \mu_1 = -1$	1
2	$\lambda_0 = 1\,,\ \lambda_1 = -1$	$\mu_0 = 1\,,\ \mu_2 = -1$	3/2
3	$\lambda_0 = 5/3\,,\ \lambda_1 = -7/3\,,\ \lambda_2 = 2/3$	$\mu_0 = 5/3\,,\ \mu_1 = -2/3\,,\ \mu_3 = -1$	5/2
4	$\lambda_0 = 35/12\,,\ \lambda_1 = -49/12\,,\ \lambda_2 = 5/12\,,$ $\lambda_3 = 3/4$	$\mu_0 = 35/12\,,\ \mu_1 = -7/6\,,\ \mu_2 = -3/4\,,$ $\mu_4 = -1$	35/8
5	$\lambda_0 = 21/4\,,\ \lambda_1 = -147/20\,,\ \lambda_2 = 3/4\,,$ $\lambda_3 = 11/20\,,\ \lambda_4 = 4/5$	$\mu_0 = 21/4\ \mu_1 = -21/10\,,$ $\mu_2 = -27/20\,,\ \mu_3 = -4/5\,,\ \mu_5 = -1$	63/8
6	$\lambda_0 = 77/8\,,\ \lambda_1 = -539/40\,,\ \lambda_2 = 11/8\,,$ $\lambda_3 = 121/120\,,\ \lambda_4 = 19/30\,,\ \lambda_5 = 5/6$	$\mu_0 = 77/8\,,\ \mu_1 = -77/20\,,\ \mu_2 = -99/40\,,$ $\mu_3 = -22/15\,,\ \mu_4 = -5/6\,,\ \mu_6 = -1$	231/16
7	$\lambda_0 = 143/8\,,\ \lambda_1 = -1001/40\,,\ \lambda_2 = 143/56\,,$ $\lambda_3 = 1573/840\,,\ \lambda_4 = 247/210\,,\ \lambda_5 = 29/42\,,$ $\lambda_6 = 6/7$	$\mu_0 = 143/8\,,\ \mu_1 = -143/20\,,$ $\mu_2 = -1287/280\,,\ \mu_3 = -286/105\,,$ $\mu_4 = -65/42\,,\ \mu_5 = -6/7\,,\ \mu_7 = -1$	429/16
8	$\lambda_0 = 429/16\,,\ \lambda_1 = -3003/64\,,$ $\lambda_2 = 2145/448\,,\ \lambda_3 = 1573/448\,,$ $\lambda_4 = 247/112\,,\ \lambda_5 = 145/112\,,$ $\lambda_6 = 41/56\,,\ \lambda_7 = 7/8$	$\mu_0 = 2145/64\,,\ \mu_1 = -429/32\,,$ $\mu_2 = -3861/448\,,\ \mu_3 = -143/28\,,$ $\mu_4 = -325/112\,,\ \mu_5 = -45/28\,,$ $\mu_6 = -7/8\,,\ \mu_8 = -1$	6435/128
9	$\lambda_0 = 12155/192\,,$ $\lambda_1 = -17017/192\,,$ $\lambda_2 = 12155/1344\,,\ \lambda_3 = 26741/4032\,,$ $\lambda_4 = 4199/1008\,,\ \lambda_5 = 2465/1008\,,$ $\lambda_6 = 697/504\,,\ \lambda_7 = 55/72\,,\ \lambda_8 = 8/9$	$\mu_0 = 12155/192\,,$ $\mu_1 = -2431/96\,,$ $\mu_2 = -7293/448\,,\ \mu_3 = -2431/252\,,$ $\mu_4 = -5525/1008\,,\ \mu_5 = -85/28\,,$ $\mu_6 = -119/72\,,\ \mu_7 = -8/9\,,\ \mu_9 = -1$	12155/128

n	λ_k	μ_k	γ
10	$\lambda_0 = 46189/384\,,$ $\lambda_1 = -323323/1920\,,$ $\lambda_2 = 46189/2688\,,$ $\lambda_3 = 508079/40320\,,$ $\lambda_4 = 79781/10080\,,\ \lambda_5 = -9367/2016\,,$ $\lambda_6 = 13243/5040\,,\ \lambda_7 = -209/144\,,$ $\lambda_8 = 71/90\,,\ \lambda_9 = 9/10$	$\mu_0 = 46189/384\,,$ $\mu_1 = -46189/960\,,$ $\mu_2 = -138567/4480\,,$ $\mu_3 = -46189/2520\,,$ $\mu_4 = -20995/2016\,,\ \mu_5 = 323/56\,,$ $\mu_6 = -2261/720\,,\ \mu_7 = -76/45\,,$ $\mu_8 = -9/10\,,\ \mu_{10} = -1$	$943/256$

$Z_\nu(x) \ = \ J_1(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = -1\,,$	–	-1
2	$\lambda_0 = -2\,,\ \lambda_1 = 2\,,\ \lambda_2 = -1$	$\mu_0 = 1\,,\ \mu_1 = -1$	-2
3	$\lambda_0 = -3\,,\ \lambda_1 = 3\,,\ \lambda_3 = -1$	$\mu_0 = 2\,,\ \mu_1 = -1\,,\ \mu_2 = -1$	$-5/2$
4	$\lambda_0 = -10/3\,,\ \lambda_1 = 8/3\,,\ \lambda_2 = 2/3\,,\ \lambda_4 = -1$	$\mu_0 = 7/3\,,\ \mu_1 = -1/3\,,\ \mu_2 = -1\,,\ \mu_3 = -1$	-2
5	$\lambda_0 = -29/12\,,\ \lambda_1 = 7/12\,,$ $\lambda_2 = 13/12\,,\ \lambda_3 = 3/4\,, \lambda_5 = -1$	$\mu_0 = 17/12\,,\ \mu_1 = 5/6\,,$ $\mu_2 = -1/4\,,\ \mu_3 = -1\ \mu_4 = -1$	$-3/8$
6	$\lambda_0 = -7/30\,,\ \lambda_1 = -79/30\,,$ $\lambda_2 = 23/30\,,\ \lambda_3 = 13/10\,,$ $\lambda_4 = 4/5\,,\ \lambda_6 = -1$	$\mu_0 = -23/30\,,\ \mu_1 = 28/15\,,$ $\mu_2 = 11/10\,,\ \mu_3 = -1/5\,,$ $\mu_4 = -1\,,\ \mu_5 = -1\,,$	$7/4$
7	$\lambda_0 = 269/120\,,\ \lambda_1 = -607/120\,,$ $\lambda_2 = -61/120\,,\ \lambda_3 = 127/120\,,$ $\lambda_4 = 43/30\,,\ \lambda_5 = 5/6\,,\ \lambda_7 = -1$	$\mu_0 = -389/120\,,\ \mu_1 = 109/60\,,$ $\mu_2 = 93/40\,,\ \mu_3 = 19/15\,,$ $\mu_4 = -1/6\,,\ \mu_5 = -1\,,\ \mu_6 = -1$	$47/16$
8	$\lambda_0 = 65/21\,,\ \lambda_1 = -85/21\,,$ $\lambda_2 = -7/3\,,\ \lambda_3 = -1/3\,,$ $\lambda_4 = 26/21\,,\ \lambda_5 = 32/21\,,$ $\lambda_6 = 6/7\,,\ \lambda_8 = -1$	$\mu_0 = -86/21\,,\ \mu_1 = -1/21\,,$ $\mu_2 = 16/7\,,\ \mu_3 = 55/21\,,$ $\mu_4 = 29/21\,,\ \mu_5 = -1/7\,,$ $\mu_6 = -1\,,\ \mu_7 = -1$	$3/2$
9	$\lambda_0 = 2449/6720\,,\ \lambda_1 = 14053/6720\,,$ $\lambda_2 = -3383/960\,,\ \lambda_3 = -2419/960\,,$ $\lambda_4 = -397/1680\,,\ \lambda_5 = 457/336\,,$ $\lambda_6 = 89/56\,,\ \lambda_7 = 7/8\,,$ $\lambda_9 = -1$	$\mu_0 = -9169/6720\,,\ \mu_1 = -11611/3360\,,$ $\mu_2 = 153/2240\,,\ \mu_3 = 1087/420\,,$ $\mu_4 = 949/336\,,\ \mu_5 = 41/28\,,$ $\mu_6 = -1/8\,,\ \mu_7 = -1\,,$ $\mu_8 = -1$	$-419/128$
10	$\lambda_0 = -63617/10080\,,\ \lambda_1 = 119851/10080\,,$ $\lambda_2 = -3641/1440\,,\ \lambda_3 = -6013/1440\,,$ $\lambda_4 = -6739/2520\,,\ \lambda_5 = -89/504\,,$ $\lambda_6 = 365/252\,,\ \lambda_7 = 59/36\,,$ $\lambda_8 = 8/9\,,\ \lambda_{10} = -1$	$\mu_0 = 53537/10080\,,\ \mu_1 = -33157/5040\,,$ $\mu_2 = -13609/3360\,,\ \mu_3 = 79/630\,,$ $\mu_4 = 1411/504\,,\ \mu_5 = 125/42\,,$ $\mu_6 = 55/36\,,\ \mu_7 = -1/9\,,$ $\mu_8 = -1\,,\ \mu_9 = -1$	$-623/64$

$Z_\nu(x) \ = \ I_1(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = 1\,,$	–	1
2	$\lambda_0 = 2\,,\ \lambda_1 = -2\,,\ \lambda_2 = 1$	$\mu_0 = -1\,,\ \mu_1 = 1$	2
3	$\lambda_0 = 3\,,\ \lambda_1 = -3\,,\ \lambda_3 = 1$	$\mu_0 = -2\,,\ \mu_1 = 1\,,\ \mu_2 = 1$	$7/2$
4	$\lambda_0 = 14/3\,,\ \lambda_1 = -16/3\,,\ \lambda_2 = 2/3\,,\ \lambda_4 = 1$	$\mu_0 = -11/3\,,\ \mu_1 = 5/3\,,\ \mu_2 = 1\,,\ \mu_3 = 1$	6
5	$\lambda_0 = 91/12\,,\ \lambda_1 = -113/12\,,$ $\lambda_2 = 13/12\,,\ \lambda_3 = 3/4\,, \lambda_5 = 1$	$\mu_0 = -79/12\,,\ \mu_1 = 17/6\,,$ $\mu_2 = 7/4\,,\ \mu_3 = 1\ \mu_4 = 1$	$83/8$
6	$\lambda_0 = 77/6\,,\ \lambda_1 = -503/30\,,$ $\lambda_2 = 11/6\,,\ \lambda_3 = 13/10\,,$ $\lambda_4 = 4/5\,,\ \lambda_6 = 1$	$\mu_0 = -71/6\,,\ \mu_1 = 74/15\,,$ $\mu_2 = 31/10\,,\ \mu_3 = 9/5\,,$ $\mu_4 = 1\,,\ \mu_5 = 1\,,$	$73/4$

n	λ_k	μ_k	γ
7	$\lambda_0 = 539/24\,,\ \lambda_1 = -3629/120\,,$ $\lambda_2 = 77/24\,,\ \lambda_3 = 277/120\,,$ $\lambda_4 = 43/30\,,\ \lambda_5 = 5/6\,,\ \lambda_7 = -1$	$\mu_0 = 277/120\,,\ \mu_1 = 527/60\,,$ $\mu_2 = 223/40\,,\ \mu_3 = 49/15\,,$ $\mu_4 = 11/6\,,\ \mu_5 = 1\,,\ \mu_6 = 1$	$523/15$
8	$\lambda_0 = 121/3\,,\ \lambda_1 = -829/15\,,$ $\lambda_2 = 121/21\,,\ \lambda_3 = 439/105\,,$ $\lambda_4 = 274/105\,,\ \lambda_5 = 32/21\,,$ $\lambda_6 = 6/7\,,\ \lambda_8 = 1$	$\mu_0 = -118/3\,,\ \mu_1 = 239/15\,,$ $\mu_2 = 356/35\,,\ \mu_3 = 629/105\,,$ $\mu_4 = 71/21\,,\ \mu_5 = 13/7\,,$ $\mu_6 = 1\,,\ \mu_7 = 1$	$119/2$
9	$\lambda_0 = 14179/192\,,\ \lambda_1 = 14179/192\,,$ $\lambda_2 = 14179/1344\,,\ \lambda_3 = 51691/6720\,,$ $\lambda_4 = 8089/1680\,,\ \lambda_5 = 947/336\,,$ $\lambda_6 = 89/56\,,\ \lambda_7 = 7/8\,,$ $\lambda_9 = 1$	$\mu_0 = -13987/192\,,\ \mu_1 = 14083/480\,,$ $\mu_2 = 42089/2240\,,\ \mu_3 = 4661/420\,,$ $\mu_4 = 2111/336\,,\ \mu_5 = 97/28\,,$ $\mu_6 = 15/8\,,\ \mu_7 = 1\,,$ $\mu_8 = 1$	$14051/128$
10	$\lambda_0 = 4389/32\,,\ \lambda_1 = -30531/160\,,$ $\lambda_2 = 627/32\,,\ \lambda_3 = 20627/1440\,,$ $\lambda_4 = 3233/360\,,\ \lambda_5 = 379/72\,,$ $\lambda_6 = 107/36\,,\ \lambda_7 = 59/36\,,$ $\lambda_8 = 8/9\,,\ \lambda_{10} = 1$	$\mu_0 = -4357/32\,,\ \mu_1 = 4373/80\,,$ $\mu_2 = 5611/160\,,\ \mu_3 = 1867/90\,,$ $\mu_4 = 1867/90\,,\ \mu_5 = 13/2\,,$ $\mu_6 = 127/36\,,\ \mu_7 = 17/9\,,$ $\mu_8 = 1\,,\ \mu_9 = 1$	$13103/64$

$Z_\nu(x) \ = \ K_1(x):$

n	λ_k	μ_k	γ
1	$\lambda_0 = -1\,,$	$-$	-1
2	$\lambda_0 = -2\,,\ \lambda_1 = 2\,,\ \lambda_2 = -1$	$\mu_0 = -1\,,\ \mu_1 = 1$	-2
3	$\lambda_0 = -3\,,\ \lambda_1 = 3\,,\ \lambda_3 = -1$	$\mu_0 = -2\,,\ \mu_1 = 1\,,\ \mu_2 = 1$	$-7/2$
4	$\lambda_0 = -14/3\,,\ \lambda_1 = 16/3\,,\ \lambda_2 = -2/3\,,$ $\lambda_4 = -1$	$\mu_0 = -11/3\,,\ \mu_1 = 5/3\,,\ \mu_2 = 1\,,$ $\mu_3 = 1$	-6
5	$\lambda_0 = -91/12\,,\ \lambda_1 = 113/12\,,$ $\lambda_2 = -13/12\,,\ \lambda_3 = -3/4\,, \lambda_5 = 1$	$\mu_0 = -79/12\,,\ \mu_1 = 17/6\,,$ $\mu_2 = 7/4\,,\ \mu_3 = 1\ \mu_4 = 1$	$-83/8$
6	$\lambda_0 = -77/6\,,\ \lambda_1 = 503/30\,,$ $\lambda_2 = -11/6\,,\ \lambda_3 = -13/10\,,$ $\lambda_4 = -4/5\,,\ \lambda_6 = -1$	$\mu_0 = -71/6\,,\ \mu_1 = 74/15\,,$ $\mu_2 = 31/10\,,\ \mu_3 = 9/5\,,$ $\mu_4 = 1\,,\ \mu_5 = 1\,,$	$-73/4$
7	$\lambda_0 = -539/24\,,\ \lambda_1 = 3629/120\,,$ $\lambda_2 = -77/24\,,\ \lambda_3 = -277/120\,,$ $\lambda_4 = -43/30\,,\ \lambda_5 = -5/6\,,\ \lambda_7 = -1$	$\mu_0 = -277/120\,,\ \mu_1 = 527/60\,,$ $\mu_2 = 223/40\,,\ \mu_3 = 49/15\,,$ $\mu_4 = 11/6\,,\ \mu_5 = 1\,,\ \mu_6 = 1$	$-523/15$
8	$\lambda_0 = -121/3\,,\ \lambda_1 = 829/15\,,$ $\lambda_2 = -121/21\,,\ \lambda_3 = -439/105\,,$ $\lambda_4 = -274/105\,,\ \lambda_5 = -32/21\,,$ $\lambda_6 = -6/7\,,\ \lambda_8 = -1$	$\mu_0 = -118/3\,,\ \mu_1 = 239/15\,,$ $\mu_2 = 356/35\,,\ \mu_3 = 629/105\,,$ $\mu_4 = 71/21\,,\ \mu_5 = 13/7\,,$ $\mu_6 = 1\,,\ \mu_7 = 1$	$-119/2$
9	$\lambda_0 = 14179/192\,,\ \lambda_1 = 14179/192\,,$ $\lambda_2 = -14179/1344\,,\ \lambda_3 = 51691/6720\,,$ $\lambda_4 = -8089/1680\,,\ \lambda_5 = -947/336\,,$ $\lambda_6 = -89/56\,,\ \lambda_7 = -7/8\,,$ $\lambda_9 = -1$	$\mu_0 = -13987/192\,,\ \mu_1 = 14083/480\,,$ $\mu_2 = 42089/2240\,,\ \mu_3 = 4661/420\,,$ $\mu_4 = 2111/336\,,\ \mu_5 = 97/28\,,$ $\mu_6 = 15/8\,,\ \mu_7 = 1\,,$ $\mu_8 = 1$	$-14051/128$
10	$\lambda_0 = -4389/32\,,\ \lambda_1 = 30531/160\,,$ $\lambda_2 = -627/32\,,\ \lambda_3 = -20627/1440\,,$ $\lambda_4 = -3233/360\,,\ \lambda_5 = -379/72\,,$ $\lambda_6 = -107/36\,,\ \lambda_7 = -59/36\,,$ $\lambda_8 = -8/9\,,\ \lambda_{10} = -1$	$\mu_0 = -4357/32\,,\ \mu_1 = 4373/80\,,$ $\mu_2 = 5611/160\,,\ \mu_3 = 1867/90\,,$ $\mu_4 = 1867/90\,,\ \mu_5 = 13/2\,,$ $\mu_6 = 127/36\,,\ \mu_7 = 17/9\,,$ $\mu_8 = 1\,,\ \mu_9 = 1$	$-13103/64$

g) Generalized Laguerre Polynomials $L_n^{(\alpha)}(x)$:

$$\int_0^\infty x^\alpha\, e^{-x}\, [L_n^{(\alpha)}(x)]^2\, dx \quad = \quad \frac{\Gamma(n+\alpha+1)}{n!}\,, \quad \alpha > -1$$

First polynomials:

$$L_0^{(\alpha)}(x) = 1\,, \quad L_1^{(\alpha)}(x) = -x + 1 + \alpha\,, \quad L_2^{(\alpha)}(x) = \frac{x^2}{2} - (2+\alpha)x + \frac{(\alpha+1)(\alpha+2)}{2}\,,$$

$$L_3^{(\alpha)}(x) = -\frac{x^3}{6} + \frac{3+\alpha}{2}\,x^2 - \frac{\alpha^2+5\alpha+6}{2}\,x + \frac{\alpha^3+6\alpha+11\alpha+6}{6}\,,$$

$$L_4^{(\alpha)}(x) = \frac{x^4}{24} - \frac{\alpha+4}{6}\,x^3 + \frac{\alpha^2+7\alpha+12}{4}\,x^2 - \frac{\alpha^3+9\alpha^2+26\alpha+24}{6}\,x + \frac{\alpha^4+10\alpha^3+35\alpha^2+50\alpha+24}{24}$$

$Z_\nu(x) \;=\; J_0(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0 = (\alpha+1)^2\,,\ \lambda_1 = -(\alpha+1)$	$\mu_0 = -(\alpha+1)\,,\ \mu_1 = 1$	$\alpha+1$
2	$\lambda_0 = (\alpha+1)^2(\alpha+2)/2\,,$ $\lambda_1 = -(\alpha+1)(\alpha+2)/2$	$\mu_0 = -(\alpha+1)(\alpha+2)/2\,,\ \mu_2 = 1$	$(\alpha^2+3\alpha+1)/2$
3	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^2+4\alpha+1)/6\,,$ $\lambda_1 = -(\alpha+2)(\alpha^2+4\alpha-1)/6\,,$ $\lambda_2 = -2/3$	$\mu_0 = -(\alpha+1)(\alpha^2+5\alpha+2)/6\,,$ $\mu_1 = -2/3\,,\ \mu_3 = 1$	$(\alpha+3)(\alpha^2+3\alpha-1)/6$
4	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^3+8\alpha^2+$ $+14\alpha-11)/24\,,$ $\lambda_1 = -\,(\alpha+2)(\alpha^3+8\alpha^2+12\alpha$ $-25)/24\,,$ $\lambda_2 = (\alpha-5)/12\,,\ \lambda_3 = -3/4$	$\mu_0 = -(\alpha+1)(\alpha^3+9\alpha^2+19\alpha-$ $-22)/24\,,$ $\mu_1 = (\alpha-14)/12\,,$ $\mu_2 = -3/4\,,\ \mu_4 = 1$	$(\alpha^4+10\alpha^3+29\alpha^2+$ $+8\alpha-39)/24$
5	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^4+13\alpha^3+50\alpha^2+19\alpha-131)/120$ $\lambda_1 = -(\alpha+2)(\alpha^4+13\alpha^3+48\alpha^2-5\alpha-193)/120$ $\lambda_2 = (\alpha^2+12\alpha+19)/60\,,\ \lambda_3 = (\alpha-11)/20\,,\ \lambda_4 = -4/5$ $\mu_0 = -(\alpha+1)(\alpha^4+14\alpha^3+60\alpha^2+29\alpha-262)/120$ $\mu_1 = (\alpha^2+15\alpha-62)/60\,,\ \mu_2 = (\alpha-27)/20\,,\ \mu_3 = -4/5\,,\ \mu_5 = 1$ $\gamma = (\alpha+5)(\alpha^4+10\alpha^3+25\alpha^2-20\alpha-51)/120$		
6	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^5+19\alpha^4+123\alpha^3+239\alpha^2-367\alpha-891)/720$ $\lambda_1 = -(\alpha+2)(\alpha^5+19\alpha^4+121\alpha^3+203\alpha^2-563\alpha-873)/720$ $\lambda_2 = (\alpha^3+18\alpha^2+86\alpha+459)/360\,,\ \lambda_3 = (\alpha^2+15\alpha+29/120\,,\ \lambda_4 = (\alpha-19)/30\,,\ \lambda_5 = -5/6$ $\mu_0 = -(\alpha+1)(\alpha^5+20\alpha^4+139\alpha^3+304\alpha^2-503\alpha-1782)/720$ $\mu_1 = (\alpha^3+21\alpha^2+143\alpha+18)/360\,,\ \mu_2 = (\alpha^2+19\alpha-147)/120\,,\ \mu_3 = (\alpha-44)/30\,,$ $\mu_4 = -5/6\,,\ \mu_6 = 1$ $\gamma = (\alpha^6+21\alpha^5+160\alpha^4+465\alpha^3-26\alpha^2-1881\alpha-855)/720$		
7	$\lambda_0 = (\alpha+1)(\alpha+2)((\alpha^6+26\alpha^5+250\alpha^4+962\alpha^3+286\alpha^2-5374\alpha-2151)/5040$ $\lambda_1 = -(\alpha+2)(\alpha^6+26\alpha^5+248\alpha^4+912\alpha^3-150\alpha^2-6464\alpha+2547)/5040$ $\lambda_2 = (\alpha^4+25\alpha^3+206\alpha^2+209\alpha+4599)/2520\,,\ \lambda_3 = (aa^3+22\alpha^2+128\alpha+1169)/840$ $\lambda_4 = (aa^2+18\alpha+41)/210\,,\ \lambda_5 = (\alpha-29)/42\,,\ \lambda_6 = -6/7$ $\mu_0 = -(\alpha+1)(\alpha^6+27\alpha^5+273\alpha^4+1133\alpha^3+485\alpha^2-7817\alpha-4302)/5040$ $\mu_1 = (\alpha^4+28\alpha^3+284\alpha^2+869\alpha+4698)/2520\,,\ \mu_2 = (\alpha^3+26\alpha^2+220\alpha+33)/840$ $\mu_3 = (\alpha^2+23\alpha-284)/210\,,\ \mu_4 = (\alpha-65)/42\,,\ \mu_5 = -6/7\,,\ \mu_7 = 1$ $\gamma = (\alpha+7)(\alpha^6+21\alpha^5+154\alpha^4+357\alpha^3-560\alpha^2-1953\alpha+1035)/5040$		

n	λ_k	μ_k	γ
8	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^7 + 34\alpha^6 + 451\alpha^5 + 2745\alpha^4 + 5609\alpha^3 - 12319\alpha^2 - 39970\alpha + 55053)/40320$ $\lambda_1 = -(\alpha+2)(\alpha^7 + 34\alpha^6 + 449\alpha^5 + 2679\alpha^4 + 4787\alpha^3 - 16477\alpha^2 - 39764\alpha + 123759)/40320$ $\lambda_2 = (\alpha^5 + 33\alpha^4 + 399\alpha^3 + 1647\alpha^2 - 5839\alpha + 24003)/20160$ $\lambda_3 = (\alpha^4 + 30\alpha^3 + 297\alpha^2 + 324\alpha + 14693)/6720\,,\ \lambda_4 = (\alpha^3 + 26\alpha^2 + 178\alpha + 2477)/1680$ $\lambda_5 = (\alpha^2 + 21\alpha + 55)/336\,,\ \lambda_6 = (\alpha - 41)/56\,,\ \lambda_7 = -7/8$ $\mu_0 = -(\alpha+1)(\alpha^7 + 35\alpha^6 + 482\alpha^5 + 3093\alpha^4 + 6980\alpha^3 - 14885\alpha^2 - 63653\alpha + 110106)/40320$ $\mu_1 = (\alpha^5 + 36\alpha^4 + 501\alpha^3 + 2910\alpha^2 - 1111\alpha + 68706)/20160$ $\mu_2 = (\alpha^4 + 34\alpha^3 + 421\alpha^2 + 1576\alpha + 14901)/6720\,,\ \mu_3 = (\alpha^3 + 31\alpha^2 + 313\alpha + 52)/1680$ $\mu_4 = (\alpha^2 + 27\alpha - 485)/336\,,\ \mu_5 = (\alpha - 90)/56\,,\ \mu_6 = -7/8\,,\ \mu_8 = 1$ $\gamma = (\alpha^8 + 36\alpha^7 + 518\alpha^6 + 3612\alpha^5 + 10619\alpha^4 - 4116\alpha^3 - 70358\alpha^2 - 792\alpha + 192465)/40320$		
9	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^8 + 43\alpha^7 + 749\alpha^6 + 6484\alpha^5 + 25514\alpha^4 + 7954\alpha^3 - 186617\alpha^2 + 35451\alpha +$ $+1211229)/362880$ $\lambda_1 = -(\alpha+2)(\alpha^8 + 43\alpha^7 + 747\alpha^6 + 6400\alpha^5 + 24114\alpha^4 - 2978\alpha^3 - 214969\alpha^2 + 152811\alpha +$ $+1777887)/362880$ $\lambda_2 = (\alpha^6 + 42\alpha^5 + 688\alpha^4 + 4926\alpha^3 + 4648\alpha^2 - 128556\alpha - 180621)/181440$ $\lambda_3 = (\alpha^5 + 39\alpha^4 + 559\alpha^3 + 2709\alpha^2 - 12647\alpha + 100149)/60480$ $\lambda_4 = (\alpha^4 + 35\alpha^3 + 404\alpha^2 + 463\alpha + 36861)/15120\,,\ \lambda_5 = (\alpha^3 + 30\alpha^2 + 236\alpha + 4647)/3024$ $\lambda_6 = (\alpha^2 + 24\alpha + 71)/504\,,\ \lambda_7 = (\alpha - 55)/72\,,\ \lambda_8 = -8/9$ $\mu_0 = -(\alpha+1)(\alpha^8 + 44\alpha^7 + 789\alpha^6 + 7103\alpha^5 + 29721\alpha^4 + 13895\alpha^3 - 248986\alpha^2 - 67707\alpha +$ $+2422458)/362880$ $\mu_1 = (\alpha^6 + 45\alpha^5 + 817\alpha^4 + 7083\alpha^3 + 19783\alpha^2 - 135621\alpha + 566658)/181440$ $\mu_2 = (\alpha^5 + 43\alpha^4 + 719\alpha^3 + 5045\alpha^2 - 2355\alpha + 249093)/60480$ $\mu_3 = (\alpha^4 + 40\alpha^3 + 584\alpha^2 + 2573\alpha + 37236)/15120\,,\ \mu_4 = (\alpha^3 + 36\alpha^2 + 422\alpha + 75)/3024$ $\mu_5 = (\alpha^2 + 31\alpha - 762)/504\,,\ \mu_6 = (\alpha - 119)/72\,,\ \mu_7 = -8/9\,,\ \mu_9 = 1$ $\gamma = (\alpha+9)(\alpha^8 + 36\alpha^7 + 510\alpha^6 + 3348\alpha^5 + 7563\alpha^4 - 16092\alpha^3 - 53686\alpha^2 + 151056\alpha +$ $+260505)/362880$		
10	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^9 + 53\alpha^8 + 1170\alpha^7 + 13524\alpha^6 + 81543\alpha^5 + 183237\alpha^4 - 359590\alpha^3 -$ $-1034264\alpha^2 + 7302861\alpha + 14094765)/3628800$ $\lambda_1 = -(\alpha+2)(\alpha^9 + 53\alpha^8 + 1168\alpha^7 + 13420\alpha^6 + 79321\alpha^5 + 159187\alpha^4 - 480648\alpha^3 -$ $-1063930\alpha^2 + 9274833\alpha + 13730895)/3628800$ $\lambda_2 = (\alpha^7 + 52\alpha^6 + 1099^5\alpha + 11365\alpha^4 + 45709\alpha^3 - 144707\alpha^2 - 1469934\alpha - 7320285)/1814400$ $\lambda_3 = (\alpha^6 + 49\alpha^5 + 940\alpha^4 + 7885\alpha^3 + 7594\alpha^2 - 304709\alpha - 477435)/604800$ $\lambda_4 = (\alpha^5 + 45\alpha^4 + 745\alpha^3 + 4125\alpha^2 - 24191\alpha + 299085)/151200$ $\lambda_5 = (\alpha^4 + 40\alpha^3 + 527\alpha^2 + 626\alpha + 79383)/30240$ $\lambda_6 = (\alpha^3 + 34\alpha^2 + 302\alpha + 7991)/5040\,,\ \lambda_7 = (\alpha^2 + 27\alpha + 89)/720$ $\lambda_8 = (\alpha - 71)/90\,,\ \lambda_9 = -9/10$ $\mu_0 = -(1+\alpha)(\alpha^9 + 54\alpha^8 + 1220\alpha^7 + 14534\alpha^6 + 91517\alpha^5 + 223796\alpha^4 - 415640\alpha^3 -$ $-1771714\alpha^2 + 8970777\alpha + 28189530)/3628800$ $\mu_1 = (\alpha^7 + 55\alpha^6 + 1258\alpha^5 + 14785\alpha^4 + 81064\alpha^3 - 26045\alpha^2 - 2439873\alpha - 363870)/1814400$ $\mu_2 = (\alpha^6 + 53\alpha^5 + 1140\alpha^4 + 11785\alpha^3 + 39554\alpha^2 - 323313\alpha + 2318805)/604800$ $\mu_3 = (\alpha^5 + 50\alpha^4 + 975\alpha^3 + 7990\alpha^2 - 4651\alpha + 699060)/151200$ $\mu_4 = (\alpha^4 + 46\alpha^3 + 773\alpha^2 + 3908\alpha + 79995)/30240$ $\mu_5 = (\alpha^3 + 41\alpha^2 + 547\alpha + 102)/5040\,,\ \mu_6 = (\alpha^2 + 35\alpha - 1127)/720$ $\mu_7 = (\alpha - 152)/90\,,\ \mu_8 = -9/10\,,\ \mu_{10} = 1$ $\gamma = (\alpha^{10} + 55\alpha^9 + 1275\alpha^8 + 15810\alpha^7 + 107373\alpha^6 + 331905\alpha^5 - 81775\alpha^4 - 1939510\alpha^3 +$ $+5912451\alpha^2 + 30976515\alpha + 13367025)/3628800$		

$Z_\nu(x) \;=\; I_0(x)\,,\; K_0(x)$ with s defined as on page 185:

n	λ_k	μ_k	γ
1	$\lambda_0=(\alpha+1)^2\,,\ \lambda_1=-(\alpha+1)$	$\mu_0=-s(\alpha+1)\,,\ \mu_1=s$	$\alpha+1$
2	$\lambda_0=(\alpha+1)^2(\alpha+2)/2\ ,$ $\lambda_1=-(\alpha+1)(\alpha+2)/2$	$\mu_0=-s(\alpha+1)(\alpha+2)/2\ ,\ \mu_2=s$	$(\alpha^2+3\,\alpha+3)/2$
3	$\lambda_0=(\alpha+1)(\alpha+2)(\alpha^2+4\,\alpha+5)/6$ $\lambda_1=-\,(\alpha+2)\left(\alpha^2+4\,\alpha+7\right)/6\,,$ $\lambda_2=2/3$	$\mu_0=-s(\alpha+1)(\alpha^2+5\,\alpha+10)/6$ $\mu_1=2s/3\,,\ \mu_3=s$	$(\alpha+3)\,\cdot$ $\cdot(\alpha^2+3\,\alpha+5)/6$
4	$\lambda_0=(\alpha+1)(\alpha+2)\cdot$ $\cdot(\alpha^3+8\,\alpha^2+24\,\alpha+35)/24\ ,$ $\lambda_1=-(\alpha+2)\cdot$ $\cdot(\alpha^3+8\,\alpha^2+26\,\alpha+49)/24\ ,$ $\lambda_2=-(\alpha-5)/12\ ,\ \lambda_3=3/4$	$\mu_0=-s(\alpha+1)\cdot$ $\cdot(\alpha^3+9\,\alpha^2+33\,\alpha+70)/24\ ,$ $\mu_1=-s(\alpha-14)/12\ ,\ \mu_2=3s/4\ ,$ $\mu_4=s$	$(\alpha^4+10\,\alpha^3+41\,\alpha^2+$ $+92\,\alpha+105)/24$
5	$\lambda_0=(\alpha+1)(\alpha+2)(\alpha^4+13\,\alpha^3+68\,\alpha^2+195\,\alpha+315)/120$ $\lambda_1=-(\alpha+2)(\alpha^4+13\,\alpha^3+70\,\alpha^2+219\,\alpha+441)/120$ $\lambda_2=-(\alpha+15)(\alpha-3)/60\,,\ \lambda_3=-(\alpha-11)/20\,,\ \lambda_4=4/5$ $\mu_0=-s(\alpha+1)(\alpha^4+14\,\alpha^3+82\,\alpha^2+279\,\alpha+630)/120$ $\mu_1=-s(\alpha+21)(\alpha-6)/60\,,\ \mu_2=-s(\alpha-27)/20\,,\ \mu_3=4s/5\,,\ \mu_5=s$ $\gamma=(\alpha+5)(\alpha^4+10\,\alpha^3+45\,\alpha^2+120\,\alpha+189)/120$		
6	$\lambda_0=(\alpha+1)(\alpha+2)(\alpha^5+19\,\alpha^4+151\,\alpha^3+683\,\alpha^2+2005\,\alpha+3465)/720$ $\lambda_1=-(\alpha+2)(\alpha^5+19\,\alpha^4+153\,\alpha^3+719\,\alpha^2+2285\,\alpha+4851)/720$ $\lambda_2=-(\alpha^3+18\,\alpha^2+152\,\alpha-495)/360\,,\ \lambda_3=-(\alpha^2+15\,\alpha-121)/120$ $\lambda_4=-(\alpha-19)/30\,,\ \lambda_5=5/6$ $\mu_0=-s(\alpha+1)(\alpha^2+11\,\alpha+45)(\alpha^3+9\,\alpha^2+27\,\alpha+154)/720$ $\mu_1=-s(\alpha^3+21\,\alpha^2+209\,\alpha-1386)/360\,,\ \mu_2=-s(\alpha^2+19\,\alpha-297)/120$ $\mu_3=-s(\alpha-44)/30\,,\ \mu_4=5s/6\,,\ \mu_6=s$ $\gamma=(\alpha^6+21\,\alpha^5+190\,\alpha^4+1005\,\alpha^3+3544\,\alpha^2+8379\,\alpha+10395)/720$		
7	$\lambda_0=(\alpha+1)(\alpha+2)(\alpha^6+26\,\alpha^5+290\,\alpha^4+1878\,\alpha^3+8154\,\alpha^2+24910\,\alpha+45045)/5040$ $\lambda_1=-(\alpha+2)(\alpha^6+26\,\alpha^5+292\,\alpha^4+1928\,\alpha^3+8698\,\alpha^2+28520\,\alpha+63063)/5040$ $\lambda_2=-(\alpha^4+25\,\alpha^3+284\,\alpha^2+2141\,\alpha-6435)/2520\,,\ \lambda_3=-(\alpha^3+22\,\alpha^2+230\,\alpha+1573)/840$ $\lambda_4=-(\alpha^2+18\,\alpha-247)/210\,,\ \lambda_5=-(\alpha-29)/42\,,\ \lambda_6=6/7$ $\mu_0=-s(\alpha+1)(\alpha^6+27\,\alpha^5+317\,\alpha^4+2197\,\alpha^3+10413\,\alpha^2+36227\,\alpha+90090)/5040$ $\mu_1=-s(\alpha^4+28\,\alpha^3+362\,\alpha^2+3107\,\alpha-18018)/2520$ $\mu_2=-s(\alpha^3+26\,\alpha^2+322\,\alpha-3861)/840\,,\ \mu_3=-s(\alpha^2+23\,\alpha-572)/210$ $\mu_4=-s(\alpha-65)/42\,,\ \mu_5=6s/7\,,\ \mu_7=s$ $\gamma=(\alpha+7)(\alpha^6+21\,\alpha^5+196\,\alpha^4+1113\,\alpha^3+4438\,\alpha^2+12411\,\alpha+19305)/5040$		
8	$\lambda_0=(\alpha+1)(\alpha+2)(\alpha^7+34\,\alpha^6+505\,\alpha^5+4415\,\alpha^4+26167\,\alpha^3+114075\,\alpha^2+$ $+362100\,\alpha+675675)/40320$ $\lambda_1=-(\alpha+2)(\alpha^7+34\,\alpha^6+507\,\alpha^5+4481\,\alpha^4+27125\,\alpha^3+122457\,\alpha^2+414654\,\alpha+$ $+945945)/40320$ $\lambda_2=-(\alpha^5+33\,\alpha^4+491\,\alpha^3+4623\,\alpha^2+33765\,\alpha-96525)/20160$ $\lambda_3=-(\alpha^4+30\,\alpha^3+413\,\alpha^2+3816\,\alpha-23595)/6720$ $\lambda_4=-(\alpha^3+26\,\alpha^2+324\,\alpha-3705)/1680\,,\ \lambda_5=-(\alpha^2+21\,\alpha-435)/336$ $\lambda_6=-(\alpha-41)/56\,,\ \lambda_7=7/8$ $\mu_0=-s(\alpha+1)(\alpha^7+35\,\alpha^6+540\,\alpha^5+4957\,\alpha^4+31202\,\alpha^3+146691\,\alpha^2+$ $+525345\,\alpha+1351350)/40320$ $\mu_1=-s(\alpha^5+36\,\alpha^4+593\,\alpha^3+6234\,\alpha^2+50721\,\alpha-270270)/20160$ $\mu_2=-s(\alpha^4+34\,\alpha^3+537\,\alpha^2+5652\,\alpha-57915)/6720$ $\mu_3=-s(\alpha^3+31\,\alpha^2+459\,\alpha-8580)/1680\,,\ \mu_4=-s(\alpha^2+27\,\alpha-975)/336$ $\mu_5=-s(\alpha-90)/56\,,\ \mu_6=7s/8\,,\ \mu_8=s$ $\gamma=(\alpha^8+36\,\alpha^7+574\,\alpha^6+5460\,\alpha^5+35539\,\alpha^4+171444\,\alpha^3+622866\,\alpha^2+$ $+1563120\,\alpha+2027025)/40320$		

n	$\lambda_k\,,\ \mu_k\,,\ \gamma$
9	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^8 + 43\,\alpha^7 + 819\,\alpha^6 + 9280\,\alpha^5 + 71694\,\alpha^4 + 412426\,\alpha^3 + 1836935\,\alpha^2 +$ $+6018255\,\alpha + 11486475)/362880$ $\lambda_1 = -(\alpha+2)(\alpha^8 + 43\,\alpha^7 + 821\,\alpha^6 + 9364\,\alpha^5 + 73262\,\alpha^4 + 430054\,\alpha^3 + 1975999\,\alpha^2 +$ $+6880887\,\alpha + 16081065)/362880$ $\lambda_2 = -(\alpha^6 + 42\,\alpha^5 + 796\,\alpha^4 + 9354\,\alpha^3 + 81004\,\alpha^2 + 593640\,\alpha - 1640925)/181440$ $\lambda_3 = -(\alpha^5 + 39\,\alpha^4 + 691\,\alpha^3 + 7821\,\alpha^2 + 69373\,\alpha - 401115)/60480$ $\lambda_4 = -(\alpha^4 + 35\,\alpha^3 + 566\,\alpha^2 + 6187\,\alpha - 62985)/15120$ $\lambda_5 = -(\alpha^3 + 30\,\alpha^2 + 434\,\alpha - 7395)/3024\,,\ \lambda_6 = -(\alpha+41)(\alpha-17)/504$ $\lambda_7 = -(\alpha-55)/72\,,\ \lambda_8 = 8/9$ $\mu_0 = -s(\alpha+1)(\alpha^8 + 44\,\alpha^7 + 863\,\alpha^6 + 10145\,\alpha^5 + 81935\,\alpha^4 + 496493\,\alpha^3 + 2363428\,\alpha^2 +$ $+8696295\,\alpha + 22972950)/362880$ $\mu_1 = -s(\alpha^6 + 45\,\alpha^5 + 925\,\alpha^4 + 11907\,\alpha^3 + 113419\,\alpha^2 + 913203\,\alpha - 4594590)/181440$ $\mu_2 = -s(\alpha^5 + 43\,\alpha^4 + 851\,\alpha^3 - 10805\,\alpha^2 + 106521\,\alpha - 984555)/60480$ $\mu_3 = -s(\alpha^4 + 40\,\alpha^3 + 746\,\alpha^2 + 9287\,\alpha - 145860)/15120$ $\mu_4 = -(\alpha^3 + 36\,\alpha^2 + 620\,\alpha - 16575)/3024\,,\ \mu_5 = -s(\alpha^2 + 31\,\alpha - 1530)/504$ $\mu_6 = -s(\alpha-119)/72\,,\ \mu_7 = 8s/9\,,\ \mu_9 = s$ $\gamma = (\alpha+9)(\alpha^8 + 36\,\alpha^7 + 582\,\alpha^6 + 5724\,\alpha^5 + 39603\,\alpha^4 + 209628\,\alpha^3 + 859202\,\alpha^2 +$ $+2485800\,\alpha + 3828825)/362880$
10	$\lambda_0 = (\alpha+1)(\alpha+2)(\alpha^9 + 53\,\alpha^8 + 1258\,\alpha^7 + 17920\,\alpha^6 + 174799\,\alpha^5 + 1273357\,\alpha^4 +$ $+7327962\,\alpha^3 + 33481160\,\alpha^2 + 112434165\,\alpha + 218243025)/3628800$ $\lambda_1 = -(\alpha+2)(\alpha^9 + 53\,\alpha^8 + 1260\,\alpha^7 + 18024\,\alpha^6 + 177225\,\alpha^5 + 1307547\,\alpha^4 + 7663160\,\alpha^3 +$ $+36009166\,\alpha^2 + 128276829\,\alpha + 305540235)/3628800$ $\lambda_2 = -(\alpha^7 + 52\,\alpha^6 + 1225\,\alpha^5 + 17755\,\alpha^4 + 184579\,\alpha^3 + 1546663\,\alpha^2 + 11552400\,\alpha -$ $-31177575)/1814400$ $\lambda_3 = -(\alpha^6 + 49\,\alpha^5 + 1090\,\alpha^4 + 15145\,\alpha^3 + 156484\,\alpha^2 + 1382191\,\alpha - 7621185)/604800$ $\lambda_4 = -(\alpha^5 + 45\,\alpha^4 + 925\,\alpha^3 + 12225\,\alpha^2 + 127369\,\alpha - 1196715)/151200$ $\lambda_5 = -(\alpha^4 + 40\,\alpha^3 + 743\,\alpha^2 + 9374\,\alpha - 140505)/30240$ $\lambda_6 = -(\alpha^3 + 34\,\alpha^2 + 560\,\alpha - 13243)/5040\,,\ \lambda_7 = -(\alpha^2 + 27\,\alpha - 1045)/720$ $\lambda_8 = -(\alpha-71)/90\,,\ \lambda_9 = 9/10$ $\mu_0 = -s(\alpha+1)(\alpha^9 + 54\,\alpha^8 + 1312\,\alpha^7 + 19234\,\alpha^6 + 194149\,\alpha^5 + 1470616\,\alpha^4 + 8850888\,\alpha^3 +$ $+42972346\,\alpha^2 + 161767125\,\alpha + 436486050)/3628800$ $\mu_1 = -s(\alpha^7 + 55\,\alpha^6 + 1384\,\alpha^5 + 21625\,\alpha^4 + 243874\,\alpha^3 + 2222155\,\alpha^2 + 18079641\,\alpha -$ $-87297210)/1814400$ $\mu_2 = -s(\alpha^6 + 53\,\alpha^5 + 1290\,\alpha^4 + 19765\,\alpha^3 + 225164\,\alpha^2 + 2175747\,\alpha - 18706545)/604800$ $\mu_3 = -s(\alpha^5 + 50\,\alpha^4 + 1155\,\alpha^3 + 17170\,\alpha^2 + 198389\,\alpha - 2771340)/151200$ $\mu_4 = -s(\alpha^4 + 46\,\alpha^3 + 989\,\alpha^2 + 14204\,\alpha - 314925)/30240$ $\mu_5 = -s(\alpha^3 + 41\,\alpha^2 + 805\,\alpha - 29070)/5040$ $\mu_6 = -s(\alpha^2 + 35\,\alpha - 2261)/720\,,\ \mu_7 = -s(\alpha-152)/90\,,\ \mu_8 = 9s/10\,,\ \mu_{10} = s$ $\gamma = (\alpha^{10} + 55\,\alpha^9 + 1365\,\alpha^8 + 20490\,\alpha^7 + 211953\,\alpha^6 + 1642305\,\alpha^5 + 10071935\,\alpha^4 +$ $+49660010\,\alpha^3 + 188887671\,\alpha^2 + 492444315\,\alpha + 654729075)/3628800$

$Z_\nu(x) \;=\; J_1(x)$:

n	λ_k	μ_k	γ
1	$\lambda_0=-(\alpha+1)$	$-$	-1
2	$\lambda_0=-(\alpha+1)(\alpha+2)\,,\ \lambda_1=\alpha+2\,,$ $\lambda_2=-1$	$\mu_0=\alpha+1\,,\ \mu_1=-1$	$-(\alpha+2)$
3	$\lambda_0=-(\alpha+1)(\alpha+2)(\alpha+3)/2\,,$ $\lambda_1=(\alpha+2)(\alpha+3)/2\,,\ \lambda_3=-1$	$\mu_0=(\alpha+1)(\alpha+4)/2\,,$ $\mu_1=-1\,,\ \mu_2=-1$	$-(\alpha^2+5\alpha+5)/2$
4	$\lambda_0=-(\alpha+1)(\alpha+2)^2(\alpha+5)/6\,,$ $\lambda_1=(\alpha+2)(\alpha^2+7\alpha+8)/6\,,\ \lambda_2=2/3\,,$ $\lambda_4=-1$	$\mu_0=(\alpha+1)(\alpha^2+8\alpha+14)/6\,,$ $\mu_1=-1/3\,,\ \mu_2=-1\,,$ $\mu_3=-1$	$-(\alpha+4)\cdot$ $\cdot(\alpha^2+8\alpha+14)/6$
5	$\lambda_0=-(\alpha+1)(\alpha+2)(\alpha^3+12\alpha^2+42\alpha+29)/24\,,\ \lambda_1=(\alpha+2)(\alpha^3+12\alpha^2+40\alpha+7)/24$ $\lambda_2=-(\alpha-13)/12\,,\ \lambda_3=3/4\,,\ \lambda_5=-1$ $\mu_0=(\alpha+1)(\alpha^3+13\alpha^2+51\alpha+34)/24\,,\ \mu_1=-(\alpha-10)/12$ $\mu_2=-1/4\,,\ \mu_3=-1\,,\ \mu_4=-1$ $\gamma=-(\alpha^4+14\alpha^3+65\alpha^2+100\alpha+9)/24$		
6	$\lambda_0=-(\alpha+1)(\alpha+2)(\alpha^4+18\alpha^3+110\alpha^2+229\alpha+14)/120$ $\lambda_1=(\alpha+2)(\alpha^4+18\alpha^3+108\alpha^2+195\alpha-158)/120\,,\ \lambda_2=-(\alpha^2+17\alpha-46)/60$ $\lambda_3=-(\alpha-26)/20\,,\ \lambda_4=4/5\,,\ \lambda_6=-1$ $\mu_0=(\alpha+1)(\alpha^4+19\alpha^3+125\alpha^2+284\alpha-92)/120\,,\ \mu_1=-(\alpha^2+20\alpha-112)/60$ $\mu_2=-(\alpha-22)/20\,,\ \mu_3=-1/5\,,\ \mu_4=-1\,,\ \mu_5=-1$ $\gamma=-(\alpha+6)(\alpha^4+14\alpha^3+61\alpha^2+64\alpha-35)/120$		
7	$\lambda_0=-(\alpha+1)(\alpha+2)(\alpha^5+25\alpha^4+231\alpha^3+899\alpha^2+1007\alpha-807)/720$ $\lambda_1=(\alpha+2)(\alpha^5+25\alpha^4+229\alpha^3+851\alpha^2+607\alpha-1821)/720$ $\lambda_2=-(\alpha^3+24\alpha^2+188\alpha+183)/360\,,\ \lambda_3=-(\alpha^2+21\alpha-127)/120$ $\lambda_4=-(\alpha-43)/30\,,\ \lambda_5=5/6\,,\ \lambda_7=-1$ $\mu_0=(\alpha+1)(\alpha^5+26\alpha^4+253\alpha^3+1054\alpha^2+1201\alpha-2334)/720$ $\mu_1=-(\alpha^3+27\alpha^2+263-654)/360\,,\ \mu_2=-(\alpha^2+25\alpha-279)/120$ $\mu_3=-(\alpha-38)/30\,,\ \mu_4=-1/6\,,\ \mu_5=-1\,,\ \mu_6=-1$ $\gamma=-(\alpha^6+27\alpha^5+280\alpha^4+1335\alpha^3+2554\alpha^2+213\alpha-2115)/720$		
8	$\lambda_0=-(\alpha+1)(\alpha+2)(\alpha^6+33\alpha^5+425\alpha^4+2579\alpha^3+6579\alpha^2+1675\alpha-7800)/5040$ $\lambda_1=(\alpha+2)(\alpha^6+33\alpha^5+423\alpha^4+2515\alpha^3+5807\alpha^2-2215\alpha-10200)/5040$ $\lambda_2=-(\alpha^4+32\alpha^3+374\alpha^2+1525\alpha+5880)/2520\,,\ \lambda_3=-(\alpha^3+29\alpha^2+275\alpha+280)/840$ $\lambda_4=-(\alpha^2+25\alpha-260)/210\,,\ \lambda_5=-(\alpha-64)/42\,,\ \lambda_6=6/7\,,\ \lambda_8=-1$ $\mu_0=(\alpha+1)(\alpha^6+34\alpha^5+455\alpha^4+2904\alpha^3+7863\alpha^2+590\alpha-20640)/5040$ $\mu_1=-(\alpha^4+35\alpha^3+473\alpha^2+2710\alpha+120)/2520\,,\ \mu_2=-(\alpha^3+33\alpha^2+395\alpha-1920)/840$ $\mu_3=-(\alpha^2+30\alpha-550)/210\,,\ \mu_4=-(\alpha-58)/42\,,\ \mu_5=-1/7\,,\ \mu_6=-1\,,\ \mu_7=-1$ $\gamma=-(\alpha+8)(\alpha^6+27\alpha^5+274\alpha^4+1203\alpha^3+1660\alpha^2-1275\alpha-945)/5040$		
9	$\lambda_0=-(1+\alpha)(\alpha+2)(\alpha^7+42\alpha^6+715\alpha^5+6145\alpha^4+26241\alpha^3+40313\alpha^2-26570\alpha-$ $-7347)/40320$ $\lambda_1=(\alpha+2)(\alpha^7+42\alpha^6+713\alpha^5+6063\alpha^4+24907\alpha^3+29979\alpha^2-57484\alpha+42159)/40320$ $\lambda_2=-(\alpha^5+41\alpha^4+655\alpha^3+4639\alpha^2+6361\alpha+71043)/20160$ $\lambda_3=-(\alpha^4+38\alpha^3+529\alpha^2+2524\alpha+16933)/6720\,,\ \lambda_4=-(\alpha^3+34\alpha^2+378\alpha+397)/1680$ $\lambda_5=-(\alpha^2+29\alpha-457)/336\,,\ \lambda_6=-(\alpha-89)/56\,,\ \lambda_7=7/8\,,\ \lambda_9=-1$ $\mu_0=(\alpha+1)(\alpha^7+43\alpha^6+754\alpha^5+6733\alpha^4+30212\alpha^3+48019\alpha^2-58933\alpha-55014)/40320$ $\mu_1=-(\alpha^5+44\alpha^4+781\alpha^3+6694\alpha^2+20569\alpha+69666)/20160$ $\mu_2=-(\alpha^4+42\alpha^3+685\alpha^2+4736\alpha-459)/6720\,,\ \mu_3=-(\alpha^3+39\alpha^2+553\alpha-4348)/1680$ $\mu_4=-(\alpha^2+35\alpha-949)/336\,,\ \mu_5=-(\alpha-82)/56\,,\ \mu_6=-1/8\,,\ \mu_7=-1\,,\ \mu_8=-1$ $\gamma=-(\alpha^8+44\alpha^7+798\alpha^6+7532\alpha^5+37779\alpha^4+86156\alpha^3+25682\alpha^2-89952\alpha+$ $+131985)/40320$		

n	$\lambda_k\,,\ \mu_k\,,\ \gamma$
10	$\lambda_0=-(\alpha+1)(\alpha+2)(\alpha^8+52\alpha^7+1127\alpha^6+12919\alpha^5+80819\alpha^4+244123\alpha^3+176200\alpha^2-$ $-203679\alpha+1145106)/362880$ $\lambda_1=(\alpha+2)(\alpha^8+52\alpha^7+1125\alpha^6+12817\alpha^5+78681\alpha^4+221185\alpha^3+54842\alpha^2-364545\alpha+$ $+2157318)/362880$ $\lambda_2=-(\alpha^6+51\alpha^5+1057\alpha^4+10821\alpha^3+46399\alpha 2-71307\alpha+458766)/181440$ $\lambda_3=-(\alpha^5+48\alpha^4+901\alpha^3+7470\alpha^2+10069\alpha+252546)/60480$ $\lambda_4=-(\alpha^4+44\alpha^3+710^2\alpha^2+3865\alpha+40434)/15120\,,\ \lambda_5=-(\alpha^3+39\alpha^2+497\alpha+534)/3024$ $\lambda_6=-(\alpha^2+33\alpha-730)/504\,,\ \lambda_7=-(\alpha-118)/72\,,\ \lambda_8=8/9\,,\ \lambda_{10}=-1$ $\mu_0=(\alpha+1)(\alpha^8+53\alpha^7+1176\alpha^6+13889\alpha^5+90318\alpha^4+285803\alpha^3+183185\alpha^2-598104\alpha+$ $+1927332)/362880$ $\mu_1=-(\alpha^6+54\alpha^5+1213\alpha^4+14112\alpha^3+80029\alpha^2+49500\alpha+1193652)/181440$ $\mu_2=-(\alpha^5+52\alpha^4+1097\alpha^3+11210\alpha^2+40269\alpha+244962)/60480$ $\mu_3=-(\alpha^4+49\alpha^3+935\alpha^2+7550\alpha-1896)/15120\,,\ \mu_4=-(\alpha^3+45\alpha^2+737\alpha-8466)/3024$ $\mu_5=-(aa^2+40\alpha-1500)/504\,,\ \mu_6=-(\alpha-110)/72\,,\ \mu_7=-1/9\,,\ \mu_8=-1\,,\ \mu_9=-1$ $\gamma=-(\alpha+10)(\alpha^8+44\alpha^7+790\alpha^6+7220\alpha^5+33283\alpha^4+59156\alpha^3-14670\alpha^2+45720\alpha+$ $+353241)/362880$

$Z_\nu(x)\ =\ I_1(x)\,,\ K_1(x)$ with s defined as on page 185:

n	λ_k	μ_k	γ
1	$\lambda_0=s(\alpha+1)$	$-$	s
2	$\lambda_0=s(\alpha+1)(\alpha+2)\,,\ \lambda_1=-s(\alpha+2)\,,$ $\lambda_2=s$	$\mu_0=-(\alpha+1)\,,\ \mu_1=1$	$s(\alpha+2)$
3	$\lambda_0=s(\alpha+1)(\alpha+2)(\alpha+3)/2\,,$ $\lambda_1=-s(\alpha+2)(\alpha+3)/2\,,\ \lambda_3=s$	$\mu_0=-(\alpha+1)(\alpha+4)/2\,,$ $\mu_1=1\,,\ \mu_2=1$	$s(\alpha^2+5\,\alpha+7)/2$
4	$\lambda_0=s(\alpha+1)(\alpha+2)(\alpha^2+7\,\alpha+14)/6$ $\lambda_1=-s(\alpha+2)(\alpha^2+7\,\alpha+16)/6\,,$ $\lambda_2=2s/3\,,\ \lambda_4=s$	$\mu_0=-(\alpha+1)(\alpha^2+8\,\alpha+22)/6\,,$ $\mu_1=5/3\,,\ \mu_2=1\,,\ \mu_3=1$	$s(\alpha+4)\cdot$ $\cdot(\alpha^2+5\,\alpha+9)/6$
5	$\lambda_0=s(\alpha+1)(\alpha+2)(\alpha^3+12\,\alpha^2+52\,\alpha+91)/24\,,\ \lambda_1=-s(\alpha+2)(\alpha^3+12\,\alpha^2+54\,\alpha+113)/24$ $\lambda_2=-s(\alpha-13)/12\,,\ \lambda_3=3s/4\,,\ \lambda_5=s$ $\mu_0=-(\alpha+1)(\alpha^3+13\,\alpha^2+65\,\alpha+158)/24\,,\ \mu_1=-(\alpha-34)/12$ $\mu_2=7/4\,,\ \mu_3=1\,,\ \mu_4=1$ $\gamma=s(\alpha^4+14\,\alpha^3+77\,\alpha^2+208\,\alpha+249)/24$		
6	$\lambda_0=s(\alpha+1)(\alpha+2)(\alpha^4+18\,\alpha^3+128\,\alpha^2+455\,\alpha+770)/120$ $\lambda_1=-s(\alpha+2)(\alpha^4+18\,\alpha^3+130\,\alpha^2+489\,\alpha+1006)/120$ $\lambda_2=-s(\alpha+22)(\alpha-5)/60\,,\ \lambda_3=-s(\alpha-26)/20\,,\ \lambda_4=4s/5\,,\ \lambda_6=s$ $\mu_0=-(\alpha+1)(\alpha^4+19\,\alpha^3+147\,\alpha^2+604\,\alpha+1420)/120\,,\ \mu_1=-(\alpha^2+20\,\alpha-296)/60$ $\mu_2=-(\alpha-62)/20\,,\ \mu_3=9/5\,,\ \mu_4=1\,,\ \mu_5=1$ $\gamma=s(\alpha+6)(\alpha^4+14\,\alpha^3+81\,\alpha^2+244\,\alpha+365)/120$		
7	$\lambda_0=s(\alpha+1)(\alpha+2)(\alpha^5+25\,\alpha^4+259\,\alpha^3+1451\,\alpha^2+4735\,\alpha+8085)/720$ $\lambda_1=-s(\alpha+2)(\alpha^2+13\,\alpha+57)(\alpha^3+12\,\alpha^2+48\,\alpha+191)/720$ $\lambda_2=-s(\alpha^3+24\,\alpha^2+254\,\alpha-1155)/360\,,\ \lambda_3=-s(\alpha^2+21\,\alpha-277)/120$ $\lambda_4=-s(\alpha-43)/30\,,\ \lambda_5=5s/6\,,\ \lambda_7=s$ $\mu_0=-(\alpha+1)(\alpha^5+26\,\alpha^4+285\,\alpha^3+1738\,\alpha^2+6533\,\alpha+15450)/720$ $\mu_1=-(\alpha-6)(\alpha^2+33\,\alpha+527)/360\,,\ \mu_2=-(\alpha^2+25\,\alpha-669)/120$ $\mu_3=-(\alpha-98)/30\,,\ \mu_4=11/6\,,\ \mu_5=1\,,\ \mu_6=1$ $\gamma=s(\alpha^6+27\,\alpha^5+310\,\alpha^4+1995\,\alpha^3+7924\,\alpha^2+19353\,\alpha+23535)/720$		
8	$\lambda_0=s(\alpha+1)(\alpha+2)(\alpha^6+33\,\alpha^5+465\,\alpha^4+3691\,\alpha^3+18311\,\alpha^2+58055\,\alpha+101640)/5040$ $\lambda_1=-s(\alpha+2)(\alpha^6+33\,\alpha^5+467\,\alpha^4+3755\,\alpha^3+19191\,\alpha^2+65053\,\alpha+139272)/5040$ $\lambda_2=-s(\alpha^4+32\,\alpha^3+452\,\alpha^2+3919\,\alpha-14520)/2520\,,\ \lambda_3=-s(\alpha^3+29\,\alpha^2+377\,\alpha-3512)/840$ $\lambda_4=-s(\alpha^2+25\,\alpha-548)/210\,,\ \lambda_5=-s(\alpha-64)/42\,,\ \lambda_6=6s/7\,,\ \lambda_8=s$ $\mu_0=-(\alpha+1)(\alpha^6+34\,\alpha^5+499\,\alpha^4+4192\,\alpha^3+22579\,\alpha^2+81958\,\alpha+198240)/5040$ $\mu_1=-(\alpha^4+35\,\alpha^3+551\,\alpha^2+5410\,\alpha-40152)/2520\,,\ \mu_2=-(\alpha^3+33\,\alpha^2+497\,\alpha-8544)/840$ $\mu_3=-(\alpha^2+30\,\alpha-1258)/210\,,\ \mu_4=-(\alpha-142)/42\,,\ \mu_5=13/7\,,\ \mu_6=1\,,\ \mu_7=1$ $\gamma=s(\alpha+8)(\alpha^6+27\,\alpha^5+316\,\alpha^4+2127\,\alpha^3+9178\,\alpha^2+25521\,\alpha+37485)/5040$		

n	$\lambda_k\,,\ \mu_k\,,\ \gamma$
9	$\lambda_0 = s(\alpha+1)(\alpha+2)(\alpha^7 + 42\,\alpha^6 + 769\,\alpha^5 + 8135\,\alpha^4 + 55695\,\alpha^3 + 260563\,\alpha^2 + 826540\,\alpha +$ $+1488795)/40320$ $\lambda_1 = -s(\alpha+2)(\alpha^7 + 42\,\alpha^6 + 771\,\alpha^5 + 8217\,\alpha^4 + 57165\,\alpha^3 + 275985\,\alpha^2 + 935078\,\alpha +$ $+2060121)/40320$ $\lambda_2 = -s(\alpha^5 + 41\,\alpha^4 + 747\,\alpha^3 + 8239\,\alpha^2 + 65117\,\alpha - 212685)/20160$ $\lambda_3 = -s(\alpha^4 + 38\,\alpha^3 + 645\,\alpha^2 + 6832\,\alpha - 51691)/6720$ $\lambda_4 = -s(\alpha^3 + 34\,\alpha^2 + 524\,\alpha - 8089)/1680\,,\ \lambda_5 = -s(\alpha^2 + 29\,\alpha - 947)/336$ $\lambda_6 = -s(\alpha - 89)/56\,,\ \lambda_7 = 7s/8\,,\ \lambda_9 = s$ $\mu_0 = -(\alpha+1)(\alpha^7 + 43\,\alpha^6 + 812\,\alpha^5 + 8949\,\alpha^4 + 64738\,\alpha^3 + 327323\,\alpha^2 + 1181009\,\alpha +$ $+2937270)/40320$ $\mu_1 = -(\alpha^5 + 44\,\alpha^4 + 873\,\alpha^3 + 10642\,\alpha^2 + 94001\,\alpha + 591486)/20160$ $\mu_2 = -(\alpha^4 + 42\,\alpha^3 + 801\,\alpha^2 + 9628\,\alpha - 126267)/6720$ $\mu_3 = -(\alpha^3 + 39\,\alpha^2 + 699\,\alpha - 18644)/1680\,,\ \mu_4 = -(\alpha^2 + 35\,\alpha - 2111)/336$ $\mu_5 = -(\alpha - 194)/56\,,\ \mu_6 = 15/8\,,\ \mu_7 = 1\,,\ \mu_8 = 1$ $\gamma = s(\alpha^8 + 44\,\alpha^7 + 854\,\alpha^6 + 9716\,\alpha^5 + 72779\,\alpha^4 + 380996\,\alpha^3 + 1414426\,\alpha^2 + 3496344\,\alpha +$ $+4426065)/40320$
10	$\lambda_0 = s(\alpha+1)(\alpha+2)(\alpha^8 + 52\,\alpha^7 + 1197\,\alpha^6 + 16201\,\alpha^5 + 144909\,\alpha^4 + 913681\,\alpha^3 + 4182002\,\alpha^2 +$ $+13457115\,\alpha + 24885630)/362880$ $\lambda_1 = -s(\alpha+2)(\alpha^8 + 52\,\alpha^7 + 1199\,\alpha^6 + 16303\,\alpha^5 + 147215\,\alpha^4 + 944539\,\alpha^3 + 4459864\,\alpha^2 +$ $+15296589\,\alpha + 34622154)/362880$ $\lambda_2 = -s(\alpha^6 + 51\,\alpha^5 + 1165\,\alpha^4 + 16077\,\alpha^3 + 155155\,\alpha^2 + 1179693\,\alpha - 3555090)/181440$ $\lambda_3 = -s(\alpha^5 + 48\,\alpha^4 + 1033\,\alpha^3 + 13626\,\alpha^2 + 130861\,\alpha - 866334)/60480$ $\lambda_4 = -s(\alpha^4 + 44\,\alpha^3 + 872\,\alpha^2 + 10903\,\alpha - 135786)/15120$ $\lambda_5 = -s(\alpha^3 + 39\,\alpha^2 + 695\,\alpha - 15918)/3024\,,\ \lambda_6 = -s(\alpha^2 + 33\,\alpha - 1498)/506$ $\lambda_7 = -s(\alpha - 118)/72\,,\ \lambda_8 = 8s/9\,,\ \lambda_{10} = s$ $\mu_0 = -(\alpha+1)(\alpha^8 + 53\,\alpha^7 + 1250\,\alpha^6 + 17453\,\alpha^5 + 162476\,\alpha^4 + 1079135\,\alpha^3 + 5309335\,\alpha^2 +$ $+19325376\,\alpha + 49408380)/362880$ $\mu_1 = -(\alpha^6 + 54\,\alpha^5 + 1321\,\alpha^4 + 19764\,\alpha^3 + 209197\,\alpha^2 + 1759212\,\alpha - 9917964)/181440$ $\mu_2 = -(\alpha^5 + 52\,\alpha^4 + 1229\,\alpha^3 + 18014\,\alpha^2 + 193173\,\alpha - 2120958)/60480$ $\mu_3 = -(\alpha^4 + 49\,\alpha^3 + 1097\,\alpha^2 + 15578\,\alpha - 313656)/15120$ $\mu_4 = -(\alpha^3 + 45\,\alpha^2 + 935\,\alpha - 35574)/3024\,,\ \mu_5 = -(\alpha^2 + 40\,\alpha - 3276)/504$ $\mu_6 = -(\alpha - 254)/72\,,\ \mu_7 = 17/9\,,\ \mu_8 = 1\,,\ \mu_9 = 1$ $\gamma = s(\alpha+10)(\alpha^8 + 44\,\alpha^7 + 862\,\alpha^6 + 10028\,\alpha^5 + 78283\,\alpha^4 + 438236\,\alpha^3 + 1792458\,\alpha^2 +$ $+5023872\,\alpha + 7429401)/362880$

g) Hermite Polynomials $H_n(x)$:

$$\int_{-\infty}^{\infty} e^{-x^2}\, H_n^2(x)\, dx \quad = \quad \sqrt{\pi}\,\cdot\, 2^n\, n! \quad = \quad \sqrt{\pi}\,\cdot\,(2n)!!$$

First polynomials:

$$H_0(x) = 1\,, \quad H_1(x) = 2x\,, \quad H_2(x) = 4x^2 - 2\,, \quad H_3(x) = 8x^3 - 12x\,, \quad H_4(x) = 16x^4 - 48x^2 + 12$$

$Z_\nu(x) \ = \ J_0(x)$:

n	λ_k	μ_k	γ
1	–	$\mu_1 = 1\,,$	0
2	$\lambda_1 = -1\,,$	$\mu_0 = 2\,,\ \mu_2 = 1$	-6
3	$\lambda_0 = 8\,,\ \lambda_2 = 4$	$\mu_1 = -16\,,\ \mu_3 = 1$	0
4	$\lambda_1 = 42\,,\ \lambda_3 = 6$	$\mu_0 = -84\,,\ \mu_2 = -36\,,\ \mu_4 = 1$	204
5	$\lambda_0 = -576\,,\ \lambda_2 = -240\,,\ \lambda_4 = 8$	$\mu_1 = 960\,,\ \mu_3 = -64\,,\ \mu_5 = 1$	0
6	$\lambda_1 = -4140\,,\ \lambda_3 = -580\,,\ \lambda_5 = 10$	$\mu_0 = 8280\,,\ \mu_2 = 3480\,,\ \mu_4 = -100\,,\ \mu_6 = 1$	-19560

n	λ_k	μ_k	γ
7	$\lambda_0 = 82176\,,\ \lambda_2 = 33600\,,\ \lambda_4 = -1128\,,$ $\lambda_6 = 12$	$\mu_1 = -134400\,,\ \mu_3 = 9024\,,\ \mu_5 = -144\,,$ $\mu_7 = 1$	0
8	$\lambda_1 = 798840\,,\ \lambda_3 = 111720\,,$ $\lambda_5 = -1932\,,\ \lambda_7 = 14$	$\mu_0 = -1597680\,,\ \mu_2 = -670320\,,$ $\mu_4 = 19320\,,\ \mu_6 = -196\,,\ \mu_8 = 1$	3764880
9	$\lambda_0 = -20803584\,,\ \lambda_2 = -8494080\,,$ $\lambda_4 = 285312\,,\ \lambda_6 = -3040\,,\ \lambda_8 = 16$	$\mu_1 = 33976320\,,\ \mu_3 = -2282496\,,$ $\mu_5 = 36480\,,\ \mu_7 = -256\,,\ \mu_9 = 1$	0
10	$\lambda_1 = -256510800\,,\ \lambda_3 = -35869680\,,$ $\lambda_5 = 620424\,,\ \lambda_7 = -4500\,,\ \lambda_9 = 18$	$\mu_0 = 513021600\,,\ \mu_2 = 215218080\,,$ $\mu_4 = -6204240\,,\ \mu_6 = 63000\,,\ \mu_8 = -324\,,$ $\mu_{10} = 1$	-1208723040

$Z_\nu(x) \;=\; I_0(x)$:

n	λ_k	μ_k	γ
1	–	$\mu_1 = 1\,,$	0
2	$\lambda_1 = -1\,,$	$\mu_0 = 2\,,\ \mu_2 = 1$	2
3	$\lambda_0 = -8\,,\ \lambda_2 = -4$	$\mu_1 = 16\,,\ \mu_3 = 1$	0
4	$\lambda_1 = -30\,,\ \lambda_3 = -6$	$\mu_0 = 60\,,\ \mu_2 = 36\,,\ \mu_4 = 1$	108
5	$\lambda_0 = -448\,,\ \lambda_2 = -272\,,\ \lambda_4 = -8$	$\mu_1 = 1088\,,\ \mu_3 = 64\,,\ \mu_5 = 1$	0
6	$\lambda_1 = -3180\,,\ \lambda_3 = -620\,,\ \lambda_5 = -10$	$\mu_0 = 6360\,,\ \mu_2 = 3720\,,\ \mu_4 = 100\,,$ $\mu_6 = 1$	10680
7	$\lambda_0 = -66816\,,\ \lambda_2 = -66816\,,$ $\lambda_4 = -1176\,,\ \lambda_6 = -12$	$\mu_1 = 158976\,,\ \mu_3 = 9408\,,\ \mu_5 = 144\,,$ $\mu_7 = 1$	0
8	$\lambda_1 = -629160\,,\ \lambda_3 = -122920\,,$ $\lambda_5 = -1988\,,\ \lambda_7 = -14$	$\mu_0 = 1258320\,,\ \mu_2 = 737520\,,$ $\mu_4 = 19880\,,\ \mu_6 = 196\,,\ \mu_8 = 1$	2125200
9	$\lambda_0 = -17240064\,,\ \lambda_2 = -17240064\,,$ $\lambda_4 = -303744\,,\ \lambda_6 = -3104\,,\ \lambda_8 = -16$	$\mu_1 = 41078784\,,\ \mu_3 = 2429952\,,$ $\mu_5 = 37248\,,\ \mu_7 = 256\,,\ \mu_9 = 1$	0
10	$\lambda_1 = -205344720\,,\ \lambda_3 = -40113360\,,$ $\lambda_5 = -648648\,,\ \lambda_7 = -4572\,,\ \lambda_9 = -18$	$\mu_0 = 410689440\,,\ \mu_2 = 240680160\,,$ $\mu_4 = 6486480\,,\ \mu_6 = 64008\,,$ $\mu_8 = 324\,,\ \mu_{10} = 1$	693372960

$Z_\nu(x) \;=\; K_0(x)$:

n	λ_k	μ_k	γ
1	–	$\mu_1 = -1\,,$	0
2	$\lambda_1 = -1\,,$	$\mu_0 = -2\,,\ \mu_2 = -1$	2
3	$\lambda_0 = -8\,,\ \lambda_2 = -4$	$\mu_1 = -16\,,\ \mu_3 = -1$	0
4	$\lambda_1 = -30\,,\ \lambda_3 = -6$	$\mu_0 = -60\,,\ \mu_2 = -36\,,\ \mu_4 = -1$	108
5	$\lambda_0 = -448\,,\ \lambda_2 = -272\,,\ \lambda_4 = -8$	$\mu_1 = -1088\,,\ \mu_3 = -64\,,\ \mu_5 = -1$	0
6	$\lambda_1 = -3180\,,\ \lambda_3 = -620\,,\ \lambda_5 = -10$	$\mu_0 = -6360\,,\ \mu_2 = -3720\,,$ $\mu_4 = -100\,,\ \mu_6 = -1$	10680
7	$\lambda_0 = -66816\,,\ \lambda_2 = -66816\,,$ $\lambda_4 = -1176\,,\ \lambda_6 = -12$	$\mu_1 = -158976\,,\ \mu_3 = -9408\,,$ $\mu_5 = -144\,,\ \mu_7 = -1$	0
8	$\lambda_1 = -629160\,,\ \lambda_3 = -122920\,,$ $\lambda_5 = -1988\,,\ \lambda_7 = -14$	$\mu_0 = -1258320\,,\ \mu_2 = -737520\,,$ $\mu_4 = -19880\,,\ \mu_6 = -196\,,\ \mu_8 = -1$	2125200
9	$\lambda_0 = -17240064\,,\ \lambda_2 = -17240064\,,$ $\lambda_4 = -303744\,,\ \lambda_6 = -3104\,,\ \lambda_8 = -16$	$\mu_1 = -41078784\,,\ \mu_3 = -2429952\,,$ $\mu_5 = -37248\,,\ \mu_7 = 2-56\,,\ \mu_9 = -1$	0
10	$\lambda_1 = -205344720\,,\ \lambda_3 = -40113360\,,$ $\lambda_5 = -648648\,,\ \lambda_7 = -4572\,,\ \lambda_9 = -18$	$\mu_0 = -410689440\,,\ \mu_2 = -240680160\,,$ $\mu_4 = -6486480\,,\ \mu_6 = -64008\,,$ $\mu_8 = -324\,,\ \mu_{10} = -1$	693372960

$Z_\nu(x) \ = \ J_1(x)$:

n	λ_k	μ_k	γ
1	–	–	2
2	$\lambda_2 = -1$	$\mu_1 = 4$	0
3	$\lambda_1 = -6\,,\ \lambda_3 = -1$	$\mu_0 = 12\,,\ \mu_2 = 6$	-36
4	$\lambda_0 = 64\,,\ \lambda_2 = 32\,,\ \lambda_4 = -1$	$\mu_1 = -128\,,\ \mu_3 = 8$	0
5	$\lambda_1 = 420\,,\ \lambda_3 = 60\,,\ \lambda_5 = -1$	$\mu_0 = -840\,,\ \mu_2 = -360\,,\ \mu_4 = 10$	2040
6	$\lambda_0 = -6912\,,\ \lambda_2 = -2880\,,\ \lambda_4 = 96\,,$ $\lambda_6 = -1$	$\mu_1 = 11520\,,\ \mu_3 = -768\,,\ \mu_5 = 12$	0
7	$\lambda_1 = -57960\,,\ \lambda_3 = -8120\,,$ $\lambda_5 = 140\,,\ \lambda_7 = -1$	$\mu_0 = 115920\,,\ \mu_2 = 48720\,,$ $\mu_4 = -1400\,,\ \mu_6 = 14$	-273840
8	$\lambda_0 = 1314816\,,\ \lambda_2 = 537600\,,$ $\lambda_4 = -18048\,,\ \lambda_6 = 192\,,\ \lambda_8 = -1$	$\mu_1 = -2150400\,,\ \mu_3 = 144384\,,$ $\mu_5 = -2304\,,\ \mu_7 = 16$	0
9	$\lambda_1 = 14379120\,,\ \lambda_3 = 2010960\,,$ $\lambda_5 = -34776\,,\ \lambda_7 = 252\,,\ \lambda_9 = -1$	$\mu_0 = -28758240\,,\ \mu_2 = -12065760\,,$ $\mu_4 = 347760\,,\ \mu_6 = -3528\,,\ \mu_8 = 18$	67767840
10	$\lambda_0 = -416071680\,,\ \lambda_2 = -169881600\,,$ $\lambda_4 = 5706240\,,\ \lambda_6 = -60800\,,$ $\lambda_8 = 320\,,\ \lambda_{10} = -1$	$\mu_1 = 679526400\,,\ \mu_3 = -45649920\,,$ $\mu_5 = 729600\,,\ \mu_7 = -5120\,,$ $\mu_9 = 20$	0

$Z_\nu(x) \ = \ I_1(x)$:

n	λ_k	μ_k	γ
1	–	–	-2
2	$\lambda_2 = 1$	$\mu_1 = -4$	0
3	$\lambda_1 = 6\,,\ \lambda_3 = 1$	$\mu_0 = -12\,,\ \mu_2 = -6$	-12
4	$\lambda_0 = 64\,,\ \lambda_2 = 32\,,\ \lambda_4 = 1$	$\mu_1 = -128\,,\ \mu_3 = -8$	0
5	$\lambda_1 = 300\,,\ \lambda_3 = 60\,,\ \lambda_5 = 1$	$\mu_0 = -600\,,\ \mu_2 = -360\,,\ \mu_4 = -10$	-1080
6	$\lambda_0 = 5376\,,\ \lambda_2 = 3264\,,\ \lambda_4 = 96\,,$ $\lambda_6 = 1$	$\mu_1 = -13056\,,\ \mu_3 = -768\,,\ \mu_5 = -12$	0
7	$\lambda_1 = 44520\,,\ \lambda_3 = 8680\,,$ $\lambda_5 = 140\,,\ \lambda_7 = 1$	$\mu_0 = 8680\,,\ \mu_2 = -52080\,,$ $\mu_4 = -1400\,,\ \mu_6 = -14$	-149520
8	$\lambda_0 = 1069056\,,\ \lambda_2 = 635904\,,$ $\lambda_4 = 18816\,,\ \lambda_6 = 192\,,\ \lambda_8 = 1$	$\mu_1 = -2543616\,,\ \mu_3 = -150528\,,$ $\mu_5 = -2304\,,\ \mu_7 = -16$	0
9	$\lambda_1 = 11324880\,,\ \lambda_3 = 2212560\,,$ $\lambda_5 = 35784\,,\ \lambda_7 = 252\,,\ \lambda_9 = 1$	$\mu_0 = -22649760\,,\ \mu_2 = -13275360\,,$ $\mu_4 = -357840\,,\ \mu_6 = -3528\,,\ \mu_8 = -18$	-38253600
10	$\lambda_0 = 344801280\,,\ \lambda_2 = 205393920\,,$ $\lambda_4 = 6074880\,,\ \lambda_6 = 62080\,,$ $\lambda_8 = 320\,,\ \lambda_{10} = 1$	$\mu_1 = -821575680\,,\ \mu_3 = -48599040\,,$ $\mu_5 = -744960\,,\ \mu_7 = -5120\,,$ $\mu_9 = -20$	0

$Z_\nu(x) \;=\; K_1(x)$:

n	λ_k	μ_k	γ
1	$-$	$-$	2
2	$\lambda_2 = -1$	$\mu_1 = -4$	0
3	$\lambda_1 = -6\,,\ \lambda_3 = -1$	$\mu_0 = -12\,,\ \mu_2 = -6$	12
4	$\lambda_0 = -64\,,\ \lambda_2 = -32\,,\ \lambda_4 = -1$	$\mu_1 = -128\,,\ \mu_3 = -8$	0
5	$\lambda_1 = -300\,,\ \lambda_3 = -60\,,\ \lambda_5 = -1$	$\mu_0 = -600\,,\ \mu_2 = -360\,,\ \mu_4 = -10$	1080
6	$\lambda_0 = -5376\,,\ \lambda_2 = -3264\,,\ \lambda_4 = -96\,,$ $\lambda_6 = 1$	$\mu_1 = -13056\,,\ \mu_3 = -768\,,\ \mu_5 = -12$	0
7	$\lambda_1 = -44520\,,\ \lambda_3 = -8680\,,$ $\lambda_5 = -140\,,\ \lambda_7 = -1$	$\mu_0 = -8680\,,\ \mu_2 = -52080\,,$ $\mu_4 = -1400\,,\ \mu_6 = -14$	149520
8	$\lambda_0 = -1069056\,,\ \lambda_2 = -635904\,,$ $\lambda_4 = -18816\,,\ \lambda_6 = -192\,,\ \lambda_8 = -1$	$\mu_1 = -2543616\,,\ \mu_3 = -150528\,,$ $\mu_5 = -2304\,,\ \mu_7 = -16$	0
9	$\lambda_1 = -11324880\,,\ \lambda_3 = -2212560\,,$ $\lambda_5 = -35784\,,\ \lambda_7 = -252\,,\ \lambda_9 = -1$	$\mu_0 = -22649760\,,\ \mu_2 = -13275360\,,$ $\mu_4 = -357840\,,\ \mu_6 = -3528\,,\ \mu_8 = -18$	38253600
10	$\lambda_0 = -344801280\,,\ \lambda_2 = -205393920\,,$ $\lambda_4 = -6074880\,,\ \lambda_6 = -62080\,,$ $\lambda_8 = -320\,,\ \lambda_{10} = -1$	$\mu_1 = -821575680\,,\ \mu_3 = -48599040\,,$ $\mu_5 = -744960\,,\ \mu_7 = -5120\,,$ $\mu_9 = -20$	0

1.3.2. Integrals of the type $\int x^n \operatorname{Ei}(x) \cdot Z_\nu(x)\,dx$

About $\operatorname{Ei}(x)$ see [1], 5.1., or [7], 8.2. In [4], page 657, is no reference to the fact, that the integral should be used as a principal value.

$$\int x \operatorname{Ei}(x)\, I_0(x)\, dx = x\operatorname{Ei}(x) I_1(x) + e^x\left[(x-1)I_0(x) - xI_1(x)\right]$$

$$\int x \operatorname{Ei}(x)\, K_0(x)\, dx = -x\operatorname{Ei}(x) K_1(x) + e^x\left[(x-1)K_0(x) + xK_1(x)\right]$$

$$\int x^2 \operatorname{Ei}(x)\, I_1(x)\, dx = \operatorname{Ei}(x)[x^2 I_0(x) - 2xI_1(x)] + \frac{e^x}{3}\left[(-x^2-6\,x+6)I_0(x) + (x^2+5\,x)I_1(x)\right]$$

$$\int x^2 \operatorname{Ei}(x)\, K_1(x)\, dx = -\operatorname{Ei}(x)[x^2 K_0(x) + 2xK_1(x)] + \frac{e^x}{3}\left[(x^2+6\,x-6)K_0(x) + (x^2+5\,x)K_1(x)\right]$$

$$\int x^3 \operatorname{Ei}(x)\, I_0(x)\, dx = \operatorname{Ei}(x)[-2x^2 I_0(x) + (x^3+4\,x)I_1(x)]+$$

$$+\frac{e^x}{15}\left[(3\,x^3+7\,x^2+60\,x-60)I_0(x) - (3\,x^3+16\,x^2+44\,x)I_1(x)\right]$$

$$\int x^3 \operatorname{Ei}(x)\, K_0(x)\, dx = -\operatorname{Ei}(x)[2x^2 K_0(x) + (x^3+4x)K_1(x)]+$$

$$+\frac{e^x}{15}\left[(3\,x^3+7\,x^2+60\,x-60)K_0(x) + (3\,x^3+16\,x^2+44\,x)K_1(x)\right]$$

$$\int x^4 \operatorname{Ei}(x)\, I_1(x)\, dx = \operatorname{Ei}(x)\left[(x^4+8\,x^2)I_0(x) - (16\,x+4\,x^3)I_1(x)\right]+$$

$$+\frac{e^x}{105}\left[-(15\,x^4+102\,x^3+178\,x^2+1680\,x-1680)I_0(x) + (15\,x^4+57\,x^3+484\,x^2+1196\,x)I_1(x)\right]$$

$$\int x^4 \operatorname{Ei}(x)\, K_1(x)\, dx = -\operatorname{Ei}(x)\left[(x^4+8\,x^2)K_0(x) + (4\,x^3+16\,x)K_1(x)\right]+$$

$$+\frac{e^x}{105}\left[(15\,x^4+102\,x^3+178\,x^2+1680\,x-1680)K_0(x) + (15\,x^4+57\,x^3+484\,x^2+1196\,x)K_1(x)\right]$$

$$\int x^5 \operatorname{Ei}(x)\, I_0(x)\, dx = \operatorname{Ei}(x)\left[-(4\,x^4+32\,x^2)I_0(x) + (x^5+16\,x^3+64\,x)I_1(x)\right]+$$

$$+\frac{e^x}{45}\left[(5\,x^5+15\,x^4+192\,x^3+288\,x^2+2880\,x-2880)\,I_0(x) - (5\,x^5+40\,x^4+72\,x^3+864\,x^2+2016\,x)I_1(x)\right]$$

$$\int x^5 \operatorname{Ei}(x)\, K_0(x)\, dx = -\operatorname{Ei}(x)\left[(4\,x^4+32\,x^2)K_0(x) + (x^5+16\,x^3+64\,x)K_1(x)\right]+$$

$$+\frac{e^x}{45}\left[(5\,x^5+15\,x^4+192\,x^3+288\,x^2+2880\,x-2880)K_0(x) + (5\,x^5+40\,x^4+72\,x^3+864\,x^2+2016\,x)K_1(x)\right]$$

$$\int x^6 \operatorname{Ei}(x)\, I_1(x)\, dx = \operatorname{Ei}(x)\left[(x^6+24\,x^4+192\,x^2)I_0(x) - (384\,x+96\,x^3+6\,x^5)I_1(x)\right]+$$

$$+\frac{e^x}{3465}\left[-(315\,x^6+3010\,x^5+5430\,x^4+91104\,x^3+130656\,x^2+1330560\,x-1330560)I_0(x)+\right.$$

$$\left.+(315\,x^6+1435\,x^5+20480\,x^4+29664\,x^3+403968\,x^2+926592\,x)I_1(x)\right]$$

$$\int x^6 \operatorname{Ei}(x)\, K_1(x)\, dx = -\operatorname{Ei}(x)\left[(x^6+24\,x^4+192\,x^2)K_0(x) + (6\,x^5+96\,x^3+384\,x)K_1(x)\right]+$$

$$+\frac{e^x}{3465}\left[(315\,x^6+3010\,x^5+5430\,x^4+91104\,x^3+130656\,x^2+1330560\,x-1330560)K_0(x)+\right.$$
$$\left.+(315\,x^6+1435\,x^5+20480\,x^4+29664\,x^3+403968\,x^2+926592\,x)K_1(x)\right]$$

$$\int x^7\,\mathrm{Ei}(x)\,I_0(x)\,dx=\mathrm{Ei}(x)\left[-(6\,x^6+144\,x^4+1152\,x^2)I_0(x)+(x^7+36\,x^5+576\,x^3+2304\,x)I_1(x)\right]+$$
$$+\frac{e^x}{15015}\left[(1155\,x^7+4515\,x^6+88060\,x^5+120180\,x^4+2402304\,x^3+3363456\,x^2+34594560\,x-34594560)I_0(x)+\right.$$
$$\left.-(1155\,x^7+12600\,x^6+25060\,x^5+560480\,x^4+720864\,x^3+10570368\,x^2+24024192\,x)I_1(x)\right]$$
$$\int x^7\,\mathrm{Ei}(x)\,K_0(x)\,dx=-\,\mathrm{Ei}(x)\left[(6\,x^6+144\,x^4+1152\,x^2)K_0(x)+(x^7+36\,x^5+576\,x^3+2304\,x)K_1(x)\right]+$$
$$+\frac{e^x}{15015}\left[(1155\,x^7+4515\,x^6+88060\,x^5+120180\,x^4+2402304\,x^3+3363456\,x^2+34594560\,x-34594560)K_0(x)+\right.$$
$$\left.+\,(1155\,x^7+12600\,x^6+25060\,x^5+560480\,x^4+720864\,x^3+10570368\,x^2+24024192\,x)K_1(x)\right]$$

$$\int x^8\,\mathrm{Ei}(x)\,I_1(x)\,dx=\mathrm{Ei}(x)[(x^8+48\,x^6+1152\,x^4+9216\,x^2)I_0(x)-(8\,x^7+288\,x^5+4608\,x^3+18432\,x)I_1(x)]+$$
$$+\frac{e^x}{15015}\left[-(1001\,x^8+12474\,x^7+25830\,x^6+731920\,x^5+902640\,x^4+19312512\,x^3+26813568\,x^2+\right.$$
$$+276756480\,x-276756480)I_0(x)+(1001\,x^8+5467\,x^7+113148\,x^6+166180\,x^5+4562240\,x^4+5625792\,x^3+$$
$$\left.+84751104\,x^2+192005376\,x)I_1(x)\right]$$
$$\int x^8\,\mathrm{Ei}(x)\,K_1(x)\,dx=-\,\mathrm{Ei}(x)[(x^8+48\,x^6+1152\,x^4+9216\,x^2)K_0(x)+(8\,x^7+288\,x^5+4608\,x^3+18432\,x)K_1(x)]+$$
$$+\frac{e^x}{15015}\left[(1001\,x^8+12474\,x^7+25830\,x^6+731920\,x^5+902640\,x^4+19312512\,x^3+26813568\,x^2+\right.$$
$$+276756480\,x-276756480)K_0(x)+(1001\,x^8+5467\,x^7+113148\,x^6+166180\,x^5+4562240\,x^4+5625792\,x^3+$$
$$\left.+84751104\,x^2+192005376\,x)K_1(x)\right]$$

Recurrence Relations: Let $\mathfrak{E}_\nu^{(m)}(x)=\int x^m\,\mathrm{Ei}(x)\,I_\nu(x)\,dx$ and $\tilde{\mathfrak{E}}_\nu^{(m)}(x)=\int x^m\,\mathrm{Ei}(x)\,K_\nu(x)\,dx$.

$$\mathfrak{E}_0^{(2n+1)}(x)=\frac{1}{4n+1}\left\{x^{2n}\,\left[2n(2n+1)\,I_0(x)+(4n+1)x\,I_1(x)\right]\,\mathrm{Ei}(x)+\right.$$
$$\left.+x^{2n}\left[(x-2n-1)\,I_0(x)-xI_1(x)\right]e^x-4n^2(2n+1)\,\mathfrak{E}_0^{(2n-1)}(x)-4n(3n+1)\,\mathfrak{E}_1^{(2n)}(x)\right\}$$
$$\tilde{\mathfrak{E}}_0^{(2n+1)}(x)=\frac{1}{4n+1}\left\{x^{2n}\,\left[2n(2n+1)\,K_0(x)-(4n+1)x\,K_1(x)\right]\,\mathrm{Ei}(x)+\right.$$
$$\left.+x^{2n}\left[(x-2n-1)\,K_0(x)+xK_1(x)\right]e^x-4n^2(2n+1)\,\tilde{\mathfrak{E}}_0^{(2n-1)}(x)+4n(3n+1)\,\tilde{\mathfrak{E}}_1^{(2n)}(x)\right\}$$
$$\mathfrak{E}_1^{(2n+2)}(x)=\frac{1}{4n+3}\left\{x^{2n}\,\left[(4n+3)x^2\,I_0(x)+2n(2n+1)x\,I_1(x)\right]\,\mathrm{Ei}(x)-\right.$$
$$\left.-x^{2n+1}\left[x\,I_0(x)+(x-2n-1)\,K_1(x)\right]e^x-4n^2(2n+1)\,\mathfrak{E}_1^{(2n)}(x)-4n(3n+1)\,\mathfrak{E}_0^{(2n+1)}(x)\right\}$$
$$\tilde{\mathfrak{E}}_1^{(2n+2)}(x)=\frac{1}{4n+3}\left\{x^{2n}\,\left[(4n+3)x^2\,K_0(x)+2n(2n+1)x\,K_1(x)\right]\,\mathrm{Ei}(x)-\right.$$
$$\left.-x^{2n+1}\left[x\,K_0(x)+(x-2n-1)\,K_1(x)\right]e^x-4n^2(2n+1)\,\tilde{\mathfrak{E}}_1^{(2n)}(x)-4n(3n+1)\,\tilde{\mathfrak{E}}_0^{(2n+1)}(x)\right\}$$

1.3.3. Integrals of the type $\int x^n \operatorname{Si}(x)\cdot J_\nu(x)\,dx$ and $\int x^n \operatorname{Ci}(x)\cdot J_\nu(x)\,dx$

Let

$$\operatorname{Si}(x)=\int_0^x \frac{\sin t\,dt}{t} \quad\text{and}\quad \operatorname{Ci}(x)=C+\ln x+\int_0^x \frac{\cos t-1}{t}\,dt\,.$$

About C see page 122.

(In [4], p. 656, the function $\operatorname{ci}(x)$ is defined by some integral which fails to converge.)

$$\int x\operatorname{Si}(x)\,J_0(x)\,dx = x\operatorname{Si}(x)\,J_1(x)+\sin x\,J_0(x)-x[\sin x\,J_1(x)+\cos x\,J_0(x)]$$

$$\int x\operatorname{Ci}(x)\,J_0(x)\,dx = x\operatorname{Ci}(x)\,J_1(x)+\cos x\,J_0(x)+x[\sin x\,J_0(x)-\cos x\,J_1(x)]$$

$$\int x^2\operatorname{Si}(x)\,J_1(x)\,dx=\frac{1}{3}\,\big[-3\,x^2\operatorname{Si}(x)\,J_0(x)+6\,x\operatorname{Si}(x)\,J_1(x)+\left(x^2+6\right)\sin x\,J_0(x)-5\,x\sin x\,J_1(x)-$$
$$-6\,x\cos x\,J_0(x)-x^2\cos x\,J_1(x)\big]$$

$$\int x^2\operatorname{Ci}(x)\,J_1(x)\,dx=\frac{1}{3}\,\big[-3\,x^2\operatorname{Ci}(x)\,J_0(x)+6\,x\operatorname{Ci}(x)\,J_1(x)+6\,x\sin x\,J_0(x)+x^2\sin x\,J_1(x)+$$
$$+\left(x^2+6\right)\cos x\,J_0(x)-5\,x\cos x\,J_1(x)\big]$$

$$\int x^3\operatorname{Si}(x)\,J_0(x)\,dx=\frac{1}{15}\,\big[30\,x^2\operatorname{Si}(x)\,J_0(x)+\left(15\,x^3-60\,x\right)\operatorname{Si}(x)\,J_1(x)+\left(-7\,x^2-60\right)\sin x\,J_0(x)+$$
$$+\left(-3\,x^3+44\,x\right)\sin x\,J_1(x)+\left(-3\,x^3+60\,x\right)\cos x\,J_0(x)+16\,x^2\cos x\,J_1(x)\big]$$

$$\int x^3\operatorname{Ci}(x)\,J_0(x)\,dx=\frac{1}{15}\,\big[30\,x^2\operatorname{Ci}(x)\,J_0(x)+\left(15\,x^3-60\,x\right)\operatorname{Ci}(x)\,J_1(x)+\left(3\,x^3-60\,x\right)\sin x\,J_0(x)-$$
$$-16\,x^2\sin x\,J_1(x)+\left(-7\,x^2-60\right)\cos x\,J_0(x)+\left(-3\,x^3+44\,x\right)\cos x\,J_1(x)\big]$$

$$\int x^4\operatorname{Si}(x)\,J_1(x)\,dx=\frac{1}{105}\,\big[\left(-105\,x^4+840\,x^2\right)\operatorname{Si}(x)\,J_0(x)+\left(420\,x^3-1680\,x\right)\operatorname{Si}(x)\,J_1(x)+$$
$$+\left(15\,x^4-178\,x^2-1680\right)\sin x\,J_0(x)+\left(-57\,x^3+1196\,x\right)\sin x\,J_1(x)+\left(-102\,x^3+1680\,x\right)\cos x\,J_0(x)+$$
$$+\left(-15\,x^4+484\,x^2\right)\cos x\,J_1(x)\big]$$

$$\int x^4\operatorname{Ci}(x)\,J_1(x)\,dx=\frac{1}{105}\,\big[\left(-105\,x^4+840\,x^2\right)\operatorname{Ci}(x)\,J_0(x)+\left(420\,x^3-1680\,x\right)\operatorname{Ci}(x)\,J_1(x)+$$
$$+\left(102\,x^3-1680\,x\right)\sin x\,J_0(x)+\left(15\,x^4-484\,x^2\right)\sin x\,J_1(x)+\left(15\,x^4-178\,x^2-1680\right)\cos x\,J_0(x)+$$
$$+\left(-57\,x^3+1196\,x\right)\cos x\,J_1(x)\big]$$

$$\int x^5\operatorname{Si}(x)\,J_0(x)\,dx=\frac{1}{45}\,\big[\left(180\,x^4-1440\,x^2\right)\operatorname{Si}(x)\,J_0(x)+\left(45\,x^5-720\,x^3+2880\,x\right)\operatorname{Si}(x)\,J_1(x)+$$
$$+\left(-15\,x^4+288\,x^2+2880\right)\sin x\,J_0(x)+\left(-5\,x^5+72\,x^3-2016\,x\right)\sin x\,J_1(x)+$$
$$+\left(-5\,x^5+192\,x^3-2880\,x\right)\cos x\,J_0(x)+\left(40\,x^4-864\,x^2\right)\cos x\,J_1(x)\big]$$

$$\int x^5\,\mathrm{Ci}(x)\,J_0(x)\,dx = \frac{1}{45}\,\Big[\big(180\,x^4-1440\,x^2\big)\,\mathrm{Ci}(x)\,J_0(x)+\big(45\,x^5-720\,x^3+2880\,x\big)\,\mathrm{Ci}(x)\,J_1(x)+$$
$$+\big(5\,x^5-192\,x^3+2880\,x\big)\sin x\,J_0(x)+\big(-40\,x^4+864\,x^2\big)\sin x\,J_1(x)+$$
$$+\big(-15\,x^4+288\,x^2+2880\big)\cos x\,J_0(x)+\big(-5\,x^5+72\,x^3-2016\,x\big)\cos x\,J_1(x)\Big]$$

$$\int x^6\,\mathrm{Si}(x)\,J_1(x)\,dx = \frac{1}{3465}\,\Big[\big(-3465\,x^6+83160\,x^4-665280\,x^2\big)\,\mathrm{Si}(x)\,J_0(x)+$$
$$+\big(20790\,x^5-332640\,x^3+1330560\,x\big)\,\mathrm{Si}(x)\,J_1(x)+$$
$$+\big(315\,x^6-5430\,x^4+130656\,x^2+1330560\big)\sin x\,J_0(x)+\big(-1435\,x^5+29664\,x^3-926592\,x\big)\sin x\,J_1(x)+$$
$$+\big(-3010\,x^5+91104\,x^3-1330560\,x\big)\cos x\,J_0(x)+\big(-315\,x^6+20480\,x^4-403968\,x^2\big)\cos x\,J_1(x)\Big]$$

$$\int x^6\,\mathrm{Ci}(x)\,J_1(x)\,dx = \frac{1}{3465}\,\Big[\big(-3465\,x^6+83160\,x^4-665280\,x^2\big)\,\mathrm{Ci}(x)\,J_1(x)+$$
$$+\big(20790\,x^5-332640\,x^3+1330560\,x\big)\,\mathrm{Ci}(x)\,J_0(x)+\big(3010\,x^5-91104\,x^3+1330560\,x\big)\sin x\,J_0(x)+$$
$$+\big(315\,x^6-20480\,x^4+403968\,x^2\big)\sin x\,J_1(x)+\big(315\,x^6-5430\,x^4+130656\,x^2+1330560\big)\cos x\,J_0(x)+$$
$$+\big(-1435\,x^5+29664\,x^3-926592\,x\big)\cos x\,J_1(x)\Big]$$

Recurrence Relations: Let $\mathfrak{S}_\nu^{(m)}(x)=\int x^m\,\mathrm{Si}(x)\,J_\nu(x)\,dx$ and $\mathfrak{C}_\nu^{(m)}(x)=\int x^m\,\mathrm{Ci}(x)\,J_\nu(x)\,dx$.

$$\mathfrak{S}_0^{(2n+1)}(x)=\frac{x^{2n}}{4n+1}\,\{[(4n+1)x\,J_1(x)-2n(2n+1)\,J_0(x)]\;\mathrm{Si}(x)+$$
$$+[(2n+1)\,J_0(x)-xJ_1(x)]\sin x-xJ_0(x)\,\cos x\}+\frac{4n^2(2n+1)}{4n+1}\,\mathfrak{S}_0^{(2n-1)}(x)-\frac{4n(3n+1)}{4n+1}\,\mathfrak{S}_1^{(2n)}(x)$$

$$\mathfrak{C}_0^{(2n+1)}(x)=\frac{x^{2n}}{4n+1}\,\{[(4n+1)x\,J_1(x)-2n(2n+1)\,J_0(x)]\;\mathrm{Ci}(x)+$$
$$+[(2n+1)\,J_0(x)-xJ_1(x)]\cos x+xJ_0(x)\,\sin x\}+\frac{4n^2(2n+1)}{4n+1}\,\mathfrak{C}_0^{(2n-1)}(x)-\frac{4n(3n+1)}{4n+1}\,\mathfrak{C}_1^{(2n)}(x)$$

$$\mathfrak{S}_1^{(2n+2)}(x)=\frac{x^{2n}}{4n+3}\,\Big\{-\,\big[(4n+3)x^2\,J_0(x)+2n(2n+1)x\,J_1(x)\big]\;\mathrm{Si}(x)+$$
$$+\big[x^2J_0(x)+(2n+1)x\,J_1(x)\big]\sin x-x^2J_1(x)\,\cos x\Big\}+\frac{4n^2(2n+1)}{4n+3}\,\mathfrak{S}_1^{(2n)}(x)+\frac{2(6n^2+8n+3)}{4n+4}\,\mathfrak{S}_0^{(2n+1)}(x)$$

$$\mathfrak{C}_1^{(2n+2)}(x)=\frac{x^{2n}}{4n+3}\,\Big\{-\,\big[(4n+3)x^2\,J_0(x)+2n(2n+1)x\,J_1(x)\big]\;\mathrm{Ci}(x)+$$
$$+\big[x^2J_0(x)+(2n+1)x\,J_1(x)\big]\cos x+x^2J_1(x)\,\sin x\Big\}+\frac{4n^2(2n+1)}{4n+3}\,\mathfrak{C}_1^{(2n)}(x)+\frac{2(6n^2+8n+3)}{4n+4}\,\mathfrak{C}_0^{(2n+1)}(x)$$

1.3.4. $\int x^n \operatorname{erf}(x)\, J_\nu(\alpha x)\, dx$ and $\int x^n \operatorname{erf}(x)\, I_\nu(\alpha x)\, dx$

a) The Case $\alpha = 1,\ J_\nu(x)$

About the basic integrals

$$F_0(x) = \int_0^x e^{-t^2}\, J_0(t)\, dt \quad \text{and} \quad F_-(x) = \int_0^x \frac{e^{-t^2}\, J_1(t)\, dt}{t}$$

see page 139 and following.

$$\int \operatorname{erf}(x)\, J_1(x)\, dx = -\operatorname{erf}(x)\, J_0(x) + \frac{2}{\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx$$

$$\int x \operatorname{erf}(x)\, J_0(x)\, dx = \frac{e^{-x^2}}{\sqrt{\pi}}\, J_1(x) + x \operatorname{erf}(x)\, J_1(x) - \frac{1}{\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx + \frac{1}{\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^2 \operatorname{erf}(x)\, J_1(x)\, dx = \frac{e^{-x^2}}{2\sqrt{\pi}} \left[-2x\, J_0(x) + 5 J_1(x)\right] + \operatorname{erf}(x) \left[-x^2\, J_0(x) + 2x\, J_1(x)\right] -$$

$$-\frac{3}{2\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx + \frac{5}{2\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^3 \operatorname{erf}(x)\, J_0(x)\, dx = \frac{e^{-x^2}}{4\sqrt{\pi}} \left[10x\, J_0(x) + (4x^2 - 19)\, J_1(x)\right] + \operatorname{erf}(x) \left[2x^2\, J_0(x) + (x^3 - 4x)\, J_1(x)\right] +$$

$$+\frac{9}{4\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx - \frac{19}{4\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^4 \operatorname{erf}(x)\, J_1(x)\, dx = \frac{e^{-x^2}}{8\sqrt{\pi}} \left[(-8\, x^3 + 70\, x)\, J_0(x) + (36\, x^2 - 145)\, J_1(x)\right] +$$

$$+ \operatorname{erf}(x) \left[(-x^4 + 8\, x^2)\, J_0(x) + (4\, x^3 - 16\, x)\, J_1(x)\right] + \frac{75}{8\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx - \frac{145}{8\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^5 \operatorname{erf}(x)\, J_0(x)\, dx = \frac{e^{-x^2}}{16\sqrt{\pi}} \left[(72\, x^3 - 538\, x)\, J_0(x) + (16\, x^4 - 268\, x^2 + 1159)\, J_1(x)\right] +$$

$$+ \operatorname{erf}(x) \left[(4\, x^4 - 32\, x^2)\, J_0(x) + (x^5 - 16\, x^3 + 64\, x)\, J_1(x)\right] - \frac{621}{16\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx + \frac{1159}{16\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^6 \operatorname{erf}(x)\, J_1(x)\, dx = \frac{e^{-x^2}}{32\sqrt{\pi}} \left[(-32\, x^5 + 792\, x^3 - 6534\, x)\, J_0(x) + (208\, x^4 - 3156\, x^2 + 13977)\, J_1(x)\right] +$$

$$+ \operatorname{erf}(x) \left[(-x^6 + 24\, x^4 - 192\, x^2)\, J_0(x) + (6\, x^5 - 96\, x^3 + 384\, x)\, J_1(x)\right] -$$

$$-\frac{7443}{32\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx + \frac{13977}{32\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^7 \operatorname{erf}(x)\, J_0(x)\, dx =$$

$$= \frac{e^{-x^2}}{64\sqrt{\pi}} \left[(416\, x^5 - 9352\, x^3 + 78706\, x)\, J_0(x) + (64\, x^6 - 2352\, x^4 + 38012\, x^2 - 167803)\, J_1(x)\right] +$$

$$+ \operatorname{erf}(x) \left[(6\, x^6 - 144\, x^4 + 1152\, x^2)\, J_0(x) + (x^7 - 36\, x^5 + 576\, x^3 - 2304\, x)\, J_1(x)\right] +$$

$$+\frac{89097}{64\sqrt{\pi}} \int e^{-x^2}\, J_0(x)\, dx - \frac{167803}{64\sqrt{\pi}} \int \frac{e^{-x^2}\, J_1(x)\, dx}{x}$$

$$\int x^8 \operatorname{erf}(x)\, J_1(x)\, dx =$$

$$= \frac{e^{-x^2}}{128\sqrt{\pi}} \left[(-128\, x^7 + 6240\, x^5 - 150488\, x^3 + 1258502\, x)\, J_0(x) + (1088\, x^6 - 37264\, x^4 + 609172\, x^2 - 2683961)\, J_1(x)\right] +$$

$$+\operatorname{erf}(x)\ \left[(-x^8+48\,x^6-1152\,x^4+9216\,x^2)\,J_0(x)+(8\,x^7-288\,x^5+4608\,x^3-18432\,x)\,J_1(x)\right]+$$
$$+\frac{1425459}{128\sqrt{\pi}}\int e^{-x^2}\,J_0(x)\,dx-\frac{2683961}{128\sqrt{\pi}}\int\frac{e^{-x^2}\,J_1(x)\,dx}{x}$$
$$\int x^9\operatorname{erf}(x)\,J_0(x)\,dx=\frac{e^{-x^2}}{256\sqrt{\pi}}\ \left[(2176\,x^7-98976\,x^5+2410792\,x^3-20131066\,x)\,J_0(x)+\right.$$
$$\left.+\,(256\,x^8-16576\,x^6+597872\,x^4-9745772\,x^2+42941383)\,J_1(x)\right]+$$
$$+\operatorname{erf}(x)\ \left[(8\,x^8-384\,x^6+9216\,x^4-73728\,x^2)\,J_0(x)+(x^9-64\,x^7+2304\,x^5-36864\,x^3+147456\,x)\,J_1(x)\right]-$$
$$-\frac{22810317}{256\sqrt{\pi}}\int e^{-x^2}\,J_0(x)\,dx+\frac{42941383}{256\sqrt{\pi}}\int\frac{e^{-x^2}\,J_1(x)\,dx}{x}$$

Recurrence relations:

$$\int x^{2n+2}\operatorname{erf}(x)\,J_1(x)\,dx=\frac{4n+1}{2}\int x^{2n}\operatorname{erf}(x)\,J_1(x)\,dx+\frac{4n+5}{2}\int x^{2n+1}\operatorname{erf}(x)\,J_0(x)\,dx-$$
$$-n(2n+1)\int x^{2n-1}\operatorname{erf}(x)\,J_0(x)\,dx-\frac{x^{2n+1}\,e^{-x^2}}{\sqrt{\pi}}\,J_0(x)+\frac{x^{2n}}{2}\ \left[(2n+1-2x^2)\,J_0(x)-x\,J_1(x)\right]\ \operatorname{erf}(x)$$
$$\int x^{2n+1}\operatorname{erf}(x)\,J_0(x)\,dx=-\frac{4n+1}{2}\int x^{2n}\operatorname{erf}(x)\,J_1(x)\,dx+(2n-1)(n-1)\int x^{2n-2}\operatorname{erf}(x)\,J_1(x)\,dx+$$
$$+\frac{4n-1}{2}\int x^{2n-1}\operatorname{erf}(x)\,J_0(x)\,dx+\frac{x^{2n}\,e^{-x^2}\,J_1(x)}{\sqrt{\pi}}-\frac{x^{2n-1}}{2}\,\{[x\,J_0(x)+(2n-1-x^2)\,J_1(x)]\operatorname{erf}(x)$$

b) The General Case, $J_\nu(x)$

About the basic integrals

$$F_0(x)=\int_0^x e^{-t^2}\,J_0(\alpha t)\,dt\quad\text{and}\quad F_-(x)=\int_0^x\frac{e^{-t^2}\,J_1(\alpha t)\,dt}{t}$$

see page 149 and following.

$$\int\operatorname{erf}(x)\,J_1(\alpha x)\,dx=-\frac{\operatorname{erf}(x)\,J_0(\alpha x)}{\alpha}+\frac{2}{\alpha\sqrt{\pi}}\int e^{-x^2}\,J_0(\alpha x)\,dx$$
$$\int x\operatorname{erf}(x)\,J_0(\alpha x)\,dx=\frac{e^{-x^2}}{\sqrt{\pi}\,\alpha}\,J_1(\alpha x)+\frac{x\operatorname{erf}(x)}{\alpha}\,J_1(\alpha x)-\frac{1}{\sqrt{\pi}}\int e^{-x^2}\,J_0(\alpha x)\,dx+\frac{1}{\sqrt{\pi}\,\alpha}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$
$$\int x^2\operatorname{erf}(x)\,J_1(\alpha x)\,dx=\frac{e^{-x^2}}{2\sqrt{\pi}\,\alpha^2}\,[-2\alpha x\,J_0(\alpha x)+(\alpha^2+4)\,J_1(\alpha x)]+\frac{\operatorname{erf}(x)}{\alpha^2}\,[-\alpha x^2\,J_0(\alpha x)+2x\,J_1(\alpha x)]-$$
$$-\frac{\alpha^2+2}{2\sqrt{\pi}\,\alpha}\int e^{-x^2}\,J_0(\alpha x)\,dx+\frac{\alpha^2+4}{2\sqrt{\pi}\,\alpha^2}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$
$$\int x^3\operatorname{erf}(x)\,J_0(\alpha x)\,dx=\frac{e^{-x^2}}{4\sqrt{\pi}\,\alpha^3}\,\{2\alpha(\alpha^2+4)x\,J_0(\alpha x)+[4\alpha^2x^2-(\alpha^4+2\,\alpha^2+16)]\,J_1(\alpha x)\}+$$
$$+\frac{\operatorname{erf}(x)}{\alpha^3}\,[2\alpha x^2\,J_0(\alpha x)+(\alpha^2x^3-4x)\,J_1(\alpha x)]+\frac{\alpha^4+8}{4\sqrt{\pi}\,\alpha^2}\int e^{-x^2}\,J_0(\alpha x)\,dx-\frac{\alpha^4+2\,\alpha^2+16}{4\sqrt{\pi}\,\alpha^3}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$
$$\int x^4\operatorname{erf}(x)\,J_1(\alpha x)\,dx=$$
$$=\frac{e^{-x^2}}{8\sqrt{\pi}\,\alpha^4}\,\{[-8\alpha^3x^3+2(\alpha^5+2\,\alpha^3+32\alpha)x]\,J_0(\alpha x)+[(4\,\alpha^4+32\,\alpha^2)x^2-\alpha^6-16\,\alpha^2-128]\,J_1(\alpha x)\}+$$
$$+\frac{\operatorname{erf}(x)}{\alpha^4}\,[(-\alpha^3x^4+8\,\alpha\,x^2)\,J_0(\alpha x)+(4\,\alpha^2x^3-16\,x)\,J_1(\alpha x)]+$$

$$+\frac{\alpha^6-2\,\alpha^4+12\,\alpha^2+64}{8\sqrt{\pi}\,\alpha^3}\int e^{-x^2}\,J_0(\alpha x)\,dx-\frac{\alpha^6+16\,\alpha^2+128}{8\sqrt{\pi}\,\alpha^4}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^5\mathrm{erf}(x)\,J_0(\alpha x)\,dx=\frac{e^{-x^2}}{16\sqrt{\pi}\,\alpha^5}\,\{[(8\,\alpha^5+64\,\alpha^3)x^3-(2\,\alpha^7-8\,\alpha^5+32\,\alpha^3+512\,\alpha)x]\,J_0(\alpha x)+$$

$$+[16\alpha^4x^4-(4\,\alpha^6+8\,\alpha^4+256\,\alpha^2)x^2+\alpha^8-6\,\alpha^6+12\,\alpha^4+128\,\alpha^2+1024]\,J_1(\alpha x)\}+$$

$$+\frac{\mathrm{erf}(x)}{\alpha^5}\,[(4\,\alpha^3x^4-32\,\alpha\,x^2)\,J_0(\alpha x)+(\alpha^4x^5-16\,\alpha^2x^3+64\,x)\,J_1(\alpha x)]-$$

$$-\frac{\alpha^8-8\,\alpha^6+20\,\alpha^4+96\,\alpha^2+512}{16\sqrt{\pi}\,\alpha^4}\int e^{-x^2}\,J_0(\alpha x)\,dx+$$

$$+\frac{\alpha^8-6\,\alpha^6+12\,\alpha^4+128\,\alpha^2+1024}{16\sqrt{\pi}\,\alpha^5}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^6\,\mathrm{erf}(x)\,J_1(\alpha x)\,dx=$$

$$=\frac{e^{-x^2}}{32\sqrt{\pi}\,\alpha^6}\,\{[-32\alpha^5x^5+(8\,\alpha^7+16\,\alpha^5+768\,\alpha^3)x^3-(2\,\alpha^9-20\,\alpha^7+24\,\alpha^5+384\,\alpha^3+6144\,\alpha)x]\,J_0(\alpha x)+$$

$$+[(16\,\alpha^6+192\,\alpha^4)x^4-(4\,\alpha^8-16\,\alpha^6+96\,\alpha^4+3072\,\alpha^2)x^2+\alpha^{10}-12\,\alpha^8+20\,\alpha^6+144\,\alpha^4+1536\,\alpha^2+12288]\,J_1(\alpha x)\}+$$

$$+\frac{\mathrm{erf}(x)}{\alpha^6}\,[-(\alpha^5x^6-24\,\alpha^3x^4+192\,\alpha\,x^2)\,J_0(\alpha x)+(6\,\alpha^4x^5-96\,\alpha^2x^3+384\,x)\,J_1(\alpha x)]-$$

$$-\frac{\alpha^{10}-14\,\alpha^8+40\,\alpha^6+120\,\alpha^4+1152\,\alpha^2+6144}{32\sqrt{\pi}\,\alpha^5}\int e^{-x^2}\,J_0(\alpha x)\,dx+$$

$$+\frac{\alpha^{10}-12\,\alpha^8+20\,\alpha^6+144\,\alpha^4+1536\,\alpha^2+12288}{32\sqrt{\pi}\,\alpha^6}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^7\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx=\frac{e^{-x^2}}{64\sqrt{\pi}\,\alpha^7}\,\{[(32\,\alpha^7+384\,\alpha^5)x^5-(8\,\alpha^9-64\,\alpha^7+192\,\alpha^5+9216\,\alpha^3)x^3+$$

$$+(2\,\alpha^{11}-40\,\alpha^9+120\,\alpha^7+288\,\alpha^5+4608\,\alpha^3+73728\,\alpha)x]\,J_0(\alpha x)+[64\alpha^6x^6-(16\,\alpha^8+32\,\alpha^6+2304\,\alpha^4)x^4+$$

$$+(4\,\alpha^{10}-56\,\alpha^8+48\,\alpha^6+1152\,\alpha^4+36864\,\alpha^2)x^2-\alpha^{12}+22\,\alpha^{10}-88\,\alpha^8-120\,\alpha^6-1728\,\alpha^4-18432\,\alpha^2-147456]\,J_1(\alpha x)\}+$$

$$+\frac{\mathrm{erf}(x)}{\alpha^7}\,[(6\,\alpha^5x^6-144\,\alpha^3x^4+1152\,\alpha\,x^2)\,J_0(\alpha x)+(\alpha^6x^7-36\,\alpha^4x^5+576\,\alpha^2x^3-2304\,x)\,J_1(\alpha x)]+$$

$$+\frac{\alpha^{12}-24\,\alpha^{10}+128\,\alpha^8+1440\,\alpha^4+13824\,\alpha^2+73728}{64\sqrt{\pi}\,\alpha^6}\int e^{-x^2}\,J_0(\alpha x)\,dx-$$

$$-\frac{\alpha^{12}-22\,\alpha^{10}+88\,\alpha^8+120\,\alpha^6+1728\,\alpha^4+18432\,\alpha^2+147456}{64\sqrt{\pi}\,\alpha^7}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^8\,\mathrm{erf}(x)\,J_1(\alpha x)\,dx=$$

$$=\frac{e^{-x^2}}{128\sqrt{\pi}\,\alpha^8}\,\{[-128\alpha^7x^7+(32\,\alpha^9+64\,\alpha^7+6144\,\alpha^5)x^5-(8\,\alpha^{11}-144\,\alpha^9+96\,\alpha^7+3072\,\alpha^5+147456\,\alpha^3)x^3+$$

$$+(2\,\alpha^{13}-60\,\alpha^{11}+336\,\alpha^9+240\,\alpha^7+4608\,\alpha^5+73728\,\alpha^3+1179648\,\alpha)x]\,J_0(\alpha x)+$$

$$+[(64\,\alpha^8+1024\,\alpha^6)x^6-(16\,\alpha^{10}-128\,\alpha^8+512\,\alpha^6+36864\,\alpha^4)x^4+$$

$$+(4\,\alpha^{12}-96\,\alpha^{10}+240\,\alpha^8+768\,\alpha^6+18432\,\alpha^4+589824\,\alpha^2)x^2-$$

$$-\alpha^{14}+32\,\alpha^{12}-216\,\alpha^{10}-1920\,\alpha^6-27648\,\alpha^4-294912\,\alpha^2-2359296]\,J_1(\alpha x)\}+$$

$$+\frac{\mathrm{erf}(x)}{\alpha^8}\,[-(\alpha^7x^8-48\,\alpha^5x^6+1152\,\alpha^3x^4-9216\,\alpha\,x^2)\,J_0(\alpha x)+(8\,\alpha^6x^7-288\,\alpha^4x^5+4608\,\alpha^2x^3-18432\,x)\,J_1(\alpha x)]+$$

$$+\frac{\alpha^{14}-34\,\alpha^{12}+276\,\alpha^{10}-336\,\alpha^8+1680\,\alpha^6+23040\,\alpha^4+221184\,\alpha^2+1179648}{128\sqrt{\pi}\,\alpha^7}\int e^{-x^2}\,J_0(\alpha x)\,dx-$$

$$-\frac{\alpha^{14}-32\,\alpha^{12}+216\,\alpha^{10}+1920\,\alpha^6+27648\,\alpha^4+294912\,\alpha^2+2359296}{128\sqrt{\pi}\,\alpha^8}\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

$$\int x^9\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx=\frac{e^{-x^2}}{256\sqrt{\pi}\,\alpha^9}\,\{[(128\,\alpha^9+2048\,\alpha^7)x^7-(32\,\alpha^{11}-384\,\alpha^9+1024\,\alpha^7+98304\,\alpha^5)x^5+$$

$$+(8\,\alpha^{13}-256\,\alpha^{11}+1056\,\alpha^9+1536\,\alpha^7+49152\,\alpha^5+2359296\,\alpha^3)x^3-$$

$$-(2\,\alpha^{15}-88\,\alpha^{13}+912\,\alpha^{11}-1344\,\alpha^9+3840\,\alpha^7+73728\,\alpha^5+1179648\,\alpha^3+18874368\,\alpha)x]\,J_0(\alpha x)+$$

$$+[256\alpha^8x^8-(64\,\alpha^{10}+128\,\alpha^8+16384\,\alpha^6)x^6+(16\,\alpha^{12}-352\,\alpha^{10}+192\,\alpha^8+8192\,\alpha^6+589824\,\alpha^4)x^4-$$

$$-(4\,\alpha^{14}-152\,\alpha^{12}+1056\,\alpha^{10}+480\,\alpha^8+12288\,\alpha^6+294912\,\alpha^4+9437184\,\alpha^2)x^2+$$

$$+\alpha^{16}-46\,\alpha^{14}+532\,\alpha^{12}-1200\,\alpha^{10}+1680\,\alpha^8+30720\,\alpha^6+442368\,\alpha^4+4718592\,\alpha^2+37748736]\,J_1(\alpha x)\}+$$

$$+\frac{\mathrm{erf}(x)}{\alpha^9}\,[(8\,\alpha^7x^8-384\,\alpha^5x^6+9216\,\alpha^3x^4-73728\,\alpha\,x^2)\,J_0(\alpha x)+$$

$$+(\alpha^8x^9-64\,\alpha^6x^7+2304\,\alpha^4x^5-36864\,\alpha^2x^3+147456\,x)\,J_1(\alpha x)]-$$

$$-\frac{\alpha^{16}-48\,\alpha^{14}+620\,\alpha^{12}-2112\,\alpha^{10}+3024\,\alpha^8+26880\,\alpha^6+368640\,\alpha^4+3538944\,\alpha^2+18874368}{256\sqrt{\pi}\,\alpha^8}\,\cdot$$

$$\cdot\int e^{-x^2}\,J_0(\alpha x)\,dx+$$

$$+\frac{\alpha^{16}-46\,\alpha^{14}+532\,\alpha^{12}-1200\,\alpha^{10}+1680\,\alpha^8+30720\,\alpha^6+442368\,\alpha^4+4718592\,\alpha^2+37748736}{256\sqrt{\pi}\,\alpha^9}\,\cdot$$

$$\cdot\int\frac{e^{-x^2}\,J_1(\alpha x)}{x}\,dx$$

Recurrence relations:

$$\int x^{2n+2}\,\mathrm{erf}(x)\,J_1(\alpha x)\,dx=\frac{\alpha^2+4n+4}{2\alpha}\int x^{2n+1}\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx-\frac{n(2n+1)}{\alpha}\int x^{2n-1}\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx+$$

$$+\frac{4n+1}{2}\int x^{2n}\,\mathrm{erf}(x)\,J_1(\alpha x)\,dx-\frac{x^{2n+1}\,e^{-x^2}}{\alpha\sqrt{\pi}}\,J_0(\alpha x)+\frac{x^{2n}}{2\alpha}\,\left[(2n+1-2x^2)\,J_0(\alpha x)-\alpha x\,J_1(\alpha x)\right]\,\mathrm{erf}(x)$$

$$\int x^{2n+1}\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx=-\frac{4n+\alpha^2}{2\alpha}\int x^{2n}\,\mathrm{erf}(x)\,J_1(\alpha x)\,dx+\frac{(2n-1)(n-1)}{\alpha}\int x^{2n-2}\,\mathrm{erf}(x)\,J_1(\alpha x)\,dx+$$

$$+\frac{4n-1}{2}\int x^{2n-1}\,\mathrm{erf}(x)\,J_0(\alpha x)\,dx+\frac{x^{2n}\,e^{-x^2}\,J_1(\alpha x)}{\sqrt{\pi}\,\alpha}-\frac{x^{2n-1}\,\mathrm{erf}(x)}{2\alpha}\,\left[\alpha x\,J_0(\alpha x)+(2n-1-2x^2)\,J_1(\alpha x)\right]$$

c) The General Case, $I_\nu(x)$

About the basic integrals

$$\tilde{F}_0(x)=\int_0^x e^{-t^2}\,I_0(\alpha t)\,dt\quad\text{and}\quad\tilde{F}_-(x)=\int_0^x\frac{e^{-t^2}\,I_1(\alpha t)\,dt}{t}$$

see page 167, the following and 172.

$$\int\mathrm{erf}(x)\,I_1(\alpha x)\,dx=\frac{\mathrm{erf}(x)\,I_0(\alpha x)}{\alpha}-\frac{2}{\alpha\sqrt{\pi}}\int e^{-x^2}\,I_0(\alpha x)\,dx$$

$$\int x\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx=\frac{e^{-x^2}}{\sqrt{\pi}\,\alpha}\,I_1(\alpha x)+\frac{x\,\mathrm{erf}(x)}{\alpha}\,I_1(\alpha x)-\frac{1}{\sqrt{\pi}}\int e^{-x^2}\,I_0(\alpha x)\,dx+\frac{1}{\sqrt{\pi}\,\alpha}\int\frac{e^{-x^2}\,I_1(\alpha x)}{x}\,dx$$

$$\int x^2\,\mathrm{erf}(x)\,I_1(\alpha x)\,dx=\frac{e^{-x^2}}{2\sqrt{\pi}\,\alpha^2}\,[2\alpha x\,I_0(\alpha x)+(\alpha^2-4)\,I_1(\alpha x)]+\frac{\mathrm{erf}(x)}{\alpha^2}\,[\alpha x^2\,I_0(\alpha x)-2x\,I_1(\alpha x)]-$$

$$-\frac{\alpha^2-2}{2\sqrt{\pi}\,\alpha}\int e^{-x^2}\,I_0(\alpha x)\,dx+\frac{\alpha^2-4}{2\sqrt{\pi}\,\alpha^2}\int\frac{e^{-x^2}\,I_1(\alpha x)}{x}\,dx$$

$$\int x^3\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx=\frac{e^{-x^2}}{4\sqrt{\pi}\,\alpha^3}\left\{2\alpha(\alpha^2-4)x\,I_0(\alpha x)+[4\alpha^2x^2+\alpha^4-2\,\alpha^2+16]\,I_1(\alpha x)\right\}+$$

$$+\frac{\mathrm{erf}(x)}{\alpha^3}\left[2\alpha x^2\,I_0(\alpha x)-(\alpha^2x^3+4x)\,I_1(\alpha x)\right]-\frac{\alpha^4+8}{4\sqrt{\pi}\,\alpha^2}\int e^{-x^2}\,I_0(\alpha x)\,dx+\frac{\alpha^4-2\,\alpha^2+16}{4\sqrt{\pi}\,\alpha^3}\int\frac{e^{-x^2}\,I_1(\alpha x)}{x}\,dx$$

$$\int x^4\mathrm{erf}(x)\,I_1(\alpha x)\,dx=$$

$$=\frac{e^{-x^2}}{8\sqrt{\pi}\,\alpha^4}\left\{[8\alpha^3x^3+2(\alpha^5-2\,\alpha^3+32\alpha)x]\,I_0(\alpha x)+[(4\,\alpha^4-32\,\alpha^2)x^2+\alpha^6+16\,\alpha^2-128]\,I_1(\alpha x)\right\}+$$

$$+\frac{\mathrm{erf}(x)}{\alpha^4}\left[(\alpha^3x^4+8\,\alpha\,x^2)\,I_0(\alpha x)-(4\,\alpha^2x^3+16\,x)\,I_1(\alpha x)\right]-$$

$$-\frac{\alpha^6+2\,\alpha^4+12\,\alpha^2-64}{8\sqrt{\pi}\,\alpha^3}\int e^{-x^2}\,I_0(\alpha x)\,dx+\frac{\alpha^6+16\,\alpha^2-128}{8\sqrt{\pi}\,\alpha^4}\int\frac{e^{-x^2}\,I_1(\alpha x)}{x}\,dx$$

$$\int x^5\mathrm{erf}(x)\,I_0(\alpha x)\,dx=\frac{e^{-x^2}}{16\sqrt{\pi}\,\alpha^5}\left\{[(8\,\alpha^5-64\,\alpha^3)x^3+(2\,\alpha^7+8\,\alpha^5+32\,\alpha^3-512\,\alpha)x]\,I_0(\alpha x)+\right.$$

$$\left.+[16\alpha^4x^4+(4\,\alpha^6-8\,\alpha^4+256\,\alpha^2)x^2+\alpha^8+6\,\alpha^6+12\,\alpha^4-128\,\alpha^2+1024]\,I_1(\alpha x)\right\}+$$

$$-\frac{\mathrm{erf}(x)}{\alpha^5}\left[(4\,\alpha^3x^4+32\,\alpha\,x^2)\,I_0(\alpha x)-(\alpha^4x^5+16\,\alpha^2x^3+64\,x)\,I_1(\alpha x)\right]-$$

$$-\frac{\alpha^8+8\,\alpha^6+20\,\alpha^4-96\,\alpha^2+512}{16\sqrt{\pi}\,\alpha^4}\int e^{-x^2}\,I_0(\alpha x)\,dx+$$

$$+\frac{\alpha^8+6\,\alpha^6+12\,\alpha^4-128\,\alpha^2+1024}{16\sqrt{\pi}\,\alpha^5}\int\frac{e^{-x^2}\,I_1(\alpha x)}{x}\,dx$$

Recurrence relations:

$$\int x^{2n+2}\,\mathrm{erf}(x)\,I_1(\alpha x)\,dx=\frac{\alpha^2-4n-4}{2\alpha}\int x^{2n+1}\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx+\frac{n(2n+1)}{\alpha}\int x^{2n-1}\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx+$$

$$+\frac{4n+1}{2}\int x^{2n}\,\mathrm{erf}(x)\,I_1(\alpha x)\,dx+\frac{x^{2n+1}\,e^{-x^2}}{\alpha\sqrt{\pi}}\,I_0(\alpha x)-\frac{x^{2n}}{2\alpha}\left[(2n+1-2x^2)\,I_0(\alpha x)+\alpha x\,I_1(\alpha x)\right]\,\mathrm{erf}(x)$$

$$\int x^{2n+1}\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx=-\frac{4n-\alpha^2}{2\alpha}\int x^{2n}\,\mathrm{erf}(x)\,I_1(\alpha x)\,dx+\frac{(2n-1)(n-1)}{\alpha}\int x^{2n-2}\,\mathrm{erf}(x)\,I_1(\alpha x)\,dx+$$

$$+\frac{4n-1}{2}\int x^{2n-1}\,\mathrm{erf}(x)\,I_0(\alpha x)\,dx+\frac{x^{2n}\,e^{-x^2}\,I_1(\alpha x)}{\sqrt{\pi}\,\alpha}+\frac{x^{2n-1}\,\mathrm{erf}(x)}{2\alpha}\left[-\alpha x\,I_0(\alpha x)+(2n-1+2x^2)\,I_1(\alpha x)\right]$$

1.3.4. Struve Functions

a) Integrals

$$\int \frac{J_0(x)\,\mathbf{H}_1(x)\,dx}{x} = x[J_0(x)\mathbf{H}_0(x) + J_1(x)\mathbf{H}_1(x)] - J_0(x)\mathbf{H}_1(x) - \frac{2xJ_1(x)}{\pi}$$

$$\int \frac{I_0(x)\,\mathbf{L}_1(x)\,dx}{x} = x[I_0(x)\mathbf{L}_0(x) - I_1(x)\mathbf{L}_1(x)] - I_0(x)\mathbf{L}_1(x) - \frac{2xI_1(x)}{\pi}$$

$$\int \frac{K_0(x)\,\mathbf{L}_1(x)\,dx}{x} = x[K_0(x)\mathbf{L}_0(x) + K_1(x)\mathbf{L}_1(x)] - K_0(x)\mathbf{L}_1(x) + \frac{2xK_1(x)}{\pi}$$

$$\int \frac{J_1(x)\,\mathbf{H}_0(x)\,dx}{x} = x[J_0(x)\mathbf{H}_0(x) + J_1(x)\mathbf{H}_1(x)] - J_1(x)\mathbf{H}_0(x) - \frac{2xJ_1(x) + 2J_0(x)}{\pi}$$

$$\int \frac{I_1(x)\,\mathbf{L}_0(x)\,dx}{x} = x[I_0(x)\mathbf{L}_0(x) - I_1(x)\mathbf{L}_1(x)] - I_1(x)\mathbf{L}_0(x) + \frac{2I_0(x) - 2xI_1(x)}{\pi}$$

$$\int \frac{K_1(x)\,\mathbf{L}_0(x)\,dx}{x} = -x[K_0(x)\mathbf{L}_0(x) + K_1(x)\mathbf{L}_1(x)] - K_1(x)\mathbf{L}_0(x) - \frac{2K_0(x) + 2xK_1(x)}{\pi}$$

$$\int \frac{J_1(x)\,\mathbf{H}_1(x)\,dx}{x} = \frac{1}{2}\left\{x[J_1(x)\mathbf{H}_0(x) - J_0(x)\mathbf{H}_1(x)] - J_0(x)\mathbf{H}_0(x) - J_1(x)\mathbf{H}_1(x)\right\} + \frac{xJ_0(x)}{\pi}$$

$$\int \frac{I_1(x)\,\mathbf{L}_1(x)\,dx}{x} = \frac{1}{2}\left\{x[I_1(x)\mathbf{L}_0(x) - I_0(x)\mathbf{L}_1(x)] + I_0(x)\mathbf{L}_0(x) - I_1(x)\mathbf{L}_1(x)\right\} - \frac{xI_0(x)}{\pi}$$

$$\int \frac{K_1(x)\,\mathbf{L}_1(x)\,dx}{x} = \frac{1}{2}\left\{x[K_1(x)\mathbf{L}_0(x) + K_0(x)\mathbf{L}_1(x)] - K_0(x)\mathbf{L}_0(x) - K_1(x)\mathbf{L}_1(x)\right\} + \frac{xK_0(x)}{\pi}$$

$$\int x\,J_0(x)\,\mathbf{H}_0(x)\,dx = \frac{x^2\,[J_0(x)\mathbf{H}_0(x) + J_1(x)\mathbf{H}_1(x)] + x[J_1(x)\mathbf{H}_0(x) - J_0(x)\mathbf{H}_1(x)]}{2} - \frac{x^2J_1(x)}{\pi}$$

$$\int x\,I_0(x)\,\mathbf{L}_0(x)\,dx = \frac{x^2\,[I_0(x)\mathbf{L}_0(x) - I_1(x)\mathbf{L}_1(x)] + x[I_1(x)\mathbf{L}_0(x) - I_0(x)\mathbf{L}_1(x)]}{2} - \frac{x^2I_1(x)}{\pi}$$

$$\int x\,K_0(x)\,\mathbf{L}_0(x)\,dx = \frac{x^2\,[K_0(x)\mathbf{L}_0(x) + K_1(x)\mathbf{L}_1(x)] - x[K_1(x)\mathbf{L}_0(x) + K_0(x)lK_1(x)]}{2} + \frac{x^2K_1(x)}{\pi}$$

$$\int x\,J_1(x)\,\mathbf{H}_1(x)\,dx = \frac{x^2\,[J_0(x)\mathbf{H}_0(x) + J_1(x)\mathbf{H}_1(x)] + x[J_1(x)\mathbf{H}_0(x) - 3J_0(x)\mathbf{H}_1(x)]}{2} - \frac{x^2J_1(x)}{\pi}$$

$$\int x\,I_1(x)\,\mathbf{L}_1(x)\,dx = \frac{x^2\,[I_1(x)\mathbf{L}_1(x) - I_0(x)\mathbf{L}_0(x)] + x[3I_0(x)\mathbf{L}_1(x) - I_1(x)\mathbf{L}_0(x)]}{2} + \frac{x^2I_1(x)}{\pi}$$

$$\int x\,K_1(x)\,\mathbf{L}_1(x)\,dx = \frac{x^2\,[K_1(x)\mathbf{L}_1(x) + K_0(x)\mathbf{L}_1(x)] - x[3K_0(x)\mathbf{L}_1(x) + K_1(x)\mathbf{L}_0(x)]}{2} + \frac{x^2K_1(x)}{\pi}$$

$$\int x^2\,J_0(x)\,\mathbf{H}_1(x)\,dx = \frac{(x^3+3x)\,[J_0(x)\mathbf{H}_1(x) - J_1(x)\mathbf{H}_0(x)]}{4} + \frac{x^2\,J_1(x)\mathbf{H}_1(x)}{2} + \frac{3x^2J_1(x) - x^3J_0(x)}{2\pi}$$

$$\int x^2\,I_0(x)\,\mathbf{L}_1(x)\,dx = \frac{(x^3-3x)\,[I_0(x)\mathbf{L}_1(x) - I_1(x)\mathbf{L}_0(x)]}{4} + \frac{x^2\,I_1(x)\mathbf{L}_1(x)}{2} - \frac{3x^2I_1(x) - x^3I_0(x)}{2\pi}$$

$$\int x^2\,K_0(x)\,\mathbf{L}_1(x)\,dx = \frac{(x^3-3x)\,[K_0(x)\mathbf{L}_1(x) + K_1(x)\mathbf{L}_0(x)]}{4} - \frac{x^2\,K_1(x)\mathbf{L}_1(x)}{2} + \frac{3x^2K_1(x) + x^3K_0(x)}{2\pi}$$

$$\int x^2\,J_1(x)\,\mathbf{H}_0(x)\,dx = \frac{(x^3+3x)\,[J_1(x)\mathbf{H}_0(x) - J_0(x)\mathbf{H}_1(x)]}{4} + \frac{x^2\,J_1(x)\mathbf{H}_1(x)}{2} + \frac{x^3J_0(x) - 3x^2J_1(x)}{2\pi}$$

$$\int x^2\,I_1(x)\,\mathbf{L}_0(x)\,dx = \frac{(x^3-3x)\,[I_1(x)\mathbf{L}_0(x) - I_0(x)\mathbf{L}_1(x)]}{4} + \frac{x^2\,I_1(x)\mathbf{L}_1(x)}{2} - \frac{x^3I_0(x) - 3x^2I_1(x)}{2\pi}$$

$$\int x^2\,K_1(x)\,\mathbf{L}_0(x)\,dx = \frac{(x^3-3x)\,[K_1(x)\mathbf{L}_0(x) + K_0(x)\mathbf{L}_1(x)]}{4} + \frac{x^2\,K_1(x)\mathbf{L}_1(x)}{2} + \frac{x^3K_0(x) + 3x^2K_1(x)}{2\pi}$$

$$\int x^3\,J_0(x)\,\mathbf{H}_0(x)\,dx =$$

$$= \frac{1}{6}\left[x^4 J_0(x)\mathbf{H}_0(x) + x(x^2+6)\,J_0(x)\mathbf{H}_1(x) + x(x^2-6)\,J_1(x)\mathbf{H}_0(x) + x^2(x^2-2)\,J_1(x)\mathbf{H}_1(x)\right] -$$

$$-\frac{2x^3\,J_0(x) + x^2(x^2-6)\,J_1(x)}{3\pi}$$

$$\int x^3\,I_0(x)\,\mathbf{L}_0(x)\,dx =$$

$$= \frac{1}{6}\left[x^4 I_0(x)\mathbf{L}_0(x) + x(x^2-6)\,I_0(x)\mathbf{L}_1(x) + x(x^2+6)\,I_1(x)\mathbf{L}_0(x) - x^2(x^2+2)\,I_1(x)\mathbf{L}_1(x)\right] +$$

$$+\frac{2x^3\,I_0(x) - x^2(x^2+6)\,I_1(x)}{3\pi}$$

$$\int x^3\,K_0(x)\,\mathbf{L}_0(x)\,dx =$$

$$= \frac{1}{6}\left[x^4 K_0(x)\mathbf{L}_0(x) + x(x^2-6)\,K_0(x)\mathbf{L}_1(x) - x(x^2+6)\,K_1(x)\mathbf{L}_0(x) + x^2(x^2+2)\,K_1(x)\mathbf{L}_1(x)\right] +$$

$$+\frac{2x^3\,K_0(x) + x^2(x^2+6)\,K_1(x)}{3\pi}$$

$$\int x^3\,J_1(x)\,\mathbf{H}_1(x)\,dx =$$

$$= \frac{1}{6}\left[x^4 J_0(x)\mathbf{H}_0(x) - x(2x^2-15)\,J_0(x)\mathbf{H}_1(x) - x(2x^2+15)\,J_1(x)\mathbf{H}_0(x) + x^2(x^2+4)\,J_1(x)\mathbf{H}_1(x)\right] -$$

$$-\frac{5x^3\,J_0(x) + x^2(x^2-15)\,J_1(x)}{3\pi}$$

$$\int x^3\,I_1(x)\,\mathbf{L}_1(x)\,dx =$$

$$= \frac{1}{6}\left[-x^4 I_0(x)\mathbf{L}_0(x) + x(2x^2+15)\,I_0(x)\mathbf{L}_1(x) + x(2x^2-15)\,I_1(x)\mathbf{L}_0(x) + x^2(x^2-4)\,I_1(x)\mathbf{L}_1(x)\right] +$$

$$+\frac{x^2(x^2+15)\,I_1(x) - 5x^3\,I_0(x)}{3\pi}$$

$$\int x^3\,K_1(x)\,\mathbf{L}_1(x)\,dx =$$

$$= \frac{1}{6}\left[x^4 K_0(x)\mathbf{L}_0(x) - x(2x^2+15)\,K_0(x)\mathbf{L}_1(x) + x(2x^2-15)\,K_1(x)\mathbf{L}_0(x) + x^2(x^2-4)\,K_1(x)\mathbf{L}_1(x)\right] +$$

$$+\frac{x^2(x^2+15)\,K_1(x) + 5x^3\,K_0(x)}{3\pi}$$

$$\int x^4\,J_0(x)\,\mathbf{H}_1(x)\,dx =$$

$$= \frac{1}{24}\left[-4x^4\,J_0(x)\mathbf{H}_0(x) + x(3x^4+8x^2-195)\,J_0(x)\mathbf{H}_1(x) - x(3x^4-8x^2-195)\,J_1(x)\mathbf{H}_0(x) +\right.$$

$$\left. +8x^2(x^2-2)\,J_1(x)\mathbf{H}_1(x)\right] + \frac{x^3(65-3x^2)\,J_0(x) + x^2(19x^2-195)\,J_1(x)}{12\pi}$$

$$\int x^4\,I_0(x)\,\mathbf{L}_1(x)\,dx =$$

$$= \frac{1}{24}\left[4x^4\,I_0(x)\mathbf{L}_0(x) + x(3x^4-8x^2-195)\,I_0(x)\mathbf{L}_1(x) - x(3x^4+8x^2-195)\,I_1(x)\mathbf{L}_0(x) +\right.$$

$$\left. +8x^2(x^2+2)\,I_1(x)\mathbf{L}_1(x)\right] + \frac{x^3(3x^2+65)\,I_0(x) - x^2(19x^2+195)\,I_1(x)}{12\pi}$$

$$\int x^4\,K_0(x)\,\mathbf{L}_1(x)\,dx =$$

$$= \frac{1}{24}\left[4x^4\,K_0(x)\mathbf{L}_0(x) + x(3x^4-8x^2-195)\,K_0(x)\mathbf{L}_1(x) + x(3x^4+8x^2-195)\,K_1(x)\mathbf{L}_0(x) -\right.$$

$$-8x^2(x^2+2)\,K_1(x)\mathbf{L}_1(x)\big] + \frac{x^3(3x^2+65)\,K_0(x) + x^2(19x^2+195)\,K_1(x)}{12\pi}$$

$$\int x^4\,J_1(x)\,\mathbf{H}_0(x)\,dx =$$

$$= \frac{1}{24}\,\big[-4x^4\,J_0(x)\mathbf{H}_0(x) - x(3x^4-8x^2-75)\,J_0(x)\mathbf{H}_1(x) + x(3x^4+8x^2-75)\,J_1(x)\mathbf{H}_0(x) +$$

$$+8x^2(x^2-2)\,J_1(x)\mathbf{H}_1(x)\big] + \frac{x^3(3x^2-25)\,J_0(x) - x^2(11x^2-75)\,J_1(x)}{12\pi}$$

$$\int x^4\,I_1(x)\,\mathbf{L}_0(x)\,dx =$$

$$= \frac{1}{24}\,\big[4x^4\,I_0(x)\mathbf{L}_0(x) - x(3x^4+8x^2-75)\,I_0(x)\mathbf{L}_1(x) + x(3x^4-8x^2-75)\,I_1(x)\mathbf{L}_0(x) +$$

$$+8x^2(x^2+2)\,I_1(x)\mathbf{L}_1(x)\big] - \frac{x^3(3x^2+25)\,I_0(x) - x^2(11x^2+75)\,I_1(x)}{12\pi}$$

$$\int x^4\,K_1(x)\,\mathbf{L}_0(x)\,dx =$$

$$= \frac{1}{24}\,\big[-4x^4\,K_0(x)\mathbf{L}_0(x) + x(3x^4+8x^2-75)\,K_0(x)\mathbf{L}_1(x) + x(3x^4-8x^2-75)\,K_1(x)\mathbf{L}_0(x) +$$

$$+8x^2(x^2+2)\,K_1(x)\mathbf{L}_1(x)\big] + \frac{x^3(3x^2+25)\,K_0(x) + x^2(11x^2+75)\,K_1(x)}{12\pi}$$

b) Recurrence relations:

I) Functions of the First Kind, $J_\nu(x)$:

$$\int x^{2n+1}\,J_0(x)\,\mathbf{H}_0(x)\,dx =$$

$$= \frac{x^{2n+1}\,[xJ_0(x)\mathbf{H}_0(x) + 2n\,J_0(x)\mathbf{H}_1(x) + x\,J_1(x)\mathbf{H}_1(x)]}{2(2n+1)} -$$

$$-\frac{2n^2}{2n+1}\int x^{2n}\,J_0(x)\mathbf{H}_1(x)\,dx - \frac{1}{(2n+1)\pi}\int x^{2n+2}\,J_0(x)\,dx$$

$$\int x^{2n+1}\,J_1(x)\,\mathbf{H}_1(x)\,dx =$$

$$= \frac{x^{2n+1}\,[x\,J_0(x)\mathbf{H}_0(x) - 2(n+1)\,J_0(x)\mathbf{H}_1(x) + x\,J_1(x)\mathbf{H}_1(x)]}{2(2n+1)} +$$

$$+\frac{2n(n+1)}{2n+1}\int x^{2n}\,J_0(x)\mathbf{H}_1(x)\,dx - \frac{1}{(2n+1)\pi}\int x^{2n+2}\,J_0(x)\,dx$$

$$\int x^{2n+2}\,J_0(x)\,\mathbf{H}_1(x)\,dx =$$

$$= \frac{x^{2n+2}\,[x\,J_0(x)\mathbf{H}_1(x) - x\,J_1(x)\mathbf{H}_0(x) + 2(n+1)\,J_1(x)\mathbf{H}_1(x)]}{4(n+1)} -$$

$$-n\int x^{2n+1}\,J_1(x)\mathbf{H}_1(x)\,dx + \frac{1}{2(n+1)\pi}\int x^{2n+3}\,J_1(x)\,dx$$

$$\int x^{2n+2}\,J_1(x)\,\mathbf{H}_0(x)\,dx =$$

$$= \frac{x^{2n+2}\,[x\,J_1(x)\mathbf{H}_0(x) - x\,J_0(x)\mathbf{H}_1(x) + 2(n+1)\,J_1(x)\mathbf{H}_1(x)]}{4(n+1)} -$$

$$-n\int x^{2n+1}\,J_1(x)\mathbf{H}_1(x)\,dx - \frac{1}{2(n+1)\pi}\int x^{2n+3}\,J_1(x)\,dx$$

II) Functions of the Second Kind, $I_\nu(x)$:

$$\int x^{2n+1}\, I_0(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{x^{2n+1}\left[xI_0(x)\mathbf{L}_0(x) + 2n\, I_0(x)\mathbf{L}_1(x) - x\, I_1(x)\mathbf{L}_1(x)\right]}{2(2n+1)} -$$

$$-\frac{2n^2}{2n+1}\int x^{2n}\, I_0(x)\mathbf{L}_1(x)\, dx - \frac{1}{(2n+1)\pi}\int x^{2n+2}\, I_0(x)\, dx$$

$$\int x^{2n+1}\, I_1(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{x^{2n+1}\left[-x\, I_0(x)\mathbf{L}_0(x) + 2(n+1)\, I_0(x)\mathbf{L}_1(x) + x\, I_1(x)\mathbf{L}_1(x)\right]}{2(2n+1)} -$$

$$-\frac{2n(n+1)}{2n+1}\int x^{2n}\, I_0(x)\mathbf{L}_1(x)\, dx + \frac{1}{(2n+1)\pi}\int x^{2n+2}\, I_0(x)\, dx$$

$$\int x^{2n+2}\, I_0(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{x^{2n+2}\left[x\, I_0(x)\mathbf{L}_1(x) - x\, I_1(x)\mathbf{L}_0(x) + 2(n+1)\, I_1(x)\mathbf{L}_1(x)\right]}{4(n+1)} -$$

$$-n\int x^{2n+1}\, I_1(x)\mathbf{L}_1(x)\, dx + \frac{1}{2(n+1)\pi}\int x^{2n+3}\, I_1(x)\, dx$$

$$\int x^{2n+2}\, I_1(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{x^{2n+2}\left[x\, I_1(x)\mathbf{L}_0(x) - x\, I_0(x)\mathbf{L}_1(x) + 2(n+1)\, I_1(x)\mathbf{L}_1(x)\right]}{4(n+1)} -$$

$$-n\int x^{2n+1}\, I_1(x)\mathbf{H}_1(x)\, dx - \frac{1}{2(n+1)\pi}\int x^{2n+3}\, I_1(x)\, dx$$

III) Functions of the Second Kind, $K_\nu(x)$:

$$\int x^{2n+1}\, K_0(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{x^{2n+1}\left[xK_0(x)\mathbf{L}_0(x) + 2n\, K_0(x)\mathbf{L}_1(x) + x\, K_1(x)\mathbf{L}_1(x)\right]}{2(2n+1)} -$$

$$-\frac{2n^2}{2n+1}\int x^{2n}\, K_0(x)\mathbf{L}_1(x)\, dx - \frac{1}{(2n+1)\pi}\int x^{2n+2}\, K_0(x)\, dx$$

$$\int x^{2n+1}\, K_1(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{x^{2n+1}\left[x\, K_0(x)\mathbf{L}_0(x) - 2(n+1)\, K_0(x)\mathbf{L}_1(x) + x\, K_1(x)\mathbf{L}_1(x)\right]}{2(2n+1)} +$$

$$+\frac{2n(n+1)}{2n+1}\int x^{2n}\, K_0(x)\mathbf{L}_1(x)\, dx - \frac{1}{(2n+1)\pi}\int x^{2n+2}\, \mathbf{K}_0(x)\, dx$$

$$\int x^{2n+2}\, K_0(x)\, \mathbf{L}_1(x)\, dx =$$

$$= \frac{x^{2n+2}\left[x\, K_0(x)\mathbf{L}_1(x) + x\, K_1(x)\mathbf{L}_0(x) - 2(n+1)\, K_1(x)\mathbf{L}_1(x)\right]}{4(2n+1)} +$$

$$+n\int x^{2n+1}\, K_1(x)\mathbf{L}_1(x)\, dx - \frac{1}{2(n+1)\pi}\int x^{2n+3}\, K_1(x)\, dx$$

$$\int x^{2n+2}\, K_1(x)\, \mathbf{L}_0(x)\, dx =$$

$$= \frac{x^{2n+2}\left[x\,K_0(x)\mathbf{L}_1(x) + x\,K_1(x)\mathbf{L}_0(x) + 2(n+1)\,K_1(x)\mathbf{L}_1(x)\right]}{4(2n+1)} -$$

$$-n\int x^{2n+1}\,K_1(x)\mathbf{L}_1(x)\,dx - \frac{1}{2(n+1)\pi}\int x^{2n+3}\,K_1(x)\,dx$$

c) Some Special Cases:

$$\int x^2\,J_0(x)\,\mathbf{H}_1(\sqrt{3}\,x)\,dx =$$

$$= \frac{-\sqrt{3}\,x^2\,J_0(x)\mathbf{H}_0(\sqrt{3}\,x) + 3x\,J_0(x)\mathbf{H}_1(\sqrt{3}\,x) - \sqrt{3}\,x\,J_1(x)\mathbf{H}_0(\sqrt{3}\,x) - x^2\,J_1(x)\mathbf{H}_1(\sqrt{3}\,x)}{2} + \frac{3x^2\,J_1(x)}{\pi}$$

$$\int x^2\,I_0(x)\,\mathbf{L}_1(\sqrt{3}\,x)\,dx =$$

$$= \frac{\sqrt{3}\,x^2\,I_0(x)\mathbf{L}_0(\sqrt{3}\,x) - 3x\,I_0(x)\mathbf{L}_1(\sqrt{3}\,x) + \sqrt{3}\,x\,I_1(x)\mathbf{L}_0(\sqrt{3}\,x) - x^2\,I_1(x)\mathbf{L}_1(\sqrt{3}\,x)}{2} - \frac{3x^2\,I_1(x)}{\pi}$$

$$\int x^2\,K_0(x)\,\mathbf{L}_1(\sqrt{3}\,x)\,dx =$$

$$= \frac{\sqrt{3}\,x^2\,K_0(x)\mathbf{L}_0(\sqrt{3}\,x) - 3x\,K_0(x)\mathbf{L}_1(\sqrt{3}\,x) - \sqrt{3}\,x\,K_1(x)\mathbf{L}_0(\sqrt{3}\,x) + x^2\,K_1(x)\mathbf{L}_1(\sqrt{3}\,x)}{2} + \frac{3x^2\,K_1(x)}{\pi}$$

$$\int x^2\,J_1(x)\,\mathbf{H}_0(\sqrt{3}\,x)\,dx =$$

$$= \frac{x^2\,J_0(x)\mathbf{H}_0(\sqrt{3}\,x) - \sqrt{3}\,x\,J_0(x)\mathbf{H}_1(\sqrt{3}\,x) + x\,J_1(x)\mathbf{H}_0(\sqrt{3}\,x) + \sqrt{3}\,x^2\,J_1(x)\mathbf{H}_1(\sqrt{3}\,x)}{2} - \frac{\sqrt{3}\,x^2\,J_1(x)}{\pi}$$

$$\int x^2\,I_1(x)\,\mathbf{L}_0(\sqrt{3}\,x)\,dx =$$

$$= \frac{-x^2\,I_0(x)\mathbf{L}_0(\sqrt{3}\,x) + \sqrt{3}\,x\,I_0(x)\mathbf{L}_1(\sqrt{3}\,x) - x\,I_1(x)\mathbf{L}_0(\sqrt{3}\,x) + \sqrt{3}\,x^2\,I_1(x)\mathbf{L}_1(\sqrt{3}\,x)}{2} + \frac{\sqrt{3}\,x^2\,I_1(x)}{\pi}$$

$$\int x^2\,K_1(x)\,\mathbf{L}_0(\sqrt{3}\,x)\,dx =$$

$$= \frac{x^2\,K_0(x)\mathbf{L}_0(\sqrt{3}\,x) - \sqrt{3}\,x\,K_0(x)\mathbf{L}_1(\sqrt{3}\,x) - x\,I_1(x)\mathbf{L}_0(\sqrt{3}\,x) + \sqrt{3}\,x^2\,K_1(x)\mathbf{L}_1(\sqrt{3}\,x)}{2} + \frac{\sqrt{3}\,x^2\,K_1(x)}{\pi}$$

$$\int x^3\,J_0(x)\,\mathbf{H}_0((\sqrt{2}\,+i)x)\,dx =$$

$$= -\frac{(1+\sqrt{2}\,i)x^2}{2}\,J_0(x)\,\mathbf{H}_0((\sqrt{2}\,+i)x) + \frac{x\left[(\sqrt{2}\,-2i)x^2+3\sqrt{2}\right]}{4}\,J_0(x)\,\mathbf{H}_1((\sqrt{2}\,+i)x) +$$

$$+\frac{x\left[\sqrt{2}\,ix^2-2+\sqrt{2}\,i\right]}{4}\,J_1(x)\,\mathbf{H}_0((\sqrt{2}\,+i)x) - \frac{(\sqrt{2}+i)\,x^2}{2}\,J_1(x)\,\mathbf{H}_1((\sqrt{2}\,+i)x) -$$

$$-\frac{(\sqrt{2}\,-2i)\,x^3\,J_0(x)}{2\pi} + \frac{3\sqrt{2}\,x^2\,J_1(x)}{2\pi}$$

$$\int x^3\,I_0(x)\,\mathbf{L}_0((\sqrt{2}\,+i)x)\,dx =$$

$$= \frac{(1+\sqrt{2}\,i)x^2}{2}\,I_0(x)\,\mathbf{L}_0((\sqrt{2}\,+i)x) + \frac{x\left[(\sqrt{2}-2i)x^2-3\sqrt{2}\right]}{4}\,I_0(x)\,\mathbf{L}_1((\sqrt{2}\,+i)x) +$$

$$+\frac{x\left[\sqrt{2}\,ix^2+(2-\sqrt{2}\,i)\right]}{4}\,I_1(x)\,\mathbf{L}_0((\sqrt{2}\,+i)x) - \frac{(\sqrt{2}+i)\,x^2}{2}\,I_1(x)\,\mathbf{L}_1((\sqrt{2}\,+i)x) +$$

$$+\frac{(\sqrt{2}\,-2i)\,x^3\,I_0(x)}{2\pi} - \frac{3\sqrt{2}\,x^2\,I_1(x)}{2\pi}$$

$$\int x^3\,K_0(x)\,\mathbf{L}_0((\sqrt{2}\,+i)x)\,dx =$$

$$= \frac{(1+\sqrt{2}\,i)x^2}{2}\,K_0(x)\,\mathbf{L}_0((\sqrt{2}\,+i)x) + \frac{x\,[(\sqrt{2}-2i)x^2-3\sqrt{2}]}{4}\,K_0(x)\,\mathbf{L}_1((\sqrt{2}\,+i)x) -$$

$$- \frac{x\,[\sqrt{2}\,ix^2+(2-\sqrt{2}\,i)]}{4}\,K_1(x)\,\mathbf{L}_0((\sqrt{2}\,+i)x) + \frac{(\sqrt{2}+i)\,x^2}{2}\,K_1(x)\,\mathbf{L}_1((\sqrt{2}\,+i)x) +$$

$$+ \frac{(\sqrt{2}\,-2i)\,x^3\,K_0(x)}{2\pi} + \frac{3\sqrt{2}\,x^2\,K_1(x)}{2\pi}$$

$$\int x^3\,J_0(x)\,\mathbf{H}_0((\sqrt{2}\,-i)x)\,dx =$$

$$= -\frac{(1-\sqrt{2}\,i)x^2}{2}\,J_0(x)\,\mathbf{H}_0((\sqrt{2}\,-i)x) + \frac{x\,[(2+\sqrt{2}\,i)x^2+3\sqrt{2}]}{4}\,J_0(x)\,\mathbf{H}_1((\sqrt{2}\,-i)x) -$$

$$- \frac{x\,[\sqrt{2}\,ix^2+(2+\sqrt{2}\,i)]}{4}\,J_1(x)\,\mathbf{H}_0((\sqrt{2}\,-i)x) - \frac{(\sqrt{2}-i)\,x^2}{2}\,J_1(x)\,\mathbf{H}_1((\sqrt{2}\,-i)x) -$$

$$- \frac{(\sqrt{2}\,+2i)\,x^3\,J_0(x)}{2\pi} + \frac{3\sqrt{2}\,x^2\,J_1(x)}{2\pi}$$

$$\int x^3\,I_0(x)\,\mathbf{L}_0((\sqrt{2}\,-i)x)\,dx =$$

$$= \frac{(1-\sqrt{2}\,i)x^2}{2}\,I_0(x)\,\mathbf{L}_0((\sqrt{2}\,-i)x) + \frac{x\,[(\sqrt{2}+2i)x^2-3\sqrt{2}]}{4}\,I_0(x)\,\mathbf{L}_1((\sqrt{2}\,-i)x) +$$

$$+ \frac{x\,[-\sqrt{2}\,ix^2+(2+\sqrt{2}\,i)]}{4}\,I_1(x)\,\mathbf{L}_0((\sqrt{2}\,-i)x) - \frac{(\sqrt{2}-i)\,x^2}{2}\,I_1(x)\,\mathbf{L}_1((\sqrt{2}\,-i)x) +$$

$$+ \frac{(\sqrt{2}\,+2i)\,x^3\,I_0(x)}{2\pi} - \frac{3\sqrt{2}\,x^2\,I_1(x)}{2\pi}$$

$$\int x^3\,K_0(x)\,\mathbf{L}_0((\sqrt{2}\,-i)x)\,dx =$$

$$= \frac{(1-\sqrt{2}\,i)x^2}{2}\,K_0(x)\,\mathbf{L}_0((\sqrt{2}\,-i)x) + \frac{x\,[(\sqrt{2}+2i)x^2-3\sqrt{2}]}{4}\,K_0(x)\,\mathbf{L}_1((\sqrt{2}\,-i)x) +$$

$$- \frac{x\,[\sqrt{2}\,ix^2-(2+\sqrt{2}\,i)]}{4}\,K_1(x)\,\mathbf{L}_0((\sqrt{2}\,-i)x) + \frac{(\sqrt{2}-i)\,x^2}{2}\,K_1(x)\,\mathbf{L}_1((\sqrt{2}\,-i)x) +$$

$$+ \frac{(\sqrt{2}\,+2i)\,x^3\,K_0(x)}{2\pi} + \frac{3\sqrt{2}\,x^2\,K_1(x)}{2\pi}$$

Let $\lambda = (\lambda_r + \lambda_i\,i)/6 = [\sqrt{18\sqrt{5}\,+30}\, + \sqrt{18\sqrt{5}\,-30}\,i]/6 = 1.39691 + 0.53357\,i$, then holds

$$\int x^3\,J_1(x)\,\mathbf{H}_1(\lambda x)\,dx =$$

$$= \frac{([\sqrt{5}-3)\lambda_r - (\sqrt{5}+3)\lambda_i\,i]\,x^3}{8}\,J_0(x)\,\mathbf{H}_0(\lambda x) + \frac{(1-\sqrt{5}\,i)x^3+3(5+\sqrt{5}\,i)x}{4}\,J_0(x)\,\mathbf{H}_1(\lambda x) +$$

$$+ \frac{[(1-\sqrt{5})\lambda_r+(1+\sqrt{5})\lambda_i\,i]x^3 - [(3+\sqrt{5})\lambda_r-(3-\sqrt{5})\lambda_i\,i]x}{16}\,J_1(x)\,\mathbf{H}_0(\lambda x) -$$

$$- \frac{(1+2\sqrt{5}\,i)x^2}{2}\,J_1(x)\,\mathbf{H}_1(\lambda x) - \frac{(5-\sqrt{5}\,i)x^3}{2\pi}\,J_0(x) + \frac{3(5+\sqrt{5}\,i)x^2}{2\pi}\,J_1(x)$$

$$\int x^3\,I_1(x)\,\mathbf{L}_1(\lambda x)\,dx =$$

$$= \frac{(\sqrt{5}-3)\lambda_r - (\sqrt{5}+3)\lambda_i\,i}{8}\,x^2\,I_0(x)\,\mathbf{L}_0(\lambda x) + \frac{(-1+\sqrt{5}\,i)x^3+3(5+\sqrt{5}\,i)x}{4}\,I_0(x)\,\mathbf{L}_1(\lambda x) +$$

$$+ \frac{[(\sqrt{5}\,-1)\lambda_r-(1+\sqrt{5})\lambda_i\,i]x^3 - [(3+\sqrt{5})\lambda_r+(3-\sqrt{5})\lambda_i\,i]x}{16}\,I_1(x)\,\mathbf{L}_0(\lambda x) +$$

$$+ \frac{1+2\sqrt{5}\,i}{2}\,x^2\,I_1(x)\,\mathbf{L}_1(\lambda x) - \frac{5-\sqrt{5}\,i}{2\pi}\,I_0(x) + \frac{3(5+\sqrt{5}\,i)}{2\pi}\,I_1(x)$$

$$\int x^3\, K_1(x)\, \mathbf{L}_1(\lambda x)\, dx =$$

$$= \frac{(3-\sqrt{5})\lambda_r + (\sqrt{5}+3)\lambda_i\, i}{8}\, x^2\, K_0(x)\, \mathbf{L}_0(\lambda x) + \frac{(1-\sqrt{5}\, i)x^3 - 3(5+\sqrt{5}\, i)x}{4}\, K_0(x)\, \mathbf{L}_1(\lambda x) -$$

$$-\frac{[(1-\sqrt{5})\lambda_r + (1+\sqrt{5})\lambda_i\, i]x^3 + [(3+\sqrt{5})\lambda_r + (3-\sqrt{5})\lambda_i\, i]x}{16}\, K_1(x)\, \mathbf{L}_0(\lambda x) +$$

$$+\frac{1+2\sqrt{5}\, i}{2}\, x^2\, K_1(x)\, \mathbf{L}_1(\lambda x) + \frac{5-\sqrt{5}\, i}{2\pi}\, K_0(x) + \frac{3(5+\sqrt{5}\, i)}{2\pi}\, K_1(x)$$

With $\mu = \bar{\lambda} = (\lambda_r - \lambda_i\, i)/6 = [\sqrt{18\sqrt{5}+30} - \sqrt{18\sqrt{5}-30}\, i]/6$ holds

$$\int x^3\, J_1(x)\, \mathbf{H}_1(\mu x)\, dx =$$

$$= \frac{(\sqrt{5}-3)\lambda_r + (\sqrt{5}+3)\lambda_i\, i}{8}\, x^2\, J_0(x)\, \mathbf{H}_0(\mu x) + \frac{(1+\sqrt{5}\, i)x^3 + 3(5-\sqrt{5}\, i)x}{4}\, J_0(x)\, \mathbf{H}_1(\mu x) -$$

$$-\frac{[(\sqrt{5}-1)\lambda_r + (1+\sqrt{5}\, i)\lambda_i\, i]x^3 + [(3+\sqrt{5})\lambda_r + (\sqrt{5}-3)\lambda_i\, i]x}{16}\, J_1(x)\, \mathbf{H}_0(\mu x) -$$

$$-\frac{1-2\sqrt{5}\, i}{2}\, x^2\, J_1(x)\, \mathbf{H}_1(\lambda x) - \frac{5+\sqrt{5}\, i}{2\pi}\, x^2\, J_0(x) + \frac{3(5-\sqrt{5}\, i)}{2\pi}\, x^3\, J_1(x)$$

$$\int x^3\, I_1(x)\, \mathbf{L}_1(\mu x)\, dx =$$

$$= \frac{(\sqrt{5}-3)\lambda_r + (\sqrt{5}+3)\lambda_i\, i}{8}\, x^2\, I_0(x)\, \mathbf{L}_0(\mu x) - \frac{(1+\sqrt{5}\, i)x^3 - 3(5-\sqrt{5}\, i)x}{4}\, I_0(x)\, \mathbf{L}_1(\mu x) +$$

$$+\frac{[(\sqrt{5}-1)\lambda_r + (\sqrt{5}+1)\lambda_i\, i]x^3 - [(3+\sqrt{5})\lambda_r - (3-\sqrt{5})\lambda_i\, i]x}{16}\, I_1(x)\, \mathbf{L}_0(\mu x) +$$

$$+\frac{1-2\sqrt{5}\, i}{2}\, x^2\, I_1(x)\, \mathbf{L}_1(\mu x) - \frac{5+\sqrt{5}\, i}{2\pi}\, x^2\, I_0(x) + \frac{3(5-\sqrt{5}\, i)}{2\pi}\, x^3\, I_1(x)$$

$$\int x^3\, K_1(x)\, \mathbf{L}_1(\mu x)\, dx =$$

$$= \frac{(3-\sqrt{5})\lambda_r - (\sqrt{5}+3)\lambda_i\, i}{8}\, x^2\, K_0(x)\, \mathbf{L}_0(\mu x) + \frac{(1+\sqrt{5}\, i)x^3 - 3(5-\sqrt{5}\, i)x}{4}\, K_0(x)\, \mathbf{L}_1(\mu x) +$$

$$+\frac{[(\sqrt{5}-1)\lambda_r + (1+\sqrt{5})\lambda_i\, i]x^3 - [(3+\sqrt{5})\lambda_r - (3-\sqrt{5})\lambda_i\, i]x}{16}\, K_1(x)\, \mathbf{L}_0(\mu x) +$$

$$+\frac{1-2\sqrt{5}\, i}{2}\, x^2\, I_1(x)\, \mathbf{L}_1(\mu x) + \frac{5+\sqrt{5}\, i}{2\pi}\, x^2\, K_0(x) + \frac{3(5-\sqrt{5}\, i)}{2\pi}\, x^3\, K_1(x)$$

$$\int x^4\, J_0(x)\, \mathbf{H}_1(\sqrt{3}\, x)\, dx =$$

$$= \frac{\sqrt{3}\, x^2(10-x^2)}{2}\, J_0(x)\, \mathbf{H}_0(\sqrt{3}\, x) + \frac{x(7x^2-30)}{2}\, J_0(x)\, \mathbf{H}_1(\sqrt{3}\, x) +$$

$$+\frac{\sqrt{3}\, x(10-3x^2)}{2}\, J_1(x)\, \mathbf{H}_0(\sqrt{3}\, x) + \frac{x^2(16-x^2)}{2}\, J_1(x)\, \mathbf{H}_1(\sqrt{3}\, x) + \frac{3x^2(x^2-10)}{\pi}\, J_1(x)$$

$$\int x^4\, I_0(x)\, \mathbf{L}_1(\sqrt{3}\, x)\, dx =$$

$$= \frac{\sqrt{3}\, x^2(x^2+10)}{2}\, I_0(x)\, \mathbf{L}_0(\sqrt{3}\, x) - \frac{x(7x^2+30)}{2}\, I_0(x)\, \mathbf{L}_1(\sqrt{3}\, x) +$$

$$+\frac{\sqrt{3}\, x(3x^2+10)}{2}\, I_1(x)\, \mathbf{L}_0(\sqrt{3}\, x) - \frac{x^2(x^2+16)}{2}\, I_1(x)\, \mathbf{L}_1(\sqrt{3}\, x) - \frac{3x^2(x^2+10)}{\pi}\, I_1(x)$$

$$\int x^4\, K_0(x)\, \mathbf{L}_1(\sqrt{3}\, x)\, dx =$$

$$= \frac{\sqrt{3}\,(x^2+10)}{2}\,x^2\,K_0(x)\,\mathbf{L}_0(\sqrt{3}\,x) - \frac{x(7x^2+30)}{2}\,K_0(x)\,\mathbf{L}_1(\sqrt{3}\,x) -$$

$$-\frac{\sqrt{3}\,x(3x^2+10)}{2}\,K_1(x)\,\mathbf{L}_0(\sqrt{3}\,x) + \frac{x^2(x^2+16)}{2}\,K_1(x)\,\mathbf{L}_1(\sqrt{3}\,x) + \frac{3x^2(x^2+10)}{\pi}\,K_1(x)$$

Let $\nu = \sqrt{5}\,(\sqrt{2}\,+i)/3 = 1.05409\,25534 + 0.74535\,59925\,i$, then holds

$$\int x^4\,J_0(x)\,\mathbf{H}_1(\nu x)\,dx =$$

$$= -\frac{x^2[(\sqrt{10}\,-4\sqrt{5}\,i)x^2+9\sqrt{5}\,(\sqrt{2}\,-4i)]}{12}\,J_0(x)\,\mathbf{H}_0(\nu x) - \frac{x[(31+10\sqrt{2}\,i)x^2-15(7+4\sqrt{2}\,i]}{12}\,J_0(x)\,\mathbf{H}_1(\nu x) +$$

$$+\frac{x[(11\sqrt{10}\,+\sqrt{5}\,i)x^2+33\sqrt{10}\,+3\sqrt{5}\,i]}{12}\,J_1(x)\,\mathbf{H}_0(\nu x) + \frac{x^2[(2+5\sqrt{2}\,i)x^2-28+65\sqrt{2}\,i]}{12}\,J_1(x)\,\mathbf{H}_1(\nu x) +$$

$$+\frac{5(13+\sqrt{2}\,i)}{6\pi}\,J_0(x) + \frac{5x^2[(2-\sqrt{2}\,i)x^2-3(7+4\sqrt{2}\,i]}{6\pi}\,J_1(x)$$

$$\int x^4\,I_0(x)\,\mathbf{L}_1(\nu x)\,dx =$$

$$= \frac{x^2[(\sqrt{10}\,-4\sqrt{5}\,i)x^2-9\sqrt{10}\,+36\sqrt{5}\,i]}{12}\,I_0(x)\,\mathbf{L}_0(\nu x) + \frac{x[(31+10\sqrt{2}\,i)x^2-105-60\sqrt{2}\,i]}{12}\,I_0(x)\,\mathbf{L}_1(\nu x) -$$

$$-\frac{x[(11\sqrt{10}\,+\sqrt{5}\,i)x^2-33\sqrt{10}\,-3\sqrt{5}\,i]}{12}\,I_1(x)\,\mathbf{L}_0(\nu x) + \frac{x^2[(2+5\sqrt{2}\,i)x^2+28-65\sqrt{2}\,i]}{12}\,I_1(x)\,\mathbf{L}_1(\nu x) +$$

$$+\frac{(65+5\sqrt{2}\,i)x^3}{6\pi}\,I_0(x) - \frac{x^2[(10-5\sqrt{2}\,i)x^2+105+60\sqrt{2}\,i]}{6\pi}\,I_0(x)$$

$$\int x^4\,K_0(x)\,\mathbf{L}_1(\nu x)\,dx =$$

$$= \frac{x^2[(\sqrt{10}\,-4\sqrt{5}\,i)x^2-9\sqrt{5}\,(\sqrt{2}\,-4i)]}{12}\,K_0(x)\,\mathbf{L}_0(\nu x) - \frac{x[(31+10\sqrt{2}\,i)x^2-15(7+4\sqrt{2}\,i)]}{12}\,K_0(x)\,\mathbf{L}_1(\nu x) +$$

$$+\frac{x[(11\sqrt{10}\,+\sqrt{5}\,i)x^2-33\sqrt{10}\,+3\sqrt{5}\,i]}{12}\,K_1(x)\,\mathbf{L}_0(\nu x) - \frac{x^2[(2+5\sqrt{2}\,i)x^2+28-65\sqrt{2}\,i]}{12}\,K_1(x)\,\mathbf{L}_1(\nu x) +$$

$$+\frac{5(13+\sqrt{2}\,i)}{6\pi}\,x^3\,K_0(x) + \frac{5x^2[(2-\sqrt{2}\,i)x^2+3(7+4\sqrt{2}\,i)]}{6\pi}\,K_1(x)$$

With $\eta = \bar{\mu} = \sqrt{5}\,(\sqrt{2}\,-i)/3$ holds

$$\int x^4\,J_0(x)\,\mathbf{H}_1(\eta x)\,dx =$$

$$= -\frac{x^2[(\sqrt{10}+4\sqrt{5}\,i)x^2+9\sqrt{10}\,+36\sqrt{5}\,i]}{12}\,J_0(x)\,\mathbf{H}_0(\eta x) - \frac{x[(31-10\sqrt{2}\,i)x^2+105-60\sqrt{2}\,i]}{12}\,J_0(x)\,\mathbf{H}_1(\eta x) +$$

$$+\frac{x[(11\sqrt{10}\,-\sqrt{5}\,i)x^2+33\sqrt{10}\,-3\sqrt{5}\,i]}{12}\,J_1(x)\,\mathbf{H}_0(\eta x) + \frac{x^2[(2-5\sqrt{2}\,i)x^2-28-65\sqrt{2}\,i]}{12}\,J_1(x)\,\mathbf{H}_1(\eta x) +$$

$$+\frac{5(13-\sqrt{2}\,i)}{6\pi}\,J_0(x) + \frac{5x^2[(2+\sqrt{2}\,i)x^2-21+12\sqrt{2}\,i]}{6\pi}\,J_1(x)$$

$$\int x^4\,I_0(x)\,\mathbf{L}_1(\eta x)\,dx =$$

$$= \frac{x^2[(\sqrt{10}\,+4\sqrt{5}\,i)x^2-9\sqrt{10}\,-36\sqrt{5}\,i]}{12}\,I_0(x)\,\mathbf{L}_0(\eta x) + \frac{x[(31-10\sqrt{2}\,i)x^2-105+60\sqrt{2}\,i]}{12}\,I_0(x)\,\mathbf{L}_1(\eta x) -$$

$$-\frac{x[(11\sqrt{10}\,-\sqrt{5}\,i)x^2-33\sqrt{10}\,+3\sqrt{5}\,i]}{12}\,I_1(x)\,\mathbf{L}_0(\eta x) + \frac{x^2[2-5\sqrt{2}\,i)x^2+28+65\sqrt{2}\,i]}{12}\,I_1(x)\,\mathbf{L}_1(\eta x) +$$

$$+\frac{(65-5\sqrt{2}\,i)x^3}{6\pi}\,I_0(x) - \frac{x^2[(10+5\sqrt{2}\,i)x^2+105-60\sqrt{2}\,i]}{6\pi}\,I_1(x)$$

$$\int x^4\, K_0(x)\, \mathbf{L}_1(\eta x)\, dx =$$

$$= \frac{x^2[(\sqrt{10}\, + 4\sqrt{5}\, i)x^2 - 9\sqrt{10}\, - 36\sqrt{5}\, i]}{12}\, K_0(x)\, \mathbf{L}_0(\eta x) + \frac{x[(31 - 10\sqrt{2}\, i)x^2 - 105 + 60\sqrt{2}\, i]}{12}\, K_0(x)\, \mathbf{L}_1(\eta x) +$$

$$+ \frac{x[(11\sqrt{10}\, - \sqrt{5}\, i)x^2 - 33\sqrt{10}\, + 3\sqrt{5}\, i]}{12}\, K_1(x)\, \mathbf{L}_0(\eta x) - \frac{x^2[2 - 5\sqrt{2}\, i)x^2 + 28 + 65\sqrt{2}\, i]}{12}\, K_1(x)\, \mathbf{L}_1(\eta x) +$$

$$+ \frac{5(13 - \sqrt{2}\, i)}{6\pi}\, K_0(x) + \frac{x^2[(10 + 5\sqrt{2}\, i)x^2 + 105 - 60\sqrt{2}\, i]}{6\pi}\, K_1(x)$$

1.3.6. Complete Elliptic Integral $\mathbf{E}(x)$

No integrals with incomplete elliptic functions of the type

$$\int \varphi(x)\,Z_\nu(x)\,\mathbf{E}(\alpha x,k)\,dx\,,\ \int \varphi(x)\,Z_\nu(x)\,\mathbf{E}(y,\alpha x)\,dx \quad \text{or}$$

$$\int \varphi(x)\,Z_\nu(x)\,\mathbf{F}(\alpha x,k)\,dx\,,\ \int \varphi(x)\,Z_\nu(x)\,\mathbf{F}(y,\alpha x)\,dx\,,$$

where $Z_\nu(x)$ denotes a Bessel function and $\varphi(x)$ a simple rational function of x, $\sqrt{1-x^2}$ and $\sqrt{1-k^2x^2}$, were found.

$$\int \frac{x^3\,J_0(x)\,\mathbf{E}(x)\,dx}{1-x^2} = J_0(x)\,[\mathbf{K}(x)-\mathbf{E}(x)] - xJ_1(x)\,\mathbf{E}(x)$$

$$\int \left[1+\frac{\alpha^2}{\alpha^2x^2-1}\right] x\,\mathbf{E}(\alpha x)\,J_0(x)\,dx = \mathbf{E}(\alpha x)\,[\,xJ_1(x)+J_0(x)] - \mathbf{K}(\alpha x)\,J_0(x)$$

$$\int (16x^4+456x^2+333)x\,\mathbf{E}\left(\frac{2\sqrt{3}\,ix}{3}\right) J_0(x)\,dx =$$

$$= (80x^4+168x^2-63)\,\mathbf{E}\left(\frac{2\sqrt{3}\,ix}{3}\right) J_0(x) + (16x^5+56x^3-159x)\,\mathbf{E}\left(\frac{2\sqrt{3}\,ix}{3}\right) J_1(x) -$$

$$-(16x^4-72x^2-63)\,\mathbf{K}\left(\frac{2\sqrt{3}\,ix}{3}\right) J_0(x) + (128x^3+96x)\,\mathbf{K}\left(\frac{2\sqrt{3}\,ix}{3}\right) J_1(x)$$

$$\int (16x^4-456x^2+333)x\,\mathbf{E}\left(\frac{2\sqrt{3}\,x}{3}\right) I_0(x)\,dx =$$

$$= -\,(80x^4-168x^2-63)\,\mathbf{E}\left(\frac{2\sqrt{3}\,x}{3}\right) I_0(x) + (16x^5-56x^3-159x)\,\mathbf{E}\left(\frac{2\sqrt{3}\,x}{3}\right) I_1(x) +$$

$$+(16x^4+72x^2-63)\,\mathbf{K}\left(\frac{2\sqrt{3}\,x}{3}\right) I_0(x) - (128x^3-96x)\,\mathbf{K}\left(\frac{2\sqrt{3}\,x}{3}\right) I_1(x)$$

$$\int [\alpha^4x^4+(27\alpha^2-2)\alpha^2x^2+18\alpha^4-3\alpha^2+1]\,x\,\mathbf{E}(\alpha x)\,J_0(x)\,dx =$$

$$= [5\alpha^4x^4+6(\alpha^2-1)\alpha^2x^2+6\alpha^2+1]\,\mathbf{E}(\alpha x)\,J_0(x) + [\alpha^4x^4+2(\alpha^2-1)\alpha^2x^2+14\alpha^2+1]\,x\,\mathbf{E}(\alpha x)\,J_1(x) -$$

$$-[\alpha^4x^4-2(3\alpha^2+1)\alpha^2x^2+6\alpha^2+1]\,\mathbf{K}(\alpha x)\,J_0(x) + 8[\alpha^2x^2-1)]\alpha^2x\,\mathbf{K}(\alpha x)\,J_1(x)$$

$$\int (4x^4+120x^2+99)\,x\,\mathbf{E}\left(\frac{\sqrt{6}\,ix}{3}\right) J_0(x)\,dx =$$

$$= (20x^4+60x^2-27)\,\mathbf{E}\left(\frac{\sqrt{6}\,ix}{3}\right) J_0(x) + (4x^5+20x^3-75x)\,\mathbf{E}\left(\frac{\sqrt{6}\,ix}{3}\right) J_1(x) -$$

$$-(4x^4-12x^2-27)\,\mathbf{K}\left(\frac{\sqrt{6}\,ix}{3}\right) J_0(x) + 16x(2x^2+3)\,\mathbf{K}\left(\frac{\sqrt{6}\,ix}{3}\right) J_1(x)$$

$$\int (4x^4-120x^2+99)\,x\,\mathbf{E}\left(\frac{\sqrt{6}\,x}{3}\right) I_0(x)\,dx =$$

$$= -\,(20x^4-60x^2-27)\,\mathbf{E}\left(\frac{\sqrt{6}\,x}{3}\right) I_0(x) - (4x^5-20x^3-75x)\,\mathbf{E}\left(\frac{\sqrt{6}\,x}{3}\right) I_1(x) -$$

$$-(4x^4+12x^2-27)\,\mathbf{K}\left(\frac{\sqrt{6}\,x}{3}\right) I_0(x) + 16x(2x^2-3)\,\mathbf{K}\left(\frac{\sqrt{6}\,x}{3}\right) I_1(x)$$

$$\int \frac{(x^2+2)x\,\mathbf{E}(ix)\,J_0(x)\,dx}{1+x^2} = \mathbf{E}(ix)\,[x\,J_1(x) - 2J_0(x)] - \mathbf{K}(ix)\,J_0(x)$$

$$\int \frac{(x^2+2)x\,\mathbf{E}(ix)\,J_0(x)\,dx}{1+x^2} = \mathbf{E}(ix)\,[x\,J_1(x) + J_0(x)] - \mathbf{K}(ix)\,J_0(x)$$

$$\int \frac{(x^2-2)x\,\mathbf{E}(x)\,I_0(x)\,dx}{x^2-1} = \mathbf{E}(x)\,[I_0(x) - x\,I_1(x)] - \mathbf{K}(x)\,I_0(x)$$

$$\int \frac{(\alpha^2x^4-x^2+1)\,\mathbf{E}(\alpha x)\,J_1(x)\,dx}{(1-\alpha^2x^2)x} = \mathbf{E}(\alpha x)\,[x\,J_0(x) - 2J_1(x)] + \mathbf{K}(\alpha x)\,J_1(x)$$

$$\int (2x^2+51)x^2\,\mathbf{E}\left(\frac{\sqrt{2}\,ix}{3}\right)\,J_1(x)\,dx = -\,(2x^4+21x^2-54)\,\mathbf{E}\left(\frac{\sqrt{2}\,ix}{3}\right)\,J_0(x)+$$

$$+(10x^3+63x)\,\mathbf{E}\left(\frac{\sqrt{2}\,ix}{3}\right)\,J_1(x) - (12x^2+54)\,\mathbf{K}\left(\frac{\sqrt{2}\,ix}{3}\right)\,J_0(x) - (2x^3+9x)\,\mathbf{K}\left(\frac{\sqrt{2}\,ix}{3}\right)\,J_1(x)$$

$$\int (2x^2+51)x^2\,\mathbf{E}\left(\frac{\sqrt{2}\,x}{3}\right)\,I_1(x)\,dx = -\,(2x^4-21x^2-54)\,\mathbf{E}\left(\frac{\sqrt{2}\,x}{3}\right)\,I_0(x)+$$

$$+(10x^3-63x)\,\mathbf{E}\left(\frac{\sqrt{2}\,x}{3}\right)\,I_1(x) + (12x^2-54)\,\mathbf{K}\left(\frac{\sqrt{2}\,x}{3}\right)\,I_0(x) - (2x^3-9x)\,\mathbf{K}\left(\frac{\sqrt{2}\,x}{3}\right)\,I_1(x)$$

$$\int (64x^4+816x^2+297)x\,\mathbf{K}\left(\frac{2\sqrt{6}\,ix}{3}\right)\,J_0(x) = (24x^2-63)\,\mathbf{E}\left(\frac{2\sqrt{6}\,ix}{3}\right)\,J_0(x)-$$

$$-144x\,\mathbf{E}\left(\frac{2\sqrt{6}\,ix}{3}\right)\,J_1(x)+(192x^4+240x^2+63)\,\mathbf{K}\left(\frac{2\sqrt{6}\,ix}{3}\right)\,J_0(x)+(64x^5+240x^3+81x)\,\mathbf{K}\left(\frac{2\sqrt{6}\,ix}{3}\right)\,J_1(x)$$

$$\int (64x^4-816x^2+297)x\,\mathbf{K}\left(\frac{2\sqrt{6}\,x}{3}\right)\,I_0(x) = (24x^2+63)\,\mathbf{E}\left(\frac{2\sqrt{6}\,x}{3}\right)\,I_0(x)-$$

$$-144x\,\mathbf{E}\left(\frac{2\sqrt{6}\,x}{3}\right)\,I_1(x)-(192x^4-240x^2+63)\,\mathbf{K}\left(\frac{2\sqrt{6}\,x}{3}\right)\,I_0(x)+(64x^5-240x^3+81x)\,\mathbf{K}\left(\frac{2\sqrt{6}\,x}{3}\right)\,I_1(x)$$

$$\int [\alpha^4x^4+2(6\alpha^2-1)\alpha^2x^2+3\alpha^4-4\alpha^2+1]x\,\mathbf{K}(\alpha x)\,J_0(x)\,dx = -\,(\alpha^2x^2-3\alpha^2-1)\,\mathbf{E}(\alpha x)\,J_0(x)+$$

$$+6\alpha^2x\,\mathbf{E}\,J_1(x)(\alpha x) + [3\alpha^4x^4+(3\alpha^2-2)\alpha^2x^2-3\alpha^2-1]\,\mathbf{K}(\alpha x)\,J_0(x)+$$
$$+[\alpha^4x^4+(3\alpha^2-2)\alpha^2x^2-3\alpha^2+1]x\,\mathbf{K}(\alpha x)\,J_1(x)$$

$$\int (x^2+9)x^2\,\mathbf{K}\left(\frac{\sqrt{2}\,ix}{2}\right)\,J_0(x) = 8\,\mathbf{E}\left(\frac{\sqrt{2}\,ix}{2}\right)\,J_0(x) + 2x\,\mathbf{E}\left(\frac{\sqrt{2}\,ix}{2}\right)\,J_1(x)-$$

$$-(x^4+6x^2+8)\,\mathbf{K}\left(\frac{\sqrt{2}\,ix}{2}\right)\,J_0(x) + (3x^3+6x)\,\mathbf{K}\left(\frac{\sqrt{2}\,ix}{2}\right)\,J_1(x)$$

$$\int (x^2-9)x^2\,\mathbf{K}\left(\frac{\sqrt{2}\,x}{2}\right)\,I_0(x) = -8\,\mathbf{E}\left(\frac{\sqrt{2}\,x}{2}\right)\,I_0(x) + 2x\,\mathbf{E}\left(\frac{\sqrt{2}\,x}{2}\right)\,I_1(x)+$$

$$+(x^4-6x^2+8)\,\mathbf{K}\left(\frac{\sqrt{2}\,x}{2}\right)\,I_0(x) - (3x^3-6x)\,\mathbf{K}\left(\frac{\sqrt{2}\,x}{2}\right)\,I_1(x)$$

1.4. Integrals $\int f(x)\cdot Z_\nu(\sqrt{x+a}\,)\,dx$ and $\int g(x)\,Z_\nu(\sqrt{x^2+ax+b}\,)\,dx$

1.4.1. Integrals with $\int x^n\,Z_0(\sqrt{x+a})\,dx$ and $\int x^n\,(x+a)^{-1/2}\,Z_1(\sqrt{x+a})\,dx$

$$\int J_0(\sqrt{x+a}\,)\,dx = 2\sqrt{x+a}\,J_1(\sqrt{x+a}\,)\,,\qquad \int I_0(\sqrt{x+a}\,)\,dx = 2\sqrt{x+a}\,I_1(\sqrt{x+a}\,)$$

$$\int K_0(\sqrt{x+a}\,)\,dx = -2\sqrt{x+a}\,K_1(\sqrt{x+a}\,)$$

$$\int \frac{J_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} = -2J_0(\sqrt{x+a}\,)\,,\qquad \int \frac{I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} = 2I_0(\sqrt{x+a}\,)$$

$$\int \frac{K_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} = -2K_0(\sqrt{x+a}\,)$$

$$\int x\,J_0(\sqrt{x+a}\,)\,dx = 4(x+a)\,J_0(\sqrt{x+a}\,) + 2(x-4)\sqrt{x+a}\,J_1(\sqrt{x+a}\,)$$

$$\int x\,I_0(\sqrt{x+a}\,)\,dx = -4(x+a)\,I_0(\sqrt{x+a}\,) + 2(x+4)\sqrt{x+a}\,I_1(\sqrt{x+a}\,)$$

$$\int x\,K_0(\sqrt{x+a}\,)\,dx = -4(x+a)\,K_0(\sqrt{x+a}\,) - 2(x+4)\sqrt{x+a}\,K_1(\sqrt{x+a}\,)$$

$$\int \frac{x\,J_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} = -2x\,J_0(\sqrt{x+a}\,) + 4\sqrt{x+a}\,)\,J_1(\sqrt{x+a}\,)$$

$$\int \frac{x\,I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} = 2x\,I_0(\sqrt{x+a}\,) - 4\sqrt{x+a}\,)\,I_1(\sqrt{x+a}\,)$$

$$\int \frac{x\,K_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} = -2x\,K_0(\sqrt{x+a}\,) - 4\sqrt{x+a}\,)\,K_1(\sqrt{x+a}\,)$$

$$\int x^2\,J_0(\sqrt{x+a}\,)\,dx = 8[x^2+(a-8)x-8a]\,J_0(\sqrt{x+a}\,) + 2(x^2-16x+64-8a)\sqrt{x+a}\,J_1(\sqrt{x+a}\,)$$

$$\int x^2\,I_0(\sqrt{x+a}\,)\,dx = -8[x^2+(a+8)x+8a]\,I_0(\sqrt{x+a}\,) + 2(x^2+16x+64+8a)\sqrt{x+a}\,I_1(\sqrt{x+a}\,)$$

$$\int x^2\,K_0(\sqrt{x+a}\,)\,dx = -8[x^2+(a+8)x+8a]\,K_0(\sqrt{x+a}\,) - 2(x^2+16x+64+8a)\sqrt{x+a}\,K_1(\sqrt{x+a}\,)$$

$$\int \frac{x^2\,J_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} = -2(x^2-8x-8a)\,J_0(\sqrt{x+a}\,) + 8(x-4)\sqrt{x+a}\,)\,J_1(\sqrt{x+a}\,)$$

$$\int \frac{x^2\,I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} = 2(x^2+8x+8a)\,I_0(\sqrt{x+a}\,) - 8(x+4)\sqrt{x+a}\,)\,I_1(\sqrt{x+a}\,)$$

$$\int \frac{x^2\,K_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} = -2(x^2+8x+8a)\,K_0(\sqrt{x+a}\,) - 8(x+4)\sqrt{x+a}\,)\,K_1(\sqrt{x+a}\,)$$

$$\int x^3\,J_0(\sqrt{x+a}\,)\,dx = 12[x^3+(a-24)x^2+32(6-a)x+8a(24-a)]\,J_0(\sqrt{x+a}\,)+$$
$$+2[x^3-36x^2+24(24-a)x+384(a-6)]\sqrt{x+a}\,J_1(\sqrt{x+a}\,)$$

$$\int x^3\,I_0(\sqrt{x+a}\,)\,dx = -12[x^3+(a+24)x^2+32(6+a)x+8a(24+a)]\,I_0(\sqrt{x+a}\,)+$$
$$+2[x^3+36x^2+24(24+a)x+384(a+6)]\sqrt{x+a}\,I_1(\sqrt{x+a}\,)$$

$$\int x^3\,K_0(\sqrt{x+a}\,)\,dx = -12[x^3+(a+24)x^2+32(6+a)x+8a(24+a)]\,K_0(\sqrt{x+a}\,)-$$

$$-2[x^3+36x^2+24(24+a)x+384(a+6)]\sqrt{x+a}\,I_1(\sqrt{x+a}\,)$$

$$\int \frac{x^3\,J_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} =$$

$$= -2[x^3-24x^2+24(8-a)x+192a]J_0(\sqrt{x+a}\,)+12[x^2-16x+8(8-a)]\sqrt{x+a}\,)\,J_1(\sqrt{x+a}\,)$$

$$\int \frac{x^3\,I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} =$$

$$= 2[x^3+24x^2+24(8+a)x+192a]I_0(\sqrt{x+a}\,)-12[x^2+16x+8(8+a)]\sqrt{x+a}\,)\,I_1(\sqrt{x+a}\,)$$

$$\int \frac{x^3\,K_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} =$$

$$= -2[x^3+24x^2+24(8+a)x+192a]K_0(\sqrt{x+a}\,)-12[x^2+16x+8(8+a)]\sqrt{x+a}\,)\,K_1(\sqrt{x+a}\,)$$

$$\int x^4\,J_0(\sqrt{x+a}\,)\,dx =$$

$$= 16[x^4+(a-48)x^3+72(16-a)x^2-24(a^2-72a+384)x+576a(a-16)]\,J_0(\sqrt{x+a}\,)+$$

$$+2[x^4-64x^3+48(48-a)x^2+2304(a-16)x+384(a^2-72a+384)]\sqrt{x+a}\,J_1(\sqrt{x+a}\,)$$

$$\int x^4\,I_0(\sqrt{x+a}\,)\,dx =$$

$$= -16[x^4+(a+48)x^3+72(16+a)x^2+24(a^2+72a+384)x+576a(a+16)]\,I_0(\sqrt{x+a}\,)+$$

$$+2[x^4+64x^3+48(48+a)x^2+2304(a+16)x+384(a^2+72a+384)]\sqrt{x+a}\,I_1(\sqrt{x+a}\,)$$

$$\int x^4\,K_0(\sqrt{x+a}\,)\,dx =$$

$$= -16[x^4+(a+48)x^3+72(16+a)x^2+24(a^2+72a+384)x+576a(a+16)]\,K_0(\sqrt{x+a}\,)-$$

$$-2[x^4+64x^3+48(48+a)x^2+2304(a+16)x+384(a^2+72a+384)]\sqrt{x+a}\,K_1(\sqrt{x+a}\,)$$

$$\int \frac{x^4\,J_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} =$$

$$= -2[x^4-48x^3+48(24-a)x^2-1536(6-a)x+384a(a-24)]J_0(\sqrt{x+a}\,)+$$

$$+16[x^3-36x^2+24(24-a)x+384(a-6)]\sqrt{x+a}\,)\,J_1(\sqrt{x+a}\,)$$

$$\int \frac{x^4\,I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} =$$

$$= 2[x^4+48x^3+48(24+a)x^2+1536(6+a)x+384a(a+24)]I_0(\sqrt{x+a}\,)+$$

$$-16[x^3+36x^2+24(24+a)x+384(a+6)]\sqrt{x+a}\,)\,I_1(\sqrt{x+a}\,)$$

$$\int \frac{x^4\,K_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}} =$$

$$= -2[x^4+48x^3+48(24+a)x^2+1536(6+a)x+384a(a+24)]K_0(\sqrt{x+a}\,)+$$

$$-16[x^3+36x^2+24(24+a)x+384(a+6)]\sqrt{x+a}\,)\,K_1(\sqrt{x+a}\,)$$

$$\int x^5\,J_0(\sqrt{x+a}\,)\,dx = 20[x^5+(a-80)x^4+128(30-a)x^3-48(a^2-144a+1920)x^2+$$

$$+1152(3a^2-128a+640)x+384a(a^2-144a+1920)]\,J_0(\sqrt{x+a}\,)+2[x^5-100x^4+80(80-a)x^3+$$

$$+7680(a-30)x^2+1920(a^2-144a+1920)x-23040(3a^2-128a+640)]\sqrt{x+a}\,J_1(\sqrt{x+a}\,)$$

$$\int x^5\,I_0(\sqrt{x+a}\,)\,dx = -20[x^5+(a+80)x^4+128(30+a)x^3+48(a^2+144a+1920)x^2+$$

$$+1152(3a^2+128a+640)x+384a(a^2+144a+1920)]\,I_0(\sqrt{x+a}\,)+2[x^5+100x^4+80(80+a)x^3+$$
$$+7680(a+30)x^2+1920(a^2+144a+1920)x+23040(3a^2+128a+640)]\sqrt{x+a}\,I_1(\sqrt{x+a}\,)$$

$$\int x^5\,K_0(\sqrt{x+a}\,)\,dx=-20[x^5+(a+80)x^4+128(30+a)x^3+48(a^2+144a+1920)x^2+$$
$$+1152(3a^2+128a+640)x+384a(a^2+144a+1920)]\,K_0(\sqrt{x+a}\,)-2[x^5+100x^4+80(80+a)x^3+$$
$$+7680(a+30)x^2+1920(a^2+144a+1920)x+23040(3a^2+128a+640)]\sqrt{x+a}\,K_1(\sqrt{x+a}\,)$$

$$\int \frac{x^5\,J_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}=-2[x^5-80x^4+80(48-a)x^3+$$
$$+5760(a-16)x^2+1920(a^2-72a+384)x-46080a(a-16)]J_0(\sqrt{x+a}\,)+$$
$$+20\sqrt{x+a}\,[x^4+64x^3+48(48-a)x^2+2304(a-16)x+384(a^2-72a+384)]\,J_1(\sqrt{x+a}\,)$$

$$\int \frac{x^5\,I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}=2[x^5+80x^4+80(48+a)x^3+$$
$$+5760(a+16)x^2+1920(a^2+72a+384)x+46080a(a+16)]I_0(\sqrt{x+a}\,)-$$
$$-20\sqrt{x+a}\,[x^4+64x^3+48(48+a)x^2+2304(a+16)x+384(a^2+72a+384)]I_1(\sqrt{x+a}\,)$$

$$\int \frac{x^5\,K_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}=\;-2[x^5+80x^4+80(48+a)x^3+$$
$$+5760(a+16)x^2+1920(a^2+72a+384)x+46080a(a+16)]K_0(\sqrt{x+a}\,)-$$
$$-20\sqrt{x+a}\,[x^4+64x^3+48(48+a)x^2+2304(a+16)x+384(a^2+72a+384)]K_1(\sqrt{x+a}\,)$$

Recurrence relations;

$$\int x^{n+1}\,J_0(\sqrt{x+a}\,)\,dx=4(n+1)(x+a)x^n\,J_0(\sqrt{x+a}\,)+2x^{n+1}\sqrt{x+a}\,J_1(\sqrt{x+a}\,)-$$
$$-4(n+1)^2\int x^n\,J_0(\sqrt{x+a}\,)\,dx-4n(n+1)a\int x^{n-1}\,J_0(\sqrt{x+a}\,)\,dx$$

$$\int x^{n+1}\,I_0(\sqrt{x+a}\,)\,dx=-4(n+1)(x+a)x^n\,I_0(\sqrt{x+a}\,)+2x^{n+1}\sqrt{x+a}\,I_1(\sqrt{x+a}\,)+$$
$$+4(n+1)^2\int x^n\,I_0(\sqrt{x+a}\,)\,dx+4n(n+1)a\int x^{n-1}\,I_0(\sqrt{x+a}\,)\,dx$$

$$\int x^{n+1}\,K_0(\sqrt{x+a}\,)\,dx=-4(n+1)(x+a)x^n\,K_0(\sqrt{x+a}\,)-2x^{n+1}\sqrt{x+a}\,I_1(\sqrt{x+a}\,)+$$
$$+4(n+1)^2\int x^n\,I_0(\sqrt{x+a}\,)\,dx+4n(n+1)a\int x^{n-1}\,I_0(\sqrt{x+a}\,)\,dx$$

$$\int \frac{x^{n+1}\,J_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}=-2x^{n+1}\,J_0(\sqrt{x+a}\,)+4(n+1)x^n\sqrt{x+a}\,J_1(\sqrt{x+a}\,)-$$
$$-4n(n+1)\int \frac{x^n\,J_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}-4n(n+1)a\int \frac{x^{n-1}\,J_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}$$

$$\int \frac{x^{n+1}\,I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}=2x^{n+1}\,I_0(\sqrt{x+a}\,)-4(n+1)x^n\sqrt{x+a}\,I_1(\sqrt{x+a}\,)+$$
$$+4n(n+1)\int \frac{x^n\,I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}+4n(n+1)a\int \frac{x^{n-1}\,I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}$$

$$\int \frac{x^{n+1}\,K_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}=-2x^{n+1}\,K_0(\sqrt{x+a}\,)-4(n+1)x^n\sqrt{x+a}\,I_1(\sqrt{x+a}\,)+$$
$$+4n(n+1)\int \frac{x^n\,I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}+4n(n+1)a\int \frac{x^{n-1}\,I_1(\sqrt{x+a}\,)\,dx}{\sqrt{x+a}}$$

$$\int x\,\sqrt{x-9}\,J_0(\sqrt{x-9}\,)\,dx = 6(x-9)\,\sqrt{x-9}\,J_0(\sqrt{x-9}\,) + 2(x-9)^2\,J_1(\sqrt{x-9}\,)$$

$$\int x\,\sqrt{x+9}\,I_0(\sqrt{x+9}\,)\,dx = -6(x+9)\,\sqrt{x+9}\,I_0(\sqrt{x+9}\,) + 2(x+9)^2\,I_1(\sqrt{x+9}\,)$$

$$\int x\,\sqrt{x+9}\,K_0(\sqrt{x+9}\,)\,dx = -6(x+9)\,\sqrt{x+9}\,K_0(\sqrt{x+9}\,) - 2(x+9)^2\,K_1(\sqrt{x+9}\,)$$

$$\int x\,J_1(\sqrt{x-3}\,)\,dx = (6-2x)\,\sqrt{x-3}\,J_0(\sqrt{x-3}\,) + 6(x-3)\,J_1(\sqrt{x-3}\,)$$

$$\int x\,I_1(\sqrt{x+3}\,)\,dx = (6+2x)\,\sqrt{x+3}\,I_0(\sqrt{x+3}\,) - 6(x+3)\,I_1(\sqrt{x+3}\,)$$

$$\int x\,K_1(\sqrt{x+3}\,)\,dx = -(6+2x)\,\sqrt{x+3}\,K_0(\sqrt{x+3}\,) - 6(x+3)\,K_1(\sqrt{x+3}\,)$$

$$\int \frac{x\,J_0(\sqrt{x-1}\,)\,dx}{\sqrt{x-1}} = 2\sqrt{x-1}\,J_0(\sqrt{x-1}\,) + 2(x-1)\,J_1(\sqrt{x-1}\,)$$

$$\int \frac{x\,I_0(\sqrt{x+1}\,)\,dx}{\sqrt{x+1}} = -2\sqrt{x+1}\,I_0(\sqrt{x+1}\,) + 2(x+1)\,I_1(\sqrt{x+1}\,)$$

$$\int \frac{x\,K_0(\sqrt{x+1}\,)\,dx}{\sqrt{x+1}} = -2\sqrt{x+1}\,K_0(\sqrt{x+1}\,) - 2(x+1)\,K_1(\sqrt{x+1}\,)$$

$$\gamma = \sqrt{73} = 8.54400\,37453\,, \quad \beta = \sqrt[3]{100+12\gamma} = 5.87257\,25412\,, \quad \alpha = \frac{\beta^2+\beta-8}{\beta} = 5.51030\,75468$$

$$\int \frac{x^3\,J_0(\sqrt{x-\alpha}\,)\,dx}{\sqrt{x-\alpha}} =$$

$$= \frac{1}{8}\,\bigl\{\bigl[80x^2 + ((3\gamma-25)\beta^2 - 16\beta - 1216)x + 6(189-23\gamma)\beta^2 + 24(39-\gamma)\beta + 4608\bigr]\,\sqrt{x-\alpha}\,J_0(\sqrt{x-\alpha}\,) +$$
$$+ \bigl[16x^3 - 400x^2 + (23(25-3\gamma)\beta^2 + 368\beta + 3968)x + 24((25\gamma-209)\beta^2 - 24(125-\gamma)\beta - 272)\bigr]\,J_1(\sqrt{x-\alpha}\,)\bigr\}$$

$$\int \frac{x^3\,I_0(\sqrt{x+\alpha}\,)\,dx}{\sqrt{x+\alpha}} =$$

$$= -\frac{1}{8}\,\bigl\{\bigl[80x^2 + ((25-3\gamma)\beta^2 + 16\beta + 1216)x + 6(189-23\gamma)\beta^2 + 24(39-\gamma)\beta + 4608\bigr]\,\sqrt{x+\alpha}\,I_0(\sqrt{x+\alpha}\,) +$$
$$+ \bigl[16x^3 + 400x^2 + (23(3\gamma-25)\beta^2 - 368\beta - 3968)x + 24((25\gamma-209)\beta^2 - 24(125+\gamma)\beta - 272)\bigr]\,I_1(\sqrt{x+\alpha}\,)\bigr\}$$

$$\int \frac{x^3\,K_0(\sqrt{x+\alpha}\,)\,dx}{\sqrt{x+\alpha}} =$$

$$= -\frac{1}{8}\,\bigl\{\bigl[80x^2 + ((25-3\gamma)\beta^2 + 16\beta + 1216)x + 6(189-23\gamma)\beta^2 + 24(39-\gamma)\beta + 4608\bigr]\,\sqrt{x+\alpha}\,K_0(\sqrt{x+\alpha}\,) +$$
$$+ \bigl[16x^3 + 400x^2 + (23(3\gamma-25)\beta^2 - 368\beta - 3968)x + 24((25\gamma-209)\beta^2 - 24(125+\gamma)\beta - 272)\bigr]\,K_1(\sqrt{x+\alpha}\,)\bigr\}$$

$$\gamma = \sqrt{13} = 4.79583\,15233\,, \quad \beta = \sqrt[3]{612+180\gamma} = 10.80367\,73874\,, \quad \alpha = \frac{\beta^2+3\beta-36}{\beta} = 10.47147\,86638$$

$$\int x^3\,J_1(\sqrt{x-\alpha}\,)\,dx = -\frac{1}{18}\,\bigl\{\bigl[36x^3 - 1260x^2 + 25((17-5\gamma)\beta^2 + 36\beta + 864)x +$$

$$+12((160\gamma-541)\beta^2+3(5\gamma-401)\beta-8424)\big]\ \sqrt{x-\alpha}\,J_0(\sqrt{x-\alpha}\,)-[252x^3+6((5\gamma-17)\beta^2-36\beta-1158)x^2+$$
$$+5(43(17-5\gamma)\beta^2+1548\beta+15984)x+6(1445\gamma-5207)\beta^2-6(901+245\gamma)\beta-16848\big]\ J_1(\sqrt{x-\alpha}\,)\,\big\}$$

$$\int x^3\,I_1(\sqrt{x+\alpha}\,)\,dx=\frac{1}{18}\,\big\{\big[36x^3+1260x^2+25((17-5\gamma)\beta^2+36\beta+864)x-$$
$$-12((160\gamma-541)\beta^2+3(5\gamma-401)\beta-8424)\big]\ \sqrt{x+\alpha}\,I_0(\sqrt{x+\alpha}\,)-[252x^3-6((5\gamma-17)\beta^2-36\beta-1158)x^2+$$
$$-5(43(17-5\gamma)\beta^2-1548\beta+15984)x+6(5207-1445\gamma)\beta^2+6(901+245\gamma)\beta+16848\big]\ I_1(\sqrt{x+\alpha}\,)\,\big\}$$

$$\int x^3\,K_1(\sqrt{x+\alpha}\,)\,dx=\frac{1}{18}\,\big\{\big[36x^3+1260x^2+25((17-5\gamma)\beta^2+36\beta+864)x-$$
$$-12((160\gamma-541)\beta^2+3(5\gamma-401)\beta-8424)\big]\ \sqrt{x+\alpha}\,K_0(\sqrt{x+\alpha}\,)-[252x^3-6((5\gamma-17)\beta^2-36\beta-1158)x^2+$$
$$-5(43(17-5\gamma)\beta^2-1548\beta+15984)x+6(5207-1445\gamma)\beta^2+6(901+245\gamma)\beta+16848\big]\ K_1(\sqrt{x+\alpha}\,)\,\big\}$$

1.4.3. Special cases of $Z_\nu(\sqrt{x+a}\,)/(x+a)$

$$\int\frac{x\,J_1(\sqrt{x+1}\,)}{x+1}=-2\sqrt{x+1}\,J_0(\sqrt{x+1}\,)+2J_1(\sqrt{x+1}\,)$$
$$\int\frac{x\,I_1(\sqrt{x-1}\,)}{x-1}=2\sqrt{x-1}\,I_0(\sqrt{x-1}\,)-2I_1(\sqrt{x-1}\,)$$
$$\int\frac{x\,K_1(\sqrt{x-1}\,)}{x-1}=-2\sqrt{x-1}\,K_0(\sqrt{x-1}\,)-2K_1(\sqrt{x-1}\,)$$
$$\int\frac{x^2\,J_1(\sqrt{x-1}\,)\,dx}{x-1}=2(2-x)\,\sqrt{x-1}\,J_0(\sqrt{x-1}\,)+2(3x-4)\,J_1(\sqrt{x-1}\,)$$
$$\int\frac{x^2\,I_1(\sqrt{x+1}\,)\,dx}{x+1}=2(2+x)\,\sqrt{x+1}\,I_0(\sqrt{x+1}\,)-2(3x+4)\,I_1(\sqrt{x+1}\,)$$
$$\int\frac{x^2\,K_1(\sqrt{x+1}\,)\,dx}{x+1}=-2(2+x)\,\sqrt{x+1}\,K_0(\sqrt{x+1}\,)-2(3x+4)\,K_1(\sqrt{x+1}\,)$$
$$\int\frac{x^2\,J_1(\sqrt{x+3}\,)\,dx}{x+3}=2(6-x)\,\sqrt{x+3}\,J_0(\sqrt{x+3}\,)+6x\,J_1(\sqrt{x+3}\,)$$
$$\int\frac{x^2\,I_1(\sqrt{x-3}\,)\,dx}{x-3}=2(6+x)\,\sqrt{x-3}\,I_0(\sqrt{x-3}\,)-6x\,I_1(\sqrt{x-3}\,)$$
$$\int\frac{x^2\,K_1(\sqrt{x-3}\,)\,dx}{x-3}=-2(6+x)\,\sqrt{x-3}\,K_0(\sqrt{x-3}\,)-6x\,K_1(\sqrt{x-3}\,)$$

$$\gamma=\sqrt{5}=2.23606\ 79775,\ \ \beta=\sqrt[3]{28+12\gamma}=3.79909\ 52539,\ \ \alpha=\frac{\beta^2+\beta+4}{\beta}=5.85197\ 75147$$

$$\int\frac{x^3\,J_1(\sqrt{x+\alpha}\,)\,dx}{x+\alpha}=$$
$$=\ -\frac{1}{2}\,\big\{[4x^2+((3\gamma-7)\beta^2-4\beta-64)x+12(\gamma-2)\beta^2+12(1-\gamma)\beta+192)]\,\sqrt{x+\alpha}\,J_0(\sqrt{x+\alpha}\,)+$$
$$-\,[20x^2-((3\gamma-7)\beta^2-4\beta+176))x+6(19\gamma-45)\beta^2+12(\gamma-15)\beta]\,J_1(\sqrt{x+\alpha}\,)\,\big\}$$
$$\int\frac{x^3\,I_1(\sqrt{x-\alpha}\,)\,dx}{x-\alpha}=$$
$$=\frac{1}{2}\,\big\{[4x^2-((3\gamma-7)\beta^2+4\beta+64)x+12(\gamma-2)\beta^2+12(1-\gamma)\beta+192)]\,\sqrt{x-\alpha}\,I_0(\sqrt{x-\alpha}\,)-$$
$$-[20x^2+((3\gamma-7)\beta^2-4\beta+176)x+6(19\gamma-45)\beta^2+12(\gamma-15)\beta]\,I_1(\sqrt{x-\alpha}\,)\,\big\}$$

$$\int \frac{x^3\, K_1(\sqrt{x-\alpha}\,)\, dx}{x-\alpha} =$$

$$= -\frac{1}{2}\, \{[4x^2 - ((3\gamma-7)\beta^2 - 4\beta - 64)x + 12(\gamma-2)\beta^2 + 12(1-\gamma)\beta + 192)]\, \sqrt{x-\alpha}\, K_0(\sqrt{x-\alpha}\,) +$$
$$+ [20x^2 + ((3\gamma-7)\beta^2 - 4\beta + 176)x + 6(19\gamma-45)\beta^2 + 12(\gamma-15)\beta]\, K_1(\sqrt{x-\alpha}\,)\,\}$$

$$\alpha = 2\,\sqrt[3]{36} + 3 = 9.60385\,44978\ :$$

$$\int \frac{x^4\, J_1(\sqrt{x+\alpha}\,)\, dx}{x+a} =$$

$$= 2\left[7x^3 - 2(86 - \sqrt[3]{36}\,)x^2 + 8(132 - 44\sqrt[3]{36} - 3\sqrt[3]{6}\,)x + 24(117 - 4\sqrt[3]{6} + 112\sqrt[3]{36}\,)\right]\, J_1(\sqrt{x+\alpha}\,) -$$
$$-2\sqrt{x+\alpha}\,\left[x^3 - 2(19 + \sqrt[3]{36}\,)x^2 + 8(63 - \sqrt[3]{36} + 3\sqrt[3]{6}\,)x - 24(33 + 2\sqrt[3]{6} - 30\sqrt[3]{36}\,)\right]\, J_0(\sqrt{x+\alpha}\,)$$

$$\int \frac{x^4\, I_1(\sqrt{x-\alpha}\,)\, dx}{x-a} =$$

$$= -2\left[7x^3 + 2(86 - \sqrt[3]{36}\,)x^2 + 8(132 - 44\sqrt[3]{36} - 3\sqrt[3]{6}\,)x - 24(117 - 4\sqrt[3]{6} + 112\sqrt[3]{36}\,)\right]\, I_1(\sqrt{x-\alpha}\,) +$$
$$+2\sqrt{x-\alpha}\,\left[x^3 + 2(19 + \sqrt[3]{36}\,)x^2 + 8(63 - \sqrt[3]{36} + 3\sqrt[3]{6}\,)x + 24(33 + 2\sqrt[3]{6} - 30\sqrt[3]{36}\,)\right]\, I_0(\sqrt{x-\alpha}\,)$$

$$\int \frac{x^4\, K_1(\sqrt{x-\alpha}\,)\, dx}{x-a} =$$

$$= -2\left[7x^3 + 2(86 - \sqrt[33]{6}\,)x^2 + 8(132 - 44\sqrt[3]{36} - 3\sqrt[3]{6}\,)x - 24(117 - 4\sqrt[3]{6} + 112\sqrt[3]{36}\,)\right]\, K_1(\sqrt{x-\alpha}\,) -$$
$$-2\sqrt{x-\alpha}\,\left[x^3 + 2(19 + \sqrt[3]{36}\,)x^2 + 8(63 - \sqrt[3]{36} + 3\sqrt[3]{6}\,)x + 24(33 + 2\sqrt[3]{6} - 30\sqrt[3]{36}\,)\right]\, K_0(\sqrt{x-\alpha}\,)$$

1.4.4. $\int x^n\, Z_0(\sqrt{x^2+a}\, dx$ and $\int x^n\, (x+a)^{-1/2}\, Z_1(\sqrt{x^2+a}\, dx$

$$\int x\, J_0(\sqrt{x^2+a}\,)\, dx = \sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int x\, I_0(\sqrt{x^2+a}\,)\, dx = \sqrt{x^2+a}\, I_1(\sqrt{x^2+a}\,)$$

$$\int x\, K_0(\sqrt{x^2+a}\,)\, dx = -\sqrt{x^2+a}\, K_1(\sqrt{x^2+a}\,)$$

$$\int x\,\sqrt{x^2+a}\cdot J_1(\sqrt{x^2+a}\,)\, dx = -(x^2+a)\, J_0(\sqrt{x^2+a}\,) + 2\sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int x\,\sqrt{x^2+a}\cdot I_1(\sqrt{x^2+a}\,)\, dx = (x^2+a)\, I_0(\sqrt{x^2+a}\,) - 2\sqrt{x^2+a}\, I_1(\sqrt{x^2+a}\,)$$

$$\int x\,\sqrt{x^2+a}\cdot K_1(\sqrt{x^2+a}\,)\, dx = -(x^2+a)\, K_0(\sqrt{x^2+a}\,) - 2\sqrt{x^2+a}\, K_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x\, J_1(\sqrt{x^2+a}\,)\, dx}{\sqrt{x^2+a}} = -\, J_0(\sqrt{x^2+a}\,)$$

$$\int \frac{x\, I_1(\sqrt{x^2+a}\,)\, dx}{\sqrt{x^2+a}} = I_0(\sqrt{x^2+a}\,)$$

$$\int \frac{x\, K_1(\sqrt{x^2+a}\,)\, dx}{\sqrt{x^2+a}} = -\, K_0(\sqrt{x^2+a}\,)$$

$$\int x^3\, J_0(\sqrt{x^2+a}\,)\, dx = 2(x^2+a)\, J_0(\sqrt{x^2+a}\,) + \sqrt{x^2+a}\,(x^2-4)\, J_1(\sqrt{x^2+a}\,)$$

$$\int x^3\, I_0(\sqrt{x^2+a}\,)\, dx = -2(x^2+a)\, I_0(\sqrt{x^2+a}\,) + \sqrt{x^2+a}\,(x^2+4)\, I_1(\sqrt{x^2+a}\,)$$

$$\int x^3\, K_0(\sqrt{x^2+a}\,)\, dx = -2(x^2+a)\, K_0(\sqrt{x^2+a}\,) - \sqrt{x^2+a}\,(x^2+4)\, K_1(\sqrt{x^2+a}\,)$$

$$\int x^3\, \sqrt{x^2+a}\cdot J_1(\sqrt{x^2+a}\,)\, dx = -[x^4-(8-a)x^2-8a]\, J_0(\sqrt{x^2+a}\,)+(4x^2-16+2a)\, \sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int x^3\, \sqrt{x^2+a}\cdot I_1(\sqrt{x^2+a}\,)\, dx = [x^4+(8+a)x^2+8a]\, I_0(\sqrt{x^2+a}\,)-(4x^2+16+2a)\, \sqrt{x^2+a}\, I_1(\sqrt{x^2+a}\,)$$

$$\int x^3\, \sqrt{x^2+a}\cdot K_1(\sqrt{x^2+a}\,)\, dx = -[x^4+(8+a)x^2+8a]\, K_0(\sqrt{x^2+a}\,)-(4x^2+16+2a)\, \sqrt{x^2+a}\, K_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^3\, J_1(\sqrt{x^2+a}\,)\, dx}{\sqrt{x^2+a}} = -x^2\, J_0(\sqrt{x^2+a}\,) + 2\sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^3\, I_1(\sqrt{x^2+a}\,)\, dx}{\sqrt{x^2+a}} = x^2\, I_0(\sqrt{x^2+a}\,) - 2\sqrt{x^2+a}\, I_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^3\, K_1(\sqrt{x^2+a}\,)\, dx}{\sqrt{x^2+a}} = -x^2\, K_0(\sqrt{x^2+a}\,) - 2\sqrt{x^2+a}\, K_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, J_0(\sqrt{x^2+a}\,)\, dx = 4\,[x^4+(a-8)x^2-8a]\, J_0(\sqrt{x^2+a}\,) + \sqrt{x^2+a}\,(x^4-16x^2-8a+64)\, J_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, I_0(\sqrt{x^2+a}\,)\, dx = -4\,[x^4+(a+8)x^2+8a]\, I_0(\sqrt{x^2+a}\,) + \sqrt{x^2+a}\,(x^4+16x^2+8a+64)\, I_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, K_0(\sqrt{x^2+a}\,)\, dx = -4\,[x^4+(a+8)x^2+8a]\, K_0(\sqrt{x^2+a}\,) - \sqrt{x^2+a}\,(x^4+16x^2+8a+64)\, K_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, \sqrt{x^2+a}\cdot J_1(\sqrt{x^2+a}\,)\, dx = -[x^6-(24-a)x^4+32(6-a)x^2+8a(24-a)]\, J_0(\sqrt{x^2+a}\,)+$$

$$+[6x^4-4(24-a)x^2+64(6-a)]\, \sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, \sqrt{x^2+a}\cdot I_1(\sqrt{x^2+a}\,)\, dx = [x^6+(24+a)x^4+32(6+a)x^2+8a(24+a)]\, I_0(\sqrt{x^2+a}\,)-$$

$$-[6x^4+4(24+a)x^2+64(6+a)]\, \sqrt{x^2+a}\, I_1(\sqrt{x^2+a}\,)$$

$$\int x^5\, \sqrt{x^2+a}\cdot K_1(\sqrt{x^2+a}\,)\, dx = -[x^6+(24+a)x^4+32(6+a)x^2+8a(24+a)]\, K_0(\sqrt{x^2+a}\,)-$$

$$-[6x^4+4(24+a)x^2+64(6+a)]\, \sqrt{x^2+a}\, K_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^5\, J_1(\sqrt{x^2+a}\,)\, dx}{\sqrt{x^2+a}} = (-x^4+8x^2+8a)\, J_0(\sqrt{x^2+a}\,) + 4(x^2-4)\, \sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^5\, I_1(\sqrt{x^2+a}\,)\, dx}{\sqrt{x^2+a}} = (x^4+8x^2+8a)\, I_0(\sqrt{x^2+a}\,) - 4(x^2+4)\, \sqrt{x^2+a}\, I_1(\sqrt{x^2+a}\,)$$

$$\int \frac{x^5\, K_1(\sqrt{x^2+a}\,)\, dx}{\sqrt{x^2+a}} = -\,(x^4+8x^2+8a)\, K_0(\sqrt{x^2+a}\,) - 4(x^2+4)\, \sqrt{x^2+a}\, J_1(\sqrt{x^2+a}\,)$$

$$\int x^7\, J_0(\sqrt{x^2+a}\,)\, dx = 6\,[x^6+(a-24)x^4+32(6-a)x^2+8a(24-a)]\, J_0(\sqrt{x^2+a}\,)+$$

$$+\sqrt{x^2+a}\,[x^6-36x^4+24(24-a)x^2+384(a-6)]\, J_1(\sqrt{x^2+a}\,)$$

$$\int x^7\, I_0(\sqrt{x^2+a}\,)\, dx = -6\,[x^6+(a+24)x^4+32(6+a)x^2+8a(24+a)]\, I_0(\sqrt{x^2+a}\,)+$$

$$+\sqrt{x^2+a}\,[x^6+36x^4+24(24+a)x^2+384(a+6)]\,I_1(\sqrt{x^2+a}\,)$$

$$\int x^7\,K_0(\sqrt{x^2+a}\,)\,dx=-6\,[x^6+(a+24)x^4+32(6+a)x^2+8a(24+a)]\,K_0(\sqrt{x^2+a}\,)-$$

$$-\sqrt{x^2+a}\,[x^6+36x^4+24(24+a)x^2+384(a+6)]\,K_1(\sqrt{x^2+a}\,)$$

$$\int x^7\,\sqrt{x^2+a}\cdot J_1(\sqrt{x^2+a}\,)\,dx=$$

$$=-[x^8-(48-a)x^6+72(16-a)x^4-24(384-72a+a^2)x^2-576a(16-a)]\,J_0(\sqrt{x^2+a}\,)+$$

$$+[8x^6-6(48-a)x^4+288(16-a)x^2-48(384-72a+a^2)]\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)$$

$$\int x^7\,\sqrt{x^2+a}\cdot I_1(\sqrt{x^2+a}\,)\,dx=$$

$$=[x^8+(48+a)x^6+72(16+a)x^4+24(384+72a+a^2)x^2+576a(16+a)]\,I_0(\sqrt{x^2+a}\,)+$$

$$-[8x^6+6(48+a)x^4+288(16+a)x^2+48(384+72a+a^2)]\,\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)$$

$$\int x^7\,\sqrt{x^2+a}\cdot K_1(\sqrt{x^2+a}\,)\,dx=$$

$$=-[x^8+(48+a)x^6+72(16+a)x^4+24(384+72a+a^2)x^2+576a(16+a)]\,K_0(\sqrt{x^2+a}\,)+$$

$$-[8x^6+6(48+a)x^4+288(16+a)x^2+48(384+72a+a^2)]\,\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)$$

$$\int\frac{x^7\,J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}}=(-x^6+24x^4+24(a-8)x^2-192a)\,J_0(\sqrt{x^2+a}\,)+$$

$$+6(x^4-16x^2+64-8a)\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)$$

$$\int\frac{x^7\,I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}}=(x^6+24x^4+24(a+8)x^2+192a)\,I_0(\sqrt{x^2+a}\,)-$$

$$-6(x^4+16x^2+64+8a)\,\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)$$

$$\int\frac{x^7\,K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}}=-\,(x^6+24x^4+24(a+8)x^2+192a)\,K_0(\sqrt{x^2+a}\,)-$$

$$-6(x^4+16x^2+64+8a)\,\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)$$

Recurrence relations:

$$\int x^{2n+1}\,J_0(\sqrt{x^2+a}\,)\,dx=2nx^{2n-2}(x^2+a)\,J_0(\sqrt{x^2+a}\,)+x^{2n}\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)-$$

$$-4n^2\int x^{2n-1}\,J_0(\sqrt{x^2+a}\,)\,dx-4n(n-1)a\int x^{2n-3}\,J_0(\sqrt{x^2+a}\,)\,dx$$

$$\int x^{2n+1}\,I_0(\sqrt{x^2+a}\,)\,dx=-2nx^{2n-2}(x^2+a)\,I_0(\sqrt{x^2+a}\,)+x^{2n}\,\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)+$$

$$+4n^2\int x^{2n-1}\,I_0(\sqrt{x^2+a}\,)\,dx+4n(n-1)a\int x^{2n-3}\,I_0(\sqrt{x^2+a}\,)\,dx$$

$$\int x^{2n+1}\,K_0(\sqrt{x^2+a}\,)\,dx=-2nx^{2n-2}(x^2+a)\,K_0(\sqrt{x^2+a}\,)-x^{2n}\,\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)+$$

$$+4n^2\int x^{2n-1}\,K_0(\sqrt{x^2+a}\,)\,dx+4n(n-1)a\int x^{2n-3}\,K_0(\sqrt{x^2+a}\,)\,dx$$

$$\int x^{2n+1}\,\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)\,dx=$$

$$= -x^{2n}(x^2+a)\,J_0(\sqrt{x^2+a}\,) + 2x^{2n-2}\,[(n+1)x^2+na]\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,) -$$
$$-4n(n+1)\int x^{2n-1}\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)\,dx - 4n(n-1)a\int x^{2n-3}\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,)\,dx$$

$$\int x^{2n+1}\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)\,dx =$$
$$= x^{2n}(x^2+a)\,I_0(\sqrt{x^2+a}\,) - 2x^{2n-2}\,[(n+1)x^2+na]\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,) +$$
$$+4n(n+1)\int x^{2n-1}\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)\,dx + 4n(n-1)a\int x^{2n-3}\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,)\,dx$$

$$\int x^{2n+1}\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)\,dx =$$
$$= -x^{2n}(x^2+a)\,K_0(\sqrt{x^2+a}\,) - 2x^{2n-2}\,[(n+1)x^2+na]\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,) +$$
$$+4n(n+1)\int x^{2n-1}\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)\,dx + 4n(n-1)a\int x^{2n-3}\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,)\,dx$$

$$\int \frac{x^{2n+1}\,J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = -x^{2n}\,J_0(\sqrt{x^2+a}\,) + 2nx^{2n-2}\sqrt{x^2+a}\,J_1(\sqrt{x^2+a}\,) -$$
$$-4n(n-1)\int \frac{x^{2n-1}\,J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} - 4n(n-1)a\int \frac{x^{2n-3}\,J_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}}$$

$$\int \frac{x^{2n+1}\,I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = x^{2n}\,I_0(\sqrt{x^2+a}\,) - 2nx^{2n-2}\sqrt{x^2+a}\,I_1(\sqrt{x^2+a}\,) +$$
$$+4n(n-1)\int \frac{x^{2n-1}\,I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} + 4n(n-1)a\int \frac{x^{2n-3}\,I_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}}$$

$$\int \frac{x^{2n+1}\,K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} = -x^{2n}\,K_0(\sqrt{x^2+a}\,) - 2nx^{2n-2}\sqrt{x^2+a}\,K_1(\sqrt{x^2+a}\,) +$$
$$+4n(n-1)\int \frac{x^{2n-1}\,K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}} + 4n(n-1)a\int \frac{x^{2n-3}\,K_1(\sqrt{x^2+a}\,)\,dx}{\sqrt{x^2+a}}$$

1.4.5. Special cases with $Z_\nu(\sqrt{x^2+a}\,)$ and $Z_\nu(\sqrt{x^2+ax+b}\,)$

$$\int x^3\sqrt{x^2-9}\cdot J_0(\sqrt{x^2-9}\,)\,dx = 3(x^2-9)\sqrt{x^2-9}\,J_0(\sqrt{x^2-9}\,) + (x^4-18x^2+81)\,J_1(\sqrt{x^2-9}\,)$$

$$\int x^3\sqrt{x^2+9}\cdot I_0(\sqrt{x^2+9}\,)\,dx = -3(x^2+9)\sqrt{x^2+9}\,J_0(\sqrt{x^2+9}\,) + (x^4+18x^2+81)\,I_1(\sqrt{x^2+9}\,)$$

$$\int x^3\sqrt{x^2+9}\cdot K_0(\sqrt{x^2+9}\,)\,dx = -3(x^2+9)\sqrt{x^2+9}\,K_0(\sqrt{x^2+9}\,) - (x^4+18x^2+81)\,K_1(\sqrt{x^2+9}\,)$$

$$\int \frac{x^3\,J_0(\sqrt{x^2-1}\,)\,dx}{\sqrt{x^2-1}} = \sqrt{x^2-1}\,J_0(\sqrt{x^2-1}\,) + (x^2-1)\,J_1(\sqrt{x^2-1}\,)$$

$$\int \frac{x^3\,I_0(\sqrt{x^2+1}\,)\,dx}{\sqrt{x^2+1}} = -\sqrt{x^2+1}\,I_0(\sqrt{x^2+1}\,) + (x^2+1)\,I_1(\sqrt{x^2+1}\,)$$

$$\int \frac{x^3\,K_0(\sqrt{x^2+1}\,)\,dx}{\sqrt{x^2+1}} = -\sqrt{x^2+1}\,K_0(\sqrt{x^2+1}\,) - (x^2+1)\,K_1(\sqrt{x^2+1}\,)$$

$$\int x^3\,J_1(\sqrt{x^2-3}\,)\,dx = -(x^2-3)\sqrt{x^2-3}\,J_0(\sqrt{x^2-3}\,) + 3(x^2-3)\,J_1(\sqrt{x^2-3}\,)$$

$$\int x^3\,I_1(\sqrt{x^2+3}\,)\,dx = (x^2+3)\sqrt{x^2+3}\,I_0(\sqrt{x^2+3}\,) - 3(x^2-3)\,I_1(\sqrt{x^2+3}\,)$$

$$\int x^3\,K_1(\sqrt{x^2+3}\,)\,dx = -(x^2+3)\sqrt{x^2+3}\,K_0(\sqrt{x^2+3}\,)-3(x^2-3)\,K_1(\sqrt{x^2+3}\,)$$

$$\int \frac{x^3\,J_1(\sqrt{x^2+1}\,)\,dx}{x^2+1} = -\sqrt{x^2+1}\,J_0(\sqrt{x^2+1}\,)+J_1(\sqrt{x^2+1}\,)$$

$$\int \frac{x^3\,I_1(\sqrt{x^2-1}\,)\,dx}{x^2-1} = \sqrt{x^2-1}\,I_0(\sqrt{x^2-1}\,)-I_1(\sqrt{x^2-1}\,)$$

$$\int \frac{x^3\,K_1(\sqrt{x^2-1}\,)\,dx}{x^2-1} = -\sqrt{x^2-1}\,K_0(\sqrt{x^2-1}\,)-K_1(\sqrt{x^2-1}\,)$$

$$\int x^4\,J_0\left(\sqrt{x^2+\sqrt{6}\,x+\frac{9}{4}}\right)\,dx = \frac{24x^3+4\sqrt{6}\,x^2-66x-45\sqrt{6}}{8}\,J_0\left(\sqrt{x^2+\sqrt{6}\,x+\frac{9}{4}}\right)+$$

$$+\frac{4x^3-2\sqrt{6}\,x^2-30x+11\sqrt{6}}{4}\sqrt{x^2+\sqrt{6}\,x+\frac{9}{4}}\,J_1\left(\sqrt{x^2+\sqrt{6}\,x+\frac{9}{4}}\right)$$

$$\int x^4\,\left(x^2+\sqrt{2}\,x+\frac{7}{12}\right)^{-1/2}\cdot J_1\left(\sqrt{x^2+\sqrt{2}\,x+\frac{7}{12}}\right)\,dx =$$

$$= -\frac{4x^3-2\sqrt{2}\,x^2-10x-7\sqrt{2}}{4}\,J_0\left(\sqrt{x^2+\sqrt{2}\,x+\frac{7}{12}}\right)+\frac{6x-5\sqrt{2}}{2}\sqrt{x^2+\sqrt{2}\,x+\frac{7}{12}}\cdot J_1\left(\sqrt{x^2+\sqrt{2}\,x+\frac{7}{12}}\right)$$

$$\int x^4\,\left(x^2-\sqrt{2}\,x+\frac{7}{12}\right)^{-1/2}\cdot J_1\left(\sqrt{x^2-\sqrt{2}\,x+\frac{7}{12}}\right)\,dx =$$

$$= -\frac{4x^3+2\sqrt{2}\,x^2-10x+7\sqrt{2}}{4}\,J_0\left(\sqrt{x^2-\sqrt{2}\,x+\frac{7}{12}}\right)+\frac{6x+5\sqrt{2}}{2}\sqrt{x^2-\sqrt{2}\,x+\frac{7}{12}}\cdot J_1\left(\sqrt{x^2-\sqrt{2}\,x+\frac{7}{12}}\right)$$

$$\int x^5\sqrt{x^2+\sqrt{30}\,x+\frac{45}{4}}\,\cdot J_1\left(\sqrt{x^2+\sqrt{30}\,x+\frac{45}{4}}\right)\,dx =$$

$$= -\frac{16x^6+8\sqrt{30}\,x^5-324x^4-198\sqrt{30}\,x^3+1722x^2+1767\sqrt{30}\,x+135}{16}\,J_0\left(\sqrt{x^2+\sqrt{30}\,x+\frac{45}{4}}\right)+$$

$$+\frac{48x^4-4\sqrt{30}\,x^3-588x^2-3\sqrt{30}\,x+1767}{8}\sqrt{x^2+\sqrt{30}\,x+\frac{45}{8}}\cdot J_1\left(\sqrt{x^2+\sqrt{30}\,x+\frac{45}{4}}\right)$$

$$\int x^5\sqrt{x^2-\sqrt{30}\,x+\frac{45}{4}}\,\cdot J_1\left(\sqrt{x^2-\sqrt{30}\,x+\frac{45}{4}}\right)\,dx =$$

$$= -\frac{16x^6-8\sqrt{30}\,x^5-324x^4+198\sqrt{30}\,x^3+1722x^2-1767\sqrt{30}\,x+135}{16}\,J_0\left(\sqrt{x^2-\sqrt{30}\,x+\frac{45}{4}}\right)+$$

$$+\frac{48x^4+4\sqrt{30}\,x^3-588x^2+3\sqrt{30}\,x+1767}{8}\sqrt{x^2-\sqrt{30}\,x+\frac{45}{8}}\cdot J_1\left(\sqrt{x^2-\sqrt{30}\,x+\frac{45}{4}}\right)$$

$$\int x^5\,\left(x^2+\sqrt{6}\,x+\frac{33}{20}\right)^{-1/2}\cdot J_1\left(\sqrt{x^2+\sqrt{6}\,x+\frac{33}{20}}\right)\,dx =$$

$$= -\frac{20x^4-10\sqrt{6}\,x^3-130x^2-25\sqrt{6}\,x+231}{20}\,J_0\left(\sqrt{x^2+\sqrt{6}\,x+\frac{33}{20}}\right)+$$

$$+\frac{8x^2-7\sqrt{6}\,x-5}{2}\sqrt{x^2+\sqrt{6}\,x+\frac{33}{20}}\cdot J_1\left(\sqrt{x^2+\sqrt{6}\,x+\frac{33}{20}}\right)$$

$$\int x^5 \left(x^2 - \sqrt{6}\,x + \frac{33}{20}\right)^{-1/2} \cdot J_1\left(\sqrt{x^2 - \sqrt{6}\,x + \frac{33}{20}}\right) dx =$$

$$= -\frac{20x^4 + 10\sqrt{6}\,x^3 - 130x^2 + 25\sqrt{6}\,x + 231}{20} J_0\left(\sqrt{x^2 - \sqrt{6}\,x + \frac{33}{20}}\right) +$$

$$+\frac{8x^2 + 7\sqrt{6}\,x - 5}{2} \sqrt{x^2 - \sqrt{6}\,x + \frac{33}{20}} \cdot J_1\left(\sqrt{x^2 - \sqrt{6}\,x + \frac{33}{20}}\right)$$

$$\int x^5\, J_0\left(\sqrt{x^2 + 3\sqrt{2}\,x + \frac{117}{20}}\right) dx =$$

$$= \frac{160x^4 + 60\sqrt{2}\,x^3 - 1244x^2 - 1437\sqrt{2}\,x + 1989}{40} J_0\left(\sqrt{x^2 + 3\sqrt{2}\,x + \frac{117}{20}}\right) +$$

$$+\frac{20x^4 - 30\sqrt{2}\,x^3 - 230x^2 + 255\sqrt{2}\,x + 479}{20} \sqrt{x^2 + 3\sqrt{2}\,x + \frac{117}{20}}\, J_1\left(\sqrt{x^2 + 3\sqrt{2}\,x + \frac{117}{20}}\right)$$

$$\int x^5\, J_0\left(\sqrt{x^2 - 3\sqrt{2}\,x + \frac{117}{20}}\right) dx =$$

$$= \frac{160x^4 - 60\sqrt{2}\,x^3 - 1244x^2 + 1437\sqrt{2}\,x + 1989}{40} J_0\left(\sqrt{x^2 - 3\sqrt{2}\,x + \frac{117}{20}}\right) +$$

$$+\frac{20x^4 + 30\sqrt{2}\,x^3 - 230x^2 - 255\sqrt{2}\,x + 479}{20} \sqrt{x^2 - 3\sqrt{2}\,x + \frac{117}{20}}\, J_1\left(\sqrt{x^2 - 3\sqrt{2}\,x + \frac{117}{20}}\right)$$

$$\int \frac{x^5\, J_1(\sqrt{x^2-1}\,)\, dx}{x^2-1} = (2-x^2)\sqrt{x^2-1}\, J_0(\sqrt{x^2-1}\,) + (3x^2-4)\, J_1(\sqrt{x^2-1}\,)$$

$$\int \frac{x^5\, I_1(\sqrt{x^2+1}\,)\, dx}{x^2+1} = (2+x^2)\sqrt{x^2+1}\, I_0(\sqrt{x^2+1}\,) - (3x^2+4)\, I_1(\sqrt{x^2+1}\,)$$

$$\int \frac{x^5\, K_1(\sqrt{x^2+1}\,)\, dx}{x^2+1} = -(2+x^2)\sqrt{x^2+1}\, K_0(\sqrt{x^2+1}\,) - (3x^2+4)\, K_1(\sqrt{x^2+1}\,)$$

$$\int \frac{x^5\, J_1(\sqrt{x^2+3}\,)\, dx}{x^2+3} = -(x^2-6)\,\sqrt{x^2+3}\, J_0(\sqrt{x^2+3}\,) + 3x^2\, J_1(\sqrt{x^2+3}\,)$$

$$\int \frac{x^5\, I_1(\sqrt{x^2-3}\,)\, dx}{x^2+3} = (x^2+6)\,\sqrt{x^2-3}\, I_0(\sqrt{x^2-3}\,) - 3x^2\, I_1(\sqrt{x^2-3}\,)$$

$$\int \frac{x^5\, K_1(\sqrt{x^2-3}\,)\, dx}{x^2+3} = -(x^2+6)\,\sqrt{x^2-3}\, K_0(\sqrt{x^2-3}\,) - 3x^2\, K_1(\sqrt{x^2-3}\,)$$

$$\gamma = \sqrt{5} = 2.23606\,79775,\ \beta = \sqrt[3]{28+12\gamma} = 3.79909\,52539,\ \alpha = \frac{\beta^2+\beta+4}{\beta} = 5.85197\,75147$$

$$\int \frac{x^7\, J_1(\sqrt{x^2+\alpha}\,)\, dx}{x^2+\alpha} =$$

$$= \frac{1}{4}\,\left\{20x^4 + [(7-3\gamma)\beta^2 + 4\beta - 176]x^2 + 6(19\gamma-45)\beta^2 + 12(\gamma-15)\beta)\right\}\, J_1(\sqrt{x^2-\alpha}\,) -$$

$$-\frac{1}{4}\,\left\{4x^4 - [(7-3\gamma)\beta^2 + 4\beta + 64]x^2 + 12(\gamma-2)\beta^2 + 12(1-\gamma)\beta + 192\right\}\, \sqrt{x^2+\alpha} \cdot J_0(\sqrt{x^2+\alpha}\,)$$

$$\int \frac{x^7\, I_1(\sqrt{x^2-\alpha}\,)\, dx}{x^2+\alpha} =$$

$$= -\frac{1}{4}\,\left\{20x^4 - [(7-3\gamma)\beta^2 + 4\beta - 176]x^2 + 6(19\gamma - 45)\beta^2 + 12(\gamma - 15)\beta)\right\}\, I_1(\sqrt{x^2+\alpha}\,) +$$

$$+\frac{1}{4}\,\left\{4x^4 + [(7-3\gamma)\beta^2 + 4\beta + 64]x^2 + 12(\gamma-2)\beta^2 + 12(1-\gamma)\beta + 192\right\}\,\sqrt{x^2-\alpha}\cdot I_0(\sqrt{x^2-\alpha}\,)$$

$$\int \frac{x^7\, K_1(\sqrt{x^2-\alpha}\,)\, dx}{x^2+\alpha} =$$

$$= -\frac{1}{4}\,\left\{20x^4 - [(7-3\gamma)\beta^2 + 4\beta - 176]x^2 + 6(19\gamma - 45)\beta^2 + 12(\gamma - 15)\beta)\right\}\, K_1(\sqrt{x^2+\alpha}\,) -$$

$$-\frac{1}{4}\,\left\{4x^4 + [(7-3\gamma)\beta^2 + 4\beta + 64]x^2 + 12(\gamma-2)\beta^2 + 12(1-\gamma)\beta + 192\right\}\,\sqrt{x^2-\alpha}\cdot K_0(\sqrt{x^2-\alpha}\,)$$

1.4.6. Integrals with $e^{\alpha x}\, Z_\nu(\sqrt{x+\beta}\,)$

Only special cases were found.

$$\int x\, e^{\alpha x}\, J_0\left(\sqrt{x - \frac{1}{\alpha} - \frac{1}{4\alpha^2}}\right)\, dx =$$

$$= \frac{e^{\alpha x}}{4\alpha^3}\,\left[(4\alpha^2 x - 1 - 4\alpha)\, J_0\left(\sqrt{x - \frac{1}{\alpha} - \frac{1}{4\alpha^2}}\right) + 2\alpha\sqrt{x - \frac{1}{\alpha} - \frac{1}{4\alpha^2}}\, J_1\left(\sqrt{x - \frac{1}{\alpha} - \frac{1}{4\alpha^2}}\right)\right]$$

$$\int x\, e^{\alpha x}\, I_0\left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}}\right)\, dx =$$

$$= \frac{e^{\alpha x}}{4\alpha^3}\,\left[(4\alpha^2 x + 1 - 4\alpha)\, I_0\left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}}\right) - 2\alpha\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}}\, I_1\left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}}\right)\right]$$

$$\int x\, e^{\alpha x}\, K_0\left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}}\right)\, dx =$$

$$= \frac{e^{\alpha x}}{4\alpha^3}\,\left[(4\alpha^2 x + 1 - 4\alpha)\, K_0\left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}}\right) + 2\alpha\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}}\, K_1\left(\sqrt{x - \frac{1}{\alpha} + \frac{1}{4\alpha^2}}\right)\right]$$

Special cases:

$$\int x\, e^{-x/2}\, J_0(\sqrt{x+1}\,)\, dx = -2e^{-x/2}\,[(x+1)\, J_0(\sqrt{x+1}\,) - \sqrt{x+1}\, J_1(\sqrt{x+1}\,)]$$

$$\int x\, e^{x/2}\, I_0(\sqrt{x-1}\,)\, dx = 2e^{x/2}\,[(x-1)\, I_0(\sqrt{x-1}\,) - \sqrt{x-1}\, I_1(\sqrt{x-1}\,)]$$

$$\int x\, e^{x/2}\, K_0(\sqrt{x-1}\,)\, dx = 2e^{x/2}\,[(x-1)\, K_0(\sqrt{x-1}\,) + \sqrt{x-1}\, K_1(\sqrt{x-1}\,)]$$

$$\int x\, e^{-x/4}\, J_0(\sqrt{x}\,)\, dx = -4e^{-x/4}\,[x\, J_0(\sqrt{x}\,) - 2\sqrt{x}\, J_1(\sqrt{x}\,)]$$

$$\int x\, e^{x/4}\, I_0(\sqrt{x}\,)\, dx = 4e^{x/4}\,[x\, I_0(\sqrt{x}\,) - 2\sqrt{x}\, I_1(\sqrt{x}\,)]$$

$$\int x\, e^{x/4}\, K_0(\sqrt{x}\,)\, dx = 4e^{x/4}\,[x\, K_0(\sqrt{x}\,) + 2\sqrt{x}\, K_1(\sqrt{x}\,)]$$

$$\int x\, e^{x/2}\, J_0(\sqrt{x-3}\,)\, dx = 2e^{x/2}\,[(x-3)\, J_0(\sqrt{x-3}\,) + \sqrt{x-3}\, J_1(\sqrt{x-3}\,)]$$

$$\int x\, e^{-x/2}\, I_0(\sqrt{x+3}\,)\, dx = -2e^{-x/2}\,[(x+3)\, I_0(\sqrt{x+3}\,) + \sqrt{x+3}\, I_1(\sqrt{x+3}\,)]$$

$$\int x\,e^{-x/2}\,K_0(\sqrt{x+3}\,)\,dx = -2e^{-x/2}\,[(x+3)\,K_0(\sqrt{x+3}\,) - \sqrt{x+3}\,K_1(\sqrt{x+3}\,)]$$

$$\int x\,e^{-x/6}\,J_0(\sqrt{x-3}\,)\,dx = -6e^{-x/6}\,[(x-3)\,J_0(\sqrt{x-3}\,) - 3\sqrt{x-3}\,J_1(\sqrt{x-3}\,)]$$

$$\int x\,e^{x/6}\,I_0(\sqrt{x+3}\,)\,dx = 6e^{-x/6}\,[(x+3)\,I_0(\sqrt{x+3}\,) - 3\sqrt{x+3}\,I_1(\sqrt{x+3}\,)]$$

$$\int x\,e^{x/6}\,K_0(\sqrt{x+3}\,)\,dx = 6e^{-x/6}\,[(x+3)\,K_0(\sqrt{x+3}\,) + 3\sqrt{x+3}\,K_1(\sqrt{x+3}\,)]$$

$$\int x\,e^{x/4}\,J_0(\sqrt{x-8}\,)\,dx = 4e^{x/4}\,[(x-8)\,J_0(\sqrt{x-8}\,) + 2\sqrt{x-8}\,J_1(\sqrt{x-8}\,)]$$

$$\int x\,e^{-x/4}\,I_0(\sqrt{x+8}\,)\,dx = -4e^{-x/4}\,[(x+8)\,I_0(\sqrt{x+8}\,) + 2\sqrt{x+8}\,I_1(\sqrt{x+8}\,)]$$

$$\int x\,e^{-x/4}\,K_0(\sqrt{x+8}\,)\,dx = -4e^{-x/4}\,[(x+8)\,K_0(\sqrt{x+8}\,) - 2\sqrt{x+8}\,K_1(\sqrt{x+8}\,)]$$

$$\int x\,e^{-x/8}\,J_0(\sqrt{x-8}\,)\,dx = -8e^{-x/8}\,[(x-8)\,J_0(\sqrt{x-8}\,) - 4\sqrt{x-8}\,J_1(\sqrt{x-8}\,)]$$

$$\int x\,e^{x/8}\,I_0(\sqrt{x+8}\,)\,dx = 8e^{x/8}\,[(x+8)\,I_0(\sqrt{x+8}\,) - 4\sqrt{x+8}\,I_1(\sqrt{x+8}\,)]$$

$$\int x\,e^{x/8}\,K_0(\sqrt{x+8}\,)\,dx = 8e^{x/8}\,[(x+8)\,K_0(\sqrt{x+8}\,) + 4\sqrt{x+8}\,K_1(\sqrt{x+8}\,)]$$

Generally: Let $r \in \{\pm 1, \pm 2, \pm 3, \ldots\}$, then one gets an integral of the type

$$\int x\,e^{x/2r}\,J_0(\sqrt{x-r(r+2)}\,)\,dx \quad \text{or} \quad \int x\,e^{x/2r} \left\{ \begin{array}{c} I_0 \\ K_0 \end{array} \right\} (\sqrt{x+r(r-2)}\,)\,dx\,.$$

$$\int x\,e^{x/2r}\,J_0(\sqrt{x-r(r+2)}\,)\,dx =$$

$$= \frac{2e^{x/2r}}{r^2}\,\left\{[x-r(r+2)]\,J_0(\sqrt{x-r(r+2)}\,) + r\,\sqrt{x-r(r+2)}\,J_1(\sqrt{x-r(r+2)}\,)\right\}$$

$$\int x\,e^{x/2r}\,I_0(\sqrt{x+r(r-2)}\,)\,dx =$$

$$= \frac{2e^{x/2r}}{r^2}\,\left\{[x+r(r-2)]\,I_0(\sqrt{x+r(r+2)}\,) - r\,\sqrt{x+r(r-2)}\,I_1(\sqrt{x+r(r-2)}\,)\right\}$$

$$\int x\,e^{x/2r}\,K_0(\sqrt{x+r(r-2)}\,)\,dx =$$

$$= \frac{2e^{x/2r}}{r^2}\,\left\{[x+r(r-2)]\,K_0(\sqrt{x+r(r+2)}\,) + r\,\sqrt{x+r(r-2)}\,K_1(\sqrt{x+r(r-2)}\,)\right\}$$

$$\int \frac{x\,e^{\alpha x}\,J_1(\sqrt{x-1/4\alpha^2}\,)\,dx}{\sqrt{x-1/4\alpha^2}} = -\frac{e^{\alpha x}}{2\alpha^2}\left[J_0\left(\sqrt{x-\frac{1}{4\alpha^2}}\right) - 2\alpha\sqrt{x-\frac{1}{4\alpha^2}}\;J_1\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)\right]$$

$$\int \frac{x\,e^{\alpha x}\,I_1(\sqrt{x+1/4\alpha^2}\,)\,dx}{\sqrt{x+1/4\alpha^2}} = -\frac{e^{\alpha x}}{2\alpha^2}\left[I_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right) - 2\alpha\sqrt{x+\frac{1}{4\alpha^2}}\;I_1\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)\right]$$

$$\int \frac{x\,e^{\alpha x}\,K_1(\sqrt{x+1/4\alpha^2}\,)\,dx}{\sqrt{x+1/4\alpha}^2} = \frac{e^{\alpha x}}{2\alpha}\left[K_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right) - 2\alpha\sqrt{x+\frac{1}{4\alpha^2}}\;K_1\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)\right]$$

$(2-\sqrt{2}\,)/8 = 0.07322\,33047\,,\ (2+\sqrt{2}\,)/8 = 0.42677\,66953$

$$\int x^2\, e^{(\sqrt{2}\,-2)x/8}\, J_0(\sqrt{x}\,)\, dx =$$

$$= -4\, e^{(\sqrt{2}\,-2)x/8}\, \left\{[(2+\sqrt{2}\,)x - 8(4+3\sqrt{2}\,)]x\, J_0(\sqrt{x}\,) - 4[(3+2\sqrt{2}\,)x - 16 - 12\sqrt{2}\,]\,\sqrt{x}\, J_1(\sqrt{x}\,)\right\}$$

$$\int x^2\, e^{(2-\sqrt{2}\,)x/8}\, I_0(\sqrt{x}\,)\, dx =$$

$$= 4\, e^{(2-\sqrt{2}\,)x/8}\, \left\{[(2+\sqrt{2}\,)x + 8(4+3\sqrt{2}\,)]x\, I_0(\sqrt{x}\,) - 4[(3+2\sqrt{2}\,)x + 16 + 12\sqrt{2}\,]\,\sqrt{x}\, I_1(\sqrt{x}\,)\right\}$$

$$\int x^2\, e^{(2-\sqrt{2}\,)x/8}\, K_0(\sqrt{x}\,)\, dx =$$

$$= 4\, e^{(2-\sqrt{2}\,)x/8}\, \left\{[(2+\sqrt{2}\,)x + 8(4+3\sqrt{2}\,)]x\, K_0(\sqrt{x}\,) + 4[(3+2\sqrt{2}\,)x + 16 + 12\sqrt{2}\,]\,\sqrt{x}\, K_1(\sqrt{x}\,)\right\}$$

$$\int x^2\, e^{-(\sqrt{2}\,+2)x/8}\, J_0(\sqrt{x}\,)\, dx =$$

$$= 4\, e^{-(\sqrt{2}\,-2)x/8}\, \left\{[(\sqrt{2}\,-2)x + 8(4-3\sqrt{2}\,)]x\, J_0(\sqrt{x}\,) + 4[(3-2\sqrt{2}\,)x - 16 + 12\sqrt{2}\,]\,\sqrt{x}\, J_1(\sqrt{x}\,)\right\}$$

$$\int x^2\, e^{(\sqrt{2}\,+2)x/8}\, I_0(\sqrt{x}\,)\, dx =$$

$$= 4\, e^{(\sqrt{2}\,+2)x/8}\, \left\{[(2-\sqrt{2}\,)x + 8(4-3\sqrt{2}\,)]x\, I_0(\sqrt{x}\,) - 4[(3-2\sqrt{2}\,)x + 16 - 12\sqrt{2}\,]\,\sqrt{x}\, I_1(\sqrt{x}\,)\right\}$$

$$\int x^2\, e^{(\sqrt{2}\,+2)x/8}\, K_0(\sqrt{x}\,)\, dx =$$

$$= 4\, e^{(\sqrt{2}\,+2)x/8}\, \left\{[(2-\sqrt{2}\,)x + 8(4-3\sqrt{2}\,)]x\, K_0(\sqrt{x}\,) + 4[(3-2\sqrt{2}\,)x + 16 - 12\sqrt{2}\,]\,\sqrt{x}\, K_1(\sqrt{x}\,)\right\}$$

$$\int \frac{x^2\, e^{-x/4}\, J_0(\sqrt{x+2}\,)\, dx}{\sqrt{x+2}} = -4e^{-x/4}\, \left[x\,\sqrt{x+2}\, J_0\left(\sqrt{x+2}\,\right) - 2(x+2)\, J_1\left(\sqrt{x+2}\,\right)\right]$$

$$\int \frac{x^2\, e^{x/4}\, I_0(\sqrt{x-2}\,)\, dx}{\sqrt{x-2}} = 4\, e^{x/4}\, \left[x\,\sqrt{x-2}\, I_0\left(\sqrt{x-2}\,\right) - 2(x-2)\, I_1\left(\sqrt{x-2}\,\right)\right]$$

$$\int \frac{x^2\, e^{x/4}\, K_0(\sqrt{x-2}\,)\, dx}{\sqrt{x-2}} = 4\, e^{x/4}\, \left[x\,\sqrt{x-2}\, K_0\left(\sqrt{x-2}\,\right) + 2(x-2)\, K_1\left(\sqrt{x-2}\,\right)\right]$$

$$\int \frac{x^2\, e^{-\alpha x^2}\, J_1\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\right)}{\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}} = -\frac{e^{-\alpha x^2}}{4\alpha^7}\left[(2\alpha^3 x + 4\alpha - \sqrt{2}\,)\, J_0\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\right) + \right.$$

$$\left. +\alpha^2(4\alpha^3 x + 4\alpha - 2\sqrt{2}\,)\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\; J_1\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\right)\right]$$

$$\int \frac{x^2\, e^{\alpha x^2}\, I_1\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)}{\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}} = -\frac{e^{\alpha x^2}}{4\alpha^7}\left[(2\alpha^3 x - 4\alpha - \sqrt{2}\,)\, I_0\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right) - \right.$$

$$\left. -\alpha^2(4\alpha^3 x - 4\alpha - 2\sqrt{2}\,)\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\; I_1\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)\right]$$

$$\int \frac{x^2\, e^{\alpha x^2}\, K_1\left(\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)}{\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}} = \frac{e^{\alpha x^2}}{4\alpha^7}\left[(2\alpha^3 x-4\alpha+\sqrt{2}\,)\,K_0\left(\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)+\right.$$

$$\left.+\alpha^2(4\alpha^3 x-4\alpha+2\sqrt{2}\,)\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\;K_1\left(\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)\right]$$

$$\int \frac{x^2\, e^{-\alpha x^2}\, J_1\left(\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\right)}{\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}} = -\frac{e^{-\alpha x^2}}{4\alpha^7}\left[(2\alpha^3 x+4\alpha+\sqrt{2}\,)\,J_0\left(\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\right)+\right.$$

$$\left.+\alpha^2(4\alpha^3 x+4\alpha+2\sqrt{2}\,)\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\;J_1\left(\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}-\frac{1}{4\alpha^4}}\right)\right]$$

$$\int \frac{x^2\, e^{\alpha x^2}\, I_1\left(\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)}{\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}} = -\frac{e^{\alpha x^2}}{4\alpha^7}\left[(2\alpha^3 x-4\alpha+\sqrt{2}\,)\,I_0\left(\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)-\right.$$

$$\left.-\alpha^2(4\alpha^3 x-4\alpha+2\sqrt{2}\,)\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\;I_1\left(\sqrt{x-\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)\right]$$

$$\int \frac{x^2\, e^{\alpha x^2}\, K_1\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)}{\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}} = \frac{e^{\alpha x^2}}{4\alpha^7}\left[(2\alpha^3 x-4\alpha-\sqrt{2}\,)\,K_0\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)+\right.$$

$$\left.+\alpha^2(4\alpha^3 x-4\alpha-{}^{2}\,\sqrt{2}\,)\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\;K_1\left(\sqrt{x+\frac{\sqrt{2}}{2\alpha^3}+\frac{1}{4\alpha^4}}\right)\right]$$

Special cases:

$$\alpha=\frac{\sqrt{2}}{2}:\quad \int \frac{x^2\, e^{-x/2}\, J_1(\sqrt{x+1}\,)\,dx}{\sqrt{x+1}} = -2e^{-x/2}\left[(x+2)J_0(\sqrt{x+3}\,)+x\sqrt{x+1}\,J_1(\sqrt{x+1}\,)\right]$$

$$\alpha=-\frac{\sqrt{2}}{2}:\quad \int \frac{x^2\, e^{-x/2}\, J_1(\sqrt{x-3}\,)\,dx}{\sqrt{x-3}} = -2e^{-x/2}\left[(x+6)J_0(\sqrt{x-3}\,)+(x+4)\sqrt{x-3}\,J_1(\sqrt{x-3}\,)\right]$$

$$\alpha=\frac{\sqrt{2}}{4}:\quad \int x^{3/2}\, e^{-x/8}\, J_1(\sqrt{x}\,)\,dx = -8e^{x/8}\left[4x\,J_0(\sqrt{x}\,)+(x-8)\sqrt{x}\,J_1(\sqrt{x}\,)\right]$$

$$\alpha=-\frac{\sqrt{2}}{4}:\quad \int x^{2}\, e^{-x/8}\, J_1(\sqrt{x-32}\,)\,dx = -8e^{-x/8}\left[(4x+128)\,J_0(\sqrt{x-32}\,)+(x+24)\sqrt{x}\,J_1(\sqrt{x+32}\,)\right]$$

$$\alpha=\frac{\sqrt{2}}{2}:\quad \int \frac{x^2\, e^{x/2}\, I_1(\sqrt{x-1}\,)\,dx}{\sqrt{x-1}} = 2e^{x/2}\left[(2-x)I_0(\sqrt{x-1}\,)+x\sqrt{x-1}\,I_1(\sqrt{x-1}\,)\right]$$

$$\alpha=-\frac{\sqrt{2}}{2}:\quad \int \frac{x^2\, e^{x/2}\, I_1(\sqrt{x+3}\,)\,dx}{\sqrt{x+3}} = 2e^{x/2}\left[(6-x)I_0(\sqrt{x+3}\,)+(x-4)\sqrt{x+3}\,I_1(\sqrt{x+3}\,)\right]$$

$$\alpha=\frac{\sqrt{2}}{4}:\quad \int x^{3/2}\, e^{x/8}\, I_1(\sqrt{x}\,)\,dx = -8e^{x/8}\left[4x\,I_0(\sqrt{x}\,)-(x+8)\sqrt{x}\,I_1(\sqrt{x}\,)\right]$$

$$\alpha = -\frac{\sqrt{2}}{4}: \quad \int x^2 e^{x/8} I_1(\sqrt{x+32})\,dx = -8e^{x/8}\left[(4x-128)\,I_0(\sqrt{x+32}) - (x-24)\sqrt{x}\,I_1(\sqrt{x+32})\right]$$

$$\alpha = \frac{\sqrt{2}}{2}: \quad \int \frac{x^2 e^{x/2} K_1(\sqrt{x+3})\,dx}{\sqrt{x+3}} = 2e^{x/2}\left[(x-6)K_0(\sqrt{x+3}) + (x-4)\sqrt{x+3}\,K_1(\sqrt{x+3})\right]$$

$$\alpha = \frac{\sqrt{2}}{2}: \quad \int \frac{x^2 e^{x/2} K_1(\sqrt{x-1})\,dx}{\sqrt{x-1}} = 2e^{x/2}\left[(x-2)K_0(\sqrt{x-1}) + x\sqrt{x-1}\,K_1(\sqrt{x-1})\right]$$

$$\alpha = \frac{\sqrt{2}}{4}: \quad \int x^{3/2} e^{x/8} K_1(\sqrt{x})\,dx = 8e^{x/8}\left[4x\,K_0(\sqrt{x}) + (x+8)\sqrt{x}\,K_1(\sqrt{x})\right]$$

$$\alpha = -\frac{\sqrt{2}}{4}: \quad \int x^2 e^{x/8} K_1(\sqrt{x+32})\,dx = 8e^{x/8}\left[(4x-128)\,K_0(\sqrt{x+32}) + (x-24)\sqrt{x}\,K_1(\sqrt{x+32})\right]$$

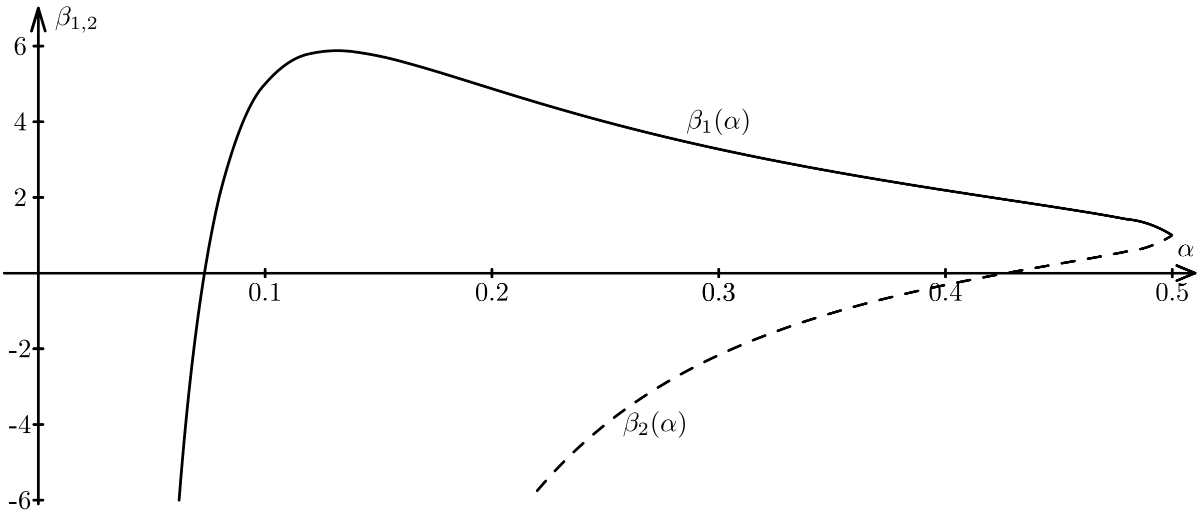
Generally , let $v_1, v_2 \in \{-1, 1\}$ and $r \in \{\pm1, \pm2, \pm3, \ldots\}$, then $\alpha = 1/\sqrt{2}\,r$ gives

$$\int \frac{x^2 e^{\pm 1/2r^2} Z_1(\sqrt{x+m})\,dx}{\sqrt{x+m}} \qquad \text{with integer coefficients,} \quad m = 2v_1r^3 + v_2r^4 = r^3(2v_1 + v_2r)\,.$$

In the case $v_1 = v_2 = 1$ one has $m \in \{-1, 0, 3, 27, 32, 128, 135, 375, 384, 864, 875, \ldots\}$.

$$0 < \alpha \le \frac{1}{2}: \ \beta_1 = \frac{1}{\alpha} - \frac{1}{4\alpha^2} + \frac{1}{2\alpha^2}\sqrt{2\alpha(1-2\alpha)}\ , \ \beta_2 = \frac{1}{\alpha} - \frac{1}{4\alpha^2} - \frac{1}{2\alpha^2}\sqrt{2\alpha(1-2\alpha)}\,,$$

$$\beta_3 = -\beta_2\,,\ \beta_4 = -\beta_1$$



Special values: $\beta_1(0.073223) = 0\,,\ \beta_2(0.42678) = 0$

α	1/20 0.05	2/29 0.0690	1/10 0.1	2/13 0.1538	1/4 0.25	8/25 0.32	9/26 0.3462	2/5 0.4	9/20 0.45	8/17 0.4708	1/2 0.5
$\beta_1(\alpha)$	-20	-29/16 -1.8125	5	91/16 5.6875	4	775/256 3.0273	221/81 2.7284	35/16 2.1875	140/81 1.7284	391/256 1.5237	1
$\beta_2(\alpha)$	-140	-1189/16 -74.3125	-35	-221/16 -13.8125	-4	-425/256 -1.6602	-91/81 -1.1235	-5/16 -0.3125	20/81 0.2469	119/256 0.4649	1

Formulas with rational coefficients may be found for $\alpha = \lambda/\mu$ with natural numbers $p_s,\ q_s,\ 0 \le p_s \le q_s$, prime factors f_s and $\lambda = 2^{p_2}3^{p_3}5^{p_5}\ldots f_m^{p_\lambda},\ \mu = [2^{q_2}3^{q_3}5^{p_5}\ldots f_\mu^{p_\mu}]^2/(2\lambda) + 2\lambda$.

$$\int x^2 e^{-\alpha x} J_0(\sqrt{x+\beta_1}\,) =$$

$$= -\frac{e^{-\alpha x}}{8\alpha^5}\left[\left(8\alpha^4x^2 + 2\alpha^2(8\alpha-1)x + \sqrt{2\alpha(1-2\alpha)} + 16\alpha^2 - 6\alpha\right) J_0(\sqrt{x+\beta_1}\,) - \right.$$
$$\left. - \left(4\alpha^3x - 2\alpha\sqrt{2\alpha(1-2\alpha)} + 8\alpha^2\right)\sqrt{x+\beta_1}\,J_1(\sqrt{x+\beta_1})\,)\right]$$

$$\int x^2\, e^{-\alpha x}\, J_0(\sqrt{x+\beta_2}\,) \;=$$

$$= -\frac{e^{-\alpha x}}{8\alpha^5}\;\Big[\Big(8\alpha^4 x^2 + 2\alpha^2(8\alpha-1)x - \sqrt{2\alpha(1-2\alpha)} + 16\alpha^2 - 6\alpha\Big)\; J_0(\sqrt{x+\beta_2}\,) -$$

$$- \Big(4\alpha^3 x + 2\alpha\sqrt{2\alpha(1-2\alpha)} + 8\alpha^2\Big)\;\sqrt{x+\beta_2}\, J_1(\sqrt{x+\beta_2})\,)\Big]$$

$$\int x^2\, e^{\,\alpha x}\, I_0(\sqrt{x+\beta_3}\,) \;=$$

$$= \frac{e^{\,\alpha x}}{8\alpha^5}\;\Big[\Big(8\alpha^4 x^2 + 2\alpha^2(1-8\alpha)x - \sqrt{2\alpha(1-2\alpha)} + 16\alpha^2 - 6\alpha\Big)\; I_0(\sqrt{x+\beta_3}\,) -$$

$$- \Big(4\alpha^3 x - 2\alpha\sqrt{2\alpha(1-2\alpha)} - 8\alpha^2\Big)\;\sqrt{x+\beta_3}\, I_1(\sqrt{x+\beta_3})\,)\Big]$$

$$\int x^2\, e^{\,\alpha x}\, I_0(\sqrt{x+\beta_4}\,) \;=$$

$$= \frac{e^{\alpha x}}{8\alpha^5}\;\Big[\Big(8\alpha^4 x^2 + 2\alpha^2(1-8\alpha)x + \sqrt{2\alpha(1-2\alpha)} + 16\alpha^2 - 6\alpha\Big)\; I_0(\sqrt{x+\beta_4}\,) -$$

$$- \Big(4\alpha^3 x + 2\alpha\sqrt{2\alpha(1-2\alpha)} - 8\alpha^2\Big)\;\sqrt{x+\beta_4}\, I_1(\sqrt{x+\beta_4})\,)\Big]$$

$$\int x^2\, e^{\,\alpha x}\, K_0(\sqrt{x+\beta_3}\,) \;=$$

$$= \frac{e^{\,\alpha x}}{8\alpha^5}\;\Big[\Big(8\alpha^4 x^2 + 2\alpha^2(1-8\alpha)x - \sqrt{2\alpha(1-2\alpha)} + 16\alpha^2 - 6\alpha\Big)\; K_0(\sqrt{x+\beta_3}\,) +$$

$$+ \Big(4\alpha^3 x - 2\alpha\sqrt{2\alpha(1-2\alpha)} - 8\alpha^2\Big)\;\sqrt{x+\beta_3}\, K_1(\sqrt{x+\beta_3})\,)\Big]$$

$$\int x^2\, e^{\,\alpha x}\, K_0(\sqrt{x+\beta_4}\,) \;=$$

$$= \frac{e^{\alpha x}}{8\alpha^5}\;\Big[\Big(8\alpha^4 x^2 + 2\alpha(1-8\alpha)x + \sqrt{2\alpha^2(1-2\alpha)} + 16\alpha^2 - 6\alpha\Big)\; K_0(\sqrt{x+\beta_4}\,) +$$

$$+ \Big(4\alpha^3 x + 2\alpha\sqrt{2\alpha(1-2\alpha)} - 8\alpha^2\Big)\;\sqrt{x+\beta_4}\, K_1(\sqrt{x+\beta_4})\,)\Big]$$

Special parameters (for K_0 compare with the upper formulas):

$$\int x^2\, e^{-x/20}\, J_0(\sqrt{x-20}\,) =$$

$$= -20\, e^{-x/20}\;\big[(x^2 - 60x + 800) J_0(\sqrt{x-20}\,) - 10(x-20)\,\sqrt{x-20}\, J_1(\sqrt{x-20}\,)\big]$$

$$\int x^2\, e^{-x/20}\, J_0(\sqrt{x-140}\,) =$$

$$= -20\, e^{-x/20}\;\big[(x^2 - 60x - 11200) J_0(\sqrt{x-140}\,) - 10(x+100)\,\sqrt{x-140}\, J_1(\sqrt{x-140}\,)\big]$$

$$\int x^2\, e^{\,x/20}\, I_0(\sqrt{x+20}\,) =$$

$$= 20\, e^{\,x/20}\;\big[(x^2 + 60x + 800) I_0(\sqrt{x+20}\,) - 10(x+20)\,\sqrt{x+20}\, I_1(\sqrt{x+20}\,)\big]$$

$$\int x^2\, e^{\,x/20}\, I_0(\sqrt{x+140}\,) =$$

$$= 20\, e^{\,x/20}\;\big[(x^2 + 60x - 11200) I_0(\sqrt{x+140}\,) - 10(x-10)\,\sqrt{x+140}\, I_1(\sqrt{x+140}\,)\big]$$

$$\int x^2\, e^{-2x/29}\, J_0\left(\sqrt{x - \frac{29}{16}}\right) = -\frac{29 e^{-2x/29}}{128}\;\left[(1856x^2 - 43732x + 73167) J_0\left(\sqrt{x - \frac{29}{16}}\right) -\right.$$

$$-(13456x-97556)\sqrt{x-\frac{29}{16}}\,J_1\left(\sqrt{x-\frac{29}{16}}\right)\Bigg]$$

$$\int x^2\,e^{-2x/29}\,J_0\left(\sqrt{x-\frac{1189}{16}}\right) = -\frac{29e^{-2x/29}}{128}\left[(64x^2-1508x-241367)J_0\left(\sqrt{x-\frac{1189}{16}}\right)-\right.$$

$$\left.-116(4x+261)\sqrt{x-\frac{1189}{16}}\,J_1\left(\sqrt{x-\frac{1189}{16}}\right)\right]$$

$$\int x^2\,e^{2x/29}\,I_0\left(\sqrt{x+\frac{29}{16}}\right) = \frac{29e^{2x/29}}{128}\left[(64x^2+1508x+2523)I_0\left(\sqrt{x+\frac{29}{16}}\right)-\right.$$

$$\left.-116(4x+29)\sqrt{x+\frac{29}{16}}\,I_1\left(\sqrt{x+\frac{29}{16}}\right)\right]$$

$$\int x^2\,e^{2x/29}\,I_0\left(\sqrt{x+\frac{1189}{16}}\right) = \frac{29e^{2x/29}}{128}\left[(64x^2+1508x-241367)I_0\left(\sqrt{x+\frac{1189}{16}}\right)-\right.$$

$$\left.-116(4x-261)\sqrt{x+\frac{1189}{16}}\,I_1\left(\sqrt{x+\frac{1189}{16}}\right)\right]$$

$$\int x^2\,e^{-x/10}\,J_0(\sqrt{x+5}\,) = -10\,e^{-x/10}\,\left[(x^2-5x-50)J_0(\sqrt{x+5}\,)-5x\sqrt{x+5}\,J_1(\sqrt{x+5}\,)\right]$$

$$\int x^2\,e^{-x/10}\,J_0(\sqrt{x-35}\,) = -10\,e^{-x/10}\,\left[(x^2-5x-1050)J_0(\sqrt{x-35}\,)-5(x+40)\sqrt{x-35}\,J_1(\sqrt{x-35}\,)\right]$$

$$\int x^2\,e^{x/10}\,I_0(\sqrt{x-5}\,) =$$

$$= 10\,e^{x/10}\,\left[(x^2+5x-50)I_0(\sqrt{x-5}\,)-5x\sqrt{x-5}\,I_1(\sqrt{x-5}\,)\right]$$

$$\int x^2\,e^{x/10}\,I_0(\sqrt{x+35}\,) =$$

$$= 10\,e^{x/10}\,\left[(x^2+5x-1050)I_0(\sqrt{x+35}\,)-5(x-40)\sqrt{x+35}\,I_1(\sqrt{x+35}\,)\right]$$

$$\int x^2\,e^{-2x/13}\,J_0\left(\sqrt{x+\frac{91}{16}}\right) = -\frac{13e^{-2x/13}}{128}\left[(64x^2+156x-1183)J_0\left(\sqrt{x+\frac{91}{16}}\right)-\right.$$

$$\left.-52(4x+13)\sqrt{x+\frac{91}{16}}\,J_1\left(\sqrt{x+\frac{91}{16}}\right)\right]$$

$$\int x^2\,e^{-2x/13}\,J_0\left(\sqrt{x-\frac{221}{16}}\right) = -\frac{13e^{-2x/13}}{128}\left[(64x^2+156x-14365)J_0\left(\sqrt{x-\frac{221}{16}}\right)-\right.$$

$$\left.-52(4x+91)\sqrt{x-\frac{221}{16}}\,J_1\left(\sqrt{x-\frac{91}{16}}\right)\right]$$

$$\int x^2\,e^{2x/13}\,I_0\left(\sqrt{x-\frac{91}{16}}\right) = \frac{13e^{2x/13}}{128}\left[(64x^2-156x-1183)I_0\left(\sqrt{x-\frac{91}{16}}\right)-\right.$$

$$\left.-52(4x-13)\sqrt{x-\frac{91}{16}}\,I_1\left(\sqrt{x-\frac{91}{16}}\right)\right]$$

$$\int x^2\,e^{2x/13}\,I_0\left(\sqrt{x+\frac{221}{16}}\right) = \frac{13e^{2x/13}}{128}\left[(64x^2-156x-14365)I_0\left(\sqrt{x+\frac{221}{16}}\right)-\right.$$

$$-52(4x-91)\sqrt{x+\frac{221}{16}}\,I_1\left(\sqrt{x+\frac{211}{16}}\right)\Bigg]$$

$$\int x^2\,e^{-x/4}\,J_0(\sqrt{x+4}\,)=\,-4\,e^{-x/4}\,\left[(x^2+4x)J_0(\sqrt{x+4}\,)+2(x+4)\sqrt{x+4}\,J_1(\sqrt{x+4}\,)\right]$$

$$\int x^2\,e^{-x/4}\,J_0(\sqrt{x-4}\,)=\,-\,\frac{e^{-x/4}}{4}\,\left[(x^2+4x-32)J_0(\sqrt{x-4}\,)+2(x+12)\sqrt{x-4}\,J_1(\sqrt{x-4}\,)\right]$$

$$\int x^2\,e^{x/4}\,I_0(\sqrt{x+4}\,)=4\,e^{x/4}\,\left[(x^2-4x-32)I_0(\sqrt{x+4}\,)-2(x-12)\sqrt{x+4}\,I_1(\sqrt{x+4}\,)\right]$$

$$\int x^2\,e^{x/4}\,I_0(\sqrt{x-4}\,)=\frac{e^{x/4}}{4}\,\left[(x^2-4x)I_0(\sqrt{x-4}\,)-2(x-4)\sqrt{x-4}\,I_1(\sqrt{x-4}\,)\right]$$

$$\int x^2\,e^{-8x/25}\,J_0\left(\sqrt{x+\frac{775}{256}}\right)=-\frac{25e^{-8x/25}}{65536}\,\Bigg[(8192x^2+31200x+19375)J_0\left(\sqrt{x+\frac{775}{256}}\right)-$$

$$-400(32x+125)\sqrt{x+\frac{775}{256}}\,J_1\left(\sqrt{x+\frac{775}{256}}\right)\Bigg]$$

$$\int x^2\,e^{-8x/25}\,J_0\left(\sqrt{x-\frac{425}{256}}\right)=-\frac{25e^{-8x/25}}{65536}\,\Bigg[(8192x^2+31200x-74375)J_0\left(\sqrt{x-\frac{425}{256}}\right)-$$

$$-400(32x+275)\sqrt{x-\frac{425}{256}}\,J_1\left(\sqrt{x-\frac{425}{256}}\right)\Bigg]$$

$$\int x^2\,e^{8x/25}\,I_0\left(\sqrt{x-\frac{775}{256}}\right)=\frac{25e^{8x/25}}{65536}\,\Bigg[(8192x^2-31200x+19375)I_0\left(\sqrt{x-\frac{775}{256}}\right)-$$

$$-400(32x-125)\sqrt{x-\frac{775}{256}}\,I_1\left(\sqrt{x-\frac{775}{256}}\right)\Bigg]$$

$$\int x^2\,e^{8x/25}\,I_0\left(\sqrt{x+\frac{425}{256}}\right)=-\frac{25e^{8x/25}}{65536}\,\Bigg[(8192x^2-31200x-74375)I_0\left(\sqrt{x+\frac{425}{256}}\right)-$$

$$-400(32x-275)\sqrt{x+\frac{425}{256}}\,I_1\left(\sqrt{x+\frac{425}{256}}\right)\Bigg]$$

$$\int x^2\,e^{-9x/26}\,J_0\left(\sqrt{x-\frac{91}{81}}\right)=-\frac{26e^{-9x/26}}{19683}\,\Bigg[(2187x^2+8073x-11830)J_0\left(\sqrt{x-\frac{91}{81}}\right)-$$

$$-117(27x+208)\sqrt{x-\frac{91}{81}}\,J_1\left(\sqrt{x-\frac{91}{81}}\right)\Bigg]$$

$$\int x^2\,e^{-9x/26}\,J_0\left(\sqrt{x+\frac{221}{81}}\right)=-\frac{26e^{-9x/26}}{19683}\,\Bigg[(2187x^2+8073x+5764)J_0\left(\sqrt{x+\frac{221}{81}}\right)-$$

$$-117(27x+104)\sqrt{x+\frac{221}{81}}\,J_1\left(\sqrt{x+\frac{221}{81}}\right)\Bigg]$$

$$\int x^2\,e^{9x/26}\,I_0\left(\sqrt{x+\frac{91}{81}}\right)=\frac{26e^{9x/26}}{19683}\,\Bigg[(2187x^2-8073x-11830)I_0\left(\sqrt{x+\frac{91}{81}}\right)-$$

$$-117(27x-208)\sqrt{x+\frac{91}{81}}\,I_1\left(\sqrt{x+\frac{91}{81}}\right)\Bigg]$$

$$\int x^2\,e^{\,9x/26}\,I_0\left(\sqrt{x-\frac{221}{81}}\right)=\frac{26e^{\,9x/26}}{19683}\,\Bigg[(2187x^2-8073x+5764)I_0\left(\sqrt{x-\frac{221}{81}}\right)-$$

$$-117(27x-208)\sqrt{x-\frac{221}{81}}\,I_1\left(\sqrt{x-\frac{221}{81}}\right)\Bigg]$$

$$\int x^2\,e^{\,-2x/5}\,J_0\left(\sqrt{x-\frac{5}{16}}\right)=-\frac{5e^{\,-2x/5}}{128}\,\Bigg[(64x^2+220x-75)\,J_0\left(\sqrt{x-\frac{5}{16}}\right)-$$

$$-20(4x+25)\sqrt{x-\frac{5}{16}}\,J_1\left(\sqrt{x-\frac{5}{16}}\right)\Bigg]$$

$$\int x^2\,e^{\,-2x/5}\,J_0\left(\sqrt{x+\frac{35}{16}}\right)=-\frac{5e^{\,-2x/5}}{128}\,\Bigg[(64x^2+220x+175)\,J_0\left(\sqrt{x+\frac{35}{16}}\right)-$$

$$-20(4x+15)\sqrt{x+\frac{35}{16}}\,J_1\left(\sqrt{x+\frac{35}{16}}\right)\Bigg]$$

$$\int x^2\,e^{\,2x/5}\,I_0\left(\sqrt{x+\frac{5}{16}}\right)=\frac{5e^{\,2x/5}}{128}\,\Bigg[(64x^2-220x-75)\,I_0\left(\sqrt{x+\frac{5}{16}}\right)-$$

$$-20(4x-25)\sqrt{x+\frac{5}{16}}\,I_1\left(\sqrt{x+\frac{5}{16}}\right)\Bigg]$$

$$\int x^2\,e^{\,2x/5}\,I_0\left(\sqrt{x-\frac{35}{16}}\right)=\frac{5e^{\,-2x/5}}{128}\,\Bigg[(64x^2-220x+175)\,I_0\left(\sqrt{x-\frac{35}{16}}\right)-$$

$$-20(4x-15)\sqrt{x-\frac{35}{16}}\,I_1\left(\sqrt{x-\frac{35}{16}}\right)\Bigg]$$

$$\int x^2\,e^{\,-9x/20}\,J_0\left(\sqrt{x+\frac{20}{81}}\right)=-\frac{20e^{\,-9x/20}}{19683}\,\Bigg[(2187x^2+7020x+1600)\,J_0\left(\sqrt{x+\frac{20}{81}}\right)-$$

$$-90(27x+140)\sqrt{x+\frac{20}{81}}\,J_1\left(\sqrt{x+\frac{20}{81}}\right)\Bigg]$$

$$\int x^2\,e^{\,-9x/20}\,J_0\left(\sqrt{x+\frac{140}{81}}\right)=-\frac{20e^{\,-9x/20}}{19683}\,\Bigg[(2187x^2+7020x+5600)\,J_0\left(\sqrt{x+\frac{140}{81}}\right)-$$

$$-90(27x+100)\sqrt{x+\frac{140}{81}}\,J_1\left(\sqrt{x+\frac{140}{81}}\right)\Bigg]$$

$$\int x^2\,e^{\,9x/20}\,I_0\left(\sqrt{x-\frac{20}{81}}\right)=\frac{20e^{\,9x/20}}{19683}\,\Bigg[(2187x^2-7020x+1600)\,I_0\left(\sqrt{x-\frac{20}{81}}\right)-$$

$$-90(27x-140)\sqrt{x-\frac{20}{81}}\,I_1\left(\sqrt{x-\frac{20}{81}}\right)\Bigg]$$

$$\int x^2\,e^{\,9x/20}\,I_0\left(\sqrt{x-\frac{140}{81}}\right)=\frac{20e^{\,9x/20}}{19683}\,\Bigg[(2187x^2-7020x+5600)\,I_0\left(\sqrt{x-\frac{140}{81}}\right)-$$

$$-90(27x-100)\sqrt{x-\frac{140}{81}}\,I_1\left(\sqrt{x-\frac{140}{81}}\right)\Bigg]$$

$$\int x^2\,e^{-8x/17}\,J_0\left(\sqrt{x+\frac{391}{256}}\right)=-\frac{17e^{-8x/17}}{65536}\left[(8192x^2+25568x+19941)\,J_0\left(\sqrt{x+\frac{391}{256}}\right)-\right.$$

$$\left.-272(32x+119)\sqrt{x+\frac{391}{256}}\,J_1\left(\sqrt{x+\frac{391}{256}}\right)\right]$$

$$\int x^2\,e^{-8x/17}\,J_0\left(\sqrt{x+\frac{119}{256}}\right)=-\frac{17e^{-8x/17}}{65536}\left[(8192x^2+25568x+10115)\,J_0\left(\sqrt{x+\frac{119}{256}}\right)-\right.$$

$$\left.-272(32x+153)\sqrt{x+\frac{119}{256}}\,J_1\left(\sqrt{x+\frac{119}{256}}\right)\right]$$

$$\int x^2\,e^{8x/17}\,I_0\left(\sqrt{x-\frac{391}{256}}\right)=\frac{17e^{8x/17}}{65536}\left[(8192x^2-25568x+19941)\,I_0\left(\sqrt{x-\frac{391}{256}}\right)-\right.$$

$$\left.-272(32x-119)\sqrt{x-\frac{391}{256}}\,J_1\left(\sqrt{x-\frac{391}{256}}\right)\right]$$

$$\int x^2\,e^{8x/17}\,I_0\left(\sqrt{x-\frac{119}{256}}\right)=\frac{17e^{8x/17}}{65536}\left[(8192x^2-25568x+10115)\,I_0\left(\sqrt{x-\frac{119}{256}}\right)-\right.$$

$$\left.-272(32x-153)\sqrt{x-\frac{119}{256}}\,J_1\left(\sqrt{x-\frac{119}{256}}\right)\right]$$

$$\int x^2\,e^{-x/2}\,J_0(\sqrt{x+1}\,)=\;-2\,e^{-x/2}\;\left[(x^2+3x+2)\,J_0(\sqrt{x+1}\,)-(x+4)\sqrt{x+1}\,J_1(\sqrt{x+1}\,)\right]$$

$$\int x^2\,e^{x/2}\,I_0(\sqrt{x-1}\,)=2\,e^{x/2}\;\left[(x^2-3x+2)\,I_0(\sqrt{x-1}\,)-(x-4)\sqrt{x-1}\,I_1(\sqrt{x-1}\,)\right]$$

$$\int x^2\,e^{-x/4}\sqrt{x+6}\,J_0(\sqrt{x+6}\,)\,dx=-4(x+6)\,e^{-x/4}\,[x\sqrt{x+6}\,J_0(\sqrt{x+6}\,)]\;-2(x+6)\,J_1(\sqrt{x+6}\,)$$

$$\int x^2\,e^{x/4}\sqrt{x-6}\,I_0(\sqrt{x-6}\,)\,dx=4(x-6)\,e^{x/4}\,[x\sqrt{x-6}\,I_0(\sqrt{x-6}\,)]\;-2(x-6)\,I_1(\sqrt{x-6}\,)$$

$$\int x^2\,e^{x/4}\sqrt{x-6}\,K_0(\sqrt{x-6}\,)\,dx=4(x-6)\,e^{x/4}\,[x\sqrt{x-6}\,K_0(\sqrt{x-6}\,)]\;+2(x-6)\,K_1(\sqrt{x-6}\,)$$

$$\int x^2\,e^{-x/28}\sqrt{x-42}\,J_0(\sqrt{x-42}\,)\,dx=$$

$$=-28(x-42)\,e^{-x/28}\,[14(x-42)\sqrt{x-42}\,J_0(\sqrt{x-42}\,)]\;-(x-84)\,J_1(\sqrt{x-42}\,)]$$

$$\int x^2\,e^{x/28}\sqrt{x+42}\,I_0(\sqrt{x+42}\,)\,dx=$$

$$=\;-28(x+42)\,e^{x/28}\,[14(x+42)\sqrt{x+42}\,I_0(\sqrt{x+42}\,)]\;-(x+84)\,I_1(\sqrt{x+42}\,)]$$

$$\int x^2\,e^{x/28}\sqrt{x+42}\,K_0(\sqrt{x+42}\,)\,dx=$$

$$=28(x+42)\,e^{x/28}\,[14(x+42)\sqrt{x+42}\,K_0(\sqrt{x+42}\,)]\;+(x+84)\,K_1(\sqrt{x+42}\,)]$$

$$\int \frac{x^2\, e^{-x/12}\, J_0(\sqrt{x-6}\,)\, dx}{\sqrt{x-6}} = e^{-x/12}\, \left[12(12-x)\sqrt{x-6}\, J_0\left(\sqrt{x-6}\,\right) + 72(x-6)\, J_1\left(\sqrt{x-6}\,\right)\right]$$

$$\int \frac{x^2\, e^{x/12}\, I_0(\sqrt{x+6}\,)\, dx}{\sqrt{x+6}} = e^{x/12}\, \left[12(12+x)\sqrt{x+6}\, I_0\left(\sqrt{x+6}\,\right) - 72(x+6)\, I_1\left(\sqrt{x+6}\,\right)\right]$$

$$\int \frac{x^2\, e^{x/12}\, K_0(\sqrt{x+6}\,)\, dx}{\sqrt{x+6}} = e^{x/12}\, \left[12(12+x)\sqrt{x+6}\, K_0\left(\sqrt{x+6}\,\right) + 72(x+6)\, K_1\left(\sqrt{x+6}\,\right)\right]$$

$$\alpha = \frac{3-\sqrt{3}}{24} = 0.05283\ 12164,\ \beta = 6(\sqrt{3}\,+1) = 16.39230\ 48454$$

$$\int x^2\, e^{-\alpha x}\, J_1(\sqrt{x-\beta}\,)\, dx = -4e^{-\alpha x}\, \Big\{12[(2+\sqrt{3}\,)x - 30 - 18\sqrt{3}\,]\sqrt{x-\beta}\, J_0(\sqrt{x-\beta}\,) -$$

$$-[(3+\sqrt{3}\,)x^2 - 12(12+7\sqrt{3}\,)(x-12)]\, J_1(\sqrt{x-\beta}\,)\Big\}$$

$$\int x^2\, e^{\alpha x}\, I_1(\sqrt{x+\beta}\,)\, dx = -4e^{\alpha x}\, \Big\{12[(2+\sqrt{3}\,)x + 30 + 18\sqrt{3}\,]\sqrt{x+\beta}\, I_0(\sqrt{x+\beta}\,) -$$

$$-[(3+\sqrt{3}\,)x^2 + 12(12+7\sqrt{3}\,)(x+12)]\, I_1(\sqrt{x+\beta}\,)\Big\}$$

$$\int x^2\, e^{\alpha x}\, K_1(\sqrt{x+\beta}\,)\, dx = 4e^{\alpha x}\, \Big\{12[(2+\sqrt{3}\,)x + 30 + 18\sqrt{3}\,]\sqrt{x+\beta}\, K_0(\sqrt{x+\beta}\,) +$$

$$+[(3+\sqrt{3}\,)x^2 + 12(12+7\sqrt{3}\,)(x+12)]\, K_1(\sqrt{x+\beta}\,)\Big\}$$

$$\gamma = \frac{3+\sqrt{3}}{24} = 0.19716\ 87836,\ \eta = 6(\sqrt{3}\,-1) = 4.39230\ 48454$$

$$\int x^2\, e^{-\gamma x}\, J_1(\sqrt{x+\eta}\,)\, dx = -4e^{-\gamma x}\, \Big\{12[(2-\sqrt{3}\,)x - 30 + 18\sqrt{3}\,]\sqrt{x+\eta}\, J_0(\sqrt{x+\eta}\,) +$$

$$+[(3-\sqrt{3}\,)x^2 - 12(12-7\sqrt{3}\,)(x-12)]\, J_1(\sqrt{x+\eta}\,)\Big\}$$

$$\int x^2\, e^{\gamma x}\, I_1(\sqrt{x-\eta}\,)\, dx = -4e^{\gamma x}\, \Big\{12[(2-\sqrt{3}\,)x + 30 - 18\sqrt{3}\,]\sqrt{x-\eta}\, I_0(\sqrt{x-\eta}\,) -$$

$$-[(3-\sqrt{3}\,)x^2 + 12(12-7\sqrt{3}\,)(x+12)]\, I_1(\sqrt{x-\eta}\,)\Big\}$$

$$\int x^2\, e^{\gamma x}\, K_1(\sqrt{x-\eta}\,)\, dx = 4e^{\gamma x}\, \Big\{12[(2-\sqrt{3}\,)x + 30 - 18\sqrt{3}\,]\sqrt{x-\eta}\, K_0(\sqrt{x-\eta}\,) +$$

$$+[(3-\sqrt{3}\,)x^2 + 12(12-7\sqrt{3}\,)(x+12)]\, K_1(\sqrt{x-\eta}\,)\Big\}$$

1.4.7. Integrals with $\sin\alpha x\, Z_\nu(\sqrt{x+\beta}\,)$ or $\sin\alpha x\, Z_\nu(\sqrt{x+\beta}\,)$

Only special cases were found.

$$\int x\, \sin\alpha x \cdot \left(x + \frac{1}{4\alpha^2}\right)^{-1/2} \cdot J_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right) dx =$$

$$= \frac{1}{2\alpha^2}\left[\sin\alpha x \cdot J_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right) - 2\alpha\, \cos\alpha x \sqrt{x+\frac{1}{4\alpha^2}} \cdot J_1\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)\right]$$

$$\int x\,\sin\alpha x\cdot\left(x-\frac{1}{4\alpha^2}\right)^{-1/2}\cdot I_0\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)dx=$$

$$=\frac{1}{2\alpha^2}\left[\sin\alpha x\cdot I_0\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)-2\alpha\,\cos\alpha x\cdot\sqrt{(x-\frac{1}{4\alpha^2}}\cdot I_1\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)\right]$$

$$\int x\,\sin\alpha x\cdot\left(x-\frac{1}{4\alpha^2}\right)^{-1/2}\cdot K_0\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)dx=$$

$$=-\frac{1}{2\alpha^2}\left[\sin\alpha x\cdot K_0\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)+2\alpha\,\cos\alpha x\cdot\sqrt{x-\frac{1}{4\alpha^2}}\cdot K_1\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)\right]$$

$$\int x\,\cos\alpha x\cdot\left(x+\frac{1}{4\alpha^2}\right)^{-1/2}\cdot J_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)dx= \qquad (16)$$

$$=\frac{1}{2\alpha^2}\left[\cos\alpha x\cdot J_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)+2\alpha\,\sin\alpha x\cdot\sqrt{x+\frac{1}{4\alpha^2}}\cdot J_1\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)\right] \qquad (17)$$

$$\int x\,\cos\alpha x\cdot\left(x-\frac{1}{4\alpha^2}\right)^{-1/2}\cdot I_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)dx=$$

$$=\frac{1}{2\alpha^2}\left[\cos\alpha x\cdot I_0\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)+2\alpha\,\sin\alpha x\cdot\sqrt{x-\frac{1}{4\alpha^2}}\cdot I_1\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)\right]$$

$$\int x\,\cos\alpha x\cdot\left(x-\frac{1}{4\alpha^2}\right)^{-1/2}\cdot K_0\left(\sqrt{x+\frac{1}{4\alpha^2}}\right)dx=$$

$$=\frac{1}{2\alpha^2}\left[-\cos\alpha x\cdot K_0\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)+2\alpha\,\sin\alpha x\cdot\sqrt{x-\frac{1}{4\alpha^2}}\cdot K_1\left(\sqrt{x-\frac{1}{4\alpha^2}}\right)\right]$$

$$\int x^2\,\sin\frac{\sqrt{2}\,x}{8}\cdot J_0(\sqrt{x+16}\,)\,dx=16\left\{\left[16(x+16)\,J_0(\sqrt{x+16}\,)-4(x-8)\,J_1(\sqrt{x+16}\,)\right]\,\sin\frac{\sqrt{2}\,x}{8}-\right.$$

$$\left.-\left[\sqrt{2}\,(x^2+8x-128)\,J_0(\sqrt{x+16}\,)+48\sqrt{2(x+16)}\,J_1(\sqrt{x+16}\,)\right]\,\cos\frac{\sqrt{2}\,x}{8}\right\}$$

$$\int x^2\,\sin\frac{\sqrt{2}\,x}{8}\cdot I_0(\sqrt{x-16}\,)\,dx=16\left\{\left[16(x-16)\,I_0(\sqrt{x-16}\,)+4(x+8)\,I_1(\sqrt{x-16}\,)\right]\,\sin\frac{\sqrt{2}\,x}{8}-\right.$$

$$\left.-\left[\sqrt{2}\,(x^2-8x-128)\,I_0(\sqrt{x-16}\,)-48\sqrt{2(x+16)}\,I_1(\sqrt{x-16}\,)\right]\,\cos\frac{\sqrt{2}\,x}{8}\right\}$$

$$\int x^2\,\sin\frac{\sqrt{2}\,x}{8}\cdot K_0(\sqrt{x-16}\,)\,dx=16\left\{\left[16(x-16)\,K_0(\sqrt{x-16}\,)-4(x+8)\,K_1(\sqrt{x-16}\,)\right]\,\sin\frac{\sqrt{2}\,x}{8}-\right.$$

$$\left.-\left[\sqrt{2}\,(x^2-8x-128)\,K_0(\sqrt{x-16}\,)+48\sqrt{2(x+16)}\,K_1(\sqrt{x-16}\,)\right]\,\cos\frac{\sqrt{2}\,x}{8}\right\}$$

$$\int x^2\,\cos\frac{\sqrt{2}\,x}{8}\cdot J_0(\sqrt{x+16}\,)\,dx=16\left\{\left[\sqrt{2}\,(x^2+8x-128)\,J_0(\sqrt{x+16}\,)+48\sqrt{2}\,J_1(\sqrt{x+16}\,)\right]\,\sin\frac{\sqrt{2}\,x}{8}+\right.$$

$$+\left[16(x+16)\,J_0(\sqrt{x+16}\,)-4(x-8)\sqrt{x+16}\,J_1(\sqrt{x+16}\,)\right]\,\cos\frac{\sqrt{2}\,x}{8}\Bigg\}$$

$$\int x^2\,\cos\frac{\sqrt{2}\,x}{8}\cdot I_0(\sqrt{x-16}\,)\,dx=16\,\Bigg\{\Big[\sqrt{2}\,(x^2-8x-128)\,I_0(\sqrt{x-16}\,)-48\sqrt{2}\,I_1(\sqrt{x-16}\,)\Big]\,\sin\frac{\sqrt{2}\,x}{8}+$$

$$+\left[16(x-16)\,I_0(\sqrt{x-16}\,)+4(x+8)\sqrt{x-16}\,I_1(\sqrt{x+16}\,)\right]\,\cos\frac{\sqrt{2}\,x}{8}\Bigg\}$$

$$\int x^2\,\cos\frac{\sqrt{2}\,x}{8}\cdot K_0(\sqrt{x-16}\,)\,dx=16\,\Bigg\{\Big[\sqrt{2}\,(x^2-8x-128)\,K_0(\sqrt{x-16}\,)+48\sqrt{2}\,K_1(\sqrt{x-16}\,)\Big]\,\sin\frac{\sqrt{2}\,x}{8}+$$

$$+\left[16(x-16)\,K_0(\sqrt{x-16}\,)-4(x+8)\sqrt{x-16}\,K_1(\sqrt{x+16}\,)\right]\,\cos\frac{\sqrt{2}\,x}{8}\Bigg\}$$

$$\frac{2\sqrt{6}+\sqrt{23}}{4}=2.423\,703,\quad 236-20\sqrt{138}=1.053\,198$$

$$\int x^4\,\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236-20\sqrt{138}}\right)=$$

$$=64\,\Big[(47-4\sqrt{138}\,)x^3+(44170-3760\sqrt{138}\,)x^2+(6183048-526336\sqrt{138}\,)x-1975942080+$$

$$+168203360\sqrt{138}\,\Big]\,\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236-20\sqrt{138}}\right)+$$

$$+8\sqrt{x+236-20\sqrt{138}}\,\Big[(4\sqrt{138}-47)x^3+(4464-380\sqrt{138}\,)x^2+(1413440-120320\sqrt{138}\,)x+982229504-$$

$$-83612928\sqrt{138}\,\Big]\,\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236-20\sqrt{138}}\right)+4\,\Big[(\sqrt{23}-2\sqrt{6}\,)\,)x^4+$$

$$+(380\sqrt{23}-744\sqrt{6}\,)\,)x^3-(54336\sqrt{23}-106384\sqrt{6}\,)\,)x^2-(22858240\sqrt{23}-44753920\sqrt{6}\,)\,)x-$$

$$-8742279680\sqrt{23}+17116422144\sqrt{6}\,\Big]\,\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236-20\sqrt{138}}\right)+$$

$$+32\sqrt{x+236-20\sqrt{138}}\,\Big[(665\sqrt{23}-1302\sqrt{6}\,)x^2+(160152\sqrt{23}-313560\sqrt{6}\,)x-36783680\sqrt{23}+$$

$$+72018400\sqrt{6}\,\Big]\,\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236-20\sqrt{138}}\right)$$

$$\int x^4\,\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236-20\sqrt{138}}\right)=$$

$$=4\,\Big[(2\sqrt{6}-\sqrt{23}\,)x^4+4(186\sqrt{6}-95\sqrt{23}\,)x^3-16(6649\sqrt{6}-3396\sqrt{23}\,)x^2-2560(17482\sqrt{6}-8929\sqrt{23}\,)x+$$

$$+8742279680\sqrt{6}-17116422144\sqrt{23}\,\Big]\,\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236-20\sqrt{138}}\right)+$$

$$+32\sqrt{x+236-20\sqrt{138}}\,\Big[7(186\sqrt{6}-95\sqrt{23}\,)x^2+24(13065\sqrt{6}-6673\sqrt{23}\,)x-72018400\sqrt{6}+$$

$$+36783680\sqrt{23}\,\Big]\,\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236-20\sqrt{138}}\right)+64\,\Big[(47-4\sqrt{138}\,)\,)x^3+$$

$$+10(4417-376\sqrt{138}\,)\,)x^2+8(772881-65792\sqrt{138}\,)\,)x-$$

$$-1975942080+168203360\sqrt{138}\Big]\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236-20\sqrt{138}}\right)-$$

$$-8\sqrt{x+236-20\sqrt{138}}\,\Big[(47-4\sqrt{138}\,)x^3-4(1116-95\sqrt{138}\,)x^2-320(4417-376\sqrt{138}\,)x-982229504+$$

$$+83612928\sqrt{138}\Big]\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236-20\sqrt{138}}\right)$$

$$\int x^4\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\right)=$$

$$=64\,\Big[(47-4\sqrt{138}\,)x^3+(44170-3760\sqrt{138}\,)x^2+(6183048-526336\sqrt{138}\,)x-1975942080+$$

$$+168203360\sqrt{138}\Big]\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\right)+$$

$$+8\sqrt{x+236-20\sqrt{138}}\,\Big[(47-4\sqrt{138}\,)x^3+(4464-380\sqrt{138}\,)x^2-(1413440-120320\sqrt{138}\,)x+982229504-$$

$$-83612928\sqrt{138}\Big]\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236+20\sqrt{138}}\right)+4\,\Big[(\sqrt{23}-2\sqrt{6}\,)\,)x^4-$$

$$-(380\sqrt{23}-744\sqrt{6}\,)\,)x^3-(54336\sqrt{23}-106384\sqrt{6}\,)\,)x^2+(22858240\sqrt{23}-44753920\sqrt{6}\,)\,)x-$$

$$-8742279680\sqrt{23}+17116422144\sqrt{6}\Big]\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\right)+$$

$$+32\sqrt{x+236-20\sqrt{138}}\,\Big[(1302\sqrt{6}-665\sqrt{23}\,)x^2+(160152\sqrt{23}-313560\sqrt{6}\,)x+36783680\sqrt{23}-$$

$$-72018400\sqrt{6}\Big]\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236+20\sqrt{138}}\right)$$

$$\int x^4\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\right)=$$

$$=4\,\Big[(2\sqrt{6}-\sqrt{23}\,)x^4-4(186\sqrt{6}-95\sqrt{23}\,)x^3+16(6649\sqrt{6}-3396\sqrt{23}\,)x^2+2560(17482\sqrt{6}-8929\sqrt{23}\,)x+$$

$$+8742279680\sqrt{6}-17116422144\sqrt{23}\Big]\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\right)-$$

$$-32\sqrt{x-236+20\sqrt{138}}\,\Big[7(186\sqrt{6}-95\sqrt{23}\,)x^2-24(13065\sqrt{6}-6673\sqrt{23}\,)x-72018400\sqrt{6}+$$

$$+36783680\sqrt{23}\Big]\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236+20\sqrt{138}}\right)+64\,\Big[(47-4\sqrt{138}\,)\,)x^3-$$

$$-10(4417-376\sqrt{138}\,)\,)x^2+8(772881-65792\sqrt{138}\,)\,)x+$$

$$+1975942080-168203360\sqrt{138}\Big]\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236+20\sqrt{138}}\right)+$$

$$+8\sqrt{x-236+20\sqrt{138}}\,\Big[(47-4\sqrt{138}\,)x^3+4(1116-95\sqrt{138}\,)x^2-320(4417-376\sqrt{138}\,)x+982229504+$$

$$-83612928\sqrt{138}\Big]\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236+20\sqrt{138}}\right)$$

$$\int x^4\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_0\left(\sqrt{x-236+20\sqrt{138}}\right)=$$

$$=64\,\Big[(47-4\sqrt{138}\,)x^3-(44170-3760\sqrt{138}\,)x^2+(6183048-526336\sqrt{138}\,)x+1975942080-$$

$$-168203360\sqrt{138}\Big]\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_0\left(\sqrt{x-236+20\sqrt{138}}\right)+$$

$$+8\sqrt{x+236-20\sqrt{138}}\,\Big[(4\sqrt{138}-47)x^3-(4464-380\sqrt{138}\,)x^2+(1413440-120320\sqrt{138}\,)x-982229504+$$

$$+83612928\sqrt{138}\Big]\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_1\left(\sqrt{x-236+20\sqrt{138}}\right)+4\,\Big[(\sqrt{23}-2\sqrt{6}\,)\,)x^4-$$

$$-(380\sqrt{23}-744\sqrt{6}\,)\,)x^3-(54336\sqrt{23}-106384\sqrt{6}\,)\,)x^2+(22858240\sqrt{23}-44753920\sqrt{6}\,)\,)x-$$

$$-8742279680\sqrt{23}+17116422144\sqrt{6}\Big]\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_0\left(\sqrt{x-236+20\sqrt{138}}\right)+$$

$$+32\sqrt{x+236-20\sqrt{138}}\,\Big[(665\sqrt{23}-1302\sqrt{6}\,)x^2-(160152\sqrt{23}-313560\sqrt{6}\,)x-36783680\sqrt{23}+$$

$$+72018400\sqrt{6}\Big]\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_1\left(\sqrt{x-236+20\sqrt{138}}\right)$$

$$\int x^4\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_0\left(\sqrt{x-236+20\sqrt{138}}\right)=$$

$$=4\,\Big[(2\sqrt{6}-\sqrt{23}\,)x^4-4(186\sqrt{6}-95\sqrt{23}\,)x^3-16(6649\sqrt{6}-3396\sqrt{23}\,)x^2+2560(17482\sqrt{6}-8929\sqrt{23}\,)x-$$

$$-8742279680\sqrt{6}+17116422144\sqrt{23}\Big]\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_0\left(\sqrt{x-236+20\sqrt{138}}\right)+$$

$$+32\sqrt{x-236+20\sqrt{138}}\,\Big[7(186\sqrt{6}-95\sqrt{23}\,)x^2-24(13065\sqrt{6}-6673\sqrt{23}\,)x-72018400\sqrt{6}+$$

$$+36783680\sqrt{23}\Big]\sin\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_1\left(\sqrt{x-236+20\sqrt{138}}\right)+64\,\Big[(47-4\sqrt{138}\,)\,)x^3-$$

$$-10(4417-376\sqrt{138}\,)\,)x^2+8(772881-65792\sqrt{138}\,)\,)x+$$

$$+1975942080-168203360\sqrt{138}\Big]\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_0\left(\sqrt{x-236+20\sqrt{138}}\right)-$$

$$-8\sqrt{x-236+20\sqrt{138}}\,\Big[(47-4\sqrt{138}\,)x^3-4(1116-95\sqrt{138}\,)x^2+320(4417-376\sqrt{138}\,)x+982229504-$$

$$-83612928\sqrt{138}\Big]\cos\frac{(2\sqrt{6}+\sqrt{23}\,)x}{4}\cdot K_1\left(\sqrt{x-236+20\sqrt{138}}\right)$$

$$\frac{2\sqrt{6}-\sqrt{23}}{4}=0.103\,148\qquad 236+20\sqrt{138}=470.946\,802$$

$$\int x^4\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236+20\sqrt{138}}\right)=$$

$$=-64\,\Big[(47+4\sqrt{138}\,)x^3+(44170+3760\sqrt{138}\,)x^2+(6183048+526336\sqrt{138}\,)x-1975942080-$$

$$-168203360\sqrt{138}\Big]\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236+20\sqrt{138}}\right)+$$

$$+8\sqrt{x+236+20\sqrt{138}}\,\Big[(4\sqrt{138}+47)x^3-(4464+380\sqrt{138}\,)x^2-(1413440+120320\sqrt{138}\,)x-982229504-$$

$$-83612928\sqrt{138}\Big]\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236+20\sqrt{138}}\right)-4\,\Big[(\sqrt{23}+2\sqrt{6}\,)\,)x^4+$$

$$+(380\sqrt{23}+744\sqrt{6}\,)\,)x^3-(54336\sqrt{23}+106384\sqrt{6}\,)\,)x^2-(22858240\sqrt{23}+44753920\sqrt{6}\,)\,)x-$$

$$-8742279680\sqrt{23}-17116422144\sqrt{6}\Big]\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236+20\sqrt{138}}\right)-$$

$$-32\sqrt{x+236+20\sqrt{138}}\,\Big[(665\sqrt{23}+1302\sqrt{6}\,)x^2+(160152\sqrt{23}+313560\sqrt{6}\,)x-36783680\sqrt{23}-$$

$$-72018400\sqrt{6}\Big]\,\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236+20\sqrt{138}}\right)$$

$$\int x^4\,\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236+20\sqrt{138}}\right)=$$

$$=4\,\Big[(2\sqrt{6}+\sqrt{23}\,)x^4+4(186\sqrt{6}+95\sqrt{23}\,)x^3-16(6649\sqrt{6}+3396\sqrt{23}\,)x^2-2560(17482\sqrt{6}+8929\sqrt{23}\,)x-$$

$$-8742279680\sqrt{6}-17116422144\sqrt{23}\Big]\,\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236+20\sqrt{138}}\right)+$$

$$+32\sqrt{x+236+20\sqrt{138}}\,\Big[7(186\sqrt{6}+95\sqrt{23}\,)x^2+24(13065\sqrt{6}+6673\sqrt{23}\,)x-72018400\sqrt{6}-$$

$$-36783680\sqrt{23}\Big]\,\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236+20\sqrt{138}}\right)+64\,\Big[(47+4\sqrt{138}\,)\,)x^3+$$

$$+10(4417+376\sqrt{138}\,)\,)x^2+8(772881+65792\sqrt{138}\,)\,)x-$$

$$-1975942080-168203360\sqrt{138}\Big]\,\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_0\left(\sqrt{x+236+20\sqrt{138}}\right)-$$

$$-8\sqrt{x+236+20\sqrt{138}}\,\Big[(47+4\sqrt{138}\,)x^3-4(1116+95\sqrt{138}\,)x^2-320(4417+376\sqrt{138}\,)x-982229504-$$

$$-83612928\sqrt{138}\Big]\,\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot J_1\left(\sqrt{x+236+20\sqrt{138}}\right)$$

$$\int x^4\,\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)=$$

$$=\;64\,\Big[(47+4\sqrt{138}\,)x^3-(44170+3760\sqrt{138}\,)x^2+(6183048+526336\sqrt{138}\,)x+1975942080+$$

$$+168203360\sqrt{138}\Big]\,\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)+$$

$$+8\sqrt{x-236-20\sqrt{138}}\,\Big[(4\sqrt{138}+47)x^3+(4464+380\sqrt{138}\,)x^2-(1413440+120320\sqrt{138}\,)x+982229504+$$

$$+83612928\sqrt{138}\Big]\,\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236-20\sqrt{138}}\right)-4\,\Big[(\sqrt{23}+2\sqrt{6}\,)\,)x^4-$$

$$-(380\sqrt{23}+744\sqrt{6}\,)\,)x^3-(54336\sqrt{23}+106384\sqrt{6}\,)\,)x^2+(22858240\sqrt{23}+44753920\sqrt{6}\,)\,)x-$$

$$-8742279680\sqrt{23}-17116422144\sqrt{6}\Big]\,\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)+$$

$$+32\sqrt{x-236-20\sqrt{138}}\,\Big[(665\sqrt{23}+1302\sqrt{6}\,)x^2-(160152\sqrt{23}+313560\sqrt{6}\,)x-36783680\sqrt{23}-$$

$$-72018400\sqrt{6}\Big]\,\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_1\left(\sqrt{x-236-20\sqrt{138}}\right)$$

$$\int x^4\,\cos\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)=$$

$$=4\,\Big[(2\sqrt{6}+\sqrt{23}\,)x^4-4(186\sqrt{6}+95\sqrt{23}\,)x^3-16(6649\sqrt{6}+3396\sqrt{23}\,)x^2+2560(17482\sqrt{6}+8929\sqrt{23}\,)x-$$

$$-8742279680\sqrt{6}-17116422144\sqrt{23}\Big]\,\sin\frac{(2\sqrt{6}-\sqrt{23}\,)x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)-$$

$$-32\sqrt{x-236-20\sqrt{138}}\,\Big[7(186\sqrt{6}+95\sqrt{23}\,)x^2-24(13065\sqrt{6}+6673\sqrt{23}\,)x-72018400\sqrt{6}-$$

$$-36783680\sqrt{23}\Big] \sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot I_1\left(\sqrt{x-236-20\sqrt{138}}\right)+64\,\Big[(47+4\sqrt{138})x^3-$$

$$-10(4417+376\sqrt{138})x^2+8(772881+65792\sqrt{138})x+$$

$$+1975942080+168203360\sqrt{138}\Big] \cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot I_0\left(\sqrt{x-236-20\sqrt{138}}\right)-$$

$$-8\sqrt{x-236-20\sqrt{138}}\,\Big[(47+4\sqrt{138})x^3+4(1116+95\sqrt{138})x^2-320(4417+376\sqrt{138})x+982229504+$$

$$+83612928\sqrt{138}\Big] \cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot I_1\left(\sqrt{x-236-20\sqrt{138}}\right)$$

$$\int x^4 \sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)=$$

$$=64\,\Big[(47+4\sqrt{138})x^3-(44170+3760\sqrt{138})x^2+(6183048+526336\sqrt{138})x+1975942080+$$

$$+168203360\sqrt{138}\Big] \sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)+$$

$$-8\sqrt{x-236-20\sqrt{138}}\,\Big[(4\sqrt{138}+47)x^3-(4464+380\sqrt{138})x^2+(1413440+120320\sqrt{138})x+982229504+$$

$$+83612928\sqrt{138}\Big] \sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_1\left(\sqrt{x-236-20\sqrt{138}}\right)-4\,\Big[(\sqrt{23}+2\sqrt{6})x^4-$$

$$-(380\sqrt{23}+744\sqrt{6})x^3-(54336\sqrt{23}+106384\sqrt{6})x^2+(22858240\sqrt{23}+44753920\sqrt{6})x-$$

$$-8742279680\sqrt{23}-17116422144\sqrt{6}\Big] \cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)-$$

$$-32\sqrt{x-236-20\sqrt{138}}\,\Big[(665\sqrt{23}+1302\sqrt{6})x^2-(160152\sqrt{23}+313560\sqrt{6})x-36783680\sqrt{23}-$$

$$-72018400\sqrt{6}\Big] \cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_1\left(\sqrt{x-236-20\sqrt{138}}\right)$$

$$\int x^4 \cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)=$$

$$=4\,\Big[(2\sqrt{6}+\sqrt{23})x^4-4(186\sqrt{6}+95\sqrt{23})x^3-16(6649\sqrt{6}+3396\sqrt{23})x^2+2560(17482\sqrt{6}+8929\sqrt{23})x-$$

$$-8742279680\sqrt{6}-17116422144\sqrt{23}\Big] \sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)+$$

$$+32\sqrt{x-236-20\sqrt{138}}\,\Big[7(186\sqrt{6}+95\sqrt{23})x^2-24(13065\sqrt{6}+6673\sqrt{23})x-72018400\sqrt{6}-$$

$$-36783680\sqrt{23}\Big] \sin\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_1\left(\sqrt{x-236-20\sqrt{138}}\right)+64\,\Big[(47+4\sqrt{138})x^3-$$

$$-10(4417+376\sqrt{138})x^2+8(772881+65792\sqrt{138})x+$$

$$+1975942080+168203360\sqrt{138}\Big] \cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_0\left(\sqrt{x-236-20\sqrt{138}}\right)-$$

$$-8\sqrt{x-236-20\sqrt{138}}\,\Big[(47+4\sqrt{138})x^3+4(1116+95\sqrt{138})x^2-320(4417+376\sqrt{138})x+982229504+$$

$$+83612928\sqrt{138}\Big] \cos\frac{(2\sqrt{6}-\sqrt{23})x}{4}\cdot K_1\left(\sqrt{x-236-20\sqrt{138}}\right)$$

$$4\sqrt{42}+\frac{76}{3}=51.256\,296,\qquad \frac{\sqrt{21}-3\sqrt{2}}{4}=0.084\,984 \quad (36)$$

$$\int x^4\,\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot\left(x+\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot J_1\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)\,dx=$$

$$=\frac{8}{27}\,\Big[9(13+2\sqrt{42}\,)x^3-12(162+25\sqrt{42}\,)x^2+768(337+52\sqrt{42}\,)x-15721344-$$

$$-2425856\sqrt{42}\Big]\,\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{16}{81}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\cdot\Big[27(13+2\sqrt{42}\,)x^2+24(849+131\sqrt{42}\,)x-737664-$$

$$-113824\sqrt{42}\,)\Big]\,\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{64}{27}\,\Big[12(81\sqrt{2}+25\sqrt{21}\,)x^2+12(1429\sqrt{2}+441\sqrt{21}\,)x-1036400\sqrt{2}-$$

$$-319840\sqrt{21}\Big]\,\cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)-$$

$$-\frac{4}{27}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\cdot\Big[3(3\sqrt{2}+\sqrt{21}\,)x^3-12(13\sqrt{2}+4\sqrt{21}\,)x^2-64(81\sqrt{2}+25\sqrt{21}\,)x-$$

$$-1076736\sqrt{2}-332288\sqrt{21}\Big]\,\cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)=$$

$$=(69.23061705\,x^3-1152.065840\,x^2-153372.1068\,x-9316351.912)\,\sin(0.084984\,x)\,J_0(\sqrt{x+51.256296}\,)+$$

$$+\sqrt{x+51.256296}\,(138.4612341\,x^2+8049.668889\,x-291422.7811)\,\sin(0.084984\,x)\,J_1(\sqrt{x+51.256296}\,)+$$

$$+(4887.801406\,x^2+114967.5253\,x-6948460.909)\,\cos(0.084984\,x)\,J_0(\sqrt{x+51.256296}\,)-$$

$$-\sqrt{x+51.256296}\,(3.922318391\,x^3-65.27125170\,x^2-2172.356181\,x-$$

$$-451180.6764)\,\cos(0.084984\,x)\,J_1(\sqrt{x+51.256296}\,)$$

$$\int x^4\,\cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot\left(x+\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot J_1\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)\,dx=$$

$$=-\frac{64}{27}\,\Big[9(25\sqrt{21}+81\sqrt{2}\,)x^2+12(441\sqrt{21}+1429\sqrt{2}\,)x-319840\sqrt{21}-$$

$$-1036400\sqrt{2}\,\Big]\,\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{9}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\,\Big[3(\sqrt{21}+3\sqrt{2}\,)x^3-12(4\sqrt{21}+13\sqrt{2}\,)x^2-64(25\sqrt{21}+81\sqrt{2}\,)x-332288\sqrt{21}-$$

$$-1076736\sqrt{2}\,\Big]\,\sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{8}{27}\,\Big[9(13+2\sqrt{42}\,)x^3-12(25+162\sqrt{42}\,)x^2-768(52+337\sqrt{42}\,)x-15721344-$$

$$-2425856\sqrt{42}\,\Big]\,\cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{16}{27}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\,\Big[27(13+2\sqrt{42}\,)x^2+24(849+131\sqrt{42}\,)x-737664-$$

$$-113824\sqrt{42}\Big] \cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)$$

$$\int x^4 \sin\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot\left(x-\frac{76}{3}-4\sqrt{42}\right)^{-1/2}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)dx=$$

$$=\frac{8}{27}\Big[9(13+2\sqrt{42})x^3+12(162+25\sqrt{42})x^2-768(337+52\sqrt{42})x+15721344+$$

$$+2425856\sqrt{42}\Big] \sin\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{16}{81}\sqrt{x-\frac{76}{3}-4\sqrt{42}}\cdot\Big[27(13+2\sqrt{42})x^2-24(849+131\sqrt{42})x-737664-$$

$$-113824\sqrt{42})\Big] \sin\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{64}{27}\Big[12(81\sqrt{2}+25\sqrt{21})x^2-12(1429\sqrt{2}+441\sqrt{21})x-1036400\sqrt{2}-$$

$$-319840\sqrt{21}\Big] \cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)-$$

$$-\frac{4}{27}\sqrt{x-\frac{76}{3}-4\sqrt{42}}\cdot\Big[3(3\sqrt{2}+\sqrt{21})x^3+12(13\sqrt{2}+4\sqrt{21})x^2-64(81\sqrt{2}+25\sqrt{21})x+$$

$$+1076736\sqrt{2}-+332288\sqrt{21}\Big] \cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)$$

$$\int x^4 \cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot\left(x+\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)dx=$$

$$=-\frac{64}{27}\Big[9(25\sqrt{21}+81\sqrt{2})x^2-12(441\sqrt{21}+1429\sqrt{2})x-319840\sqrt{21}-$$

$$-1036400\sqrt{2}\Big] \sin\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{9}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\Big[3(\sqrt{21}+3\sqrt{2})x^3+12(4\sqrt{21}+13\sqrt{2})x^2-64(25\sqrt{21}+81\sqrt{2})x+332288\sqrt{21}+$$

$$+1076736\sqrt{2}\Big] \sin\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{8}{27}\Big[9(13+2\sqrt{42})x^3+12(25+162\sqrt{42})x^2-768(52+337\sqrt{42})x+15721344+$$

$$+2425856\sqrt{42}\Big] \cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{16}{27}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\Big[27(13+2\sqrt{42})x^2-24(849+131\sqrt{42})x-737664-$$

$$-113824\sqrt{42}\Big] \cos\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)$$

$$\int x^4 \sin\frac{(\sqrt{21}-3\sqrt{2})x}{4}\cdot\left(x-\frac{76}{3}-4\sqrt{42}\right)^{-1/2}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)dx=$$

$$=-\frac{8}{27}\Big[9(13+2\sqrt{42})x^3+12(162+25\sqrt{42})x^2-768(337+52\sqrt{42})x+15721344+$$

$$+2425856\sqrt{42}\Big]\ \sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{16}{81}\sqrt{x-\frac{76}{3}-4\sqrt{42}}\cdot\Big[27(13+2\sqrt{42}\,)x^2-24(849+131\sqrt{42}\,)x-737664-$$

$$-113824\sqrt{42}\,)\Big]\ \sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)-$$

$$-\frac{64}{27}\ \Big[12(81\sqrt{2}+25\sqrt{21}\,)x^2-12(1429\sqrt{2}+441\sqrt{21}\,)x-1036400\sqrt{2}-$$

$$-319840\sqrt{21}\Big]\ \cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)-$$

$$-\frac{4}{27}\sqrt{x-\frac{76}{3}-4\sqrt{42}}\cdot\Big[3(3\sqrt{2}+\sqrt{21}\,)x^3+12(13\sqrt{2}+4\sqrt{21}\,)x^2-64(81\sqrt{2}+25\sqrt{21}\,)x+$$

$$+1076736\sqrt{2}-+332288\sqrt{21}\Big]\ \cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)$$

$$\int x^4\ \cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot\left(x+\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)\,dx=$$

$$=\ \frac{64}{27}\ \Big[9(25\sqrt{21}+81\sqrt{2}\,)x^2-12(441\sqrt{21}+1429\sqrt{2}\,)x-319840\sqrt{21}-$$

$$-1036400\sqrt{2}\,\Big]\ \sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot K_0\left(\sqrt{x+\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{9}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\ \Big[3(\sqrt{21}+3\sqrt{2}\,)x^3+12(4\sqrt{21}+13\sqrt{2}\,)x^2-64(25\sqrt{21}+81\sqrt{2}\,)x+332288\sqrt{21}+$$

$$+1076736\sqrt{2}\,\Big]\ \sin\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)-$$

$$-\frac{8}{27}\ \Big[9(13+2\sqrt{42}\,)x^3+12(25+162\sqrt{42}\,)x^2-768(52+337\sqrt{42}\,)x+15721344+$$

$$+2425856\sqrt{42}\,\Big]\ \cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{16}{27}\sqrt{x+\frac{76}{3}+4\sqrt{42}}\ \Big[27(13+2\sqrt{42}\,)x^2-24(849+131\sqrt{42}\,)x-737664-$$

$$-113824\sqrt{42}\,\Big]\ \cos\frac{(\sqrt{21}-3\sqrt{2}\,)x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}-4\sqrt{42}}\right)$$

$$\frac{76}{3}-4\sqrt{42}=-0.589\,629,\qquad\frac{\sqrt{21}+3\sqrt{2}}{4}=2.206\,304$$

$$\int x^4\ \sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot\left(x+\frac{76}{3}-4\sqrt{42}\right)^{-1/2}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)\,dx=$$

$$=\ -\frac{8}{27}\ \Big[9(13+2\sqrt{42}\,)x^3+12(162+25\sqrt{42}\,)x^2-768(337+52\sqrt{42}\,)x+15721344+$$

$$+2425856\sqrt{42}\Big]\ \sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{16}{81}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\cdot\Big[27(13+2\sqrt{42}\,)x^2-24(849+131\sqrt{42}\,)x-737664-$$

$$-113824\sqrt{42}\,)\Big]\,\sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)-$$

$$-\frac{64}{27}\,\Big[12(81\sqrt{2}+25\sqrt{21}\,)x^2-12(1429\sqrt{2}+441\sqrt{21}\,)x-1036400\sqrt{2}-$$

$$-319840\sqrt{21}\Big]\,\cos\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)-$$

$$-\frac{4}{27}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\cdot\Big[3(3\sqrt{2}+\sqrt{21}\,)x^3+12(13\sqrt{2}+4\sqrt{21}\,)x^2-64(81\sqrt{2}+25\sqrt{21}\,)x+$$

$$+1076736\sqrt{2}+332288\sqrt{21}\Big]\,\cos\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)$$

$$\int x^4\,\cos\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot\left(x+\frac{76}{3}-4\sqrt{42}\right)^{-1/2}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)\,dx=$$

$$=\frac{64}{27}\,\Big[9(25\sqrt{21}-81\sqrt{2}\,)x^2-12(441\sqrt{21}-1429\sqrt{2}\,)x-319840\sqrt{21}+$$

$$+1036400\sqrt{2}\,\Big]\,\sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{4}{9}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\,\Big[3(\sqrt{21}-3\sqrt{2}\,)x^3-12(4\sqrt{21}-13\sqrt{2}\,)x^2-64(25\sqrt{21}-81\sqrt{2}\,)x-332288\sqrt{21}+$$

$$+1076736\sqrt{2}\,\Big]\,\sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{8}{27}\,\Big[9(13-2\sqrt{42}\,)x^3-12(25-162\sqrt{42}\,)x^2-768(52-337\sqrt{42}\,)x-15721344+$$

$$+2425856\sqrt{42}\,\Big]\,\cos\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot J_0\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)+$$

$$+\frac{16}{27}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\,\Big[27(13-2\sqrt{42}\,)x^2-24(849-131\sqrt{42}\,)x-737664+$$

$$+113824\sqrt{42}\,\Big]\,\cos\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot J_1\left(\sqrt{x+\frac{76}{3}-4\sqrt{42}}\right)$$

$$\int x^4\,\sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot\left(x-\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)\,dx=$$

$$=\,-\frac{8}{27}\,\Big[9(13-2\sqrt{42}\,)x^3+12(162-25\sqrt{42}\,)x^2+768(337-52\sqrt{42}\,)x+15721344-$$

$$-2425856\sqrt{42}\Big]\,\sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{16}{81}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\cdot\Big[27(13-2\sqrt{42}\,)x^2-24(849-131\sqrt{42}\,)x-737664+$$

$$+113824\sqrt{42}\,)\Big]\,\sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)-$$

$$-\frac{64}{27}\,\Big[12(81\sqrt{2}-25\sqrt{21}\,)x^2-12(1429\sqrt{2}-441\sqrt{21}\,)x-1036400\sqrt{2}+$$

$$+319840\sqrt{21}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{27}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\cdot\Big[3(3\sqrt{2}-\sqrt{21})x^3+12(13\sqrt{2}-4\sqrt{21})x^2-64(81\sqrt{2}-25\sqrt{21})x+$$

$$+1076736\sqrt{2}-332288\sqrt{21}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)$$

$$\int x^4\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot\left(x-\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)dx=$$

$$=-\frac{64}{27}\Big[9(25\sqrt{21}-81\sqrt{2})x^2-12(441\sqrt{21}-1429\sqrt{2})x-319840\sqrt{21}+$$

$$+1036400\sqrt{2}\Big]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{9}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\Big[3(\sqrt{21}-3\sqrt{2})x^3+12(4\sqrt{21}-13\sqrt{2})x^2-64(25\sqrt{21}-81\sqrt{2})x+332288\sqrt{21}-$$

$$-1076736\sqrt{2}\Big]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{8}{27}\Big[9(13-2\sqrt{42})x^3+12(25-162\sqrt{42})x^2-768(52-337\sqrt{42})x+15721344-$$

$$-2425856\sqrt{42}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{16}{27}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\Big[27(13-2\sqrt{42})x^2-24(849-131\sqrt{42})x-737664+$$

$$+113824\sqrt{42}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot I_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)$$

$$\int x^4\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot\left(x-\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)dx=$$

$$=-\frac{8}{27}\Big[9(13-2\sqrt{42})x^3+12(162-25\sqrt{42})x^2-768(337-52\sqrt{42})x+15721344-$$

$$-2425856\sqrt{42}\Big]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{16}{81}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\cdot\Big[27(13-2\sqrt{42})x^2-24(849-131\sqrt{42})x-737664+$$

$$+113824\sqrt{42})\Big]\sin\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{64}{27}\Big[12(81\sqrt{2}-25\sqrt{21})x^2-12(1429\sqrt{2}-441\sqrt{21})x-1036400\sqrt{2}+$$

$$+319840\sqrt{21}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{27}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\cdot\Big[3(3\sqrt{2}-\sqrt{21})x^3+12(13\sqrt{2}-4\sqrt{21})x^2-64(81\sqrt{2}-25\sqrt{21})x+$$

$$+1076736\sqrt{2}-332288\sqrt{21}\Big]\cos\frac{(\sqrt{21}+3\sqrt{2})x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)$$

$$\int x^4\, \cos\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot\left(x-\frac{76}{3}+4\sqrt{42}\right)^{-1/2}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)\,dx=$$

$$=\frac{64}{27}\,\Big[9(25\sqrt{21}-81\sqrt{2}\,)x^2-12(441\sqrt{21}-1429\sqrt{2}\,)x-319840\sqrt{21}+$$

$$+1036400\sqrt{2}\,\Big]\,\sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{4}{9}\sqrt{x+\frac{76}{3}-4\sqrt{42}}\,\Big[3(\sqrt{21}-3\sqrt{2}\,)x^3+12(4\sqrt{21}-13\sqrt{2}\,)x^2-64(25\sqrt{21}-81\sqrt{2}\,)x+332288\sqrt{21}-$$

$$-1076736\sqrt{2}\,\Big]\,\sin\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)-$$

$$-\frac{8}{27}\,\Big[9(13-2\sqrt{42}\,)x^3+12(25-162\sqrt{42}\,)x^2-768(52-337\sqrt{42}\,)x+15721344-$$

$$-2425856\sqrt{42}\,\Big]\,\cos\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot K_0\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)+$$

$$+\frac{16}{27}\sqrt{x-\frac{76}{3}+4\sqrt{42}}\,\Big[27(13-2\sqrt{42}\,)x^2-24(849-131\sqrt{42}\,)x-737664+$$

$$+113824\sqrt{42}\,\Big]\,\cos\frac{(\sqrt{21}+3\sqrt{2}\,)x}{4}\cdot K_1\left(\sqrt{x-\frac{76}{3}+4\sqrt{42}}\right)$$

Part II

Integrals with two or more Bessel Functions

Gaussian Quadrature Formulas

2. Products of two Bessel functions

2.1. Bessel Functions with the the same Argument x :

See also [10], 3. .

2.1.1. Integrals of the type $\int x^{2n+1} Z_\nu^2(x)\,dx$

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x),\ p=1,2$.

$$\int \frac{J_1^2(x)}{x}\,dx = -\frac{1}{2}\left[J_0^2(x)+J_1^2(x)\right]$$

$$\int \frac{I_1^2(x)}{x}\,dx = \frac{1}{2}\left[I_0^2(x)-I_1^2(x)\right]$$

$$\int \frac{K_1^2(x)}{x}\,dx = \frac{1}{2}\left[K_0^2(x)-K_1^2(x)\right]$$

$$\int xJ_0^2(x)\,dx = \frac{x^2}{2}\left[J_0^2(x)+J_1^2(x)\right]$$

$$\int xJ_1^2(x)\,dx = \frac{x}{2}\left[xJ_0^2(x)+xJ_1^2(x)-2J_0(x)\cdot J_1(x)\right]$$

$$\int xI_0^2(x)\,dx = \frac{x^2}{2}\left[I_0^2(x)-I_1^2(x)\right]$$

$$\int xI_1^2(x)\,dx = \frac{x}{2}\left[xI_1^2(x)-xI_0^2(x)-2I_0(x)\cdot I_1(x)\right]$$

$$\int xK_0^2(x)\,dx = \frac{x^2}{2}\left[K_0^2(x)-K_1^2(x)\right]$$

$$\int xK_1^2(x)\,dx = \frac{x}{2}\left[xK_1^2(x)-xK_0^2(x)+2K_0(x)\cdot K_1(x)\right]$$

$$\int x^3J_0^2(x)\,dx = \frac{x^4}{6}J_0^2(x)+\frac{x^3}{3}J_0(x)J_1(x)+\left(\frac{x^4}{6}-\frac{x^2}{3}\right)J_1^2(x)$$

$$\int x^3J_1^2(x)\,dx = \frac{x^4}{6}J_0^2(x)-\frac{2x^3}{3}J_0(x)J_1(x)+\left(\frac{x^4}{6}+\frac{2x^2}{3}\right)J_1^2(x)$$

$$\int x^3I_0^2(x)\,dx = \frac{x^4}{6}I_0^2(x)+\frac{x^3}{3}I_0(x)I_1(x)-\left(\frac{x^4}{6}+\frac{x^2}{3}\right)I_1^2(x)$$

$$\int x^3I_1^2(x)\,dx = -\frac{x^4}{6}I_0^2(x)+\frac{2x^3}{3}I_0(x)I_1(x)+\left(\frac{x^4}{6}-\frac{2x^2}{3}\right)I_1^2(x)$$

$$\int x^3K_0^2(x)\,dx = \frac{x^4}{6}K_0^2(x)-\frac{x^3}{3}K_0(x)K_1(x)-\left(\frac{x^4}{6}+\frac{x^2}{3}\right)K_1^2(x)$$

$$\int x^3K_1^2(x)\,dx = -\frac{x^4}{6}K_0^2(x)-\frac{2x^3}{3}K_0(x)K_1(x)+\left(\frac{x^4}{6}-\frac{2x^2}{3}\right)K_1^2(x)$$

$$\int x^5J_0^2(x)\,dx = \left(\frac{x^6}{10}+\frac{4x^4}{15}\right)J_0^2(x)+\left(\frac{2x^5}{5}-\frac{16x^3}{15}\right)J_0(x)J_1(x)+\left(\frac{x^6}{10}-\frac{8x^4}{15}+\frac{16x^2}{15}\right)J_1^2(x)$$

$$\int x^5J_1^2(x)\,dx = \left(\frac{x^6}{10}-\frac{2x^4}{5}\right)J_0^2(x)+\left(-\frac{3x^5}{5}+\frac{8x^3}{5}\right)J_0(x)J_1(x)+\left(\frac{x^6}{10}+\frac{4x^4}{5}-\frac{8x^2}{5}\right)J_1^2(x)$$

$$\int x^5I_0^2(x)\,dx = \left(\frac{x^6}{10}-\frac{4x^4}{15}\right)I_0^2(x)+\left(\frac{2x^5}{5}+\frac{16x^3}{15}\right)I_0(x)I_1(x)-\left(\frac{x^6}{10}+\frac{8x^4}{15}+\frac{16x^2}{15}\right)I_1^2(x)$$

$$\int x^5I_1^2(x)\,dx = -\left(\frac{x^6}{10}+\frac{2x^4}{5}\right)I_0^2(x)+\left(\frac{3x^5}{5}+\frac{8x^3}{5}\right)I_0(x)I_1(x)+\left(\frac{x^6}{10}-\frac{4x^4}{5}-\frac{8x^2}{5}\right)I_1^2(x)$$

$$\int x^5 K_0^2(x)\,dx = \left(\frac{x^6}{10} - \frac{4x^4}{15}\right) K_0^2(x) - \left(\frac{2x^5}{5} + \frac{16x^3}{15}\right) K_0(x)K_1(x) - \left(\frac{x^6}{10} + \frac{8x^4}{15} + \frac{16x^2}{15}\right) K_1^2(x)$$

$$\int x^5 K_1^2(x)\,dx = -\left(\frac{x^6}{10} + \frac{2x^4}{5}\right) K_0^2(x) - \left(\frac{3x^5}{5} + \frac{8x^3}{5}\right) K_0(x)K_1(x) + \left(\frac{x^6}{10} - \frac{4x^4}{5} - \frac{8x^2}{5}\right) K_1^2(x)$$

$$\int x^7 J_0^2(x)\,dx = \left(\frac{x^8}{14} + \frac{18x^6}{35} - \frac{72x^4}{35}\right) J_0^2(x) + \left(\frac{3x^7}{7} - \frac{108x^5}{35} + \frac{288x^3}{35}\right) J_0(x)J_1(x) + \\ + \left(\frac{x^8}{14} - \frac{27x^6}{35} + \frac{144x^4}{35} - \frac{288x^2}{35}\right) J_1^2(x)$$

$$\int x^7 J_1^2(x)\,dx = \left(\frac{x^8}{14} - \frac{24x^6}{35} + \frac{96x^4}{35}\right) J_0^2(x) + \left(-\frac{4x^7}{7} + \frac{144x^5}{35} - \frac{384x^3}{35}\right) J_0(x)J_1(x) + \\ + \left(\frac{x^8}{14} + \frac{36x^6}{35} - \frac{192x^4}{35} + \frac{384x^2}{35}\right) J_1^2(x)$$

$$\int x^7 I_0^2(x)\,dx = \left(\frac{x^8}{14} - \frac{18x^6}{35} - \frac{72x^4}{35}\right) I_0^2(x) + \left(\frac{3x^7}{7} + \frac{108x^5}{35} + \frac{288x^3}{35}\right) I_0(x)I_1(x) - \\ - \left(\frac{x^8}{14} + \frac{27x^6}{35} + \frac{144x^4}{35} + \frac{288x^2}{35}\right) I_1^2(x)$$

$$\int x^7 I_1^2(x)\,dx = -\left(\frac{x^8}{14} + \frac{24x^6}{35} + \frac{96x^4}{35}\right) I_0^2(x) + \left(\frac{4x^7}{7} + \frac{144x^5}{35} + \frac{384x^3}{35}\right) I_0(x)I_1(x) + \\ + \left(\frac{x^8}{14} - \frac{36x^6}{35} - \frac{192x^4}{35} - \frac{384x^2}{35}\right) I_1^2(x)$$

$$\int x^7 K_0^2(x)\,dx = \left(\frac{x^8}{14} - \frac{18x^6}{35} - \frac{72x^4}{35}\right) K_0^2(x) - \left(\frac{3x^7}{7} + \frac{108x^5}{35} + \frac{288x^3}{35}\right) K_0(x)K_1(x) - \\ - \left(\frac{x^8}{14} + \frac{27x^6}{35} + \frac{144x^4}{35} + \frac{288x^2}{35}\right) K_1^2(x)$$

$$\int x^7 K_1^2(x)\,dx = -\left(\frac{x^8}{14} + \frac{24x^6}{35} + \frac{96x^4}{35}\right) K_0^2(x) - \left(\frac{4x^7}{7} + \frac{144x^5}{35} + \frac{384x^3}{35}\right) K_0(x)K_1(x) + \\ + \left(\frac{x^8}{14} - \frac{36x^6}{35} - \frac{192x^4}{35} - \frac{384x^2}{35}\right) K_1^2(x)$$

$$\int x^9 J_0^2(x)\,dx = \left(\frac{x^{10}}{18} + \frac{16x^8}{21} - \frac{256x^6}{35} + \frac{1024x^4}{35}\right) J_0^2(x) + \\ + \left(\frac{4x^9}{9} - \frac{128x^7}{21} + \frac{1536x^5}{35} - \frac{4096x^3}{35}\right) J_0(x)J_1(x) + \\ + \left(\frac{x^{10}}{18} - \frac{64x^8}{63} + \frac{384x^6}{35} - \frac{2048x^4}{35} + \frac{4096x^2}{35}\right) J_1^2(x)$$

$$\int x^9 J_1^2(x)\,dx = \left(\frac{x^{10}}{18} - \frac{20x^8}{21} + \frac{64x^6}{7} - \frac{256x^4}{7}\right) J_0^2(x) + \\ + \left(-\frac{5x^9}{9} + \frac{160x^7}{21} - \frac{384x^5}{7} + \frac{1024x^3}{7}\right) J_0(x)J_1(x) + \\ + \left(\frac{x^{10}}{18} + \frac{80x^8}{63} - \frac{96x^6}{7} + \frac{512x^4}{7} - \frac{1024x^2}{7}\right) J_1^2(x)$$

$$\int x^9 I_0^2(x)\,dx = \left(\frac{x^{10}}{18} - \frac{16x^8}{21} - \frac{256x^6}{35} - \frac{1024x^4}{35}\right) I_0^2(x) + \\ + \left(\frac{4x^9}{9} + \frac{128x^7}{21} + \frac{1536x^5}{35} + \frac{4096x^3}{35}\right) I_0(x)I_1(x) -$$

$$-\left(\frac{x^{10}}{18}+\frac{64x^8}{63}+\frac{384x^6}{35}+\frac{2048x^4}{35}+\frac{4096x^2}{35}\right)I_1^2(x)$$

$$\int x^9 I_1^2(x)\,dx \;=\; -\left(\frac{x^{10}}{18}+\frac{20x^8}{21}+\frac{64x^6}{7}+\frac{256x^4}{7}\right)I_0^2(x)+$$
$$+\left(\frac{5x^9}{9}+\frac{160x^7}{21}+\frac{384x^5}{7}+\frac{1024x^3}{7}\right)I_0(x)I_1(x)+$$
$$+\left(\frac{x^{10}}{18}-\frac{80x^8}{63}-\frac{96x^6}{7}-\frac{512x^4}{7}-\frac{1024x^2}{7}\right)I_1^2(x)$$

$$\int x^9 K_0^2(x)\,dx \;=\; \left(\frac{x^{10}}{18}-\frac{16x^8}{21}-\frac{256x^6}{35}-\frac{1024x^4}{35}\right)K_0^2(x)-$$
$$-\left(\frac{4x^9}{9}+\frac{128x^7}{21}+\frac{1536x^5}{35}+\frac{4096x^3}{35}\right)K_0(x)K_1(x)-$$
$$-\left(\frac{x^{10}}{18}+\frac{64x^8}{63}+\frac{384x^6}{35}+\frac{2048x^4}{35}+\frac{4096x^2}{35}\right)K_1^2(x)$$

$$\int x^9 K_1^2(x)\,dx \;=\; -\left(\frac{x^{10}}{18}+\frac{20x^8}{21}+\frac{64x^6}{7}+\frac{256x^4}{7}\right)K_0^2(x)-$$
$$-\left(\frac{5x^9}{9}+\frac{160x^7}{21}+\frac{384x^5}{7}+\frac{1024x^3}{7}\right)K_0(x)K_1(x)+$$
$$+\left(\frac{x^{10}}{18}-\frac{80x^8}{63}-\frac{96x^6}{7}-\frac{512x^4}{7}-\frac{1024x^2}{7}\right)K_1^2(x)$$

Let

$$\int x^m\cdot J_0^2(x)\,dx = A_m(x)\cdot J_0^2(x)+B_m(x)\cdot J_0(x)\cdot J_1(x)+C_m(x)\cdot J_1^2(x)\;,$$
$$\int x^m\cdot J_1^2(x)\,dx = D_m(x)\cdot J_0^2(x)+E_m(x)\cdot J_0(x)\cdot J_1(x)+F_m(x)\cdot J_1^2(x)\;,$$
$$\int x^m\cdot I_0^2(x)\,dx = A_m^*(x)\cdot I_0^2(x)+B_m^*(x)\cdot I_0(x)\cdot I_1(x)+C_m^*(x)\cdot I_1^2(x)\;,$$
$$\int x^m\cdot I_1^2(x)\,dx = D_m^*(x)\cdot I_0^2(x)+E_m^*(x)\cdot I_0(x)\cdot I_1(x)+F_m^*(x)\cdot I_1^2(x)$$

and

$$\int x^m\cdot K_0^2(x)\,dx = A_m^*(x)\cdot K_0^2(x)-B_m^*(x)\cdot K_0(x)\cdot K_1(x)+C_m^*(x)\cdot K_1^2(x)\;,$$
$$\int x^m\cdot K_1^2(x)\,dx = D_m^*(x)\cdot K_0^2(x)-E_m^*(x)\cdot K_0(x)\cdot K_1(x)+F_m^*(x)\cdot K_1^2(x)\;,$$

then holds

$$A_{11}=\frac{1}{22}\,x^{12}+\frac{100}{99}\,x^{10}-\frac{4000}{231}\,x^8+\frac{12800}{77}\,x^6-\frac{51200}{77}\,x^4$$
$$B_{11}=\frac{5}{11}\,x^{11}-\frac{1000}{99}\,x^9+\frac{32000}{231}\,x^7-\frac{76800}{77}\,x^5+\frac{204800}{77}\,x^3$$
$$C_{11}=\frac{1}{22}\,x^{12}-\frac{125}{99}\,x^{10}+\frac{16000}{693}\,x^8-\frac{19200}{77}\,x^6+\frac{102400}{77}\,x^4-\frac{204800}{77}\,x^2$$
$$D_{11}=\frac{1}{22}\,x^{12}-\frac{40}{33}\,x^{10}+\frac{1600}{77}\,x^8-\frac{15360}{77}\,x^6+\frac{61440}{77}\,x^4$$
$$E_{11}=-\frac{6}{11}\,x^{11}+\frac{400}{33}\,x^9-\frac{12800}{77}\,x^7+\frac{92160}{77}\,x^5-\frac{245760}{77}\,x^3$$
$$F_{11}=\frac{1}{22}\,x^{12}+\frac{50}{33}\,x^{10}-\frac{6400}{231}\,x^8+\frac{23040}{77}\,x^6-\frac{122880}{77}\,x^4+\frac{245760}{77}\,x^2$$
$$A_{11}^*=\frac{1}{22}\,x^{12}-\frac{100}{99}\,x^{10}-\frac{4000}{231}\,x^8-\frac{12800}{77}\,x^6-\frac{51200}{77}\,x^4$$

$$B_{11}^* = \frac{5}{11}x^{11} + \frac{1000}{99}x^9 + \frac{32000}{231}x^7 + \frac{76800}{77}x^5 + \frac{204800}{77}x^3$$
$$C_{11}^* = -\frac{1}{22}x^{12} - \frac{125}{99}x^{10} - \frac{16000}{693}x^8 - \frac{19200}{77}x^6 - \frac{102400}{77}x^4 - \frac{204800}{77}x^2$$
$$D_{11}^* = -\frac{1}{22}x^{12} - \frac{40}{33}x^{10} - \frac{1600}{77}x^8 - \frac{15360}{77}x^6 - \frac{61440}{77}x^4$$
$$E_{11}^* = \frac{6}{11}x^{11} + \frac{400}{33}x^9 + \frac{12800}{77}x^7 + \frac{92160}{77}x^5 + \frac{245760}{77}x^3$$
$$F_{11}^* = \frac{1}{22}x^{12} - \frac{50}{33}x^{10} - \frac{6400}{231}x^8 - \frac{23040}{77}x^6 - \frac{122880}{77}x^4 - \frac{245760}{77}x^2$$

$$A_{13} = \frac{1}{26}x^{14} + \frac{180}{143}x^{12} - \frac{4800}{143}x^{10} + \frac{576000}{1001}x^8 - \frac{5529600}{1001}x^6 + \frac{22118400}{1001}x^4$$
$$B_{13} = \frac{6}{13}x^{13} - \frac{2160}{143}x^{11} + \frac{48000}{143}x^9 - \frac{4608000}{1001}x^7 + \frac{33177600}{1001}x^5 - \frac{88473600}{1001}x^3$$
$$C_{13} = \frac{1}{26}x^{14} - \frac{216}{143}x^{12} + \frac{6000}{143}x^{10} - \frac{768000}{1001}x^8 + \frac{8294400}{1001}x^6 - \frac{44236800}{1001}x^4 + \frac{88473600}{1001}x^2$$
$$D_{13} = \frac{1}{26}x^{14} - \frac{210}{143}x^{12} + \frac{5600}{143}x^{10} - \frac{96000}{143}x^8 + \frac{921600}{143}x^6 - \frac{3686400}{143}x^4$$
$$E_{13} = -\frac{7}{13}x^{13} + \frac{2520}{143}x^{11} - \frac{56000}{143}x^9 + \frac{768000}{143}x^7 - \frac{5529600}{143}x^5 + \frac{14745600}{143}x^3$$
$$F_{13} = \frac{1}{26}x^{14} + \frac{252}{143}x^{12} - \frac{7000}{143}x^{10} + \frac{128000}{143}x^8 - \frac{1382400}{143}x^6 + \frac{7372800}{143}x^4 - \frac{14745600}{143}x^2$$
$$A_{13}^* = \frac{1}{26}x^{14} - \frac{180}{143}x^{12} - \frac{4800}{143}x^{10} - \frac{576000}{1001}x^8 - \frac{5529600}{1001}x^6 - \frac{22118400}{1001}x^4$$
$$B_{13}^* = \frac{6}{13}x^{13} + \frac{2160}{143}x^{11} + \frac{48000}{143}x^9 + \frac{4608000}{1001}x^7 + \frac{33177600}{1001}x^5 + \frac{88473600}{1001}x^3$$
$$C_{13}^* = -\frac{1}{26}x^{14} - \frac{216}{143}x^{12} - \frac{6000}{143}x^{10} - \frac{768000}{1001}x^8 - \frac{8294400}{1001}x^6 - \frac{44236800}{1001}x^4 - \frac{88473600}{1001}x^2$$
$$D_{13}^* = -\frac{1}{26}x^{14} - \frac{210}{143}x^{12} - \frac{5600}{143}x^{10} - \frac{96000}{143}x^8 - \frac{921600}{143}x^6 - \frac{3686400}{143}x^4$$
$$E_{13}^* = \frac{7}{13}x^{13} + \frac{2520}{143}x^{11} + \frac{56000}{143}x^9 + \frac{768000}{143}x^7 + \frac{5529600}{143}x^5 + \frac{14745600}{143}x^3$$
$$F_{13}^* = \frac{1}{26}x^{14} - \frac{252}{143}x^{12} - \frac{7000}{143}x^{10} - \frac{128000}{143}x^8 - \frac{1382400}{143}x^6 - \frac{7372800}{143}x^4 - \frac{14745600}{143}x^2$$

$$A_{15} = \frac{1}{30}x^{16} + \frac{98}{65}x^{14} - \frac{8232}{143}x^{12} + \frac{219520}{143}x^{10} - \frac{3763200}{143}x^8 + \frac{36126720}{143}x^6 - \frac{144506880}{143}x^4$$
$$B_{15} = \frac{7}{15}x^{15} - \frac{1372}{65}x^{13} + \frac{98784}{143}x^{11} - \frac{2195200}{143}x^9 + \frac{30105600}{143}x^7 - \frac{216760320}{143}x^5 + \frac{578027520}{143}x^3$$
$$C_{15} = \frac{x^{16}}{30} - \frac{343}{195}x^{14} + \frac{49392}{715}x^{12} - \frac{274400}{143}x^{10} + \frac{5017600}{143}x^8 - \frac{54190080}{143}x^6 + \frac{289013760}{143}x^4 - \frac{578027520}{143}x^2$$
$$D_{15} = \frac{x^{16}}{30} - \frac{112}{65}x^{14} + \frac{9408}{143}x^{12} - \frac{250880}{143}x^{10} + \frac{4300800}{143}x^8 - \frac{41287680}{143}x^6 + \frac{165150720}{143}x^4$$
$$E_{15} = -\frac{8}{15}x^{15} + \frac{1568}{65}x^{13} - \frac{112896}{143}x^{11} + \frac{2508800}{143}x^9 - \frac{34406400}{143}x^7 + \frac{247726080}{143}x^5 - \frac{660602880}{143}x^3$$
$$F_{15} = \frac{x^{16}}{30} + \frac{392}{195}x^{14} - \frac{56448}{715}x^{12} + \frac{313600}{143}x^{10} - \frac{5734400}{143}x^8 + \frac{61931520}{143}x^6 - \frac{330301440}{143}x^4 + \frac{660602880}{143}x^2$$
$$A_{15}^* = \frac{x^{16}}{30} - \frac{98}{65}x^{14} - \frac{8232}{143}x^{12} - \frac{219520}{143}x^{10} - \frac{3763200}{143}x^8 - \frac{36126720}{143}x^6 - \frac{144506880}{143}x^4$$

$$B_{15}^{*}=\frac{7}{15}\,x^{15}+\frac{1372}{65}\,x^{13}+\frac{98784}{143}\,x^{11}+\frac{2195200}{143}\,x^{9}+\frac{30105600}{143}\,x^{7}+\frac{216760320}{143}\,x^{5}+\frac{578027520}{143}\,x^{3}$$

$$C_{15}^{*}=-\frac{x^{16}}{30}-\frac{343}{195}\,x^{14}-\frac{49392}{715}\,x^{12}-\frac{274400}{143}\,x^{10}-\frac{5017600}{143}\,x^{8}-\frac{54190080}{143}\,x^{6}-\frac{289013760}{143}\,x^{4}-\frac{578027520}{143}\,x^{2}$$

$$D_{15}^{*}=-\frac{x^{16}}{30}-\frac{112}{65}\,x^{14}-\frac{9408}{143}\,x^{12}-\frac{250880}{143}\,x^{10}-\frac{4300800}{143}\,x^{8}-\frac{41287680}{143}\,x^{6}-\frac{165150720}{143}\,x^{4}$$

$$E_{15}^{*}=\frac{8}{15}\,x^{15}+\frac{1568}{65}\,x^{13}+\frac{112896}{143}\,x^{11}+\frac{2508800}{143}\,x^{9}+\frac{34406400}{143}\,x^{7}+\frac{247726080}{143}\,x^{5}+\frac{660602880}{143}\,x^{3}$$

$$F_{15}^{*}=\frac{x^{16}}{30}-\frac{392}{195}\,x^{14}-\frac{56448}{715}\,x^{12}-\frac{313600}{143}\,x^{10}-\frac{5734400}{143}\,x^{8}-\frac{61931520}{143}\,x^{6}-\frac{330301440}{143}\,x^{4}-\frac{660602880}{143}\,x^{2}$$

Recurrence Formulas:

$$\int x^{2n+1}J_0^2(x)\,dx=\frac{x^{2n}}{4n+2}\left\{(x^2+2n^2)J_0^2(x)+x^2J_1^2(x)+2nx\,J_0(x)J_1(x)\right\}-\frac{2n^3}{2n+1}\int x^{2n-1}J_0^2(x)\,dx$$

$$\int x^{2n+1}J_1^2(x)\,dx=$$

$$=\frac{x^{2n}}{4n+2}\left\{x^2\,J_0^2(x)+\left[x^2+2n(n+1)\right]\,J_1^2(x)-2(n+1)xJ_0(x)J_1(x)\right\}-\frac{2n(n^2-1)}{2n+1}\int x^{2n-1}J_1^2(x)\,dx$$

$$\int x^{2n+1}I_0^2(x)\,dx=\frac{x^{2n}}{4n+2}\left\{(x^2-2n^2)I_0^2(x)-x^2I_1^2(x)+2nx\,I_0(x)I_1(x)\right\}+\frac{2n^3}{2n+1}\int x^{2n-1}I_0^2(x)\,dx$$

$$\int x^{2n+1}I_1^2(x)\,dx=$$

$$=\frac{x^{2n}}{4n+2}\left\{-x^2\,I_0^2(x)+\left[x^2-2n(n+1)\right]\,I_1^2(x)+2(n+1)xI_0(x)I_1(x)\right\}+\frac{2n(n^2-1)}{2n+1}\int x^{2n-1}I_1^2(x)\,dx$$

$$\int x^{2n+1}K_0^2(x)\,dx=\frac{x^{2n}}{4n+2}\left\{(x^2-2n^2)K_0^2(x)-x^2K_1^2(x)-2nx\,K_0(x)K_1(x)\right\}+\frac{2n^3}{2n+1}\int x^{2n-1}K_0^2(x)\,dx$$

$$\int x^{2n+1}K_1^2(x)\,dx=$$

$$=\frac{x^{2n}}{4n+2}\left\{-x^2\,K_0^2(x)+\left[x^2-2n(n+1)\right]\,K_1^2(x)-2(n+1)xK_0(x)K_1(x)\right\}+\frac{2n(n^2-1)}{2n+1}\int x^{2n-1}K_1^2(x)\,dx$$

2.1.2. Integrals of the type $\int x^{-2n} Z_\nu^2(x)\,dx$

See also [4], 1.8.3.
Concerning the case $x^{+2n} Z_\nu^2(x)$ see 2.1.3., p. 271 .

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x),\ p=1,2$.

$$\int \frac{J_0^2(x)}{x^2}\,dx = -\left(2x+\frac{1}{x}\right) J_0^2(x) + 2\,J_0(x)\,J_1(x) - 2x\,J_1^2(x)$$

$$\int \frac{I_0^2(x)}{x^2}\,dx = \left(2x-\frac{1}{x}\right) I_0^2(x) - 2\,I_0(x)\,I_1(x) - 2x\,I_1^2(x)$$

$$\int \frac{K_0^2(x)}{x^2}\,dx = \left(2x-\frac{1}{x}\right) K_0^2(x) + 2\,K_0(x)\,K_1(x) - 2x\,K_1^2(x)$$

$$\int \frac{J_1^2(x)}{x^2}\,dx = \frac{2x}{3}\,J_0^2(x) - \frac{2}{3}\,J_0(x)\,J_1(x) + \left(\frac{2x}{3}-\frac{1}{3x}\right) J_1^2(x)$$

$$\int \frac{I_1^2(x)}{x^2}\,dx = \frac{2x}{3}\,I_0^2(x) - \frac{2}{3}\,I_0(x)\,I_1(x) - \left(\frac{2x}{3}+\frac{1}{3x}\right) I_1^2(x)$$

$$\int \frac{K_1^2(x)}{x^2}\,dx = \frac{2x}{3}\,K_0^2(x) + \frac{2}{3}\,K_0(x)\,K_1(x) - \left(\frac{2x}{3}+\frac{1}{3x}\right) K_1^2(x)$$

$$\int \frac{J_0^2(x)}{x^4}\,dx = \frac{1}{27x^3}\,(16\,x^4+6\,x^2-9)\,J_0^2(x) + \frac{1}{27x^2}\,(-16\,x^2+6)\,J_0(x)\,J_1(x) + \frac{1}{27x}\,(16\,x^2-2)\,J_1^2(x)$$

$$\int \frac{I_0^2(x)}{x^4}\,dx = \frac{1}{27x^3}\,(16\,x^4-6\,x^2-9)\,I_0^2(x) - \frac{1}{27x^2}\,(16\,x^2+6)\,I_0(x)\,I_1(x) - \frac{1}{27x}\,(16\,x^2+2)\,I_1^2(x)$$

$$\int \frac{K_0^2(x)}{x^4}\,dx = \frac{1}{27x^3}\,(16\,x^4-6\,x^2-9)\,K_0^2(x) + \frac{1}{27x^2}\,(16\,x^2+6)\,K_0(x)\,K_1(x) - \frac{1}{27x}\,(16\,x^2+2)\,K_1^2(x)$$

$$\int \frac{J_1^2(x)}{x^4}\,dx = \frac{-16\,x^2-6}{45x}\,J_0^2(x) + \frac{16\,x^2-6}{45x^2}\,J_0(x)\,J_1(x) + \frac{-16\,x^4+2\,x^2-9}{45x^3}\,J_1^2(x)$$

$$\int \frac{I_1^2(x)}{x^4}\,dx = \frac{16\,x^2-6}{45x}\,I_0^2(x) - \frac{16\,x^2+6}{45x^2}\,I_0(x)\,I_1(x) - \frac{16\,x^4+2\,x^2+9}{45x^3}\,I_1^2(x)$$

$$\int \frac{K_1^2(x)}{x^4}\,dx = \frac{16\,x^2-6}{45x}\,K_0^2(x) + \frac{16\,x^2+6}{45x^2}\,K_0(x)\,K_1(x) - \frac{16\,x^4+2\,x^2+9}{45x^3}\,K_1^2(x)$$

$$\int \frac{J_0^2(x)}{x^6}\,dx = \frac{-256\,x^6-96\,x^4+90\,x^2-675}{3375x^5}\,J_0^2(x) + \frac{256\,x^4-96\,x^2+270}{3375x^4}\,J_0(x)\,J_1(x) + \\ + \frac{-256\,x^4+32\,x^2-54}{3375x^3}\,J_1^2(x)$$

$$\int \frac{I_0^2(x)}{x^6}\,dx = \frac{256\,x^6-96\,x^4-90\,x^2-675}{3375x^5}\,I_0^2(x) - \frac{256\,x^4+96\,x^2+270}{3375x^4}\,I_0(x)\,I_1(x) - \\ - \frac{256\,x^4+32\,x^2+54}{3375x^3}\,I_1^2(x)$$

$$\int \frac{K_0^2(x)}{x^6}\,dx = \frac{256\,x^6-96\,x^4-90\,x^2-675}{3375x^5}\,K_0^2(x) + \frac{256\,x^4+96\,x^2+270}{3375x^4}\,K_0(x)\,K_1(x) - \\ - \frac{256\,x^4+32\,x^2+54}{3375x^3}\,K_1^2(x)$$

$$\int \frac{J_1^2(x)}{x^6}\,dx = \frac{1}{4725x^5}\,\left[x^2(256\,x^4+96\,x^2-90)\,J_0^2(x) + x(-256\,x^4+96\,x^2-270)\,J_0(x)\,J_1(x) + \right. \\ \left. +(256\,x^6-32\,x^4+54\,x^2-675)\,J_1^2(x)\right]$$

$$\int \frac{I_1^2(x)}{x^6}\,dx = \frac{1}{4725x^5}\,\left[x^2(256\,x^4-96\,x^2-90)\,I_0^2(x) - x(256\,x^4+96\,x^2+270)\,I_0(x)\,I_1(x) - \right.$$

$$-(256\,x^6+32\,x^4+54\,x^2+675)\,I_1^2(x)\big]$$

$$\int\frac{K_1^2(x)}{x^6}\,dx \;=\; \frac{1}{4725x^5}\,\big[\,x^2(256\,x^4-96\,x^2-90)\,K_0^2(x)+x(256\,x^4+96\,x^2+270)\,K_0(x)\,K_1(x)-$$
$$-(256\,x^6+32\,x^4+54\,x^2+675)\,K_1^2(x)\big]$$

$$\int\frac{J_0^2(x)}{x^8}\,dx \;=\; \frac{1}{385875x^7}\,\big[\,(2048\,x^8+768\,x^6-720\,x^4+3150\,x^2-55125)\,J_0^2(x)+$$
$$+x(-2048\,x^6+768\,x^4-2160\,x^2+15750)\,J_0(x)\,J_1(x)+x^2\,(2048\,x^6-256\,x^4+432\,x^2-2250)\,J_1^2(x)\big]$$

$$\int\frac{I_0^2(x)}{x^8}\,dx \;=\; \frac{1}{385875x^7}\,\big[\,(2048\,x^8-768\,x^6-720\,x^4-3150\,x^2-55125)\,I_0^2(x)-$$
$$-x(2048\,x^6+768\,x^4+2160\,x^2+15750)\,I_0(x)\,I_1(x)-x^2\,(2048\,x^6+256\,x^4+432\,x^2+2250)\,I_1^2(x)\big]$$

$$\int\frac{K_0^2(x)}{x^8}\,dx \;=\; \frac{1}{385875x^7}\,\big[\,(2048\,x^8-768\,x^6-720\,x^4-3150\,x^2-55125)\,K_0^2(x)+$$
$$+x(2048\,x^6+768\,x^4+2160\,x^2+15750)\,K_0(x)\,K_1(x)-x^2\,(2048\,x^6+256\,x^4+432\,x^2+2250)\,K_1^2(x)\big]$$

$$\int\frac{J_1^2(x)}{x^8}\,dx \;=\; \frac{1}{496125x^7}\,\big[x^2(-2048\,x^6-768\,x^4+720\,x^2-3150)\,J_0^2(x)+$$
$$+x(2048\,x^6-768\,x^4+2160\,x^2-15750)\,J_0(x)\,J_1(x)+(-2048\,x^8+256\,x^6-432\,x^4+2250\,x^2-55125)\,J_1^2(x)\big]$$

$$\int\frac{I_1^2(x)}{x^8}\,dx \;=\; \frac{1}{496125x^7}\,\big[x^2(2048\,x^6-768\,x^4-720\,x^2-3150)\,I_0^2(x)-$$
$$-x(2048\,x^6+768\,x^4+2160\,x^2+15750)\,I_0(x)\,I_1(x)-(2048\,x^8+256\,x^6+432\,x^4+2250\,x^2+55125)\,I_1^2(x)\big]$$

$$\int\frac{K_1^2(x)}{x^8}\,dx \;=\; \frac{1}{496125x^7}\,\big[x^2(2048\,x^6-768\,x^4-720\,x^2-3150)\,K_0^2(x)+$$
$$+x(2048\,x^6+768\,x^4+2160\,x^2+15750)\,K_0(x)\,K_1(x)-(2048\,x^8+256\,x^6+432\,x^4+2250\,x^2+55125)\,K_1^2(x)\big]$$

$$\int\frac{J_0^2(x)}{x^{10}}\,dx \;=\; \frac{1}{281302875x^9}\,\big[(-65536\,x^{10}-24576\,x^8+23040\,x^6-100800\,x^4+992250\,x^2-31255875)\,J_0^2(x)+$$
$$+x(65536\,x^8-24576\,x^6+69120\,x^4-504000\,x^2+6945750)\,J_0(x)\,J_1(x)+$$
$$+x^2\,(-65536\,x^8+8192\,x^6-13824\,x^4+72000\,x^2-771750)\,J_1^2(x)\big]$$

$$\int\frac{I_0^2(x)}{x^{10}}\,dx \;=\; \frac{1}{281302875x^9}\,\big[(65536\,x^{10}-24576\,x^8-23040\,x^6-100800\,x^4-992250\,x^2-31255875)\,I_0^2(x)-$$
$$-x(65536\,x^8+24576\,x^6+69120\,x^4+504000\,x^2+6945750)\,I_0(x)\,I_1(x)-$$
$$-x^2\,(65536\,x^8+8192\,x^6+13824\,x^4+72000\,x^2+771750)\,I_1^2(x)\big]$$

$$\int\frac{K_0^2(x)}{x^{10}}\,dx \;=\; \frac{1}{281302875x^9}\,\big[(65536\,x^{10}-24576\,x^8-23040\,x^6-100800\,x^4-992250\,x^2-31255875)\,K_0^2(x)+$$
$$+x(65536\,x^8+24576\,x^6+69120\,x^4+504000\,x^2+6945750)\,K_0(x)\,K_1(x)-$$
$$-x^2\,(65536\,x^8+8192\,x^6+13824\,x^4+72000\,x^2+771750)\,K_1^2(x)\big]$$

$$\int\frac{J_1^2(x)}{x^{10}}\,dx \;=\; \frac{1}{343814625x^9}\,\big[x^2(65536\,x^8+24576\,x^6-23040\,x^4+100800\,x^2-992250)\,J_0^2(x)+$$
$$+x(-65536\,x^8+24576\,x^6-69120\,x^4+504000\,x^2-6945750)\,J_0(x)\,J_1(x)+$$
$$+(65536\,x^{10}-8192\,x^8+13824\,x^6-72000\,x^4+771750\,x^2-31255875)\,J_1^2(x)\big]$$

$$\int\frac{I_1^2(x)}{x^{10}}\,dx \;=\; \frac{1}{343814625x^9}\,\big[x^2(65536\,x^8-24576\,x^6-23040\,x^4-100800\,x^2-992250)\,I_0^2(x)-$$
$$-x(65536\,x^8+24576\,x^6+69120\,x^4+504000\,x^2+6945750)\,I_0(x)\,I_1(x)-$$
$$-(65536\,x^{10}+8192\,x^8+13824\,x^6+72000\,x^4+771750\,x^2-31255875)\,I_1^2(x)\big]$$

$$\int \frac{K_1^2(x)}{x^{10}}\,dx \;=\; \frac{1}{343814625x^9}\,\big[x^2(65536\,x^8 - 24576\,x^6 - 23040\,x^4 - 100800\,x^2 - 992250)\,K_0^2(x) +$$
$$+x(65536\,x^8 + 24576\,x^6 + 69120\,x^4 + 504000\,x^2 + 6945750)\,K_0(x)\,K_1(x) -$$
$$-(65536\,x^{10} + 8192\,x^8 + 13824\,x^6 + 72000\,x^4 + 771750\,x^2 - 31255875)\,K_1^2(x)\big]$$

Recurrence formulas:

$$\int \frac{J_0^2(x)}{x^{2n+2}}\,dx = (2n+1)^{-3}\cdot$$
$$\cdot\left[\frac{-[2x^2+(2n+1)^2]\,J_0^2(x) + (4n+2)xJ_0(x)J_1(x) - 2x^2\,J_1^2(x)}{x^{2n+1}} - 8n\int \frac{J_0^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{I_0^2(x)}{x^{2n+2}}\,dx = (2n+1)^{-3}\cdot$$
$$\cdot\left[\frac{[2x^2-(2n+1)^2]\,I_0^2(x) - (4n+2)xI_0(x)I_1(x) - 2x^2\,I_1^2(x)}{x^{2n+1}} + 8n\int \frac{I_0^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{K_0^2(x)}{x^{2n+2}}\,dx = (2n+1)^{-3}\cdot$$
$$\cdot\left[\frac{[2x^2-(2n+1)^2]\,K_0^2(x) + (4n+2)xK_0(x)K_1(x) - 2x^2\,K_1^2(x)}{x^{2n+1}} + 8n\int \frac{K_0^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{J_1^2(x)}{x^{2n+2}}\,dx = -\frac{1}{(2n+3)(2n+1)(2n-1)}\cdot$$
$$\cdot\left[\frac{2x^2\,J_0^2(x) + (4n-2)xJ_0(x)J_1(x) + (4n^2-1+2x^2)\,J_1^2(x)}{x^{2n+1}} + 8n\int \frac{J_1^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{I_1^2(x)}{x^{2n+2}}\,dx = -\frac{1}{(2n+3)(2n+1)(2n-1)}\cdot$$
$$\cdot\left[\frac{2x^2\,I_0^2(x) + (4n-2)xI_0(x)I_1(x) + (4n^2-1-2x^2)\,I_1^2(x)}{x^{2n+1}} - 8n\int \frac{I_1^2(x)}{x^{2n}}\,dx\right]$$

$$\int \frac{K_1^2(x)}{x^{2n+2}}\,dx = -\frac{1}{(2n+3)(2n+1)(2n-1)}\cdot$$
$$\cdot\left[\frac{2x^2\,K_0^2(x) - (4n-2)xK_0(x)K_1(x) + (4n^2-1-2x^2)\,K_1^2(x)}{x^{2n+1}} - 8n\int \frac{K_1^2(x)}{x^{2n}}\,dx\right]$$

2.1.3. Integrals of the type $\int x^{2n}\, Z_\nu^2(x)\, dx$

a) The functions $\Theta(x)$ and $\Omega(x)$:

From HANKEL's asymptotic expansion of $J_\nu(x)$ and $Y_\nu(x)$ (see [1], 9.2, or [5], XIII. A. 4) and such of $\mathbf{H}_\nu(x)$ follows, that no finite representations of the integrals $\int Z_\nu^2(x)\, dx$ by functions of the type

$$A(x)\, J_0^2(x) + B(x)\, J_0(x)\, J_1(x) + C(x)\, J_1^2(x) + [D(x)\, J_0(x) + E(x)\, J_1(x)]\, \Phi(x) + F(x)\, \Phi^2(x)$$

with

$$A(x) = \sum_{i=-m}^{n} a_i\, x^i\,,\ \ldots\,,$$

can be expected. Indeed, one has

$$\lim_{x\to+\infty} \frac{1}{\ln x} \int_0^x J_\nu^2(t)\, dt = \frac{1}{\pi}$$

and this contradicts the upper statement.
At least should be given some other representations or approximations.

$$\Theta(x) = \int_0^x J_0^2(t)\, dt = 2\sum_{k=0}^\infty \frac{(-1)^k\cdot(2k)!}{(2k+1)\cdot(k!)^4}\cdot\left(\frac{x}{2}\right)^{2k+1}\,,$$

$$\Omega(x) = \int_0^x I_0^2(t)\, dt = 2\sum_{k=0}^\infty \frac{(2k)!}{(2k+1)\cdot(k!)^4}\cdot\left(\frac{x}{2}\right)^{2k+1}\,.$$

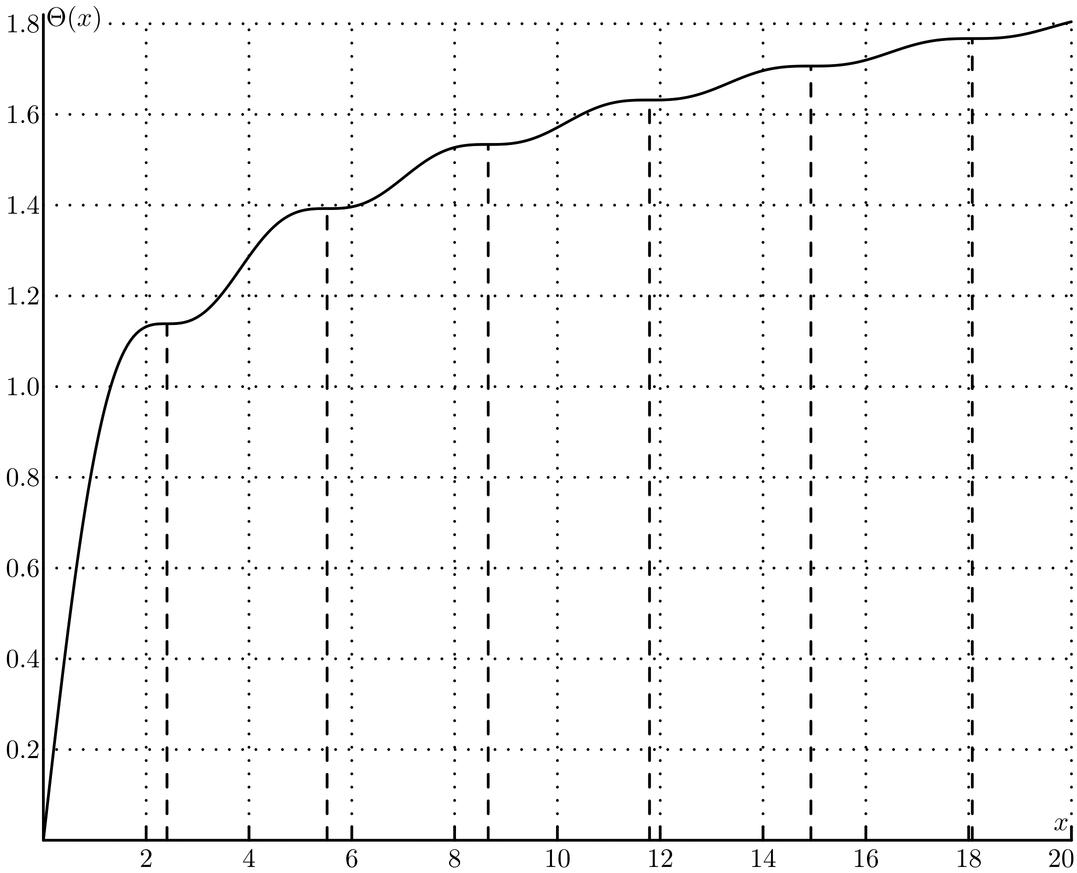


FIGURE 6 : *Function* $\Theta\,(x)$

The dashed lines are located in the zeros of $J_0(x)$.

If $\Theta(x)$ is computed by its series expansion with floating point numbers with n decimal digits, then the rounding error is (roughly spoken) about $10^{-n} \cdot \Omega(x)$. The computation of $\Omega(x)$ does not cause problems.

x	$\Theta(x)$	$\Omega(x)$	x	$\Theta(x)$	$\Omega(x)$
1	0.850 894 480	1.186 711 080	11	1.623 448 675	27 934 437.937
2	1.132 017 958	4.122 544 686	12	1.631 897 146	187 937 123.616
3	1.153 502 059	16.143 998 37	13	1.653 795 366	1 274 682 776.62
4	1.286 956 020	77.509 947 74	14	1.696 509 451	8 704 524 383.83
5	1.386 983 380	425.031 292 0	15	1.706 616 878	59 786 647 515.3
6	1.396 339 284	2 509.864 255	16	1.719 735 792	412 698 941 831.
7	1.460 064 224	15 483.965 76	17	1.755 251 443	2 861 234 688 170
8	1.527 171 173	98 307.748 55	18	1.767 226 854	19 912 983 676 244
9	1.534 810 723	637 083.688 6	19	1.774 861 457	139 056 981 172 080
10	1.571 266 461	4 193 041.1057	20	1.804 335 251	974 012 122 207 867

Differential equations:

$$2x\Theta''' \cdot \Theta' - 2\Theta'' \cdot \Theta' - x\Theta''^2 + 4x\Theta'^2 = 0$$
$$2x\Omega''' \cdot \Omega' - 2\Omega'' \cdot \Omega' - x\Omega''^2 - 4x\Omega'^2 = 0$$

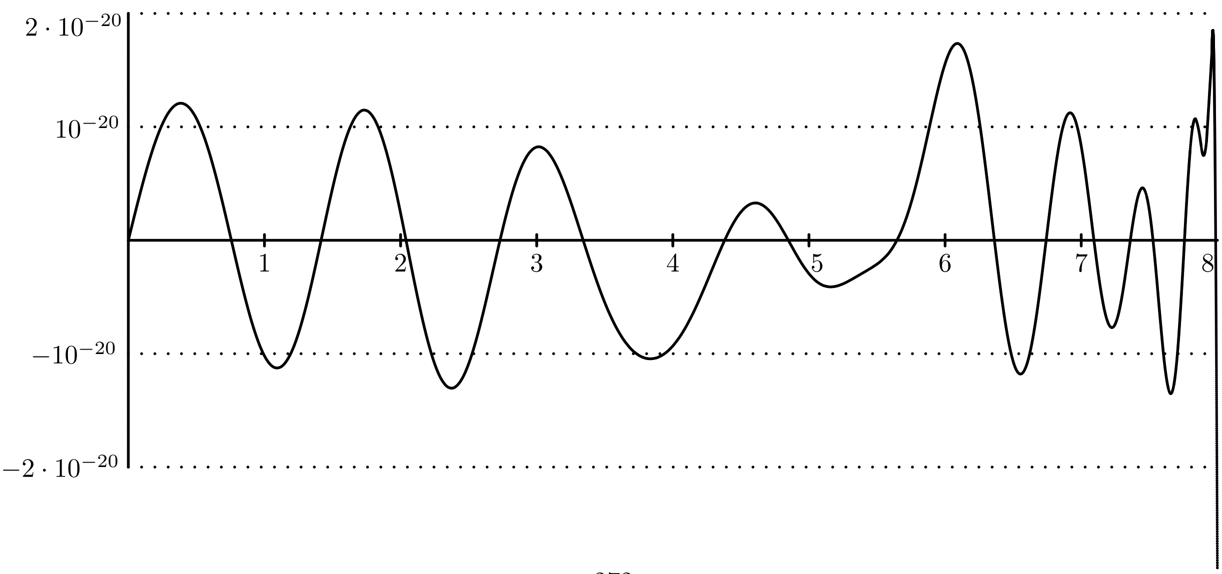
Approximation by Chebyshev polynomials:

From [2], table 9.1., follows, that in the case $-8 \le x \le 8$ holds

$$\Theta(x) \quad \approx \quad \sum_{k=0}^{23} c_k \, T_{2k+1}\left(\frac{x}{8}\right) . \qquad (*)$$

k	c_k	k	c_k
0	1.80296 30053 02073 59034	12	0.00000 59437 33032 29013
1	-0.41322 52443 66465 65056	13	-0.00000 05921 36076 87261
2	0.21926 25129 41565 79685	14	0.00000 00501 50462 51669
3	-0.12660 62713 07010 86382	15	-0.00000 00036 59712 13919
4	0.08920 60707 21441 83736	16	0.00000 00002 32718 06621
5	-0.08107 02495 61597 55273	17	-0.00000 00000 13019 50604
6	0.05544 00433 79678 61623	18	0.00000 00000 00646 15991
7	-0.02523 73073 64048 13366	19	-0.00000 00000 00028 65553
8	0.00802 84592 74213 97781	20	0.00000 00000 00001 14279
9	-0.00188 87924 36267 70784	21	-0.00000 00000 00000 04122
10	0.00034 34505 49931 43439	22	0.00000 00000 00000 00135
11	-0.00004 99025 73931 36611	23	-0.00000 00000 00000 00004

This approximation differs from $\Theta(x)$ as shown in the following figure:



Asymptotic series of $\Theta(x)$ for $x \to +\infty$:

$$\Theta(x) \quad \sim \quad \frac{1}{\pi} \left[\ln 8x + \mathbf{C} + \mathcal{A}(x)\cos 2x + \mathcal{B}(x)\sin 2x + \mathcal{C}(x)\right]$$

with Euler's constant $\mathbf{C} = 0.577\ 215\ 664\ 901\ 533\ ...$ and

$$\mathcal{A}(x) = -\frac{1}{2x} + \frac{29}{64x^3} - \frac{6747}{4096x^5} + \frac{1796265}{131072x^7} - \frac{3447866835}{16777216x^9} + \frac{2611501938675}{536870912x^{11}} - \frac{5739627264576975}{34359738368x^{13}} +$$
$$+\frac{8634220069330080225}{1099511627776x^{15}} - \frac{1363263923927901083838 75}{281474976710656x^{17}} + \frac{3417525716134419776210 07375}{9007199254740992x^{19}} - \ldots ,$$
$$\mathcal{B}(x) = -\frac{3}{8x^2} + \frac{195}{256x^4} - \frac{71505}{16384x^6} + \frac{26103735}{524288x^8} - \frac{63761381145}{67108864x^{10}} + \frac{58671892003725}{2147483648x^{12}} - \frac{151798966421827725}{137438953472x^{14}} +$$
$$+\frac{262762002151603329375}{4398046511104x^{16}} - \frac{469243026363058463378 3625}{1125899906842624x^{18}} + \frac{1312688010142958160034 8860625}{36028797018963968x^{20}} - \ldots ,$$
$$\mathcal{C}(x) = \frac{1}{16x^2} - \frac{27}{512x^4} + \frac{375}{2048x^6} - \frac{385875}{262144x^8} + \frac{11252115}{524288x^{10}} - \frac{8320313925}{16777216x^{12}} -$$
$$+\frac{1119167124075}{67108864x^{14}} - \frac{26440323306271875}{34359738368x^{16}} + \frac{1603719856835971875}{34359738368x^{18}} - \frac{395996921929365519262 5}{1099511627776x^{20}} + \ldots .$$

The asymptotic series

$$\mathcal{A}(x) = \sum_{k=1}^{\infty} \frac{a_k}{x^{2k-1}} \ , \qquad \mathcal{B}(x) = \sum_{k=1}^{\infty} \frac{b_k}{x^{2k}} \ , \qquad \mathcal{C}(x) = \sum_{k=0}^{\infty} \frac{c_k}{x^{2k}}$$

begin with

k	a_k	b_k	c_k
1	-0.500 000 000 000 000	-0.375 000 000 000 000	0.062 500 000 000 000
2	0.453 125 000 000 000	0.761 718 750 000 000	-0.052 734 375 000 000
3	-1.647 216 796 875 000	-4.364 318 847 656 250	0.183 105 468 750 000
4	13.704 414 367 675 78	49.788 923 263 549 80	-1.471 996 307 373 047
5	-205.508 877 933 025 4	-950.118 618 384 003 6	21.461 706 161 499 02
6	4 864.301 418 280 229	27 321.228 759 235 24	-495.929 355 919 361 1
7	-167 045.138 793 094 5	-1 104 482.845 562 072	16 676.889 718 696 48
8	7 852 777.406 997 193	59 745 162.196 032 50	-769 514.686 726 961 4
9	-484 328 639.035 390 4	-4 167 715 296.104 454	46 674 390.813 451 37
10	37 942 157 373.010 10	364 344 113 252.523 3	-3 601 571 024.131 458

Let x_k denote the k-th positive zero of $J_0(x)$, then holds

$$\Theta(x_k) \ \sim \ \frac{1}{\pi} \left[\ln x_k + \frac{5}{2^4 \cdot x_k^2} - \frac{331}{2^9 \cdot x_k^4} + \frac{7\,987}{2^{11} \cdot x_k^6} - \frac{753\,375}{2^{14} \cdot x_k^8} + \frac{246\,293\,295}{2^{18} \cdot x_k^{10}} - \ldots\right] =$$
$$= \frac{1}{\pi} \left[\ln x_k + \frac{0.312\,500}{x_k^2} - \frac{0.646\,484}{x_k^4} + \frac{3.899\,902}{x_k^6} - \frac{45.982\,36}{x_k^8} + \frac{939.5344}{x_k^{10}} - \ldots\right] .$$

Simple approximation: $\Theta(x) \approx (\ln 8x + \mathbf{C})/\pi$:

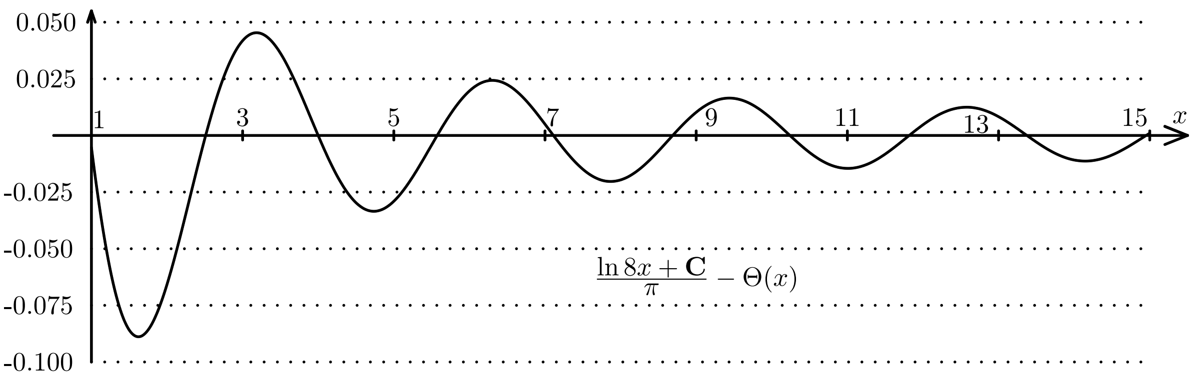


Figure 7

Let

$$\Delta_n(x) = \frac{1}{\pi}\left[\ln 8x + \mathbf{C} + \sum_{k=1}^{n} \frac{a_k x \cos 2x + b_k \sin 2x + c_k}{x^{2k}}\right] - \Theta(x)$$

with $\Delta_0(x) = (\ln 8x + \mathbf{C})/\pi$.
In the following table are given some first consecutive maxima and minima $\Delta^*_{n,k}$ of the differences $\Delta_n(x)$

$$\Delta^*_{n,k} = \Delta_n(x_{n,k}) .$$

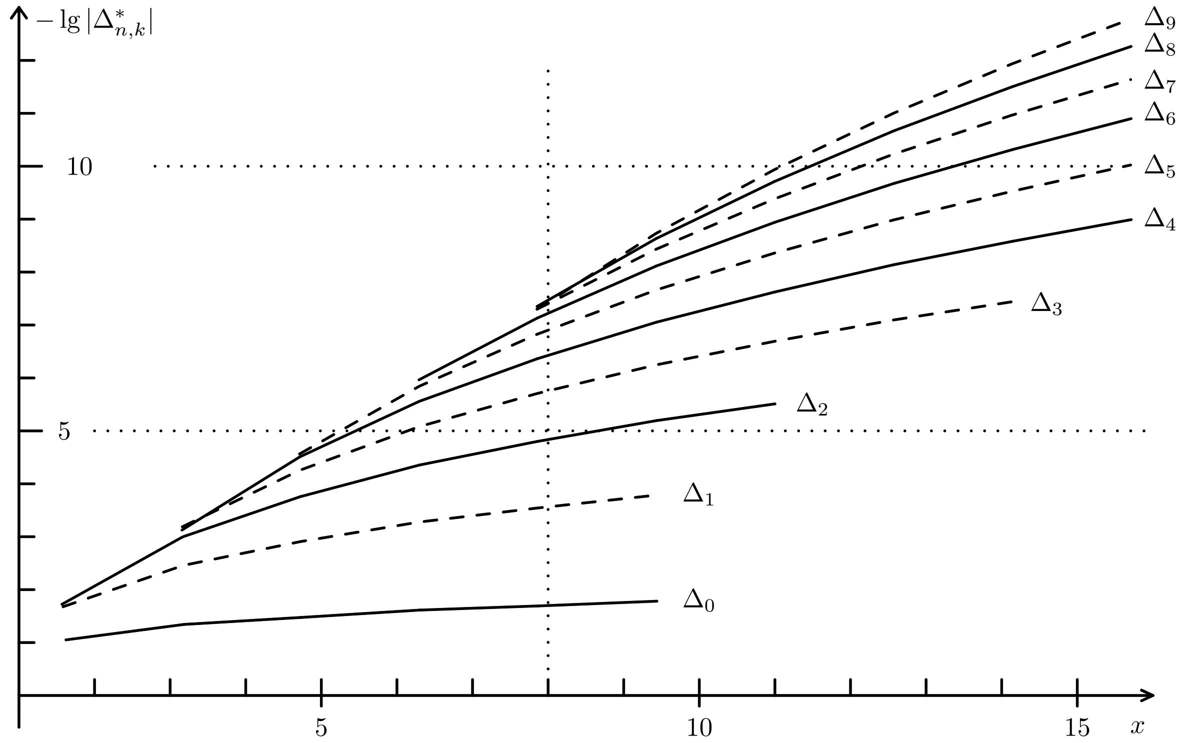
Values	$x_{n,1},\ \Delta^*_{n,1}$	$x_{n,2},\ \Delta^*_{n,2}$	$x_{n,3},\ \Delta^*_{n,3}$	$x_{n,4},\ \Delta^*_{n,4}$	$x_{n,5},\ \Delta^*_{n,5}$	$x_{n,6},\ \Delta^*_{n,6}$	$x_{n,7},\ \Delta^*_{n,7}$
x	1.6216	3.1847	4.7356	6.3043	7.8688	9.4386	11.0064
$\Delta_0(x)$	-8.886E-02	4.536E-02	-3.347E-02	2.432E-02	-2.031E-02	1.651E-02	-1.454E-02
x	1.5839	3.1735	4.7253	6.2993	7.8633	9.4353	11.0027
$\Delta_1(x)$	2.104E-02	-3.483E-03	1.237E-03	-5.267E-04	2.863E-04	-1.638E-04	1.066E-04
x	1.5694	3.1664	4.7195	6.2969	7.8602	9.4338	11.0007
$\Delta_2(x)$	-1.894E-02	1.007E-03	-1.756E-04	4.430E-05	-1.583E-05	6.405E-06	-3.092E-06
x	1.5650	3.1612	4.7160	6.2952	7.8583	9.4330	10.9995
$\Delta_3(x)$	3.978E-02	-6.484E-04	5.541E-05	-8.322E-06	1.965E-06	-5.644E-07	2.029E-07
x	1.5642	3.1574	4.7138	6.2939	7.8569	9.4324	10.9986
$\Delta_4(x)$	-1.578E-01	7.486E-04	-3.105E-05	2.773E-06	-4.335E-07	8.855E-08	-2.376E-08
x	1.5644	3.1545	4.7124	6.2928	7.8559	9.4319	10.9979
$\Delta_5(x)$	1.041E+00	-1.376E-03	2.730E-05	-1.444E-06	1.494E-07	-2.172E-08	4.352E-09
x	1.5649	3.1523	4.7115	6.2918	7.8551	9.4314	10.9973
$\Delta_6(x)$	-1.045E+01	3.722E-03	-3.486E-05	1.085E-06	-7.420E-08	7.673E-09	-1.149E-09
x	1.5655	3.1507	4.7110	6.2909	7.8545	9.4310	10.9968
$\Delta_7(x)$	1.493E+02	-1.403E-02	6.123E-05	-1.115E-06	5.025E-08	-3.692E-09	4.132E-10
x	1.5660	3.1494	4.7107	6.2902	7.8540	9.4305	10.9964
$\Delta_8(x)$	-2.891E+03	7.055E-02	-1.421E-04	1.506E-06	-4.457E-08	2.324E-09	-1.942E-10
x	1.5664	3.1483	4.7105	6.2895	7.8537	9.4302	10.9961
$\Delta_9(x)$	7.303E+04	-4.582E-01	4.224E-04	-2.590E-06	5.022E-08	-1.854E-09	1.156E-10

If $x > 8$, then $|\Delta_n(x)|$ is restricted by $|\Delta_n(x)| \le |\Delta^*_{n,5}|$. More accurate:

n	0	1	2	3	4
$\|\Delta_n(x)\| <$	1.9625E-02	2.7573E-04	1.5193E-05	1.8792E-06	4.1271E-07
n	5	6	7	8	9
$\|\Delta_n(x)\| <$	1.4150E-07	6.9856E-08	4.6990E-08	4.1368E-08	4.6224E-08

Therefore the formula (*) from page 272 may be continued to $x > 8$ by an asymptotic formula ($n = 7$, 8 or 9) with a uniformly bounded absolute error less then 0.5E-07.

The following figure shows the values of $-\lg|\Delta^*_{n,k}|$ from the preceding table, connected by a polygonal line. It gives the intervals where some special asymptotic formula is preferable.



The differences $\Delta_n(x)$ are shown in the following figures:

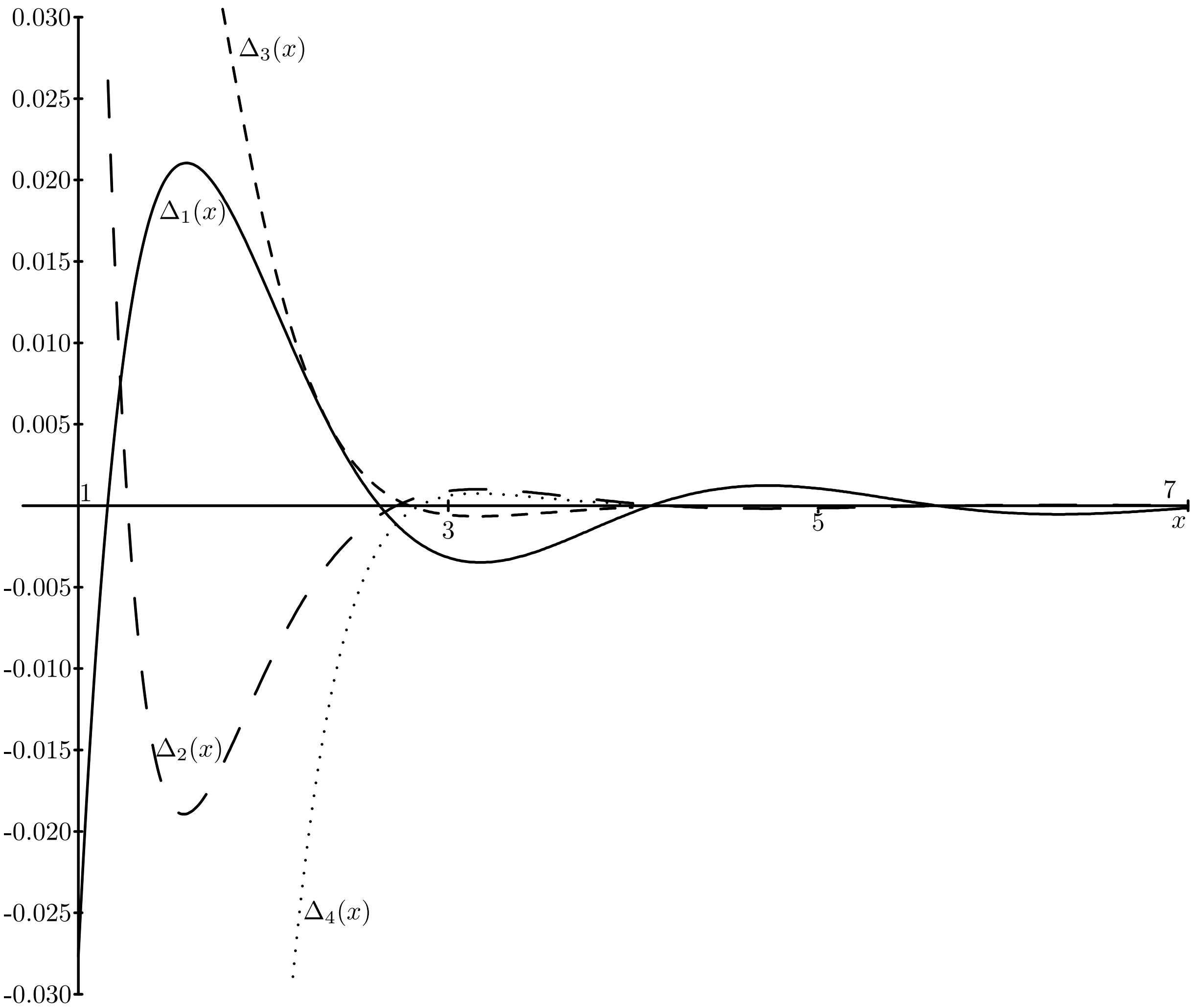


FIGURE 8 : *Differences* $\Delta_{1\ldots4}(x),\ 1 \le x \le 7$

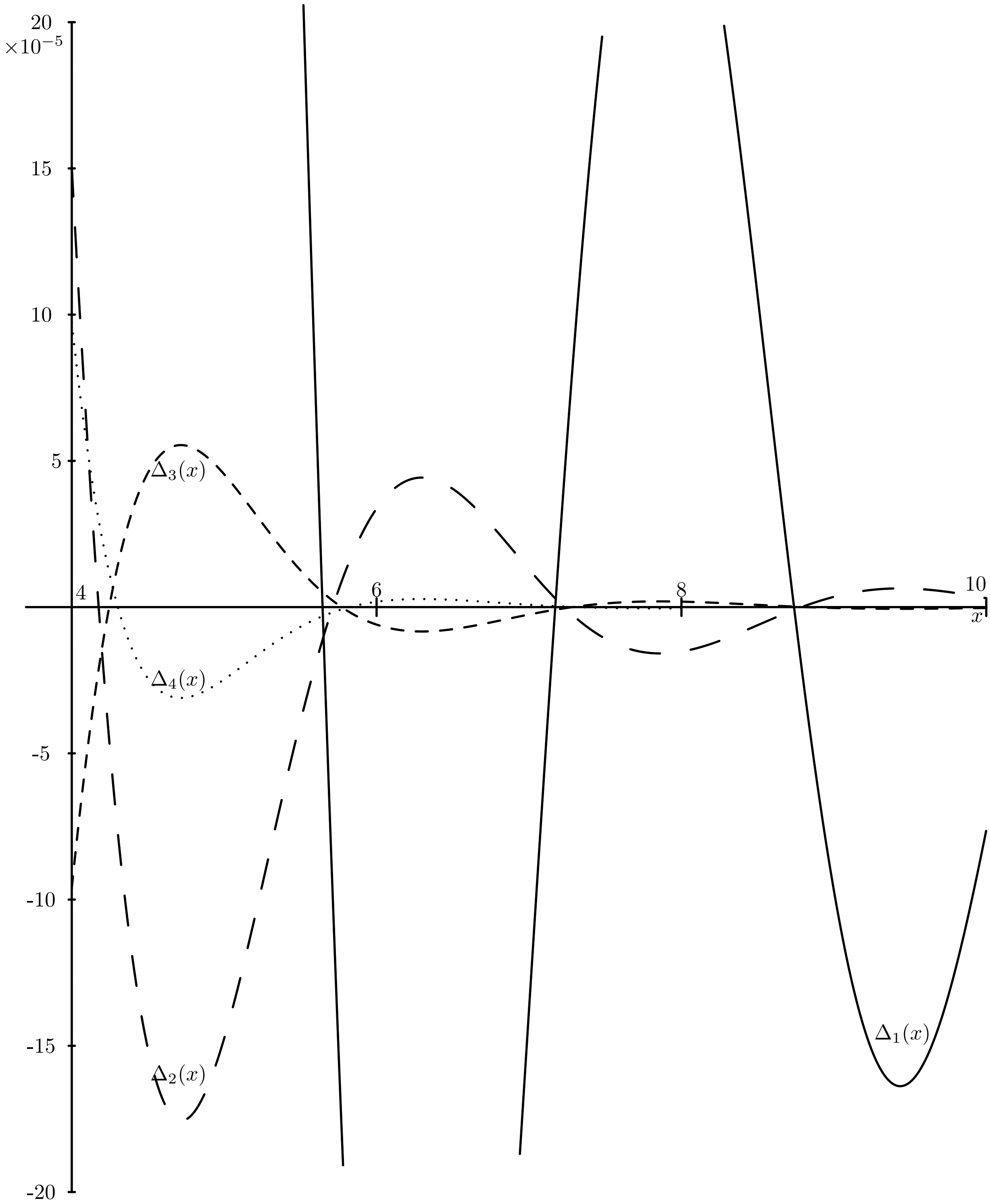


FIGURE 9 : *Differences* $\Delta_{1\ldots4}(x),\ 4 \le x \le 10$

Asymptotic behaviour of $\Omega(x)$ for $x \to \infty$:

$$\Omega(x) \sim$$

$$\sim \frac{e^{2x}}{4\pi x}\left[1+\frac{3}{4x}+\frac{29}{32x^2}+\frac{195}{128x^3}+\frac{6\,747}{2\,048x^4}+\frac{71\,505}{8\,192x^5}+\frac{1\,796\,265}{65\,536x^6}+\frac{26\,103\,735}{262\,144x^7}+\frac{430\,983\,354}{1\,048\,576x^8}+\ldots\right]$$

$$=\frac{e^{2x}}{4\pi x}\left[1+\frac{0.75}{x}+\frac{0.90625}{x^2}+\frac{1.5234}{x^3}+\frac{3.2944}{x^4}+\frac{8.7286}{x^5}+\frac{27.409}{x^6}+\frac{99.578}{x^7}+\frac{411.02}{x^8}+\ldots\right]$$

b) Integrals:

Holds

$$\int J_0^2(x)\,dx = x[J_0^2(x)+J_1^2(x)] + \int J_1^2(x)\,dx \;,$$

$$\int I_0^2(x)\,dx = x[I_0^2(x)-I_1^2(x)] - \int I_1^2(x)\,dx \;.$$

These formulas express every integral by the other one. Therefore the next integrals are given with $\int J_0^2(x)\,dx$ or $\int I_0^2(x)\,dx$. The last integrals are represented by the functions $\Theta(x)$ respectively $\Omega(x)$, see page 271 .

$$\int x^2\, J_0^2(x)\,dx = \frac{1}{8}\,\left[(2\,x^3+x)\,J_0^2(x)+2x^2 J_0(x)J_1(x)+2x^3\,J_1^2(x)\right] - \frac{1}{8}\int J_0^2(x)\,dx$$

$$\int x^2\, I_0^2(x)\,dx = \frac{1}{8}\,\left[(2\,x^3-x)\,I_0^2(x)+2x^2 I_0(x)I_1(x)-2x^3\,I_1^2(x)\right] + \frac{1}{8}\int I_0^2(x)\,dx$$

$$\int x^2\, J_1^2(x)\,dx = \frac{1}{8}\,\left[(2\,x^3-3x)\,J_0^2(x)-6x^2 J_0(x)J_1(x)+2x^3\,J_1^2(x)\right] + \frac{3}{8}\int J_0^2(x)\,dx$$

$$\int x^2\, I_1^2(x)\,dx = \frac{1}{8}\,\left[(-2\,x^3-3x)\,I_0^2(x)+6x^2 I_0(x)I_1(x)+2x^3\,I_1^2(x)\right] + \frac{3}{8}\int I_0^2(x)\,dx$$

$$\int x^4\, J_0^2(x)\,dx =$$

$$= \frac{1}{128}\,\left[(16\,x^5+18\,x^3-27\,x)\,J_0^2(x)+(48\,x^4-54\,x^2)J_0(x)J_1(x)+(16\,x^5-54\,x^3)\,J_1^2(x)\right] + \frac{27}{128}\int J_0^2(x)\,dx$$

$$\int x^4\, I_0^2(x)\,dx =$$

$$= \frac{1}{128}\,\left[(16\,x^5-18\,x^3-27\,x)\,I_0^2(x)+(48\,x^4+54\,x^2)I_0(x)I_1(x)-(16\,x^5+54\,x^3)\,I_1^2(x)\right] + \frac{27}{128}\int I_0^2(x)\,dx$$

$$\int x^4\, J_1^2(x)\,dx =$$

$$= \frac{1}{128}\,\left[(16\,x^5-30\,x^3+45\,x)\,J_0^2(x)+(-80\,x^4+90\,x^2)J_0(x)J_1(x)+(16\,x^5+90\,x^3)\,J_1^2(x)\right] -$$

$$-\frac{45}{128}\int J_0^2(x)\,dx$$

$$\int x^4\, I_1^2(x)\,dx =$$

$$= \frac{1}{128}\,\left[(-16\,x^5-30\,x^3-45\,x)\,I_0^2(x)+(80\,x^4+90\,x^2)I_0(x)I_1(x)+(16\,x^5-90\,x^3)\,I_1^2(x)\right] +$$

$$+\frac{45}{128}\int I_0^2(x)\,dx$$

With

$$\int x^{2n}\, J_0^2(x)\,dx = \frac{1}{\beta_n}\left[A_n(x)\,J_0^2(x)+B_n(x)J_0(x)J_1(x)+C_n(x)\,J_1^2(x)+\gamma_n\int J_0^2(x)\,dx\right]$$

$$\int x^{2n}\, I_0^2(x)\,dx = \frac{1}{\beta_n^*}\left[A_n^*(x)\,I_0^2(x)+B_n^*(x)I_0(x)I_1(x)+C_n^*(x)\,I_1^2(x)+\gamma_n^*\int I_0^2(x)\,dx\right]$$

$$\int x^{2n}\, J_1^2(x)\,dx = \frac{1}{\xi_n}\left[P_n(x)\,J_0^2(x)+Q_n(x)J_0(x)J_1(x)+R_n(x)\,J_1^2(x)+\varrho_n\int J_0^2(x)\,dx\right]$$

$$\int x^{2n}\, I_1^2(x)\,dx = \frac{1}{\xi_n^*}\left[P_n^*(x)\,I_0^2(x)+Q_n^*(x)I_0(x)I_1(x)+R_n^*(x)\,I_1^2(x)+\varrho_n^*\int I_0^2(x)\,dx\right]$$

holds

$$\beta_3 = 3072\,, \quad \gamma_3 = -3375$$
$$A_3(x) = 256\,x^7 + 1200\,x^5 - 2250\,x^3 + 3375\,x$$
$$B_3(x) = 1280\,x^6 - 6000\,x^4 + 6750\,x^2\,, \quad C_3(x) = 256\,x^7 - 2000\,x^5 + 6750\,x^3$$

$$\beta_3^* = 3072\,, \quad \gamma_3^* = 3375$$
$$A_3^*(x) = 256\,x^7 - 1200\,x^5 - 2250\,x^3 - 3375\,x$$
$$B_3^*(x) = 1280\,x^6 + 6000\,x^4 + 6750\,x^2\,, \quad C_3^*(x) = -256\,x^7 - 2000\,x^5 - 6750\,x^3$$

$$\xi_3 = 3072\,, \quad \varrho_3 = 4725$$
$$P_3(x) = 256\,x^7 - 1680\,x^5 + 3150\,x^3 - 4725\,x$$
$$Q_3(x) = -1792\,x^6 + 8400\,x^4 - 9450\,x^2\,, \quad R_3(x) = 256\,x^7 + 2800\,x^5 - 9450\,x^3$$

$$\xi_3^* = 3072\,, \quad \varrho_3^* = 4725$$
$$P_3^*(x) = -256\,x^7 - 1680\,x^5 - 3150\,x^3 - 4725\,x$$
$$Q_3^*(x) = 1792\,x^6 + 8400\,x^4 + 9450\,x^2\,, \quad R_3^*(x) = 256\,x^7 - 2800\,x^5 - 9450\,x^3$$

$$\beta_4 = 98304\,, \quad \gamma_4 = 1157625$$
$$A_4(x) = 6144\,x^9 + 62720\,x^7 - 411600\,x^5 + 771750\,x^3 - 1157625\,x$$
$$B_4(x) = 43008\,x^8 - 439040\,x^6 + 2058000\,x^4 - 2315250\,x^2$$
$$C_4(x) = 6144\,x^9 - 87808\,x^7 + 686000\,x^5 - 2315250\,x^3$$

$$\beta_4^* = 98304\,, \quad \gamma_4^* = 1157625$$
$$A_4^*(x) = 6144\,x^9 - 62720\,x^7 - 411600\,x^5 - 771750\,x^3 - 1157625\,x$$
$$B_4^*(x) = 43008\,x^8 + 439040\,x^6 + 2058000\,x^4 + 2315250\,x^2$$
$$C_4^*(x) = -6144\,x^9 - 87808\,x^7 - 686000\,x^5 - 2315250\,x^3$$

$$\xi_4 = 32768\,, \quad \varrho_4 = -496125$$
$$P_4(x) = 2048\,x^9 - 26880\,x^7 + 176400\,x^5 - 330750\,x^3 + 496125\,x$$
$$Q_4(x) = -18432\,x^8 + 188160\,x^6 - 882000\,x^4 + 992250\,x^2$$
$$R_4(x) = 2048\,x^9 + 37632\,x^7 - 294000\,x^5 + 992250\,x^3$$

$$\xi_4^* = 32768\,, \quad \varrho_4^* = 496125$$
$$P_4^*(x) = -2048\,x^9 - 26880\,x^7 - 176400\,x^5 - 330750\,x^3 - 496125\,x$$
$$Q_4^*(x) = 18432\,x^8 + 188160\,x^6 + 882000\,x^4 + 992250\,x^2$$
$$R_4^*(x) = 2048\,x^9 - 37632\,x^7 - 294000\,x^5 - 992250\,x^3$$

$$\beta_5 = 1310720\,, \quad \gamma_5 = -281302875$$
$$A_5(x) = 65536\,x^{11} + 1161216\,x^9 - 15240960\,x^7 + 100018800\,x^5 - 187535250\,x^3 + 281302875\,x$$
$$B_5(x) = 589824\,x^{10} - 10450944\,x^8 + 106686720\,x^6 - 500094000\,x^4 + 562605750\,x^2$$
$$C_5(x) = 65536\,x^{11} - 1492992\,x^9 + 21337344\,x^7 - 166698000\,x^5 + 562605750\,x^3$$

$$\beta_5^* = 1310720\,, \quad \gamma_5^* = 281302875^*$$

$$A_5(x)^* = 65536\,x^{11} - 1161216\,x^9 - 15240960\,x^7 - 100018800\,x^5 - 187535250\,x^3 - 281302875\,x$$
$$B_5(x)^* = 589824\,x^{10} + 10450944\,x^8 + 106686720\,x^6 + 500094000\,x^4 + 562605750\,x^2$$
$$C_5(x)^* = -65536\,x^{11} - 1492992\,x^9 - 21337344\,x^7 - 166698000\,x^5 - 562605750\,x^3$$

$$\xi_5 = 1310720\,, \quad \varrho_5 = 343814625$$
$$P_5(x) = 65536\,x^{11} - 1419264\,x^9 + 18627840\,x^7 - 122245200\,x^5 + 229209750\,x^3 - 343814625\,x$$
$$Q_5(x) = -720896\,x^{10} + 12773376\,x^8 - 130394880\,x^6 + 611226000\,x^4 - 687629250\,x^2$$
$$R_5(x) = 65536\,x^{11} + 1824768\,x^9 - 26078976\,x^7 + 203742000\,x^5 - 687629250\,x^3$$

$$\xi_5^* = 1310720\,, \quad \varrho_5^* = 343814625$$
$$P_5^*(x) = -65536\,x^{11} - 1419264\,x^9 - 18627840\,x^7 - 122245200\,x^5 - 229209750\,x^3 - 343814625\,x$$
$$Q_5^*(x) = 720896\,x^{10} + 12773376\,x^8 + 130394880\,x^6 + 611226000\,x^4 + 687629250\,x^2$$
$$R_5^*(x) = 65536\,x^{11} - 1824768\,x^9 - 26078976\,x^7 - 203742000\,x^5 - 687629250\,x^3$$

Recurrence relations:

$$\int x^{2n+2}\,J_0^2(x)\,dx =$$
$$= \frac{x^{2n+1}}{8(n+1)}\,\left\{[x^2+(2n+1)^2]\,J_0^2(x) + 2x^2\,J_1^2(x) + 2(2n+1)x\,J_0(x)\,J_1(x)\right\} - \frac{(2n+1)^3}{8(n+1)}\int x^{2n}\,J_0^2(x)\,dx$$

$$\int x^{2n+2}\,I_0^2(x)\,dx =$$
$$= \frac{x^{2n+1}}{8(n+1)}\,\left\{[x^2-(2n+1)^2]\,I_0^2(x) - 2x^2\,I_1^2(x) + 2(2n+1)x\,I_0(x)\,I_1(x)\right\} + \frac{(2n+1)^3}{8(n+1)}\int x^{2n}\,I_0^2(x)\,dx$$

$$\int x^{2n+2}\,J_1^2(x)\,dx = \frac{x^{2n+1}}{8(n+1)}\,\left\{[x^2+(2n+1)(2n+3)]\,J_1^2(x) + 2x^2\,J_0^2(x) - 2(2n+3)x\,J_0(x)\,J_1(x)\right\} -$$
$$-\frac{(2n-1)(2n+1)(2n+3)}{8(n+1)}\int x^{2n}\,J_0^2(x)\,dx$$

$$\int x^{2n+2}\,I_1^2(x)\,dx = \frac{x^{2n+1}}{8(n+1)}\,\left\{[x^2-(2n+1)(2n+3)]\,I_1^2(x) - 2x^2\,I_0^2(x) + 2(2n+3)x\,I_0(x)\,I_1(x)\right\} +$$
$$+\frac{(2n-1)(2n+1)(2n+3)}{8(n+1)}\int x^{2n}\,I_0^2(x)\,dx$$

2.1.4. Integrals of the type $\int x^{2n} Z_0(x) Z_1(x)\,dx$

In the following formulas both $J_0(x)$ and $J_1(x)$ together may be substituted by $Y_0(x)$ and $Y_1(x)$ respectively or $H_0^{(p)}(x),\ H_1^{(p)}(x),\ p = 1, 2.$

$$\int J_0(x)J_1(x)\,dx = -\frac{1}{2}J_0^2(x)$$

$$\int I_0(x)I_1(x)\,dx = \frac{1}{2}I_0^2(x)$$

$$\int K_0(x)K_1(x)\,dx = -\frac{1}{2}K_0^2(x)$$

$$\int x^2 \cdot J_0(x)J_1(x)\,dx = \frac{x^2}{2}J_1^2(x)$$

$$\int x^2 \cdot I_0(x)I_1(x)\,dx = \frac{x^2}{2}I_1^2(x)$$

$$\int x^2 \cdot K_0(x)I_1(x)\,dx = -\frac{x^2}{2}K_1^2(x)$$

$$\int x^4 \cdot J_0(x)J_1(x)\,dx = -\frac{x^4}{6}J_0^2(x) + \frac{2x^3}{3}J_0(x)J_1(x) + \left(\frac{x^4}{3} - \frac{2x^2}{3}\right)J_1^2(x)$$

$$\int x^4 \cdot I_0(x)I_1(x)\,dx = \frac{x^4}{6}I_0^2(x) - \frac{2x^3}{3}I_0(x)I_1(x) + \left(\frac{x^4}{3} + \frac{2x^2}{3}\right)I_1^2(x)$$

$$\int x^4 \cdot K_0(x)K_1(x)\,dx = -\frac{x^4}{6}K_0^2(x) - \frac{2x^3}{3}K_0(x)K_1(x) - \left(\frac{x^4}{3} + \frac{2x^2}{3}\right)K_1^2(x)$$

$$\int x^6 \cdot J_0(x)J_1(x)\,dx = \left(-\frac{x^6}{5} + \frac{4x^4}{5}\right)J_0^2(x) + \left(\frac{6x^5}{5} - \frac{16x^3}{5}\right)J_0(x)J_1(x) + \left(\frac{3x^6}{10} - \frac{8x^4}{5} + \frac{16x^2}{5}\right)J_1^2(x)$$

$$\int x^6 \cdot I_0(x)I_1(x)\,dx = \left(\frac{x^6}{5} + \frac{4x^4}{5}\right)I_0^2(x) - \left(\frac{6x^5}{5} + \frac{16x^3}{5}\right)I_0(x)I_1(x) + \left(\frac{3x^6}{10} + \frac{8x^4}{5} + \frac{16x^2}{5}\right)I_1^2(x)$$

$$\int x^6 \cdot K_0(x)K_1(x)\,dx =$$

$$= -\left(\frac{x^6}{5} + \frac{4x^4}{5}\right)K_0^2(x) - \left(\frac{6x^5}{5} + \frac{16x^3}{5}\right)K_0(x)K_1(x) - \left(\frac{3x^6}{10} + \frac{8x^4}{5} + \frac{16x^2}{5}\right)K_1^2(x)$$

$$\int x^8 \cdot J_0(x)J_1(x)\,dx = \left(-\frac{3x^8}{14} + \frac{72x^6}{35} - \frac{288x^4}{35}\right)J_0^2(x) + \left(\frac{12x^7}{7} - \frac{432x^5}{35} + \frac{1152x^3}{35}\right)J_0(x)J_1(x) +$$

$$+ \left(\frac{2x^8}{7} - \frac{108x^6}{35} + \frac{576x^4}{35} - \frac{1152x^2}{35}\right)J_1^2(x)$$

$$\int x^8 \cdot I_0(x)I_1(x)\,dx = \left(\frac{3x^8}{14} + \frac{72x^6}{35} + \frac{288x^4}{35}\right)I_0^2(x) - \left(\frac{12x^7}{7} + \frac{432x^5}{35} + \frac{1152x^3}{35}\right)I_0(x)I_1(x) +$$

$$+ \left(\frac{2x^8}{7} + \frac{108x^6}{35} + \frac{576x^4}{35} + \frac{1152x^2}{35}\right)I_1^2(x)$$

$$\int x^8 \cdot K_0(x)K_1(x)\,dx = -\left(\frac{3x^8}{14} + \frac{72x^6}{35} + \frac{288x^4}{35}\right)K_0^2(x) - \left(\frac{12x^7}{7} + \frac{432x^5}{35} + \frac{1152x^3}{35}\right)K_0(x)K_1(x) -$$

$$- \left(\frac{2x^8}{7} + \frac{108x^6}{35} + \frac{576x^4}{35} + \frac{1152x^2}{35}\right)K_1^2(x)$$

$$\int x^{10} \cdot J_0(x)J_1(x)\,dx = \left(-\frac{2x^{10}}{9} + \frac{80x^8}{21} - \frac{256x^6}{7} + \frac{1024x^4}{7}\right)J_0^2(x) +$$

$$+ \left(\frac{20x^9}{9} - \frac{640x^7}{21} + \frac{1536x^5}{7} - \frac{4096x^3}{7}\right)J_0(x)J_1(x) +$$

$$+\left(\frac{5x^{10}}{18}-\frac{320x^8}{63}+\frac{384x^6}{7}-\frac{2048x^4}{7}+\frac{4096x^2}{7}\right)J_1^2(x)$$

$$\int x^{10}\cdot I_0(x)I_1(x)\,dx=\left(\frac{2x^{10}}{9}+\frac{80x^8}{21}+\frac{256x^6}{7}+\frac{1024x^4}{7}\right)I_0^2(x)-$$

$$-\left(\frac{20x^9}{9}+\frac{640x^7}{21}+\frac{1536x^5}{7}+\frac{4096x^3}{7}\right)I_0(x)I_1(x)+$$

$$+\left(\frac{5x^{10}}{18}+\frac{320x^8}{63}+\frac{384x^6}{7}+\frac{2048x^4}{7}+\frac{4096x^2}{7}\right)I_1^2(x)$$

$$\int x^{10}\cdot K_0(x)K_1(x)\,dx=-\left(\frac{2x^{10}}{9}+\frac{80x^8}{21}+\frac{256x^6}{7}+\frac{1024x^4}{7}\right)K_0^2(x)-$$

$$-\left(\frac{20x^9}{9}+\frac{640x^7}{21}+\frac{1536x^5}{7}+\frac{4096x^3}{7}\right)K_0(x)K_1(x)-$$

$$-\left(\frac{5x^{10}}{18}+\frac{320x^8}{63}+\frac{384x^6}{7}+\frac{2048x^4}{7}+\frac{4096x^2}{7}\right)K_1^2(x)$$

Let

$$\int x^m J_0(x)J_1(x)\,dx=P_m(x)J_0^2(x)+Q_m(x)J_0(x)J_1(x)+R_m(x)J_1^2(x)\,,$$

$$\int x^m I_0(x)I_1(x)\,dx=P_m^*(x)I_0^2(x)+Q_m^*(x)I_0(x)I_1(x)+R_m^*(x)I_1^2(x)\,,$$

$$\int x^m K_0(x)K_1(x)\,dx=-P_m^*(x)K_0^2(x)+Q_m^*(x)K_0(x)K_1(x)-R_m^*(x)K_1^2(x)\,,$$

then holds

$$P_{12}=-\frac{5}{22}x^{12}+\frac{200}{33}x^{10}-\frac{8000}{77}x^8+\frac{76800}{77}x^6-\frac{307200}{77}x^4$$

$$Q_{12}=\frac{30}{11}x^{11}-\frac{2000}{33}x^9+\frac{64000}{77}x^7-\frac{460800}{77}x^5+\frac{1228800}{77}x^3$$

$$R_{12}=\frac{3}{11}x^{12}-\frac{250}{33}x^{10}+\frac{32000}{231}x^8-\frac{115200}{77}x^6+\frac{614400}{77}x^4-\frac{1228800}{77}x^2$$

$$P_{12}^*=\frac{5}{22}x^{12}+\frac{200}{33}x^{10}+\frac{8000}{77}x^8+\frac{76800}{77}x^6+\frac{307200}{77}x^4$$

$$Q_{12}^*=-\frac{30}{11}x^{11}-\frac{2000}{33}x^9-\frac{64000}{77}x^7-\frac{460800}{77}x^5-\frac{1228800}{77}x^3$$

$$R_{12}^*=\frac{3}{11}x^{12}+\frac{250}{33}x^{10}+\frac{32000}{231}x^8+\frac{115200}{77}x^6+\frac{614400}{77}x^4+\frac{1228800}{77}x^2$$

$$P_{14}=-\frac{3}{13}x^{14}+\frac{1260}{143}x^{12}-\frac{33600}{143}x^{10}+\frac{576000}{143}x^8-\frac{5529600}{143}x^6+\frac{22118400}{143}x^4$$

$$Q_{14}=\frac{42}{13}x^{13}-\frac{15120}{143}x^{11}+\frac{336000}{143}x^9-\frac{4608000}{143}x^7+\frac{33177600}{143}x^5-\frac{88473600}{143}x^3$$

$$R_{14}=\frac{7}{26}x^{14}-\frac{1512}{143}x^{12}+\frac{42000}{143}x^{10}-\frac{768000}{143}x^8+\frac{8294400}{143}x^6-\frac{44236800}{143}x^4+\frac{88473600}{143}x^2$$

$$P_{14}^*=\frac{3}{13}x^{14}+\frac{1260}{143}x^{12}+\frac{33600}{143}x^{10}+\frac{576000}{143}x^8+\frac{5529600}{143}x^6+\frac{22118400}{143}x^4$$

$$Q_{14}^*=-\frac{42}{13}x^{13}-\frac{15120}{143}x^{11}-\frac{336000}{143}x^9-\frac{4608000}{143}x^7-\frac{33177600}{143}x^5-\frac{88473600}{143}x^3$$

$$R_{14}^*=\frac{7}{26}x^{14}+\frac{1512}{143}x^{12}+\frac{42000}{143}x^{10}+\frac{768000}{143}x^8+\frac{8294400}{143}x^6+\frac{44236800}{143}x^4+\frac{88473600}{143}x^2$$

$$P_{16}=-\frac{7}{30}x^{16}+\frac{784}{65}x^{14}-\frac{65856}{143}x^{12}+\frac{1756160}{143}x^{10}-\frac{30105600}{143}x^8+\frac{289013760}{143}x^6-\frac{1156055040}{143}x^4$$

$$Q_{16} = \frac{56}{15}\,x^{15} - \frac{10976}{65}\,x^{13} + \frac{790272}{143}\,x^{11} - \frac{17561600}{143}\,x^{9} + \frac{240844800}{143}\,x^{7} - \frac{1734082560}{143}\,x^{5} + \frac{4624220160}{143}\,x^{3}$$

$$R_{16} =$$

$$= \frac{4}{15}\,x^{16} - \frac{2744}{195}\,x^{14} + \frac{395136}{715}\,x^{12} - \frac{2195200}{143}\,x^{10} + \frac{40140800}{143}\,x^{8} - \frac{433520640}{143}\,x^{6} + \frac{2312110080}{143}\,x^{4} - \frac{4624220160}{143}\,x^{2}$$

$$P^*_{16} = \frac{7}{30}\,x^{16} + \frac{784}{65}\,x^{14} + \frac{65856}{143}\,x^{12} + \frac{1756160}{143}\,x^{10} + \frac{30105600}{143}\,x^{8} + \frac{289013760}{143}\,x^{6} + \frac{1156055040}{143}\,x^{4}$$

$$Q^*_{16} = -\frac{56}{15}\,x^{15} - \frac{10976}{65}\,x^{13} - \frac{790272}{143}\,x^{11} - \frac{17561600}{143}\,x^{9} - \frac{240844800}{143}\,x^{7} - \frac{1734082560}{143}\,x^{5} - \frac{4624220160}{143}\,x^{3}$$

$$R^*_{16} =$$

$$= \frac{4}{15}\,x^{16} + \frac{2744}{195}\,x^{14} + \frac{395136}{715}\,x^{12} + \frac{2195200}{143}\,x^{10} + \frac{40140800}{143}\,x^{8} + \frac{433520640}{143}\,x^{6} + \frac{2312110080}{143}\,x^{4} + \frac{4624220160}{143}\,x^{2}$$

Recurrence Formulas:

$$\int x^{2n+2} J_0(x) J_1(x)\,dx =$$

$$= \frac{x^{2n+1}}{4n+2}\left[-nxJ_0^2(x) + 2n(n+1)J_0(x)J_1(x) + (n+1)xJ_1^2(x)\right] - \frac{2n^2(n+1)}{2n+1}\int x^{2n} J_0(x) J_1(x)\,dx$$

$$\int x^{2n+2} I_0(x) I_1(x)\,dx =$$

$$= \frac{x^{2n+1}}{4n+2}\left[nxI_0^2(x) - 2n(n+1)I_0(x)I_1(x) + (n+1)xI_1^2(x)\right] + \frac{2n^2(n+1)}{2n+1}\int x^{2n} I_0(x) I_1(x)\,dx$$

$$\int x^{2n+2} K_0(x) K_1(x)\,dx =$$

$$= -\frac{x^{2n+1}}{4n+2}\left[nxK_0^2(x) + 2n(n+1)K_0(x)K_1(x) + (n+1)xK_1^2(x)\right] + \frac{2n^2(n+1)}{2n+1}\int x^{2n} K_0(x) K_1(x)\,dx$$

2.1.5. Integrals of the type $\int x^{2n+1} Z_0(x) Z_1(x)\,dx$

The integrals $\int J_0^2(x)\,dx$ and $\int I_0^2(x)\,dx$ may be defined as the functions $\Theta(x)$ and $\Omega(x)$ in 2.1.3., page 272
.

$$\int x\,J_0(x)J_1(x)\,dx = -\frac{x}{2}\,J_0^2(x) + \frac{1}{2}\int J_0^2(x)\,dx$$

$$\int x\,I_0(x)I_1(x)\,dx = \frac{x}{2}\,I_0^2(x) - \frac{1}{2}\int I_0^2(x)\,dx$$

$$\int x^3\,J_0(x)J_1(x)\,dx = \frac{1}{16}\left[(-2\,x^3+3\,x)\,J_0^2(x) + 6\,x^2 J_0(x)J_1(x) + 6\,x^3\,J_1^2(x)\right] - \frac{3}{16}\int J_0^2(x)\,dx$$

$$\int x^3\,I_0(x)I_1(x)\,dx = \frac{1}{16}\left[(2\,x^3+3\,x)\,I_0^2(x) - 6x^2 I_0(x)I_1(x) + 6x^3\,I_1^2(x)\right] - \frac{3}{16}\int I_0^2(x)\,dx$$

With

$$\int x^{2n+1}\,J_0(x)\cdot J_1(x)\,dx = \frac{1}{\beta_n}\left[A_n(x)\,J_0^2(x) + B_n(x)J_0(x)J_1(x) + C_n(x)\,J_1^2(x) + \gamma_n\int J_0^2(x)\,dx\right]$$

$$\int x^{2n+1}\,I_0(x)\cdot I_1(x)\,dx = \frac{1}{\beta_n^*}\left[A_n^*(x)\,I_0^2(x) + I_n^*(x)I_0(x)I_1(x) + C_n^*(x)\,I_1^2(x) + \gamma_n^*\int I_0^2(x)\,dx\right]$$

holds

$$\beta_2 = 256\,, \qquad \gamma_2 = 135$$
$$A_2(x) = -48\,x^5 + 90\,x^3 - 135\,x\,, \quad B_2(x) = 240\,x^4 - 270\,x^2\,, \quad C_2(x) = 80\,x^5 - 270\,x^3$$

$$\beta_2^* = 256\,, \qquad \gamma_2^* = -135$$
$$A_2^*(x) = 48\,x^5 + 90\,x^3 + 135\,x\,, \quad B_2^*(x) = -240\,x^4 - 270\,x^2\,, \quad C_2^*(x) = 80\,x^5 + 270\,x^3$$

$$\beta_3 = 6144\,, \qquad \gamma_3 = -23625$$
$$A_3(x) = -1280\,x^7 + 8400\,x^5 - 15750\,x^3 + 23625\,x$$
$$B_3(x) = 8960\,x^6 - 42000\,x^4 + 47250\,x^2\,, \quad C_3(x) = 1792\,x^7 - 14000\,x^5 + 47250\,x^3$$

$$\beta_3^* = 6144\,, \qquad \gamma_3^* = -23625$$
$$A_3^*(x) = 1280\,x^7 + 8400\,x^5 + 15750\,x^3 + 23625\,x$$
$$B_3^*(x) = -8960\,x^6 - 42000\,x^4 - 47250\,x^2\,, \quad C_3^*(x) = 1792\,x^7 + 14000\,x^5 + 47250\,x^3$$

$$\beta_4 = 65536\,, \qquad \gamma_4 = 3472875$$
$$A_4(x) = -14336\,x^9 + 188160\,x^7 - 1234800\,x^5 + 2315250\,x^3 - 3472875\,x$$
$$B_4(x) = 129024\,x^8 - 1317120\,x^6 + 6174000\,x^4 - 6945750\,x^2$$
$$C_4(x) = 18432\,x^9 - 263424\,x^7 + 2058000\,x^5 - 6945750\,x^3$$

$$\beta_4^* = 65536\,, \qquad \gamma_4^* = -3472875$$
$$A_4^*(x) = 14336\,x^9 + 188160\,x^7 + 1234800\,x^5 + 2315250\,x^3 + 3472875\,x$$
$$B_4^*(x) = -129024\,x^8 - 1317120\,x^6 - 6174000\,x^4 - 6945750\,x^2$$
$$C_4^*(x) = 18432\,x^9 + 263424\,x^7 + 2058000\,x^5 + 6945750\,x^3$$

$$\beta_5 = 2621440\,, \qquad \gamma_5 = -3094331625$$
$$A_5(x) = -589824\,x^{11} + 12773376\,x^9 - 167650560\,x^7 + 1100206800\,x^5 - 2062887750\,x^3 + 3094331625\,x$$

$$B_5(x) = 6488064\,x^{10} - 114960384\,x^8 + 1173553920\,x^6 - 5501034000\,x^4 + 6188663250\,x^2$$
$$C_5(x) = 720896\,x^{11} - 16422912\,x^9 + 234710784\,x^7 - 1833678000\,x^5 + 6188663250\,x^3$$

$$\beta_5^* = 2621440\ , \qquad \gamma_5^* = -3094331625$$
$$A_5^*(x) = 589824\,x^{11} + 12773376\,x^9 + 167650560\,x^7 + 1100206800\,x^5 + 2062887750\,x^3 + 3094331625\,x$$
$$B_5^*(x) = -6488064\,x^{10} - 114960384\,x^8 - 1173553920\,x^6 - 5501034000\,x^4 - 6188663250\,x^2$$
$$C_5^*(x) = 720896\,x^{11} + 16422912\,x^9 + 234710784\,x^7 + 1833678000\,x^5 + 6188663250\,x^3$$

$$\beta_6 = 125829120\ , \qquad \gamma_6 = 4867383646125$$
$$A_6(x) = -28835840\,x^{13} + 927793152\,x^{11} - 20092520448\,x^9 + 263714330880\,x^7 - 1730625296400\,x^5 + \\ +3244922430750\,x^3 - 4867383646125\,x$$
$$B_6(x) = 374865920\,x^{12} - 10205724672\,x^{10} + 180832684032\,x^8 - 1846000316160\,x^6 + 8653126482000\,x^4 - \\ -9734767292250\,x^2$$
$$C_6(x) = 34078720\,x^{13} - 1133969408\,x^{11} + 25833240576\,x^9 - 369200063232\,x^7 + 2884375494000\,x^5 - \\ -9734767292250\,x^3$$

$$\beta_6^* = 125829120\ , \qquad \gamma_6^* = 4867383646125$$
$$A_6^*(x) = 28835840\,x^{13} + 927793152\,x^{11} + 20092520448\,x^9 + 263714330880\,x^7 + 1730625296400\,x^5 + \\ +3244922430750\,x^3 + 4867383646125\,x$$
$$B_6^*(x) = -374865920\,x^{12} - 10205724672\,x^{10} - 180832684032\,x^8 - 1846000316160\,x^6 - 8653126482000\,x^4 - \\ -9734767292250\,x^2$$
$$C_6^*(x) = 34078720\,x^{13} + 1133969408\,x^{11} + 25833240576\,x^9 + 369200063232\,x^7 + 2884375494000\,x^5 + \\ +9734767292250\,x^3$$

Recurrence formulas:

$$\int x^{2n+1}\,J_0(x)\,J_1(x)\,dx =$$
$$= -\frac{x^{2n}}{8n}\,\left[(2n-1)xJ_0^2(x) - (2n+1)xJ_1^2(x) - (4n^2-1)\,J_0(x)J_1(x)\right] - (2n-1)^2(2n+1)\int x^{2n-1}\,J_0(x)\,J_1(x)\,dx$$
$$\int x^{2n+1}\,I_0(x)\,I_1(x)\,dx =$$
$$= \frac{x^{2n}}{8n}\,\left[(2n-1)xI_0^2(x) + (2n+1)xI_1^2(x) - (4n^2-1)\,I_0(x)I_1(x)\right] + (2n-1)^2(2n+1)\int x^{2n-1}\,I_0(x)\,I_1(x)\,dx$$

2.1.6. Integrals of the type $\int x^{-(2n+1)} Z_0(x) Z_1(x)\, dx$

See also [4], 1.8.3..

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x} = x[J_0^2(x) + J_1^2(x)] - J_0(x)\, J_1(x)$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x} = x[I_0^2(x) - I_1^2(x)] - I_0(x)\, I_1(x)$$

$$\int \frac{K_0(x)\, K_1(x)\, dx}{x} = x[-K_0^2(x) + K_1^2(x)] - K_0(x)\, K_1(x)$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x} = -\, x[K_0(x)\, I_0(x) + K_1(x)\, I_1(x)] - I_0(x)\, K_1(x)$$

$$\int \frac{K_0(x)\, I_1(x)\, dx}{x} = x[K_0(x)\, I_0(x) + K_1(x)\, I_1(x)] - K_0(x)\, I_1(x)$$

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x^3} = \frac{1}{9x^2} \left[x(-8\,x^2 - 3)\, J_0^2(x) + (8\,x^2 - 3)\, J_0(x)\, J_1(x) + x(-8\,x^2 + 1)\; J_1^2(x)\right]$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x^3} = \frac{1}{9x^2} \left[x(8\,x^2 - 3)\, I_0^2(x) - (8\,x^2 + 3)\, I_0(x)\, J_1(x) - x(8\,x^2 + 1)\; I_1^2(x)\right]$$

$$\int \frac{K_0(x)\, K_1(x)\, dx}{x^3} = \frac{1}{9x^2} \left[x(-8\,x^2 + 3)\, K_0^2(x) + (-8\,x^2 - 3)\, K_0(x)\, K_1(x) + x(8\,x^2 + 1)\; K_1^2(x)\right]$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x^3} =$$

$$= \frac{1}{9x^2} \left[(-8\,x^3 + 3\,x)\, I_0(x) K_0(x) + (-4\,x^2 - 3)\, I_0(x) K_1(x) + 4\,x^2\, I_1(x) K_0(x) - (8\,x^3 + x)\, I_1(x) K_1(x)\right]$$

$$\int \frac{I_1(x)\, K_0(x)\, dx}{x^3} =$$

$$= \frac{1}{9x^2} \left[(8\,x^3 - 3\,x)\, I_0(x) K_0(x) + 4\,x^2\, I_0(x) K_1(x) - (4\,x^2 + 3)\, I_1(x) K_0(x) + (8\,x^3 + x)\, I_1(x) K_1(x)\right]$$

$$\int \frac{J_0(x)\, J_1(x)\, dx}{x^5} = \frac{1}{675x^4} \left[x(128\,x^4 + 48\,x^2 - 45)\, J_0^2(x) + (-128\,x^4 + 48\,x^2 - 135)\, J_0(x)\, J_1(x) + \right.$$
$$\left. + \,x(128\,x^4 - 16\,x^2 + 27)\; J_1^2(x)\right]$$

$$\int \frac{I_0(x)\, I_1(x)\, dx}{x^5} = \frac{1}{675x^4} \left[x(128\,x^4 - 48\,x^2 - 45)\, I_0^2(x) - (128\,x^4 + 48\,x^2 + 135)\, I_0(x)\, I_1(x) - \right.$$
$$\left. -\, x(128\,x^4 + 16\,x^2 + 27)\; I_1^2(x)\right]$$

$$\int \frac{K_0(x)\, K_1(x)\, dx}{x^5} = \frac{1}{675x^4} \left[x(-128\,x^4 + 48\,x^2 + 45)\, K_0^2(x) + (-128\,x^4 - 48\,x^2 - 135)\, K_0(x)\, K_1(x) + \right.$$
$$\left. +\; x(128\,x^4 + 16\,x^2 + 27)\; K_1^2(x)\right]$$

$$\int \frac{I_0(x)\, K_1(x)\, dx}{x^5} = \frac{1}{675x^4} \left[(-128\,x^5 + 48\,x^3 + 45\,x)\, I_0(x) K_0(x) + (-64\,x^4 - 24\,x^2 - 135)\, I_0(x) K_1(x) + \right.$$
$$\left. + (64\,x^4 + 24\,x^2)\, I_1(x) K_0(x) + (-128\,x^5 - 16\,x^3 - 27\,x)\, I_1(x) K_1(x)\right]$$

$$\int \frac{I_1(x)\, K_0(x)\, dx}{x^5} = \frac{1}{675x^4} \left[(128\,x^5 - 48\,x^3 - 45\,x)\, I_0(x) K_0(x) + (64\,x^4 + 24\,x^2)\, I_0(x) K_1(x) + \right.$$
$$\left. + (-64\,x^4 - 24\,x^2 - 135)\, I_1(x) K_0(x) + (128\,x^5 + 16\,x^3 + 27\,x)\, I_1(x) K_1(x)\right]$$

The integrals for $K_0(x)K_1(x)$ may be found in a simple way from such for $I_0(x)I_1(x)$. The same holds for $I_1(x)K_0(x)$, which is similar to $I_0(x)K_1(x)$.

$$\int \frac{J_0(x)\,J_1(x)\,dx}{x^7} = \frac{1}{55\,125x^6}\,\big[x(-1\,024\,x^6-384\,x^4+360\,x^2-1\,575)\,J_0^2(x)+$$
$$+(1\,024\,x^6-384\,x^4+1\,080\,x^2-7\,875)\,J_0(x)\,J_1(x)+x\,(-1\,024\,x^6+128\,x^4-216\,x^2+1\,125)\;J_1^2(x)\big]$$

$$\int \frac{I_0(x)\,I_1(x)\,dx}{x^7} = \frac{1}{55\,125x^6}\,\big[x(1\,024\,x^6-384\,x^4-360\,x^2-1\,575)\,I_0^2(x)-$$
$$-(1\,024\,x^6+384\,x^4+1\,080\,x^2+7\,875)\,I_0(x)\,I_1(x)-x\,(1\,024\,x^6+128\,x^4+216\,x^2+1\,125)\;I_1^2(x)\big]$$

$$\int \frac{I_0(x)\,K_1(x)\,dx}{x^7} = \frac{1}{55125x^6}\,\big[(-1024\,x^7+384\,x^5+360\,x^3+1575\,x)\,I_0(x)K_0(x)-$$
$$-(512\,x^6+192\,x^4+540\,x^2+7875)\,I_0(x)K_1(x)+(512\,x^6+192\,x^4+540\,x^2)\,I_1(x)K_0(x)-$$
$$-(1024\,x^7+128\,x^5+216\,x^3+1125\,x)\,I_1(x)K_1(x)\big]$$

$$\int \frac{J_0(x)\,J_1(x)\,dx}{x^9} = \frac{1}{31\,255\,875x^8}\,\big[x\,(32\,768\,x^8+12\,288\,x^6-11\,520\,x^4+50\,400\,x^2-496\,125)\,J_0^2(x)+$$
$$+(-32\,768\,x^8+12\,288\,x^6-34\,560\,x^4+252\,000\,x^2-3\,472\,875)\,J_0(x)\,J_1(x)+$$
$$+x\,(32\,768\,x^8-4\,096\,x^6+6\,912\,x^4-36\,000\,x^2+385\,875)\;J_1^2(x)\big]$$

$$\int \frac{I_0(x)\,I_1(x)\,dx}{x^9} = \frac{1}{31\,255\,875x^8}\,\big[x\,(32\,768\,x^8-12\,288\,x^6-11\,520\,x^4-50\,400\,x^2-496\,125)\,I_0^2(x)-$$
$$-(32\,768\,x^8+12\,288\,x^6+34\,560\,x^4+252\,000\,x^2+3\,472\,875)\,I_0(x)\,I_1(x)-$$
$$-x\,(32\,768\,x^8+4\,096\,x^6+6\,912\,x^4+36\,000\,x^2+385\,875)\;I_1^2(x)\big]$$

$$\int \frac{I_0(x)\,K_1(x)\,dx}{x^9} = \frac{1}{31255875x^8}\,\big[(-32768\,x^9+12288\,x^7+11520\,x^5+50400\,x^3+496125\,x)\,I_0(x)K_0(x)-$$
$$-(16384\,x^8+6144\,x^6+17280\,x^4+126000\,x^2+3472875)\,I_0(x)K_1(x)+$$
$$+(16384\,x^8+6144\,x^6+17280\,x^4+126000\,x^2)\,I_1(x)K_0(x)-$$
$$-(32768\,x^9+4096\,x^7+6912\,x^5+36000\,x^3+385875\,x)\,I_1(x)K_1(x)\big]$$

$$\int \frac{J_0(x)\,J_1(x)\,dx}{x^{11}} = \frac{1}{3\,403\,7647\,875x^{10}}\,\big[\,x\,(-1\,310\,720\,x^{10}-$$
$$-491\,520\,x^8+460\,800\,x^6-2\,016\,000\,x^4+19\,845\,000\,x^2-343\,814\,625)\,J_0^2(x)+$$
$$+(1\,310\,720\,x^{10}-491\,520\,x^8+1\,382\,400\,x^6-10\,080\,000\,x^4+138\,915\,000\,x^2-3\,094\,331\,625)\,J_0(x)\,J_1(x)+$$
$$+x\,(-1\,310\,720\,x^{10}+163\,840\,x^8-276\,480\,x^6+1\,440\,000\,x^4-15\,435\,000\,x^2+281\,302\,875)\,J_1^2(x)\big]$$

$$\int \frac{I_0(x)\,I_1(x)\,dx}{x^{11}} = \frac{1}{34\,037\,647\,875x^{10}}\,\big[x\,(1\,310\,720\,x^{10}-$$
$$-491\,520\,x^8-460\,800\,x^6-2\,016\,000\,x^4-19\,845\,000\,x^2-343\,814\,625)\,I_0^2(x)-$$
$$-(1\,310\,720\,x^{10}+491\,520\,x^8+1\,382\,400\,x^6+10\,080\,000\,x^4+138\,915\,000\,x^2+3\,094\,331\,625)\,I_0(x)\,I_1(x)-$$
$$-x\,(1\,310\,720\,x^{10}+163\,840\,x^8+276\,480\,x^6+1\,440\,000\,x^4+15\,435\,000\,x^2+281\,302\,875)\;I_1^2(x)\big]$$

$$\int \frac{I_0(x)\,K_1(x)\,dx}{x^{11}} = \frac{1}{6807529575x^{10}}\,\cdot$$
$$\cdot\;\big[(-262144\,x^{11}+98304\,x^9+92160\,x^7+403200\,x^5+3969000\,x^3+68762925\,x)\,I_0(x)K_0(x)-$$
$$-(131072\,x^{10}+49152\,x^8+138240\,x^6+1008000\,x^4+13891500\,x^2+618866325)\,I_0(x)K_1(x)+$$

$$+(131072\,x^{10}+49152\,x^{8}+138240\,x^{6}+1008000\,x^{4}+13891500\,x^{2})\,I_1(x)K_0(x)-$$
$$-(262144\,x^{11}+32768\,x^{9}+55296\,x^{7}+288000\,x^{5}+3087000\,x^{3}+56260575\,x)\,I_1(x)K_1(x)\big]$$

$$\int\frac{J_0(x)\,J_1(x)\,dx}{x^{13}} = \frac{1}{4\,218\,399\,159\,975x^{12}}\,\big[x\,(4\,194\,304\,x^{12}+1\,572\,864\,x^{10}-1\,474\,560\,x^{8}+$$
$$+6\,451\,200\,x^{6}-63\,504\,000\,x^{4}+1\,100\,206\,800\,x^{2}-29\,499\,294\,825)\,J_0^2(x)+$$
$$+(-4\,194\,304\,x^{12}+1\,572\,864\,x^{10}-4\,423\,680\,x^{8}+32\,256\,000\,x^{6}-$$
$$-444\,528\,000\,x^{4}+9\,901\,861\,200\,x^{2}-324\,492\,243\,075)\,J_0(x)\,J_1(x)+$$
$$+x\,(4\,194\,304\,x^{12}-524\,288\,x^{10}+884\,736\,x^{8}-4\,608\,000\,x^{6}+$$
$$+49\,392\,000\,x^{4}-900\,169\,200\,x^{2}+24\,960\,941\,775)\;J_1^2(x)\big]$$

$$\int\frac{I_0(x)\,I_1(x)\,dx}{x^{13}} = \frac{1}{4\,218\,399\,159\,975x^{12}}\,\big[x\,(4\,194\,304\,x^{12}-1\,572\,864\,x^{10}-1\,474\,560\,x^{8}-$$
$$-6\,451\,200\,x^{6}-63\,504\,000\,x^{4}-1\,100\,206\,800\,x^{2}-29\,499\,294\,825)\,I_0^2(x)-$$
$$-(4\,194\,304\,x^{12}+1\,572\,864\,x^{10}+4\,423\,680\,x^{8}+32\,256\,000\,x^{6}+$$
$$+444\,528\,000\,x^{4}+9\,901\,861\,200\,x^{2}+324\,492\,243\,075)\,I_0(x)\,I_1(x)-$$
$$-x\,(4\,194\,304\,x^{12}+524\,288\,x^{10}+884\,736\,x^{8}+4\,608\,000\,x^{6}+$$
$$+49\,392\,000\,x^{4}+900\,169\,200\,x^{2}+24\,960\,941\,775)\;I_1^2(x)\big]$$

$$\int\frac{I_0(x)\,K_1(x)\,dx}{x^{13}} = \frac{1}{4218399159975x^{12}}\cdot$$
$$\cdot\,\big[(-4194304\,x^{13}+1572864\,x^{11}+1474560\,x^{9}+6451200\,x^{7}+63504000\,x^{5}+1100206800\,x^{3}+$$
$$+29499294825\,x)\,I_0(x)K_0(x)-(2097152\,x^{12}+786432\,x^{10}+2211840\,x^{8}+16128000\,x^{6}+222264000\,x^{4}+$$
$$*4950930600\,x^{2}+324492243075)\,I_0(x)K_1(x)+(2097152\,x^{12}+786432\,x^{10}+2211840\,x^{8}+16128000\,x^{6}+$$
$$+222264000\,x^{4}+4950930600\,x^{2})\,I_1(x)K_0(x)-(4194304\,x^{13}+524288\,x^{11}+884736\,x^{9}+4608000\,x^{7}+$$
$$+49392000\,x^{5}+900169200\,x^{3}+24960941775\,x)\,I_1(x)K_1(x)\big]$$

Recurrence Relations:

$$\int\frac{J_0(x)\,J_1(x)\,dx}{x^{2n+1}} =$$
$$= \frac{1}{(2n+1)x^{2n}}\left[-\frac{x\,J_0^2(x)}{2n-1}-J_0(x)\,J_1(x)+\frac{x\,J_1^2(x)}{2n+1}\right]-\frac{8n}{(2n+1)^2(2n-1)}\int\frac{J_0(x)\,J_1(x)\,dx}{x^{2n-1}}$$

$$\int\frac{I_0(x)\,I_1(x)\,dx}{x^{2n+1}} =$$
$$= -\frac{1}{(2n+1)x^{2n}}\left[\frac{x\,I_0^2(x)}{2n-1}+I_0(x)\,I_1(x)+\frac{x\,I_1^2(x)}{2n+1}\right]+\frac{8n}{(2n+1)^2(2n-1)}\int\frac{I_0(x)\,I_1(x)\,dx}{x^{2n-1}}$$

$$\int\frac{K_0(x)\,K_1(x)\,dx}{x^{2n+1}} =$$
$$= \frac{1}{(2n+1)x^{2n}}\left[\frac{x\,K_0^2(x)}{2n-1}-K_0(x)\,K_1(x)+\frac{x\,K_1^2(x)}{2n+1}\right]+\frac{8n}{(2n+1)^2(2n-1)}\int\frac{K_0(x)\,K_1(x)\,dx}{x^{2n-1}}$$

$$\int\frac{I_0(x)\,K_1(x)\,dx}{x^{2n+1}} = \frac{1}{(2n+1)x^{2n}}\left[\frac{x\,I_0(x)\,K_0(x)}{2n-1}-I_0(x)\,K_1(x)-\frac{x\,I_1(x)\,K_1(x)}{2n+1}\right]+$$
$$+\frac{4n}{(2n+1)^2(2n-1)}\int\frac{I_0(x)\,K_1(x)-I_1(x)\,K_0(x)}{x^{2n-1}}\,dx$$

$$\int\frac{I_1(x)\,K_0(x)\,dx}{x^{2n+1}} = \frac{1}{(2n+1)x^{2n}}\left[-\frac{x\,I_0(x)\,K_0(x)}{2n-1}-I_1(x)\,K_0(x)+\frac{x\,I_1(x)\,K_1(x)}{2n+1}\right]+$$
$$+\frac{4n}{(2n+1)^2(2n-1)}\int\frac{I_1(x)\,K_0(x)-I_0(x)\,K_1(x)}{x^{2n-1}}\,dx$$

2.1.7. Integrals of the type $\int x^{2n+1} \cdot J_\nu(x) \cdot \left\{ \begin{array}{c} I_\nu(x) \\ K_\nu(x) \end{array} \right\} dx$

a) $\nu = 0$:

$$\int x \cdot J_0(x) \cdot I_0(x)\,dx = \frac{x}{2}\,[J_0(x) \cdot I_1(x) + J_1(x) \cdot I_0(x)]$$

$$\int x \cdot J_0(x) \cdot K_0(x)\,dx = \frac{x}{2}\,[-J_0(x) \cdot K_1(x) + J_1(x) \cdot K_0(x)]$$

$$\int x^3 \cdot J_0(x) \cdot I_0(x)\,dx = \frac{1}{2}\,\left[x^3 J_0(x) \cdot I_1(x) + x^3 J_1(x) \cdot I_0(x) - 2x^2 J_1(x) \cdot I_1(x)\right]$$

$$\int x^3 \cdot J_0(x) \cdot K_0(x)\,dx = \frac{1}{2}\,\left[-x^3 J_0(x) \cdot K_1(x) + x^3 J_1(x) \cdot K_0(x) + 2x^2 J_1(x) \cdot K_1(x)\right]$$

$$\int x^5 \cdot J_0(x) \cdot I_0(x)\,dx =$$

$$= \frac{1}{2}\,\left[8\,x^2 J_0(x) \cdot I_0(x) + (x^5 - 4\,x^3 - 8\,x) J_0(x) \cdot I_1(x) + (x^5 + 4\,x^3 - 8\,x) J_1(x) \cdot I_0(x) - 4\,x^4 J_1(x) \cdot I_1(x)\right]$$

$$\int x^5 \cdot J_0(x) \cdot K_0(x)\,dx =$$

$$= \frac{1}{2}\,\left[8\,x^2 J_0(x) \cdot K_0(x) - (x^5 - 4\,x^3 - 8\,x) J_0(x) \cdot K_1(x) + (x^5 + 4\,x^3 - 8\,x) J_1(x) \cdot K_0(x) + 4\,x^4 J_1(x) \cdot I_K(x)\right]$$

$$\int x^7 \cdot J_0(x) \cdot I_0(x)\,dx = \frac{1}{2}\,\left[48\,x^4 J_0(x) \cdot I_0(x) + (x^7 - 12\,x^5 - 96\,x^3) J_0(x) \cdot I_1(x) + \right.$$
$$\left. + (x^7 + 12\,x^5 - 96\,x^3) J_1(x) \cdot I_0(x) + (-6\,x^6 + 192\,x^2) J_1(x) \cdot I_1(x)\right]$$

$$\int x^7 \cdot J_0(x) \cdot K_0(x)\,dx = \frac{1}{2}\,\left[48\,x^4 J_0(x) \cdot K_0(x) - (x^7 - 12\,x^5 - 96\,x^3) J_0(x) \cdot K_1(x) + \right.$$
$$\left. + (x^7 + 12\,x^5 - 96\,x^3) J_1(x) \cdot K_0(x) + (6\,x^6 - 192\,x^2) J_1(x) \cdot K_1(x)\right]$$

$$\int x^9 \cdot J_0(x) \cdot I_0(x)\,dx =$$

$$= \frac{1}{2}\,\left[(144\,x^6 - 3456\,x^2) J_0(x) \cdot I_0(x) + (x^9 - 24\,x^7 - 432\,x^5 + 1728\,x^3 + 3456\,x) J_0(x) \cdot I_1(x) + \right.$$
$$\left. + (x^9 + 24\,x^7 - 432\,x^5 - 1728\,x^3 + 3456\,x) J_1(x) \cdot I_0(x) + (-8\,x^8 + 1728\,x^4) J_1(x) \cdot I_1(x)\right]$$

$$\int x^9 \cdot J_0(x) \cdot K_0(x)\,dx =$$

$$= \frac{1}{2}\,\left[(144\,x^6 - 3456\,x^2) J_0(x) \cdot K_0(x) - (x^9 - 24\,x^7 - 432\,x^5 + 1728\,x^3 + 3456\,x) J_0(x) \cdot K_1(x) + \right.$$
$$\left. + (x^9 + 24\,x^7 - 432\,x^5 - 1728\,x^3 + 3456\,x) J_1(x) \cdot K_0(x) + (8\,x^8 - 1728\,x^4) J_1(x) \cdot K_1(x)\right]$$

With

$$\int x^n \cdot J_0(x) \cdot I_0(x)\,dx = \frac{1}{2}\,[P_n(x) J_0(x) \cdot I_0(x) + Q_n(x) J_0(x) \cdot I_1(x) + R_n(x) J_1(x) \cdot I_0(x) + S_n(x) J_1(x) \cdot I_1(x)]$$

holds

$$\int x^n \cdot J_0(x) \cdot K_0(x)\,dx =$$

$$= \frac{1}{2}\,[P_n(x) J_0(x) \cdot K_0(x) - Q_n(x) J_0(x) \cdot K_1(x) + R_n(x) J_1(x) \cdot K_0(x) - S_n(x) J_1(x) \cdot K_1(x)]\ .$$

$$P_{11}(x) = 320\,x^8 - 61440\,x^4$$
$$Q_{11}(x) = x^{11} - 40\,x^9 - 1280\,x^7 + 15360\,x^5 + 122880\,x^3$$

$$R_{11}(x) = x^{11} + 40\,x^9 - 1280\,x^7 - 15360\,x^5 + 122880\,x^3$$
$$S_{11}(x) = -10\,x^{10} + 7680\,x^6 - 245760\,x^2$$

$$P_{13}(x) = 600\,x^{10} - 432000\,x^6 + 10368000\,x^2$$
$$Q_{13}(x) = x^{13} - 60\,x^{11} - 3000\,x^9 + 72000\,x^7 + 1296000\,x^5 - 5184000\,x^3 - 10368000\,x$$
$$R_{13}(x) = x^{13} + 60\,x^{11} - 3000\,x^9 - 72000\,x^7 + 1296000\,x^5 + 5184000\,x^3 - 10368000\,x$$
$$S_{13}(x) = -12\,x^{12} + 24000\,x^8 - 5184000\,x^4$$

$$P_{15}(x) = 1008\,x^{12} - 1935360\,x^8 + 371589120\,x^4$$
$$Q_{15}(x) = x^{15} - 84\,x^{13} - 6048\,x^{11} + 241920\,x^9 + 7741440\,x^7 - 92897280\,x^5 - 743178240\,x^3$$
$$R_{15}(x) = x^{15} + 84\,x^{13} - 6048\,x^{11} - 241920\,x^9 + 7741440\,x^7 + 92897280\,x^5 - 743178240\,x^3$$
$$S_{15}(x) = -14\,x^{14} + 60480\,x^{10} - 46448640\,x^6 + 1486356480\,x^2$$

b) $\nu = 1$:

$$\int x \cdot J_1(x) \cdot I_1(x)\,dx = \frac{x}{2}\,\left[-J_0(x) \cdot I_1(x) + J_1(x) \cdot I_0(x)\right]$$
$$\int x \cdot J_1(x) \cdot K_1(x)\,dx = -\,\frac{x}{2}\,\left[J_0(x) \cdot K_1(x) + J_1(x) \cdot K_0(x)\right]$$
$$\int x^3 \cdot J_1(x) \cdot I_1(x)\,dx = \frac{1}{2}\,\left[2\,x^2 J_0(x) \cdot I_0(x) - (x^3 + 2\,x) J_0(x) \cdot I_1(x) + (x^3 - 2\,x) J_1(x) \cdot I_0(x)\right]$$
$$\int x^3 \cdot J_1(x) \cdot K_1(x)\,dx = \frac{1}{2}\,\left[-2\,x^2 J_0(x) \cdot K_0(x) - (x^3 + 2\,x) J_0(x) \cdot K_1(x) - (x^3 - 2\,x) J_1(x) \cdot K_0(x)\right]$$

With

$$\int x^n \cdot J_1(x) \cdot I_1(x)\,dx = \frac{1}{2}\,\left[P_n(x) J_0(x) \cdot I_0(x) + Q_n(x) J_0(x) \cdot I_1(x) + R_n(x) J_1(x) \cdot I_0(x) + S_n(x) J_1(x) \cdot I_1(x)\right]$$

holds

$$\int x^n \cdot J_1(x) \cdot K_1(x)\,dx =$$
$$= \frac{1}{2}\,\left[-P_n(x) J_0(x) \cdot K_0(x) + Q_n(x) J_0(x) \cdot K_1(x) - R_n(x) J_1(x) \cdot K_0(x) + S_n(x) J_1(x) \cdot K_1(x)\right]\;.$$

$$P_5(x) = 4\,x^4\;, \quad Q_5(x) = -x^5 - 8\,x^3\;, \quad R_5(x) = x^5 - 8\,x^3\;, \quad S_5(x) = 16\,x^2$$

$$P_7(x) = 6\,x^6 - 144\,x^2\;, \quad Q_7(x) = -x^7 - 18\,x^5 + 72\,x^3 + 144\,x$$
$$R_7(x) = x^7 - 18\,x^5 - 72\,x^3 + 144\,x\;, \quad S_7(x) = 72\,x^4$$

$$P_9(x) = 8\,x^8 - 1536\,x^4\;, \quad Q_9(x) = -x^9 - 32\,x^7 + 384\,x^5 + 3072\,x^3$$
$$R_9(x) = x^9 - 32\,x^7 - 384\,x^5 + 3072\,x^3\;, \quad S_9(x) = 192\,x^6 - 6144\,x^2$$

$$P_{11}(x) = 10\,x^{10} - 7200\,x^6 + 172800\,x^2$$
$$Q_{11}(x) = -x^{11} - 50\,x^9 + 1200\,x^7 + 21600\,x^5 - 86400\,x^3 - 172800\,x$$
$$R_{11}(x) = x^{11} - 50\,x^9 - 1200\,x^7 + 21600\,x^5 + 86400\,x^3 - 172800\,x$$
$$S_{11}(x) = 400\,x^8 - 86400\,x^4$$

$$P_{13}(x) = 12\,x^{12} - 23040\,x^8 + 4423680\,x^4$$
$$Q_{13}(x) = -x^{13} - 72\,x^{11} + 2880\,x^9 + 92160\,x^7 - 1105920\,x^5 - 8847360\,x^3$$
$$R_{13}(x) = x^{13} - 72\,x^{11} - 2880\,x^9 + 92160\,x^7 + 1105920\,x^5 - 8847360\,x^3$$
$$S_{13}(x) = 720\,x^{10} - 552960\,x^6 + 17694720\,x^2$$

$$P_{15}(x) = 14\,x^{14} - 58800\,x^{10} + 42336000\,x^6 - 1016064000\,x^2$$
$$Q_{15}(x) = -x^{15} - 98\,x^{13} + 5880\,x^{11} + 294000\,x^9 - 7056000\,x^7 - 127008000\,x^5 + 508032000\,x^3 + 1016064000\,x$$
$$R_{15}(x) = x^{15} - 98\,x^{13} - 5880\,x^{11} + 294000\,x^9 + 7056000\,x^7 - 127008000\,x^5 - 508032000\,x^3 + 1016064000\,x$$
$$S_{15}(x) = 1176\,x^{12} - 2352000\,x^8 + 508032000\,x^4$$

c) Recurrence relations:

$$\int x^{2n+1}\,J_0(x)\,I_0(x)\,dx =$$
$$= \frac{x^{2n}}{2}\,\left[x\,J_0(x)\,I_1(x) + x\,J_1(x)\,I_0(x) - 2n\,J_1(x)\,I_1(x)\right] + 2n(n-1)\int x^{2n-1}\,J_1(x)\,I_1(x)\,dx$$
$$\int x^{2n+1}\,J_1(x)\,I_1(x)\,dx =$$
$$= \frac{x^{2n}}{2}\,\left[-x\,J_0(x)\,I_1(x) + x\,J_1(x)\,I_0(x) + 2n\,J_0(x)\,I_0(x)\right] - 2n^2\int x^{2n-1}\,J_0(x)\,I_0(x)\,dx$$

$$\int x^{2n+1}\,K_0(x)\,I_0(x)\,dx =$$
$$= \frac{x^{2n}}{2}\,\left[-x\,J_0(x)\,K_1(x) + x\,J_1(x)\,K_0(x) + 2n\,J_1(x)\,K_1(x)\right] - 2n(n-1)\int x^{2n-1}\,J_1(x)\,K_1(x)\,dx$$
$$\int x^{2n+1}\,J_1(x)\,K_1(x)\,dx =$$
$$= -\,\frac{x^{2n}}{2}\,\left[x\,J_0(x)\,K_1(x) + x\,J_1(x)\,K_0(x) + 2n\,J_0(x)\,K_0(x)\right] + 2n^2\int x^{2n-1}\,J_0(x)\,K_0(x)\,dx$$

2.1.8. Integrals of the type $\int x^{2n} \cdot J_\nu(x) \cdot \left\{ \begin{array}{c} I_{1-\nu}(x) \\ K_{1-\nu}(x) \end{array} \right\} dx$

a) $\nu = 0$:

$$\int x^2 \cdot J_0(x) \cdot I_1(x)\, dx = \frac{1}{2} \left[x^2 J_0(x) \cdot I_0(x) - x J_0(x) \cdot I_1(x) - x J_1(x) \cdot I_0(x) + x^2 J_1(x) \cdot I_1(x) \right]$$

$$\int x^2 \cdot J_0(x) \cdot K_1(x)\, dx = \frac{1}{2} \left[-x^2 J_0(x) \cdot K_0(x) - x J_0(x) \cdot K_1(x) + x J_1(x) \cdot K_0(x) + x^2 J_1(x) \cdot K_1(x) \right]$$

$$\int x^4 \cdot J_0(x) \cdot I_1(x)\, dx =$$

$$= \frac{1}{2} \left[(x^4 - 2\,x^2) J_0(x) \cdot I_0(x) + (-x^3 + 2\,x) J_0(x) \cdot I_1(x) + (-3\,x^3 + 2\,x) J_1(x) \cdot I_0(x) + (x^4 + 4\,x^2) J_1(x) \cdot I_1(x) \right]$$

$$\int x^4 \cdot J_0(x) \cdot K_1(x)\, dx =$$

$$= \frac{1}{2} \left[-(x^4 - 2\,x^2) J_0(x) \cdot K_0(x) - (x^3 - 2\,x) J_0(x) \cdot K_1(x) + (3\,x^3 - 2\,x) J_1(x) \cdot K_0(x) + (x^4 + 4\,x^2) J_1(x) \cdot K_1(x) \right]$$

With

$$\int x^n \cdot J_0(x) \cdot I_1(x)\, dx = \frac{1}{2} \left[P_n(x) J_0(x) \cdot I_0(x) + Q_n(x) J_0(x) \cdot I_1(x) + R_n(x) J_1(x) \cdot I_0(x) + S_n(x) J_1(x) \cdot I_1(x) \right]$$

holds

$$\int x^n \cdot J_0(x) \cdot K_1(x)\, dx =$$

$$= \frac{1}{2} \left[-P_n(x) J_0(x) \cdot K_0(x) + Q_n(x) J_0(x) \cdot K_1(x) - R_n(x) J_1(x) \cdot K_0(x) + S_n(x) J_1(x) \cdot K_1(x) \right] .$$

$$P_6(x) = x^6 - 8\,x^4 - 24\,x^2 , \quad Q_6(x) = -x^5 + 28\,x^3 + 24\,x$$

$$R_6(x) = -5\,x^5 + 4\,x^3 + 24\,x , \quad S_6(x) = x^6 + 12\,x^4 - 32\,x^2$$

$$P_8(x) = x^8 - 18\,x^6 - 192\,x^4 + 432\,x^2 , \quad Q_8(x) = -x^7 + 102\,x^5 + 168\,x^3 - 432\,x$$

$$R_8(x) = -7\,x^7 + 6\,x^5 + 600\,x^3 - 432\,x , \quad S_8(x) = x^8 + 24\,x^6 - 216\,x^4 - 768\,x^2$$

$$P_{10}(x) = x^{10} - 32\,x^8 - 720\,x^6 + 6144\,x^4 + 17280\,x^2$$

$$Q_{10}(x) = -x^9 + 248\,x^7 + 624\,x^5 - 20928\,x^3 - 17280\,x$$

$$R_{10}(x) = -9\,x^9 + 8\,x^7 + 3696\,x^5 - 3648\,x^3 - 17280\,x$$

$$S_{10}(x) = x^{10} + 40\,x^8 - 768\,x^6 - 8640\,x^4 + 24576\,x^2$$

$$P_{12}(x) = x^{12} - 50\,x^{10} - 1920\,x^8 + 36000\,x^6 + 368640\,x^4 - 864000\,x^2$$

$$Q_{12}(x) = -x^{11} + 490\,x^9 + 1680\,x^7 - 200160\,x^5 - 305280\,x^3 + 864000\,x$$

$$R_{12}(x) = -11\,x^{11} + 10\,x^9 + 13680\,x^7 - 15840\,x^5 - 1169280\,x^3 + 864000\,x$$

$$S_{12}(x) = x^{12} + 60\,x^{10} - 2000\,x^8 - 46080\,x^6 + 432000\,x^4 + 1474560\,x^2$$

$$P_{14}(x) = x^{14} - 72\,x^{12} - 4200\,x^{10} + 138240\,x^8 + 3024000\,x^6 - 26542080\,x^4 - 72576000\,x^2$$

$$Q_{14}(x) = -x^{13} + 852\,x^{11} + 3720\,x^9 - 1056960\,x^7 - 2436480\,x^5 + 89372160\,x^3 + 72576000\,x$$

$$R_{14}(x) = -13\,x^{13} + 12\,x^{11} + 38280\,x^9 - 48960\,x^7 - 15707520\,x^5 + 16796160\,x^3 + 72576000\,x$$

$$S_{14}(x) = x^{14} + 84\,x^{12} - 4320\,x^{10} - 168000\,x^8 + 3317760\,x^6 + 36288000\,x^4 - 106168320\,x^2$$

$$P_{16}(x) =$$
$$= x^{16} - 98\,x^{14} - 8064\,x^{12} + 411600\,x^{10} + 15482880\,x^8 - 296352000\,x^6 - 2972712960\,x^4 + 7112448000\,x^2$$
$$Q_{16}(x) =$$
$$= -x^{15} + 1358\,x^{13} + 7224\,x^{11} - 3993360\,x^9 - 12539520\,x^7 + 1632234240\,x^5 + 2389201920\,x^3 - 7112448000\,x$$
$$R_{16}(x) =$$
$$= -15\,x^{15} + 14\,x^{13} + 89544\,x^{11} - 122640\,x^9 - 111323520\,x^7 + 145877760\,x^5 + 9501649920\,x^3 - 7112448000\,x$$
$$S_{16}(x) =$$
$$= x^{16} + 112\,x^{14} - 8232\,x^{12} - 483840\,x^{10} + 16464000\,x^8 + 371589120\,x^6 - 3556224000\,x^4 - 11890851840\,x^2$$

b) $\nu = 1$:

$$\int x^2 \cdot J_1(x) \cdot I_0(x)\,dx = \frac{1}{2}\,\left[-x^2 J_0(x) \cdot I_0(x) + xJ_0(x) \cdot I_1(x) + xJ_1(x) \cdot I_0(x) + x^2 J_1(x) \cdot I_1(x)\right]$$
$$\int x^2 \cdot J_1(x) \cdot K_0(x)\,dx = \frac{1}{2}\,\left[-x^2 J_0(x) \cdot K_0(x) - xJ_0(x) \cdot K_1(x) + xJ_1(x) \cdot K_0(x) - x^2 J_1(x) \cdot K_1(x)\right]$$
$$\int x^4 \cdot J_1(x) \cdot I_0(x)\,dx =$$
$$= \frac{1}{2}\,\left[(-x^4 - 2\,x^2)J_0(x) \cdot I_0(x) + (3\,x^3 + 2\,x)J_0(x) \cdot I_1(x) + (x^3 + 2\,x)J_1(x) \cdot I_0(x) + (x^4 - 4\,x^2)J_1(x) \cdot I_1(x)\right]$$
$$\int x^4 \cdot J_1(x) \cdot K_0(x)\,dx =$$
$$= \frac{1}{2}\,\left[(-x^4 - 2\,x^2)J_0(x) \cdot K_0(x) - (3\,x^3 + 2\,x)J_0(x) \cdot K_1(x) + (x^3 + 2\,x)J_1(x) \cdot K_0(x) - (x^4 - 4\,x^2)J_1(x) \cdot K_1(x)\right]$$

With

$$\int x^n \cdot J_1(x) \cdot I_0(x)\,dx = \frac{1}{2}\,\left[P_n(x)J_0(x) \cdot I_0(x) + Q_n(x)J_0(x) \cdot I_1(x) + R_n(x)J_1(x) \cdot I_0(x) + S_n(x)J_1(x) \cdot I_1(x)\right]$$

holds

$$\int x^n \cdot J_1(x) \cdot K_0(x)\,dx = \frac{1}{2}\,\left[P_n(x)J_0(x) \cdot K_0(x) - Q_n(x)J_0(x) \cdot K_1(x) + R_n(x)J_1(x) \cdot K_0(x) - S_n(x)J_1(x) \cdot K_1(x)\right]\ .$$

$$P_6(x) = -x^6 - 8\,x^4 + 24\,x^2\ , \quad Q_6(x) = 5\,x^5 + 4\,x^3 - 24\,x$$
$$R_6(x) = x^5 + 28\,x^3 - 24\,x\ , \quad S_6(x) = x^6 - 12\,x^4 - 32\,x^2$$

$$P_8(x) = -x^8 - 18\,x^6 + 192\,x^4 + 432\,x^2\ , \quad Q_8(x) = 7\,x^7 + 6\,x^5 - 600\,x^3 - 432\,x$$
$$R_8(x) = x^7 + 102\,x^5 - 168\,x^3 - 432\,x\ , \quad S_8(x) = x^8 - 24\,x^6 - 216\,x^4 + 768\,x^2$$

$$P_{10}(x) = -x^{10} - 32\,x^8 + 720\,x^6 + 6144\,x^4 - 17280\,x^2$$
$$Q_{10}(x) = 9\,x^9 + 8\,x^7 - 3696\,x^5 - 3648\,x^3 + 17280\,x$$
$$R_{10}(x) = x^9 + 248\,x^7 - 624\,x^5 - 20928\,x^3 + 17280\,x$$
$$S_{10}(x) = x^{10} - 40\,x^8 - 768\,x^6 + 8640\,x^4 + 24576\,x^2$$

$$P_{12}(x) = -x^{12} - 50\,x^{10} + 1920\,x^8 + 36000\,x^6 - 368640\,x^4 - 864000\,x^2$$
$$Q_{12}(x) = 11\,x^{11} + 10\,x^9 - 13680\,x^7 - 15840\,x^5 + 1169280\,x^3 + 864000\,x$$
$$R_{12}(x) = x^{11} + 490\,x^9 - 1680\,x^7 - 200160\,x^5 + 305280\,x^3 + 864000\,x$$

$$S_{12}(x) = x^{12} - 60\,x^{10} - 2000\,x^8 + 46080\,x^6 + 432000\,x^4 - 1474560\,x^2$$

$$P_{14}(x) = -x^{14} - 72\,x^{12} + 4200\,x^{10} + 138240\,x^8 - 3024000\,x^6 - 26542080\,x^4 + 72576000\,x^2$$

$$Q_{14}(x) = 13\,x^{13} + 12\,x^{11} - 38280\,x^9 - 48960\,x^7 + 15707520\,x^5 + 16796160\,x^3 - 72576000\,x$$

$$R_{14}(x) = x^{13} + 852\,x^{11} - 3720\,x^9 - 1056960\,x^7 + 2436480\,x^5 + 89372160\,x^3 - 72576000\,x$$

$$S_{14}(x) = x^{14} - 84\,x^{12} - 4320\,x^{10} + 168000\,x^8 + 3317760\,x^6 - 36288000\,x^4 - 106168320\,x^2$$

$$P_{16}(x) =$$
$$= -x^{16} - 98\,x^{14} + 8064\,x^{12} + 411600\,x^{10} - 15482880\,x^8 - 296352000\,x^6 + 2972712960\,x^4 + 7112448000\,x^2$$

$$Q_{16}(x) =$$
$$= 15\,x^{15} + 14\,x^{13} - 89544\,x^{11} - 122640\,x^9 + 111323520\,x^7 + 145877760\,x^5 - 9501649920\,x^3 - 7112448000\,x$$

$$R_{16}(x) =$$
$$= x^{15} + 1358\,x^{13} - 7224\,x^{11} - 3993360\,x^9 + 12539520\,x^7 + 1632234240\,x^5 - 2389201920\,x^3 - 7112448000\,x$$

$$S_{16}(x) =$$
$$= x^{16} - 112\,x^{14} - 8232\,x^{12} + 483840\,x^{10} + 16464000\,x^8 - 371589120\,x^6 - 3556224000\,x^4 + 11890851840\,x^2$$

c) Recurrence Relations:

$$\int x^{2n+2}\,J_0(x)\,I_1(x)\,dx = \frac{x^{2n+1}}{2}\,[x\,J_0(x)\,I_0(x) - J_0(x)\,I_1(x) - (2n+1)\,J_1(x)\,I_0(x) + x\,J_1(x)\,I_1(x)] +$$
$$+ n\int x^{2n}\,J_0(x)\,I_1(x)\,dx + n(2n+1)\int x^{2n}\,J_1(x)\,I_0(x)\,dx$$

$$\int x^{2n+2}\,J_1(x)\,I_0(x)\,dx = \frac{x^{2n+1}}{2}\,[-x\,J_0(x)\,I_0(x) + (2n+1)\,J_0(x)\,I_1(x) + J_1(x)\,I_0(x) + x\,J_1(x)\,I_1(x)] -$$
$$- n(2n+1)\int x^{2n}\,J_0(x)\,I_1(x)\,dx - n\int x^{2n}\,J_1(x)\,I_0(x)\,dx$$

$$\int x^{2n+2}\,J_0(x)\,K_1(x)\,dx = \frac{x^{2n+1}}{2}\,[-x\,J_0(x)\,K_0(x) - J_0(x)\,K_1(x) + (2n+1)\,J_1(x)\,K_0(x) + x\,J_1(x)\,K_1(x)] +$$
$$+ n\int x^{2n}\,J_0(x)\,K_1(x)\,dx - n(2n+1)\int x^{2n}\,J_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+2}\,J_1(x)\,K_0(x)\,dx = \frac{x^{2n+1}}{2}\,[-x\,J_0(x)\,K_0(x) - (2n+1)\,J_0(x)\,K_1(x) + J_1(x)\,K_0(x) - x\,J_1(x)\,K_1(x)] +$$
$$+ n(2n+1)\int x^{2n}\,J_0(x)\,K_1(x)\,dx - n\int x^{2n}\,J_1(x)\,K_0(x)\,dx$$

2.1.9. Integrals of the type $\int x^{2n[+1]} J_\mu(x)\, Y_\nu\, dx$, :

a) $\int x^{2n+1}\, J_0(x)Y_0(x)\, dx$:

$$\int x\, J_0(x)Y_0(x)\, dx = \frac{x^2}{2}\, [J_0(x)Y_0(x) + J_1(x)Y_1(x)]$$

$$\int x^3\, J_0(x)Y_0(x)\, dx =$$
$$= \frac{x^4}{6} J_0(x)Y_0(x) + \frac{x^3}{6}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \left(\frac{x^4}{6} - \frac{x^2}{3}\right)\, J_1(x)Y_1(x)$$

$$\int x^5\, J_0(x)Y_0(x)\, dx = \left(\frac{x^6}{10} + \frac{4}{15}\, x^4\right) J_0(x)Y_0(x) +$$
$$+ \left(\frac{x^5}{5} - \frac{8}{15}\, x^3\right) [J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \left(\frac{x^6}{10} - \frac{8}{15}\, x^4 + \frac{16}{15}\, x^2\right)\, J_1(x)Y_1(x)$$

$$\int x^7\, J_0(x)Y_0(x)\, dx = \left(\frac{x^8}{14} + \frac{18}{35}\, x^6 - \frac{72}{35}\, x^4\right) J_0(x)Y_0(x) +$$
$$+ \left(\frac{3}{14}\, x^7 - \frac{54}{35}\, x^5 + \frac{144}{35}\, x^3\right) [J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$
$$+ \left(\frac{x^8}{14} - \frac{27}{35}\, x^6 + \frac{144}{35}\, x^4 - \frac{288}{35}\, x^2\right)\, J_1(x)Y_1(x)$$

$$\int x^9\, J_0(x)Y_0(x)\, dx = \left(\frac{x^{10}}{18} + \frac{16}{21}\, x^8 - \frac{256}{35}\, x^6 + \frac{1024}{35}\, x^4\right) J_0(x)Y_0(x) +$$
$$+ \left(\frac{2}{9}\, x^9 - \frac{64}{21}\, x^7 + \frac{768}{35}\, x^5 - \frac{2048}{35}\, x^3\right) [J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$
$$+ \left(\frac{x^{10}}{18} - \frac{64}{63}\, x^8 + \frac{384}{35}\, x^6 - \frac{2048}{35}\, x^4 + \frac{4096}{35}\, x^2\right)\, J_1(x)Y_1(x)$$

$$\int x^{11}\, J_0(x)Y_0(x)\, dx = \left(\frac{x^{12}}{22} + \frac{100}{99}\, x^{10} - \frac{4000}{231}\, x^8 + \frac{12800}{77}\, x^6 - \frac{51200}{77}\, x^4\right) J_0(x)Y_0(x) +$$
$$+ \left(\frac{5}{22}\, x^{11} - \frac{500}{99}\, x^9 + \frac{16000}{231}\, x^7 - \frac{38400}{77}\, x^5 + \frac{102400}{77}\, x^3\right) [J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$
$$+ \left(\frac{x^{12}}{22} - \frac{125}{99}\, x^{10} + \frac{16000}{693}\, x^8 - \frac{19200}{77}\, x^6 + \frac{102400}{77}\, x^4 - \frac{204800}{77}\, x^2\right)\, J_1(x)Y_1(x)$$

$$\int x^{13}\, J_0(x)Y_0(x)\, dx =$$
$$= \left(\frac{x^{14}}{26} + \frac{180}{143}\, x^{12} - \frac{4800}{143}\, x^{10} + \frac{576000}{1001}\, x^8 - \frac{5529600}{1001}\, x^6 + \frac{22118400}{1001}\, x^4\right) J_0(x)Y_0(x) +$$
$$+ \left(\frac{3}{13}\, x^{13} - \frac{1080}{143}\, x^{11} + \frac{24000}{143}\, x^9 - \frac{2304000}{1001}\, x^7 + \frac{16588800}{1001}\, x^5 - \frac{44236800}{1001}\, x^3\right) [J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$
$$+ \left(\frac{x^{14}}{26} - \frac{216}{143}\, x^{12} + \frac{6000}{143}\, x^{10} - \frac{768000}{1001}\, x^8 + \frac{8294400}{1001}\, x^6 - \frac{44236800}{1001}\, x^4 + \frac{88473600}{1001}\, x^2\right)\, J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+1}\, J_0(x)Y_0(x)\, dx =$$
$$= \frac{x^{2n}}{4n+2}\, \left\{(2n^2 + x^2)J_0(x)Y_0(x) + nx[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + x^2 J_1(x)Y_1(x)\right\} -$$
$$- \frac{2n^3}{2n+1} \int x^{2n-1}\, J_0(x)Y_0(x)\, dx$$

b) $\int x^{-2n}\, J_0(x)Y_0(x)\, dx$:

$$\int \frac{J_0(x)Y_0(x)}{x^2}\, dx = -\frac{2x^2+1}{x}\, J_0(x)Y_0(x) + J_0(x)Y_1(x) + J_1(x)Y_0(x) - 2x\, J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_0(x)}{x^4}\, dx = \frac{1}{27\,x^3}\,\big\{(16\,x^4+6\,x^2-9)\,J_0(x)Y_0(x) + (-8\,x^3+3\,x)\,[J_0(x)Y_1(x)+J_1(x)Y_0(x)] + $$
$$+(16\,x^4-2\,x^2)\,J_1(x)Y_1(x)\big\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^6}\, dx = \frac{1}{3375\,x^5}\,\big\{(-256\,x^6-96\,x^4+90\,x^2-675)\,J_0(x)Y_0(x) +$$
$$+(128\,x^5-48\,x^3+135\,x)\,[J_0(x)Y_1(x)+J_1(x)Y_0(x)]+$$
$$+(-256\,x^6+32\,x^4-54\,x^2)\,J_1(x)Y_1(x)\big\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^8}\, dx = \frac{1}{385875\,x^7}\,\big\{(2048\,x^8+768\,x^6-720\,x^4+3150\,x^2-55125)\,J_0(x)Y_0(x)+$$
$$+(-1024\,x^7+384\,x^5-1080\,x^3+7875\,x)\,[J_0(x)Y_1(x)+J_1(x)Y_0(x)]+$$
$$+(2048\,x^8-256\,x^6+432\,x^4-2250\,x^2)\,J_1(x)Y_1(x)\big\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^{10}}\, dx = \frac{1}{281302875\,x^9}\cdot$$
$$\cdot\big\{(-65536\,x^{10}-24576\,x^8+23040\,x^6-100800\,x^4+992250\,x^2-31255875)\,J_0(x)Y_0(x)+$$
$$+(32768\,x^9-12288\,x^7+34560\,x^5-252000\,x^3+3472875\,x)\,[J_0(x)Y_1(x)+J_1(x)Y_0(x)]+$$
$$+\,(-65536\,x^{10}+8192\,x^8-13824\,x^6+72000\,x^4-771750\,x^2)\,J_1(x)Y_1(x)\big\}$$

$$\int \frac{J_0(x)Y_0(x)}{x^{12}}\, dx = \frac{1}{74882825325\,x^{11}}\cdot$$
$$\cdot\big\{(524288\,x^{12}+196608\,x^{10}-184320\,x^8+806400\,x^6-7938000\,x^4+137525850\,x^2-6807529575)\,J_0(x)Y_0(x)+$$
$$+(-262144\,x^{11}+98304\,x^9-276480\,x^7+2016000\,x^5-27783000\,x^3+618866325\,x)\,[J_0(x)Y_1(x)+J_1(x)Y_0(x)]+$$
$$+(524288\,x^{12}-65536\,x^{10}+110592\,x^8-576000\,x^6+6174000\,x^4-112521150\,x^2)\,J_1(x)Y_1(x)\big\}$$

Recurrence Formula:

$$\int \frac{J_0(x)Y_0(x)}{x^{2n+2}}\, dx = \frac{1}{(2n+1)^3}\cdot$$
$$\cdot\left\{\frac{-(4n^2+4n+1+2x^2)J_0(x)Y_0(x)+(2n+1)x\,[J_0(x)Y_1(x)+J_1(x)Y_0(x)]-2x^2J_1(x)Y_1(x)}{x^{2n+1}}-\right.$$
$$\left. -\,8n\int \frac{J_0(x)Y_0(x)}{x^{2n}}\, dx\right\}$$

c) $\int x^{2n}\, J_0(x) Y_1(x)\, dx$:

$$\int x^2\, J_0(x)Y_1(x)\, dx = \frac{x^3}{4} J_0(x)Y_1(x) - \frac{x^3}{4} J_1(x)Y_0(x) + \frac{x^2}{2} J_1(x)Y_1(x)$$

$$\int x^4\, J_0(x)Y_1(x)\, dx =$$

$$= -\frac{x^4}{6} J_0(x)Y_0(x) + \frac{3\,x^5 + 8\,x^3}{24} J_0(x)Y_1(x) - \frac{3\,x^5 - 8\,x^3}{24} J_1(x)Y_0(x) + \frac{x^4 - 2\,x^2}{3} J_1(x)Y_1(x)$$

$$\int x^6\, J_0(x)Y_1(x)\, dx = -\frac{x^6 - 4\,x^4}{5} J_0(x)Y_0(x) + \frac{5\,x^7 + 36\,x^5 - 96\,x^3}{60} J_0(x)Y_1(x) -$$

$$-\frac{5\,x^7 - 36\,x^5 + 96\,x^3}{60} J_1(x)Y_0(x) + \frac{3\,x^6 - 16\,x^4 + 32\,x^2}{10} J_1(x)Y_1(x)$$

$$\int x^8\, J_0(x)Y_1(x)\, dx = -\frac{15\,x^8 - 144\,x^6 + 576\,x^4}{70} J_0(x)Y_0(x) +$$

$$+\frac{35\,x^9 + 480\,x^7 - 3456\,x^5 + 9216\,x^3}{560} J_0(x)Y_1(x) -$$

$$-\frac{35\,x^9 - 480\,x^7 + 3456\,x^5 - 9216\,x^3}{560} J_1(x)Y_0(x) +$$

$$+\frac{10\,x^8 - 108\,x^6 + 576\,x^4 - 1152\,x^2}{35} J_1(x)Y_1(x)$$

$$\int x^{10}\, J_0(x)Y_1(x)\, dx = -\frac{14\,x^{10} - 240\,x^8 + 2304\,x^6 - 9216\,x^4}{63} J_0(x)Y_0(x) +$$

$$+\frac{63\,x^{11} + 1400\,x^9 - 19200\,x^7 + 138240\,x^5 - 368640\,x^3}{1260} J_0(x)Y_1(x) -$$

$$-\frac{63\,x^{11} - 1400\,x^9 + 19200\,x^7 - 138240\,x^5 + 368640\,x^3}{1260} J_1(x)Y_0(x) +$$

$$+\frac{35\,x^{10} - 640\,x^8 + 6912\,x^6 - 36864\,x^4 + 73728\,x^2}{126} J_1(x)Y_1(x)$$

$$\int x^{12}\, J_0(x)Y_1(x)\, dx = -\frac{105\,x^{12} - 2800\,x^{10} + 48000\,x^8 - 460800\,x^6 + 1843200\,x^4}{462} J_0(x)Y_0(x) +$$

$$+\frac{77\,x^{13} + 2520\,x^{11} - 56000\,x^9 + 768000\,x^7 - 5529600\,x^5 + 14745600\,x^3}{1848} J_0(x)Y_1(x) -$$

$$-\frac{77\,x^{13} - 2520\,x^{11} + 56000\,x^9 - 768000\,x^7 + 5529600\,x^5 - 14745600\,x^3}{1848} J_1(x)Y_0(x) +$$

$$+\frac{63\,x^{12} - 1750\,x^{10} + 32000\,x^8 - 345600\,x^6 + 1843200\,x^4 - 3686400\,x^2}{231} J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+2}\, J_0(x)Y_1(x)\, dx = x^{2n} \left[-\frac{nx^2}{2(2n+1)} J_0(x)Y_0(x) + \left(\frac{n(n+1)}{2n+1} + \frac{x^2}{4(n+1)} \right) x J_0(x)Y_1(x) + \right.$$

$$\left. -\frac{x^3}{4(n+1)} J_1(x)Y_0(x) + \frac{(n+1)x^2}{2(2n+1)} J_1(x)Y_1(x) \right] - \frac{2n^2(n+1)}{2n+1} \int x^{2n}\, J_0(x)Y_1(x)\, dx$$

d) $\int x^{-2n-1}\, J_0(x)Y_1(x)\,dx$:

$$\int \frac{J_0(x)Y_1(x)\,dx}{x} = xJ_0(x)Y_0(x) - J_0(x)Y_1(x) + xJ_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^3} =$$

$$= -\frac{8\,x^2+3}{9x}J_0(x)Y_0(x) + \frac{4\,x^2-3}{9x^2}J_0(x)Y_1(x) + \frac{4}{9}J_1(x)Y_0(x) - \frac{8\,x^2-1}{9x}J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^5} = \frac{128\,x^4+48\,x^2-45}{675\,x^3}J_0(x)Y_0(x) - \frac{64\,x^4-24\,x^2+135}{675\,x^4}J_0(x)Y_1(x) -$$

$$-\frac{64\,x^2-24}{675\,x^2}J_1(x)Y_0(x) + \frac{128\,x^4-16\,x^2+27}{675\,x^3}J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^7} = -\frac{1024\,x^6+384\,x^4-360\,x^2+1575}{55125\,x^5}J_0(x)Y_0(x) +$$

$$+\frac{512\,x^6-192\,x^4+540\,x^2-7875}{55125\,x^6}J_0(x)Y_1(x) + \frac{512\,x^4-192\,x^2+540}{55125\,x^4}J_1(x)Y_0(x) -$$

$$-\frac{1024\,x^6-128\,x^4+216\,x^2-1125}{55125\,x^5}J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^9} = \frac{32768\,x^8+12288\,x^6-11520\,x^4+50400\,x^2-496125}{31255875\,x^7}J_0(x)Y_0(x) -$$

$$-\frac{16384\,x^8-6144\,x^6+17280\,x^4-126000\,x^2+3472875}{31255875\,x^8}J_0(x)Y_1(x) -$$

$$-\frac{16384\,x^6-6144\,x^4+17280\,x^2-126000}{31255875\,x^6}J_1(x)Y_0(x) +$$

$$+\frac{32768\,x^8-4096\,x^6+6912\,x^4-36000\,x^2+385875}{31255875\,x^7}J_1(x)Y_1(x)$$

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^{11}} =$$

$$-\frac{262144\,x^{10}+98304\,x^8-92160\,x^6+403200\,x^4-3969000\,x^2+68762925}{6807529575\,x^9}J_0(x)Y_0(x) +$$

$$+\frac{131072\,x^{10}-49152\,x^8+138240\,x^6-1008000\,x^4+13891500\,x^2-618866325}{6807529575\,x^{10}}J_0(x)Y_1(x) +$$

$$+\frac{131072\,x^8-49152\,x^6+138240\,x^4-1008000\,x^2+13891500}{6807529575\,x^8}J_1(x)Y_0(x) -$$

$$-\frac{262144\,x^{10}-32768\,x^8+55296\,x^6-288000\,x^4+3087000\,x^2-56260575}{6807529575\,x^9}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int \frac{J_0(x)Y_1(x)\,dx}{x^{2n+1}} = \frac{1}{x^{2n+1}}\left\{-\frac{x^2}{4n^2-1}J_0(x)Y_0(x) - \left[\frac{x}{2n+1} + \frac{4nx^3}{(4n^2-1)^2}\right]J_0(x)Y_1(x) +\right.$$

$$\left. + \frac{4nx^3}{(4n^2-1)^2}J_1(x)Y_0(x) + \frac{x^2}{(2n+1)^2}J_1(x)Y_1(x)\right\} - \frac{8n}{(2n+1)^2(2n-1)}\int \frac{J_0(x)Y_1(x)\,dx}{x^{2n-1}}$$

e) $\int x^{2n}\, J_1(x)Y_0(x)\, dx$:

$$\int x^2\, J_1(x)Y_0(x)\, dx = -\frac{x^3}{4} J_0(x)Y_1(x) + \frac{x^3}{4} J_1(x)Y_0(x) + \frac{x^2}{2} J_1(x)Y_1(x)$$

$$\int x^4\, J_1(x)Y_0(x)\, dx =$$

$$= -\frac{x^4}{6} J_0(x)Y_0(x) - \frac{3\,x^5 - 8\,x^3}{24} J_0(x)Y_1(x) + \frac{3\,x^5 + 8\,x^3}{24} J_1(x)Y_0(x) + \frac{x^4 - 2\,x^2}{3} J_1(x)Y_1(x)$$

$$\int x^6\, J_1(x)Y_0(x)\, dx = -\frac{x^6 - 4\,x^4}{5} J_0(x)Y_0(x) - \frac{5\,x^7 - 36\,x^5 + 96\,x^3}{60} J_0(x)Y_1(x) +$$

$$+\frac{5\,x^7 + 36\,x^5 - 96\,x^3}{60} J_1(x)Y_0(x) + \frac{3\,x^6 - 16\,x^4 + 32\,x^2}{10} J_1(x)Y_1(x)$$

$$\int x^8\, J_1(x)Y_0(x)\, dx = -\frac{15\,x^8 - 144\,x^6 + 576\,x^4}{70} J_0(x)Y_0(x) -$$

$$-\frac{35\,x^9 - 480\,x^7 + 3456\,x^5 - 9216\,x^3}{560} J_0(x)Y_1(x) +$$

$$+\frac{35\,x^9 + 480\,x^7 - 3456\,x^5 + 9216\,x^3}{560} J_1(x)Y_0(x) +$$

$$+\frac{10\,x^8 - 108\,x^6 + 576\,x^4 - 1152\,x^2}{35} J_1(x)Y_1(x)$$

$$\int x^{10}\, J_1(x)Y_0(x)\, dx = -\frac{14\,x^{10} - 240\,x^8 + 2304\,x^6 - 9216\,x^4}{63} J_0(x)Y_0(x) -$$

$$-\frac{63\,x^{11} - 1400\,x^9 + 19200\,x^7 - 138240\,x^5 + 368640\,x^3}{1260} J_0(x)Y_1(x) +$$

$$+\frac{63\,x^{11} + 1400\,x^9 - 19200\,x^7 + 138240\,x^5 - 368640\,x^3}{1260} J_1(x)Y_0(x) +$$

$$+\frac{35\,x^{10} - 640\,x^8 + 6912\,x^6 - 36864\,x^4 + 73728\,x^2}{126} J_1(x)Y_1(x)$$

$$\int x^{12}\, J_1(x)Y_0(x)\, dx = -\frac{105\,x^{12} - 2800\,x^{10} + 48000\,x^8 - 460800\,x^6 + 1843200\,x^4}{462} J_0(x)Y_0(x) -$$

$$-\frac{77\,x^{13} - 2520\,x^{11} + 56000\,x^9 - 768000\,x^7 + 5529600\,x^5 - 14745600\,x^3}{1848} J_0(x)Y_1(x) +$$

$$+\frac{77\,x^{13} + 2520\,x^{11} - 56000\,x^9 + 768000\,x^7 - 5529600\,x^5 + 14745600\,x^3}{1848} J_1(x)Y_0(x) +$$

$$+\frac{63\,x^{12} - 1750\,x^{10} + 32000\,x^8 - 345600\,x^6 + 1843200\,x^4 - 3686400\,x^2}{231} J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+2}\, J_1(x)Y_0(x)\, dx = x^{2n} \left[-\frac{nx^2}{2(2n+1)} J_0(x)Y_0(x) - \frac{x^3}{4(n+1)} J_0(x)Y_1(x) - \right.$$

$$\left. + \left(\frac{n(n+1)}{2n+1} + \frac{x^2}{4(n+1)} \right) x J_1(x)Y_0(x) + \frac{(n+1)x^2}{2(2n+1)} J_1(x)Y_1(x) \right] - \frac{2n^2(n+1)}{2n+1} \int x^{2n}\, J_1(x)Y_0(x)\, dx$$

f) $\int x^{-2n-1}\, J_1(x)Y_0(x)\, dx$:

$$\int \frac{J_1(x)Y_0(x)\,dx}{x} = xJ_0(x)Y_0(x) - J_1(x)Y_0(x) + xJ_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^3} =$$

$$= -\frac{8\,x^2+3}{9x}J_0(x)Y_0(x) + \frac{4}{9}J_0(x)Y_1(x) + \frac{4\,x^2-3}{9x^2}J_1(x)Y_0(x) - \frac{8\,x^2-1}{9x}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^5} = \frac{128\,x^4+48\,x^2-45}{675\,x^3}J_0(x)Y_0(x) - \frac{64\,x^2-24}{675\,x^2}J_0(x)Y_1(x) -$$

$$-\frac{64\,x^4-24\,x^2+135}{675\,x^4}J_1(x)Y_0(x) + \frac{128\,x^4-16\,x^2+27}{675\,x^3}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^7} = -\frac{1024\,x^6+384\,x^4-360\,x^2+1575}{55125\,x^5}J_0(x)Y_0(x) +$$

$$+\frac{512\,x^4-192\,x^2+540}{55125\,x^4}J_0(x)Y_1(x) + \frac{512\,x^6-192\,x^4+540\,x^2-7875}{55125\,x^6}J_1(x)Y_0(x) -$$

$$-\frac{1024\,x^6-128\,x^4+216\,x^2-1125}{55125\,x^5}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^9} = \frac{32768\,x^8+12288\,x^6-11520\,x^4+50400\,x^2-496125}{31255875\,x^7}J_0(x)Y_0(x) -$$

$$-\frac{16384\,x^6-6144\,x^4+17280\,x^2-126000}{31255875\,x^6}J_0(x)Y_1(x) -$$

$$-\frac{16384\,x^8-6144\,x^6+17280\,x^4-126000\,x^2+3472875}{31255875\,x^8}J_1(x)Y_0(x) +$$

$$+\frac{32768\,x^8-4096\,x^6+6912\,x^4-36000\,x^2+385875}{31255875\,x^7}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^{11}} =$$

$$= -\frac{262144\,x^{10}+98304\,x^8-92160\,x^6+403200\,x^4-3969000\,x^2+68762925}{6807529575\,x^9}J_0(x)Y_0(x) +$$

$$+\frac{131072\,x^8-49152\,x^6+138240\,x^4-1008000\,x^2+13891500}{6807529575\,x^8}J_0(x)Y_1(x) +$$

$$+\frac{131072\,x^{10}-49152\,x^8+138240\,x^6-1008000\,x^4+13891500\,x^2-618866325}{6807529575\,x^{10}}J_1(x)Y_0(x) -$$

$$-\frac{262144\,x^{10}-32768\,x^8+55296\,x^6-288000\,x^4+3087000\,x^2-56260575}{6807529575\,x^9}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int \frac{J_1(x)Y_0(x)\,dx}{x^{2n+1}} = \frac{1}{x^{2n+1}}\left\{-\frac{x^2}{4n^2-1}J_0(x)Y_0(x) + \frac{4nx^3}{(4n^2-1)^2}J_0(x)Y_1(x) -\right.$$

$$\left. -\left[\frac{x}{2n+1} + \frac{4nx^3}{(4n^2-1)^2}\right]J_1(x)Y_0(x) + \frac{x^2}{(2n+1)^2}J_1(x)Y_1(x)\right\} - \frac{8n}{(2n-1)(2n+1)^2}\int \frac{J_1(x)Y_0(x)\,dx}{x^{2n-1}}$$

g) $\int x^{2n+1}\, J_1(x)Y_1(x)\, dx$:

$$\int x\, J_1(x)Y_1(x)\, dx = \frac{x^2}{2} J_0(x)Y_0(x) - xJ_0(x)Y_1(x) + \frac{x^2}{2} J_1(x)Y_1(x)$$

$$\int x^3\, J_1(x)Y_1(x)\, dx =$$

$$= \frac{x^4}{6} J_0(x)Y_0(x) - \frac{x^3}{3}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{x^4 + 4\,x^2}{6} J_1(x)Y_1(x)$$

$$\int x^5\, J_1(x)Y_1(x)\, dx = \frac{x^6 - 4\,x^4}{10} J_0(x)Y_0(x) -$$

$$- \frac{3\,x^5 - 8\,x^3}{10}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{x^6 + 8\,x^4 - 16\,x^2}{10} J_1(x)Y_1(x)$$

$$\int x^7\, J_1(x)Y_1(x)\, dx = \frac{5\,x^8 - 48\,x^6 + 192\,x^4}{70} J_0(x)Y_0(x) -$$

$$- \frac{10\,x^7 - 72\,x^5 + 192\,x^3}{35}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + \frac{5\,x^8 + 72\,x^6 - 384\,x^4 + 768\,x^2}{70} J_1(x)Y_1(x)$$

$$\int x^9\, J_1(x)Y_1(x)\, dx = \frac{7\,x^{10} - 120\,x^8 + 1152\,x^6 - 4608\,x^4}{126} J_0(x)Y_0(x) -$$

$$- \frac{35\,x^9 - 480\,x^7 + 3456\,x^5 - 9216\,x^3}{126}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$+ \frac{7\,x^{10} + 160\,x^8 - 1728\,x^6 + 9216\,x^4 - 18432\,x^2}{126} J_1(x)Y_1(x)$$

$$\int x^{11}\, J_1(x)Y_1(x)\, dx = \frac{21\,x^{12} - 560\,x^{10} + 9600\,x^8 - 92160\,x^6 + 368640\,x^4}{462} J_0(x)Y_0(x) -$$

$$- \frac{63\,x^{11} - 1400\,x^9 + 19200\,x^7 - 138240\,x^5 + 368640\,x^3}{231}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$+ \frac{21\,x^{12} + 700\,x^{10} - 12800\,x^8 + 138240\,x^6 - 737280\,x^4 + 1474560\,x^2}{462} J_1(x)Y_1(x)$$

$$\int x^{13}\, J_1(x)Y_1(x)\, dx =$$

$$= \frac{11\,x^{14} - 420\,x^{12} + 11200\,x^{10} - 192000\,x^8 + 1843200\,x^6 - 7372800\,x^4}{286} J_0(x)Y_0(x) -$$

$$- \frac{77\,x^{13} - 2520\,x^{11} + 56000\,x^9 - 768000\,x^7 + 5529600\,x^5 - 14745600\,x^3}{286}[J_0(x)Y_1(x) + J_1(x)Y_0(x)] +$$

$$+ \frac{11\,x^{14} + 504\,x^{12} - 14000\,x^{10} + 256000\,x^8 - 2764800\,x^6 + 14745600\,x^4 - 29491200\,x^2}{286} J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int x^{2n+1}\, J_1(x)Y_1(x)\, dx =$$

$$= \frac{x^{2n}}{4n+2} \left\{ x^2 J_0(x)Y_0(x) - (n+1)x\,[J_0(x)Y_1(x) + J_1(x)Y_0(x)] + [2n(n+1) + x^2]J_1(x)Y_1(x) \right\} -$$

$$- \frac{2n(n^2-1)}{2n+1} \int x^{2n-1}\, J_1(x)Y_1(x)\, dx$$

h) $\int x^{-2n}\, J_1(x)Y_1(x)\, dx$:

Exception:

$$\int \frac{J_1(x)Y_1(x)\,dx}{x} = -\frac{J_0(x)Y_0(x)+J_1(x)Y_1(x)}{2}$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^2} = \frac{2x}{3}J_0(x)Y_0(x) - \frac{1}{3}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] + \frac{2x^2-1}{3x}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^4} =$$

$$= -\frac{16\,x^2+6}{45x}J_0(x)Y_0(x) + \frac{8\,x^2-3}{45x^2}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] - \frac{16\,x^4-2\,x^2+9}{45x^3}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^6} = \frac{256\,x^4+96\,x^2-90}{4725\,x^3}J_0(x)Y_0(x) -$$

$$-\frac{128\,x^4-48\,x^2+135}{4725\,x^4}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] + \frac{256\,x^6-32\,x^4+54\,x^2-675}{4725\,x^5}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^8} = -\frac{2048\,x^6+768\,x^4-720\,x^2+3150}{496125\,x^5}J_0(x)Y_0(x) +$$

$$+\frac{1024\,x^6-384\,x^4+1080\,x^2-7875}{496125\,x^6}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] -$$

$$-\frac{2048\,x^8-256\,x^6+432\,x^4-2250\,x^2+55125}{496125\,x^7}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^{10}} = \frac{65536\,x^8+24576\,x^6-23040\,x^4+100800\,x^2-992250}{343814625\,x^7}J_0(x)Y_0(x) -$$

$$-\frac{32768\,x^8-12288\,x^6+34560\,x^4-252000\,x^2+3472875}{343814625\,x^8}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] +$$

$$+\frac{65536\,x^{10}-8192\,x^8+13824\,x^6-72000\,x^4+771750\,x^2-31255875}{343814625\,x^9}J_1(x)Y_1(x)$$

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^{12}} =$$

$$= -\frac{524288\,x^{10}+196608\,x^8-184320\,x^6+806400\,x^4-7938000\,x^2+137525850}{88497884475\,x^9}J_0(x)Y_0(x) +$$

$$+\frac{262144\,x^{10}-98304\,x^8+276480\,x^6-2016000\,x^4+27783000\,x^2-618866325}{88497884475\,x^{10}}[J_0(x)Y_1(x)+J_1(x)Y_0(x)] -$$

$$-\frac{524288\,x^{12}-65536\,x^{10}+110592\,x^8-576000\,x^6+6174000\,x^4-112521150\,x^2+6807529575}{88497884475\,x^{11}}J_1(x)Y_1(x)$$

Recurrence Formula:

$$\int \frac{J_1(x)Y_1(x)\,dx}{x^{2n+2}} =$$

$$= -\frac{2x^2J_0(x)Y_0(x)+(2n-1)x[J_0(x)Y_1(x)+J_1(x)Y_0(x)]+(4n^2-1+2x^2)J_1(x)Y_1(x)}{(2n+3)(4n^2-1)\,x^{2n+1}} -$$

$$-\frac{8n}{(2n+3)(4n^2-1)}\int \frac{J_1(x)Y_1(x)\,dx}{x^{2n}}$$

2.2. Bessel Functions with different Arguments αx and βx :

See also [10], 4. - 6. .

2.2.1. One-step recurrence formulas

Let

$$\alpha^2+\beta^2=\sigma \quad \text{and} \quad \alpha^2-\beta^2=\Delta$$

and

$$Z_{\mu\nu,UW}^{(m)}=\int x^m U_\mu(\alpha x) W_\nu(\beta x)\,dx\ ,$$

then the following systems hold:

$$Z_{00,JJ}^{(2n+3)}=x^{2n+2}\cdot Z_{00,JJ}^{(1)}-\frac{2(n+1)}{\Delta}\left[\alpha Z_{10,JJ}^{(2n+2)}-\beta Z_{01,JJ}^{(2n+2)}\right]$$

$$Z_{11,JJ}^{(2n+3)}=x^{2n+2}\cdot Z_{11,JJ}^{(1)}-\frac{2(n+1)}{\Delta}\left[\beta Z_{10,JJ}^{(2n+2)}-\alpha Z_{01,JJ}^{(2n+2)}\right]$$

$$Z_{01,JJ}^{(2n+2)}=x^{2n}\cdot Z_{01,JJ}^{(2)}-\frac{4n\beta}{\Delta^2}\left[\beta Z_{01,JJ}^{(2n)}-\alpha Z_{10,JJ}^{(2n)}\right]-\frac{2n}{\Delta}\left[\beta Z_{00,JJ}^{(2n+1)}+\alpha Z_{11,JJ}^{(2n+1)}\right]$$

$$Z_{10,JJ}^{(2n+2)}=x^{2n}\cdot Z_{10,JJ}^{(2)}+\frac{4n\alpha}{\Delta^2}\left[\beta Z_{01,JJ}^{(2n)}-\alpha Z_{10,JJ}^{(2n)}\right]+\frac{2n}{\Delta}\left[\alpha Z_{00,JJ}^{(2n+1)}+\beta Z_{11,JJ}^{(2n+1)}\right]$$

$$Z_{00,II}^{(2n+3)}=x^{2n+2}\cdot Z_{00,II}^{(1)}-\frac{2(n+1)}{\Delta}\left[\alpha Z_{10,II}^{(2n+2)}-\beta Z_{01,II}^{(2n+2)}\right]$$

$$Z_{11,II}^{(2n+3)}=x^{2n+2}\cdot Z_{11,II}^{(1)}+\frac{2(n+1)}{\Delta}\left[\beta Z_{10,II}^{(2n+2)}-\alpha Z_{01,II}^{(2n+2)}\right]$$

$$Z_{01,II}^{(2n+2)}=x^{2n}\cdot Z_{01,II}^{(2)}+\frac{4n\beta}{\Delta^2}\left[\beta Z_{01,II}^{(2n)}-\alpha Z_{10,II}^{(2n)}\right]+\frac{2n}{\Delta}\left[\beta Z_{00,II}^{(2n+1)}-\alpha Z_{11,II}^{(2n+1)}\right]$$

$$Z_{10,II}^{(2n+2)}=x^{2n}\cdot Z_{10,II}^{(2)}-\frac{4n\alpha}{\Delta^2}\left[\beta Z_{01,II}^{(2n)}-\alpha Z_{10,II}^{(2n)}\right]-\frac{2n}{\Delta}\left[\alpha Z_{00,II}^{(2n+1)}-\beta Z_{11,II}^{(2n+1)}\right]$$

$$Z_{00,KK}^{(2n+3)}=x^{2n+2}\cdot Z_{00,KK}^{(1)}+\frac{2(n+1)}{\Delta}\left[\alpha Z_{10,KK}^{(2n+2)}-\beta Z_{01,KK}^{(2n+2)}\right]$$

$$Z_{11,KK}^{(2n+3)}=x^{2n+2}\cdot Z_{11,KK}^{(1)}-\frac{2(n+1)}{\Delta}\left[\beta Z_{10,KK}^{(2n+2)}-\alpha Z_{01,KK}^{(2n+2)}\right]$$

$$Z_{01,KK}^{(2n+2)}=x^{2n}\cdot Z_{01,KK}^{(2)}+\frac{4n\beta}{\Delta^2}\left[\beta Z_{01,KK}^{(2n)}-\alpha Z_{10,KK}^{(2n)}\right]-\frac{2n}{\Delta}\left[\beta Z_{00,KK}^{(2n+1)}-\alpha Z_{11,KK}^{(2n+1)}\right]$$

$$Z_{10,KK}^{(2n+2)}=x^{2n}\cdot Z_{10,KK}^{(2)}-\frac{4n\alpha}{\Delta^2}\left[\beta Z_{01,KK}^{(2n)}-\alpha Z_{10,KK}^{(2n)}\right]+\frac{2n}{\Delta}\left[\alpha Z_{00,KK}^{(2n+1)}-\beta Z_{11,KK}^{(2n+1)}\right]$$

$$Z_{00,JI}^{(2n+3)}=x^{2n+2}\cdot Z_{00,JI}^{(1)}-\frac{2(n+1)}{\sigma}\left[\alpha Z_{10,JI}^{(2n+2)}+\beta Z_{01,JI}^{(2n+2)}\right]$$

$$Z_{11,JI}^{(2n+3)}=x^{2n+2}\cdot Z_{11,JI}^{(1)}-\frac{2(n+1)}{\sigma}\left[\beta Z_{10,JI}^{(2n+2)}-\alpha Z_{01,JI}^{(2n+2)}\right]$$

$$Z_{01,JI}^{(2n+2)}=x^{2n}\cdot Z_{01,JI}^{(2)}+\frac{4n\beta}{\sigma^2}\left[\beta Z_{01,JI}^{(2n)}+\alpha Z_{10,JI}^{(2n)}\right]-\frac{2n}{\sigma}\left[\beta Z_{00,JI}^{(2n+1)}+\alpha Z_{11,JI}^{(2n+1)}\right]$$

$$Z_{10,JI}^{(2n+2)}=x^{2n}\cdot Z_{10,JI}^{(2)}-\frac{4n\alpha}{\sigma^2}\left[\beta Z_{01,JI}^{(2n)}+\alpha Z_{10,JI}^{(2n)}\right]+\frac{2n}{\sigma}\left[\alpha Z_{00,JI}^{(2n+1)}-\beta Z_{11,JI}^{(2n+1)}\right]$$

$$Z_{00,JK}^{(2n+3)}=x^{2n+2}\cdot Z_{00,JK}^{(1)}-\frac{2(n+1)}{\sigma}\left[\alpha Z_{10,JK}^{(2n+2)}-\beta Z_{01,JK}^{(2n+2)}\right]$$

$$Z_{11,JK}^{(2n+3)} = x^{2n+2} \cdot Z_{11,JK}^{(1)} + \frac{2(n+1)}{\sigma} \left[\beta Z_{10,JK}^{(2n+2)} + \alpha Z_{01,JK}^{(2n+2)}\right]$$

$$Z_{01,JK}^{(2n+2)} = x^{2n} \cdot Z_{01,JK}^{(2)} + \frac{4n\beta}{\sigma^2} \left[\beta Z_{01,JK}^{(2n)} - \alpha Z_{10,JK}^{(2n)}\right] + \frac{2n}{\sigma} \left[\beta Z_{00,JK}^{(2n+1)} - \alpha Z_{11,JK}^{(2n+1)}\right]$$

$$Z_{10,JK}^{(2n+2)} = x^{2n} \cdot Z_{10,JK}^{(2)} + \frac{4n\alpha}{\sigma^2} \left[\beta Z_{01,JK}^{(2n)} - \alpha Z_{10,JK}^{(2n)}\right] + \frac{2n}{\sigma} \left[\alpha Z_{00,JK}^{(2n+1)} + \beta Z_{11,JK}^{(2n+1)}\right]$$

$$Z_{00,IK}^{(2n+3)} = x^{2n+2} \cdot Z_{00,IK}^{(1)} - \frac{2(n+1)}{\Delta} \left[\alpha Z_{10,IK}^{(2n+2)} + \beta Z_{01,IK}^{(2n+2)}\right]$$

$$Z_{11,IK}^{(2n+3)} = x^{2n+2} \cdot Z_{11,IK}^{(1)} - \frac{2(n+1)}{\Delta} \left[\beta Z_{10,IK}^{(2n+2)} + \alpha Z_{01,IK}^{(2n+2)}\right]$$

$$Z_{01,IK}^{(2n+2)} = x^{2n} \cdot Z_{01,IK}^{(2)} + \frac{4n\beta}{\Delta^2} \left[\beta Z_{01,IK}^{(2n)} + \alpha Z_{10,IK}^{(2n)}\right] - \frac{2n}{\Delta} \left[\beta Z_{00,IK}^{(2n+1)} + \alpha Z_{11,IK}^{(2n+1)}\right]$$

$$Z_{10,IK}^{(2n+2)} = x^{2n} \cdot Z_{10,IK}^{(2)} + \frac{4n\alpha}{\Delta^2} \left[\beta Z_{01,IK}^{(2n)} + \alpha Z_{10,IK}^{(2n)}\right] - \frac{2n}{\Delta} \left[\alpha Z_{00,IK}^{(2n+1)} + \beta Z_{11,IK}^{(2n+1)}\right]$$

2.2.2. Integrals of the type $\int x^{2n} \cdot J_\nu(\alpha x) \cdot J_\nu(\beta x)\, dx$ and $\int x^{2n} \cdot I_\nu(\alpha x) \cdot I_\nu(\beta x)\, dx$

a) Basic Integrals:

In the case $\alpha = \beta$ it was necessary to define the new functions $\Theta(x)$ and $\Omega(x)$ (see page 271). The more there is no solution with already defined functions expected in the described class of functions if $\alpha \neq \beta$. Let $0 < \beta < \alpha$ and $\gamma = \beta/\alpha < 1$. The integrals may be reduced to the single parameter γ by

$$\int x^{2n} \cdot Z_\nu(\alpha x) \cdot Z_\nu(\beta x)\, dx = \alpha^{-2n-1} \int t^{2n} \cdot Z_\nu(t) \cdot Z_\nu(\gamma t)\, dt\,, \quad t = \alpha x\,.$$

The functions $\Theta(x)$ and $\Omega(x)$ from page 271 are generalized to

$$\Theta_0(x;\gamma) = \int_0^x J_0(s) \cdot J_0(\gamma s)\, ds \qquad \text{and} \qquad \Omega_0(x;\gamma) = \int_0^x I_0(s) \cdot I_0(\gamma s)\, ds\,.$$

From (for instance)

$$I_0(s) \cdot I_0(\gamma s) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\sum_{l=0}^{k} \binom{k}{l}^2 \gamma^2 \right) \left(\frac{s}{2}\right)^{2k}$$

one gets with [1], 22.3.1 the power series

$$\Theta_0(x;\gamma) = \sum_{k=0}^{\infty} \frac{(\gamma^2-1)^k}{(k!)^2 \cdot 4^k \cdot (2k+1)} \cdot P_k\left(\frac{1+\gamma^2}{1-\gamma^2}\right) x^{2k+1}$$

and

$$\Omega_0(x;\gamma) = \sum_{k=0}^{\infty} \frac{(1-\gamma^2)^k}{(k!)^2 \cdot 4^k \cdot (2k+1)} \cdot P_k\left(\frac{1+\gamma^2}{1-\gamma^2}\right) x^{2k+1}\,,$$

where

$$P_n(x) = \frac{(2n)!}{2^n \cdot (n!)^2}\, x^n + \ldots$$

denotes the Legendre polynomials. Their values may be found by the recurrence relation

$$P_{n+1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{2n+1}{n+1} \cdot \frac{1+\gamma^2}{1-\gamma^2} \cdot P_n\left(\frac{1+\gamma^2}{1-\gamma^2}\right) - \frac{n}{n+1} P_{n-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right)$$

with

$$P_0\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = 1 \qquad \text{and} \qquad P_1\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{1+\gamma^2}{1-\gamma^2}\,.$$

Some first terms of the power series:

$$\Theta_0(x;\gamma) = x - \frac{\gamma^2+1}{12}\, x^3 + \frac{\gamma^4+4\,\gamma^2+1}{320}\, x^5 - \frac{\gamma^6+9\,\gamma^4+9\,\gamma^2+1}{16128}\, x^7 + \frac{\gamma^8+16\,\gamma^6+36\,\gamma^4+16\,\gamma^2+1}{1327104}\, x^9 -$$

$$-\frac{\gamma^{10}+25\,\gamma^8+100\,\gamma^6+100\,\gamma^4+25\,\gamma^2+1}{162201600}\,x^{11}+\frac{\gamma^{12}+36\,\gamma^{10}+225\,\gamma^8+400\,\gamma^6+225\,\gamma^4+36\,\gamma^2+1}{27603763200}\,x^{13}-$$

$$-\frac{\gamma^{14}+49\,\gamma^{12}+441\,\gamma^{10}+1225\,\gamma^8+1225\,\gamma^6+441\,\gamma^4+49\,\gamma^2+1}{6242697216000}\,x^{15}+$$

$$+\frac{\gamma^{16}+64\,\gamma^{14}+784\,\gamma^{12}+3136\,\gamma^{10}+4900\,\gamma^8+3136\,\gamma^6+784\,\gamma^4+64\,\gamma^2+1}{1811214552268800}\,x^{17}-\ldots$$

If $x > \gamma x >> 1$ one has

$$\Omega_0(x;\gamma) \quad \approx \quad \frac{e^{(1+\gamma)x}}{2\pi\sqrt{\gamma}(1+\gamma)x}\ .$$

Let $\Theta_0(x;\gamma)$ be computed with n decimal signs, then in the case $x > \gamma x >> 1$ the loss of significant digits can be expected. Only about

$$n - \lg\,\frac{e^{(1+\gamma)x}}{2\pi\sqrt{\gamma}(1+\gamma)x}$$

significant digits are left.
The upper integral with primary parameters:

$$\int_0^x J_0(\alpha s)\cdot J_0(\beta s)\,ds = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2\cdot 4^k\cdot(2k+1)}\cdot(\alpha^2-\beta^2)^k\cdot P_k\left(\frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}\right)\,x^{2k+1}$$

Asymptotic series for $x > \gamma x >> 1$:

$$\Theta_0(x;\gamma) \sim \frac{2}{\pi}\,\mathbf{K}(\gamma)+\frac{1}{\pi\sqrt{\gamma}\,x}\left[\frac{\sin(1-\gamma)x}{1-\gamma}-\frac{\cos(\gamma+1)x}{\gamma+1}\right]+$$

$$+\frac{1}{8\pi\gamma^{3/2}\,x^2}\left[\frac{\gamma^2-10\,\gamma+1}{(1-\gamma)^2}\,\cos(1-\gamma)x-\frac{\gamma^2+10\,\gamma+1}{(\gamma+1)^2}\,\sin(\gamma+1)x\right]+$$

$$+\frac{1}{128\pi\gamma^{5/2}\,x^3}\left[\frac{9\,\gamma^4+52\,\gamma^3+342\,\gamma^2+52\,\gamma+9}{(\gamma+1)^3}\,\cos(\gamma+1)x-\frac{9\,\gamma^4-52\,\gamma^3+342\,\gamma^2-52\,\gamma+9}{(1-\gamma)^3}\,\sin(1-\gamma)x\right]+$$

$$+\frac{3}{1024\pi\gamma^{7/2}\,x^4}\left[\frac{25\,\gamma^6+150\,\gamma^5+503\,\gamma^4+2804\,\gamma^3+503\,\gamma^2+150\,\gamma+25}{(\gamma+1)^4}\,\sin(\gamma+1)x-\right.$$

$$\left.-\frac{25\,\gamma^6-150\,\gamma^5+503\,\gamma^4-2804\,\gamma^3+503\,\gamma^2-150\,\gamma+25}{(1-\gamma)^4}\,\cos(1-\gamma)x\right]+\ldots\ ,$$

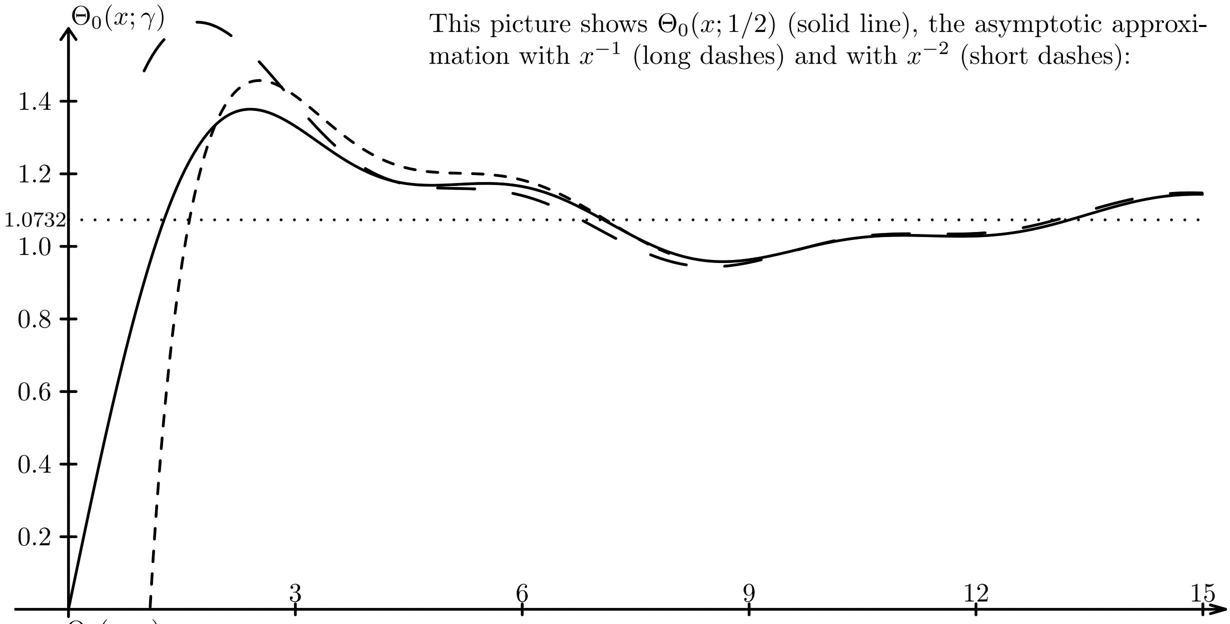
where $\mathbf{K}$ denotes the complete elliptic integral of the first kind, see [1] or [5]. Particulary follows

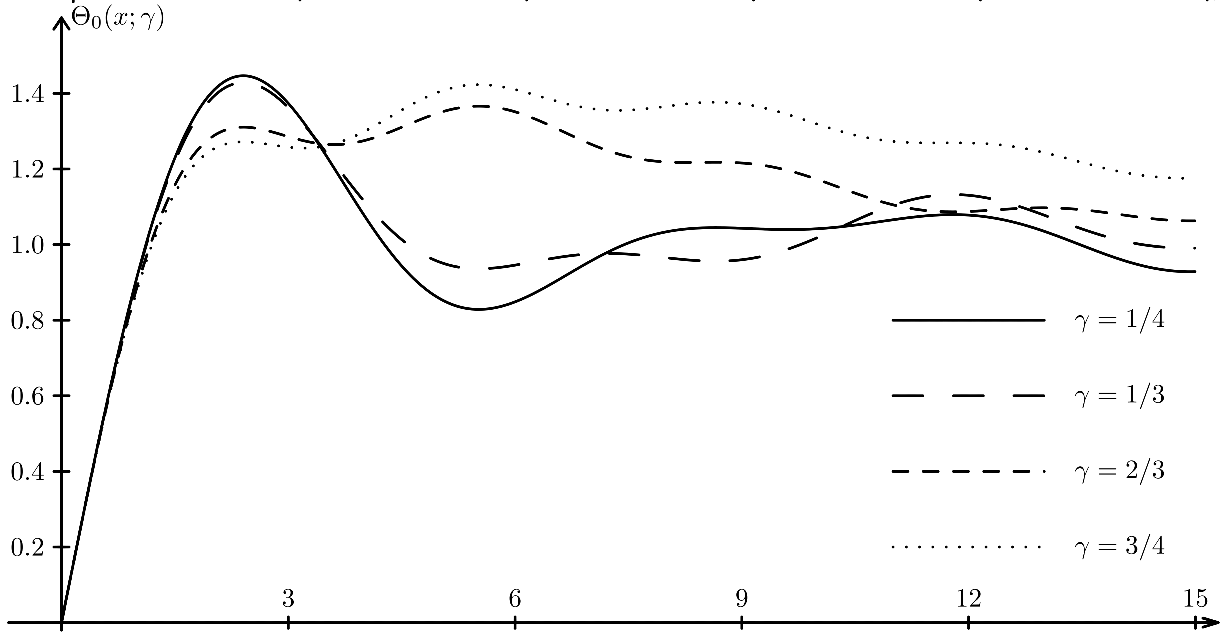
$$\lim_{x\to\infty}\Theta_0(x;\gamma)=\frac{2}{\pi}\,\mathbf{K}(\gamma) \qquad ([4],\ 2.12.31.1.)\ .$$

Some values of this limit:

γ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	1.0000	1.0000	1.0001	1.0002	1.0004	1.0006	1.0009	1.0012	1.0016	1.0020
0.1	1.0025	1.0030	1.0036	1.0043	1.0050	1.0057	1.0065	1.0073	1.0083	1.0092
0.2	1.0102	1.0113	1.0124	1.0136	1.0149	1.0162	1.0176	1.0190	1.0205	1.0221
0.3	1.0237	1.0254	1.0272	1.0290	1.0309	1.0329	1.0350	1.0371	1.0394	1.0417
0.4	1.0441	1.0465	1.0491	1.0518	1.0545	1.0574	1.0603	1.0634	1.0665	1.0698
0.5	1.0732	1.0767	1.0803	1.0841	1.0880	1.0920	1.0962	1.1006	1.1051	1.1097
0.6	1.1146	1.1196	1.1248	1.1302	1.1359	1.1417	1.1479	1.1542	1.1609	1.1678
0.7	1.1750	1.1826	1.1905	1.1988	1.2074	1.2166	1.2262	1.2363	1.2470	1.2583
0.8	1.2702	1.2830	1.2965	1.3110	1.3265	1.3432	1.3613	1.3809	1.4023	1.4258
0.9	1.4518	1.4810	1.5139	1.5517	1.5959	1.6489	1.7145	1.8004	1.9232	2.1369

This picture shows $\Theta_0(x; 1/2)$ (solid line), the asymptotic approximation with x^{-1} (long dashes) and with x^{-2} (short dashes):





Let

$$\Theta_1(x;\gamma) = \int_0^x J_1(s)\cdot J_1(\gamma s)\,ds \qquad \text{and} \qquad \Omega_1(x;\gamma) = \int_0^x I_1(s)\cdot I_1(\gamma s)\,ds\ .$$

Power series: From (for instance)

$$I_1(x)\,I_1(\gamma x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\sum_{l=1}^{k} \binom{k}{l}\binom{k}{l-1} \gamma^{2l-1} \right) \left(\frac{x}{2}\right)^{2k}$$

and

$$\sum_{l=1}^{k} \binom{k}{l}\binom{k}{l-1} \gamma^{2l-1} = k\,(1-\gamma^2)^k\, L_k^{-1}\left(\frac{1+\gamma^2}{1+\gamma^2}\right)$$

follows

$$\Theta_1(x;\gamma) = -\sum_{k=1}^{\infty} \frac{2k\,(\gamma^2-1)^k}{(k!)^2\cdot(2k+1)}\, P_{k+1}^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) \left(\frac{x}{2}\right)^{2k+1}$$

and

$$\Omega_1(x;\gamma) = \sum_{k=1}^{\infty} \frac{2k\,(1-\gamma^2)^k}{(k!)^2\cdot(2k+1)}\, P_{k+1}^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) \left(\frac{x}{2}\right)^{2k+1}\ ,$$

where $P_n^{-1}(x)$ denotes the associated Legendre functions of the first kind. Their values may be found by the recurrence relation, starting with $n = 2$:

$$P_{n+1}^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{2n+1}{n+2}\cdot\frac{1+\gamma^2}{1-\gamma^2}\cdot P_n^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) - \frac{n-1}{n+2}\,P_{n-1}^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right)$$

from

$$P_1^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{\gamma}{1-\gamma^2} \qquad \text{and} \qquad P_2^{-1}\left(\frac{1+\gamma^2}{1-\gamma^2}\right) = \frac{\gamma^3+\gamma}{(1-\gamma^2)^2}\,.$$

Some first terms of the power series:

$$\Theta_1(x;\gamma) = \frac{\gamma}{12}\,x^3 - \frac{\gamma^3+\gamma}{160}\,x^5 + \frac{\gamma^5+3\,\gamma^3+\gamma}{5376}\,x^7 - \frac{\gamma^7+6\,\gamma^5+6\,\gamma^3+\gamma}{331776}\,x^9 + \frac{\gamma^9+10\,\gamma^7+20\,\gamma^5+10\,\gamma^3+\gamma}{32440320}\,x^{11} -$$
$$-\frac{\gamma^{11}+15\,\gamma^9+50\,\gamma^7+50\,\gamma^5+15\,\gamma^3+\gamma}{4600627200}\,x^{13} + \frac{\gamma^{13}+21\,\gamma^{11}+105\,\gamma^9+175\,\gamma^7+105\,\gamma^5+21\,\gamma^3+\gamma}{891813888000}\,x^{15} -$$
$$-\frac{\gamma^{15}+28\,\gamma^{13}+196\,\gamma^{11}+490\,\gamma^9+490\,\gamma^7+196\,\gamma^5+28\,\gamma^3+\gamma}{226401819033600}\,x^{17} + \dots$$

In the case $x > \gamma x >> 1$ one has once again $\Omega_1(x;\gamma) \approx e^{(1+\gamma)x}/[2\pi\sqrt{\gamma}(1+\gamma)x]$.
Asymptotic series for $x > \gamma x >> 1$ ([4], 2.12.31.1. and [5], IX, Definitions):

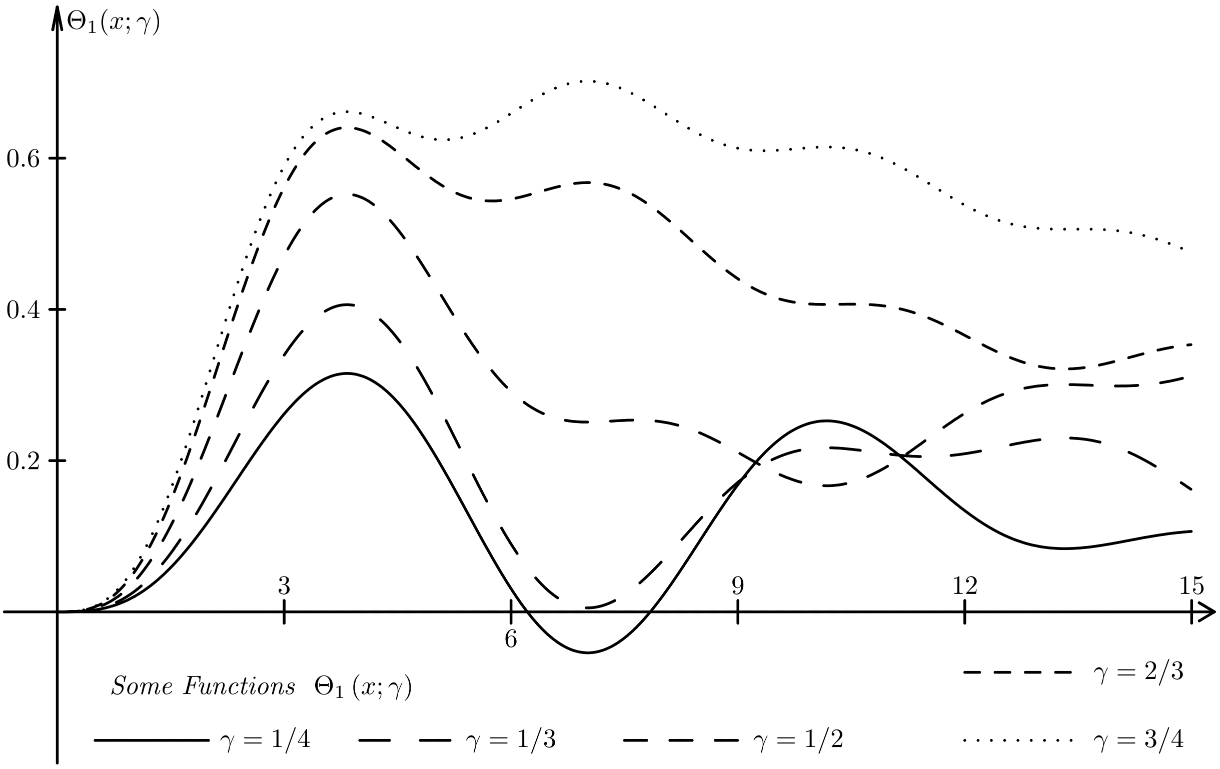
$$\Theta_1(x;\gamma) \sim \frac{2}{\pi\gamma}\,[\,\mathbf{K}(\gamma) - \mathbf{E}(\gamma)\,] + \frac{1}{\pi\sqrt{\gamma}}\left[\frac{1}{x}\left(\frac{\cos(1+\gamma)x}{1+\gamma} - \frac{\sin(1-\gamma)x}{1-\gamma}\right) -\right.$$
$$-\frac{1}{8g\,x^2}\left(\frac{3\gamma^2-2\gamma+3}{(1+\gamma)^2}\sin(1+\gamma)x + \frac{3\gamma^2+2\gamma+3}{(1-\gamma)^2}\cos(1-\gamma)x\right) +$$
$$+\frac{1}{128g^3\,x^3}\left(\frac{15\gamma^4+108\gamma^3-70\gamma^2+108\gamma+15}{(1+\gamma)^3}\cos(1+\gamma)x - \frac{15\gamma^4-108\gamma^3-70\gamma^2-108\gamma+15}{(1-\gamma)^3}\sin(1-\gamma)x\right) -$$
$$-\frac{3}{1024\gamma^3\,x^4}\left(\frac{35\gamma^6+210\gamma^5+909\gamma^4-580\gamma^3+909\gamma^2+210\gamma+35}{(1+\gamma)^4}\sin(1+\gamma)x +\right.$$
$$\left.\left. + \frac{35\gamma^6-210\gamma^5+909\gamma^4+580\gamma^3+909\gamma^2-210\gamma+35}{(1-\gamma)^4}\cos(1-\gamma)x\right) + \dots\right]$$

with the complete elliptic integrals of the first and second kind.
Particulary follows

$$\lim_{x\to\infty}\Theta_1(x;\gamma) = \frac{2}{\pi\gamma}\,[\mathbf{K}(\gamma) - \mathbf{E}(\gamma)] \qquad \text{([4], 2.12.31.1.)}\,.$$

Some values of this limit:

γ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0050	0.0100	0.0150	0.0200	0.0250	0.0300	0.0351	0.0401	0.0451
0.1	0.0502	0.0553	0.0603	0.0654	0.0705	0.0756	0.0808	0.0859	0.0911	0.0963
0.2	0.1015	0.1068	0.1121	0.1174	0.1227	0.1280	0.1334	0.1389	0.1443	0.1498
0.3	0.1554	0.1609	0.1666	0.1722	0.1780	0.1837	0.1895	0.1954	0.2013	0.2073
0.4	0.2134	0.2195	0.2257	0.2319	0.2382	0.2446	0.2511	0.2577	0.2643	0.2711
0.5	0.2779	0.2849	0.2919	0.2991	0.3064	0.3138	0.3214	0.3290	0.3369	0.3448
0.6	0.3530	0.3613	0.3698	0.3785	0.3873	0.3964	0.4058	0.4153	0.4252	0.4353
0.7	0.4457	0.4564	0.4674	0.4789	0.4907	0.5029	0.5157	0.5289	0.5427	0.5571
0.8	0.5721	0.5879	0.6046	0.6221	0.6407	0.6605	0.6816	0.7042	0.7287	0.7552
0.9	0.7844	0.8165	0.8525	0.8934	0.9407	0.9967	1.0654	1.1543	1.2803	1.4971



The value of x may be too large to use the power series for $\Theta_0(x;\gamma)$ and γx may be too small to apply the asymptotic formula. In this case

$$\Theta_0(x;\gamma) \sim \frac{2}{\pi}\,\mathbf{K}(\gamma) +$$

$$+\frac{A_0(x;\gamma)\ \cos x\, J_0(\gamma x) + A_1(x;\gamma)\ \cos x\, J_1(\gamma x) + B_0(x;\gamma)\ \sin x\, J_0(\gamma x) + B_1(x;\gamma)\ \sin x\, J_1(\gamma x)}{\sqrt{\pi x}}$$

is applicable. Let

$$A_\mu(x;\gamma) = \sum_{k=0}^{\infty} \frac{a_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k} \qquad \text{and} \qquad B_\mu(x;\gamma) = \sum_{k=0}^{\infty} \frac{b_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k}\ ,$$

then holds

$$a_0^{(0)}(x;\gamma) = -1\ , \quad a_1^{(0)}(x;\gamma) = -\frac{11\gamma^2+5}{8}\ , \quad a_2^{(0)}(x;\gamma) = -\frac{31\,\gamma^4 - 926\,\gamma^2 - 129}{128}\ ,$$

$$a_3^{(0)}(x;\gamma) = \frac{3\,\left(\gamma^2+15\right)\left(59\,\gamma^4 + 906\,\gamma^2 + 59\right)}{1024}\ ,$$

$$a_4^{(0)}(x;\gamma) = \frac{7125\,\gamma^8 + 15468\,\gamma^6 - 4088898\,\gamma^4 - 8215572\,\gamma^2 - 301035}{32768}\ ,$$

$$a_5^{(0)}(x;\gamma) = -\frac{102165\,\gamma^{10} - 208569\,\gamma^8 + 25390098\,\gamma^6 + 501398862\,\gamma^4 + 469053609\,\gamma^2 + 10896795}{262144}\ ,$$

$$a_6^{(0)}(x;\gamma) =$$

$$-\frac{45\,[84231\,\gamma^{12} - 348490\,\gamma^{10} + 2847497\,\gamma^8 - 451498956\,\gamma^6 - 2481377623\,\gamma^4 - 1343311306\,\gamma^2 - 21362649]}{4194304}$$

$$a_0^{(1)}(x;\gamma) = -\gamma\ , \quad a_1^{(1)}(x;\gamma) = -\frac{\gamma\left(\gamma^2-17\right)}{8}\ , \quad a_2^{(1)}(x;\gamma) = \frac{\gamma\,\left(9\,\gamma^4 + 206\,\gamma^2 + 809\right)}{128}\ ,$$

$$a_3^{(1)}(x;\gamma) = \frac{\gamma\,\left(75\,\gamma^6 + 143\,\gamma^4 - 24063\,\gamma^2 - 25307\right)}{1024}\ ,$$

$$a_4^{(1)}(x;\gamma) = -\frac{3\gamma\,\left(1225\,\gamma^8 - 1892\,\gamma^6 + 201078\,\gamma^4 + 2678812\,\gamma^2 + 1315081\right)}{32768}\ ,$$

$$a_5^{(1)}(x;\gamma) = -\frac{3\gamma\,\left(19845\,\gamma^{10} - 67625\,\gamma^8 + 467314\,\gamma^6 - 58112658\,\gamma^4 - 216355367\,\gamma^2 - 61495829\right)}{262144}\ ,$$

$$a_6^{(1)}(x;\gamma) =$$

$$\frac{3\gamma\,\left(800415\,\gamma^{12} - 3869530\,\gamma^{10} + 12921201\,\gamma^8 + 680167252\,\gamma^6 + 20763422609\,\gamma^4 + 36255061542\,\gamma^2 + 6716005951\right)}{4194304}$$

$$b_0^{(0)}(x;\gamma) = 1\ ,\quad b_1^{(0)}(x;\gamma) = -\frac{11\gamma^2+5}{8} = a_1^{(0)}\ ,\quad b_2^{(0)}(x;\gamma) = \frac{31\,\gamma^4 - 926\,\gamma^2 - 129}{128} = -a_2^{(0)}\ ,$$

$$b_3^{(0)}(x;\gamma) = \frac{3\,\left(\gamma^2+15\right)\left(59\,\gamma^4 + 906\,\gamma^2 + 59\right)}{1024} = a_3^{(0)}\ ,$$

$$b_4^{(0)}(x;\gamma) = -\frac{7125\,\gamma^8 + 15468\,\gamma^6 - 4088898\,\gamma^4 - 8215572\,\gamma^2 - 301035}{32768} = -a_4^{(0)}\ ,$$

$$b_5^{(0)}(x;\gamma) = -\frac{102165\,\gamma^{10} - 208569\,\gamma^8 + 25390098\,\gamma^6 + 501398862\,\gamma^4 + 469053609\,\gamma^2 + 10896795}{262144} = a_5^{(0)}\ ,$$

$$b_6^{(0)}(x;\gamma) = -a_6^{(0)} =$$

$$\frac{45\,[84231\,\gamma^{12} - 348490\,\gamma^{10} + 2847497\,\gamma^8 - 451498956\,\gamma^6 - 2481377623\,\gamma^4 - 1343311306\,\gamma^2 - 21362649]}{4194304}$$

$$b_0^{(1)}(x;\gamma) = -\gamma\ ,\quad b_1^{(1)}(x;\gamma) = \frac{\gamma\,\left(\gamma^2-17\right)}{8} = a_1^{(1)}\ ,\quad b_2^{(1)}(x;\gamma) = \frac{\gamma\,\left(9\,\gamma^4 + 206\,\gamma^2 + 809\right)}{128} = a_2^{(1)}\ ,$$

$$b_3^{(1)}(x;\gamma) = -\frac{\gamma\,\left(75\,\gamma^6 + 143\,\gamma^4 - 24063\,\gamma^2 - 25307\right)}{1024} = -a_3^{(0)}\ ,$$

$$b_4^{(1)}(x;\gamma) = -\frac{3\gamma\,\left(1225\,\gamma^8 - 1892\,\gamma^6 + 201078\,\gamma^4 + 2678812\,\gamma^2 + 1315081\right)}{32768} = a_4^{(1)}\ ,$$

$$b_5^{(1)}(x;\gamma) = \frac{3\gamma\,\left(19845\,\gamma^{10} - 67625\,\gamma^8 + 467314\,\gamma^6 - 58112658\,\gamma^4 - 216355367\,\gamma^2 - 61495829\right)}{262144} = -a_5^{(1)}\ ,$$

$$b_6^{(1)}(x;\gamma) = a_6^{(1)} =$$

$$\frac{3\gamma\,\left(800415\,\gamma^{12} - 3869530\,\gamma^{10} + 12921201\,\gamma^8 + 680167252\,\gamma^6 + 20763422609\,\gamma^4 + 36255061542\,\gamma^2 + 6716005951\right)}{4194304}$$

When $\gamma << 1$ one has approximately

$$A_0(x;\gamma) \approx A_0(x;0) = -1 - \frac{0.625}{x} + \frac{1.0078}{x^2} + \frac{2.5928}{x^3} - \frac{9.1869}{x^4} - \frac{41.568}{x^5} + \frac{229.20}{x^6} + \ldots$$

$$B_0(x;\gamma) \approx B_0(x;0) = 1 - \frac{0.625}{x} - \frac{1.0078}{x^2} + \frac{2.5928}{x^3} + \frac{9.1869}{x^4} - \frac{41.568}{x^5} - \frac{229.20}{x^6} + \ldots$$

$$A_1(x;\gamma) \approx \gamma\,\frac{\partial A_1}{\partial\gamma}\,(x;0) = \gamma\,\left[-1 + \frac{2.125}{x} + \frac{6.3203}{x^2} - \frac{24.714}{x^3} - \frac{120.40}{x^4} + \frac{703.76}{x^5} + \frac{4803.7}{x^6} + \ldots\right]$$

$$B_1(x;\gamma) \approx \gamma\,\frac{\partial B_1}{\partial\gamma}\,(x;0) = \gamma\,\left[-1 - \frac{2.125}{x} + \frac{6.3203}{x^2} + \frac{24.714}{x^3} - \frac{120.40}{x^4} - \frac{703.76}{x^5} + \frac{4803.7}{x^6} + \ldots\right]$$

$$\left|\frac{a_3^{(0)}(x;0)}{a_2^{(0)}(x;0)}\right| = \left|\frac{b_3^{(0)}(x;0)}{b_2^{(0)}(x;0)}\right| = 2.57\ ,\quad \left|\frac{a_4^{(0)}(x;0)}{a_3^{(0)}(x;0)}\right| = 3.54\ ,\quad \left|\frac{a_5^{(0)}(x;0)}{a_4^{(0)}(x;0)}\right| = 4.52\ ,\quad \left|\frac{a_6^{(0)}(x;0)}{a_5^{(0)}(x;0)}\right| = 5.51$$

The summand $a_k^{(0)}(x;\gamma)/[(1-\gamma^2)^{k+1}\,x^k]$ can be used if $|x| > |a_k^{(0)}(x;\gamma)/a_{k-1}^{(0)}(x;\gamma)|$.
The same holds for $b_k^{(0)}(x;\gamma)$.

Let

$$\Delta_n(x;\gamma) = -\Theta_0(x;\gamma)+$$
$$+\frac{1}{\sqrt{\pi x}}\left[A_0^{(n)}(x;\gamma)\ \cos x\, J_0(\gamma x) + A_1^{(n)}(x;\gamma)\ \cos x\, J_1(\gamma x) + B_0^{(n)}(x;\gamma)\ \sin x\, J_0(\gamma x) + B_1^{(n)}(x;\gamma)\ \sin x\, J_1(\gamma x)\right]$$

with

$$A_\mu^{(n)}(x;\gamma) = \sum_{k=0}^{n} \frac{a_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k} \qquad \text{and} \qquad B_\mu^{(n)}(x;\gamma) = \sum_{k=0}^{n} \frac{b_k^{(\mu)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k}\ .$$

For the case $\gamma = 0.1$ some of these differences are shown:

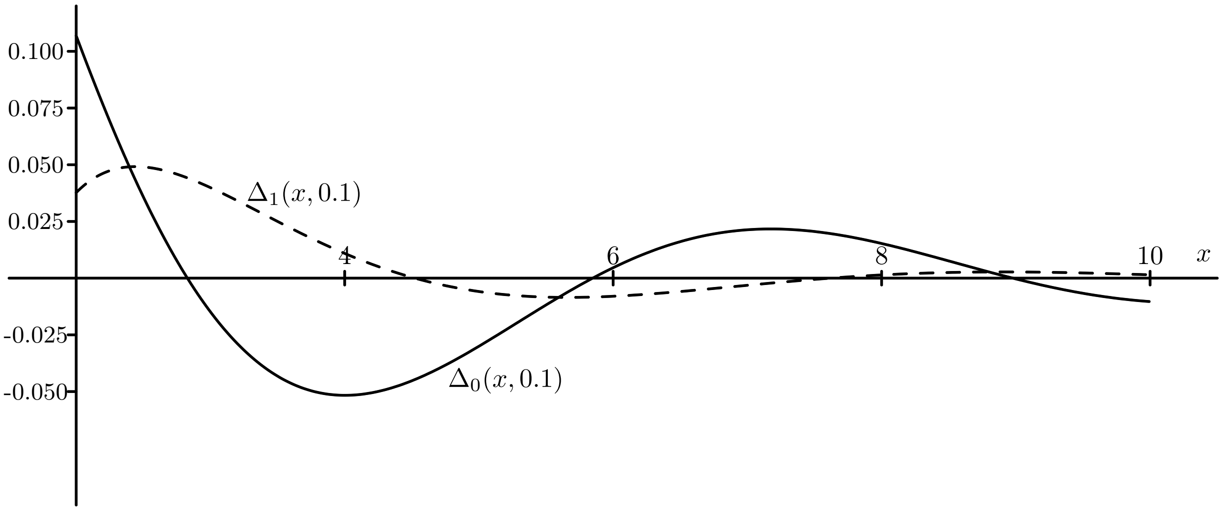


FIGURE 10 : *Differences* $\Delta_0\,(x;\gamma)$ *and* $\Delta_1\,(x;\gamma)$ *with* $\gamma = 0.1$

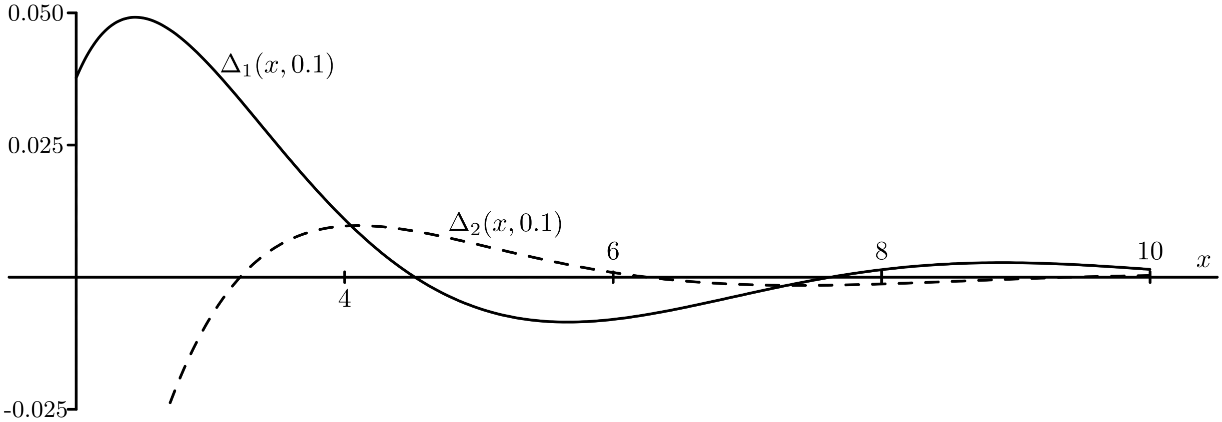


FIGURE 11 : *Differences* $\Delta_1\,(x;\gamma)$ *and* $\Delta_2\,(x;\gamma)$ *with* $\gamma = 0.1$

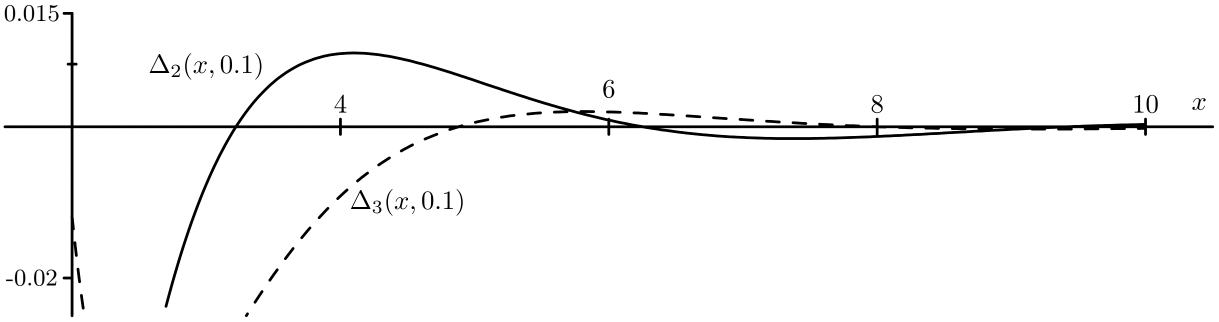


FIGURE 12 : *Differences* $\Delta_2\,(x;\gamma)$ *and* $\Delta_3\,(x;\gamma)$ *with* $\gamma = 0.1$

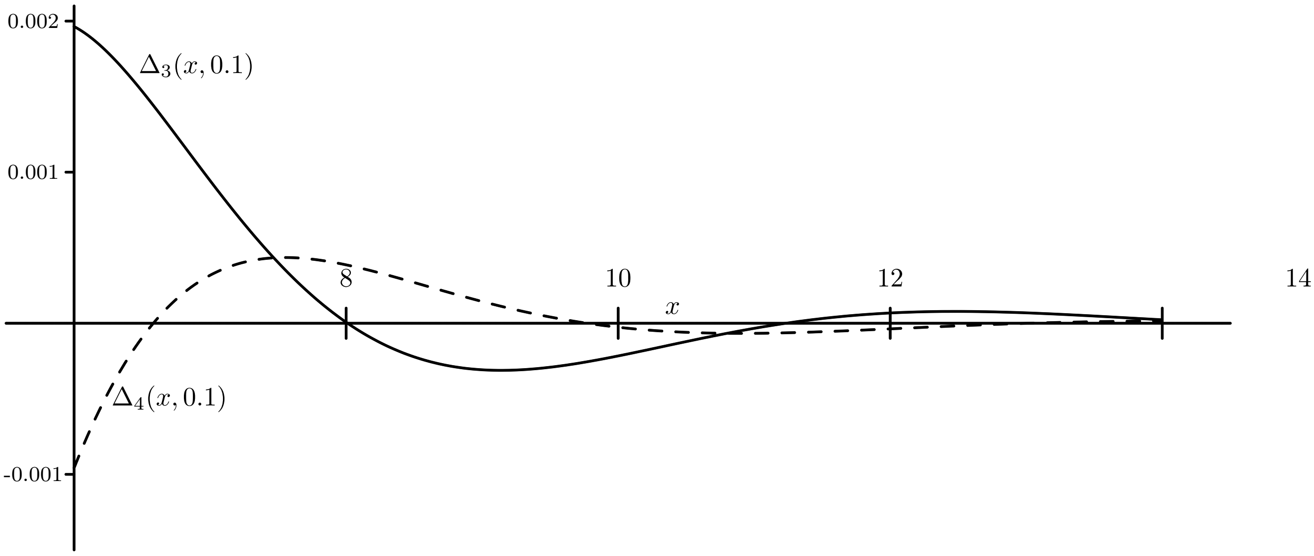


FIGURE 13 : *Differences* $\Delta_3\,(x;\gamma)$ *and* $\Delta_4\,(x;\gamma)$ *with* $\gamma = 0.1$

The same way the asymptotic expansion

$$\Theta_1(x;\gamma) \sim \frac{2}{\pi\gamma}\,[\mathbf{K}(\gamma) - \mathbf{E}(\gamma)] +$$

$$+\frac{A_0^*(x;\gamma)\;\cos x\,J_0(\gamma x) + A_1^*(x;\gamma)\;\cos x\,J_1(\gamma x) + B_0^*(x;\gamma)\;\sin x\,J_0(\gamma x) + B_1^*(x;\gamma)\;\sin x\,J_1(\gamma x)}{\sqrt{\pi x}}$$

is applicable in the case $x >> 1$ and $\gamma x \approx 1$. Let

$$A_\mu^*(x;\gamma) = \sum_{k=0}^{\infty} \frac{a_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k} \qquad \text{and} \qquad B_\mu^*(x;\gamma) = \sum_{k=0}^{\infty} \frac{b_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\,x^k}\;,$$

then holds

$$a_0^{(0,*)}(x;\gamma) = -\gamma\;,\quad a_1^{(0,*)}(x;\gamma) = -\frac{\gamma\,\left(3\,\gamma^2+13\right)}{8}\;,\quad a_2^{(0,*)}(x;\gamma) = -\frac{\gamma\,\left(15\,\gamma^4-382\,\gamma^2-657\right)}{128}\;,$$

$$a_3^{(0,*)}(x;\gamma) = \frac{3\,\gamma\,\left(35\,\gamma^6+327\,\gamma^4+8457\,\gamma^2+7565\right)}{1024}$$

$$a_4^{(0,*)}(x;\gamma) = \frac{3\,\gamma\,\left(1575\,\gamma^8-860\,\gamma^6-455382\,\gamma^4-2435292\,\gamma^2-1304345\right)}{32768}$$

$$a_5^{(0,*)}(x;\gamma) = -\frac{3\,\gamma\,\left(24255\,\gamma^{10}-74795\,\gamma^8+1750326\,\gamma^6+76252650\,\gamma^4+190548859\,\gamma^2+67043025\right)}{262144}$$

$$a_6^{(0,*)}(x;\gamma) =$$

$$\frac{45\,\gamma\,\left(63063\,\gamma^{12}-295722\,\gamma^{10}+1295545\,\gamma^8-137114124\,\gamma^6-1469273511\,\gamma^4-2157119402\,\gamma^2-532523145\right)}{4194304}$$

$$a_0^{(1,*)}(x;\gamma) = -1\;,\quad a_1^{(1,*)}(x;\gamma) = \frac{7\,\gamma^2+9}{8}\;,\quad a_2^{(1,*)}(x;\gamma) = \frac{57\,\gamma^4+622\,\gamma^2+345}{128}\;,$$

$$a_3^{(1,*)}(x;\gamma) = \frac{195\,\gamma^6-8921\,\gamma^4-30871\,\gamma^2-9555}{1024}$$

$$a_4^{(1,*)}(x;\gamma) = -\frac{7035\,\gamma^8+100692\,\gamma^6+4097826\,\gamma^4+7006164\,\gamma^2+1371195}{32768}$$

$$a_5^{(1,*)}(x;\gamma) = -\frac{97335\,\gamma^{10}-38595\,\gamma^8-54339354\,\gamma^6-442588230\,\gamma^4-449504301\,\gamma^2-60259815}{262144}$$

$$a_6^{(1,*)}(x;\gamma) =$$

$$\frac{3565485\,\gamma^{12} - 12841710\,\gamma^{10} + 423532419\,\gamma^8 + 25838749116\,\gamma^6 + 96291507171\,\gamma^4 + 64464832914\,\gamma^2 + 6264182925}{4194304}$$

$$b_0^{(0,*)}(x;\gamma) = \gamma\ , \quad b_1^{(0,*)}(x;\gamma) = -\frac{\gamma\,(3\,\gamma^2+13)}{8} = a_1^{(0,*)}\ , \quad b_2^{(0,*)}(x;\gamma) = \frac{\gamma\,\left(15\,\gamma^4 - 382\,\gamma^2 - 657\right)}{128} = -a_2^{(0,*)}\ ,$$

$$b_3^{(0,*)}(x;\gamma) = \frac{3\,\gamma\,\left(35\,\gamma^6 + 327\,\gamma^4 + 8457\,\gamma^2 + 7565\right)}{1024} = a_3^{(0,*)}$$

$$b_4^{(0,*)}(x;\gamma) = -\frac{3\,\gamma\,\left(1575\,\gamma^8 - 860\,\gamma^6 - 455382\,\gamma^4 - 2435292\,\gamma^2 - 1304345\right)}{32768} = -a_4^{(0,*)}$$

$$b_5^{(0,*)}(x;\gamma) = -\frac{3\,\gamma\,\left(24255\,\gamma^{10} - 74795\,\gamma^8 + 1750326\,\gamma^6 + 76252650\,\gamma^4 + 190548859\,\gamma^2 + 67043025\right)}{262144} = a_5^{(0,*)}$$

$$b_6^{(0,*)}(x;\gamma) = -a_6^{(0,*)} =$$

$$\frac{45\,\gamma\,\left(63063\,\gamma^{12} - 295722\,\gamma^{10} + 1295545\,\gamma^8 - 137114124\,\gamma^6 - 1469273511\,\gamma^4 - 2157119402\,\gamma^2 - 532523145\right)}{4194304}$$

$$b_0^{(1,*)}(x;\gamma) = -1\ , \quad b_1^{(1,*)}(x;\gamma) = -\frac{7\,\gamma^2+9}{8} = -a_1^{(1,*)}\ , \quad b_2^{(1,*)}(x;\gamma) = \frac{57\,\gamma^4 + 622\,\gamma^2 + 345}{128} = a_2^{(1,*)}\ ,$$

$$b_3^{(1,*)}(x;\gamma) = \frac{195\,\gamma^6 - 8921\,\gamma^4 - 30871\,\gamma^2 - 9555}{1024} = -a_3^{(1,*)}$$

$$b_4^{(1,*)}(x;\gamma) = -\frac{7035\,\gamma^8 + 100692\,\gamma^6 + 4097826\,\gamma^4 + 7006164\,\gamma^2 + 1371195}{32768} = a_4^{(1,*)}$$

$$b_5^{(1,*)}(x;\gamma) = \frac{97335\,\gamma^{10} - 38595\,\gamma^8 - 54339354\,\gamma^6 - 442588230\,\gamma^4 - 449504301\,\gamma^2 - 60259815}{262144} = -a_5^{(1,*)}$$

$$b_6^{(1,*)}(x;\gamma) = a_6^{(1,*)} =$$

$$\frac{3565485\,\gamma^{12} - 12841710\,\gamma^{10} + 423532419\,\gamma^8 + 25838749116\,\gamma^6 + 96291507171\,\gamma^4 + 64464832914\,\gamma^2 + 6264182925}{4194304}$$

When $\gamma << 1$ one has approximately

$$A_0^*(x;\gamma) \approx \gamma\,\frac{\partial A_0^*}{\partial\gamma}\,(x;0) = \gamma\,\left[-1 - \frac{1.625}{x} + \frac{5.1328}{x^2} + \frac{22.163}{x^3} - \frac{119.42}{x^4} - \frac{767.25}{x^5} + \frac{5713.4}{x^6} + \ldots\right]$$

$$B_0^*(x;\gamma) \approx \gamma\,\frac{\partial B_0^*}{\partial\gamma}\,(x;0) = \gamma\,\left[1 - \frac{1.625}{x} - \frac{5.1328}{x^2} + \frac{22.163}{x^3} + \frac{119.42}{x^4} - \frac{767.25}{x^5} - \frac{5713.4}{x^6} + \ldots\right]$$

$$A_1^*(x;\gamma) \approx A_1^*(x;0) = -1 + \frac{1.125}{x} + \frac{2.6953}{x^2} - \frac{9.3311}{x^3} - \frac{41.846}{x^4} + \frac{229.87}{x^5} + \frac{1493.5}{x^6} + \ldots$$

$$B_1^*(x;\gamma) \approx B_1^*(x;0) = -1 - \frac{1.125}{x} + \frac{2.6953}{x^2} + \frac{9.3311}{x^3} - \frac{41.846}{x^4} - \frac{229.87}{x^5} + \frac{1493.5}{x^6} + \ldots$$

$$\left|\frac{a_3^{(1,*)}(x;0)}{a_2^{(1,*)}(x;0)}\right| = \left|\frac{b_3^{(1,*)}(x;0)}{b_2^{(1,*)}(x;0)}\right| = 3.46\ , \quad \left|\frac{a_4^{(1,*)}(x;0)}{a_3^{(1,*)}(x;0)}\right| = 4.48\ , \quad \left|\frac{a_5^{(1,*)}(x;0)}{a_4^{(1,*)}(x;0)}\right| = 5.49\ , \quad \left|\frac{a_6^{(1,*)}(x;0)}{a_5^{(1,*)}(x;0)}\right| = 6.50$$

The summand $a_k^{(0,*)}(x;\gamma)/[(1-\gamma^2)^{k+1}\,x^k]$ can be used if $|x| > |a_k^{(0,*)}(x;\gamma)/a_{k-1}^{(0)}(x;\gamma)|$.
The same holds for $b_k^{(0,*)}(x;\gamma)$.
Let

$$\Delta_n^*(x;\gamma) = -\Theta_1(x;\gamma) +$$

$$+\frac{1}{\sqrt{\pi x}}\left[A_0^{(n,*)}(x;\gamma)\ \cos x\,J_0(\gamma x) + A_1^{(n,*)}(x;\gamma)\ \cos x\,J_1(\gamma x) + B_0^{(n,*)}(x;\gamma)\ \sin x\,J_0(\gamma x) + B_1^{(n,*)}(x;\gamma)\ \sin x\,J_1(\gamma x)\right]$$

with

$$A_\mu^{(n,*)}(x;\gamma) = \sum_{k=0}^{n} \frac{a_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k} \qquad \text{and} \qquad B_\mu^{(n,*)}(x;\gamma) = \sum_{k=0}^{n} \frac{b_k^{(\mu,*)}(x;\gamma)}{(1-\gamma^2)^{k+1}\, x^k} \; .$$

For the case $\gamma = 0.1$ some of these differences are shown:

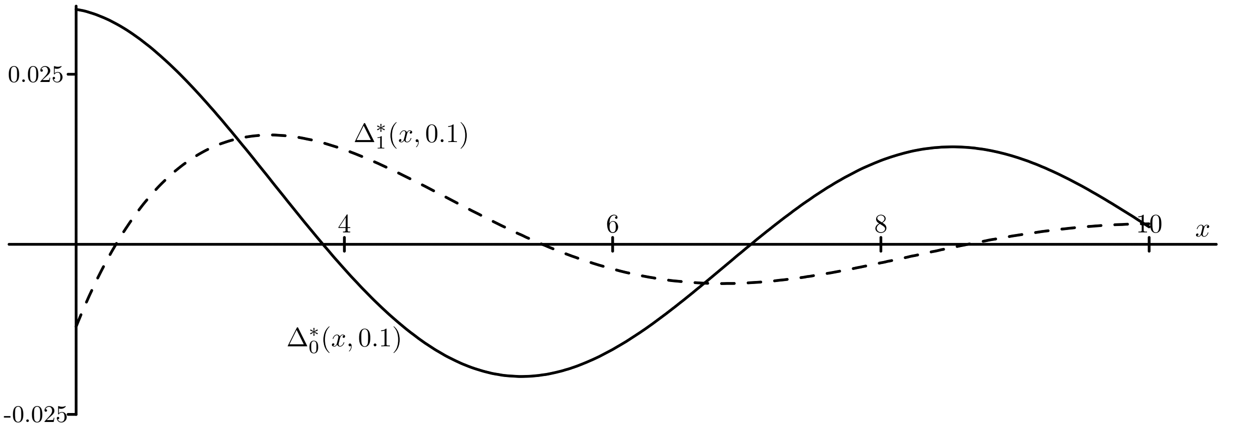


FIGURE 14 : *Differences* $\Delta_0^*(x;\gamma)$ *and* $\Delta_1^*(x;\gamma)$ *with* $\gamma = 0.1$

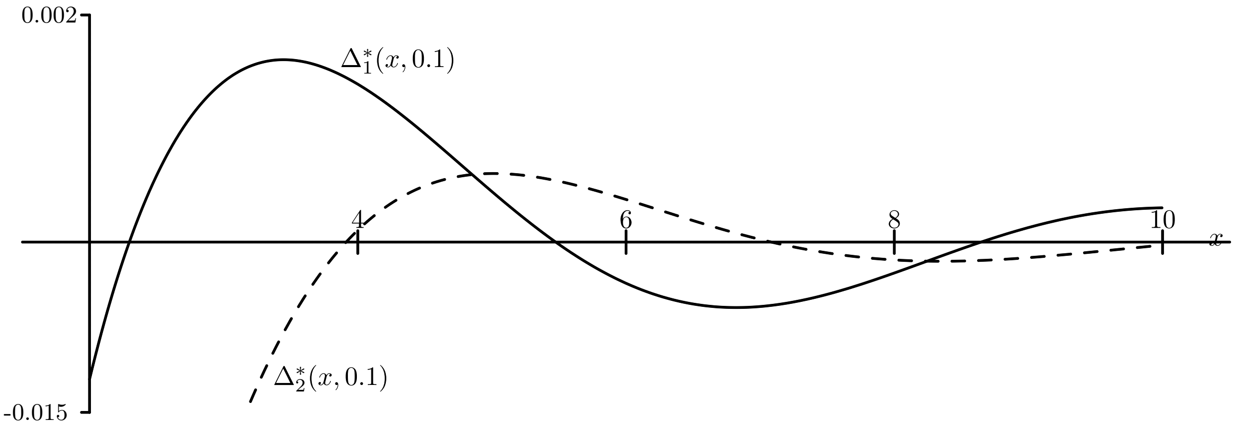


FIGURE 15 : *Differences* $\Delta_1^*(x;\gamma)$ *and* $\Delta_2^*(x;\gamma)$ *with* $\gamma = 0.1$

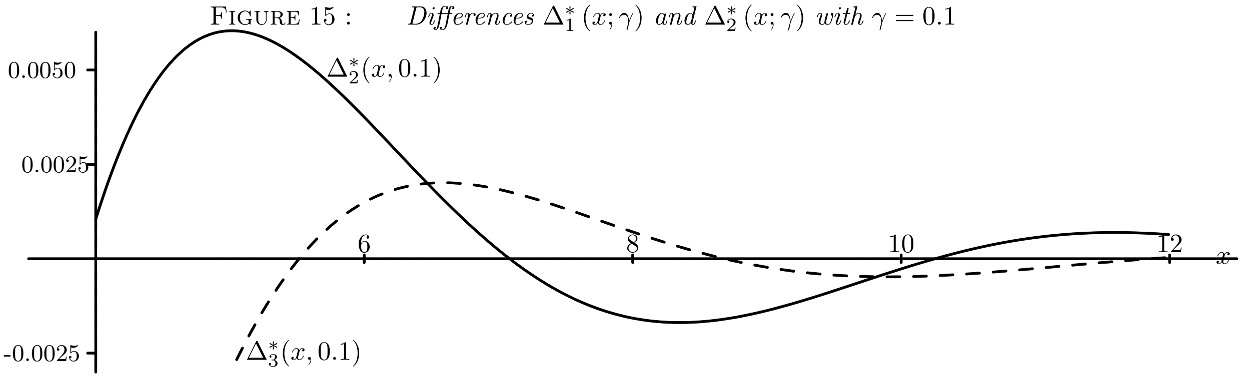


FIGURE 16 : *Differences* $\Delta_2^*(x;\gamma)$ *and* $\Delta_3^*(x;\gamma)$ *with* $\gamma = 0.1$

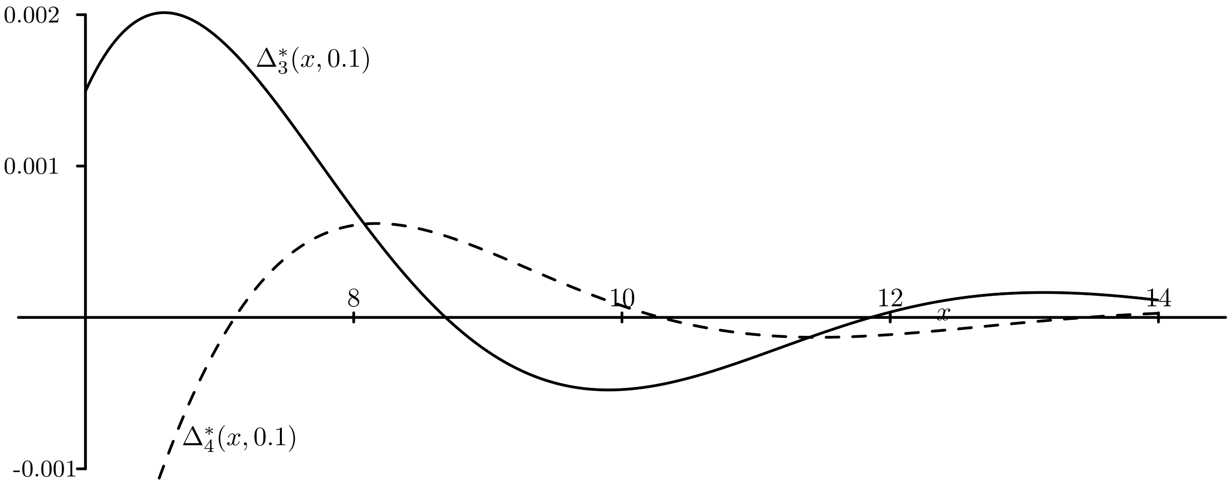


FIGURE 17 : *Differences* $\Delta_3^*(x;\gamma)$ *and* $\Delta_4^*(x;\gamma)$ *with* $\gamma = 0.1$

b) Integrals:

Holds (with $0 < \beta < \alpha$ and $\beta/\alpha = \gamma < 1$)

$$\int J_0(\alpha x)\, J_0(\beta x)\, dx = \frac{1}{\alpha}\, \Theta_0\left(ax; \frac{b}{a}\right) , \qquad \int J_1(\alpha x)\, J_1(\beta x)\, dx = \frac{1}{\alpha}\, \Theta_1\left(ax; \frac{b}{a}\right)$$

Let $\boxed{\alpha^2 + \beta^2 = \sigma \quad \text{and} \quad \alpha^2 - \beta^2 = \Delta\ .}$

$$\int I_0(\alpha x)\, I_0(\beta x)\, dx = \frac{1}{\alpha}\, \Omega_0\left(ax; \frac{b}{a}\right) , \qquad \int I_1(\alpha x)\, I_1(\beta x)\, dx = \frac{1}{\alpha}\, \Omega_1\left(ax; \frac{b}{a}\right)$$

(Θ_ν and Ω_ν as definded on pages 303 and 305.)

$$\int x^2 \cdot J_0(\alpha x)\, J_0(\beta x)\, dx = \frac{\sigma x}{\Delta^2}\, J_0(\alpha x)\, J_0(\beta x) - \frac{\beta\, x^2}{\Delta}\, J_0(\alpha x)\, J_1(\beta x) + \frac{\alpha\, x^2}{\Delta}\, J_1(\alpha x)\, J_0(\beta x) +$$
$$+ \frac{2\,\alpha\,\beta x}{\Delta^2} J_1(\alpha x)\, J_1(\beta x) - \frac{\sigma}{\Delta^2} \int J_0(\alpha x)\, J_0(\beta x)\, dx + \frac{2\,\alpha\,\beta}{\Delta^2} \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^2 \cdot J_1(\alpha x)\, J_1(\beta x)\, dx = \frac{2\,\alpha\,\beta\, x}{\Delta^2}\, J_0(\alpha x)\, J_0(\beta x) - \frac{\alpha\, x^2}{\Delta}\, J_0(\alpha x)\, J_1(\beta x) + \frac{\beta\, x^2}{\Delta}\, J_1(\alpha x)\, J_0(\beta x) +$$
$$+ \frac{\sigma\, x}{\Delta^2}\, J_1(\alpha x)\, J_1(\beta x) - \frac{2\,\alpha\,\beta}{\Delta^2} \int J_0(\alpha x)\, J_0(\beta x)\, dx + \frac{\sigma}{\Delta^2} \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^2 \cdot I_0(\alpha x)\, I_0(\beta x)\, dx = -\frac{\sigma\, x}{\Delta^2}\, I_0(\alpha x)\, I_0(\beta x) - \frac{\beta\, x^2}{\Delta}\, I_0(\alpha x)\, I_1(\beta x) + \frac{\alpha\, x^2}{\Delta}\, I_1(\alpha x)\, I_0(\beta x) +$$
$$+ \frac{2\,\alpha\,\beta\, x}{\Delta^2}\, I_1(\alpha x)\, I_1(\beta x) + \frac{\sigma}{\Delta^2} \int I_0(\alpha x)\, I_0(\beta x)\, dx + \frac{2\,\alpha\,\beta}{\Delta^2} \int I_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x^2 \cdot I_1(\alpha x)\, I_1(\beta x)\, dx = \frac{2\alpha\,\beta\, x}{\Delta^2}\, I_0(\alpha x)\, I_0(\beta x) + \frac{\alpha\, x^2}{\Delta}\, I_0(\alpha x)\, I_1(\beta x) - \frac{\beta\, x^2}{\Delta}\, I_1(\alpha x)\, I_0(\beta x) -$$
$$- \frac{\sigma\, x}{\Delta^2}\, I_1(\alpha x)\, I_1(\beta x) - \frac{2\,\alpha\,\beta}{\Delta^2} \int I_0(\alpha x)\, I_0(\beta x)\, dx - \frac{\sigma}{\Delta^2} \int I_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x^4 \cdot J_0(\alpha x)\, J_0(\beta x)\, dx = \frac{3\,[\sigma\,\Delta^2\,x^2 - 3\,\sigma^2 - 4\alpha^2\beta^2]\,x}{\Delta^4}\, J_0(\alpha x)\, J_0(\beta x) -$$

$$-\frac{(\Delta^2\,x^2 - 15\,\alpha^2 - 9\,\beta^2)\beta\,x^2}{\Delta^3}\, J_0(\alpha x)\, J_1(\beta x) + \frac{(\Delta^2\,x^2 - 9\,\alpha^2 - 15\,\beta^2)\,\alpha\,x^2}{\Delta^3}\, J_1(\alpha x)\, J_0(\beta x) +$$

$$+\frac{6\,(\Delta^2\,x^2 - 4\,\sigma)\alpha\,\beta\,x}{\Delta^4}\, J_1(\alpha x)\, J_1(\beta x) + \frac{9\,\sigma^2 + 12\,\alpha^2\beta^2}{\Delta^4} \int J_0(\alpha x)\, J_0(\beta x)\, dx - \frac{24\,\alpha\,\beta\,\sigma}{\Delta^4} \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^4 \cdot J_1(\alpha x)\, J_1(\beta x)\, dx = \frac{6\,(\Delta^2\,x^2\alpha^4 - 4\,\sigma)\,\beta\,\alpha\,x}{\Delta^4}\, J_0(\alpha x)\, J_0(\beta x) -$$

$$-\frac{(\Delta^2\,x^2 - 3\,\alpha^2 - 21\,\beta^2)\alpha\,x^2}{\Delta^3}\, J_0(\alpha x)\, J_1(\beta x) + \frac{(\Delta^2\,x^2 - 21\,\alpha^2 - 3\,\beta^2)\,\beta\,x^2}{\Delta}\, J_1(\alpha x)\, J_0(\beta x) +$$

$$+\frac{3\,[\sigma\,\Delta^2\,x^2 - \sigma^2 - 12\,\alpha^2\beta^2]\,x}{\Delta^4}\, J_1(\alpha x)\, J_1(\beta x) -$$

$$+\frac{24\,\alpha\,\beta\,\sigma}{\Delta^4} \int J_0(\alpha x)\, J_0(\beta x)\, dx - \frac{3\,\sigma^2 + 36\,\alpha^2\beta^2}{\Delta^4} \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^4 \cdot I_0(\alpha x)\, I_0(\beta x)\, dx = -\frac{3[\sigma\,\Delta^2\,x^2 + 3\,\sigma^2 + 4\,\alpha^2\beta^2]\,x}{\Delta^4}\, I_0(\alpha x)\, I_0(\beta x) -$$

$$-\frac{(\Delta^2\,x^2 + 15\,\alpha^2 + 9\,\beta^2)\,\beta\,x^2}{\Delta^3}\, I_0(\alpha x)\, I_1(\beta x) + \frac{(\Delta^2\,x^2 + 9\,\alpha^2 + 15\,\beta^2)\,\alpha\,x^2}{\Delta^3}\, I_1(\alpha x)\, I_0(\beta x) +$$

$$+\frac{6\,(\Delta^2\,x^2 + 4\,\sigma)\,\alpha\,\beta\,x}{\Delta^4}\, I_1(\alpha x)\, I_1(\beta x) -$$

$$+\frac{9\,\sigma^2 + 12\,\alpha^2\beta^2)}{\Delta^4} \int I_0(\alpha x)\, I_0(\beta x)\, dx + \frac{24\,\alpha\,\beta\,\sigma}{\Delta^4} \int I_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x^4 \cdot I_1(\alpha x)\, I_1(\beta x)\, dx = \frac{6\,(\Delta^2\,x^2 + 4\,\sigma)\,\alpha\,\beta\,x}{\Delta^4}\, I_0(\alpha x)\, I_0(\beta x) +$$

$$+\frac{(\Delta^2\,x^2 + 3\,\alpha^2 + 21\,\beta^2)\alpha\,x^2}{\Delta^3}\, I_0(\alpha x)\, I_1(\beta x) - \frac{(\Delta^2\,x^2 + 21\,\alpha^2 + 3\,\beta^2)\,\beta\,x^2}{\Delta^3}\, I_1(\alpha x)\, I_0(\beta x) -$$

$$-\frac{3\,(\sigma\,\Delta^2\,x^2 + \sigma^2 + 12\,\alpha^2\beta^2)\,x}{\Delta^4}\, I_1(\alpha x)\, I_1(\beta x) -$$

$$-\frac{24\,\alpha\,\beta\,\sigma}{\Delta^4} \int I_0(\alpha x)\, I_0(\beta x)\, dx - \frac{3\,\sigma^2 + 36\,\alpha^2\beta^2}{\Delta^4} \int I_1(\alpha x)\, I_1(\beta x)\, dx$$

Let

$$\int x^n \cdot J_\nu(\alpha x)\, J_\nu(\beta x)\, dx = \frac{P_n^{(\nu)}(x)}{\Delta^n}\, J_0(\alpha x)\, J_0(\beta x) + \frac{Q_n^{(\nu)}(x)}{\Delta^{n-1}}\, J_0(\alpha x)\, J_1(\beta x) +$$

$$+\frac{R_n^{(\nu)}(x)}{\Delta^{n-1}}\, J_1(\alpha x)\, J_0(\beta x) + \frac{S_n^{(\nu)}(x)}{\Delta^n}\, J_1(\alpha x)\, J_1(\beta x) +$$

$$+\frac{U_n^{(\nu)}(x)}{\Delta^n} \int J_0(\alpha x)\, J_0(\beta x)\, dx + \frac{V_n^{(\nu)}(x)}{\Delta^n} \int J_1(\alpha x)\, J_1(\beta x)\, dx$$

and

$$\int x^n \cdot I_\nu(\alpha x)\, I_\nu(\beta x)\, dx = \frac{\bar{P}_n^{(\nu)}(x)}{\Delta^n}\, I_0(\alpha x)\, I_0(\beta x) + \frac{\bar{Q}_n^{(\nu)}(x)}{\Delta^{n-1}}\, I_0(\alpha x)\, I_1(\beta x) +$$

$$+\frac{\bar{R}_n^{(\nu)}(x)}{\Delta^{n-1}}\, I_1(\alpha x)\, I_0(\beta x) + \frac{\bar{S}_n^{(\nu)}(x)}{\Delta^n}\, I_1(\alpha x)\, I_1(\beta x) +$$

$$+\frac{\bar{U}_n^{(\nu)}}{\Delta^n} \int I_0(\alpha x)\, I_0(\beta x)\, dx + \frac{\bar{V}_n^{(\nu)}}{\Delta^n} \int I_1(\alpha x)\, I_1(\beta x)\, dx\,.$$

$$P_6^{(0)} = 5\,x[\sigma\,\Delta^4\,x^4 - 3\,(5\,\sigma^2 + 12\,\alpha^2\,\beta^2)\Delta^2 x^2 + 3\,\sigma\,(15\,\sigma^2 + 68\,\alpha^2\,\beta^2)]$$

$$Q_6^{(0)} = -\beta\, x^2[\Delta^4 x^4 - 5\,(11\,\alpha^2 + 5\,\beta^2)\Delta^2 x^2 + 465\,\alpha^4 + 1230\,\alpha^2\,\beta^2 + 225\,\beta^4]$$

$$R_6^{(0)} = \alpha\, x^2[\Delta^4 x^4 - 5\,(5\,\alpha^2 + 11\,\beta^2)\Delta^2 x^2 + 225\,\alpha^4 + 1230\alpha^2\,\beta^2 + 465\,\beta^4]$$

$$S_6^{(0)} = 10\,\alpha\,\beta\, x[\Delta^4 x^4 - 24\,\sigma\Delta^2 x^2 + 69\,\sigma^2 + 108\alpha^2\,\beta^2]$$

$$U_6^{(0)} = -15\,\sigma(15\,\sigma^2 + 68\alpha^2\,\beta^2)\ , \quad V_6^{(0)} = 30\,\alpha\,\beta\,(23\,\sigma^2 + 36\,\alpha^2\,\beta^2)$$

$$P_6^{(1)} = 10\,\alpha\,\beta\,[\Delta^4 x^4 - 24\,\sigma\Delta^2 x^2 + 81\,\sigma^2 + 60\,\alpha^2\beta^2]\,x$$

$$Q_6^{(1)} = -\alpha\,[\Delta^4 x^4 - 5\,(3\,\alpha^2 + 13\,\beta^2)\,\Delta^2\,x^2 + 45\,\alpha^4 + 1110\,\alpha^2\,\beta^2 + 765\,\beta^4]\,x^2$$

$$R_6^{(1)} = \beta\,[\Delta^4\,x^4 - 5\,(13\,\alpha^2 + 3\,\beta^2)\,\Delta^2\,x^2 + 765\,\alpha^4 + 1110\,\alpha^2\,\beta^2 + 45\,\beta^4]\,x^2$$

$$S_6^{(1)} = 5\,[\sigma\,\Delta^4\,x^4 - 3\,(3\,\sigma^2 + 20\,\alpha^2\,\beta^2)\Delta^2\,x^2 + 3\,\sigma\,(3\,\sigma^2 + 116\,\alpha^2\,\beta^2)]\,x$$

$$U_6^{(1)} = -30\,(27\,\sigma^2 + 20\,\alpha^2\,\beta^2)\alpha\,\beta\ , \quad V_6^{(1)} = 15\,\sigma(3\,\sigma^2 + 116\,\alpha^2\,\beta^2)$$

$$\bar{P}_6^{(0)} = -5\,[\sigma\Delta^4\,x^4 + 3\,(5\,\sigma^2 + 12\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 + 3\,\sigma(15\,\sigma^2 + 68\,\alpha^2\,\beta^2)]\,x$$

$$\bar{Q}_6^{(0)} = -\beta\,[\Delta^4\,x^4 + 5\,(11\,\alpha^2 + 5\,\beta^2)\,\Delta^2\,x^2 + 465\,\alpha^4 + 1230\,\alpha^2\,\beta^2 + 225\,\beta^4]\,x^2$$

$$\bar{R}_6^{(0)} = \alpha\,[\Delta^4\,x^4 + 5\,(5\,\alpha^2 + 11\,\beta^2)\,\Delta^2\,x^2 + 225\,\alpha^4 + 1230\,\alpha^2\,\beta^2 + 465\,\beta^4]\,x^2$$

$$\bar{S}_6^{(0)} = 10\,\alpha\,\beta\,[\Delta^4\,x^4 + 24\,\sigma\,\Delta^2\,x^2 + 69\,\sigma^2 + 108\,\alpha^2\,\beta^2]\,x$$

$$\bar{U}_6^{(0)} = 15\,\sigma(15\,\sigma^2 + 68\,\alpha^2\,\beta^2)\ , \quad \bar{V}_6^{(0)} = 30\,\alpha\,\beta\,(23\,\sigma^2 + 36\,\alpha^2\,\beta^2)$$

$$\bar{P}_6^{(1)} = 10\,\alpha\,\beta\,[\Delta^4 x^4 + 24\,\sigma\Delta^2 x^2 + 81\,\sigma^2 + 60\,\alpha^2\,\beta^2]\,x$$

$$\bar{Q}_6^{(1)} = \alpha\,[\Delta^4\,x^4 + 5\,(3\,\alpha^2 + 13\,\beta^2)\,\Delta^2\,x^2 + 45\,\alpha^4 + 1110\,\alpha^2\,\beta^2 + 765\,\beta^4]\,x^2$$

$$\bar{R}_6^{(1)} = -\beta[\Delta^4\,x^4 + 5\,(13\,\alpha^2 + 3\,\beta^2)\,\Delta^2\,x^2 + 765\,\alpha^4 + 1110\,\alpha^2\,\beta^2 + 45\,\beta^4]\,x^2$$

$$\bar{S}_6^{(1)} = -5\,x[\sigma\,\Delta^4\,x^4 + 3\,(3\,\sigma^2 + 20\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 + 3\,\sigma(3\,\sigma^2 + 116\,\alpha^2\,\beta^2)]$$

$$\bar{U}_6^{(1)} = -30\,(27\,\sigma^2 + 20\,\alpha^2\,\beta^2)\alpha\,\beta\ , \quad \bar{V}_6^{(1)} = -15\,\sigma(3\,\sigma^2 + 116\,\alpha^2\,\beta^2)$$

$$P_8^{(0)} = 7\,[\sigma\Delta^6\,x^6 - 5\,(7\,\sigma^2 + 20\,\alpha^2\,\beta^2)\,\Delta^4\,x^4 + 15\,\sigma\,(35\,\sigma^2 + 244\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 -$$
$$-1575\,\sigma^4 - 15240\,\alpha^2\,\beta^2\,\sigma^2 - 6000\,\alpha^4\,\beta^4]\,x$$

$$Q_8^{(0)} = -\beta\,[\Delta^6 x^6 - 7\,(17\,\alpha^2 + 7\,\beta^2)\Delta^4 x^4 + 35\,(107\,\alpha^4 + 242\,\alpha^2\,\beta^2 + 35\,\beta^4)\Delta^2 x^2 -$$
$$-25935\,\alpha^6 - 160755\,\alpha^4\beta^2 - 124845\,\alpha^2\beta^4 - 11025\,\beta^6]\,x^2$$

$$R_8^{(0)} = \alpha\,[\Delta^6 x^6 - 7\,(7\,\alpha^2 + 17\,\beta^2)\,\Delta^4\,x^4 + 35\,(35\,\alpha^4 + 242\,\alpha^2\,\beta^2 + 107\,\beta^4)\Delta^2 x^2 -$$
$$-11025\alpha^6 - 124845\,\alpha^4\beta^2 - 160755\,\alpha^2\beta^4 - 25935\,\beta^6]\,x^2$$

$$S_8^{(0)} = 14\,\alpha\,\beta\,[\Delta^6 x^6 - 60\,\sigma\Delta^4 x^4 + 15\,(71\,\sigma^2 + 100\,\alpha^2\,\beta^2)\Delta^2\,x^2 - 240\,\sigma(11\,\sigma^2 + 52\,\alpha^2\,\beta^2)]\,x$$

$$U_8^{(0)} = 11025\,\sigma^4 + 106680\,\alpha^2\beta^2\sigma^2 + 42000\,\alpha^4\beta^4$$

$$V_8^{(0)} = -3360\,\alpha\,\beta\,\sigma(11\,\sigma^2 + 52\,\alpha^2\,\beta^2)$$

$$P_8^{(1)} = 14\,\alpha\,\beta\,[\Delta^6 x^6 - 60\,\sigma\Delta^4 x^4 + 45\,(25\,\sigma^2 + 28\,\alpha^2\,\beta^2)\,\Delta^2\,x^2 - 720\,\sigma\,(5\,\sigma^2 + 12\,\alpha^2\,\beta^2)]\,x$$

$$Q_8^{(1)} = -\alpha\,[\Delta^6 x^6 - 7\,(5\,\alpha^2 + 19\,\beta^2)\Delta^4 x^4 +$$
$$+105\,(5\,\alpha^2 + 3\,\beta^2)(\alpha^2 + 15\,\beta^2)\,\Delta^2\,x^2 - 1575\,\alpha^6 - 85995\,\alpha^4\beta^2 - 186165\,\alpha^2\beta^4 - 48825\,\beta^6]\,x^2$$

$$R_8^{(1)} = \beta\,[\Delta^6 x^6 - 7\,(19\,\alpha^2 + 5\,\beta^2)\,\Delta^4\,x^4 +$$

$$+105\,(3\,\alpha^2+5\,\beta^2)(15\,\alpha^2+\beta^2)\,\Delta^2\,x^2-48825\,\alpha^6-186165\,\alpha^4\beta^2-85995\,\alpha^2\beta^4-1575\,\beta^6]\,x^2$$

$$S_8^{(1)}=7\,[\sigma\,\Delta^6\,x^6-5\,(5\,\sigma^2+28\,\alpha^2\,\beta^2)\,\Delta^4\,x^4+45\,\sigma\,(5\,\sigma^2+108\,\alpha^2\beta^2)\,\Delta^2\,x^2-$$
$$-225\,\sigma^4-18360\,\alpha^2\beta^2\sigma^2-15120\,\alpha^4\beta^4]\,x$$

$$U_8^{(1)}=10080\,\alpha\,\beta\,\sigma(5\,\sigma^2+12\,\alpha^2\,\beta^2)$$

$$V_8^{(1)}=-(1575\,\sigma^4+128520\,\alpha^2\beta^2\sigma^2+105840\,\alpha^4\beta^4)$$

$$\bar P_8^{(0)}=-7\,[\sigma\Delta^6x^6+5\,(7\,\sigma^2+20\,\alpha^2\,\beta^2)\,\Delta^4\,x^4+15\,\sigma\,(35\,\sigma^2+244\,\alpha^2\,\beta^2)\,\Delta^2\,x^2+$$
$$+1575\,\sigma^4+15240\,\alpha^2\beta^2\sigma^2+6000\,\alpha^4\,\beta^4]\,x$$

$$\bar Q_8^{(0)}=-\beta\,[\Delta^6x^6+7\,(17\,\alpha^2+7\,\beta^2)\,\Delta^4\,x^4+35\,(107\,\alpha^4+242\,\alpha^2\,\beta^2+35\,\beta^4)\,\Delta^2\,x^2+$$
$$+25935\,\alpha^6+160755\,\alpha^4\beta^2+124845\,\alpha^2\beta^4+11025\,\beta^6]\,x^2$$

$$\bar R_8^{(0)}=\alpha\,[\Delta^6x^6+7\,(7\,\alpha^2+17\,\beta^2)\,\Delta^4\,x^4+35\,(35\,\alpha^4+242\,\alpha^2\,\beta^2+107\,\beta^4)\,\Delta^2\,x^2+$$
$$+11025\,\alpha^6+124845\,\alpha^4\beta^2+160755\,\alpha^2\beta^4+25935\,\beta^6]\,x^2$$

$$\bar S_8^{(0)}=14\,\alpha\,\beta\,[\Delta^6x^6+60\,\sigma\,\Delta^4\,x^4+15\,(71\,\sigma^2+100\,\alpha^2\,\beta^2)\,\Delta^2\,x^2+240\,\sigma(11\,\sigma^2+52\,\alpha^2\beta^2)]\,x$$

$$\bar U_8^{(0)}=11025\,\sigma^4+106680\,\alpha^2\beta^2\sigma^2+42000\,\alpha^4\beta^4$$

$$\bar V_8^{(0)}=3360\,\alpha\,\beta\,\sigma\,(11\,\sigma^2+52\,\alpha^2\beta^2)$$

$$\bar P_8^{(1)}=14\,\alpha\,\beta\,[\Delta^6x^6+60\,\sigma\Delta^4x^4+45\,(25\,\sigma^2+28\,\alpha^2\,\beta^2)\,\Delta^2\,x^2+720\,\sigma\,(5\,\sigma^2+12\,\alpha^2\beta^2)]\,x$$

$$\bar Q_8^{(1)}=\alpha\,[\Delta^6\,x^6+7\,(5\,\alpha^2+19\,\beta^2)\Delta^4x^4+$$
$$+105\,(5\,\alpha^2+3\,\beta^2)(\alpha^2+15\,\beta^2)\,\Delta^2\,x^2+1575\,\alpha^6+85995\,\alpha^4\beta^2+186165\,\alpha^2\beta^4+48825\,\beta^6]\,x^2$$

$$\bar R_8^{(1)}=-\beta\,[\Delta^6x^6+7\,(19\,\alpha^2+5\,\beta^2)\,\Delta^4\,x^4+$$
$$+105\,(3\,\alpha^2+5\,\beta^2)(15\,\alpha^2+\beta^2)\,\Delta^2\,x^2+48825\,\alpha^6+186165\,\alpha^4\beta^2+85995\,\alpha^2\beta^4+1575\,\beta^6]\,x^2$$

$$\bar S_8^{(1)}=-7\,[\sigma\,\Delta^6\,x^6+5\,(5\,\sigma^2+28\,\alpha^2\,\beta^2)\,\Delta^4\,x^4+45\,\sigma\,(5\,\sigma^2+108\,\alpha^2\beta^2)\,\Delta^2\,x^2+$$
$$+225\,\sigma^4+18360\,\alpha^2\beta^2\sigma^2+15120\,\alpha^4\beta^4]\,x$$

$$\bar U_8^{(1)}=-10080\,\alpha\,\beta\,\sigma\,(5\,\sigma^2+12\,\alpha^2\,\beta^2)$$

$$\bar V_8^{(1)}=-(1575\,\sigma^4+128520\,\alpha^2\beta^2\sigma^2+105840\,\alpha^4\beta^4)$$

Recurrence relations: (see also page 302)

$$\int x^{2n+2}\,J_0(\alpha x)\,J_0(\beta x)\,dx=\frac{(2n+1)\,\sigma\,x^{2n+1}}{\Delta^2}\,J_0(\alpha x)\,J_0(\beta x)-\frac{\beta\,x^{2n+2}}{\Delta}\,J_0(\alpha x)\,J_1(\beta x)+$$
$$+\frac{\alpha\,x^{2n+2}}{\Delta}\,J_1(\alpha x)\,J_0(\beta x)+\frac{(4n+2)\,\alpha\,\beta\,x^{2n+1}}{\Delta^2}\,J_1(\alpha x)\,J_1(\beta x)-$$
$$-\frac{(2n+1)^2\,\sigma}{\Delta^2}\int x^{2n}\,J_0(\alpha x)\,J_0(\beta x)\,dx-\frac{(8n^2-2)\,\alpha\,\beta}{\Delta^2}\int x^{2n}\,J_1(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^{2n+2}\,J_1(\alpha x)\,J_1(\beta x)\,dx=\frac{(4n+2)\,\alpha\,\beta\,x^{2n+1}}{\Delta^2}\,J_0(\alpha x)\,J_0(\beta x)-\frac{\alpha\,x^{2n+2}}{\Delta}\,J_0(\alpha x)\,J_1(\beta x)+$$
$$+\frac{\beta\,x^{2n+2}}{\Delta}\,J_1(\alpha x)\,J_0(\beta x)+\frac{(2n+1)\,\sigma\,x^{2n+1}}{\Delta^2}\,J_1(\alpha x)\,J_1(\beta x)-$$
$$-\frac{2\,(2n+1)^2\,\alpha\,\beta}{\Delta^2}\int x^{2n}\,J_0(\alpha x)\,J_0(\beta x)\,dx-\frac{(4n^2-1)\,\sigma}{\Delta^2}\int x^{2n}\,J_1(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^{2n+2}\, I_0(\alpha x)\, I_0(\beta x)\, dx = -\frac{(2n+1)\,\sigma\, x^{2n+1}}{\Delta^2}\, I_0(\alpha x)\, I_0(\beta x) - \frac{\beta\, x^{2n+2}}{\Delta}\, I_0(\alpha x)\, I_1(\beta x) +$$

$$+\frac{\alpha\, x^{2n+2}}{\Delta}\, I_1(\alpha x)\, I_0(\beta x) + \frac{(4n+2)\,\alpha\,\beta\, x^{2n+1}}{\Delta^2}\, I_1(\alpha x)\, I_1(\beta x) +$$

$$+\frac{(2n+1)^2\,\sigma}{\Delta^2}\int x^{2n}\, I_0(\alpha x)\, I_0(\beta x)\, dx - \frac{(8n^2-2)\,\alpha\,\beta}{\Delta^2}\int x^{2n}\, I_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x^{2n+2}\, I_1(\alpha x)\, I_1(\beta x)\, dx = \frac{(4n+2)\,\alpha\,\beta\, x^{2n+1}}{\Delta^2}\, I_0(\alpha x)\, I_0(\beta x) + \frac{\alpha\, x^{2n+2}}{\Delta}\, I_0(\alpha x)\, I_1(\beta x) -$$

$$-\frac{\beta\, x^{2n+2}}{\Delta}\, I_1(\alpha x)\, I_0(\beta x) - \frac{(2n+1)\,\sigma\, x^{2n+1}}{\Delta^2}\, I_1(\alpha x)\, I_1(\beta x) -$$

$$-\frac{2\,(2n+1)^2\,\alpha\,\beta}{\Delta^2}\int x^{2n}\, I_0(\alpha x)\, I_0(\beta x)\, dx + \frac{(4n^2-1)\,\sigma}{\Delta^2}\int x^{2n}\, I_1(\alpha x)\, I_1(\beta x)\, dx$$

2.2.3. Integrals of the type $\int x^{2n} Z_0(\alpha x)\, Z_1(\beta x)\, dx$ and $\int x^{2n} V_0(\alpha x)\, W_1(\beta x)\, dx$, $\alpha^2 \neq \beta^2$

See the remark in 2.2.4, page 325.

Let

$$\boxed{\alpha^2 + \beta^2 = \sigma \qquad \text{and} \qquad \alpha^2 - \beta^2 = \Delta\ .}$$

$$\int x^2 \cdot J_0(\alpha x) J_1(\beta x)\, dx \ =$$

$$= \ \frac{2\beta x}{\Delta^2} \left[\beta J_0(\alpha x) J_1(\beta x) - \alpha J_1(\alpha x) J_0(\beta x) \right] + \frac{x^2}{\Delta} \left[\beta J_0(\alpha x) J_0(\beta x) + \alpha J_1(\alpha x) J_1(\beta x) \right]$$

$$\int x^2 \cdot I_0(\alpha x) I_1(\beta x)\, dx \ =$$

$$= \ -\frac{2\beta x}{\Delta^2} \left[\beta I_0(\alpha x) I_1(\beta x) - \alpha I_1(\alpha x) I_0(\beta x) \right] - \frac{x^2}{\Delta} \left[\beta I_0(\alpha x) I_0(\beta x) - \alpha I_1(\alpha x) I_1(\beta x) \right]$$

$$\int x^2 \cdot K_0(\alpha x) K_1(\beta x)\, dx \ =$$

$$= \ -\frac{2\beta x}{\Delta^2} \left[\beta K_0(\alpha x) K_1(\beta x) - \alpha K_1(\alpha x) K_0(\beta x) \right] + \frac{x^2}{\Delta} \left[\beta K_0(\alpha x) K_0(\beta x) - \alpha K_1(\alpha x) K_1(\beta x) \right]$$

$$\int x^2 \cdot J_0(\alpha x) I_1(\beta x)\, dx \ =$$

$$= \ -\frac{2\beta x}{\sigma^2} \left[\beta J_0(\alpha x) I_1(\beta x) + \alpha J_1(\alpha x) I_0(\beta x) \right] + \frac{x^2}{\sigma} \left[\beta J_0(\alpha x) I_0(\beta x) + \alpha J_1(\alpha x) I_1(\beta x) \right]$$

$$\int x^2 \cdot J_1(\alpha x) I_0(\beta x)\, dx \ =$$

$$= \ \frac{2\alpha x}{\sigma^2} \left[\beta J_0(\alpha x) I_1(\beta x) + \alpha J_1(\alpha x) I_0(\beta x) \right] - \frac{x^2}{\sigma} \left[\alpha J_0(\alpha x) I_0(\beta x) - \beta J_1(\alpha x) I_1(\beta x) \right]$$

$$\int x^2 \cdot J_0(\alpha x) K_1(\beta x)\, dx \ =$$

$$= \ -\frac{2\beta x}{\sigma^2} \left[\beta J_0(\alpha x) K_1(\beta x) - \alpha J_1(\alpha x) K_0(\beta x) \right] - \frac{x^2}{\sigma} \left[\beta J_0(\alpha x) K_0(\beta x) - \alpha J_1(\alpha x) K_1(\beta x) \right]$$

$$\int x^2 \cdot J_1(\alpha x) K_0(\beta x)\, dx \ =$$

$$= \ -\frac{2\alpha x}{\sigma^2} \left[\beta J_0(\alpha x) K_1(\beta x) - \alpha J_1(\alpha x) K_0(\beta x) \right] - \frac{x^2}{\sigma} \left[\alpha J_0(\alpha x) K_0(\beta x) - \beta J_1(\alpha x) K_1(\beta x) \right]$$

$$\int x^2 \cdot I_0(\alpha x) K_1(\beta x)\, dx \ =$$

$$= \ -\frac{2\beta x}{\Delta^2} \left[\beta I_0(\alpha x) K_1(\beta x) + \alpha I_1(\alpha x) K_0(\beta x) \right] + \frac{x^2}{\Delta} \left[\beta I_0(\alpha x) K_0(\beta x) + \alpha I_1(\alpha x) K_1(\beta x) \right]$$

$$\int x^2 \cdot I_1(\alpha x) K_0(\beta x)\, dx \ =$$

$$= \ -\frac{2\alpha x}{\Delta^2} \left[\beta I_0(\alpha x) K_1(\beta x) + \alpha I_1(\alpha x) K_0(\beta x) \right] + \frac{x^2}{\Delta} \left[\alpha I_0(\alpha x) K_0(\beta x) + \beta I_1(\alpha x) K_1(\beta x) \right]$$

Let

$$\int x^m\, U_0(\alpha x) W_1(\beta x)\, dx =$$

$$= P_m^{[UW]}(x) U_0(\alpha x) W_0(\beta x) + Q_m^{[UW]}(x) U_0(\alpha x) W_1(\beta x) + R_m^{[UW]}(x) U_1(\alpha x) W_0(\beta x) + S_m^{[UW]}(x) U_1(\alpha x) W_1(\beta x)\ .$$

One has

$$P_m^{[JJ]} = P_m^{[YY]} = P_m^{[H^{(1)}H^{(1)}]} = P_m^{[H^{(2)}H^{(2)}]} = P_m^{[JY]} = P_m^{[JH^{(1)}]} = P_m^{[JH^{(2)}]} = P_m^{[YH^{(1)}]} = P_m^{[YH^{(2)}]} = P_m^{[H^{(1)}H^{(2)}]}\ ,$$

$$P_m^{[JI]} = P_m^{[YI]} = P_m^{[H^{(1)}I]} = P_m^{[H^{(2)}I]} \quad \text{and} \quad P_m^{[JK]} = P_m^{[YK]} = P_m^{[H^{(1)}K]} = P_m^{[H^{(2)}K]} .$$

The same holds for the polynomials $Q_m(x),\ R_m(x)$ and $S_m(x)$.

$$P_4^{[JJ]}(x) = \frac{\beta x^4}{\Delta} - \frac{8\beta\,(2\alpha^2+\beta^2)\,x^2}{\Delta^3} \quad , \qquad Q_4^{[JJ]}(x) = \frac{2\,(\alpha^2+2\,\beta^2)\,x^3}{\Delta^2} - \frac{16\beta^2(2\alpha^2+\beta^2)\,x}{\Delta^4}$$

$$R_4^{[JJ]}(x) = -\frac{6\,\alpha\,\beta\,x^3}{\Delta^2} + \frac{16\,\alpha\,\beta\,(2\alpha^2+\beta^2)\,x}{\Delta^4} \quad , \qquad S_4^{[JJ]}(x) = \frac{\alpha x^4}{\Delta} - \frac{4\,\alpha\,(\alpha^2+5\,\beta^2)\,x^2}{\Delta^3}$$

$$P_4^{[II]}(x) = -P_4^{[KK]}(x) = -\frac{\beta\,x^4}{\Delta} - \frac{8\,\beta\,(2\alpha^2+\beta^2)\,x^2}{\Delta^3}$$

$$Q_4^{[II]}(x) = Q_4^{[KK]}(x) = -\frac{2\,(\alpha^2+2\,\beta^2)\,x^3}{\Delta^2} - \frac{16\,\beta^2\,(2\,\alpha^2+\beta^2)\,x}{\Delta^4}$$

$$R_4^{[II]}(x) = R_4^{[KK]}(x) = \frac{6\,\alpha\,\beta\,x^3}{\Delta^2} + \frac{16\,\alpha\,\beta\,(2\,\alpha^2+\beta^2)\,x}{\Delta^4}$$

$$S_4^{[II]}(x) = -S_4^{[KK]}(x) = \frac{\alpha\,x^4}{\Delta} + \frac{4\,\alpha\,(\alpha^2+5\,\beta^2)\,x^2}{\Delta^3}$$

$$P_4^{[JI]}(x) = -P_4^{[JK]}(x) = \frac{\beta\,x^4}{\sigma} - \frac{8\,\beta\,(2\,\alpha^2-\beta^2)\,x^2}{\sigma^3}$$

$$Q_4^{[JI]}(x) = Q_4^{[JK]}(x) = \frac{2\,(\alpha^2-2\,\beta^2)\,x^3}{\sigma^2} + \frac{16\,\beta^2(2\alpha^2-\beta^2)\,x}{\sigma^4}$$

$$R_4^{[JI]}(x) = -R_4^{[JK]}(x) = -\frac{6\,\alpha\,\beta\,x^3}{\sigma^2} + \frac{16\,\alpha\,\beta\,(2\,\alpha^2-\beta^2)\,x}{\sigma^4}$$

$$S_4^{[JI]}(x) = S_4^{[JK]}(x) = \frac{\alpha\,x^4}{\sigma} - \frac{4\,\alpha\,(\alpha^2-5\,\beta^2)\,x^2}{\sigma^3}$$

$$P_4^{[IJ]}(x) = P_4^{[KJ]}(x) = -\frac{\beta\,x^4}{\sigma} - \frac{8\,\beta\,(2\,\alpha^2-\beta^2)\,x^2}{\sigma^3}$$

$$Q_4^{[IJ]}(x) = Q_4^{[KJ]}(x) = -\frac{2\,(\alpha^2-2\,\beta^2)\,x^3}{\sigma^2} + \frac{16\,\beta^2(2\,\alpha^2-\beta^2)\,x}{\sigma^4}$$

$$R_4^{[IJ]}(x) = -R_4^{[KJ]}(x) = \frac{6\,\alpha\,\beta\,x^3}{\sigma^2} + \frac{16\,\alpha\,\beta\,(2\,\alpha^2-\beta^2)\,x}{\sigma^4}$$

$$S_4^{[IJ]}(x) = -S_4^{[KJ]}(x) = \frac{\alpha\,x^4}{\sigma} + \frac{4\,\alpha\,(\alpha^2-5\,\beta^2)\,x^2}{\sigma^3}$$

$$P_4^{[IK]}(x) = -P_4^{[KI]}(x) = \frac{\beta\,x^4}{\Delta} + \frac{8\,\beta\,(2\alpha^2+\beta^2)\,x^2}{\Delta^3}$$

$$Q_4^{[IK]}(x) = Q_4^{[KI]}(x) = -\frac{2\,(\alpha^2+2\,\beta^2)\,x^3}{\Delta^2} - \frac{16\,\beta^2(2\alpha^2+\beta^2)\,x}{\Delta^4}$$

$$R_4^{[IK]}(x) = R_4^{[KI]}(x) = -\frac{6\,\alpha\,\beta\,x^3}{\Delta^2} - \frac{16\,\alpha\,\beta\,(2\,\alpha^2+\beta^2)\,x}{\Delta^4}$$

$$S_4^{[IK]}(x) = -S_4^{[KI]}(x) = \frac{\alpha\,x^4}{\Delta} + \frac{4\,\alpha\,(\alpha^2+5\,\beta^2)\,x^2}{\Delta^3}$$

$$P_6^{[JJ]}(x) = \frac{\beta\,x^6}{\Delta} - \frac{8\,\beta\,(7\,\alpha^2+3\,\beta^2)\,x^4}{\Delta^3} + \frac{192\,\beta\,(3\,\sigma^2-2\alpha^4\beta^4)\,x^2}{\Delta^5}$$

$$Q_6^{[JJ]}(x) = \frac{2\,(2\,\alpha^2+3\,\beta^2)\,x^5}{\Delta^2} - \frac{32\,(\alpha^4+11\,\alpha^2\beta^2+3\,\beta^4)\,x^3}{\Delta^4} + \frac{384\,\beta^2(3\,\sigma^2-2\beta^4)\,x}{\Delta^6}$$

$$R_6^{[JJ]}(x) = -\frac{10\,\alpha\,\beta\,x^5}{\Delta^2} + \frac{32\,\alpha\,\beta\,(8\,\alpha^2+7\,\beta^2)\,x^3}{\Delta^4} - \frac{384\,\alpha\,\beta\,(3\,\sigma^2-2\beta^4)\,x}{\Delta^6}$$

$$S_6^{[JJ]}(x) = \frac{\alpha\, x^6}{\Delta} - \frac{16\,\alpha\,(\alpha^2+4\,\beta^2)\,x^4}{\Delta^3} + \frac{64\,\alpha\,(\alpha^4+19\,\alpha^2\beta^2+10\,\beta^4)\,x^2}{\Delta^5}$$

$$P_6^{[II]}(x) = -\frac{\beta\, x^6}{\Delta} - \frac{8\,\beta\,(7\,\alpha^2+3\,\beta^2)\,x^4}{\Delta^3} - \frac{192\,\beta\,(3\,\sigma^2-2\,\beta^4)\,x^2}{\Delta^5}$$

$$P_6^{[KK]}(x) = \frac{\beta\, x^6}{\Delta} + \frac{8\,\beta\,(7\,\alpha^2+3\,\beta^2)\,x^4}{\Delta^3} - \frac{192\,\beta\,(3\,\sigma^2-2\,\beta^4)\,x^2}{\Delta^5}$$

$$Q_6^{[II]}(x) = Q_6^{[KK]}(x) = -\frac{2\,(2\,\alpha^2+3\,\beta^2)\,x^5}{\Delta^2} - \frac{32\,(\alpha^4+11\,\alpha^2\beta^2+3\,\beta^4)\,x^3}{\Delta^4} - \frac{384\,\beta^2(3\,\sigma^2-2\,\beta^4)\,x}{\Delta^6}$$

$$R_6^{[II]}(x) = R_6^{[KK]}(x) = \frac{10\,\alpha\,\beta\,x^5}{\Delta^2} + \frac{32\,\alpha\,\beta\,(8\,\alpha^2+7\,\beta^2)\,x^3}{\Delta^4} + \frac{384\,\alpha\,\beta\,(3\,\sigma^2-2\,\beta^4)\,x}{\Delta^6}$$

$$S_6^{[II]}(x) = -S_6^{[KK]}(x) = \frac{\alpha\, x^6}{\Delta} + \frac{16\,\alpha\,(\alpha^2+4\,\beta^2)\,x^4}{\Delta^3} + \frac{64\,\alpha\,(\alpha^4+19\,\alpha^2\beta^2+10\,\beta^4)\,x^2}{\Delta^5}$$

$$P_6^{[JI]}(x) = \frac{\beta\, x^6}{\sigma} - \frac{8\,\beta\,(7\,\alpha^2-3\,\beta^2)\,x^4}{\sigma^3} + \frac{192\,\beta\,(3\,\Delta^2-2\beta^4)\,x^2}{\sigma^5}$$

$$P_6^{[JK]}(x) = -\frac{\beta\, x^6}{\sigma} + \frac{8\,\beta\,(7\,\alpha^2-3\,\beta^2)\,x^4}{\sigma^3} + \frac{192\,\beta\,(3\,\Delta^2-2\beta^4)\,x^2}{\sigma^5}$$

$$Q_6^{[JI]}(x) = Q_6^{[JK]}(x) = \frac{2\,(2\,\alpha^2-3\,\beta^2)\,x^5}{\sigma^2} - \frac{32\,(\alpha^4-11\,\alpha^2\beta^2+3\,\beta^4)\,x^3}{\sigma^4} - \frac{384\,\beta^2(3\,\Delta^2-2\beta^4)\,x}{\sigma^6}$$

$$R_6^{[JI]}(x) = -\frac{10\,\alpha\,\beta\,x^5}{\sigma^2} + \frac{32\,\alpha\,\beta\,(8\,\alpha^2-7\,\beta^2)\,x^3}{\sigma^4} - \frac{384\,\alpha\,\beta\,(3\,\Delta^2-2\beta^4)\,x}{\sigma^6}$$

$$R_6^{[JK]}(x) = \frac{10\,\alpha\,\beta\,x^5}{\sigma^2} - \frac{32\,\alpha\,\beta\,(8\,\alpha^2-7\,\beta^2)\,x^3}{\sigma^4} - \frac{384\,\alpha\,\beta\,(3\,\Delta^2-2\beta^4)\,x}{\sigma^6}$$

$$S_6^{[JI]}(x) = S_6^{[JK]}(x) = \frac{\alpha\, x^6}{\sigma} - \frac{16\,\alpha\,(\alpha^2-4\,\beta^2)\,x^4}{\sigma^3} + \frac{64\,\alpha\,(\alpha^4-19\,\alpha^2\beta^2+10\,\beta^4)\,x^2}{\sigma^5}$$

$$P_6^{[IJ]}(x) = P_6^{[KJ]}(x) = -\frac{\beta\, x^6}{\sigma} - \frac{8\,\beta\,(7\,\alpha^2-3\,\beta^2)\,x^4}{\sigma^3} - \frac{192\,\beta\,(3\,\Delta^2-2\beta^4)\,x^2}{\sigma^5}$$

$$Q_6^{[IJ]}(x) = Q_6^{[KJ]}(x) = -\frac{2\,(2\,\alpha^2-3\,\beta^2)\,x^5}{\sigma^2} - \frac{32\,(\alpha^4-11\,\alpha^2\,\beta^2+3\,\beta^4)\,x^3}{\sigma^4} + \frac{384\,\beta^2(3\,\Delta^2-2\beta^4)\,x}{\sigma^6}$$

$$R_6^{[IJ]}(x) = \frac{10\,\alpha\,\beta\,x^5}{\sigma^2} + \frac{32\,\alpha\,\beta\,(8\,\alpha^2-7\,\beta^2)\,x^3}{\sigma^4} + \frac{384\,\alpha\,\beta\,(3\,\Delta^2-2\beta^4)\,x}{\sigma^6}$$

$$R_6^{[KJ]}(x) = -\frac{10\,\alpha\,\beta\,x^5}{\sigma^2} - \frac{32\,\alpha\,\beta\,(8\,\alpha^2-7\,\beta^2)\,x^3}{\sigma^4} + \frac{384\,\alpha\,\beta\,(3\,\Delta^2-2\beta^4)\,x}{\sigma^6}$$

$$S_6^{[IJ]}(x) = -S_6^{[KJ]}(x) = \frac{\alpha\, x^6}{\sigma} + \frac{16\,\alpha\,(\alpha^2-4\,\beta^2)\,x^4}{\sigma^3} + \frac{64\,\alpha\,(\alpha^4-19\,\alpha^2\,\beta^2+10\,\beta^4)\,x^2}{\sigma^5}$$

$$P_6^{[IK]}(x) = \frac{\beta\, x^6}{\Delta} + \frac{8\,\beta\,(7\,\alpha^2+3\,\beta^2)\,x^4}{\Delta^3} + \frac{192\,\beta\,(3\,\sigma^2-2\beta^2)\,x^2}{\Delta^5}$$

$$P_6^{[KI]}(x) = -\frac{\beta\, x^6}{\Delta} - \frac{8\,\beta\,(7\,\alpha^2+3\,\beta^2)\,x^4}{\Delta^3} + \frac{192\,\beta\,(3\,\sigma^2-2\beta^2)\,x^2}{\Delta^5}$$

$$Q_6^{[IK]}(x) = Q_6^{[KI]}(x) = -\frac{2\,(2\,\alpha^2+3\,\beta^2)\,x^5}{\Delta^2} - \frac{32\,(\alpha^4+11\,\alpha^2\beta^2+3\,\beta^4)\,x^3}{\Delta^4} - \frac{384\,\beta^2(3\,\sigma^2-2\beta^2)\,x}{\Delta^6}$$

$$R_6^{[IK]}(x) = R_6^{[KI]}(x) = -\frac{10\,\alpha\,\beta\,x^5}{\Delta^2} - \frac{32\,\alpha\,\beta\,(8\,\alpha^2+7\,\beta^2)\,x^3}{\Delta^4} - \frac{384\,\alpha\,\beta\,(3\,\sigma^2-2\beta^2)\,x}{\Delta^6}$$

$$S_6^{[IK]}(x) = -S_6^{[KI]}(x) = \frac{\alpha\, x^6}{\Delta} + \frac{16\,\alpha\,(\alpha^2+4\,\beta^2)\,x^4}{\Delta^3} + \frac{64\,\alpha\,(\alpha^4+19\,\alpha^2\beta^2+10\,\beta^4)\,x^2}{\Delta^5}$$

$$P_8^{[JJ]}(x) = \frac{\beta}{\Delta}\,x^8 - \frac{24\,\beta\,(5\,\alpha^2+2\,\beta^2)\,x^6}{\Delta^3} + \frac{192\,\beta\,(21\,\alpha^4+43\,\alpha^2\beta^2+6\,\beta^4)\,x^4}{\Delta^5} -$$

$$-\frac{9216\,\beta\,(4\,\alpha^6+18\,\alpha^4\beta^2+12\,\alpha^2\beta^4+\beta^6)\,x^2}{\Delta^7}$$

$$Q_8^{[JJ]}(x)=\frac{2\,(3\,\alpha^2+4\,\beta^2)\,x^7}{\Delta^2}-\frac{48\,(3\,\alpha^4+26\,\alpha^2\beta^2+6\,\beta^4)\,x^5}{\Delta^4}+\frac{384\,(3\,\alpha^6+86\,\alpha^4\beta^2+109\,\alpha^2\beta^4+12\,\beta^6)\,x^3}{\Delta^6}-$$

$$-\frac{18432\,\beta^2(4\,\alpha^6+18\,\alpha^4\beta^2+12\,\alpha^2\beta^4+\beta^6)\,x}{\Delta^8}$$

$$R_8^{[JJ]}(x)=-\frac{14\,\alpha\,\beta\,x^7}{\Delta^2}+\frac{48\,\alpha\,\beta\,(18\,\alpha^2+17\,\beta^2)\,x^5}{\Delta^4}-\frac{1920\,\alpha\,\beta\,(9\,\alpha^4+26\,\alpha^2\beta^2+7\,\beta^4)\,x^3}{\Delta^6}+$$

$$+\frac{18432\,\alpha\,\beta\,(4\,\alpha^6+18\,\alpha^4\beta^2+12\,\alpha^2\beta^4+\beta^6)\,x}{\Delta^8}$$

$$S_8^{[JJ]}(x)=\frac{\alpha\,x^8}{\Delta}-\frac{12\,\alpha\,(3\,\alpha^2+11\,\beta^2)\,x^6}{\Delta^3}+\frac{192\,\alpha\,(3\,\alpha^4+44\,\alpha^2\beta^2+23\,\beta^4)\,x^4}{\Delta^5}-$$

$$-\frac{768\,\alpha\,(3\,\alpha^6+131\,\alpha^4\beta^2+239\,\alpha^2\beta^4+47\,\beta^6)\,x^2}{\Delta^7}$$

$$P_8^{[II]}(x)=-\frac{\beta\,x^8}{\Delta}-\frac{24\,\beta\,(5\,\alpha^2+2\,\beta^2)\,x^6}{\Delta^3}-\frac{192\,\beta\,(21\,\alpha^4+43\,\alpha^2\beta^2+6\,\beta^4)\,x^4}{\Delta^5}-$$

$$-\frac{9216\,\beta\,(4\,\alpha^6+18\,\alpha^4\beta^2+12\,\alpha^2\beta^4+\beta^6)\,x^2}{\Delta^7}$$

$$P_8^{[KK]}(x)=\frac{\beta\,x^8}{\Delta}+\frac{24\,\beta\,(5\,\alpha^2+2\,\beta^2)\,x^6}{\Delta^3}+\frac{192\,\beta\,(21\,\alpha^4+43\,\alpha^2\beta^2+6\,\beta^4)\,x^4}{\Delta^5}+$$

$$+\frac{9216\,\beta\,(4\,\alpha^6+18\,\alpha^4\beta^2+12\,\alpha^2\beta^4+\beta^6)\,x^2}{\Delta^7}$$

$$Q_8^{[II]}(x)=Q_8^{[KK]}(x)=-\frac{2\,(3\,\alpha^2+4\,\beta^2)\,x^7}{\Delta^2}-\frac{48\,(3\,\alpha^4+26\,\alpha^2\beta^2+6\,\beta^4)\,x^5}{\Delta^4}-$$

$$-\frac{384\,(3\,\alpha^6+86\,\alpha^4\beta^2+109\,\alpha^2\beta^4+12\,\beta^6)\,x^3}{\Delta^6}-\frac{18432\,\beta^2(4\,\alpha^6+18\,\alpha^4\beta^2+12\,\alpha^2\beta^4+\beta^6)\,x}{\Delta^8}$$

$$R_8^{[II]}(x)=R_8^{[KK]}(x)=\frac{14\,\alpha\,\beta\,x^7}{\Delta^2}+\frac{48\,\alpha\,\beta\,(18\,\alpha^2+17\,\beta^2)\,x^5}{\Delta^4}+\frac{1920\,\alpha\,\beta\,(9\,\alpha^4+26\,\alpha^2\beta^2+7\,\beta^4)\,x^3}{\Delta^6}+$$

$$+\frac{18432\,\alpha\,\beta\,(4\,\alpha^6+18\,\alpha^4\beta^2+12\,\alpha^2\beta^4+\beta^6)\,x}{\Delta^8}$$

$$S_8^{[II]}(x)=-S_8^{[KK]}(x)=\frac{\alpha\,x^8}{\Delta}+\frac{12\,\alpha\,(3\,\alpha^2+11\,\beta^2)\,x^6}{\Delta^3}+\frac{192\,\alpha\,(3\,\alpha^4+44\,\alpha^2\beta^2+23\,\beta^4)\,x^4}{\Delta^5}+$$

$$\frac{+768\,\alpha\,(3\,\alpha^6+131\,\alpha^4\beta^2+239\,\alpha^2\beta^4+47\,\beta^6)\,x^2}{\Delta^7}$$

$$P_8^{[JI]}(x)=-P_8^{[JK]}(x)=\frac{\beta\,x^8}{\sigma}-\frac{24\,\beta\,(5\,\alpha^2-2\,\beta^2)\,x^6}{\sigma^3}+\frac{192\,\beta\,(21\,\alpha^4-43\,\alpha^2\,\beta^2+6\,\beta^4)\,x^4}{\sigma^5}-$$

$$-\frac{9216\,\beta\,(4\,\alpha^6-18\,\alpha^4\,\beta^2+12\,\alpha^2\beta^4-\beta^6)\,x^2}{\sigma^7}$$

$$Q_8^{[JI]}(x)=Q_8^{[JK]}(x)=\frac{2\,(3\,\alpha^2-4\,\beta^2)\,x^7}{\sigma^2}-\frac{48\,(3\,\alpha^4-26\,\alpha^2\,\beta^2+6\,\beta^4)\,x^5}{\sigma^4}+$$

$$+\frac{384\,(3\,\alpha^6-86\,\alpha^4\,\beta^2+109\,\alpha^2\beta^4-12\,\beta^6)\,x^3}{\sigma^6}+\frac{18432\,\beta^2(4\,\alpha^6-18\,\alpha^4\,\beta^2+12\,\alpha^2\beta^4-\beta^6)\,x}{\sigma^8}$$

$$R_8^{[JI]}(x)=-R_8^{[JK]}(x)=-\frac{14\,\alpha\,\beta\,x^7}{\sigma^2}+\frac{48\,\alpha\,\beta\,(18\,\alpha^2-17\,\beta^2)\,x^5}{\sigma^4}-\frac{1920\,\alpha\,\beta\,(9\,\alpha^4-26\,\alpha^2\,\beta^2+7\,\beta^4)\,x^3}{\sigma^6}+$$

$$+\frac{18432\,\alpha\,\beta\,(4\,\alpha^6-18\,\alpha^4\,\beta^2+12\,\alpha^2\,\beta^4-\beta^6)\,x}{\sigma^8}$$

$$S_8^{[JI]}(x)=S_8^{[JK]}(x)=\frac{\alpha\,x^8}{\sigma}-\frac{12\,\alpha\,(3\,\alpha^2-11\,\beta^2)\,x^6}{\sigma^3}+\frac{192\,\alpha\,(3\,\alpha^4-44\,\alpha^2\,\beta^2+23\,\beta^4)\,x^4}{\sigma^5}-$$

$$-\frac{768\,\alpha\,(3\,\alpha^6-131\,\alpha^4\,\beta^2+239\,\alpha^2\,\beta^4-47\,\beta^6)\,x^2}{\sigma^7}$$

$$P_8^{[IJ]}(x)=P_8^{[KJ]}(x)=-\frac{\beta\,x^8}{\sigma}-\frac{24\,\beta\,(5\,\alpha^2-2\,\beta^2)\,x^6}{\sigma^3}-\frac{192\,\beta\,(21\,\alpha^4-43\,\alpha^2\,\beta^2+6\,\beta^4)\,x^4}{\sigma^5}-$$
$$-\frac{9216\,\beta\,(4\,\alpha^6-18\alpha^4\,\beta^2+12\,\alpha^2\,\beta^4-\beta^6)\,x^2}{\sigma^7}$$

$$Q_8^{[IJ]}(x)=Q_8^{[KJ]}(x)=-\frac{2\,(3\,\alpha^2-4\,\beta^2)\,x^7}{\sigma^2}-\frac{48\,(3\,\alpha^4-26\,\alpha^2\,\beta^2+6\,\beta^4)\,x^5}{\sigma^4}-$$
$$-\frac{384\,(3\,\alpha^6-86\alpha^4\,\beta^2+109\,\alpha^2\,\beta^4-12\,\beta^6)\,x^3}{\sigma^6}+\frac{18432\,\beta^2(4\,\alpha^6-18\alpha^4\,\beta^2+12\,\alpha^2\,\beta^4-\beta^6)\,x}{\sigma^8}$$

$$R_8^{[IJ]}(x)=-R_8^{[KJ]}(x)=\frac{14\,\alpha\,\beta\,x^7}{\sigma^2}+\frac{48\,\alpha\,\beta\,(18\,\alpha^2-17\,\beta^2)\,x^5}{\sigma^4}+\frac{1920\,\alpha\,\beta\,(9\,\alpha^4-26\,\alpha^2\,\beta^2+7\,\beta^4)\,x^3}{\sigma^6}+$$
$$+\frac{18432\,\alpha\,\beta\,(4\,\alpha^6-18\alpha^4\,\beta^2+12\,\alpha^2\,\beta^4-\beta^6)\,x}{\sigma^8}$$

$$S_8^{[IJ]}(x)=-S_8^{[KJ]}(x)=\frac{\alpha\,x^8}{\sigma}+\frac{12\,\alpha\,(3\,\alpha^2-11\,\beta^2)\,x^6}{\sigma^3}+\frac{192\,\alpha\,(3\,\alpha^4-44\,\alpha^2\,\beta^2+23\,\beta^4)\,x^4}{\sigma^5}+$$
$$+\frac{768\,\alpha\,(3\,\alpha^6-131\alpha^4\,\alpha^2+239\,\alpha^2\,\beta^4-47\,\beta^6)\,x^2}{\sigma^7}$$

$$P_8^{[IK]}(x)=\frac{\beta\,x^8}{\Delta}+\frac{24\,\beta\,(5\,\alpha^2+2\,\beta^2)\,x^6}{\Delta^3}+\frac{192\,\beta\,(21\,\alpha^4+43\,\alpha^2\,\beta^2+6\,\beta^4)\,x^4}{\Delta^5}+$$
$$+\frac{9216\,\beta\,(4\,\alpha^6+18\,\alpha^4\,\beta^2+12\,\alpha^2\beta^4+\beta^6)\,x^2}{\Delta^7}$$

$$P_8^{[KI]}(x)=-\frac{\beta\,x^8}{\Delta}-\frac{24\,\beta\,(5\,\alpha^2+2\,\beta^2)\,x^6}{\Delta^3}-\frac{192\,\beta\,(21\,\alpha^4+43\,\alpha^2\,\beta^2+6\,\beta^4)\,x^4}{\Delta^5}-$$
$$-\frac{9216\,\beta\,(4\,\alpha^6+18\,\alpha^4\,\beta^2+12\,\alpha^2\beta^4+\beta^6)\,x^2}{\Delta^7}$$

$$Q_8^{[IK]}(x)=Q_8^{[KI]}(x)=-\frac{2\,(3\,\alpha^2+4\,\beta^2)\,x^7}{\Delta^2}-\frac{48\,(3\,\alpha^4+26\,\alpha^2\,\beta^2+6\,\beta^4)\,x^5}{\Delta^4}-$$
$$-\frac{384\,(3\,\alpha^6+86\,\alpha^4\,\beta^2+109\,\alpha^2\beta^4+12\,\beta^6)\,x^3}{\Delta^6}-\frac{18432\,\beta^2(4\,\alpha^6+18\,\alpha^4\,\beta^2+12\,\alpha^2\beta^4+\beta^6)\;x}{\Delta^8}$$

$$R_8^{[IK]}(x)=R_8^{[KI]}(x)=-\frac{14\,\alpha\,\beta\,x^7}{\Delta^2}-\frac{48\,\alpha\,\beta\,(18\,\alpha^2+17\,\beta^2)\,x^5}{\Delta^4}-$$
$$-\frac{1920\,\alpha\,\beta\,(9\,\alpha^4+26\,\alpha^2\,\beta^2+7\,\beta^4)\,x^3}{\Delta^6}-\frac{18432\,\alpha\,\beta\,(4\,\alpha^6+18\,\alpha^4\,\beta^2+12\,\alpha^2\beta^4+\beta^6)\,x}{\Delta^8}$$

$$S_8^{[IK]}(x)=-S_8^{[KI]}(x)=\frac{\alpha\,x^8}{\Delta}+\frac{12\,\alpha\,(3\,\alpha^2+11\,\beta^2)\,x^6}{\Delta^3}+\frac{192\,\alpha\,(3\,\alpha^4+44\,\alpha^2\,\beta^2+23\,\beta^4)\,x^4}{\Delta^5}+$$
$$+\frac{768\,\alpha\,(3\,\alpha^6+131\,\alpha^4\,\beta^2+239\,\alpha^2\beta^4+47\,\beta^6)\,x^2}{\Delta^7}$$

Recurrence relations: See also page 302.

$$\int x^{2n+2}\,J_0(\alpha x)\,J_1(\beta x)\,dx=x^{2n-1}\,\left\{\left[\frac{\beta\,x^3}{\Delta}-\frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\Delta^2}\right]\,J_0(\alpha x)\,J_0(\beta x)+\right.$$
$$+\left[\frac{2\,(n\,\sigma+\beta^2)\,x^2}{\Delta^2}+\frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\Delta^2}\right]\,J_0(\alpha x)\,J_1(\beta x)-\frac{2\,(2n+1)\alpha\beta\,x^2}{\Delta^2}\,J_1(\alpha x)\,J_0(\beta x)+$$
$$\left.+\left[\frac{\alpha\,x^3}{\Delta}+\frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\Delta^2}\right]\,J_1(\alpha x)\,J_1(\beta x)\right\}-$$

$$-\frac{8\,(2n^2\,\sigma-\beta^2)n}{(2n-1)\,\Delta^2}\int x^{2n}\,J_0(\alpha x)\,J_1(\beta x)\,dx-\frac{16\,(n-1)^2n^2(2n+1)}{(2n-1)\,\Delta^2}\int x^{2n-2}\,J_0(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^{2n+2}\,J_0(\alpha x)\,I_1(\beta x)\,dx=x^{2n-1}\left\{\left[\frac{\beta\,x^3}{\sigma}-\frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\sigma^2}\right]J_0(\alpha x)\,I_0(\beta x)+\right.$$

$$+\left[\frac{2\,(n\,\Delta-\beta^2)\,x^2}{\sigma^2}+\frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\sigma^2}\right]J_0(\alpha x)\,I_1(\beta x)-\frac{2\,(2n+1)\alpha\beta\,x^2}{\sigma^2}J_1(\alpha x)\,I_0(\beta x)+$$

$$\left.+\left[\frac{\alpha\,x^3}{\sigma}+\frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\sigma^2}\right]J_1(\alpha x)\,I_1(\beta x)\right\}-$$

$$-\frac{8\,(2n^2\,\Delta+\beta^2)n}{(2n-1)\,\sigma^2}\int x^{2n}\,J_0(\alpha x)\,I_1(\beta x)\,dx-\frac{16\,(n-1)^2n^2(2n+1)}{(2n-1)\,\sigma^2}\int x^{2n-2}\,J_0(\alpha x)\,I_1(\beta x)\,dx$$

$$\int x^{2n+2}\,J_0(\alpha x)\,K_1(\beta x)\,dx=x^{2n-1}\left\{-\left[\frac{\beta\,x^3}{\sigma}-\frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\sigma^2}\right]J_0(\alpha x)\,K_0(\beta x)+\right.$$

$$+\left[\frac{2\,(n\,\sigma-\beta^2)\,x^2}{\sigma^2}+\frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\sigma^2}\right]J_0(\alpha x)\,K_1(\beta x)+\frac{2\,(2n+1)\alpha\beta\,x^2}{\sigma^2}J_1(\alpha x)\,K_0(\beta x)+$$

$$\left.+\left[\frac{\alpha\,x^3}{\sigma}+\frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\sigma^2}\right]J_1(\alpha x)\,K_1(\beta x)\right\}-$$

$$-\frac{8\,(2n^2\,\sigma+\beta^2)n}{(2n-1)\,\sigma^2}\int x^{2n}\,J_0(\alpha x)\,K_1(\beta x)\,dx-\frac{16\,(n-1)^2n^2(2n+1)}{(2n-1)\,\sigma^2}\int x^{2n-2}\,J_0(\alpha x)\,K_1(\beta x)\,dx$$

$$\int x^{2n+2}\,I_0(\alpha x)\,J_1(\beta x)\,dx=x^{2n-1}\left\{-\left[\frac{\beta\,x^3}{\sigma}+\frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\sigma^2}\right]I_0(\alpha x)\,J_0(\beta x)-\right.$$

$$-\left[\frac{2\,(n\,\Delta-\beta^2)\,x^2}{\sigma^2}-\frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\sigma^2}\right]I_0(\alpha x)\,J_1(\beta x)+\frac{2\,(2n+1)\alpha\beta\,x^2}{\sigma^2}I_1(\alpha x)\,J_0(\beta x)+$$

$$\left.+\left[\frac{\alpha\,x^3}{\sigma}-\frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\sigma^2}\right]I_1(\alpha x)\,J_1(\beta x)\right\}+$$

$$+\frac{8\,(2n^2\,\Delta+\beta^2)n}{(2n-1)\,\sigma^2}\int x^{2n}\,I_0(\alpha x)\,J_1(\beta x)\,dx-\frac{16\,(n-1)^2n^2(2n+1)}{(2n-1)\,\sigma^2}\int x^{2n-2}\,I_0(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^{2n+2}\,I_0(\alpha x)\,I_1(\beta x)\,dx=x^{2n-1}\left\{-\left[\frac{\beta\,x^3}{\Delta}+\frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\Delta^2}\right]I_0(\alpha x)\,I_0(\beta x)-\right.$$

$$-\left[\frac{2\,(n\,\sigma+\beta^2)\,x^2}{\Delta^2}-\frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\Delta^2}\right]I_0(\alpha x)\,I_1(\beta x)+\frac{2\,(2n+1)\alpha\beta\,x^2}{\Delta^2}I_1(\alpha x)\,I_0(\beta x)+$$

$$\left.+\left[\frac{\alpha\,x^3}{\Delta}-\frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\Delta^2}\right]I_1(\alpha x)\,I_1(\beta x)\right\}+$$

$$+\frac{8\,(2n^2\,\sigma-\beta^2)n}{(2n-1)\,\Delta^2}\int x^{2n}\,I_0(\alpha x)\,I_1(\beta x)\,dx-\frac{16\,(n-1)^2n^2(2n+1)}{(2n-1)\,\Delta^2}\int x^{2n-2}\,I_0(\alpha x)\,I_1(\beta x)\,dx$$

$$\int x^{2n+2}\,I_0(\alpha x)\,K_1(\beta x)\,dx=x^{2n-1}\left\{\left[\frac{\beta\,x^3}{\Delta}+\frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\Delta^2}\right]I_0(\alpha x)\,K_0(\beta x)-\right.$$

$$-\left[\frac{2\,(n\,\sigma+\beta^2)\,x^2}{\Delta^2}-\frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\Delta^2}\right]I_0(\alpha x)\,K_1(\beta x)-\frac{2\,(2n+1)\alpha\beta\,x^2}{\Delta^2}I_1(\alpha x)\,K_0(\beta x)+$$

$$\left.+\left[\frac{\alpha\,x^3}{\Delta}-\frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\Delta^2}\right]I_1(\alpha x)\,K_1(\beta x)\right\}+$$

$$+\frac{8\,(2n^2\,\sigma-\beta^2)n}{(2n-1)\,\Delta^2}\int x^{2n}\,I_0(\alpha x)\,K_1(\beta x)\,dx-\frac{16\,(n-1)^2n^2(2n+1)}{(2n-1)\,\Delta^2}\int x^{2n-2}\,I_0(\alpha x)\,K_1(\beta x)\,dx$$

$$\int x^{2n+2}\,K_0(\alpha x)\,J_1(\beta x)\,dx=x^{2n-1}\,\left\{-\left[\frac{\beta\,x^3}{\sigma}+\frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\sigma^2}\right]\,K_0(\alpha x)\,J_0(\beta x)-\right.$$

$$-\left[\frac{2\,(n\,\sigma-\beta^2)\,x^2}{\sigma^2}-\frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\sigma^2}\right]\,K_0(\alpha x)\,J_1(\beta x)-\frac{2\,(2n+1)\alpha\beta\,x^2}{\sigma^2}\,K_1(\alpha x)\,J_0(\beta x)-$$

$$\left.-\left[\frac{\alpha\,x^3}{\sigma}-\frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\sigma^2}\right]\,K_1(\alpha x)\,J_1(\beta x)\right\}+$$

$$+\frac{8\,(2n^2\,\Delta+\beta^2)n}{(2n-1)\,\sigma^2}\int x^{2n}\,K_0(\alpha x)\,J_1(\beta x)\,dx-\frac{16\,(n-1)^2n^2(2n+1)}{(2n-1)\,\sigma^2}\int x^{2n-2}\,K_0(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x^{2n+2}\,K_0(\alpha x)\,I_1(\beta x)\,dx=x^{2n-1}\,\left\{-\left[\frac{\beta\,x^3}{\Delta}+\frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\Delta^2}\right]\,K_0(\alpha x)\,I_0(\beta x)-\right.$$

$$-\left[\frac{2\,(n\,\sigma+\beta^2)\,x^2}{\Delta^2}-\frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\Delta^2}\right]\,K_0(\alpha x)\,I_1(\beta x)-\frac{2\,(2n+1)\alpha\beta\,x^2}{\Delta^2}\,K_1(\alpha x)\,I_0(\beta x)-$$

$$\left.-\left[\frac{\alpha\,x^3}{\Delta}-\frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\Delta^2}\right]\,K_1(\alpha x)\,I_1(\beta x)\right\}-$$

$$+\frac{8\,(2n^2\,\sigma-\beta^2)n}{(2n-1)\,\Delta^2}\int x^{2n}\,K_0(\alpha x)\,I_1(\beta x)\,dx-\frac{16\,(n-1)^2n^2(2n+1)}{(2n-1)\,\Delta^2}\int x^{2n-2}\,K_0(\alpha x)\,I_1(\beta x)\,dx$$

$$\int x^{2n+2}\,K_0(\alpha x)\,K_1(\beta x)\,dx=x^{2n-1}\,\left\{\left[\frac{\beta\,x^3}{\Delta}+\frac{4\,(2n+1)(n-1)n\beta\,x}{(2n-1)\Delta^2}\right]\,K_0(\alpha x)\,K_0(\beta x)-\right.$$

$$-\left[\frac{2\,(n\,\sigma+\beta^2)\,x^2}{\Delta^2}-\frac{8\,n^2\,(2n+1)(n-1)}{(2n-1)\,\Delta^2}\right]\,K_0(\alpha x)\,K_1(\beta x)+\frac{2\,(2n+1)\alpha\beta\,x^2}{\Delta^2}\,K_1(\alpha x)\,K_0(\beta x)-$$

$$\left.-\left[\frac{\alpha\,x^3}{\Delta}-\frac{4\,(2n+1)n^2\,\alpha\,x}{(2n-1)\,\Delta^2}\right]\,K_1(\alpha x)\,K_1(\beta x)\right\}+$$

$$+\frac{8\,(2n^2\,\sigma-\beta^2)n}{(2n-1)\,\Delta^2}\int x^{2n}\,K_0(\alpha x)\,K_1(\beta x)\,dx-\frac{16\,(n-1)^2n^2(2n+1)}{(2n-1)\,\Delta^2}\int x^{2n-2}\,K_0(\alpha x)\,K_1(\beta x)\,dx$$

2.2.4. Integrals of the type $\int x^{2n+1} Z_\nu(\alpha x)\, Z_\nu(\beta x)\, dx$ and $x^{2n+1} V_\nu(\alpha x)\, W_\nu(\beta x)$, $\alpha^2 \neq \beta^2$

Let

$$\int x^{2n+1} J_\nu(\alpha x)\, J_\nu(\beta x)\, dx =$$

$$= A_\nu(x)\, J_0(\alpha x)\, J_0(\beta x) + B_\nu(x)\, J_0(\alpha x)\, J_1(\beta x) + C_\nu(x)\, J_1(\alpha x)\, J_0(\beta x) + D_\nu(x)\, J_1(\alpha x)\, J_1(\beta x)\ ,$$

then in this formula $J_\mu(\alpha x)$ or $J_\mu(\beta x)$ may be substituted by $Y_\mu(\alpha x)$, $H_\mu^{(p)}(\alpha x)$, $p = 1,2$, or $Y_\mu(\beta x)$, $H_\mu^{(p)}(\beta x)$, $p = 1,2$, respectively. The functions $A_\mu(x)$, $B_\mu(x)$, $C_\mu(x)$ and $D_\mu(x)$ are always the same in all cases. Therefore the integrals are given with $J_\nu(\alpha x)\, J_\nu(\beta x)$ only.

The same way in

$$\int x^{2n+1} J_\nu(\alpha x)\, I_\nu(\beta x)\, dx =$$

$$= P_\nu(x)\, J_0(\alpha x)\, I_0(\beta x) + Q_\nu(x)\, J_0(\alpha x)\, I_1(\beta x) + R_\nu(x)\, J_1(\alpha x)\, I_0(\beta x) + S_\nu(x)\, J_1(\alpha x)\, I_1(\beta x)\ ,$$

$J_\mu(\alpha x)$ may be substituted by $Y_\mu(\alpha x)$ or $H_\mu^{(p)}(\alpha x)$ without changing the coefficients $P_\nu(x)$ (and so on). The same holds for the integrals $\int x^{2n+1} J_\nu(\alpha x)\, K_\nu(\beta x)\, dx$. In both cases the integrals are given with $J_\nu(\alpha x)$ only.

Let

$$\Delta = \alpha^2 - \beta^2 \qquad \text{and} \qquad \sigma = \alpha^2 + \beta^2\ .$$

a) $\nu = 0$:

$$\int x \cdot J_0(\alpha x) J_0(\beta x)\, dx = \frac{\alpha x J_1(\alpha x) J_0(\beta x) - \beta x J_0(\alpha x) J_1(\beta x)}{\Delta}$$

$$\int x \cdot I_0(\alpha x) I_0(\beta x)\, dx = \frac{\alpha x I_1(\alpha x) I_0(\beta x) - \beta x I_0(\alpha x) I_1(\beta x)}{\Delta}$$

$$\int x \cdot K_0(\alpha x) K_0(\beta x)\, dx = -\frac{\alpha x K_1(\alpha x) K_0(\beta x) - \beta x K_0(\alpha x) K_1(\beta x)}{\Delta}$$

$$\int x \cdot J_0(\alpha x) I_0(\beta x)\, dx = \frac{\alpha x J_1(\alpha x) I_0(\beta x) + \beta x J_0(\alpha x) I_1(\beta x)}{\sigma}$$

$$\int x \cdot J_0(\alpha x) K_0(\beta x)\, dx = \frac{\alpha x J_1(\alpha x) K_0(\beta x) - \beta x J_0(\alpha x) K_1(\beta x)}{\sigma}$$

$$\int x \cdot I_0(\alpha x) K_0(\beta x)\, dx = \frac{\alpha x I_1(\alpha x) K_0(\beta x) + \beta x I_0(\alpha x) k_1(\beta x)}{\Delta}$$

$$\int x^3 \cdot J_0(\alpha x) J_0(\beta x)\, dx =$$

$$= \frac{2x^2}{\Delta^2} \left[\sigma J_0(\alpha x) J_0(\beta x) + 2\alpha\beta J_1(\alpha x) J_1(\beta x)\right] + \left[\frac{4\sigma x}{\Delta^3} - \frac{x^3}{\Delta}\right] \cdot \left[\beta J_0(\alpha x) J_1(\beta x) - \alpha J_1(\alpha x) J_0(\beta x)\right]$$

$$\int x^3 \cdot I_0(\alpha x) I_0(\beta x)\, dx =$$

$$= -\frac{2\,x^2}{\Delta^2} \left[\sigma I_0(\alpha x) I_0(\beta x) - 2\alpha\beta I_1(\alpha x) I_1(\beta x)\right] - \left[\frac{4\sigma x}{\Delta^3} + \frac{x^3}{\Delta}\right] \cdot \left[\beta I_0(\alpha x) I_1(\beta x) - \alpha I_1(\alpha x) I_0(\beta x)\right]$$

$$\int x^3 \cdot K_0(\alpha x) K_0(\beta x)\, dx =$$

$$= -\frac{2\,x^2}{\Delta^2} \left[\sigma K_0(\alpha x) K_0(\beta x) - 2\alpha\beta K_1(\alpha x) K_1(\beta x)\right] + \left[\frac{4\sigma x}{\Delta^3} + \frac{x^3}{\Delta}\right] \cdot \left[\beta K_0(\alpha x) K_1(\beta x) - \alpha K_1(\alpha x) K_0(\beta x)\right]$$

$$\int x^3 \cdot J_0(\alpha x) I_0(\beta x)\, dx =$$

$$= \frac{2x^2}{\sigma^2}\left[\Delta\, J_0(\alpha x)I_0(\beta x) - 2\alpha\beta J_1(\alpha x)I_1(\beta x)\right] - \left[\frac{4\Delta\, x}{\sigma^3} - \frac{x^3}{\sigma}\right]\cdot\left[\beta J_0(\alpha x)I_1(\beta x) + \alpha J_1(\alpha x)I_0(\beta x)\right]$$

$$\int x^3 \cdot J_0(\alpha x)K_0(\beta x)\,dx \;=$$

$$= \frac{2x^2}{\sigma^2}\left[\Delta\, J_0(\alpha x)K_0(\beta x) + 2\alpha\beta J_1(\alpha x)K_1(\beta x)\right] + \left[\frac{4\Delta\, x}{\sigma^3} - \frac{x^3}{\sigma}\right]\cdot\left[\beta J_0(\alpha x)K_1(\beta x) - \alpha J_1(\alpha x)K_0(\beta x)\right]$$

$$\int x^3 \cdot I_0(\alpha x)K_0(\beta x)\,dx \;=$$

$$= -\frac{2x^2}{\Delta^2}\left[\sigma I_0(\alpha x)K_0(\beta x) + 2\alpha\beta I_1(\alpha x)K_1(\beta x)\right] + \left[\frac{4\sigma x}{\Delta^3} + \frac{x^3}{\Delta}\right]\cdot\left[\beta I_0(\alpha x)K_1(\beta x) + \alpha I_1(\alpha x)K_0(\beta x)\right]$$

Let

$$\int x^m\, F_0(\alpha x)G_0(\beta x)\,dx =$$

$$= P_m^{[FG]}(x)F_0(\alpha x)G_0(\beta x) + Q_m^{[FG]}(x)F_0(\alpha x)G_1(\beta x) + R_m^{[FG]}(x)F_1(\alpha x)G_0(\beta x) + S_m^{[FG]}(x)F_1(\alpha x)G_1(\beta x)\,.$$

One has

$$P_m^{[JJ]} = P_m^{[YY]} = P_m^{[H^{(1)}H^{(1)}]} = P_m^{[H^{(2)}H^{(2)}]} = P_m^{[JY]} = P_m^{[JH^{(1)}]} = P_m^{[JH^{(2)}]} = P_m^{[YH^{(1)}]} = P_m^{[YH^{(2)}]} = P_m^{[H^{(1)}H^{(2)}]}\,,$$

$$P_m^{[JI]} = P_m^{[YI]} = P_m^{[H^{(1)}I]} = P_m^{[H^{(2)}I]} \quad \text{and} \quad P_m^{[JK]} = P_m^{[YK]} = P_m^{[H^{(1)}K]} = P_m^{[H^{(2)}K]}\,.$$

The same holds analogous for the polynomials $Q_m(x)$, $R_m(x)$ and $S_m(x)$.

$$P_5^{[JJ]}(x) = \frac{4\sigma\, x^4}{\Delta^2} - \frac{32\,(\sigma^2 + 2\,\alpha^2\beta^2)\,x^2}{\Delta^4}$$

$$Q_5^{[JJ]}(x) = -\frac{\beta\, x^5}{\Delta} + \frac{16\,\beta\,(2\,\alpha^2 + \beta^2)\,x^3}{\Delta^3} - \frac{64\,\beta\,(\sigma^2 + 2\,\alpha^2\beta^2)\,x}{\Delta^5}$$

$$R_5^{[JJ]}(x) = \frac{\alpha\, x^5}{\Delta} - \frac{16\,\alpha\,(\alpha^2 + 2\,\beta^2)\,x^3}{\Delta^3} + \frac{64\,\alpha\,(\sigma^2 + 2\,\alpha^2\beta^2)\,x}{\Delta^5}$$

$$S_5^{[JJ]}(x) = \frac{8\,\alpha\,\beta\, x^4}{\Delta^2} - \frac{96\,\alpha\,\beta\,\sigma\, x^2}{\Delta^4}$$

$$P_5^{[II]}(x) = P_5^{[KK]}(x) = -\frac{4\,\sigma\, x^4}{\Delta^2} - \frac{32\,(\sigma^2 + \alpha^2\beta^2)\,x^2}{\Delta^4}$$

$$Q_5^{[II]}(x) = -Q_5^{[KK]}(x) = -\frac{\beta\, x^5}{\Delta} - \frac{16\,\beta\,(2\alpha^2 + \beta^2)\,x^3}{\Delta^3} - \frac{64\,\beta\,(\sigma^2 + 2\alpha^2\beta^2)}{\Delta^5}$$

$$R_5^{[II]}(x) = -R_5^{[KK]}(x) = \frac{\alpha\, x^5}{\Delta} + \frac{16\,\alpha\,(\alpha^2 + 2\,\beta^2)\,x^3}{\Delta^3} + \frac{64\,\alpha\,(\sigma^2 + 2\alpha^2\beta^2)\,x}{\Delta^5}$$

$$S_5^{[II]}(x) = S_5^{[KK]}(x) = \frac{8\,\alpha\,\beta\, x^4}{\Delta^2} + \frac{96\,\alpha\,\beta\,\sigma\, x^2}{\Delta^4}$$

$$P_5^{[JI]}(x) = \frac{4\,\Delta\, x^4}{\sigma^2} - \frac{32\,(\Delta^2 - 2\alpha^2\beta^2)\,x^2}{\sigma^4}$$

$$Q_5^{[JI]}(x) = \frac{\beta\, x^5}{\sigma} - \frac{16\,\beta\,(2\alpha^2 - \beta^2)\,x^3}{\sigma^3} + \frac{64\,\beta\,(\Delta^2 - 2\alpha^2\beta^2)\,x}{\sigma^5}$$

$$R_5^{[JI]}(x) = \frac{\alpha\, x^5}{\sigma} - \frac{16\,\alpha\,(\alpha^2 - 2\beta^2)\,x^3}{\sigma^3} + \frac{64\,\alpha\,(\Delta^2 - 2\alpha^2\beta^2)\,x}{\sigma^5}$$

$$S_5^{[JI]}(x) = -\frac{8\,\alpha\,\beta\, x^4}{\sigma^2} + \frac{96\,\alpha\,\beta\,\Delta\, x^2}{\sigma^4}$$

$$P_5^{[JK]}(x) = \frac{4\,\Delta\,x^4}{\sigma^2} - \frac{32\,(\Delta^2 - 2\alpha^2\beta^2)x^2}{\sigma^4}$$

$$Q_5^{[JK]}(x) = -\frac{\beta\,x^5}{\sigma} + \frac{16\,\beta\,(2\,\alpha^2 - \beta^2)\,x^3}{\sigma^3} - \frac{64\,\beta\,(\Delta^2 - 2\alpha^2\beta^2)\,x}{\sigma^5}$$

$$R_5^{[JK]}(x) = \frac{\alpha\,x^5}{\sigma} - \frac{16\,\alpha\,(\alpha^2 - 2\beta^2)\,x^3}{\sigma^3} + \frac{64\,\alpha\,(\Delta^2 - 2\alpha^2\beta^2)\,x}{\sigma^5}$$

$$S_5^{[JK]}(x) = \frac{8\,\alpha\,\beta\,x^4}{\sigma^2} - \frac{96\,\alpha\,\beta\,\Delta\,x^2}{\sigma^4}$$

$$P_5^{[IK]}(x) = -\frac{4\,\sigma\,x^4}{\Delta^2} - \frac{32\,(\sigma^2 + 2\alpha^2\beta^2)\,x^2}{\Delta^4}$$

$$Q_5^{[IK]}(x) = \frac{\beta\,x^5}{\Delta} + \frac{16\;\beta\,(2\alpha^2 + \beta^2)\,x^3}{\Delta^3} + \frac{64\,\beta\,(\sigma^2 + 2\alpha^2\beta^2)\,x}{\Delta^5}$$

$$R_5^{[IK]}(x) = \frac{\alpha\,x^5}{\Delta} + \frac{16\,\alpha\,(\alpha^2 + 2\beta^2)\,x^3}{\Delta^3} + \frac{64\,\alpha\,(\sigma^2 + 2\alpha^2\beta^2)\,x}{\Delta^5}$$

$$S_5^{[IK]}(x) = -\frac{8\,\alpha\,\beta\,x^4}{\Delta^2} - \frac{96\,\alpha\,\beta\,\sigma\,x^2}{\Delta^4}$$

$$P_7^{[JJ]}(x) = \frac{6\,\sigma\,x^6}{\Delta^2} - \frac{48\,(3\,\sigma^2 + 8\,\alpha^2\beta^2)\,x^4}{\Delta^4} + \frac{1152\,\sigma(\sigma^2 + 6\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$Q_7^{[JJ]}(x) = -\frac{\beta\,x^7}{\Delta} + \frac{12\,\beta\,(7\,\alpha^2 + 3\,\beta^2)\,x^5}{\Delta^3} -$$

$$-\frac{192\,\beta\,(8\,\alpha^4 + 19\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\Delta^5} + \frac{2304\,\beta\,\sigma(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$R_7^{[JJ]}(x) = \frac{\alpha\,x^7}{\Delta} - \frac{12\,\alpha\,(3\,\alpha^2 + 7\,\beta^2)\,x^5}{\Delta^3} +$$

$$+\frac{192\,\alpha\,(3\,\alpha^4 + 19\,\alpha^2\beta^2 + 8\,\beta^4)\,x^3}{\Delta^5} - \frac{2304\,\alpha\,\sigma(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$S_7^{[JJ]}(x) = \frac{12\,\alpha\,\beta\,x^6}{\Delta^2} - \frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4} + \frac{384\,\alpha\,\beta\,(11\,\sigma^2 + 16\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$P_7^{[II]}(x) = P_7^{[KK]}(x) = -\frac{6\,\sigma\,x^6}{\Delta^2} - \frac{48\,(3\,\sigma^2 + 8\,\alpha^2\beta^2)\,x^4}{\Delta^4} - \frac{1152\,\sigma\,(\sigma^2 + 6\,\alpha^2)\,x^2}{\Delta^6}$$

$$Q_7^{[II]}(x) = -Q_7^{[KK]}(x) = -\frac{\beta\,x^7}{\Delta} - \frac{12\,\beta\,(7\,\alpha^2 + 3\,\beta^2)\,x^5}{\Delta^3} -$$

$$-\frac{192\,\beta\,(8\,\alpha^4 + 19\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\Delta^5} - \frac{2304\,\beta\,\sigma(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$R_7^{[II]}(x) = -R_7^{[KK]}(x) = \frac{\alpha\,x^7}{\Delta} + \frac{12\,\alpha\,(3\,\alpha^2 + 7\,\beta^2)\,x^5}{\Delta^3} +$$

$$+\frac{192\,\alpha\,(3\,\alpha^4 + 19\,\alpha^2\beta^2 + 8\,\beta^4)\,x^3}{\Delta^5} + \frac{2304\,\alpha\,\sigma\,(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$S_7^{[II]}(x) = S_7^{[KK]}(x) = \frac{12\,\alpha\,\beta\,x^6}{\Delta^2} + \frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4} + \frac{384\,\alpha\,\beta\,(11\,\sigma^2 + 16\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$P_7^{[JI]}(x) = \frac{6\,\Delta\,x^6}{\sigma^2} - \frac{48\,(3\,\Delta^2 - 8\,\alpha^2\beta^2)\,x^4}{\sigma^4} + \frac{1152\,\Delta\,(\Delta^2 - 6\alpha^2\,\beta^2)\,x^2}{\sigma^6}$$

$$Q_7^{[JI]}(x) = \frac{\beta\,x^7}{\sigma} - \frac{12\,\beta\,(7\,\alpha^2 - 3\,\beta^2)\,x^5}{\sigma^3} + \frac{192\,\beta\,(8\,\alpha^4 - 19\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\sigma^5} - \frac{2304\,\beta\,\Delta\,(\Delta^2 - 6\,\alpha^2\beta^2)\,x}{\sigma^7}$$

$$R_7^{[JI]}(x) = \frac{\alpha\,x^7}{\sigma} - \frac{12\,\alpha\,(3\,\alpha^2 - 7\,\beta^2)\,x^5}{\sigma^3} + \frac{192\,\alpha\,(3\,\alpha^4 - 19\,\alpha^2\beta^2 + 8\,\beta^4)\,x^3}{\sigma^5} - \frac{2304\,\alpha\,\Delta\,(\Delta^2 - 6\,\alpha^2\beta^2)\,x}{\sigma^7}$$

$$S_7^{[JI]}(x) = -\frac{12\,\alpha\,\beta\,x^6}{\sigma^2} + \frac{480\,\alpha\,\beta\,\Delta\,x^4}{\sigma^4} - \frac{384\,\alpha\,\beta\,(11\,\Delta^2 - 16\alpha^2\,\beta^2)\,x^2}{\sigma^6}$$

$$P_7^{[JK]}(x) = \frac{6\,\Delta x^6}{\sigma^2} - \frac{48\,(3\,\Delta^2 - 8\,\alpha^2\beta^2)\,x^4}{\sigma^4} + \frac{1152\,\Delta\,(\Delta^2 - 6\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$Q_7^{[JK]}(x) = -\frac{\beta\,x^7}{\sigma} + \frac{12\,\beta\,(7\,\alpha^2 - 3\,\beta^2)\,x^5}{\sigma^3} - \frac{192\,\beta\,(8\,\alpha^4 - 19\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\sigma^5} + \frac{2304\,\beta\,\Delta\,(\Delta^2 - 6\,\alpha^2\beta^2)\,x}{\sigma^7}$$

$$R_7^{[JK]}(x) = \frac{\alpha\,x^7}{\sigma} - \frac{12\,\alpha\,(3\,\alpha^2 - 7\,\beta^2)\,x^5}{\sigma^3} + \frac{192\,\alpha\,(3\,\alpha^4 - 19\,\alpha^2\beta^2 + 8\,\beta^4)\,x^3}{\sigma^5} - \frac{2304\,\alpha\,\Delta\,(\Delta^2 - 6\,\alpha^2\beta^2)\,x}{\sigma^7}$$

$$S_7^{[JK]}(x) = \frac{12\,\alpha\,\beta\,x^6}{\sigma^2} - \frac{480\,\alpha\,\beta\,\Delta\,x^4}{\sigma^4} + \frac{384\,\alpha\,\beta\,(11\,\Delta^2 - 16\,\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$P_7^{[IK]}(x) = -\frac{6\,\sigma\,x^6}{\Delta^2} - \frac{48\,(3\,\sigma^2 + 8\,\alpha^2\beta^2)\,x^4}{\Delta^4} - \frac{1152\,\sigma\,(\sigma^2 + 6\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$Q_7^{[IK]}(x) = \frac{\beta\,x^7}{\Delta} + \frac{12\,\beta\,(7\,\alpha^2 + 3\,\beta^2)\,x^5}{\Delta^3} + \frac{192\,\beta\,(8\,\alpha^4 + 19\,\alpha^2\beta^2 + 3\,\beta^4)\,x^3}{\Delta^5} + \frac{2304\,\beta\,\sigma\,(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$R_7^{[IK]}(x) = \frac{\alpha\,x^7}{\Delta} + \frac{12\,\alpha\,(3\,\alpha^2 + 7\,\beta^2)x^5}{\Delta^3} + \frac{12\,\alpha\,(3\,\alpha^4 + 19\,\alpha^2\beta^2 + 8\,\beta^4)\,x^3}{\Delta^5} + \frac{2304\,\alpha\,\sigma\,(\sigma^2 + 6\,\alpha^2\beta^2)\,x}{\Delta^7}$$

$$S_7^{[IK]}(x) = -\frac{12\,\alpha\,\beta\,x^6}{\Delta^2} - \frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4} - \frac{384\,\alpha\,\beta\,(11\,\sigma^2 + 16\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

<u>Recurrence formulas:</u> See also page 302.

$$\int x^{2n+1}\,J_0(\alpha x)\,J_0(\beta x)\,dx = x^{2n-2}\,\left\{\frac{\left[2\,n\sigma\,x^2 + 8\,(n-1)^2 n\right]\,J_0(\alpha x)\,J_0(\beta x)}{\Delta^2} - \right.$$

$$-\left[\frac{\beta x^3}{\Delta}-\frac{4n(n-1)\beta x}{\Delta^2}\right]J_0(\alpha x)J_1(\beta x)+\left[\frac{\alpha x^3}{\Delta}+\frac{4n(n-1)\alpha x}{\Delta^2}\right]J_1(\alpha x)J_0(\beta x)+\frac{4\alpha\beta n x^2 J_1(\alpha x)J_1(\beta x)}{\Delta^2}\Bigg\}-$$

$$-\frac{4(2n-1)n\sigma}{\Delta^2}\int x^{2n-1}J_0(\alpha x)J_0(\beta x)\,dx-\frac{16(n-1)^3n}{\Delta^2}\int x^{2n-3}J_0(\alpha x)J_0(\beta x)\,dx$$

$$\int x^{2n+1}I_0(\alpha x)I_0(\beta x)\,dx=x^{2n-2}\left\{-\frac{\left[2n\sigma x^2-8(n-1)^2\right]I_0(\alpha x)I_0(\beta x)}{\Delta^2}-\right.$$

$$-\left[\frac{\beta x^3}{\Delta}+\frac{4n(n-1)\beta x}{\Delta^2}\right]I_0(\alpha x)I_1(\beta x)+\left[\frac{\alpha x^3}{\Delta}-\frac{4n(n-1)\alpha x}{\Delta^2}\right]I_1(\alpha x)I_0(\beta x)+\frac{4\alpha\beta n x^2 I_1(\alpha x)I_1(\beta x)}{\Delta^2}\Bigg\}+$$

$$+\frac{4(2n-1)n\sigma}{\Delta^2}\int x^{2n-1}I_0(\alpha x)I_0(\beta x)\,dx-\frac{16(n-1)^3n}{\Delta^2}\int x^{2n-3}I_0(\alpha x)I_0(\beta x)\,dx$$

$$\int x^{2n+1}K_0(\alpha x)K_0(\beta x)\,dx=x^{2n-2}\left\{-\frac{\left[2n\sigma x^2-8(n-1)^2\right]K_0(\alpha x)K_0(\beta x)}{\Delta^2}+\right.$$

$$+\left[\frac{\beta x^3}{\Delta}+\frac{4n(n-1)\beta x}{\Delta^2}\right]K_0(\alpha x)K_1(\beta x)-\left[\frac{\alpha x^3}{\Delta}-\frac{4n(n-1)\alpha x}{\Delta^2}\right]K_1(\alpha x)K_0(\beta x)+\frac{4\alpha\beta n x^2 K_1(\alpha x)K_1(\beta x)}{\Delta^2}\Bigg\}+$$

$$+\frac{4(2n-1)n\sigma}{\Delta^2}\int x^{2n-1}K_0(\alpha x)K_0(\beta x)\,dx-\frac{16(n-1)^3n}{\Delta^2}\int x^{2n-3}K_0(\alpha x)K_0(\beta x)\,dx$$

$$\int x^{2n+1}J_0(\alpha x)I_0(\beta x)\,dx=x^{2n-2}\left\{\frac{\left[2n\Delta x^2+8(n-1)^2\right]J_0(\alpha x)I_0(\beta x)}{\sigma^2}+\right.$$

$$+\left[\frac{\beta x^3}{\sigma}-\frac{4n(n-1)\beta x}{\sigma^2}\right]J_0(\alpha x)I_1(\beta x)+\left[\frac{\alpha x^3}{\sigma}+\frac{4n(n-1)\alpha x}{\sigma^2}\right]J_1(\alpha x)I_0(\beta x)-\frac{4\alpha\beta n x^2 J_1(\alpha x)I_1(\beta x)}{\sigma^2}\Bigg\}-$$

$$-\frac{4(2n-1)n\Delta}{\sigma^2}\int x^{2n-1}J_0(\alpha x)I_0(\beta x)\,dx-\frac{16(n-1)^3n}{\sigma^2}\int x^{2n-3}J_0(\alpha x)I_0(\beta x)\,dx$$

$$\int x^{2n+1}J_0(\alpha x)K_0(\beta x)\,dx=x^{2n-2}\left\{\frac{\left[2n\Delta x^2+8(n-1)^2\right]J_0(\alpha x)K_0(\beta x)}{\sigma^2}-\right.$$

$$-\left[\frac{\beta x^3}{\sigma}-\frac{4n(n-1)\beta x}{\sigma^2}\right]J_0(\alpha x)K_1(\beta x)+\left[\frac{\alpha x^3}{\sigma}+\frac{4n(n-1)\alpha x}{\sigma^2}\right]J_1(\alpha x)K_0(\beta x)+\frac{4\alpha\beta n x^2 J_1(\alpha x)I_K(\beta x)}{\sigma^2}\Bigg\}-$$

$$-\frac{4(2n-1)n\Delta}{\sigma^2}\int x^{2n-1}J_0(\alpha x)K_0(\beta x)\,dx-\frac{16(n-1)^3n}{\sigma^2}\int x^{2n-3}J_0(\alpha x)K_0(\beta x)\,dx$$

$$\int x^{2n+1}I_0(\alpha x)K_0(\beta x)\,dx=x^{2n-2}\left\{-\frac{\left[2n\sigma x^2-8(n-1)^2\right]I_0(\alpha x)K_0(\beta x)}{\Delta^2}+\right.$$

$$+\left[\frac{\beta x^3}{\Delta}+\frac{4n(n-1)\beta x}{\Delta^2}\right]I_0(\alpha x)K_1(\beta x)+\left[\frac{\alpha x^3}{\Delta}-\frac{4n(n-1)\alpha x}{\Delta^2}\right]I_1(\alpha x)K_0(\beta x)-\frac{4\alpha\beta n x^2 I_1(\alpha x)K_1(\beta x)}{\Delta^2}\Bigg\}+$$

$$+\frac{4(2n-1)n\sigma}{\Delta^2}\int x^{2n-1}I_0(\alpha x)K_0(\beta x)\,dx-\frac{16(n-1)^3n}{\Delta^2}\int x^{2n-3}I_0(\alpha x)K_0(\beta x)\,dx$$

b) $\nu=1$:

Let again

$$\Delta=\alpha^2-\beta^2 \qquad \text{and} \qquad \sigma=\alpha^2+\beta^2\ .$$

$$\int x \cdot J_1(\alpha x) J_1(\beta x)\, dx \;=\; \frac{x}{\Delta}\, [\beta J_1(\alpha x) J_0(\beta x) - \alpha J_0(\alpha x) J_1(\beta x)]$$

$$\int x \cdot I_1(\alpha x) I_1(\beta x)\, dx \;=\; \frac{x}{\Delta}\, [\alpha I_0(\alpha x) I_1(\beta x) - \beta I_1(\alpha x) I_0(\beta x)]$$

$$\int x \cdot K_1(\alpha x) K_1(\beta x)\, dx \;=\; \frac{x}{\Delta}\, [\beta K_1(\alpha x) K_0(\beta x) - \alpha K_0(\alpha x) K_1(\beta x)]$$

$$\int x \cdot J_1(\alpha x) I_1(\beta x)\, dx \;=\; \frac{x}{\sigma}\, [\beta J_1(\alpha x) I_0(\beta x) - \alpha J_0(\alpha x) I_1(\beta x)]$$

$$\int x \cdot J_1(\alpha x) K_1(\beta x)\, dx \;=\; -\frac{x}{\sigma}\, [\beta J_1(\alpha x) K_0(\beta x) + \alpha J_0(\alpha x) K_1(\beta x)]$$

$$\int x \cdot I_1(\alpha x) K_1(\beta x)\, dx \;=\; \frac{x}{\Delta}\, [\beta I_1(\alpha x) K_0(\beta x) + \alpha I_0(\alpha x) K_1(\beta x)]$$

$$\int x^3 \cdot J_1(\alpha x) J_1(\beta x)\, dx \;=\; \frac{2x^2}{\Delta^2}\, [2\alpha\beta J_0(\alpha x) J_0(\beta x) + \sigma\, J_1(\alpha x) J_1(\beta x)] +$$

$$+\frac{8\alpha\beta x}{\Delta^3} \cdot [\beta J_0(\alpha x) J_1(\beta x) - \alpha J_1(\alpha x) J_0(\beta x)] - \frac{x^3}{\Delta} \cdot [\alpha J_0(\alpha x) J_1(\beta x) - \beta J_1(\alpha x) J_0(\beta x)]$$

$$\int x^3 \cdot I_1(\alpha x) I_1(\beta x)\, dx \;=\; \frac{2x^2}{\Delta^2}\, [2\alpha\beta I_0(\alpha x) I_0(\beta x) - \sigma\, I_1(\alpha x) I_1(\beta x)] +$$

$$+\frac{8\alpha\beta x}{\Delta^3} \cdot [\beta I_0(\alpha x) I_1(\beta x) - \alpha I_1(\alpha x) I_0(\beta x)] + \frac{x^3}{\Delta} \cdot [\alpha I_0(\alpha x) I_1(\beta x) - \beta I_1(\alpha x) I_0(\beta x)]$$

$$\int x^3 \cdot K_1(\alpha x) K_1(\beta x)\, dx \;=\; \frac{2x^2}{\Delta^2}\, [2\alpha\beta\, K_0(\alpha x) K_0(\beta x) - \sigma\, K_1(\alpha x) K_1(\beta x)] -$$

$$-\frac{8\alpha\beta x}{\Delta^3} \cdot [\beta K_0(\alpha x) K_1(\beta x) - \alpha K_1(\alpha x) K_0(\beta x)] - \frac{x^3}{\Delta} \cdot [\alpha K_0(\alpha x) K_1(\beta x) - \beta K_1(\alpha x) K_0(\beta x)]$$

$$\int x^3 \cdot J_1(\alpha x) I_1(\beta x)\, dx \;=\; \frac{2x^2}{\sigma^2}\, [2\alpha\beta J_0(\alpha x) I_0(\beta x) + \Delta\, J_1(\alpha x) I_1(\beta x)] -$$

$$-\frac{8\alpha\beta x}{\sigma^3} \cdot [\beta J_0(\alpha x) I_1(\beta x) + \alpha J_1(\alpha x) I_0(\beta x)] - \frac{x^3}{\sigma} \cdot [\alpha J_0(\alpha x) I_1(\beta x) - \beta J_1(\alpha x) I_0(\beta x)]$$

$$\int x^3 \cdot J_1(\alpha x) K_1(\beta x)\, dx \;=\; \frac{2x^2}{\sigma^2}\, [-\,2\alpha\beta J_0(\alpha x) K_0(\beta x) + \Delta\, J_1(\alpha x) K_1(\beta x)] -$$

$$-\frac{8\alpha\beta x}{\sigma^3} \cdot [\beta J_0(\alpha x) K_1(\beta x) - \alpha J_1(\alpha x) K_0(\beta x)] - \frac{x^3}{\sigma} \cdot [\alpha J_0(\alpha x) K_1(\beta x) + \beta J_1(\alpha x) K_0(\beta x)]$$

$$\int x^3 \cdot I_1(\alpha x) K_1(\beta x)\, dx \;=\; -\frac{2x^2}{\Delta^2}\, [2\alpha\beta I_0(\alpha x) K_0(\beta x) + \sigma\, I_1(\alpha x) K_1(\beta x)] +$$

$$+\frac{8\alpha\beta x}{\Delta^3} \cdot [\beta I_0(\alpha x) K_1(\beta x) + \alpha I_1(\alpha x) K_0(\beta x)] + \frac{x^3}{\Delta} \cdot [\alpha I_0(\alpha x) K_1(\beta x) + \beta I_1(\alpha x) K_0(\beta x)]$$

Let

$$\int x^m\, F_1(\alpha x) G_1(\beta x)\, dx =$$

$$= T_m^{[FG]}(x) F_0(\alpha x) G_0(\beta x) + U_m^{[FG]}(x) F_0(\alpha x) G_1(\beta x) + V_m^{[FG]}(x) F_1(\alpha x) G_0(\beta x) + W_m^{[FG]}(x) F_1(\alpha x) G_1(\beta x)\,.$$

One has

$$T_m^{[JJ]} = T_m^{[YY]} = T_m^{[H^{(1)}H^{(1)}]} = T_m^{[H^{(2)}H^{(2)}]} = T_m^{[JY]} = T_m^{[JH^{(1)}]} = T_m^{[JH^{(2)}]} = T_m^{[YH^{(1)}]} = T_m^{[YH^{(2)}]} = T_m^{[H^{(1)}H^{(2)}]}\,,$$

$$T_m^{[JI]} = T_m^{[YI]} = T_m^{[H^{(1)}I]} = T_m^{[H^{(2)}I]} \quad \text{and} \quad T_m^{[JK]} = T_m^{[YK]} = T_m^{[H^{(1)}K]} = T_m^{[H^{(2)}K]} \,.$$

The same holds analogous for the polynomials $U_m(x)$, $V_m(x)$ and $W_m(x)$.

$$T_5^{[JJ]}(x) = \frac{8\,\alpha\,\beta\,x^4}{\Delta^2} - \frac{96\,\alpha\,\beta\,\sigma\,x^2}{\Delta^4}\,, \quad U_5^{[JJ]}(x) = -\frac{\alpha\,x^5}{\Delta} + \frac{8\,\alpha\,(\alpha^2+5\,\beta^2)\,x^3}{\Delta^3} - \frac{192\,\alpha\,\beta^2\sigma\,x}{\Delta^5}$$

$$V_5^{[JJ]}(x) = \frac{\beta\,x^5}{\Delta} - \frac{8\,\beta\,(5\,\alpha^2+\beta^2)\,x^3}{\Delta^3} + \frac{192\,\alpha^2\beta\,\sigma\,x}{\Delta^5}\,, \quad W_5^{[JJ]}(x) = \frac{4\,\sigma\,x^4}{\Delta^2} - \frac{16\,(\sigma^2+8\,\alpha^2\beta^2)\,x^2}{\Delta^4}$$

$$T_5^{[II]}(x) = T_5^{[KK]}(x) = \frac{8\,\alpha\,\beta\,x^4}{\Delta^2} + \frac{96\,\alpha\,\beta\,\sigma\,x^2}{\Delta^4}\,, \quad U_5^{[II]}(x) = -U_5^{[KK]}(x) = \frac{\alpha\,x^5}{\Delta} + \frac{8\,\alpha\,(\alpha^2+5\,\beta^2)\,x^3}{\Delta^3} + \frac{192\,\alpha\,\beta^2\sigma\,x}{\Delta^5}$$

$$V_5^{[II]}(x) = -V_5^{[KK]}(x) = -\frac{\beta\,x^5}{\Delta} - \frac{8\,\beta\,(5\,\alpha^2+\beta^2)\,x^3}{\Delta^3} - \frac{192\,\alpha^2\beta\,\sigma\,x}{\Delta^5}$$

$$W_5^{[II]}(x) = W_5^{[KK]}(x) = -\frac{4\,\sigma\,x^4}{\Delta^2} - \frac{16\,(\sigma^2+8\,\alpha^2\beta^2)\,x^2}{\Delta^4}$$

$$T_5^{[JI]}(x) = \frac{8\,\alpha\beta\,x^4}{\sigma^2} - \frac{96\,\alpha\beta\,\Delta\,x^2}{\sigma^4}\,, \quad U_5^{[JI]}(x) = -\frac{\alpha\,x^5}{\sigma} + \frac{8\,\alpha\,(\alpha^2-5\,\beta^2)\,x^3}{\sigma^3} + \frac{192\,\alpha\,\beta^2\Delta\,x}{\sigma^5}$$

$$V_5^{[JI]}(x) = \frac{\beta\,x^5}{\sigma} - \frac{8\,\beta\,(5\,\alpha^2-\beta^2)\,x^3}{\sigma^3} + \frac{192\,\alpha^2\beta\,\Delta\,x}{\sigma^5}\,, \quad W_5^{[JI]}(x) = \frac{4\,\Delta\,x^4}{\sigma^2} - \frac{16\,(\Delta^2-8\,\alpha^2\beta^2)\,x^2}{\sigma^4}$$

$$T_5^{[JK]}(x) = -\frac{8\,\alpha\,\beta\,x^4}{\sigma^2} + \frac{96\,\alpha\,\beta\,\Delta\,x^2}{\sigma^4}\,, \quad U_5^{[JK]}(x) = -\frac{\alpha\,x^5}{\sigma} + \frac{8\,\alpha\,(\alpha^2-5\,\beta^2)\,x^3}{\sigma^3} + \frac{192\,\alpha\,\beta^2\Delta\,x}{\sigma^5}$$

$$V_5^{[JK]}(x) = -\frac{\beta\,x^5}{\sigma} + \frac{8\,\beta\,(5\,\alpha^2-\beta^2)\,x^3}{\sigma^3} - \frac{192\,\alpha^2\beta\,\Delta\,x}{\sigma^5}\,, \quad W_5^{[JK]}(x) = \frac{4\,\Delta\,x^4}{\sigma^2} - \frac{16\,(\Delta^2-8\,\alpha^2\beta^2)\,x^2}{\sigma^4}$$

$$T_5^{[IK]}(x) = -\frac{8\,\alpha\,\beta\,x^4}{\Delta^2} - \frac{96\,\alpha\,\beta\,\sigma\,x^2}{\Delta^4}\,, \quad U_5^{[IK]}(x) = \frac{\alpha\,x^5}{\Delta} + \frac{8\,\alpha\,(\alpha^2+5\,\beta^2)\,x^3}{\Delta^3} + \frac{192\,\alpha\,\beta^2\sigma\,x}{\Delta^5}$$

$$V_5^{[IK]}(x) = \frac{\beta\,x^5}{\Delta} + \frac{8\,\beta\,(5\,\alpha^2+\beta^2)\,x^3}{\Delta^3} + \frac{192\,\alpha^2\beta\,\sigma\,x}{\Delta^5}\,, \quad W_5^{[IK]}(x) = -\frac{4\,\sigma\,x^4}{\Delta^2} - \frac{16\,(\sigma^2+8\,\alpha^2\beta^2)\,x^2}{\Delta^4}$$

$$T_7^{[JJ]}(x) = \frac{12\,\alpha\,\beta\,x^6}{\Delta^2} - \frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4} + \frac{4608\,\alpha\,\beta\,(\sigma^2+\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$U_7^{[JJ]}(x) = -\frac{\alpha\,x^7}{\Delta} + \frac{24\,\alpha\,(\alpha^2+4\,\beta^2)\,x^5}{\Delta^3} -$$

$$-\frac{192\,\alpha\,(\alpha^4+18\,\alpha^2\beta^2+11\,\beta^4)\,x^3}{\Delta^5} + \frac{9216\,\alpha\,\beta^2(\sigma^2+\alpha^2\beta^2)\,x}{\Delta^7}$$

$$V_7^{[JJ]}(x) = \frac{\beta\,x^7}{\Delta} - \frac{24\,\beta\,(4\alpha^2+\beta^2)\,x^5}{\Delta^3} +$$

$$+\frac{192\,\beta\,(11\,\alpha^4+18\,\alpha^2\beta^2+\beta^4)\,x^3}{\Delta^5} - \frac{9216\,\alpha^2\beta\,(\sigma^2+\alpha^2\beta^2)\,x}{\Delta^7}$$

$$W_7^{[JJ]}(x) = \frac{6\,\sigma\,x^6}{\Delta^2} - \frac{96\,(\sigma^2+6\,\alpha^2\beta^2)\,x^4}{\Delta^4} + \frac{384\,\sigma\,(\sigma^2+26\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$T_7^{[II]}(x) = T_7^{[KK]}(x) = \frac{12\,\alpha\,\beta\,x^6}{\Delta^2} + \frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4} + \frac{4608\,\alpha\,\beta\,(\sigma^2+\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$U_7^{[II]}(x) = -U_7^{[KK]}(x) = \frac{\alpha\,x^7}{\Delta} + \frac{24\,\alpha\,(\alpha^2+4\,\beta^2)\,x^5}{\Delta^3} +$$

$$+\frac{192\,\alpha\,(\sigma^2+16\,\alpha^2\beta^2)\,x^3}{\Delta^5}+\frac{9216\,\alpha\,\beta^2\,(\sigma^2+3\alpha^2\beta^2)\,x}{\Delta^7}$$

$$V_7^{[II]}(x)=-V_7^{[KK]}(x)=-\frac{\beta\,x^7}{\Delta}-\frac{24\,\beta\,(4\,\alpha^2+\beta^2)\,x^5}{\Delta^3}-$$

$$-\frac{192\,\beta\,(11\,\alpha^4+18\,\alpha^2\beta^2+\beta^4)\,x^3}{\Delta^5}-\frac{9216\,\alpha^2\beta\,(\sigma^2+\alpha^2\beta^2)\,x}{\Delta^7}$$

$$W_7^{[II]}(x)=W_7^{[KK]}(x)=-\frac{6\,\sigma\,x^6}{\Delta^2}-\frac{96\,(\sigma^2+6\,\alpha^2\beta^2)\,x^4}{\Delta^4}-\frac{384\,\sigma\,(\sigma^2+26\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$T_7^{[JI]}(x)=\frac{12\,\alpha\,\beta\,x^6}{\sigma^2}-\frac{480\,\alpha\,\beta\,\Delta\,x^4}{\sigma^4}+\frac{4608\,\alpha\,\beta\,(\Delta^2-\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$U_7^{[JI]}(x)=-\frac{\alpha\,x^7}{\sigma}+\frac{24\,\alpha\,(\alpha^2-4\,\beta^2)\,x^5}{\sigma^3}-\frac{192\,\alpha\,(\alpha^4-18\,\alpha^2\beta^2+11\,\beta^4)\,x^3}{\sigma^5}-$$

$$-\frac{9216\,\alpha\,\beta^2(\Delta^2-\alpha^2\beta^2)\,x}{\sigma^7}$$

$$V_7^{[JI]}(x)=\frac{\beta\,x^7}{\sigma}-\frac{24\,\beta\,(4\,\alpha^2-\beta^2)\,x^5}{\sigma^3}+\frac{192\,\beta\,(11\,\alpha^4-18\,\alpha^2\beta^2+\beta^4)\,x^3}{\sigma^5}-$$

$$-\frac{9216\,\alpha^2\beta(\Delta^2-\alpha^2\beta^2)\,x}{\sigma^7}$$

$$W_7^{[JI]}(x)=\frac{6\,\Delta\,x^6}{\sigma^2}-\frac{96\,(\Delta^2-6\,\alpha^2\beta^2)\,x^4}{\sigma^4}+\frac{384\,\Delta\,(\Delta^2-26\,\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$T_7^{[JK]}(x)=-\frac{12\,\alpha\,\beta\,x^6}{\sigma^2}+\frac{480\,\alpha\,\beta\,\Delta\,x^4}{\sigma^4}-\frac{4608\,\alpha\,\beta\,(\Delta^2-\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$U_7^{[JK]}(x)=-\frac{\alpha\,x^7}{\sigma}+\frac{24\,\alpha\,(\alpha^2-4\,\beta^2)\,x^5}{\sigma^3}-\frac{192\,\alpha\,(\alpha^4-18\,\alpha^2\beta^2+11\,\beta^4)\,x^3}{\sigma^5}-$$

$$-\frac{9216\,\alpha\,\beta^2\,(\Delta^2-\alpha^2\beta^2)\,x}{\sigma^7}$$

$$V_7^{[JK]}(x)=-\frac{\beta\,x^7}{\sigma}+\frac{24\,\beta\,(4\,\alpha^2-\beta^2)\,x^5}{\sigma^3}-\frac{192\,\beta\,(11\,\alpha^4-18\,\alpha^2\beta^2+\beta^4)\,x^3}{\sigma^5}+$$

$$+\frac{9216\,\alpha^2\beta\,(\Delta^2-\,\alpha^2\beta^2)\,x}{\sigma^7}$$

$$W_7^{[JK]}(x)=\frac{6\,\Delta\,x^6}{\sigma^2}-\frac{96\,(\Delta^2-6\,\alpha^2\beta^2)\,x^4}{\sigma^4}+\frac{384\,\Delta\,(\Delta^2-26\,\alpha^2\beta^2)\,x^2}{\sigma^6}$$

$$T_7^{[IK]}(x)=-\frac{12\,\alpha\,\beta\,x^6}{\Delta^2}-\frac{480\,\alpha\,\beta\,\sigma\,x^4}{\Delta^4}-\frac{4608\,\alpha\,\beta\,(\sigma^2+\alpha^2\beta^2)\,x^2}{\Delta^6}$$

$$U_7^{[IK]}(x)=\frac{\alpha\,x^7}{\Delta}+\frac{24\,\alpha\,(\alpha^2+4\,\beta^2)\,x^5}{\Delta^3}+\frac{192\,\alpha\,(\alpha^4+18\,\alpha^2\beta^2+11\,\beta^4)\,x^3}{\Delta^5}+$$

$$+\frac{9216\,\alpha\,\beta^2\,(\sigma^2+\alpha^2\beta^2)\,x}{\Delta^7}$$

$$V_7^{[IK]}(x)=\frac{\beta\,x^7}{\Delta}+\frac{24\,\beta\,(4\,\alpha^2+\beta^2)\,x^5}{\Delta^3}+\frac{192\,\beta\,(11\,\alpha^4+18\,\alpha^2\beta^2+\beta^4)\,x^3}{\Delta^5}+$$

$$+\frac{9216\,\alpha^2\beta\,(\sigma^2+\alpha^2\beta^2)\,x}{\Delta^7}$$

$$W_7^{[IK]}(x)=-\frac{6\,\sigma\,x^6}{\Delta^2}-\frac{96\,(\sigma^2+6\,\alpha^2\beta^2)\,x^4}{\Delta^4}-\frac{384\,\sigma\,(\sigma^3+26\,\alpha^2\beta^2)\,x^2}{\Delta^6}$$

Recurrence formulas: See also page 302.

$$\int x^{2n+1}\, J_1(\alpha x)\, J_1(\beta x)\, dx = x^{2n-2}\ \left\{\frac{4\,n\alpha\beta\, x^2\, J_0(\alpha x)\, J_0(\beta x)}{\Delta^2} - \left(\frac{\alpha\, x^3}{\Delta} + \frac{4\, n^2 \alpha\, x}{\Delta^2}\right) J_0(\alpha x)\, J_1(\beta x) + \right.$$
$$\left. + \left(\frac{\beta\, x^3}{\Delta} - \frac{4\, n^2 \beta\, x}{\Delta^2}\right) J_1(\alpha x)\, J_0(\beta x) + \frac{[2\, n\sigma\, x^2 + 8\, n^2(n-1)]\, J_1(\alpha x)\, J_1(\beta x)}{\Delta^2}\right\} -$$
$$- \frac{4\,(2n-1)n\,\sigma}{\Delta^2} \int x^{2n-1}\, J_1(\alpha x)\, J_1(\beta x)\, dx - \frac{16\, n^2(n-1)(n-2)}{\Delta^2} \int x^{2n-3}\, J_1(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, dx = x^{2n-2}\ \left\{\frac{4\,n\alpha\beta\, x^2\, I_0(\alpha x)\, I_0(\beta x)}{\Delta^2} + \left(\frac{\alpha\, x^3}{\Delta} - \frac{4\, n^2 \alpha\, x}{\Delta^2}\right) I_0(\alpha x)\, I_1(\beta x) - \right.$$
$$\left. - \left(\frac{\beta\, x^3}{\Delta} + \frac{4\, n^2 \beta\, x}{\Delta^2}\right) I_1(\alpha x)\, I_0(\beta x) - \frac{[2\, n\sigma\, x^2 - 8\, n^2(n-1)]\, I_1(\alpha x)\, I_1(\beta x)}{\Delta^2}\right\} +$$
$$+ \frac{4\,(2n-1)n\,\sigma}{\Delta^2} \int x^{2n-1}\, I_1(\alpha x)\, I_1(\beta x)\, dx - \frac{16\, n^2(n-1)(n-2)}{\Delta^2} \int x^{2n-3}\, I_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x^{2n+1}\, K_1(\alpha x)\, K_1(\beta x)\, dx = x^{2n-2}\ \left\{\frac{4\,n\alpha\beta\, x^2\, K_0(\alpha x)\, K_0(\beta x)}{\Delta^2} - \left(\frac{\alpha\, x^3}{\Delta} - \frac{4\, n^2 \alpha\, x}{\Delta^2}\right) K_0(\alpha x)\, K_1(\beta x) + \right.$$
$$\left. + \left(\frac{\beta\, x^3}{\Delta} + \frac{4\, n^2 \beta\, x}{\Delta^2}\right) K_1(\alpha x)\, K_0(\beta x) - \frac{[2\, n\sigma\, x^2 - 8\, n^2(n-1)]\, K_1(\alpha x)\, K_1(\beta x)}{\Delta^2}\right\} +$$
$$+ \frac{4\,(2n-1)n\,\sigma}{\Delta^2} \int x^{2n-1}\, K_1(\alpha x)\, K_1(\beta x)\, dx - \frac{16\, n^2(n-1)(n-2)}{\Delta^2} \int x^{2n-3}\, K_1(\alpha x)\, K_1(\beta x)\, dx$$

$$\int x^{2n+1}\, J_1(\alpha x)\, I_1(\beta x)\, dx = x^{2n-2}\ \left\{\frac{4\,n\alpha\beta\, x^2\, J_0(\alpha x)\, I_0(\beta x)}{\sigma^2} - \left(\frac{\alpha\, x^3}{\sigma} + \frac{4\, n^2 \alpha\, x}{\sigma^2}\right) J_0(\alpha x)\, I_1(\beta x) + \right.$$
$$\left. + \left(\frac{\beta\, x^3}{\sigma} - \frac{4\, n^2 \beta\, x}{\sigma^2}\right) J_1(\alpha x)\, I_0(\beta x) + \frac{[2\, n\Delta\, x^2 + 8\, n^2(n-1)]\, J_1(\alpha x)\, I_1(\beta x)}{\sigma^2}\right\} -$$
$$- \frac{4\,(2n-1)n\,\Delta}{\sigma^2} \int x^{2n-1}\, J_1(\alpha x)\, I_1(\beta x)\, dx - \frac{16\, n^2(n-1)(n-2)}{\sigma^2} \int x^{2n-3}\, J_1(\alpha x)\, I_1(\beta x)\, dx$$

$$\int x^{2n+1}\, J_1(\alpha x)\, K_1(\beta x)\, dx = x^{2n-2}\ \left\{-\frac{4\,n\alpha\beta\, x^2\, J_0(\alpha x)\, K_0(\beta x)}{\sigma^2} - \left(\frac{\alpha\, x^3}{\sigma} + \frac{4\, n^2 \alpha\, x}{\sigma^2}\right) J_0(\alpha x)\, K_1(\beta x) - \right.$$
$$\left. - \left(\frac{\beta\, x^3}{\sigma} - \frac{4\, n^2 \beta\, x}{\sigma^2}\right) J_1(\alpha x)\, K_0(\beta x) + \frac{[2\, n\Delta\, x^2 + 8\, n^2(n-1)]\, J_1(\alpha x)\, I_1(\beta x)}{\sigma^2}\right\} -$$
$$- \frac{4\,(2n-1)n\,\Delta}{\sigma^2} \int x^{2n-1}\, J_1(\alpha x)\, K_1(\beta x)\, dx - \frac{16\, n^2(n-1)(n-2)}{\sigma^2} \int x^{2n-3}\, J_1(\alpha x)\, K_1(\beta x)\, dx$$

$$\int x^{2n+1}\, I_1(\alpha x)\, K_1(\beta x)\, dx = x^{2n-2}\ \left\{-\frac{4\,n\alpha\beta\, x^2\, I_0(\alpha x)\, K_0(\beta x)}{\Delta^2} + \left(\frac{\alpha\, x^3}{\Delta} - \frac{4\, n^2 \alpha\, x}{\Delta^2}\right) I_0(\alpha x)\, K_1(\beta x) + \right.$$
$$\left. + \left(\frac{\beta\, x^3}{\Delta} + \frac{4\, n^2 \beta\, x}{\Delta^2}\right) I_1(\alpha x)\, K_0(\beta x) - \frac{[2\, n\sigma\, x^2 - 8\, n^2(n-1)]\, I_1(\alpha x)\, K_1(\beta x)}{\Delta^2}\right\} +$$
$$+ \frac{4\,(2n-1)n\,\sigma}{\Delta^2} \int x^{2n-1}\, I_1(\alpha x)\, K_1(\beta x)\, dx - \frac{16\, n^2(n-1)(n-2)}{\Delta^2} \int x^{2n-3}\, I_1(\alpha x)\, K_1(\beta x)\, dx$$

2.2.5. Integrals of the type $\int x^{2n+1} \cdot J_\nu(\alpha x)\, Y_\nu(\beta x)\, dx$

Compare with 2.2.3.b) .

Let $\boxed{\alpha^2 + \beta^2 = \sigma \qquad \text{and} \qquad \alpha^2 - \beta^2 = \Delta\ .}$

$$\int x\, J_0(\alpha x)\, Y_0(\beta x)\, dx = \frac{x}{\Delta} \left[\alpha\, J_1(\alpha x)\, Y_0(\beta x) - \beta J_0(\alpha x)\, Y_1(\beta x)\right]$$

$$\int x\, J_1(\alpha x)\, Y_1(\beta x)\, dx = \frac{x}{\Delta} \left[\beta\, J_1(\alpha x)\, Y_0(\beta x) - \alpha J_0(\alpha x)\, Y_1(\beta x)\right]$$

$$\int x^3\, J_0(\alpha x)\, Y_0(\beta x)\, dx = \frac{2\,\sigma\, x^2}{\Delta^2}\, J_0(\alpha x)\, Y_0(\beta x) + \frac{4\,\alpha\,\beta x^2}{\Delta^2}\, J_1(\alpha x)\, Y_1(\beta x) +$$

$$+ \frac{\Delta^2 x^3 - 4\,\sigma x}{\Delta^3} \left[\alpha\, J_1(\alpha x)\, Y_0(\beta x) - \beta\, J_0(\alpha x)\, Y_1(\beta x)\right]$$

$$\int x^3\, J_1(\alpha x)\, Y_1(\beta x)\, dx = \frac{4\,\alpha\,\beta\, x^2}{\Delta^2}\, J_0(\alpha x)\, Y_0(\beta x) + \frac{2\,\sigma\, x^2}{\Delta^2}\, J_1(\alpha x)\, Y_1(\beta x) -$$

$$- \frac{\alpha[\Delta^2\, x^3 - 8\beta^2\, x]}{\Delta^3}\, J_0(\alpha x)\, Y_1(\beta x) + \frac{\beta[\Delta^2\, x^3 - 8\alpha^2\, x]}{\Delta^3}\, J_1(\alpha x)\, Y_0(\beta x)$$

$$\int x^5\, J_0(\alpha x)\, Y_0(\beta x)\, dx = \frac{4\,\sigma \Delta^2 x^4 - 32(\,\sigma^2 + 2\,\alpha^2\beta^2) x^2}{\Delta^4}\, J_0(\alpha x)\, Y_0(\beta x) -$$

$$- \frac{\beta\,[\Delta^4 x^5 - 16\,(2\,\alpha^2 + \beta^2)\Delta^2 x^3 + 64\,(\sigma^2 + 2\,\alpha^2\beta^2) x]}{\Delta^5}\, J_0(\alpha x)\, Y_1(\beta x) +$$

$$+ \frac{\alpha\,[\Delta^4 x^5 - 16\,(\alpha^2 + 2\,\beta^2)\Delta^2 x^3 + 64\,(\sigma^2 + 2\,\alpha^2\beta^2)\, x]}{\Delta^5}\, J_1(\alpha x)\, Y_0(\beta x) +$$

$$+ \frac{8\,\alpha\,\beta\,\Delta^2 x^4 - 96\,\alpha\,\beta\,\sigma\, x^2}{\Delta^4}\, J_1(\alpha x)\, Y_1(\beta x)$$

$$\int x^5\, J_1(\alpha x)\, Y_1(\beta x)\, dx = \frac{8\,\alpha\,\beta\,\Delta^2\, x^4 - 96\,\alpha\,\beta\,\sigma\, x^2}{\Delta^4}\, J_0(\alpha x)\, Y_0(\beta x) -$$

$$- \frac{\alpha[\Delta^4\, x^5 - 8\,(\alpha^2 + 5\,\beta^2)\Delta^2\, x^3 + 192\,\beta^2\,\sigma\, x]}{\Delta^5}\, J_0(\alpha x)\, Y_1(\beta x)] +$$

$$+ \frac{\beta[\Delta^4\, x^5 - 8\,(5\,\alpha^2 + \beta^2)\Delta^2\, x^3 + 192\,\alpha^2\,\sigma\, x]}{\Delta^5}\, J_1(\alpha x)\, Y_0(\beta x) +$$

$$+ \frac{4\,\sigma\,\Delta^2\, x^4 - 16\,(\Delta^2 + 8\,\alpha^2\beta^2)\, x^2}{\Delta^4}\, J_1(\alpha x)\, Y_1(\beta x)$$

2.2.6. Integrals of the type $\int x^{2n+1}\cdot J_0(\alpha x)\cdot J_1(\beta x)\,dx$ and $\int x^{2n+1}\cdot I_0(\alpha x)\cdot I_1(\beta x)\,dx$

Holds (for $\beta\neq\alpha$)

$$\int J_0(\alpha x)\,J_0(\beta x)\,dx=\frac{1}{\alpha}\,\Theta_0\left(\alpha x;\frac{\beta}{\alpha}\right)=\frac{1}{\beta}\,\Theta_0\left(\beta x;\frac{\alpha}{\beta}\right)\ ,$$

$$\int J_1(\alpha x)\,J_1(\beta x)\,dx=\frac{1}{\alpha}\,\Theta_1\left(\alpha x;\frac{\beta}{\alpha}\right)=\frac{1}{\beta}\,\Theta_1\left(\beta x;\frac{\alpha}{\beta}\right)\ ,$$

$$\int I_0(\alpha x)\,I_0(\beta x)\,dx=\frac{1}{\alpha}\,\Omega_0\left(\alpha x;\frac{\beta}{\alpha}\right)=\frac{1}{\beta}\,\Omega_0\left(\beta x;\frac{\alpha}{\beta}\right)\ ,$$

$$\int I_1(\alpha x)\,I_1(\beta x)\,dx=\frac{1}{\alpha}\,\Omega_1\left(\alpha x;\frac{\beta}{\alpha}\right)=\frac{1}{\beta}\,\Omega_1\left(\beta x;\frac{\alpha}{\beta}\right)\ .$$

Θ_ν and Ω_ν as definded on pages 303 and 305. In these integrals both Bessel functions are of the same order, so one can suppose $\beta<\alpha$. This relation is not presumed for the product $Z_0(\alpha x)\cdot Z_1(\beta x)$, that means for the following integrals. They were expressed by Θ_ν and Ω_ν, so one can use the according right side of the previous equations.

Let $\boxed{\alpha^2+\beta^2=\sigma \qquad \text{and} \qquad \alpha^2-\beta^2=\Delta\ .}$

$$\int\frac{J_0(\alpha x)\,J_1(\beta x)\,dx}{x}=-J_0(\alpha x)\,J_1(\beta x)+\beta\int J_0(\alpha x)\,J_0(\beta x)\,dx-\alpha\int J_1(\alpha x)\,J_1(\beta x)\,dx$$

$$\int\frac{I_0(\alpha x)\,I_1(\beta x)\,dx}{x}=-I_0(\alpha x)\,I_1(\beta x)+\beta\int I_0(\alpha x)\,I_1(\beta x)\,dx+\alpha\int I_1(\alpha x)\,I_1(\beta x)\,dx$$

$$\int x\,J_0(\alpha x)\,J_1(\beta x)\,dx=$$

$$=\frac{x}{\Delta}\left[\beta\,J_0(\alpha x)\,J_0(\beta x)+\alpha\,J_1(\alpha x)\,J_1(\beta x)\right]-\frac{\beta}{\Delta}\int J_0(\alpha x)\,J_0(\beta x)\,dx+\frac{\alpha}{\Delta}\int J_1(\alpha x)\,J_1(\beta x)\,dx$$

$$\int x\,I_0(\alpha x)\,I_1(\beta x)\,dx=$$

$$=\frac{x}{\Delta}\left[\alpha\,I_1(\alpha x)\,I_1(\beta x)-\beta\,I_0(\alpha x)\,I_0(\beta x)\right]+\frac{\beta}{\Delta}\int I_0(\alpha x)\,I_0(\beta x)\,dx+\frac{\alpha}{\Delta}\int I_1(\alpha x)\,I_1(\beta x)\,dx$$

Let $n=2m+1$ and

$$\int x^n\cdot J_0(\alpha x)\,J_1(\beta x)\,dx=\frac{P_n(x)}{\Delta^n}\,J_0(\alpha x)\,J_0(\beta x)+\frac{Q_n(x)}{\Delta^{n-1}}\,J_0(\alpha x)\,J_1(\beta x)+\frac{R_n(x)}{\Delta^{n-1}}\,J_1(\alpha x)\,J_0(\beta x)+$$

$$+\frac{S_n(x)}{\Delta^n}\,J_1(\alpha x)\,J_1(\beta x)+\frac{U_n}{\Delta^n}\int J_0(\alpha x)\,J_0(\beta x)\,dx+\frac{V_n}{\Delta^n}\int J_1(\alpha x)\,J_1(\beta x)\,dx$$

and

$$\int x^n\cdot I_0(\alpha x)\,I_1(\beta x)\,dx=\frac{\bar P_n(x)}{\Delta^n}\,I_0(\alpha x)\,I_0(\beta x)+\frac{\bar Q_n(x)}{\Delta^{n-1}}\,I_0(\alpha x)\,I_1(\beta x)++\frac{\bar R_n(x)}{\Delta^{n-1}}\,I_1(\alpha x)\,I_0(\beta x)+$$

$$+\frac{\bar S_n(x)}{\Delta^n}\,I_1(\alpha x)\,I_1(\beta x)+\frac{\bar U_n}{\Delta^n}\int I_0(\alpha x)\,I_0(\beta x)\,dx+\frac{\bar V_n}{\Delta^n}\int I_1(\alpha x)\,I_1(\beta x)\,dx\ ,$$

then holds

$$P_3=\beta\,x(\Delta^2x^2-5\,\alpha^2-3\,\beta^2)\ ,\quad Q_3=x^2(\alpha^2+3\,\beta^2)\ ,\quad R_3=-4\,\alpha\,\beta\,x^2\ ,$$
$$S_3=\alpha\,x\,(\Delta^2x^2-\alpha^2-7\,\beta^2)\ ,\quad U_3=\beta(5\,\alpha^2+3\,\beta^2)\ ,\quad V_3=-\alpha\,(\alpha^2+7\,\beta^2)$$

$$\bar P_3=-\beta\,x\,(\Delta^2\,x^2+5\,\alpha^2+3\,\beta^2)\ ,\quad \bar Q_3=-(\alpha^2+3\,\beta^2)\,x^2\ ,\quad \bar R_3=4\,\alpha\,\beta\,x^2$$
$$\bar S_3=\alpha\,x\,(\Delta^2\,x^2+\alpha^2+7\,\beta^2)\ ,\quad \bar U_3=\beta(5\,\alpha^2+3\,\beta^2)\ ,\quad \bar V_3=\alpha\,(\alpha^2+7\,\beta^2)$$

$$P_5 = \beta\, x\, [\Delta^4\, x^4 - 3\,(11\,\alpha^2 + 5\,\beta^2)\Delta^2\, x^2 + 117\,\alpha^4 + 222\,\alpha^2\beta^2 + 45\,\beta^4]$$
$$Q_5 = x^2[(3\,\alpha^2 + 5\,\beta^2)\Delta^2\, x^2 - 9\,\alpha^4 - 138\,\alpha^2\beta^2 - 45\,\beta^4]$$
$$R_5 = -4\,\alpha\,\beta\, x^2[2\,\Delta^2\, x^2 - 27\,\alpha^2 - 21\,\beta^2]$$
$$S_5 = \alpha\, x\, [\Delta^4\, x^4 - 3\,(3\,\alpha^2 + 13\,\beta^2)\Delta^2\, x^2 + 9\,\alpha^4 + 246\,\alpha^2\beta^2 + 129\,\beta^4]$$
$$U_5 = -3\beta\,(39\,\alpha^4 + 74\,\alpha^2\beta^2 + 15\,\beta^4)\,, \quad V_5 = -\,\alpha\,(3\,\alpha^4 + 82\,\alpha^2\beta^2 + 43\,\beta^4)$$

$$\bar{P}_5 = -\beta\, x\,(\Delta^4 x^4 + 3\,(11\,\alpha^2 + 5\,\beta^2)\Delta^2\, x^2 + 117\,\alpha^4 + 222\,\alpha^2\beta^2 + 455\,\beta^4)$$
$$\bar{Q}_5 = -x^2\,[(3\,\alpha^2 + 5\,\beta^2)\Delta^2\, x^2 + 9\,\alpha^4 + 138\,\beta^2\alpha^2 + 45\,\beta^4)]$$
$$\bar{R}_5 = 4\,\alpha\,\beta\, x^2\,[2\,\Delta^2\, x^2 + 27\,\alpha^2 + 21\,\beta^2]$$
$$\bar{S}_5 = \alpha\, x\,[\Delta^4\, x^4 + 3\,(3\,\alpha^2 + 13\,\beta^2)\Delta^2\, x^2 + 9\,\alpha^4 + 246\,\alpha^2\beta^2 + 129\,\beta^4]$$
$$\bar{U}_5 = 3\,\beta\,(39\,\alpha^4 + 74\,\alpha^2\beta^2 + 15\,\beta^4)\,, \quad \bar{V}_5 = 3\,\alpha\,(3\,\alpha^4 + 82\,\alpha^2\beta^2 + 43\,\beta^4)$$

$$P_7 = \beta\, x\,[\Delta^6\, x^6 - 5\,(17\,\alpha^2 + 7\,\beta^2)\Delta^4\, x^4 +$$
$$+15\,(115\,\alpha^4 + 234\,\alpha^2\beta^2 + 35\,\beta^4)\,\Delta^2\, x^2 - 5625\,\alpha^6 - 22965\,\alpha^4\beta^2 - 15915\,\alpha^2\beta^4 - 1575\,\beta^6]$$
$$Q_7 = x^2\,[(5\,\alpha^2 + 7\,\beta^2)\,\Delta^4\, x^4 - 5\,(15\,\alpha^4 + 142\,\alpha^2\beta^2 + 35\,\beta^4)\,\Delta^2\, x^2 +$$
$$225\,\alpha^6 + 8805\,\alpha^4\beta^2 + 12435\,\alpha^2\beta^4 + 1575\,\beta^6]$$
$$R_7 = -4\,\alpha\,\beta\, x^2\,[3\,\Delta^4\, x^4 - 5\,(25\,\alpha^2 + 23\,\beta^2)\Delta^2\, x^2 + 1350\,\alpha^4 + 3540\,\alpha^2\beta^2 + 870\,\beta^4]$$
$$S_7 = \alpha\, x\,[\Delta^6\, x^6 - 5\,(5\,\alpha^2 + 19\,\beta^2)\Delta^4\, x^4 +$$
$$+15\,(15\,\alpha^4 + 242\,\alpha^2\beta^2 + 127\,\beta^4)\Delta^2\, x^2 - 225\,\alpha^6 - 14205\,\alpha^4\beta^2 - 26595\,\alpha^2\beta^4 - 5055\,\beta^6]$$
$$U_7 = 15\,\beta\,(375\,\alpha^6 + 1531\,\alpha^4\beta^2 + 1061\,\alpha^2\beta^4 + 105\,\beta^6)$$
$$V_7 = -15\,\alpha\,(15\,\alpha^6 + 947\,\alpha^4\beta^2 + 1773\,\alpha^2\beta^4 + 337\,\beta^6)$$

$$\bar{P}_7 = -\beta\, x\,[\Delta^6\, x^6 + 5\,(17\,\alpha^2 + 7\,\beta^2)\Delta^4\, x^4 +$$
$$+15\,(115\,\alpha^4 + 234\,\alpha^2\beta^2 + 35\,\beta^4)\,\Delta^2\, x^2 + 5625\,\alpha^6 + 22965\,\alpha^4\beta^2 + 15915\,\alpha^2\beta^4 + 1575\,\beta^6]$$
$$\bar{Q}_7 = -x^2\,[(5\,\alpha^2 + 7\,\beta^2)\,\Delta^4\, x^4 + 5\,(15\,\alpha^4 + 142\,\alpha^2\beta^2 + 35\,\beta^4)\,\Delta^2\, x^2 +$$
$$+225\,\alpha^6 + 8805\,\alpha^4\beta^2 + 12435\,\alpha^2\beta^4 + 1575\,\beta^6]$$
$$\bar{R}_7 = 4\,\alpha\,\beta\, x^2\,[3\,\Delta^4\, x^4 + 5\,(25\,\alpha^2 + 23\,\beta^2)\,\Delta^2\, x^2 + 1350\,\alpha^4 + 3540\,\alpha^2\beta^2 + 870\,\beta^4]$$
$$\bar{S}_7 = \alpha\, x\,[\Delta^6\, x^6 + 5\,(5\,\alpha^2 + 19\,\beta^2)\,\Delta^4\, x^4 +$$
$$+15\,(15\,\alpha^4 + 242\,\alpha^2\beta^2 + 127\,\beta^4)\,\Delta^2\, x^2 + 225\,\alpha^6 + 14205\,\alpha^4\beta^2 + 26595\,\alpha^2\beta^4 + 5055\,\beta^6]$$
$$\bar{U}_7 = 15\,\beta\,(375\,\alpha^6 + 1531\,\alpha^4\beta^2 + 1061\,\alpha^2\beta^4 + 105\,\beta^6)$$
$$\bar{V}_7 = 15\,\alpha\,(15\,\alpha^6 + 947\,\alpha^4\beta^2 + 1773\,\alpha^2\beta^4 + 337\,\beta^6)$$

Recurrence relations:

$$\int x^{2n+1}\, J_0(\alpha x)\, J_1(\beta x)\, dx = \frac{x^{2n-2}}{(n-1)\,\Delta^2}\, \Big\{bx\,[(n-1)\,\Delta\, x^2 - (2n-1)(2n-3)n]\, J_0(\alpha x)\, J_0(\beta x) +$$
$$+[(2\,\sigma\, n - \Delta)\,(n-1)\, x^2 + (2n-1)^2(2n-3)n]\, J_0(\alpha x)\, J_1(\beta x) -$$
$$-\,4\,\alpha\beta\,(n-1)n\, x^2\, J_1(\alpha x)\, J_0(\beta x) + ax\,[(n-1)\,\Delta\, x^2 + (2n-1)^2 n]\, J_1(\alpha x)\, J_1(\beta x)\Big\} -$$
$$-\frac{(2n-1)\,[4\, n(n-1)\,\sigma + \Delta]}{(n-1)\,\Delta^2} \int x^{2n-1}\, J_0(\alpha x)\, J_1(\beta x)\, dx - \frac{(2n-1)^2\,(2n-3)^2\, n}{(n-1)\,\Delta^2} \int x^{2n-3}\, J_0(\alpha x)\, J_1(\beta x)\, dx$$

$$\int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, dx = \frac{x^{2n-2}}{(n-1)\,\Delta^2}\,\Big\{ - bx\,[(n-1)\,\Delta\, x^2 + (2n-1)(2n-3)n]\, I_0(\alpha x)\, I_0(\beta x) -$$
$$-[(2\,\sigma\, n - \Delta)\,(n-1)\,x^2 - (2n-1)^2(2n-3)n]\, I_0(\alpha x)\, I_1(\beta x) +$$
$$+\, 4\,\alpha\beta\,(n-1)n\,x^2\, I_1(\alpha x)\, I_0(\beta x) + \alpha\, x\,[(n-1)\,\Delta\, x^2 - (2n-1)^2\, n]\, I_1(\alpha x)\, I_1(\beta x)\Big\} +$$
$$+\frac{(2n-1)\,[4\,n(n-1)\,\sigma + \Delta]}{(n-1)\,\Delta^2}\int x^{2n-1}\, I_0(\alpha x)\, I_1(\beta x)\, dx - \frac{(2n-1)^2\,(2n-3)^2\, n}{(n-1)\,\Delta^2}\int x^{2n-3}\, I_0(\alpha x)\, I_1(\beta x)\, dx$$

2.3. Bessel Functions with different Arguments x and $x+\alpha$:

2.3.1. Integrals of the type $\int x^{-1}\, Z_\nu(x+\alpha) Z_1(x)\, dx$ and $\int [x(x+\alpha)]^{-1}\, Z_1(x+\alpha) Z_1(x)\, dx$

$$\int \frac{J_1(x) J_0(x+\alpha)}{x}\, dx = \frac{x+\alpha}{\alpha}\Big(J_0(x) J_1(x+\alpha) - J_1(x) J_0(x+\alpha)\Big)$$
$$\int \frac{I_1(x) I_0(x+\alpha)}{x}\, dx = \frac{x+\alpha}{\alpha}\Big(I_0(x) I_1(x+\alpha) - I_1(x) I_0(x+\alpha)\Big)$$
$$\int \frac{K_1(x) K_0(x+\alpha)}{x}\, dx = \frac{x+\alpha}{\alpha}\Big(K_0(x) K_1(x+\alpha) - K_1(x) K_0(x+\alpha)\Big)$$
$$\int \frac{I_1(x) K_0(x+\alpha)}{x}\, dx = -\frac{x+\alpha}{\alpha}\Big(I_0(x) K_1(x+\alpha) + I_1(x) K_0(x+\alpha)\Big)$$
$$\int \frac{J_1(x) J_1(x+\alpha)}{x}\, dx =$$
$$= -\frac{x+\alpha}{\alpha} J_0(x) J_0(x+\alpha) + \frac{x+\alpha}{\alpha^2} J_0(x) J_1(x+\alpha) - \frac{x}{\alpha^2} J_1(x) J_0(x+\alpha) - \frac{x+\alpha}{\alpha} J_1(x) J_1(x+\alpha)$$
$$\int \frac{I_1(x) I_1(x+\alpha)}{x}\, dx =$$
$$= \frac{x+\alpha}{\alpha} I_0(x) I_0(x+\alpha) - \frac{x+\alpha}{\alpha^2} I_0(x) I_1(x+\alpha) + \frac{x}{\alpha^2} I_1(x) I_0(x+\alpha) - \frac{x+\alpha}{\alpha} I_1(x) I_1(x+\alpha)$$
$$\int \frac{K_1(x) K_1(x+\alpha)}{x}\, dx =$$
$$= \frac{x+\alpha}{\alpha} K_0(x) K_0(x+\alpha) + \frac{x+\alpha}{\alpha^2} K_0(x) K_1(x+\alpha) - \frac{x}{\alpha^2} K_1(x) K_0(x+\alpha) - \frac{x+\alpha}{\alpha} K_1(x) K_1(x+\alpha)$$
$$\int \frac{I_1(x) K_1(x+\alpha)}{x}\, dx =$$
$$= -\frac{x+\alpha}{\alpha} I_0(x) K_0(x+\alpha) - \frac{x+\alpha}{\alpha^2} I_0(x) K_1(x+\alpha) - \frac{x}{\alpha^2} I_1(x) K_0(x+\alpha) - \frac{x+\alpha}{\alpha} I_1(x) K_1(x+\alpha)$$
$$\int \frac{J_1(x)\, J_1(x+\alpha)\, dx}{x(x+\alpha)} =$$
$$= \frac{1}{\alpha^3}\,\{2(x+\alpha) J_0(x)\, J_1(x+\alpha) - 2x\, J_1(x)\, J_0(x+\alpha) - \alpha(2x+\alpha)\,[\, J_0(x)\, J_0(x+\alpha) + J_1(x)\, J_1(x+\alpha)]\}$$
$$\int \frac{I_1(x)\, I_1(x+\alpha)\, dx}{x(x+\alpha)} =$$
$$= \frac{1}{\alpha^3}\,\{ -2(x+\alpha) I_0(x)\, I_1(x+\alpha) + 2x\, I_1(x)\, I_0(x+\alpha) + \alpha(2x+\alpha)\,[\, I_0(x)\, I_0(x+\alpha) - I_1(x)\, I_1(x+\alpha)]\}$$
$$\int \frac{K_1(x)\, K_1(x+\alpha)\, dx}{x(x+\alpha)} =$$
$$= \frac{1}{\alpha^3}\,\{2(x+\alpha) K_0(x)\, K_1(x+\alpha) - 2x\, K_1(x)\, K_0(x+\alpha) + \alpha(2x+\alpha)\,[\, K_0(x)\, K_0(x+\alpha) - K_1(x)\, K_1(x+\alpha)]\}$$
$$\int \frac{I_1(x)\, K_1(x+\alpha)\, dx}{x(x+\alpha)} =$$
$$= -\frac{1}{\alpha^3}\,\{2(x+\alpha) I_0(x)\, K_1(x+\alpha) + 2x\, I_1(x)\, K_0(x+\alpha) + \alpha(2x+\alpha)\,[\, I_0(x)\, K_0(x+\alpha) + I_1(x)\, K_1(x+\alpha)]\}$$

2.4. Elementary Function and two Bessel Functions:

2.4.1. Integrals of the type $\int x^{2n+1} \ln x\, Z_\nu^2(x)\, dx$ and $\int x^{2n} \ln x\, Z_0(x)\, Z_1(x)\, dx$

In the following integrals $J_\nu(x)$ may be substituted by $Y_\nu(x)$, $H_\nu^{(1)}(x)$ or $H_\nu^{(2)}(x)$.

$$\int x \ln x\, J_0^2(x)\, dx = \frac{x^2(\ln x - 1)}{2} \left[J_0^2(x) + J_1^2(x)\right] + \frac{x}{2} J_0(x)\, J_1(x)$$

$$\int x \ln x\, I_0^2(x)\, dx = \frac{x^2(\ln x - 1)}{2} \left[I_0^2(x) - I_1^2(x)\right] + \frac{x}{2} I_0(x)\, I_1(x)$$

$$\int x \ln x\, K_0^2(x)\, dx = \frac{x^2(\ln x - 1)}{2} \left[K_0^2(x) - K_1^2(x)\right] - \frac{x}{2} K_0(x)\, K_1(x)$$

$$\int x \ln x\, J_1^2(x)\, dx = \frac{x^2(\ln x - 1) - 1}{2} J_0^2(x) + \frac{x(1 - 2\ln x)}{2} J_0(x)\, J_1(x) + \frac{x^2(\ln x - 1)}{2} J_1^2(x)$$

$$\int x \ln x\, I_1^2(x)\, dx = \frac{x^2(1 - \ln x) - 1}{2} I_0^2(x) + \frac{x(2\ln x - 1)}{2} I_0(x)\, I_1(x) + \frac{x^2(\ln x - 1)}{2} I_1^2(x)$$

$$\int x \ln x\, K_1^2(x)\, dx = \frac{x^2(1 - \ln x) - 1}{2} K_0^2(x) - \frac{x(2\ln x - 1)}{2} K_0(x)\, K_1(x) + \frac{x^2(\ln x - 1)}{2} K_1^2(x)$$

$$\int x^2 \ln x\, J_0(x)\, J_1(x)\, dx = -\frac{x^2}{4} J_0^2(x) + \frac{x}{2} J_0(x)\, J_1(x) + \frac{x^2(2\ln x - 1)}{4} J_1^2(x)$$

$$\int x^2 \ln x\, I_0(x)\, I_1(x)\, dx = \frac{x^2}{4} I_0^2(x) - \frac{x}{2} I_0(x)\, I_1(x) + \frac{x^2(2\ln x - 1)}{4} I_1^2(x)$$

$$\int x^2 \ln x\, K_0(x)\, K_1(x)\, dx = -\frac{x^2}{4} K_0^2(x) - \frac{x}{2} K_0(x)\, K_1(x) + \frac{x^2(1 - 2\ln x)}{4} K_1^2(x)$$

$$\int x^3 \ln x\, J_0^2(x)\, dx =$$

$$= \frac{3x^2 - x^4 + 3x^4 \ln x}{18} J_0^2(x) + \frac{x^3 - 6x + 6x^3 \ln x}{18} J_0(x)\, J_1(x) - \frac{x^4 + x^2 + (6x^2 - 3x^4) \ln x}{18} J_1^2(x)$$

$$\int x^3 \ln x\, I_0^2(x)\, dx =$$

$$= \frac{-3x^2 - x^4 + 3x^4 \ln x}{18} I_0^2(x) + \frac{x^3 + 6x + 6x^3 \ln x}{18} I_0(x)\, I_1(x) - \frac{x^2 - x^4 + (6x^2 + 3x^4) \ln x}{18} I_1^2(x)$$

$$\int x^3 \ln x\, K_0^2(x)\, dx =$$

$$= \frac{-3x^2 - x^4 + 3x^4 \ln x}{18} K_0^2(x) - \frac{x^3 + 6x + 6x^3 \ln x}{18} K_0(x)\, K_1(x) - \frac{x^2 - x^4 + (6x^2 + 3x^4) \ln x}{18} K_1^2(x)$$

$$\int x^3 \ln x\, J_1^2(x)\, dx =$$

$$= -\frac{6x^2 + x^4 - 3x^4 \ln x}{18} J_0^2(x) + \frac{12x + x^3 - 12x^3 \ln x}{18} J_0(x)\, J_1(x) - \frac{x^2 + x^4 - (12x^2 + 3x^4) \ln x}{18} J_1^2(x)$$

$$\int x^3 \ln x\, I_1^2(x)\, dx =$$

$$= \frac{-6x^2 + x^4 - 3x^4 \ln x}{18} I_0^2(x) + \frac{12x - x^3 + 12x^3 \ln x}{18} I_0(x)\, I_1(x) + \frac{x^2 - x^4 - (12x^2 - 3x^4) \ln x}{18} I_1^2(x)$$

$$\int x^3 \ln x\, K_1^2(x)\, dx =$$

$$= \frac{-6\,x^2 + x^4 - 3\,x^4 \ln x}{18}\, K_0^2(x) - \frac{12\,x - x^3 + 12\,x^3 \ln x}{18}\, K_0(x)\, K_1(x) + \frac{x^2 - x^4 - (12x^2 - 3\,x^4) \ln x}{18}\, K_1^2(x)$$

$$\int x^4\, \ln x\, J_0(x)\, J_1(x)\, dx = \frac{12\,x^2 - x^4 - 6\,x^4 \ln x}{36}\, J_0^2(x) +$$

$$+ \frac{5\,x^3 - 12\,x + 12\,x^3 \ln x}{18}\, J_0(x)\, J_1(x) - \frac{10\,x^2 + x^4 + (24x^2 - 12\,x^4) \ln x}{36}\, J_1^2(x)$$

$$\int x^4\, \ln x\, I_0(x)\, I_1(x)\, dx = \frac{12\,x^2 + x^4 + 6\,x^4 \ln x}{36}\, I_0^2(x) -$$

$$- \frac{5\,x^3 + 12\,x + 12\,x^3 \ln x}{18}\, I_0(x)\, I_1(x) + \frac{10\,x^2 - x^4 + (24x^2 + 12\,x^4) \ln x}{36}\, I_1^2(x)$$

$$\int x^4\, \ln x\, K_0(x)\, K_1(x)\, dx = -\frac{12\,x^2 + x^4 + 6\,x^4 \ln x}{36}\, K_0^2(x) -$$

$$- \frac{5\,x^3 + 12\,x + 12\,x^3 \ln x}{18}\, K_0(x)\, K_1(x) - \frac{10\,x^2 - x^4 + (24x^2 + 12\,x^4) \ln x}{36}\, K_1^2(x)$$

Let

$$\int x^{2n+1}\, \ln x\, J_0^2(x)\, dx =$$

$$= \frac{A_n(x) + B_n(x)\, \ln x}{N_n^{(20,20)}}\, J_0^2(x) + \frac{C_n(x) + D_n(x)\, \ln x}{N_n^{(20,11)}}\, J_0(x)\, J_1(x) + \frac{E_n(x) + F_n(x)\, \ln x}{N_n^{(20,02)}}\, J_1^2(x)\,,$$

$$\int x^{2n}\, \ln x\, J_0(x)\, J_1(x)\, dx =$$

$$= \frac{G_n(x) + H_n(x)\, \ln x}{N_n^{(11,20)}}\, J_0^2(x) + \frac{I_n(x) + K_n(x)\, \ln x}{N_n^{(11,11)}}\, J_0(x)\, J_1(x) + \frac{L_n(x) + M_n(x)\, \ln x}{N_n^{(11,02)}}\, J_1^2(x)\,,$$

$$\int x^{2n+1}\, \ln x\, J_1^2(x)\, dx =$$

$$= \frac{P_n(x) + Q_n(x)\, \ln x}{N_n^{(02,20)}}\, J_0^2(x) + \frac{R_n(x) + S_n(x)\, \ln x}{N_n^{(02,11)}}\, J_0(x)\, J_1(x) + \frac{T_n(x) + U_n(x)\, \ln x}{N_n^{(02,02)}}\, J_1^2(x)\,,$$

and let the integrals with $I_\nu(x)$ be described with the polynomials $A_n^*(x), \ldots$ and such with $K_\nu(x)$ written with $A_n^{**}(x)$ ... (the denominators $N_n^{(\mu,\nu,\lambda,\kappa)}$ are the same), then holds

$$N_2^{(20,20)} = 450\,, \quad A_2(x) = -9\,x^6 + 56\,x^4 - 240\,x^2\,, \quad B_2(x) = 45\,x^6 + 120\,x^4\,,$$

$$N_2^{(20,11)} = 450\,, \quad C_2(x) = 9\,x^5 - 344\,x^3 + 480\,x\,, \quad D_2(x) = 180\,x^5 - 480\,x^3\,,$$

$$N_2^{(20,02)} = 450\,, \quad E_2(x) = -9\,x^6 - 52\,x^4 + 344\,x^2\,, \quad F_2(x) = 45\,x^6 - 240\,x^4 + 480\,x^2\,,$$

$$A_2^*(x) = -9\,x^6 - 56\,x^4 - 240\,x^2\,, \quad B_2^*(x) = 45\,x^6 - 120\,x^4\,,$$

$$C_2^*(x) = 9\,x^5 + 344\,x^3 + 480\,x\,, \quad D_2^*(x) = 180\,x^5 + 480\,x^3\,,$$

$$E_2^*(x) = 9\,x^6 - 52\,x^4 - 344\,x^2\,, \quad F_2^*(x) = -45\,x^6 - 240\,x^4 - 480\,x^2\,,$$

$$A_2^{**}(x) = -9\,x^6 - 56\,x^4 - 240\,x^2\,, \quad B_2^{**}(x) = 45\,x^6 - 120\,x^4\,,$$

$$C_2^{**}(x) = -9\,x^5 - 344\,x^3 - 480\,x\,, \quad D_2^{**}(x) = -180\,x^5 - 480\,x^3\,,$$

$$E_2^{**}(x) = 9\,x^6 - 52\,x^4 - 344\,x^2\,, \quad F_2^{**}(x) = -45\,x^6 - 240\,x^4 - 480\,x^2\,,$$

$\int x^4\, \ln x\, Z_0(x)\, Z_1(x)\, dx$ see before.

$$N_2^{(02,02)} = 150\,, \quad P_2(x) = -3x^6 - 23\,x^4 + 120\,x^2\,, \quad Q_2(x) = 15\,x^6 - 60\,x^4\,,$$

$$N_2^{(02,11)} = 150\,, \quad R_2(x) = 3\,x^5 + 152\,x^3 - 240\,x\,, \quad S_2(x) = -90\,x^5 + 240\,x^3\,,$$

$$N_2^{(02,02)} = 150\,, \quad T_2(x) = -3\,x^6 + 16\,x^4 - 152\,x^2\,, \quad U_2(x) = 15\,x^6 + 120\,x^4 - 240\,x^2\,,$$

$$P_2^*(x) = 3\,x^6 - 23\,x^4 - 120\,x^2\,, \quad Q_2^*(x) = -15\,x^6 - 60\,x^4\,,$$
$$R_2^*(x) = -3\,x^5 + 152\,x^3 + 240\,x\,, \quad S_2^*(x) = 90\,x^5 + 240\,x^3\,,$$
$$T_2^*(x) = -3\,x^6 - 16\,x^4 - 152\,x^2\,, \quad U_2^*(x) = 15\,x^6 - 120\,x^4 - 240\,x^2\,,$$

$$P_2^{**}(x) = 3\,x^6 - 23\,x^4 - 120\,x^2\,, \quad Q_2^{**}(x) = -15\,x^6 - 60\,x^4\,,$$
$$R_2^{**}(x) = 3\,x^5 - 152\,x^3 - 240\,x\,, \quad S_2^{**}(x) = -90\,x^5 - 240\,x^3\,,$$
$$T_2^{**}(x) = -3\,x^6 - 16\,x^4 - 152\,x^2\,, \quad U_2^{**}(x) = 15\,x^6 - 120\,x^4 - 240\,x^2\,,$$

$$N_3^{(20,20)} = 2450\,, \quad A_3(x) = -25\,x^8 + 303\,x^6 - 4152\,x^4 + 10080\,x^2\,, \quad B_3(x) = 175\,x^8 + 1260\,x^6 - 5040\,x^4\,,$$
$$N_3^{(20,11)} = 2450\,, \quad C_3(x) = 25\,x^7 - 3078\,x^5 + 21648\,x^3 - 20160\,x\,, \quad D_3(x) = 1050\,x^7 - 7560\,x^5 + 20160\,x^3\,,$$
$$N_3^{(20,02)} = 2450\,, \quad E_3(x) = -25\,x^8 - 297\,x^6 + 5784\,x^4 - 21648\,x^2\,, \quad F_3(x) = 175\,x^8 - 1890\,x^6 + 10080\,x^4 - 20160\,x^2\,,$$

$$A_3^*(x) = -25\,x^8 - 303\,x^6 - 4152\,x^4 - 10080\,x^2\,, \quad B_3^*(x) = 175\,x^8 - 1260\,x^6 - 5040\,x^4\,,$$
$$C_3^*(x) = 25\,x^7 + 3078\,x^5 + 21648\,x^3 + 20160\,x\,, \quad D_3^*(x) = 1050\,x^7 + 7560\,x^5 + 20160\,x^3\,,$$
$$E_3^*(x) = 25\,x^8 - 297\,x^6 - 5784\,x^4 - 21648\,x^2\,, \quad F_3^*(x) = -175\,x^8 - 1890\,x^6 - 10080\,x^4 - 20160\,x^2\,,$$

$$A_3^{**}(x) = -25\,x^8 - 303\,x^6 - 4152\,x^4 - 10080\,x^2\,, \quad B_3^{**}(x) = 175\,x^8 - 1260\,x^6 - 5040\,x^4\,,$$
$$C_3^{**}(x) = -25\,x^7 - 3078\,x^5 - 21648\,x^3 - 20160\,x\,, \quad D_3^{**}(x) = -1050\,x^7 - 7560\,x^5 - 20160\,x^3\,,$$
$$E_3^{**}(x) = 25\,x^8 - 297\,x^6 - 5784\,x^4 - 21648\,x^2\,, \quad F_3^{**}(x) = -175\,x^8 - 1890\,x^6 - 10080\,x^4 - 20160\,x^2\,,$$

$$N_3^{(11,20)} = 300\,, \quad G_3(x) = -3\,x^6 + 152\,x^4 - 480\,x^2\,, \quad H_3(x) = -60\,x^6 + 240\,x^4\,,$$
$$N_3^{(11,11)} = 150\,, \quad I_3(x) = 39\,x^5 - 424\,x^3 + 480\,x\,, \quad K_3(x) = 180\,x^5 - 480\,x^3\,,$$
$$N_3^{(11,02)} = 300\,, \quad L_3(x) = -3\,x^6 - 184\,x^4 + 848\,x^2\,, \quad M_3(x) = 90\,x^6 - 480\,x^4 + 960\,x^2\,,$$

$$G_3^*(x) = 3\,x^6 + 152\,x^4 + 480\,x^2\,, \quad H_3^*(x) = 60\,x^6 + 240\,x^4\,,$$
$$I_3^*(x) = -39\,x^5 - 424\,x^3 - 480\,x\,, \quad K_3^*(x) = -180\,x^5 - 480\,x^3\,,$$
$$L_3^*(x) = -3\,x^6 + 184\,x^4 + 848\,x^2\,, \quad M_3^*(x) = 90\,x^6 + 480\,x^4 + 960\,x^2\,,$$

$$G_3^{**}(x) = -3\,x^6 - 152\,x^4 - 480\,x^2\,, \quad H_3^{**}(x) = -60\,x^6 - 240\,x^4\,,$$
$$I_3^{**}(x) = -39\,x^5 - 424\,x^3 - 480\,x\,, \quad K_3^{**}(x) = -180\,x^5 - 480\,x^3\,,$$
$$L_3^{**}(x) = 3\,x^6 - 184\,x^4 - 848\,x^2\,, \quad M_3^{**}(x) = -90\,x^6 - 480\,x^4 - 960\,x^2\,,$$

$$N_3^{(02,20)} = 2450\,, \quad P_3(x) = -25\,x^8 - 334\,x^6 + 5256\,x^4 - 13440\,x^2\,, \quad Q_3(x) = 175\,x^8 - 1680\,x^6 + 6720\,x^4\,,$$
$$N_3^{(02,11)} = 2450\,, \quad R_3(x) = 25\,x^7 + 3684\,x^5 - 27744\,x^3 + 26880\,x\,, \quad S_3(x) = -1400\,x^7 + 10080\,x^5 - 26880\,x^3\,,$$
$$N_3^{(02,02)} = 2450\,, \quad T_3(x) = -25\,x^8 + 291\,x^6 - 7152\,x^4 + 27744\,x^2\,, \quad U_3(x) = 175\,x^8 + 2520\,x^6 - 13440\,x^4 + 26880\,x^2\,,$$

$$P_3^*(x) = 25\,x^8 - 334\,x^6 - 5256\,x^4 - 13440\,x^2\,, \quad Q_3^*(x) = -175\,x^8 - 1680\,x^6 - 6720\,x^4\,,$$
$$R_3^*(x) = -25\,x^7 + 3684\,x^5 + 27744\,x^3 + 26880\,x\,, \quad S_3^*(x) = 1400\,x^7 + 10080\,x^5 + 26880\,x^3\,,$$

$$T_3^*(x) = -25\,x^8 - 291\,x^6 - 7152\,x^4 - 27744\,x^2\,, \quad U_3^*(x) = 175\,x^8 - 2520\,x^6 - 13440\,x^4 - 26880\,x^2\,,$$

$$P_3^{**}(x) = 25\,x^8 - 334\,x^6 - 5256\,x^4 - 13440\,x^2\,, \quad Q_3^{**}(x) = -175\,x^8 - 1680\,x^6 - 6720\,x^4\,,$$
$$R_3^{**}(x) = 25\,x^7 - 3684\,x^5 - 27744\,x^3 - 26880\,x\,, \quad S_3^{**}(x) = -1400\,x^7 - 10080\,x^5 - 26880\,x^3\,,$$
$$T_3^{**}(x) = -25\,x^8 - 291\,x^6 - 7152\,x^4 - 27744\,x^2\,, \quad U_3^{**}(x) = 175\,x^8 - 2520\,x^6 - 13440\,x^4 - 26880\,x^2\,,$$

Recurrence relations:

$$\int x^{2n+1}\,\ln x\,J_0^2(x)\,dx =$$
$$= -\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 + n\left(4\,n^2+n-2\right) - \left[(2n+1)x^2 + 2\,n^2\left(4\,n^2+n-2\right)\right]\ln x\right\}\,J_0^2(x) -$$
$$-\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 + (4\,n+3)\,n^2 - \left[(2n+1)x^2 + 2\,(4\,n+3)\,(n-1)\,n^2\right]\ln x\right\}\,J_1^2(x) +$$
$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1 + 2n(2n+1)\,\ln x\right]J_0(x)\,J_1(x) -$$
$$-\frac{2\left(4\,n^2+n-2\right)n^3}{(2n+1)^2}\int x^{2n-1}\,\ln x\,J_0^2(x)\,dx - \frac{2\,(4\,n+3)\,(n-1)^2\,n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\,J_1^2(x)\,dx$$

$$\int x^{2n}\,\ln x\,J_0(x)\,J_1(x)\,dx = \frac{x^{2n}}{4}\,\left\{(1-2n\ln x)\,J_0^2(x) + \left[1-2(n-1)\ln x\right]J_1^2(x)\right\} +$$
$$+n^2\int x^{2n-1}\,\ln x\,J_0^2(x)\,dx + (n-1)^2\int x^{2n-1}\,\ln x\,J_1^2(x)\,dx$$

$$\int x^{2n+1}\,\ln x\,J_1^2(x)\,dx =$$
$$= \frac{x^{2n}}{2(2n+1)^2}\left\{-x^2 + 4\,n^3 + 3\,n^2 - 1 + \left[(2n+1)x^2 - 2\,n\left(4\,n^3+3\,n^2-1\right)\right]\ln x\right\}\,J_0^2(x) +$$
$$+\frac{x^{2n}}{2(2n+1)^2}\left\{-x^2 + \left(4\,n^2+5\,n+2\right)n + \left[(2n+1)x^2 - 2\,(n-1)\left(4\,n^2+5\,n+2\right)n\right]\ln x\right\}\,J_1^2(x) +$$
$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1 - 2\,(n+1)\,(2\,n+1)\,\ln x\right]J_0(x)\,J_1(x) +$$
$$+\frac{2\left(4\,n^3+3\,n^2-1\right)n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\,J_0^2(x)\,dx + \frac{2\left(4\,n^2+5\,n+2\right)(n-1)^2\,n}{(2n+1)^2}\int x^{2n-1}\,\ln x\,J_1^2(x)\,dx$$

$$\int x^{2n+1}\,\ln x\,I_0^2(x)\,dx =$$
$$= -\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 - n\left(4\,n^2+n-2\right) + \left[-(2n+1)x^2 + 2\,n^2\left(4\,n^2+n-2\right)\right]\ln x\right\}\,I_0^2(x) +$$
$$+\frac{x^{2n}}{2(2n+1)^2}\left\{x^2 - (4\,n+3)\,n^2 - \left[(2n+1)x^2 - 2\,(4\,n+3)\,(n-1)\,n^2\right]\ln x\right\}\,I_1^2(x) +$$
$$+\frac{x^{2n+1}}{2(2n+1)^2}\left[1 + 2n(2n+1)\,\ln x\right]I_0(x)\,I_1(x) +$$
$$+\frac{2\left(4\,n^2+n-2\right)n^3}{(2n+1)^2}\int x^{2n-1}\,\ln x\,I_0^2(x)\,dx - \frac{2\,(4\,n+3)\,(n-1)^2\,n^2}{(2n+1)^2}\int x^{2n-1}\,\ln x\,I_1^2(x)\,dx$$

$$\int x^{2n}\,\ln x\,I_0(x)\,I_1(x)\,dx = -\frac{x^{2n}}{4}\,\left\{(1-2n\ln x)\,I_0^2(x) - \left[1-2(n-1)\ln x\right]I_1^2(x)\right\} -$$

$$-n^2 \int x^{2n-1} \ln x\, I_0^2(x)\, dx - (n-1)^2 \int x^{2n-1} \ln x\, I_1^2(x)\, dx$$

$$\int x^{2n+1} \ln x\, I_1^2(x)\, dx =$$

$$= \frac{x^{2n}}{2(2n+1)^2} \left\{x^2 + 4\,n^3 + 3\,n^2 - 1 - \left[(2n+1)x^2 + 2\,n\left(4\,n^3 + 3\,n^2 - 1\right)\right] \ln x\right\} I_0^2(x) -$$

$$- \frac{x^{2n}}{2(2n+1)^2} \left\{x^2 + \left(4\,n^2 + 5\,n + 2\right) n - \left[(2n+1)x^2 + 2\,(n-1)\left(4\,n^2 + 5\,n + 2\right) n\right] \ln x\right\} I_1^2(x) +$$

$$- \frac{x^{2n+1}}{2(2n+1)^2} \left[1 - 2\,(n+1)\,(2\,n+1)\,\ln x\right] I_0(x)\, I_1(x) +$$

$$+ \frac{2\,\left(4\,n^3 + 3\,n^2 - 1\right) n^2}{(2n+1)^2} \int x^{2n-1} \ln x\, I_0^2(x)\, dx - \frac{2\,\left(4\,n^2 + 5\,n + 2\right)(n-1)^2\, n}{(2n+1)^2} \int x^{2n-1} \ln x\, I_1^2(x)\, dx$$

$$\int x^{2n+1} \ln x\, K_0^2(x)\, dx =$$

$$= -\frac{x^{2n}}{2(2n+1)^2} \left\{x^2 - n\left(4\,n^2 + n - 2\right) - \left[(2n+1)x^2 - 2\,n^2\left(4\,n^2 + n - 2\right)\right] \ln x\right\} K_0^2(x) +$$

$$+ \frac{x^{2n}}{2(2n+1)^2} \left\{x^2 - (4\,n+3)\,n^2 - \left[(2n+1)x^2 - 2\,(4\,n+3)\,(n-1)\,n^2\right] \ln x\right\} K_1^2(x) +$$

$$- \frac{x^{2n+1}}{2(2n+1)^2} \left[1 + 2n(2n+1)\,\ln x\right] K_0(x)\, K_1(x) +$$

$$+ \frac{2\,\left(4\,n^2 + n - 2\right) n^3}{(2n+1)^2} \int x^{2n-1} \ln x\, K_0^2(x)\, dx - \frac{2\,(4\,n+3)\,(n-1)^2\, n^2}{(2n+1)^2} \int x^{2n-1} \ln x\, K_1^2(x)\, dx$$

$$\int x^{2n} \ln x\, K_0(x)\, K_1(x)\, dx = \frac{x^{2n}}{4} \left\{(1 - 2n \ln x)\, K_0^2(x) - [1 - 2(n-1)\ln x]\, K_1^2(x)\right\} +$$

$$+ n^2 \int x^{2n-1} \ln x\, K_0^2(x)\, dx - (n-1)^2 \int x^{2n-1} \ln x\, K_1^2(x)\, dx$$

$$\int x^{2n+1} \ln x\, K_1^2(x)\, dx =$$

$$= \frac{x^{2n}}{2(2n+1)^2} \left\{x^2 + 4\,n^3 + 3\,n^2 - 1 - \left[(2n+1)x^2 + 2\,n\left(4\,n^3 + 3\,n^2 - 1\right)\right] \ln x\right\} K_0^2(x) -$$

$$- \frac{x^{2n}}{2(2n+1)^2} \left\{x^2 + \left(4\,n^2 + 5\,n + 2\right) n - \left[(2n+1)x^2 + 2\,(n-1)\left(4\,n^2 + 5\,n + 2\right) n\right] \ln x\right\} K_1^2(x) +$$

$$+ \frac{x^{2n+1}}{2(2n+1)^2} \left[1 - 2\,(n+1)\,(2\,n+1)\,\ln x\right] K_0(x)\, K_1(x) +$$

$$+ \frac{2\,\left(4\,n^3 + 3\,n^2 - 1\right) n^2}{(2n+1)^2} \int x^{2n-1} \ln x\, K_0^2(x)\, dx - \frac{2\,\left(4\,n^2 + 5\,n + 2\right)(n-1)^2\, n}{(2n+1)^2} \int x^{2n-1} \ln x\, K_1^2(x)\, dx$$

2.4.2. Integrals of the Type $\int x^n \ln x\, Z_\mu(x)\, Z_\nu^*(x)\, dx$:

Integrals were found in the following cases:

n	J_0I_0	J_0K_0	J_1I_1	J_1K_1	I_0K_0	I_0K_1	I_1K_0	I_1K_1
1			*	*	*			*
2						*	*	
3	*	*			*			*
4						*	*	
5			*	*	*			*
6						*	*	
7	*	*			*			*
8						*	*	
9			*	*	*			*
10						*	*	

Holds $x[I_0(x)K_1(x) + I_1(x)K_0(x)] = 1$ ([5], XIII B. 2.), so any multiple of this expression may be added to these antiderivatives.

a) Integrals with $J_0(x)\, Z_0(x)$:

$$\int x^3 \ln x\, J_0(x)\, I_0(x)\, dx = \frac{x^2 \ln x}{2}\, [x\, J_0(x)I_1(x) + x\, J_1(x)I_0(x) - 2\, J_1(x)I_1(x)] -$$

$$-\frac{x}{2}\, [J_0(x)I_1(x) - J_1(x)I_0(x) + x\, J_1(x)I_1(x)]$$

$$\int x^3 \ln x\, J_0(x)\, K_0(x)\, dx = \frac{x^2 \ln x}{2}\, [-x\, J_0(x)K_1(x) + x\, J_1(x)K_0(x) + 2\, J_1(x)K_1(x)] +$$

$$+\frac{x}{2}\, [J_0(x)K_1(x) + J_1(x)K_0(x) + x\, J_1(x)K_1(x)]$$

$$\int x^7 \ln x\, J_0(x)\, I_0(x)\, dx = \frac{x^2 \ln x}{2}\, \left[48x^2\, J_0(x)\, I_0(x) + x\left(x^4 - 12\,x^2 - 96\right)\, J_0(x)\, I_1(x) + \right.$$

$$\left. + x\left(x^4 + 12\,x^2 - 96\right)\, J_1(x)\, I_0(x) + (192 - 6\,x^4)\, J_1(x)\, I_1(x)\right] + \frac{x}{2}\, \left[32x^3\, J_0(x)\, I_0(x) - \right.$$

$$\left. -(5\,x^4 + 88\,x^2 - 96)\, J_0(x)\, I_1(x) + (5\,x^4 - 96 - 88\,x^2)\, J_1(x)\, I_0(x) + x\left(272 - x^4\right)\, J_1(x)\, I_1(x)\right]$$

$$\int x^7 \ln x\, J_0(x)\, K_0(x)\, dx = \frac{x^2 \ln x}{2}\, \left[48x^2\, J_0(x)\, K_0(x) - x\left(x^4 - 12\,x^2 - 96\right)\, J_0(x)\, K_1(x) + \right.$$

$$\left. + x\left(x^4 + 12\,x^2 - 96\right)\, J_1(x)\, K_0(x) + (6\,x^4 - 192)\, J_1(x)\, K_1(x)\right] + \frac{x}{2}\, \left[32x^3\, J_0(x)\, K_0(x) + \right.$$

$$\left. +(5\,x^4 + 88\,x^2 - 96)\, J_0(x)\, K_1(x) + (5\,x^4 - 88\,x^2 - 96)\, J_1(x)\, K_0(x) + x\left(x^4 - 272\right)\, J_1(x)\, K_1(x)\right]$$

About recurrence formulas see the next page.

b) Integrals with $J_1(x)\, Z_1(x)$:

$$\int x \ln x\, J_1(x)\, I_1(x)\, dx = \frac{x \ln x}{2}\, [J_1(x)I_0(x) - J_0(x)I_1(x)] + \frac{J_0(x)I_0(x)}{2}$$

$$\int x \ln x\, J_1(x)\, K_1(x)\, dx = -\frac{x \ln x}{2}\, [J_1(x)K_0(x) + J_0(x)K_1(x)] - \frac{J_0(x)K_0(x)}{2}$$

$$\int x^5 \ln x\, J_1(x)\, I_1(x)\, dx =$$

$$= \frac{x^2\,\ln x}{2}\,\left[4x^2\,J_0(x)\,I_0(x) - x(x^2+8)\,J_0(x)\,I_1(x) + x(x^2-8)\,J_1(x)\,I_0(x) + 16\,J_1(x)\,I_1(x)\right] +$$

$$+\frac{x}{2}\,\left[x^3\,J_0(x)\,I_0(x) + (8-4x^2)\,J_0(x)\,I_1(x) - (4x^2+8)\,J_1(x)\,I_0(x) + 16x\,J_1(x)\,I_1(x)\right]$$

$$\int x^5\,\ln x\,J_1(x)\,K_1(x)\,dx =$$

$$= \frac{x^2\,\ln x}{2}\,\left[-4x^2\,J_0(x)\,K_0(x) - x(x^2+8)\,J_0(x)\,K_1(x) - x(x^2-8)\,J_1(x)\,K_0(x) + 16\,J_1(x)\,K_1(x)\right] +$$

$$+\frac{x}{2}\,\left[-x^3\,J_0(x)\,K_0(x) + (8-4x^2)\,J_0(x)\,K_1(x) + (8+4x^2)\,J_1(x)\,K_0(x) + 16x\,J_1(x)\,K_1(x)\right]$$

$$\int x^9\,\ln x\,J_1(x)\,I_1(x)\,dx = \frac{x^2\,\ln x}{2}\,\left[8\,x^2\left(x^4-192\right)\,J_0(x)\,I_0(x)+\right.$$

$$+x\left(3072+384\,x^2-32\,x^4-x^6\right)\,J_0(x)\,I_1(x) + x\left(3072-384\,x^2-32\,x^4+x^6\right)\,J_1(x)\,I_0(x)-$$

$$\left.-(6144-192\,x^4)\,J_1(x)\,I_1(x)\right] + \frac{x}{2}\,\left[x^3\left(x^4-1408\right)\,J_0(x)\,I_0(x)-\right.$$

$$-(3072-3584\,x^2-256\,x^4+8\,x^6)\,J_0(x)\,I_1(x) + (3072+3584\,x^2-256\,x^4-8\,x^6)\,J_1(x)\,I_0(x)+$$

$$\left.+80\,x\left(x^4-128\right)\,J_1(x)\,I_1(x)\right]$$

$$\int x^9\,\ln x\,J_1(x)\,K_1(x)\,dx = \frac{x^2\,\ln x}{2}\,\left[8\,x^2\left(192-x^4\right)\,J_0(x)\,K_0(x)+\right.$$

$$+x\left(3072+384\,x^2-32\,x^4-x^6\right)\,J_0(x)\,K_1(x) - x\left(3072-384\,x^2-32\,x^4+x^6\right)\,J_1(x)\,K_0(x)-$$

$$\left.-(6144-192\,x^4)\,J_1(x)\,K_1(x)\right] + \frac{x}{2}\,\left[x^3\left(1408-x^4\right)\,J_0(x)\,K_0(x)-\right.$$

$$-(3072-3584\,x^2-256\,x^4+8\,x^6)\,J_0(x)\,K_1(x) - (3072+3584\,x^2-256\,x^4-8\,x^6)\,J_1(x)\,K_0(x)+$$

$$\left.+80\,x\left(x^4-128\right)\,J_1(x)\,K_1(x)\right]$$

Recurrence Relations:

$$\int x^{4n+3}\,\ln x\,J_0(x)\,I_0(x)\,dx = x^{4n-1}\,\left\{\left[-4\,n^2(4n+1)x\,J_0(x)\,I_0(x) + \left(\frac{x^4}{2}+n(4\,n+1)x^2\right)\,J_0(x)\,I_1(x)\;+\right.\right.$$

$$\left.+\left(\frac{x^4}{2}-n(4\,n+1)x^2\right)\,J_1(x)\,I_0(x) - (2\,n+1)x^3\,J_1(x)\,I_1(x)\right]\,\ln x + n(4\,n+1)x\,J_0(x)\,I_0(x)-$$

$$\left.-\frac{(4\,n+1)x^2}{2}\,\left[J_0(x)\,I_1(x) - J_1(x)\,I_0(x)\right] - \frac{x^3}{2}\,J_1(x)\,I_1(x)\right\} + 16\,n^3(4\,n+1)\,\int x^{4n-1}\,\ln x\,J_0(x)\,I_0(x)\,dx+$$

$$+(16\,n^2+6\,n)\int x^{4n+1}\,\ln x\,J_1(x)\,I_1(x)\,dx$$

$$\int x^{4n+3}\,\ln x\,J_0(x)\,K_0(x)\,dx = x^{4n-1}\,\left\{\left[-4\,n^2(4n+1)x\,J_0(x)\,K_0(x) - \left(\frac{x^4}{2}+n(4\,n+1)x^2\right)\,J_0(x)\,K_1(x)\;+\right.\right.$$

$$\left.+\left(\frac{x^4}{2}-n(4\,n+1)x^2\right)\,J_1(x)\,K_0(x) + (2\,n+1)x^3\,J_1(x)\,K_1(x)\right]\,\ln x + n(4\,n+1)x\,J_0(x)\,K_0(x)+$$

$$\left.+\frac{(4\,n+1)x^2}{2}\,\left[J_0(x)\,K_1(x) + J_1(x)\,K_0(x)\right] + \frac{x^3}{2}\,J_1(x)\,I_K(x)\right\}+16\,n^3(4\,n+1)\,\int x^{4n-1}\,\ln x\,J_0(x)\,K_0(x)\,dx-$$

$$-(16\,n^2+6\,n)\int x^{4n+1}\,\ln x\,J_1(x)\,K_1(x)\,dx$$

$$\int x^{4n+1}\,\ln x\,J_1(x)\,I_1(x)\,dx = x^{4n-3}\,\left\{\left[2\,nx^3\,J_0(x)\,I_0(x) - \left(\frac{x^4}{2}-\frac{8\,n(2n-1)(n-1)x^2}{4\,n-3}\right)\,J_0(x)\,I_1(x)+\right.\right.$$

$$+\left(\frac{x^4}{2}+\frac{8\,n(2n-1)(n-1)x^2}{4\,n-3}\right)\,J_1(x)\,I_0(x)-\frac{16\,(n-1)n(2n-1)^2x}{4\,n-3}\,J_1(x)\,I_1(x)\Big]\,\ln x+\frac{x^3}{2}\,J_0(x)\,I_0(x)-$$

$$-2\,nx^2\,[J_0(x)\,I_1(x)+J_1(x)\,I_0(x)]+\frac{4\,n(2n-1)^2x}{4\,n-3}\,J_1(x)\,I_1(x)+$$

$$+\frac{64\,n(2n-1)^2(n-1)^2}{4n-3}\int x^{4n-3}\,\ln x\,J_1(x)\,K_1(x)\,dx-\frac{8\,n(8\,n^2-9\,n+2)}{4n-3}\int x^{4n-1}\,\ln x\,J_0(x)\,K_0(x)\,dx$$

$$\int x^{4n+1}\,\ln x\,J_1(x)\,K_1(x)\,dx=x^{4n-3}\,\left\{\Big[-2\,nx^3\,J_0(x)\,K_0(x)-\left(\frac{x^4}{2}-\frac{8\,n(2n-1)(n-1)x^2}{4\,n-3}\right)\,J_0(x)\,K_1(x)-\right.$$

$$-\left(\frac{x^4}{2}+\frac{8\,n(2n-1)(n-1)x^2}{4\,n-3}\right)\,J_1(x)\,K_0(x)-\frac{16\,(n-1)n(2n-1)^2x}{4\,n-3}\,J_1(x)\,K_1(x)\Big]\,\ln x-\frac{x^3}{2}\,J_0(x)\,K_0(x)-$$

$$-2\,nx^2\,[J_0(x)\,K_1(x)-J_1(x)\,K_0(x)]+\frac{4\,n(2n-1)^2x}{4\,n-3}\,J_1(x)\,K_1(x)+$$

$$+\frac{64\,n(2n-1)^2(n-1)^2}{4n-3}\int x^{4n-3}\,\ln x\,J_1(x)\,K_1(x)\,dx+\frac{8\,n(8\,n^2-9\,n+2)}{4n-3}\int x^{4n-1}\,\ln x\,J_0(x)\,K_0(x)\,dx$$

c) Integrals with $I_\nu(x)\,K_\nu(x)$:

$$\int x\,\ln x\,I_0(x)\,K_0(x)\,dx=$$

$$=\frac{x^2\,\ln x}{2}\,[I_0(x)K_0(x)+I_1(x)K_1(x)]-\frac{x}{2}\,[x\,I_0(x)K_0(x)-I_1(x)K_0(x)+x\,I_1(x)K_1(x)]$$

$$\int x\,\ln x\,I_1(x)\,K_1(x)\,dx=\frac{x\,\ln x}{2}\,[x\,I_0(x)K_0(x)+I_0(x)K_1(x)-I_1(x)K_0(x)+x\,I_1(x)K_1(x)]+$$

$$+\frac{1}{2}\,\left[(1-x^2)\,I_0(x)K_0(x)-x\,I_0(x)K_1(x)-x^2\,I_1(x)K_1(x)\right]$$

$$\int x^3\,\ln x\,I_0(x)\,K_0(x)\,dx=$$

$$=\frac{x^2\,\ln x}{6}\,\left[\,x^2\,I_0(x)K_0(x)-x\,I_0(x)K_1(x)+x\,I_1(x)K_0(x)+(x^2+2)\,I_1(x)K_1(x)\right]-$$

$$-\frac{x}{36}\,\left[2x(x^2+3)\,I_0(x)K_0(x)+(x^2-4)\,I_0(x)K_1(x)-(x^2+16)\,I_1(x)K_0(x)+2x(x^2-1)\,I_1(x)K_1(x)\right]$$

$$\int x^3\,\ln x\,I_1(x)\,K_1(x)\,dx=$$

$$=\frac{x^2\,\ln x}{6}\,\left[\,x^2\,I_0(x)K_0(x)+2x\,I_0(x)K_1(x)-2x\,I_1(x)K_0(x)+(x^2-4)\,I_1(x)K_1(x)\right]-$$

$$-\frac{x}{36}\,\left[2x(x^2-6)\,I_0(x)K_0(x)+(x^2-4)\,I_0(x)K_1(x)-(x^2-20)\,I_1(x)K_0(x)+2x(x^2-1)\,I_1(x)K_1(x)\right]$$

$$\int x^5\,\ln x\,I_0(x)\,K_0(x)\,dx=$$

$$=\frac{x^2\,\ln x}{30}\,\left[x^2\left(3\,x^2-8\right)\,I_0(x)\,K_0(x)-2\,x\left(8+3\,x^2\right)\,I_0(x)\,K_1(x)+\right.$$

$$\left.+2\,x\left(8+3\,x^2\right)\,I_1(x)\,K_0(x)+(32+16\,x^2+3\,x^4)\,I_1(x)\,K_1(x)\right]+$$

$$+\frac{x}{900}\,\left[-2\,x\left(240+56\,x^2+9\,x^4\right)\,I_0(x)\,K_0(x)+(-9\,x^4-344\,x^2+1376)\,I_1(x)\,K_0(x)+\right.$$

$$\left.+(9\,x^4+344\,x^2+2336)\,I_0(x)\,K_1(x)-2\,x\left(9\,x^4-52\,x^2-344\right)\,I_1(x)\,K_1(x)\right]$$

$$\int x^5 \, \ln x \, I_1(x)\, K_1(x)\, dx =$$

$$= \frac{x^2 \, \ln x}{10} \, \left[x^2(x^2+4)\, I_0(x)\, K_0(x) + x(3x^2+8)\, I_0(x)\, K_1(x) - x(3x^2+8)\, I_1(x)\, K_0(x) + \right.$$

$$\left. +(x^4-8x^2-16)\, I_1(x)\, K_1(x)\right] + \frac{x}{300} \, \left[2\,x\left(120+23\,x^2-3\,x^4\right)\, I_0(x)\, K_0(x) - \right.$$

$$\left. -(3\,x^4+608-152\,x^2)\, I_0(x)\, K_1(x) + (3\,x^4-152\,x^2-1088)\, I_1(x)\, K_0(x) - 2x\left(152+16\,x^2+3x^4\right)\, I_1(x)\, K_1(x)\right]$$

About recurrence relations see page 347.

d) Integrals with $I_\nu(x)\, K_{1-\nu}(x)$:

$$\int x^2 \, \ln x \, I_0(x)\, K_1(x)\, dx = \frac{x^2 \, \ln x}{4} \left[x\, I_0(x)K_1(x) + x\, I_1(x)K_0(x) + 2\, I_1(x)K_1(x)\right] -$$

$$- \frac{x}{8} \left[2x\, I_0(x)K_0(x) + (x^2+4)\, I_0(x)K_1(x) + x^2\, I_1(x)K_0(x) + 2x\, I_1(x)K_1(x)\right]$$

$$\int x^2 \, \ln x \, I_1(x)\, K_0(x)\, dx = \frac{x^2 \, \ln x}{4} \left[x\, I_1(x)K_0(x) + x\, I_0(x)K_1(x) - 2\, I_1(x)K_1(x)\right] +$$

$$+ \frac{x}{8} \left[2x\, I_0(x)K_0(x) - (x^2+4)\, I_1(x)K_0(x) - x^2\, I_0(x)K_1(x) + 2x\, I_1(x)K_1(x)\right]$$

$$\int x^4 \, \ln x \, I_0(x)\, K_1(x)\, dx =$$

$$= \frac{x^2 \, \ln x}{24} \left[-4x^2\, I_0(x)K_0(x) + x(3x^2-8)\, I_0(x)K_1(x) + x(3x^2+8)\, I_1(x)K_0(x) + 8(x^2+2)\, I_1(x)K_1(x)\right] -$$

$$- \frac{x}{288} \left[8x(x^2+12)\, I_0(x)K_0(x) + (9x^4+40x^2-160)\, I_0(x)K_1(x) + \right.$$

$$\left. +(9x^4-40x^2-352)\, I_1(x)K_0(x) + 8x(x^2-10)\, I_1(x)K_1(x)\right]$$

$$\int x^4 \, \ln x \, I_1(x)\, K_0(x)\, dx =$$

$$= \frac{x^2 \, \ln x}{24} \left[4x^2\, I_0(x)K_0(x) + x(3x^2+8)\, I_0(x)K_1(x) + x(3x^2-8)\, I_1(x)K_0(x) - 8(x^2+2)\, I_1(x)K_1(x)\right] +$$

$$+ \frac{x}{288} \left[8x(x^2+12)\, I_0(x)K_0(x) - (160-40x^2+9x^4)\, I_0(x)K_1(x) - \right.$$

$$\left. -(9x^4+40x^2+352)\, I_1(x)K_0(x) + 8x(x^2-10)\, I_1(x)K_1(x)\right]$$

$$\int x^6 \, \ln x \, I_0(x)\, K_1(x)\, dx =$$

$$= \frac{x^2 \, \ln x}{60} \, \left[-12\,x^2\left(4+x^2\right)\, I_0(x)\, K_0(x) + x\left(-96-36\,x^2+5\,x^4\right)\, I_0(x)\, K_1(x) + \right.$$

$$\left. +x\left(96+36\,x^2+5\,x^4\right)\, I_1(x)\, K_0(x) + (192+96\,x^2+18\,x^4)\, I_1(x)\, K_1(x)\right] +$$

$$+ \frac{x}{1800} \left[-6\,x\left(480+152\,x^2+3\,x^4\right)\, I_0(x)\, K_0(x) - (25\,x^6+234\,x^4+2544\,x^2-10176)\, I_1(x)\, K_0(x) - \right.$$

$$\left. -(25\,x^6-234\,x^4-2544\,x^2-15936)\, I_0(x)\, K_1(x) + 6\,x\left(848+184\,x^2-3\,x^4\right)\, I_1(x)\, K_1(x)\right]$$

$$\int x^6 \, \ln x \, I_1(x)\, K_0(x)\, dx =$$

$$= \frac{x^2 \, \ln x}{60} \, \left[12\,x^2\left(4+x^2\right)\, I_0(x)\, K_0(x) + x\left(5\,x^4+36\,x^2+96\right)\, I_0(x)\, K_1(x) + \right.$$

$$\left. +x\left(5\,x^4-36\,x^2-96\right)\, I_1(x)\, K_0(x) - (192+96\,x^2+18\,x^4)\, I_1(x)\, K_1(x)\right] +$$

$$+\frac{x}{1800}\left[6x\left(480+152\,x^2+3\,x^4\right)\,I_0(x)\,K_0(x)+(-10176+2544\,x^2+234\,x^4-25\,x^6)\,I_0(x)\,K_1(x)-\right.$$
$$\left.-(15936+2544\,x^2+234\,x^4+25\,x^6)\,I_1(x)\,K_0(x)+6\,x\left(3\,x^4-848-184\,x^2\right)\,I_1(x)\,K_1(x)\right]$$

Recurrence Relations:

$$\int x^{2n+1}\,\ln x\,I_0(x)\,K_0(x)\,dx=\frac{x^{2n-1}}{2(2n+1)^2}\,\left\{\left[((2n+1)x^3+4n^2(n+1)x)\,I_0(x)\,K_0(x)-2n^2x^2\,I_0(x)\,K_1(x)+\right.\right.$$
$$\left.+2n(n+1)x^2\,I_1(x)\,K_0(x)+(2n+1)x^3\,I_1(x)\,K_1(x)\right]\,\ln x-x\,[x^2+2n(n+1)]\,I_0(x)\,K_0(x)-x^2\,I_0(x)\,K_1(x)-$$
$$\left.-\,x^3\,I_1(x)\,K_1(x)\right\}-\frac{4n^3(n+1)}{(2n+1)^2}\int x^{2n+1}\,\ln x\,I_0(x)\,K_0(x)\,dx+\frac{2n^2}{2n-1}\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx-$$
$$-\frac{4n^2(n+1)}{(2n+1)^2}\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+1}\,\ln x\,I_1(x)\,K_1(x)\,dx=\frac{x^{2n-1}}{2(2n+1)^2}\,\left\{\left[((2n+1)x^3-2n(2n^2+2n+1)x)\,I_0(x)\,K_0(x)+\right.\right.$$
$$\left.+(2n^2+4n+1)x^2\,I_0(x)\,K_1(x)-(2n^2+2n+1)x^2\,I_1(x)\,K_0(x)+(2n+1)x^3\,I_1(x)\,K_1(x)\right]\,\ln x-$$
$$\left.-[x^3-(2n^2+2n+1)x]\,I_0(x)\,K_0(x)\,-x^2\,I_0(x)\,K_1(x)-x^3\,I_1(x)\,K_1(x)\right\}+$$
$$+\frac{2n^2(2n^2+2n+1)}{(2n+1)^2}\int x^{2n-1}\,\ln x\,I_0(x)\,K_0(x)\,dx-\frac{2n(n+1)}{2n+1}\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx+$$
$$+\frac{2n(2n^2+2n+1)}{(2n+1)^2}\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+2}\,\ln x\,I_0(x)\,K_1(x)\,dx=\frac{x^{2n-1}}{2(2n+1)^2}\,\left\{\left[(2n^2(4n^2+5n+2)x-n(2n+1)x^3)\,I_0(x)\,K_0(x)+\right.\right.$$
$$+\left(\frac{(2n+1)^2x^2}{2(n+1)}-n^2\right)\,x^2\,I_0(x)\,K_1(x)+\left(\frac{(2n+1)^2x^2}{2(n+1)}+n(4n^2+5n+2)\right)\,x^2\,I_1(x)\,K_0(x)+$$
$$\left.+\,(2n+1)(n+1)x^3\,I_1(x)\,K_1(x)\right]\,\ln x-\frac{x^3+2n(4n^2+5n+2)x}{2}\,I_0(x)\,K_0(x)-$$
$$\left.-\left(\frac{(2n+1)^2\,x^2}{4(n+1)^2}+2n^2+2n+1\right)\,x^2\;I_0(x)\,K_1(x)-\frac{(2n+1)^2x^4}{4(n+1)^2}\,I_1(x)\,K_0(x)-\frac{x^3}{2}\,I_1(x)\,K_1(x)\right\}-$$
$$-\frac{2n^3(4n^2+5n+2)}{(2n+1)^2}\int x^{2n-1}\,\ln x\,I_0(x)\,K_0(x)\,dx+$$
$$+\frac{2n^2(n+1)}{2n+1}\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx-\frac{2n^2(4n^2+5n+2)}{(2n+1)^2}\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

$$\int x^{2n+2}\,\ln x\,I_1(x)\,K_0(x)\,dx=\frac{x^{2n-1}}{2(2n+1)^2}\,\left\{\left[[(2n+1)nx^3-2n^2(4n^2+5n+2)x]\,I_0(x)\,K_0(x)+\right.\right.$$
$$+\left(\frac{(2n+1)^2}{2(n+1)}\,x^2+n^2\right)\,x^2\,I_0(x)\,K_1(x)+\left(\frac{(2n+1)^2}{2(n+1)}\,x^2-n(4n^2+5n+2)\right)\,x^2\,I_1(x)\,K_0(x)-$$
$$\left.-\,(2n+1)(n+1)x^3\,I_1(x)\,K_1(x)\right]\,\ln x+\left(\frac{x^3}{2}+n(4n^2+5n+2)x\right)\,I_0(x)\,K_0(x)+$$
$$+\left(2n^2+2n+1-\frac{(2n+1)^2\,x^2}{4(n+1)^2}\right)\,x^2\,I_0(x)\,K_1(x)-\frac{(2n+1)^2x^4}{(2n+2)^2}\,I_1(x)\,K_0(x)+$$
$$\left.+\,\frac{x^3}{2}\,I_1(x)\,K_1(x)\right\}+\frac{2n^3(4n^2+5n+2)}{(2n+1)^2}\int x^{2n-1}\,\ln x\,I_0(x)\,K_0(x)\,dx-$$
$$-\frac{2n^2(n+1)}{2n+1}\int x^{2n}\,\ln x\,I_0(x)\,K_1(x)\,dx+\frac{2n^2(4n^2+5n+2)}{(2n+1)^2}\int x^{2n}\,\ln x\,I_1(x)\,K_0(x)\,dx$$

2.4.3. Some Cases of $\int x^n \,\ln x\, Z_\mu(x)\, Z_\nu^*(\alpha x)\, dx$:

n = 4

$$\int x^4 \ln x\, J_1(x)\, I_0\left(\frac{x}{\sqrt{2}}\right)\, dx =$$

$$= \frac{x^2\, \ln x}{3}\, \left[-2x^2\, J_0(x)\, I_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\, x\, J_0(x)\, I_1\left(\frac{x}{\sqrt{2}}\right) + 4x\, J_1(x)\, I_0\left(\frac{x}{\sqrt{2}}\right) + \sqrt{2}\, (x^2-8)\, J_1(x)\, I_1\left(\frac{x}{\sqrt{2}}\right)\right] -$$

$$-\frac{x}{81}\, \left[24x\, J_0(x)\, I_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\, (32-9x^2)\, J_0(x)\, I_1\left(\frac{x}{\sqrt{2}}\right) - (18x^2+176)\, J_1(x)\, I_0\left(\frac{x}{\sqrt{2}}\right) +\right.$$

$$\left. +168\sqrt{2}\, x\, J_1(x)\, I_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4 \ln x\, J_1(x)\, K_0\left(\frac{x}{\sqrt{2}}\right)\, dx =$$

$$= -\frac{x^2\, \ln x}{3}\, \left[2x^2\, J_0(x)\, K_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\, x\, J_0(x)\, K_1\left(\frac{x}{\sqrt{2}}\right) - 4x\, J_1(x)\, K_0\left(\frac{x}{\sqrt{2}}\right) +\right.$$

$$\left. +\sqrt{2}\, (x^2-8)\, J_1(x)\, K_1\left(\frac{x}{\sqrt{2}}\right)\right] - \frac{x}{81}\, \left[24x\, J_0(x)\, K_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\, (9x^2-32)\, J_0(x)\, K_1\left(\frac{x}{\sqrt{2}}\right) -\right.$$

$$\left. -(18x^2+176)\, J_1(x)\, K_0\left(\frac{x}{\sqrt{2}}\right) - 168\sqrt{2}\, x\, J_1(x)\, K_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4 \ln x\, J_0\left(\frac{x}{\sqrt{2}}\right)\, I_1(x)\, dx =$$

$$= \frac{x^2\, \ln x}{3}\, \left[2x^2\, I_0(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) - 4\sqrt{2}\, x\, I_1(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) - 4x\, I_0(x)\, J_1\left(\frac{x}{\sqrt{2}}\right) +\right.$$

$$\left. +\sqrt{2}\, (x^2+8)\, I_1(x)\, J_1\left(\frac{x}{\sqrt{2}}\right)\right] - \frac{x}{81}\, \left[24x\, I_0(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\, (32+9x^2)\, I_1(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) +\right.$$

$$\left. +(18x^2-176)\, I_0(x)\, J_1\left(\frac{x}{\sqrt{2}}\right) - 168\sqrt{2}\, x\, I_1(x)\, J_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4 \ln x\, K_1(x)\, J_0\left(\frac{x}{\sqrt{2}}\right)\, dx = \frac{x^2\, \ln x}{3}\cdot$$

$$\cdot \left[-2x^2\, K_0(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\, x\, K_0(x)\, J_1\left(\frac{x}{\sqrt{2}}\right) - 4x\, K_1(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) + \sqrt{2}\, (x^2+8)\, K_1(x)\, J_1\left(\frac{x}{\sqrt{2}}\right)\right] +$$

$$+\frac{x}{81}\, \left[24x\, K_0(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) + 4\sqrt{2}\, (9x^2+32)\, K_0(x)\, J_1\left(\frac{x}{\sqrt{2}}\right) + (176-18x^2)\, K_1(x)\, J_0\left(\frac{x}{\sqrt{2}}\right) +\right.$$

$$\left. +\, 168\sqrt{2}\, x\, K_1(x)\, J_1\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\int x^4\, \ln x\, J_0(x)\, I_1\left(\sqrt{2}\, x\right)\, dx =$$

$$= \frac{x^2\, \ln x}{3}\left[\sqrt{2}\, x^2\, J_0(x)\, I_0\left(\sqrt{2}\, x\right) - 2x\, J_0(x)\, I_1\left(\sqrt{2}\, x\right) - 2\sqrt{2}\, x\, J_1(x)\, I_0\left(\sqrt{2}\, x\right) + (x^2+4)\, J_1(x)\, I_1\left(\sqrt{2}\, x\right)\right] -$$

$$-\frac{81}{x}\, \left[6\sqrt{2}\, x\, J_0(x)\, I_0\left(\sqrt{2}\, x\right) + (9x^2-44)\, J_0(x)\, I_1\left(\sqrt{2}\, x\right) +\right.$$

$$\left. +2\sqrt{2}\, (9x^2+16)\, J_1(x)\, I_0\left(\sqrt{2}\, x\right) - 84x\, J_1(x)\, I_1\left(\sqrt{2}\, x\right)\right]$$

$$\int x^4\, \ln x\, J_0(x)\, K_1\left(\sqrt{2}\, x\right)\, dx = \frac{x^2\, \ln x}{3}\, \left[-\sqrt{2}\, x^2\, J_0(x)\, K_0\left(\sqrt{2}\, x\right) - 2x\, J_0(x)\, K_1\left(\sqrt{2}\, x\right) +\right.$$

$$\left. +\, 2\sqrt{2}\, x\, J_1(x)\, K_0\left(\sqrt{2}\, x\right) + (x^2+4)\, J_1(x)\, K_1\left(\sqrt{2}\, x\right)\right] + \frac{x}{81}\, \left[6\sqrt{2}\, x\, J_0(x)\, K_0\left(\sqrt{2}\, x\right) +\right.$$

$$+ (44-9x^2)\, J_0(x)\, K_1\left(\sqrt{2}\,x\right) + 2\sqrt{2}\,(9x^2+16)\, J_1(x)\, K_0\left(\sqrt{2}\,x\right) + 84x\, J_1(x)\, K_1\left(\sqrt{2}\,x\right)\Big]$$

$$\int x^4 \ln x\, I_0(x)\, J_1\left(\sqrt{2}\,x\right)\,dx = \frac{x^2 \ln x}{3}\left[-2\sqrt{2}\,x\, I_0(x)\, J_0\left(\sqrt{2}\,x\right) + \right.$$

$$\left. + 2x\, I_0(x)\, J_0\left(\sqrt{2}\,x\right) + 2\sqrt{2}\,x\, I_1(x)\, J_0\left(\sqrt{2}\,x\right) + (x^2-4)\, I_1(x)\, J_1\left(\sqrt{2}\,x\right)\right] + \frac{x}{81}\left[-6\sqrt{2}\,x\, I_0(x)\, J_0\left(\sqrt{2}\,x\right) + \right.$$

$$\left. + (9x^2+44)\, I_0(x)\, J_1\left(\sqrt{2}\,x\right) + 2\sqrt{2}\,(9x^2-16)\, I_1(x)\, J_0\left(\sqrt{2}\,x\right) - 84x\, I_1(x)\, J_1\left(\sqrt{2}\,x\right)\right]$$

$$\int x^4 \ln x\, K_0(x)\, J_1\left(\sqrt{2}\,x\right)\,dx = \frac{x^2 \ln x}{3}\left[-\sqrt{2}\,x^2\, K_0(x)\, J_0\left(\sqrt{2}\,x\right) + 2x\, K_0(x)\, J_1\left(\sqrt{2}\,x\right) - \right.$$

$$\left. -2\sqrt{2}\,x\, K_1(x)\, J_0\left(\sqrt{2}\,x\right) - (x^2-4)\, K_1(x)\, J_1\left(\sqrt{2}\,x\right)\right] + \frac{x}{81}\left[-6\sqrt{2}\,x\, K_0(x)\, J_0\left(\sqrt{2}\,x\right) + \right.$$

$$\left. +(9x^2+44)\, K_0(x)\, J_1\left(\sqrt{2}\,x\right) - 2\sqrt{2}\,(9x^2-16)\, K_1(x)\, J_0\left(\sqrt{2}\,x\right) + 84x\, K_1(x)\, J_1\left(\sqrt{2}\,x\right)\right]$$

n = 5

Let $\lambda = (\sqrt{3}+1)/\sqrt{2} = \sqrt{2+\sqrt{3}} = 1.93185\ 16526$ and $\mu = (\sqrt{3}-1)/\sqrt{2} = \sqrt{2-\sqrt{3}} = 0.51763\ 80902$.

$$\int x^5 \ \ln x\, J_0(x)\cdot I_0(\lambda\, x)\,dx = \frac{x^2 \ln x}{18}\left[-12\,x^2\left(\sqrt{3}-1\right)\, J_0(x)\cdot I_0(\lambda\, x) + \right.$$

$$+3\,x\sqrt{2}\left(-8\sqrt{3}+16+\sqrt{3}x^2\right)\, J_0(x)\cdot I_1(\lambda\, x) -$$

$$-3\,x\left(8-8\sqrt{3}-3\,x^2+\sqrt{3}x^2\right)\, J_1(x)\cdot I_0(\lambda\, x) -$$

$$\left. -12\sqrt{2}\left(-4\sqrt{3}+8+\sqrt{3}x^2-x^2\right)\, J_1(x)\cdot I_1(\lambda\, x)\right] +$$

$$+\frac{x}{594}\left[33\left(\sqrt{3}-1\right)x\left(4\sqrt{3}-4-3\,x^2\right)\, J_0(x)\cdot I_0(\lambda\, x) - \right.$$

$$-2\sqrt{2}\left(8\sqrt{3}-15\right)\left(4\sqrt{3}-64+33\,x^2\right)\, J_0(x)\cdot I_1(\lambda\, x) -$$

$$-2\left(-3+5\sqrt{3}\right)\left(-116+56\sqrt{3}-33\,x^2\right)\, J_1(x)\cdot I_0(\lambda\, x) -$$

$$\left. -33\sqrt{2}\left(\sqrt{3}-1\right)x\left(26\sqrt{3}-26+3\,x^2\right)\, J_1(x)\cdot I_1(\lambda\, x)\right]$$

$$\int x^5 \ \ln x\, J_0(x)\cdot K_0(\lambda\, x)\,dx = \frac{x^2 \ln x}{18}\left[-12\,x^2\left(\sqrt{3}-1\right)\, J_0(x)\cdot K_0(\lambda\, x) + \right.$$

$$-\sqrt{2}\sqrt{3}x\left(16\sqrt{3}-24+3\,x^2\right)\, J_0(x)\cdot K_1(\lambda\, x) + \left(3-\sqrt{3}\right)x\left(8\sqrt{3}+3\,x^2\right)\, J_1(x)\cdot K_0(\lambda\, x) +$$

$$\left. +12\sqrt{2}\left(\sqrt{3}-1\right)\left(x^2+2\sqrt{3}-2\right)\, J_1(x)\cdot K_1(\lambda\, x)\right] \ln x +$$

$$+\frac{x}{594}\left[(33\left(\sqrt{3}-1\right)x\left(4\sqrt{3}-4-3\,x^2\right))\, J_0(x)\cdot K_0(\lambda\, x) + \right.$$

$$+2\sqrt{2}\left(-15+8\sqrt{3}\right)\left(4\sqrt{3}-64+33\,x^2\right)\, J_0(x)\cdot K_1(\lambda\, x) -$$

$$-2\left(-3+5\sqrt{3}\right)\left(-116+56\sqrt{3}-33\,x^2\right)\, J_1(x)\cdot K_0(\lambda\, x) +$$

$$\left. +33\sqrt{2}\left(\sqrt{3}-1\right)x\left(-26+26\sqrt{3}+3\,x^2\right)\, J_1(x)\cdot K_1(\lambda\, x)\right]$$

$$\int x^5 \ \ln x\, J_0(x)\cdot I_0(\mu\, x)\,dx = \frac{x^2 \ln x}{18}\left[12\,x^2\left(1+\sqrt{3}\right)\, J_0(x)\cdot I_0(\mu\, x) + \right.$$

$$+3\,x\sqrt{2}\left(-8\sqrt{3}-16+\sqrt{3}x^2\right)\, J_0(x)\cdot I_1(\mu\, x) +$$

$$+3\,x\left(-8-8\sqrt{3}+3\,x^2+\sqrt{3}x^2\right)\, J_1(x)\cdot I_0(\mu\, x) -$$

$$-12\sqrt{2}\left(-4\sqrt{3}-8+\sqrt{3}x^2+x^2\right)\,J_1(x)\cdot I_1(\mu x)\Big]+$$

$$+\frac{x}{594}\,\Big[33\,\left(1+\sqrt{3}\right)\left(4\sqrt{3}+4+3x^2\right)x\,J_0(x)\cdot I_0(\mu x)+$$

$$+2\sqrt{2}\left(15+8\sqrt{3}\right)\left(4\sqrt{3}+64-33x^2\right)\,J_0(x)\cdot I_1(\mu x)-$$

$$-2\,\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}+33x^2\right)\,J_1(x)\cdot I_0(\mu x)+$$

$$+33\sqrt{2}\left(1+\sqrt{3}\right)\left(26+26\sqrt{3}-3x^2\right)x\,J_1(x)\cdot I_1(\mu x)\Big]$$

$$\int x^5\,\ln x\,J_0(x)\cdot K_0(\mu x)\,dx=\frac{x^2\,\ln x}{18}\Big[12x^2\left(1+\sqrt{3}\right)\,J_0(x)\cdot K_0(\mu x)-$$

$$-3x\sqrt{2}\left(-8\sqrt{3}-16+\sqrt{3}x^2\right)\,J_0(x)\cdot K_1(\mu x)+$$

$$+3x\left(-8-8\sqrt{3}+3x^2+\sqrt{3}x^2\right)\,J_1(x)\cdot K_0(\mu x)+$$

$$+12\sqrt{2}\left(-4\sqrt{3}-8+\sqrt{3}x^2+x^2\right)\,J_1(x)\cdot K_1(\mu x)\Big]\,\ln x+$$

$$+\frac{x}{594}\,\Big[33\,\left(1+\sqrt{3}\right)\left(4\sqrt{3}+4+3x^2\right)x\,J_0(x)\cdot K_0(\mu x)-$$

$$-2\sqrt{2}\left(15+8\sqrt{3}\right)\left(4\sqrt{3}+64-33x^2\right)\,J_0(x)\cdot K_1(\mu x)-$$

$$-2\,\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}+33x^2\right)\,J_1(x)\cdot K_0(\mu x)-$$

$$-33\sqrt{2}\left(1+\sqrt{3}\right)x\left(26+26\sqrt{3}-3x^2\right)\,J_1(x)\cdot K_1(\mu x)\Big]$$

$$\int x^5\,\ln x\,I_0(x)\cdot J_0(\lambda x)\,dx=\frac{x^2\,\ln x}{6}\Big[4\left(\sqrt{3}-1\right)\,x^2\,I_0(x)\cdot J_0(\lambda x)+$$

$$+\sqrt{2}\left(8\sqrt{3}-16+\sqrt{3}x^2\right)\,x\,I_0(x)\cdot J_1(\lambda x)-\left(-8+8\sqrt{3}-3x^2+\sqrt{3}x^2\right)\,x\,I_1(x)\cdot J_0(\lambda x)-$$

$$-4\sqrt{2}\left(4\sqrt{3}-8+\sqrt{3}x^2-x^2\right)\,I_1(x)\cdot J_1(\lambda x)\Big]+$$

$$+\frac{x}{594}\,\Big[33\,\left(\sqrt{3}-1\right)x\left(4\sqrt{3}-4+3x^2\right)I_0(x)\cdot J_0(\lambda x)-$$

$$-2\sqrt{2}\left(8\sqrt{3}-15\right)\left(4\sqrt{3}-64-33x^2\right)\,I_0(x)\cdot J_1(\lambda x)-$$

$$-2\,\left(-3+5\sqrt{3}\right)\left(-116+56\sqrt{3}+33x^2\right)\,I_1(x)\cdot J_0(\lambda x)+$$

$$+33\sqrt{2}\left(\sqrt{3}-1\right)\left(-26+26\sqrt{3}-3x^2\right)x\,I_1(x)\cdot J_1(\lambda x)\Big]$$

$$\int x^5\,\ln x\,I_0(x)\cdot J_0(\mu x)\,dx=\frac{x^2\,\ln x}{18}\Big[-12x^2\left(1+\sqrt{3}\right)\,I_0(x)\cdot J_0(\mu x)+$$

$$+\sqrt{6}\,x\left(24+16\sqrt{3}+3x^2\right)\,I_0(x)\cdot J_1(\mu x)+\left(3+\sqrt{3}\right)\left(8\sqrt{3}+3x^2\right)x\,I_1(x)\cdot J_0(\mu x)-$$

$$-\,12\sqrt{2}\left(1+\sqrt{3}\right)\left(2+2\sqrt{3}+x^2\right)\,I_1(x)\cdot J_1(\mu x)\Big]+$$

$$+\frac{x}{594}\,\Big[33\,\left(1+\sqrt{3}\right)x\left(4\sqrt{3}+4-3x^2\right)\,I_0(x)\cdot J_0(\mu x)+$$

$$+2\sqrt{2}\left(8\sqrt{3}+15\right)\left(4\sqrt{3}+64+33x^2\right)\,I_0(x)\cdot J_1(\mu x)-$$

$$-2\,\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}-33x^2\right)\,I_1(x)\cdot J_0(\mu x)-$$

$$-33\sqrt{2}\left(1+\sqrt{3}\right)\left(26+26\sqrt{3}+3x^2\right)x\,I_1(x)\cdot J_1(\mu x)\Big]$$

$$\int x^5 \ln x\, K_0(x)\cdot J_0(\lambda x)\, dx = \frac{x^2 \ln x}{18}\Big[12\left(\sqrt{3}-1\right) x^2 K_0(x)\cdot J_0(\lambda x) -$$

$$-\sqrt{6}\left(-24+16\sqrt{3}-3x^2\right) x\, K_0(x)\cdot J_1(\lambda x) - \left(-3+\sqrt{3}\right)\left(8\sqrt{3}-3x^2\right) x\, K_1(x)\cdot J_0(\lambda x) -$$

$$-12\sqrt{2}\left(\sqrt{3}-1\right)\left(-2+2\sqrt{3}-x^2\right) K_1(x)\cdot J_1(\lambda x)\Big] +$$

$$+\frac{x}{594}\Big[33\left(\sqrt{3}-1\right)\left(4\sqrt{3}-4+3x^2\right) x\, K_0(x)\cdot J_0(\lambda x) -$$

$$-2\sqrt{2}\left(-15+8\sqrt{3}\right)\left(4\sqrt{3}-64-33x^2\right) K_0(x)\cdot J_1(\lambda x) +$$

$$+2\left(-3+5\sqrt{3}\right)\left(-116+56\sqrt{3}+33x^2\right) K_1(x)\cdot J_0(\lambda x) -$$

$$-33\sqrt{2}\left(\sqrt{3}-1\right)\left(-26+26\sqrt{3}-3x^2\right) x\, K_1(x)\cdot J_1(\lambda x)\Big]$$

$$\int x^5 \ln x\, K_0(x)\cdot J_0(\mu x)\, dx = \frac{x^2 \ln x}{18}\Big[-12x^2\left(1+\sqrt{3}\right) K_0(x)\cdot J_0(\mu x) +$$

$$+\sqrt{6}\left(24+16\sqrt{3}+3x^2\right) x\, K_0(x)\cdot J_1(\mu x) - \left(3+\sqrt{3}\right) x\left(8\sqrt{3}+3x^2\right) K_1(x)\cdot J_0(\mu x) +$$

$$+12\sqrt{2}\left(1+\sqrt{3}\right)\left(2+2\sqrt{3}+x^2\right) K_1(x)\cdot J_1(\mu x)\Big] +$$

$$+\frac{x}{594}\Big[33\left(1+\sqrt{3}\right)\left(4\sqrt{3}+4-3x^2\right) x\, K_0(x)\cdot J_0(\mu x) +$$

$$+2\sqrt{2}\left(15+8\sqrt{3}\right)\left(4\sqrt{3}+64+33x^2\right) K_0(x)\cdot J_0(\mu x) +$$

$$+2\left(3+5\sqrt{3}\right)\left(116+56\sqrt{3}-33x^2\right) K_0(x)\cdot J_0(\mu x) +$$

$$+33\sqrt{2}\left(1+\sqrt{3}\right) x\left(26\sqrt{3}+26+3x^2\right) K_0(x)\cdot J_0(\mu x)\Big]$$

$\mathbf{n = 6}$

Let $\eta = \sqrt{3+\sqrt{6}} = 2.33441\ 42183$ and $\sigma = \sqrt{3+\sqrt{6}} = 0.74196\ 37843$.

$$\int x^6 \ln x\, J_0(x)\cdot I_1(\eta x)\, dx =$$

$$= \frac{x^2 \ln x}{70+30\sqrt{6}}\Big[5\eta\left(2+\sqrt{6}\right)\left(-8+x^2+4\sqrt{6}\right) x^2 J_0(x)\cdot I_0(\eta x) +$$

$$+10\left(5+2\sqrt{6}\right) x\left(-40-x^2+16\sqrt{6}\right) J_0(x)\cdot I_1(\eta x) -$$

$$-2\eta\left(1+\sqrt{6}\right) x\left(-8+8\sqrt{6}+5x^2\right) J_1(x)\cdot I_0(\eta x) +$$

$$+5\left(2+\sqrt{6}\right)\left(-32+8x^2+x^4+16\sqrt{6}\right) J_1(x)\cdot I_1(\eta x)\Big] +$$

$$+\frac{x}{625\left(4+\sqrt{6}\right)^3\left(2+\sqrt{6}\right)\left(11+4\sqrt{6}\right)}\cdot$$

$$\cdot\Big[-\frac{100}{73}\eta\left(331+134\sqrt{6}\right) x\left(-1825x^2+436\sqrt{6}+176\right) J_0(x)\cdot I_0(\eta x) +$$

$$+2\left(664+271\sqrt{6}\right)\left(-18552\sqrt{6}+47168+11850x^2\sqrt{6}-30400x^2-625x^4\right) J_0(x)\cdot I_1(\eta x) -$$

$$-4\eta\left(149+61\sqrt{6}\right)\left(5472\sqrt{6}-11448-2850x^2+3650x^2\sqrt{6}+625x^4\right) J_1(x)\cdot I_0(\eta x) +$$

$$+\frac{100}{67}\left(1672+683\sqrt{6}\right) x\left(1675x^2-21804+10706\sqrt{6}\right) J_1(x)\cdot I_1(\eta x)\Big]$$

$$\int x^6 \ln x\, J_0(x)\cdot I_1(\sigma x)\, dx =$$

$$= \frac{x^2\,\ln x}{70-30\sqrt{6}}\,\left[5\,\sigma\left(2-\sqrt{6}\right)\left(-8+x^2+4\,\sqrt{6}\right)x^2\,J_0(x)\cdot I_0\left(\sigma\,x\right)-\right.$$

$$-10\,\left(5-2\,\sqrt{6}\right)x\left(40+x^2+16\,\sqrt{6}\right)\,J_0(x)\cdot I_1\left(\sigma\,x\right)+$$

$$+2\,\sigma\left(1-\sqrt{6}\right)x\left(8+8\,\sqrt{6}-5\,x^2\right)J_1(x)\cdot I_0\left(\sigma\,x\right)+$$

$$\left.+5\,\left(2-\sqrt{6}\right)\left(-32+8\,x^2+x^4-16\,\sqrt{6}\right)\,J_1(x)\cdot I_1\left(\sigma\,x\right)\right]+$$

$$+\frac{x}{625\,(4-\sqrt{6})^4\,(-2+\sqrt{6})}\cdot$$

$$\cdot\left[\frac{100}{73}\,\sigma\left(-104+41\,\sqrt{6}\right)x\left(1825\,x^2-176+436\,\sqrt{6}\right)\,J_0(x)\cdot I_0\left(\sigma\,x\right)-\right.$$

$$-4\,\left(-103+42\,\sqrt{6}\right)\left(-18552\,\sqrt{6}-47168+11850\,x^2\sqrt{6}+30400\,x^2+625\,x^4\right)\,J_0(x)\cdot I_1\left(\sigma\,x\right)+$$

$$+4\,\sigma\left(-46+19\,\sqrt{6}\right)\left(11448+5472\,\sqrt{6}+3650\,x^2\sqrt{6}+2850\,x^2-625\,x^4\right)\,J_1(x)\cdot I_0\left(\sigma\,x\right)-$$

$$-\frac{200}{67}\,\left(-259+106\,\sqrt{6}\right)x\left(-1675\,x^2+10706\,\sqrt{6}+21804\right)\,J_1(x)\cdot I_1\left(\sigma\,x\right)$$

$$\int x^6\,\ln x\,J_0(x)\cdot K_1\left(\eta\,x\right)\,dx=$$

$$= \frac{x^2\,\ln x}{70+30\sqrt{6}}\,\left[-5\,\eta\left(2+\sqrt{6}\right)\left(-8+x^2+4\,\sqrt{6}\right)x^2\,J_0(x)\cdot K_0\left(\eta\,x\right)+\right.$$

$$+10\,\left(5+2\,\sqrt{6}\right)x\left(-40-x^2+16\,\sqrt{6}\right)\,J_0(x)\cdot K_1\left(\eta\,x\right)+$$

$$+2\,\eta\left(\sqrt{6}+1\right)x\left(8\,\sqrt{6}-8+5\,x^2\right)\,J_1(x)\cdot K_0\left(\eta\,x\right)+$$

$$\left.+5\,\left(2+\sqrt{6}\right)\left(-32+8\,x^2+x^4+16\,\sqrt{6}\right)\,J_1(x)\cdot K_1\left(\eta\,x\right)\right]+$$

$$+\frac{x}{152843750}\cdot\left[670\,\eta\left(-337+143\,\sqrt{6}\right)x\left(176+436\,\sqrt{6}-1825\,x^2\right)\,J_0(x)\cdot K_0\left(\eta\,x\right)+\right.$$

$$+4891\,\left(3\,\sqrt{6}-2\right)\left(47168-18552\,\sqrt{6}+11850\,x^2\sqrt{6}-30400\,x^2-625\,x^4\right)\,J_0(x)\cdot K_1\left(\eta\,x\right)-$$

$$-9782\,\eta\left(4\,\sqrt{6}-11\right)\left(-11448+5472\,\sqrt{6}-2850\,x^2+3650\,x^2\sqrt{6}+625\,x^4\right)\,J_1(x)\cdot K_0\left(\eta\,x\right)-$$

$$\left.-730\,\left(37\,\sqrt{6}-158\right)x\left(-21804+10706\,\sqrt{6}+1675\,x^2\right)\,J_1(x)\cdot K_1\left(\eta\,x\right)\right]$$

$$\int x^6\,\ln x\,J_0(x)\cdot K_1\left(\sigma\,x\right)\,dx=$$

$$= \frac{x^2\,\ln x}{70-30\sqrt{6}}\,\left[-5\,\sigma\left(-2+\sqrt{6}\right)\left(8-x^2+4\,\sqrt{6}\right)x^2\,J_0(x)\cdot K_0\left(\sigma\,x\right)+\right.$$

$$+10\,\left(-5+2\,\sqrt{6}\right)\left(40+x^2+16\,\sqrt{6}\right)x\,J_0(x)\cdot K_1\left(\sigma\,x\right)+$$

$$+2\,\sigma\left(-1+\sqrt{6}\right)\left(8-5\,x^2+8\,\sqrt{6}\right)x\,J_1(x)\cdot K_0\left(\sigma\,x\right)+$$

$$\left.+5\,\left(-2+\sqrt{6}\right)\left(32-8\,x^2-x^4+16\,\sqrt{6}\right)\,J_1(x)\cdot K_1\left(\sigma\,x\right)\right]+\frac{x}{61180296250-24977725625\sqrt{6}}\cdot$$

$$\cdot\left[1675\,\sigma\left(867\,\sqrt{6}-2128\right)\left(436\,\sqrt{6}-176+1825\,x^2\right)x\,J_0(x)\cdot K_0\left(\sigma\,x\right)+\right.$$

$$+4891\,\left(874\,\sqrt{6}-2141\right)\left(-47168-18552\,\sqrt{6}+30400\,x^2+11850\,x^2\sqrt{6}+625\,x^4\right)\,J_0(x)\cdot K_1\left(\sigma\,x\right)+$$

$$+4891\,\sigma\left(393\,\sqrt{6}-962\right)\left(11448+5472\,\sqrt{6}+3650\,x^2\sqrt{6}+2850\,x^2-625\,x^4\right)\,J_1(x)\cdot K_0\left(\sigma\,x\right)+$$

$$+3650\left(2202\sqrt{6}-5393\right)\left(10706\sqrt{6}+21804-1675\,x^2\right)x\,J_1(x)\cdot K_1\left(\sigma\,x\right)\Big]$$

$$\int x^6\,\ln x\,K_0(x)\cdot J_1\left(\eta\,x\right)\,dx=$$

$$=\frac{x^2\,\ln x}{70+30\sqrt{6}}\,\Big[5\,\eta\left(2+\sqrt{6}\right)\left(-8-x^2+4\sqrt{6}\right)x^2\,K_0(x)\cdot J_0\left(\eta\,x\right)-$$

$$-10\left(5+2\sqrt{6}\right)\left(x-4+2\sqrt{6}\right)\left(-x-4+2\sqrt{6}\right)x\,K_0(x)\cdot J_1\left(\eta\,x\right)-$$

$$-2\,\eta\left(1+\sqrt{6}\right)\left(8\sqrt{6}-8-5\,x^2\right)x\,K_1(x)\cdot J_0\left(\eta\,x\right)-$$

$$-5\left(2+\sqrt{6}\right)\left(-32-8\,x^2+x^4+16\sqrt{6}\right)K_1(x)\cdot J_1\left(\eta\,x\right)\Big]+$$

$$+\frac{x}{152843750}\,\Big[670\,\eta\left(-337+143\sqrt{6}\right)\left(176+436\sqrt{6}+1825\,x^2\right)x\,K_0(x)\cdot J_0\left(\eta\,x\right)+$$

$$+4891\left(3\sqrt{6}-2\right)\left(-47168+18552\sqrt{6}-30400\,x^2+11850\,x^2\sqrt{6}+625\,x^4\right)K_0(x)\cdot J_1\left(\eta\,x\right)-$$

$$-9782\eta\left(-11+4\sqrt{6}\right)\left(-5472\sqrt{6}+11448-2850x^2+3650x^2\sqrt{6}-625x^4\right)K_1(x)\cdot J_0\left(\eta\,x\right)+$$

$$+730\left(37\sqrt{6}-158\right)\left(-21804+10706\sqrt{6}-1675\,x^2\right)x\,K_1(x)\cdot J_1\left(\eta\,x\right)\Big]$$

$$\int x^6\,\ln x\,K_0(x)\cdot J_1\left(\sigma\,x\right)\,dx=$$

$$=\frac{x^2\,\ln x}{70-30\sqrt{6}}\,\Big[5\,\sigma\left(-2+\sqrt{6}\right)\left(8+x^2+4\sqrt{6}\right)x^2\,K_0(x)\cdot J_0\left(\sigma\,x\right)+$$

$$+10\left(-5+2\sqrt{6}\right)\left(-x+4+2\sqrt{6}\right)x\left(x+4+2\sqrt{6}\right)K_0(x)\cdot J_1\left(\sigma\,x\right)+$$

$$+2\,\sigma\left(-1+\sqrt{6}\right)x\left(8+5\,x^2+8\sqrt{6}\right)K_1(x)\cdot J_0\left(\sigma\,x\right)-$$

$$-5\left(-2+\sqrt{6}\right)\left(32+8\,x^2-x^4+16\sqrt{6}\right)K_1(x)\cdot J_1\left(\sigma\,x\right)\Big]+$$

$$+\frac{x}{152843750}\,\Big[670\,\sigma\left(337+143\sqrt{6}\right)\left(436\sqrt{6}-176-1825\,x^2\right)x\,K_0(x)\cdot J_0\left(\sigma\,x\right)+$$

$$+4891\left(3\sqrt{6}+2\right)\left(47168+18552\sqrt{6}+30400\,x^2+11850\,x^2\sqrt{6}-625\,x^4\right)K_0(x)\cdot J_1\left(\sigma\,x\right)-$$

$$-9782\,\sigma\left(11+4\sqrt{6}\right)\left(-11448-5472\sqrt{6}+3650x^2\sqrt{6}+2850x^2+625\,x^4\right)K_1(x)\cdot J_0\left(\sigma\,x\right)+$$

$$+730\left(158+37\sqrt{6}\right)x\left(10706\sqrt{6}+21804+1675\,x^2\right)K_1(x)\cdot J_1\left(\sigma\,x\right)\Big]$$

2.4.4. Integrals of the type $\int x^{-1} \cdot \exp/\sinh/\cosh/\sin/\cos(2x)\, Z_\nu(x)\, Z_1(x)\, dx$

$$\int \frac{e^{2x}\, I_0(x)\, I_1(x)\, dx}{x} = e^{2x} \left[(1-x)\, I_0^2(x) + (2x-1) I_0(x) I_1(x) - x\, I_1^2(x)\right]$$

$$\int \frac{e^{2x}\, K_0(x)\, K_1(x)\, dx}{x} = e^{2x} \left[(x-1)\, K_0^2(x) + (2x-1) K_0(x) K_1(x) + x\, K_1^2(x)\right]$$

$$\int \frac{e^{-2x}\, I_0(x)\, I_1(x)\, dx}{x} = -\, e^{-2x} \left[(1+x)\, I_0^2(x) + (2x+1) I_0(x) I_1(x) + x\, I_1^2(x)\right]$$

$$\int \frac{e^{-2x}\, K_0(x)\, K_1(x)\, dx}{x} = e^{-2x} \left[(1+x)\, K_0^2(x) - (2x+1) K_0(x) K_1(x) + x\, K_1^2(x)\right]$$

$$\int \frac{e^{2x}\, I_1^2(x)\, dx}{x} = \frac{e^{2x}}{2} \left[(1-2x)\, I_0^2(x) + 4x I_0(x) I_1(x) - (2x+1)\, I_1^2(x)\right]$$

$$\int \frac{e^{2x}\, K_1^2(x)\, dx}{x} = \frac{e^{2x}}{2} \left[(1-2x)\, K_0^2(x) - 4x K_0(x) K_1(x) - (2x+1)\, K_1^2(x)\right]$$

$$\int \frac{e^{-2x}\, I_1^2(x)\, dx}{x} = \frac{e^{-2x}}{2} \left[(1+2x)\, I_0^2(x) + 4x I_0(x) I_1(x) + (2x-1)\, I_1^2(x)\right]$$

$$\int \frac{e^{-2x}\, K_1^2(x)\, dx}{x} = \frac{e^{-2x}}{2} \left[(1+2x)\, K_0^2(x) - 4x K_0(x) K_1(x) + (2x-1)\, K_1^2(x)\right]$$

$$\int \frac{\sinh 2x\, I_0(x)\, I_1(x)\, dx}{x} = -\sinh 2x \left[x\, I_0^2(x) + I_0(x) I_1(x) + x\, I_1^2(x)\right] + \cosh 2x \left[2x\, I_0(x) I_1(x) + I_0^2(x)\right]$$

$$\int \frac{\sinh 2x\, K_0(x)\, K_1(x)\, dx}{x} = \sinh 2x \left[x\, K_0^2(x) - K_0(x) K_1(x) + x\, K_1^2(x)\right] + \cosh 2x \left[2x\, K_0(x) K_1(x) - K_0^2(x)\right]$$

$$\int \frac{\cosh 2x\, I_0(x)\, I_1(x)\, dx}{x} = \sinh 2x \left[I_0^2(x) + 2x\, I_0(x) I_1(x)\right] - \cosh 2x \left[x\, I_0^2(x) + I_0(x) I_1(x) + x\, I_1^2(x)\right]$$

$$\int \frac{\cosh 2x\, K_0(x)\, K_1(x)\, dx}{x} = \sinh 2x \left[-K_0^2(x) + 2x\, K_0(x) K_1(x)\right] + \cosh 2x \left[x\, K_0^2(x) - K_0(x) K_1(x) + x\, K_1^2(x)\right]$$

$$\int \frac{\sinh 2x\, I_1^2(x)\, dx}{x} = \frac{\sinh 2x}{2} \left[I_0^2(x) + 4x\, I_0(x) I_1(x) - I_1^2(x)\right] - x\, \cosh 2x \left[I_0^2(x) + I_1^2(x)\right]$$

$$\int \frac{\sinh 2x\, K_1^2(x)\, dx}{x} = \frac{\sinh 2x}{2} \left[K_0^2(x) - 4x\, K_0(x) K_1(x) - K_1^2(x)\right] - x\, \cosh 2x \left[K_0^2(x) + K_1^2(x)\right]$$

$$\int \frac{\cosh 2x\, I_1^2(x)\, dx}{x} = \frac{\cosh 2x}{2} \left[I_0^2(x) + 4x\, I_0(x) I_1(x) - I_1^2(x)\right] - x\, \sinh 2x \left[I_0^2(x) + I_1^2(x)\right]$$

$$\int \frac{\cosh 2x\, K_1^2(x)\, dx}{x} = \frac{\cosh 2x}{2} \left[K_0^2(x) - 4x\, K_0(x) K_1(x) - K_1^2(x)\right] - x\, \sinh 2x \left[K_0^2(x) + K_1^2(x)\right]$$

$$\int \frac{\sinh^2 x\, I_0(x)\, I_1(x)\, dx}{x} = -\frac{1}{2} \left[x I_0^2(x) - I_0(x) I_1(x) - x I_1^2(x)\right] +$$

$$+ \frac{\sinh 2x}{2} \left[\, I_0^2(x) + 2x\, I_0(x) I_1(x)\right] - \frac{\cosh 2x}{2} \left[x\, I_0^2(x) + I_0(x) I_1(x) + x\, I_1^2(x)\right]$$

$$\int \frac{\sinh^2 x\, K_0(x)\, K_1(x)\, dx}{x} = \frac{1}{2} \left[x K_1^2(x) + K_0(x) I_1(x) - x K_0^2(x)\right] -$$

$$- \frac{\sinh 2x}{2} \left[\, K_0^2(x) - 2x\, K_0(x) K_1(x)\right] + \frac{\cosh 2x}{2} \left[x\, K_0^2(x) - K_0(x) K_1(x) + x\, K_1^2(x)\right]$$

$$\int \frac{\cosh^2 x\, I_0(x)\, I_1(x)\, dx}{x} = \frac{1}{2}\left[xI_0^2(x) - I_0(x)I_1(x) - xI_1^2(x)\right] +$$
$$+\frac{\sinh 2x}{2}\ \left[I_0^2(x) + 2x\, I_0(x)I_1(x)\right] - \frac{\cosh 2x}{2}\ \left[x\, I_0^2(x) + I_0(x)I_1(x) + x\, I_0^2(x)\right]$$
$$\int \frac{\cosh^2 x\, K_0(x)\, K_1(x)\, dx}{x} = \frac{1}{2}\left[xK_1^2(x) - K_0(x)K_1(x) - xK_0^2(x)\right] +$$
$$-\frac{\sinh 2x}{2}\ \left[K_0^2(x) - 2x\, K_0(x)K_1(x)\right] + \frac{\cosh 2x}{2}\ \left[x\, K_0^2(x) - K_0(x)K_1(x) + x\, K_0^2(x)\right]$$
$$\int \frac{\sinh^2 x\, I_1^2(x)\, dx}{x} =$$
$$= \frac{I_1^2(x) - I_0^2(x)}{4} + \frac{\cosh 2x}{4}\ \left[I_0^2(x) + 4x\, I_0(x)I_1(x) - I_1^2(x)\right] - \frac{x\, \sinh 2x}{2}\ \left[I_0^2(x) + I_1^2(x)\right]$$
$$\int \frac{\sinh^2 x\, K_1^2(x)\, dx}{x} =$$
$$= \frac{K_1^2(x) - K_0^2(x)}{4} + \frac{\cosh 2x}{4}\ \left[K_0^2(x) - 4x\, K_0(x)K_1(x) - K_1^2(x)\right] - \frac{x\, \sinh 2x}{2}\ \left[K_0^2(x) + K_1^2(x)\right]$$
$$\int \frac{\cosh^2 x\, I_1^2(x)\, dx}{x} =$$
$$= \frac{I_0^2(x) - I_1^2(x)}{4} + \frac{\cosh 2x}{4}\ \left[I_0^2(x) + 4x\, I_0(x)I_1(x) - I_1^2(x)\right] - \frac{x\, \sinh 2x}{2}\ \left[I_0^2(x) + I_1^2(x)\right]$$
$$\int \frac{\cosh^2 x\, K_1^2(x)\, dx}{x} =$$
$$= \frac{K_0^2(x) - K_1^2(x)}{4} + \frac{\cosh 2x}{4}\ \left[K_0^2(x) - 4x\, K_0(x)K_1(x) - K_1^2(x)\right] - \frac{x\, \sinh 2x}{2}\ \left[K_0^2(x) + K_1^2(x)\right]$$

$$\int \frac{\sin 2x\, J_0(x)\, J_1(x)\, dx}{x} = \sin 2x\ \left[-x\, J_0^2(x) - J_0(x)J_1(x) + x\, J_1^2(x)\right] + \cos 2x\ \left[2x\, J_0(x)J_1(x) - J_0^2(x)\right]$$
$$\int \frac{\cos 2x\, J_0(x)\, J_1(x)\, dx}{x} = \sin 2x\ \left[J_0^2(x) - 2x\, J_0(x)J_1(x)\right] - \cos 2x\ \left[x\, J_0^2(x) + J_0(x)J_1(x) - x\, J_1^2(x)\right]$$
$$\int \frac{\sin 2x\, J_1^2(x)\, dx}{x} = \frac{\sin 2x}{2}\ \left[-J_0^2(x) + 4x\, J_0(x)J_1(x) - J_1^2(x)\right] + x\, \cos 2x\ \left[J_0^2(x) - J_1^2(x)\right]$$
$$\int \frac{\cos 2x\, J_1^2(x)\, dx}{x} = \frac{\cos 2x}{2}\ \left[-J_0^2(x) + 4x\, J_0(x)J_1(x) - J_1^2(x)\right] - x\, \sin 2x\ \left[J_0^2(x) - J_1^2(x)\right]$$
$$\int \frac{\sin^2 x\, J_0(x)\, J_1(x)\, dx}{x} = \frac{1}{2}\left[xJ_0^2(x) - J_0(x)J_1(x) + xJ_1^2(x)\right] -$$
$$-\frac{\sin 2x}{2}\ \left[J_0^2(x) - 2x\, J_0(x)J_1(x)\right] + \frac{\cos 2x}{2}\ \left[x\, J_0^2(x) + J_0(x)J_1(x) - x\, J_1^2(x)\right]$$
$$\int \frac{\cos^2 x\, J_0(x)\, J_1(x)\, dx}{x} = \frac{1}{2}\left[xJ_0^2(x) - J_0(x)J_1(x) + xJ_1^2(x)\right] +$$
$$+\frac{\sin 2x}{2}\ \left[J_0^2(x) - 2x\, J_0(x)J_1(x)\right] - \frac{\cos 2x}{2}\ \left[x\, J_0^2(x) + J_0(x)J_1(x) - x\, J_0^2(x)\right]$$
$$\int \frac{\sin^2 x\, J_1^2(x)\, dx}{x} =$$
$$= -\frac{J_0^2(x) + J_1^2(x)}{4} + \frac{\cos 2x}{4}\ \left[J_0^2(x) - 4x\, J_0(x)J_1(x) + J_1^2(x)\right] + \frac{x\, \sin 2x}{2}\ \left[J_0^2(x) - J_1^2(x)\right]$$
$$\int \frac{\cos^2 x\, J_1^2(x)\, dx}{x} =$$
$$= -\frac{J_0^2(x) + J_1^2(x)}{4} - \frac{\cos 2x}{4}\ \left[J_0^2(x) - 4x\, J_0(x)J_1(x) + J_1^2(x)\right] - \frac{x\, \sin 2x}{2}\ \left[J_0^2(x) - J_1^2(x)\right]$$

2.3.5. Some Cases of $\int x^n \cdot \exp(\alpha x) \cdot Z_\mu(x)\, Z_\nu(\beta x)\, dx$

<u>**n = 1 :**</u>

$$\int x\, e^{4x}\, I_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[-4x\, I_0(x)\, I_0(3x) + (4x+3)\, I_0(x)\, I_1(3x) + (4x-1)\, I_0(x)\, I_1(3x) - 4x\, I_1(x)\, I_1(3x)\right]$$

$$\int x\, e^{-4x}\, I_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[4x\, I_0(x)\, I_0(3x) + (4x-3)\, I_0(x)\, I_1(3x) + (4x+1)\, I_0(x)\, I_1(3x) + 4x\, I_1(x)\, I_1(3x)\right]$$

$$\int x\, e^{4x}\, K_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[4x\, K_0(x)\, K_0(3x) + (4x+3)\, K_0(x)\, K_1(3x) + (4x-1)\, K_0(x)\, K_1(3x) + 4x\, K_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{-4x}\, K_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[-4x\, K_0(x)\, K_0(3x) + (4x-3)\, K_0(x)\, K_1(3x) + (4x+1)\, K_0(x)\, K_1(3x) - 4x\, K_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{4x}\, I_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[4x\, I_0(x)\, K_0(3x) + (4x+3)\, I_0(x)\, K_1(3x) - (4x-1)\, I_0(x)\, K_1(3x) - 4x\, I_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{-4x}\, I_0(x)\, K_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[-4x\, I_0(x)\, K_0(3x) + (4x-3)\, I_0(x)\, K_1(3x) - (4x+1)\, I_0(x)\, K_1(3x) + 4x\, I_1(x)\, K_1(3x)\right]$$

$$\int x\, e^{4x}\, K_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{4x}}{16}\left[-4x\, K_0(x)\, I_0(3x) + (4x+3)\, K_0(x)\, I_1(3x) - (4x-1)\, K_0(x)\, I_1(3x) + 4x\, K_1(x)\, I_1(3x)\right]$$

$$\int x\, e^{-4x}\, K_0(x)\, I_1(3x)\, dx =$$

$$= \frac{e^{-4x}}{16}\left[4x\, K_0(x)\, I_0(3x) + (4x-3)\, K_0(x)\, I_1(3x) - (4x+1)\, K_0(x)\, I_1(3x) - 4x\, K_1(x)\, I_1(3x)\right]$$

<u>**n = 2 :**</u>

$$\int x^2 \exp\left(\frac{8x}{3}\right) \cdot I_0(x)\, I_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512}\exp\left(\frac{5x}{3}\right)\left[(-64\,x^2 + 24\,x)\, I_0(x)\, I_0\left(\frac{5x}{3}\right) + \right.$$

$$\left. +(64\,x^2 + 120\,x - 45)\, I_0(x)\, I_1\left(\frac{5x}{3}\right) + (64\,x^2 - 72\,x + 27)\, I_1(x)\, I_0\left(\frac{5x}{3}\right) + (-64\,x^2 + 24\,x)\, I_1(x)\, I_1\left(\frac{5x}{3}\right)\right]$$

$$\int x^2 \exp\left(-\frac{8x}{3}\right) \cdot I_0(x)\, I_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512}\exp\left(-\frac{8x}{3}\right)\left[(64\,x^2 + 24\,x)\, I_0(x)\, I_0\left(\frac{5x}{3}\right) + \right.$$

$$\left. +(64\,x^2 - 120\,x - 45)\, I_0(x)\, I_1\left(\frac{5x}{3}\right) + (64\,x^2 + 72\,x + 27)\, I_1(x)\, I_0\left(\frac{5x}{3}\right) + (64\,x^2 + 24\,x)\, I_1(x)\, I_1\left(\frac{5x}{3}\right)\right]$$

$$\int x^2 \exp\left(\frac{8x}{3}\right) \cdot I_0(x)\, K_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512}\exp\left(\frac{8x}{3}\right)\left[(64\,x^2 - 24\,x)\, I_0(x)\, K_0\left(\frac{5x}{3}\right) + \right.$$

$$+(64\,x^2+120\,x-45)\,I_0(x)\,K_1\left(\frac{5x}{3}\right)-(64\,x^2-72\,x+27)\,I_1(x)\,K_0\left(\frac{5x}{3}\right)+(-64\,x^2+24\,x)\,I_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2\exp\left(-\frac{8x}{3}\right)\cdot I_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\exp\left(-\frac{8x}{3}\right)\,\Bigg[-(64\,x^2+24\,x)\,I_0(x)\,K_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2-120\,x-45)\,I_0(x)\,K_1\left(\frac{5x}{3}\right)-(64\,x^2+72\,x+27)\,I_1(x)\,K_0\left(\frac{5x}{3}\right)+(64\,x^2+24\,x)\,I_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2\exp\left(\frac{8x}{3}\right)\cdot K_0(x)\,I_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\exp\left(\frac{8x}{3}\right)\,\Bigg[(-64\,x^2+24\,x)\,K_0(x)\,I_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2+120\,x-45)\,K_0(x)\,I_1\left(\frac{5x}{3}\right)-(64\,x^2-72\,x+27)\,K_1(x)\,I_0\left(\frac{5x}{3}\right)+(64\,x^2-24\,x)\,K_1(x)\,I_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2\exp\left(-\frac{8x}{3}\right)\cdot K_0(x)\,I_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\exp\left(-\frac{8x}{3}\right)\,\Bigg[(64\,x^2+24\,x)\,K_0(x)\,I_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2-120\,x-45)\,K_0(x)\,I_1\left(\frac{5x}{3}\right)-(64\,x^2+72\,x+27)\,K_1(x)\,I_0\left(\frac{5x}{3}\right)-(64\,x^2+24\,x)\,K_1(x)\,I_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2\exp\left(\frac{8x}{3}\right)\cdot K_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\exp\left(\frac{8x}{3}\right)\,\Bigg[(64\,x^2-24\,x)\,K_0(x)\,K_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2+120\,x-45)\,K_0(x)\,K_1\left(\frac{5x}{3}\right)+(64\,x^2-72\,x+27)\,K_1(x)\,K_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2-24\,x)\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

$$\int x^2\exp\left(-\frac{8x}{3}\right)\cdot K_0(x)\,K_1\left(\frac{5x}{3}\right)\,dx=\frac{x}{512}\exp\left(-\frac{8x}{3}\right)\,\Bigg[-(64\,x^2+24\,x)\,K_0(x)\,K_0\left(\frac{5x}{3}\right)+$$

$$+(64\,x^2-120\,x-45)\,K_0(x)\,K_1\left(\frac{5x}{3}\right)+(64\,x^2+72\,x+27)\,K_1(x)\,K_0\left(\frac{5x}{3}\right)-$$

$$-(64\,x^2+24\,x)\,K_1(x)\,K_1\left(\frac{5x}{3}\right)\Bigg]$$

n = 3 : $8/\sqrt{51}=1.12022\;40672,\;\;\sqrt{35/51}=0.82841\;68696$

$$\int x^3\exp\left(\frac{8x}{\sqrt{51}}\right)\cdot J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\,dx=$$

$$=\frac{x}{20480}\exp\left(\frac{8x}{\sqrt{51}}\right)\,\Bigg[\sqrt{35}\,(-64\,\sqrt{51}x^2+408\,x)\,J_0(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)+$$

$$+(1600\,\sqrt{51}x^2-14280\,x+1785\,\sqrt{51})\,J_0(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)+$$

$$+\sqrt{35}\,(-1088\,x^2+408\,\sqrt{51}x-2601\,)\,J_1(x)\,J_0\left(\sqrt{\frac{35}{51}}\,x\right)+(5440\,x^2-680\,\sqrt{51}x)\,J_1(x)\,J_1\left(\sqrt{\frac{35}{51}}\,x\right)\Bigg]$$

n = 4 : $2\sqrt{3}\,/5=0.69282\;03230,\;\;\sqrt{13}\,/5=0.72111\;02551$

$$\int x^4\exp\left(\frac{2\sqrt{3}\,x}{5}\right)\cdot J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)\,dx=$$

$$=\frac{x}{96}\exp\left(\frac{2\sqrt{3}\,x}{5}\right)\,\Bigg[(40\,\sqrt{3}x^3-180\,x^2+150\,\sqrt{3}x)\,J_0(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right)+$$

$$+\sqrt{13}\,(20\,\sqrt{3}x^2 - 150\,x + 125\,\sqrt{3})\,J_0(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right) +$$

$$+(48\,x^3 - 140\,\sqrt{3}x^2 + 750\,x - 625\,\sqrt{3})\,J_1(x)\,J_0\left(\frac{\sqrt{13}\,x}{5}\right) +$$

$$+\sqrt{13}\,(8\,\sqrt{3}x^3 - 60\,x^2 + 50\,\sqrt{3}x)\,J_1(x)\,J_1\left(\frac{\sqrt{13}\,x}{5}\right)\Bigg]$$

$4/\sqrt{5} = 1.78885\;43820,\;\;\sqrt{7/15} = 0.68313\;00511$

$$\int x^4 \exp\left(\frac{4x}{\sqrt{5}}\right) \cdot J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\,dx =$$

$$= \frac{x}{7168} \exp\left(\frac{4x}{\sqrt{5}}\right)\,\Bigg[\sqrt{21}\,(-64\,\sqrt{5}x^3 + 240\,x^2 - 60\,\sqrt{5}x)\,J_0(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right) +$$

$$+(1344\,\sqrt{5}x^3 - 3360\,x^2 + 1260\,\sqrt{5}x - 1575)\,J_0(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right) +$$

$$+\sqrt{21}\,(-192\,x^3 + 288\,\sqrt{5}x^2 - 900\,x + 225\,\sqrt{5})\,J_1(x)\,J_0\left(\sqrt{\frac{7}{15}}\,x\right) +$$

$$+(1344\,x^3 - 1008\,\sqrt{5}x^2 + 1260\,x)\,J_1(x)\,J_1\left(\sqrt{\frac{7}{15}}\,x\right)\Bigg]$$

2.4.6. Some Cases of $\int x^n \cdot \left\{ \begin{array}{c} \sin\;/\;\cos \\ \sinh/\cosh \end{array} \right\} \alpha x \cdot Z_\mu(x)\, Z_\nu(\beta x)\, dx$

Some integrals, are left out, where α and β are roots of cubic equations.
With the integral $\int w(\alpha x)\, Z_\nu(x)\, Z_\nu(\beta x)\, dx$ the integral

$$\int w\left(\frac{\alpha}{\beta}\, x\right)\, Z_\nu(x)\, Z_\nu\left(\frac{x}{\beta}\right)\, dx$$

may be found. So in the following tables only one of both integrals is given.

Numerical values of the coefficients:

α		β		α/β	$1/\beta$
$8/\sqrt{51}$	1.12022 40672	$\sqrt{35/51}$	0.82841 68696	1.35224 68076	1.20712 17242
$2\sqrt{3/13}$	0.96076 89228	$5/\sqrt{13}$	1.38675 04906	0.69282 03230	0.72111 02551
$4/\sqrt{5}$	1.78885 43820	$\sqrt{7/15}$	0.68313 00511	2.61861 46828	1.46385 01094
$2/\sqrt{7}$	0.75592 89460	$\sqrt{3/7}$	0.65465 36707	1.15470 05384	1.52752 52317
$2\sqrt{3/5}$	0.69282 03230	$\sqrt{13/5}$	0.72111 02551	0.96076 89228	1.38675 04906
$4/\sqrt{11}$	1.20604 53783	$\sqrt{3/11}$	0.52223 29679	2.30940 10768	1.91485 42155

2.4.6 a) $\int x^n \cdot \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} \alpha x \cdot Z_\mu(x)\, Z_\nu(\beta x)\, dx$:

<u>**n = 1 :**</u>

$$\int x \sin 4x \cdot J_0(x)\, J_1(3x)\, dx = \frac{x^2 \sin 4x}{4}\, [J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x)] +$$

$$+\frac{x \cos 4x}{16}\, [4x\, J_0(x)\, J_0(3x) - 3\, J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x) - 4x\, J_1(x)\, J_1(3x)]$$

$$\int x \cos 4x \cdot J_0(x)\, J_1(3x)\, dx = \frac{x^2 \cos 4x}{4}\, [J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x)] +$$

$$-\frac{x \sin 4x}{16}\, [4x\, J_0(x)\, J_0(3x) - 3\, J_0(x)\, J_1(3x) + J_1(x)\, J_0(3x) - 4x\, J_1(x)\, J_1(3x)]$$

<u>**n = 2 :**</u>

$$\int x^2 \sin \frac{8x}{3} \cdot J_0(x)\, J_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512} \cdot \sin \frac{8x}{3} \cdot$$

$$\cdot \left[-24x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) + (64x^2+45)\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + (64x^2-27)\, J_1(x)\, J_0\left(\frac{5x}{3}\right) + 24x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right] +$$

$$+\frac{x^2}{64} \cdot \cos \frac{8x}{3} \left[8x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) - 15\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + 9\, J_1(x)\, J_0\left(\frac{5x}{3}\right) - 8x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right]$$

$$\int x^2 \cos \frac{8x}{3} \cdot J_0(x)\, J_1\left(\frac{5x}{3}\right)\, dx = \frac{x}{512} \cdot \cos \frac{8x}{3} \cdot$$

$$\cdot \left[-24x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) + (64x^2+45)\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + (64x^2-27)\, J_1(x)\, J_0\left(\frac{5x}{3}\right) + 24x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right] -$$

$$-\frac{x^2}{64} \cdot \sin \frac{8x}{3} \left[8x\, J_0(x)\, J_0\left(\frac{5x}{3}\right) - 15\, J_0(x)\, J_1\left(\frac{5x}{3}\right) + 9\, J_1(x)\, J_0\left(\frac{5x}{3}\right) - 8x\, J_1(x)\, J_1\left(\frac{5x}{3}\right)\right]$$

<u>**n = 3 :**</u> $\lambda = 8/\sqrt{51}\,,\ \mu = \sqrt{35/51}$

$$\int x^3 \sin \lambda x \cdot I_0(x)\, I_1(\mu x)\, dx = \frac{x^2}{2560}\, \left[8\sqrt{1785}\, x\, I_0(x)\, I_0(\mu x) + 1785\, I_0(x)\, I_1(\mu x) - \right.$$

$$\left. -51\sqrt{1785}\, I_1(x)\, I_0(\mu x) + 680x\, I_1(x)\, I_1(\mu x)\right]\, \sin \lambda x +$$

$$+\frac{x}{20480}\left[408\sqrt{35}\,x\,I_0(x)\,I_0(\mu x)+\sqrt{51}\,(1785-1600\,x^2)\,I_0(x)\,I_1(\mu x)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,I_0(\mu x)+680\sqrt{51}\,x\,I_1(x)\,I_1(\mu x)\right]\cos\lambda x$$

$$\int x^3\cos\lambda x\cdot I_0(x)\,I_1(\mu x)\,dx=\frac{x}{20480}\left[-408\sqrt{35}x\;I_0(x)\,I_0(\mu x)+\right.$$

$$+\sqrt{51}\,(1600\,x^2-1785)\,I_0(x)\,I_1(\mu x)+\sqrt{35}\,(2601-1088\,x^2)\,I_1(x)\,I_0(\mu x)-$$

$$\left.-680\sqrt{51}\,x\,I_1(x)\,I_1(\mu x)\right]\sin\lambda x+\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,I_0(x)\,I_0(\mu x)+\right.$$

$$\left.+1785\,I_0(x)\,I_1(\mu x)-51\sqrt{1785}\,I_1(x)\,I_0(\mu x)+680\,x\,I_1(x)\,I_1(\mu x)\right]\cos\lambda x$$

$$\int x^3\sin\lambda x\cdot K_0(x)\,K_1(\mu x)\,dx=-\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,K_0(\mu x)-\right.$$

$$\left.-1785\,K_0(x)\,K_1(\mu x)+51\sqrt{1785}\,K_1(x)\,K_0(\mu x)+680x\,K_1(x)\,K_1(\mu x)\right]\sin\lambda x-$$

$$-\frac{x}{20480}\left[408\sqrt{35}\,x\,K_0(x)\,K_0(\mu x)-\sqrt{51}\,(1785-1600\,x^2)K_0(x)\,K_1(\mu x)+\right.$$

$$\left.-\sqrt{35}\,(1088\,x^2-2601)K_1(x)\,K_0(\mu x)+680\sqrt{51}\,x\,K_1(x)\,K_1(\mu x)\right]\cos\lambda x$$

$$\int x^3\cos\lambda x\cdot K_0(x)\,K_1(\mu x)\,dx=\frac{x}{20480}\left[408\sqrt{35}x\;K_0(x)\,K_0(\mu x)+\right.$$

$$+\sqrt{51}\,(1600\,x^2-1785)\,K_0(x)\,K_1(\mu x)+\sqrt{35}\,(2601-1088\,x^2)\,K_1(x)\,K_0(\mu x)+$$

$$\left.+680\sqrt{51}\,x\,K_1(x)\,K_1(\mu x)\right]\sin\lambda x-\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,K_0(\mu x)-\right.$$

$$\left.-1785\,K_0(x)\,K_1(\mu x)+51\sqrt{1785}\,K_1(x)\,K_0(\mu x)+680\,x\,K_1(x)\,K_1(\mu x)\right]\cos\lambda x$$

$$\int x^3\sin\lambda x\cdot I_0(x)\,K_1(\mu x)\,dx=\frac{x^2}{2560}\left[-8\sqrt{1785}\,x\,I_0(x)\,K_0(\mu x)+\right.$$

$$\left.+1785\,I_0(x)\,K_1(\mu x)+51\sqrt{1785}\,I_1(x)\,K_0(\mu x)+680x\,I_1(x)\,K_1(\mu x)\right]\sin\lambda x-$$

$$-\frac{x}{20480}\left[408\sqrt{35}\,x\,I_0(x)\,K_0(\mu x)-\sqrt{51}\,(1785-1600\,x^2)I_0(x)\,K_1(\mu x)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,K_0(\mu x)-680\sqrt{51}\,x\,I_1(x)\,K_1(\mu x)\right]\cos\lambda x$$

$$\int x^3\cos\lambda x\cdot I_0(x)\,K_1(\mu x)\,dx=\frac{x^2}{2560}\left[-8\sqrt{1785}\,x\,I_0(x)\,K_0(\mu x)+\right.$$

$$\left.+1785\,I_0(x)\,K_1(\mu x)+51\sqrt{1785}\,I_1(x)\,K_0(\mu x)+680x\,I_1(x)\,K_1(\mu x)\right]\cos\lambda x+$$

$$+\frac{x}{20480}\left[408\sqrt{35}\,x\,I_0(x)\,K_0(\mu x)-\sqrt{51}\,(1785-1600\,x^2)I_0(x)\,K_1(\mu x)+\right.$$

$$\left.+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,K_0(\mu x)-680\sqrt{51}\,x\,I_1(x)\,K_1(\mu x)\right]\sin\lambda x$$

$$\int x^3\sin\lambda x\cdot K_0(x)\,I_1(\mu x)\,dx=\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,I_0(\mu x)+\right.$$

$$\left.+1785\,K_0(x)\,I_1(\mu x)+51\sqrt{1785}\,K_1(x)\,I_0(\mu x)-680x\,K_1(x)\,I_1(\mu x)\right]\sin\lambda x+$$

$$+\frac{x}{20480}\left[408\sqrt{35}\,x\,K_0(x)\,I_0(\mu x)+\sqrt{51}\,(1785-1600\,x^2)K_0(x)\,I_1(\mu x)-\right.$$

$$\left.-\sqrt{35}\,(1088\,x^2-2601)\,K_1(x)\,I_0(\mu x)-680\sqrt{51}\,x\,K_1(x)\,I_1(\mu x)\right]\cos\lambda x$$

$$\int x^3\cos\lambda x\cdot K_0(x)\,I_1(\mu x)\,dx=\frac{x^2}{2560}\left[8\sqrt{1785}\,x\,K_0(x)\,I_0(\mu x)+\right.$$

$$+1785\,K_0(x)\,I_1(\mu x)+51\sqrt{1785}\,K_1(x)\,I_0(\mu x)-680x\,K_1(x)\,I_1(\mu x)\Big]\,\cos\lambda x+$$

$$+\frac{x}{20480}\,\Big[-408\,\sqrt{35}\,x\,K_0(x)\,I_0(\mu x)-\sqrt{51}\,(1785-1600\,x^2)\,K_0(x)\,I_1(\mu x)+$$

$$+\sqrt{35}\,(1088\,x^2-2601)I_1(x)\,K_0(\mu x)+680\,\sqrt{51}\,x\,I_1(x)\,K_1(\mu x)\Big]\,\sin\lambda x$$

n = 4 : $\sigma=2\,\sqrt{3/13}\,,\ \varrho=5/\sqrt{13}$

$$\int x^4\,\sin\,\sigma x\,I_0(x)\,I_0(\varrho x)\,dx=\frac{x^2}{80}\,\Big[78x\,I_0(x)\,I_0(\varrho x)+\sqrt{13}(8\,x^2-65)\,I_0(x)\,I_1(\varrho x)+$$

$$+169\,I_1(x)\,I_0(\varrho x)-26\,\sqrt{13}x\,I_1(x)\,I_1(\varrho x)\Big]\,\sin\,\sigma x+\frac{x}{480}\,\Big[\sqrt{39}\,x(78-40\,x^2)\,I_0(x)\,I_0(\varrho x)+$$

$$+\sqrt{3}\,(364\,x^2-845)\,I_0(x)\,I_1(\varrho x)+\sqrt{39}(169-52\,x^2)\,I_1(x)\,I_0(\varrho x)+$$

$$+\sqrt{3}\,x(104\,x^2-338)\,I_1(x)\,I_1(\varrho x)\Big]\,\cos\,\sigma x$$

$$\int x^4\,\cos\,\sigma x\,I_0(x)\,I_0(\varrho x)\,dx=\frac{x^2}{80}\,\Big[78x\,I_0(x)\,I_0(\varrho x)+\sqrt{13}(8\,x^2-65)\,I_0(x)\,I_1(\varrho x)+$$

$$+169\,I_1(x)\,I_0(\varrho x)-26\,\sqrt{13}x\,I_1(x)\,I_1(\varrho x)\Big]\,\cos\,\sigma x+\frac{x}{480}\,\Big[\sqrt{39}\,x(40\,x^2-78)\,I_0(x)\,I_0(\varrho x)-$$

$$-\sqrt{3}\,(364\,x^2-845)\,I_0(x)\,I_1(\varrho x)-\sqrt{39}(169-52\,x^2)\,I_1(x)\,I_0(\varrho x)-$$

$$-\sqrt{3}\,x(104\,x^2-338)\,I_1(x)\,I_1(\varrho x)\Big]\,\sin\,\sigma x$$

$$\int x^4\,\sin\sigma x\,K_0(x)\,K_0(\varrho x)\,dx=\frac{x^2}{80}\,[78x\,K_0(x)\,K_0(\varrho x)+$$

$$+\sqrt{13}\,(65-8x^2)\,K_0(x)\,K_1(\varrho x)-169\,K_1(x)\,K_0(\varrho x)-26\,\sqrt{13}\,x\,K_1(x)\,K_1(\varrho x)\Big]\,\sin\sigma x+$$

$$+\frac{x}{480}\,\Big[\sqrt{39}\,x(78-40x^2)\,K_0(x)\,K_0(\varrho x)+\sqrt{3}\,(845-364x^2)\,K_0(x)\,K_1(\varrho x)+$$

$$-\sqrt{39}\,(169-52x^2)\,K_1(x)\,K_0(\varrho x)-\sqrt{3}\,x(338-104x^2)\,K_1(x)\,K_1(\varrho x)\Big]\,\cos\sigma x$$

$$\int x^4\,\cos\sigma x\,K_0(x)\,K_0(\varrho x)\,dx=\frac{x^2}{80}\,[78x\,K_0(x)\,K_0(\varrho x)+$$

$$+\sqrt{13}\,(65-8x^2)\,K_0(x)\,K_1(\varrho x)-169\,K_1(x)\,K_0(\varrho x)-26\,\sqrt{13}\,x\,K_1(x)\,K_1(\varrho x)\Big]\,\cos\sigma x-$$

$$-\frac{x}{480}\,\Big[\sqrt{39}\,x(78-40x^2)\,K_0(x)\,K_0(\varrho x)-\sqrt{3}\,(845-364x^2)\,K_0(x)\,K_1(\varrho x)-$$

$$-\sqrt{39}\,(169-52x^2)\,K_1(x)\,K_0(\varrho x)-\sqrt{3}\,x(338-104x^2)\,K_1(x)\,K_1(\varrho x)\Big]\,\sin\sigma x$$

$\gamma=4/\sqrt{5}\,,\ \delta=\sqrt{7/15}$

$$\int x^4\,\sin\,\gamma x\cdot I_0(x)\,I_1(\delta x)\,dx=\frac{x}{7168}\,\Big[\sqrt{105}\,(64\,x^3-60\,x)\,I_0(x)\,I_0(\delta x)+$$

$$+(3360\,x^2-1575)\,I_0(x)\,I_1(\delta x)+\sqrt{105}\,(225-288\,x^2)\,I_1(x)\,I_0(\delta x)+$$

$$+(1344\,x^3-1260\,x)\,I_1(x)\,I_1(\delta x)]\,\sin\,\gamma x+\frac{x^2}{1792}\,\Big[60\,\sqrt{21}\,x\,I_0(x)\,I_0(\delta x)+$$

$$+\sqrt{5}\,(315-336\,x^2)\,I_0(x)\,I_1(\delta x)+\sqrt{21}\,(48\,x^2-225)\,I_1(x)\,I_0(\delta x)+$$

$$+252\,\sqrt{5}\,x\,I_1(x)\,I_1(\delta x)\Big]\,\cos\,\gamma x$$

$$\int x^4\,\cos\,\gamma x\cdot I_0(x)\,I_1(\delta x)\,dx=\frac{x^2}{1792}\,\Big[-60\,\sqrt{21}\,x\,I_0(x)\,I_0(\delta x)+$$

$$+\sqrt{5}\,(336\,x^2-315)\,I_0(x)\,I_1(\delta x)+\sqrt{21}\,(225-48\,x^2)\,I_1(x)\,I_0(\delta x)-$$
$$-252\,\sqrt{5}\,x\,I_1(x)\,I_1(\delta x)\Big]\ \sin\,\gamma x+\frac{x}{7168}\,\Big[\sqrt{105}\,x(64x^2-60)\,I_0(x)\,I_0(\delta x)+$$
$$+(3360\,x^2-1575)\,I_0(x)\,I_1(\delta x)+\sqrt{105}\,(225-288x^2)\,I_1(x)\,I_0(\delta x)+$$
$$+(1344\,x^3-1260\,x)\,I_1(x)\,I_1(\delta x)\Big]\ \cos\,\gamma x$$
$$\int x^4\ \sin\,\gamma x\cdot K_0(x)\,K_1(\delta x)\,dx=\frac{x}{7168}\,\Big[-\sqrt{105}\,(64\,x^3-60\,x)\,K_0(x)\,K_0(\delta x)+$$
$$+(3360\,x^2-1575)\,K_0(x)\,K_1(\delta x)+\sqrt{105}\,(225-288\,x^2)\,K_1(x)\,K_0(\delta x)+$$
$$-(1344\,x^3-1260\,x)\,K_1(x)\,K_1(\delta x)\Big]\ \sin\,\gamma x-\frac{x^2}{1792}\,\Big[60\,\sqrt{21}\,x\,K_0(x)\,K_0(\delta x)+$$
$$-\sqrt{5}\,(315-336\,x^2)\,K_0(x)\,K_1(\delta x)-\sqrt{21}\,(48\,x^2-225)\,K_1(x)\,K_0(\delta x)+$$
$$+252\,\sqrt{5}\,x\,K_1(x)\,K_1(\delta x)\Big]\ \cos\,\gamma x$$
$$\int x^4\ \cos\,\gamma x\cdot K_0(x)\,K_1(\delta x)\,dx=\frac{x^2}{1792}\,\Big[60\,\sqrt{21}\,x\,K_0(x)\,K_0(\delta x)+$$
$$+\sqrt{5}\,(336\,x^2-315)\,K_0(x)\,K_1(\delta x)+\sqrt{21}\,(225-48\,x^2)\,K_1(x)\,K_0(\delta x)+$$
$$+252\,\sqrt{5}\,x\,K_1(x)\,K_1(\delta x)\Big]\ \sin\,\gamma x+\frac{x}{7168}\,\Big[-\sqrt{105}\,x(64x^2-60)\,K_0(x)\,K_0(\delta x)+$$
$$+(3360\,x^2-1575)\,K_0(x)\,K_1(\delta x)+\sqrt{105}\,(225-288x^2)\,K_1(x)\,K_0(\delta x)-$$
$$-(1344\,x^3-1260\,x)\,K_1(x)\,K_1(\delta x)\Big]\ \cos\,\gamma x$$

$\kappa=2/\sqrt{7}\,,\ \nu=\sqrt{3/7}$

$$\int x^4\ \sin\,\kappa x\cdot I_1(x)\,I_1(\nu x)\,dx=\frac{x^2}{16}\,\Big[-6\,\sqrt{21}\,x\,I_0(x)\,I_0(\nu x)+(8\,x^2-63)\,I_0(x)\,I_1(\nu x)+$$
$$+21\,\sqrt{21}\,I_1(x)\,I_0(\nu x)+14\,x\,I_1(x)\,I_1(\nu x)\Big]\ \sin\,\kappa x+\frac{x}{32}\,\Big[\sqrt{3}\,(8\,x^3-42x)\,I_0(x)\,I_0(\nu x)+$$
$$+\sqrt{7}\,(20\,x^2-63)\,I_0(x)\,I_1(\nu x)+\sqrt{3}\,(147-28\,x^2)\,I_1(x)\,I_0(\nu x)+$$
$$+\sqrt{7}\,(14x-8\,x^3)\,I_1(x)\,I_1(\nu x)\Big]\ \cos\,\kappa x$$
$$\int x^4\ \cos\,\kappa x\cdot I_1(x)\,I_1(\nu x)\,dx=\frac{x}{32}\,\Big[\sqrt{3}\,(42x-8x^3)\,I_0(x)\,I_0(\nu x)+$$
$$+\sqrt{7}\,(63-20x^2)\,I_0(x)\,I_1(\nu x)+\sqrt{3}\,(28x^2-147)\,I_1(x)\,I_0(\nu x)+$$
$$+\sqrt{7}\,(8x^3-14x)\,I_1(x)\,I_1(\nu x)\Big]\ \sin\,\kappa x+\frac{x^2}{16}\,\Big[-6\sqrt{21}\,x\,I_0(x)\,I_0(\nu x)+$$
$$+(8x^2-63)\,I_0(x)\,I_1(\nu x)+21\sqrt{21}\,I_1(x)\,I_0(\nu x)+14x\,I_1(x)\,I_1(\nu x)\Big]\ \cos\,\kappa x$$
$$\int x^4\ \sin\,\kappa x\cdot K_1(x)\,K_1(\nu x)\,dx=\frac{x^2}{16}\,\Big[-6\,\sqrt{21}\,x\,K_0(x)\,K_0(\nu x)-(8\,x^2-63)\,K_0(x)\,K_1(\nu x)-$$
$$-21\,\sqrt{21}\,K_1(x)\,K_0(\nu x)+14\,x\,K_1(x)\,K_1(\nu x)\Big]\ \sin\,\kappa x+\frac{x}{32}\,\Big[\sqrt{3}\,(8\,x^3-42x)\,K_0(x)\,K_0(\nu x)-$$
$$-\sqrt{7}\,(20\,x^2-63)\,K_0(x)\,K_1(\nu x)-\sqrt{3}\,(147-28\,x^2)\,K_1(x)\,K_0(\nu x)+$$
$$+\sqrt{7}\,(14x-8\,x^3)\,K_1(x)\,K_1(\nu x)\Big]\ \cos\,\kappa x$$
$$\int x^4\ \cos\,\kappa x\cdot K_1(x)\,K_1(\nu x)\,dx=\frac{x}{32}\,\Big[\sqrt{3}\,(42x-8x^3)\,K_0(x)\,K_0(\nu x)+$$
$$-\sqrt{7}\,(63-20x^2)\,K_0(x)\,K_1(\nu x)-\sqrt{3}\,(28x^2-147)\,K_1(x)\,K_0(\nu x)+$$

$$+\sqrt{7}\,(8x^3-14x)\,K_1(x)\,K_1(\nu x)\Big]\,\sin\,\kappa x-\frac{x^2}{16}\,\Big[6\sqrt{21}\,x\,K_0(x)\,K_0(\nu x)+$$

$$+(8x^2-63)\,K_0(x)\,K_1(\nu x)+21\sqrt{21}\,K_1(x)\,K_0(\nu x)-14x\,K_1(x)\,K_1(\nu x)\Big]\,\cos\,\kappa x$$

<u>**n = 5 :**</u> $\xi=4/\sqrt{11}\,,\ \omega=\sqrt{3/11}$

$$\int x^5\,\sin\,\xi x\cdot I_1(x)\,I_1(\omega x)\,dx=\frac{x}{24576}\,\Big[\sqrt{33}\,x(7260-4224\,x^2)\,I_0(x)\,I_0(\omega x)+$$

$$+(8448\,x^4-52272\,x^2+59895)\,I_0(x)\,I_1(\omega x)+\sqrt{33}\,(256\,x^4+11088x^2-19965x)\,I_1(x)\,I_0(\omega x)+$$

$$+(12672\,x^3-4356\,x)\,I_1(x)\,I_1(\omega x)\Big]\,\sin\,\xi x+\frac{x^2}{6144}\,\Big[\sqrt{3}\,(704\,x^3-7260x)\,I_0(x)\,I_0(\omega x)+$$

$$+\sqrt{11}\,(2112\,x^2-5445)\,I_0(x)\,I_1(\omega x)+\sqrt{3}\,(19965-1408\,x^2)\,I_1(x)\,I_0(\omega x)+$$

$$+\sqrt{11}\,(396x-960\,x^3)\,I_1(x)\,I_1(\omega x)\Big]\,\cos\,\xi x$$

$$\int x^5\,\cos\,\xi x\cdot I_1(x)\,I_1(\omega x)\,dx=\frac{x^2}{6144}\,\Big[\sqrt{3}\,(7260x-704\,x^3)I_0(x)\,I_0(\omega x)+$$

$$+\sqrt{11}\,(5445-2112\,x^2)\,I_0(x)\,I_1(\omega x)+\sqrt{3}\,(1408\,x^2-19965)\,I_1(x)\,I_0(\omega x)+$$

$$+\sqrt{11}\,(960\,x^3-396x)\,I_1(x)\,I_1(\omega x)\Big]\,\sin\,\xi x+\frac{x}{24576}\,\Big[\sqrt{33}\,(7260x-4224\,x^3)I_0(x)\,I_0(\omega x)+$$

$$+(8448\,x^4-52272\,x^2+59895)\,I_0(x)\,I_1(\omega x)+\sqrt{33}\,(256x^4+11088x^2-19965)\,I_1(x)\,I_0(\omega x)+$$

$$+(12672\,x^3-4356\,x)\,I_1(x)\,I_1(\omega x)\,\Big]\,\cos\,\xi x$$

$$\int x^5\,\sin\,\xi x\cdot K_1(x)\,K_1(\omega x)\,dx=\frac{x}{24576}\,\Big[-\sqrt{33}\,x(4224\,x^2-7260)\,K_0(x)\,K_0(\omega x)-$$

$$-(8448\,x^4-52272\,x^2+59895)\,K_0(x)\,K_1(\omega x)-\sqrt{33}\,(256\,x^4+11088x^2-19965x)\,K_1(x)\,K_0(\omega x)+$$

$$+(12672\,x^3-4356\,x)\,K_1(x)\,K_1(\omega x)\Big]\,\sin\,\xi x+\frac{x^2}{6144}\,\Big[\sqrt{3}\,(704\,x^3-7260x)\,K_0(x)\,K_0(\omega x)-$$

$$-\sqrt{11}\,(2112\,x^2-5445)\,K_0(x)\,K_1(\omega x)-\sqrt{3}\,(19965-1408\,x^2)\,K_1(x)\,K_0(\omega x)+$$

$$+\sqrt{11}\,(396x-960\,x^3)\,K_1(x)\,K_1(\omega x)\Big]\,\cos\,\xi x$$

$$\int x^5\,\cos\,\xi x\cdot K_1(x)\,K_1(\omega x)\,dx=\frac{x}{24576}\,\Big[\sqrt{33}\,x(7260-4224\,x^2)\,K_0(x)\,K_0(\omega x)-$$

$$-(8448\,x^4-52272\,x^2+59895)\,K_0(x)\,K_1(\omega x)-\sqrt{33}\,(256\,x^4+11088x^2-19965x)\,K_1(x)\,K_0(\omega x)+$$

$$+(12672\,x^3-4356\,x)\,K_1(x)\,K_1(\omega x)\Big]\,\cos\,\xi x-\frac{x^2}{6144}\,\Big[\sqrt{3}\,(704\,x^3-7260x)\,K_0(x)\,K_0(\omega x)-$$

$$-\sqrt{11}\,(2112\,x^2-5445)\,K_0(x)\,K_1(\omega x)-\sqrt{3}\,(19965-1408\,x^2)\,K_1(x)\,K_0(\omega x)+$$

$$+\sqrt{11}\,(396x-960\,x^3)\,K_1(x)\,K_1(\omega x)\Big]\,\sin\,\xi x$$

2.4.6 b) $\int x^n\cdot\left\{\begin{array}{c}\sinh\\ \cosh\end{array}\right\}\alpha x\cdot Z_\mu(x)\,Z_\nu(\beta x)\,dx$:

<u>**n = 1 :**</u>

$$\int x\,\sinh 4x\cdot I_0(x)\,I_1(3x)\,dx=\frac{x^2}{4}\,[I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)]\,\sinh 4x-$$

$$-\frac{x}{16}\,[4x\,I_0(x)\,I_0(3x)-3\,I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)+4x\,I_1(x)\,I_1(3x)]\,\cosh 4x$$

$$\int x\,\cosh 4x\cdot I_0(x)\,I_1(3x)\,dx=\frac{x^2}{4}\,[I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)]\,\cosh 4x-$$

$$-\frac{x}{16}\,\left[4x\,I_0(x)\,I_0(3x)-3\,I_0(x)\,I_1(3x)+I_1(x)\,I_0(3x)+4x\,I_1(x)\,I_1(3x)\right]\,\sinh 4x$$

$$\int x\,\sinh 4x\cdot K_0(x)\,K_1(3x)\,dx=\frac{x^2}{4}\,\left[K_0(x)\,K_1(3x)+K_1(x)\,K_0(3x)\right]\,\sinh 4x+$$

$$+\frac{x}{16}\,\left[4x\,K_0(x)\,K_0(3x)+3\,K_0(x)\,K_1(3x)-K_1(x)\,K_0(3x)+4x\,K_1(x)\,K_1(3x)\right]\,\cosh 4x$$

$$\int x\,\cosh 4x\cdot K_0(x)\,K_1(3x)\,dx=\frac{x^2}{4}\,\left[K_0(x)\,K_1(3x)+K_1(x)\,K_0(3x)\right]\,\cosh 4x+$$

$$+\frac{x}{16}\,\left[4x\,K_0(x)\,K_0(3x)+3\,K_0(x)\,K_1(3x)-K_1(x)\,K_0(3x)+4x\,K_1(x)\,K_1(3x)\right]\,\sinh 4x$$

n = 2 : $\alpha=8/3\,,\ \beta=5/3$

$$\int x^2\,\sinh\alpha x\cdot I_0(x)\,I_1(\beta x)\,dx=\frac{x}{512}\,\Big[24x\,I_0(x)\,I_0(\beta x)+(64\,x^2-45)\,I_0(x)\,I_1(\beta x)+$$

$$+(64\,x^2+27)\,I_1(x)\,I_0(\beta x)+24x\,I_1(x)\,I_1(\beta x)\Big]\,\sinh\alpha x-$$

$$-\frac{x^2}{64}\,\left[8x\,I_0(x)\,I_0(\beta x)-15\,I_0(x)\,I_1(\beta x)+9\,I_1(x)\,I_0(\beta x)+8x\,I_1(x)\,I_1(\beta x)\right]\,\cosh\alpha x$$

$$\int x^2\,\cosh\alpha x\cdot I_0(x)\,I_1(\beta x)\,dx=\frac{x}{512}\,\Big[24x\,I_0(x)\,I_0(\beta x)+(64\,x^2-45)\,J_0(x)\,J_1(\beta x)+$$

$$+(64\,x^2+27)\,I_1(x)\,I_0(\beta x)+24x\,I_1(x)\,I_1(\beta x)\Big]\,\cosh\alpha x-$$

$$-\frac{x^2}{64}\,\left[8x\,I_0(x)\,I_0(\beta x)-15\,I_0(x)\,I_1(\beta x)+9\,I_1(x)\,I_0(\beta x)+8x\,I_1(x)\,I_1(\beta x)\right]\,\sinh\alpha x$$

$$\int x^2\,\sinh\alpha x\cdot K_0(x)\,K_1(\beta x)\,dx=\frac{x}{512}\,\Big[-24x\,K_0(x)\,K_0(\beta x)+(64\,x^2-45)\,K_0(x)\,K_1(\beta x)+$$

$$+(64\,x^2+27)\,K_1(x)\,K_0(\beta x)-24x\,K_1(x)\,K_1(\beta x)\Big]\,\sinh\alpha x+$$

$$+\frac{x^2}{64}\,\left[8x\,K_0(x)\,K_0(\beta x)+15\,K_0(x)\,K_1(\beta x)-9\,K_1(x)\,K_0(\beta x)+8x\,K_1(x)\,K_1(\beta x)\right]\,\cosh\alpha x$$

$$\int x^2\,\cosh\alpha x\cdot K_0(x)\,K_1(\beta x)\,dx=\frac{x}{512}\,\Big[-24x\,K_0(x)\,K_0(\beta x)+(64\,x^2-45)\,K_0(x)\,K_1(\beta x)+$$

$$+(64\,x^2+27)\,K_1(x)\,K_0(\beta x)-24x\,K_1(x)\,K_1(\beta x)\Big]\,\cosh\alpha x+$$

$$+\frac{x^2}{64}\,\left[8x\,K_0(x)\,K_0(\beta x)+15\,K_0(x)\,K_1(\beta x)-9\,K_1(x)\,K_0(\beta x)+8x\,K_1(x)\,K_1(\beta x)\right]\,\sinh\alpha x$$

n = 3 : $\lambda=8/\sqrt{51}\,,\ \mu=\sqrt{35/51}$

$$\int x^3\,\sinh\lambda x\cdot J_0(x)\,J_1(\mu x)\,dx=\frac{x^2}{2560}\,\Big[-8\sqrt{1785}\,x\,J_0(x)\,J_0(\mu x)-1785\,J_0(x)\,J_1(\mu x)+$$

$$+51\,\sqrt{1785}\,J_1(x)\,J_0(\mu x)+680x\,J_1(x)\,J_1(\mu x)\Big]\,\sinh\lambda x+$$

$$+\frac{x}{20480}\,\Big[408\sqrt{35}\,x\,J_0(x)\,J_0(\mu x)+\sqrt{51}\,(1600x^2+1785)\,J_0(x)\,J_1(\mu x)-$$

$$-\sqrt{35}\,(1088x^2+2601)\,J_1(x)\,J_0(\mu x)-680\sqrt{51}\,x\,J_1(x)\,J_1(\mu x)\Big]\,\cosh\lambda x$$

$$\int x^3\,\cosh\lambda x\cdot J_0(x)\,J_1(\mu x)\,dx=\frac{x}{20480}\,\Big[408\,\sqrt{35}\,x\,\ J_0(x)\,J_0(\mu x)+$$

$$+\sqrt{51}\,(1600\,x^2+1785)\,J_0(x)\,J_1(\mu x)-\sqrt{35}\,(1088\,x^2+2601)\,J_1(x)\,J_0(\mu x)-$$

$$-680\,\sqrt{51}\,x\ \,J_1(x)\,J_1(\mu x)\Big]\,\sinh\lambda x+\frac{x^2}{2560}\,\Big[-8\,\sqrt{1785}\,x\,J_0(x)\,J_0(\mu x)-1785\,J_0(x)\,J_1(\mu x)+$$

$$+51\sqrt{1785}\,J_1(x)\,J_0(\mu x)+680\,x\,J_1(x)\,J_1(\mu x)\Big]\;\cosh\lambda x$$

<u>**n = 4 :**</u> $\eta = 2\sqrt{3}/5\,,\;\theta=\sqrt{13}/5$

$$\int x^4\,\sinh\eta x\,J_0(x)\,J_0(\theta x)\,dx=\Bigg[-\frac{15x^3}{8}\,J_0(x)\,J_0(\theta x)-\frac{25\sqrt{13}\,x^2}{16}\,J_0(x)\,J_1(\theta x)+$$

$$+\frac{8x^4+125x^2}{16}\,J_1(x)\,J_0(\theta x)-\frac{5\sqrt{13}\,x^3}{8}\,J_1(x)\,J_1(\theta x)\Bigg]\;\sinh\eta x+$$

$$+\Bigg[\frac{5\sqrt{3}\,(4x^4+15x^2)}{48}\,J_0(x)\,J_0(\theta x)+\frac{5\sqrt{39}\,(4x^3+25x)}{96}\,J_0(x)\,J_1(\theta x)-$$

$$-\frac{5\sqrt{3}\,(28x^3+125x}{96}\,J_1(x)\,J_0(\theta x)+\frac{\sqrt{39}\,(4x^4+25x^2)}{48}\,J_1(x)\,J_1(\theta x)\Bigg]\;\cosh\eta x$$

$$\int x^4\,\cosh\eta x\,J_0(x)\,J_0(\theta x)\,dx=\frac{\sqrt{3}\,x}{96}\;\big[(40x^3+150x)\,J_0(x)\,J_0(\theta x)+$$

$$+\sqrt{13}\,(20x^2+125)\,J_0(x)\,J_1(\theta x)-(140\,x^2+625)\,J_1(x)\,J_0(\theta x)+$$

$$+\sqrt{13}\,(8\,x^3+50\,x)\,J_1(x)\,J_1(\theta x)\Big]\;\sinh\eta x-\frac{x^2}{16}\;\Big[30\,x\,J_0(x)\,J_0(\theta x)+25\sqrt{13}\,J_0(x)\,J_1(\theta x)-$$

$$-(8\,x^2+125)\,J_1(x)\,J_0(\theta x)+10\sqrt{13}x\,J_1(x)\,J_1(\theta x)\Big]\;\cosh\eta x$$

$\gamma=4/\sqrt{5}\,,\;\delta=\sqrt{7/15}$

$$\int x^4\,\sinh\gamma x\cdot J_0(x)\,J_1(\delta x)\,dx=\frac{x}{7168}\;\Big[-\sqrt{105}\,(64x^3+60\,x)J_0(x)\,J_0(\delta x)-$$

$$-(3360\,x^2+1575)\,J_0(x)\,J_1(\delta x)+\sqrt{105}\,(288x^2+225)\,J_1(x)\,J_0(\delta x)+$$

$$+(1344\,x^3+1260\,x)\,J_1(x)\,J_1(\delta x)\big]\;\sinh\gamma x+\frac{x^2}{1792}\;\Big[60\sqrt{21}x\,J_0(x)\,J_0(\delta x)+$$

$$+\sqrt{5}\,(336\,x^2+315)\,J_0(x)\,J_1(\delta x)-\sqrt{21}\,(48\,x^2+225)\,J_1(x)\,J_0(\delta x)-$$

$$-252\sqrt{5}x\,J_1(x)\,J_1(\delta x)\Big]\;\cosh\gamma x$$

$$\int x^4\,\cosh\gamma x\cdot J_0(x)\,J_1(\delta x)\,dx=\frac{x^2}{1792}\;\Big[60\sqrt{21}\,x\,J_0(x)\,J_0(\delta x)+$$

$$+\sqrt{5}\,(336\,x^2+315)\,J_0(x)\,J_1(\delta x)-\sqrt{21}\,(48\,x^2+225)\,J_1(x)\,J_0(\delta x)-$$

$$-252\sqrt{5}\,x\,J_1(x)\,J_1(\delta x)\Big]\;\sinh\gamma x+\frac{x}{7168}\;\Big[-\sqrt{105}\,(64\,x^3+60x)\,J_0(x)\,J_0(\delta x)-$$

$$-(3360\,x^2+1575)\,J_0(x)\,J_1(\delta x)+\sqrt{105}\,(288\,x^2+225)\,J_1(x)\,J_0(\delta x)+$$

$$+(1344\,x^3+1260\,x)\,J_1(x)\,J_1(\delta x)\big]\;\cosh\gamma x$$

$\kappa=2/\sqrt{7}\,,\;\nu=\sqrt{3/7}$

$$\int x^4\,\sinh\kappa x\cdot J_1(x)\,J_1(\nu x)\,dx=\frac{x^2}{16}\;\Big[-6\sqrt{21}\,x\,J_0(x)\,J_0(\nu x)-$$

$$-(8\,x^2+63)\,J_0(x)\,J_1(\nu x)+21\sqrt{21}\,J_1(x)\,J_0(\nu x)-14\,x\,J_1(x)\,J_1(\nu x)\Big]\;\sinh\kappa x+$$

$$+\frac{x}{32}\;\Big[\sqrt{3}\,(8\,x^3+42\,x)\,J_0(x)\,J_0(\nu x)+\sqrt{7}\,(20x^2+63\,)\,J_0(x)\,J_1(\nu x)-$$

$$-\sqrt{3}\,(28\,x^2+147\,)\,J_1(x)\,J_0(\nu x)+\sqrt{7}\,(8\,x^3+14\,x)\,J_1(x)\,J_1(\nu x)\Big]\;\cosh\kappa x$$

$$\int x^4\,\cosh\kappa x\cdot J_1(x)\,J_1(\nu x)\,dx=\frac{x^2}{16}\,\Big[-6\sqrt{21}\,x\,J_0(x)\,J_0(\nu x)-$$

$$-(8\,x^2+63)\,J_0(x)\,J_1(\nu x)+21\sqrt{21}\,J_1(x)\,J_0(\nu x)-14\,x\,J_1(x)\,J_1(\nu x)\Big]\,\cosh\kappa x+$$

$$+\frac{x}{32}\,\Big[\sqrt{3}\,(8\,x^3+42\,x)\,J_0(x)\,J_0(\nu x)+\sqrt{7}\,(20\,x^2+63\,)\,J_0(x)\,J_1(\nu x)-$$

$$-\sqrt{3}\,(28x^2+147)\,J_1(x)\,J_0(\nu x)+\sqrt{7}\,(8x^3+14\,x)\,J_1(x)\,J_1(\nu x)\Big]\,\sinh\kappa x$$

n = 5 : $\xi=4/\sqrt{11}\,,\ \omega=\sqrt{3/11}$

$$\int x^5\,\sinh\xi x\,\cdot J_1(x)\,J_1(\omega x)\,dx=\frac{x}{24\,576}\,\Big[-\sqrt{33}\,(4224\,x^3+7260x)\,J_0(x)\,J_0(\omega x)-$$

$$-(8448\,x^4+52272\,x^2+59895)\,J_0(x)\,J_1(\omega x)+$$

$$+\sqrt{33}\,(-256\,x^4+11088\,x^2+19965)\,J_1(x)\,J_0(\omega x)-$$

$$-(12672\,x^3+4356\,x)\,J_1(x)\,J_1(\omega x)\Big]\,\sinh\xi x+$$

$$+\frac{x^2}{6144}\,\Big[\sqrt{3}\,(704\,x^3+7260\,x)\,J_0(x)\,J_0(\omega x)+\sqrt{11}\,(2112\,x^2+5445)\,J_0(x)\,J_1\,(\omega x)-$$

$$-\sqrt{3}\,(1408\,x^2+19965)\,J_1(x)\,J_0(\omega x)+\sqrt{11}\,(960\,x^3+396\,x)\,J_1(x)\,J_1(\omega x)\Big]\,\cosh\xi x$$

$$\int x^5\,\cosh\xi x\,\cdot J_1(x)\,J_1(\omega x)\,dx=\frac{x^2}{6144}\,\Big[\sqrt{3}\,(704\,x^3+7260\,x)\,J_0(x)\,J_0(\omega x)+$$

$$+\sqrt{11}\,(2112\,x^2+5445)\,J_0(x)\,J_1(\omega x)-\sqrt{3}\,(1408\,x^2+19965)\,J_1(x)\,J_0(\omega x)+$$

$$+\sqrt{11}\,(960\,x^3+396\,x)\,J_1(x)\,J_1(\omega x)\Big]\,\sinh\xi x-$$

$$-\frac{x}{24576}\,\Big[\sqrt{33}\,(4224\,x^3+7260\,x)\,J_0(x)\,J_0(\omega x)+(8448\,x^4+52272\,x^2+59895)\,J_0(x)\,J_1(\omega x)-$$

$$-\sqrt{33}\,(-256\,x^4+11088\,x^2+19965)\,J_1(x)\,J_0(\omega x)+(12672\,x^3+4356\,x)\,J_1(x)\,J_1(\omega x)\Big]\,\cosh\xi x$$

2.5. Cross Products

2.4.1. Integrals of the type $\int x^n\,[V_0\,(x) W_1(x) \pm V_1(x)\,W_0(x)]\,dx$

$$\int [J_0(x)\,Y_1(x) + J_1(x)\,Y_0(x)]\;dx = -J_0(x)\,Y_0(x)$$

$$\int [J_0(x)\,I_1(x) - J_1(x)\,I_0(x)]\;dx = J_0(x)\,I_0(x)$$

$$\int [J_0(x)\,K_1(x) + J_1(x)\,K_0(x)]\;dx = -J_0(x)\,K_0(x)$$

$$\int [I_0(x)\,K_1(x) - I_1(x)\,K_0(x)]\;dx = -I_0(x)\,K_0(x)$$

$$\int x\;[J_0(x)\,Y_1(x) - J_1(x)\,Y_0(x)]\;dx = x^2\;[J_0(x)\,Y_1(x) - J_1(x)\,Y_0(x)]$$

$$\int x\;[I_0(x)\,K_1(x) + I_1(x)\,K_0(x)]\;dx = x^2\;[I_0(x)\,K_1(x) + I_1(x)\,K_0(x)]$$

$$\int x^2\;[J_0(x)\,Y_1(x) + J_1(x)\,Y_0(x)]\;dx = x^2\,J_1(x)\,Y_1(x)$$

$$\int x^2\;[J_0(x)\,Y_1(x) - J_1(x)\,Y_0(x)]\;dx = \frac{x^3}{2}\;[J_0(x)\,Y_1(x) - J_1(x)\,Y_0(x)]$$

$$\int x^2\;[J_0(x)\,I_1(x) + J_1(x)\,I_0(x)]\;dx = x^2\,J_1(x)\,I_1(x)$$

$$\int x^2\;[J_0(x)\,I_1(x) - J_1(x)\,I_0(x)]\;dx = x^2\,J_0(x)\,I_0(x) - x\,J_0(x)\,I_1(x) - x\,J_1(x)\,I_0(x)$$

$$\int x^2\;[J_0(x)\,K_1(x) + J_1(x)\,K_0(x)]\;dx = -x^2\,J_0(x)\,K_0(x) - x\,J_0(x)\,K_1(x) + x\,J_1(x)\,K_0(x)$$

$$\int x^2\;[J_0(x)\,K_1(x) - J_1(x)\,K_0(x)]\;dx = x^2\,J_1(x)\,K_1(x)$$

$$\int x^2\;[I_0(x)\,K_1(x) + I_1(x)\,K_0(x)]\;dx = \frac{x^3}{2}\;[I_0(x)\,K_1(x) + I_1(x)\,K_0(x)]$$

$$\int x^2\;[I_0(x)\,K_1(x) - I_1(x)\,K_0(x)]\;dx = x^2\,I_1(x)\,K_1(x)$$

Generally:

$$\int x^n\;[J_0(x)\,Y_1(x) - J_1(x)\,Y_0(x)]\;dx = \frac{x^{n+1}}{n}\;[J_0(x)\,Y_1(x) - J_1(x)\,Y_0(x)]$$

$$\int x^n\;[I_0(x)\,K_1(x) + I_1(x)\,K_0(x)]\;dx = \frac{x^{n+1}}{n}\;[I_0(x)\,K_1(x) + I_1(x)\,K_0(x)]$$

2.4.2. Integrals of the type $\int x^n\,[V_0\,(x) W_1(\lambda x) \pm V_1(x)\,W_0(\lambda x)]\,dx$

$$\int x^2\;[J_0(x)\,J_1(\lambda x) + J_1(x)\,J_0(\lambda x)]\;dx =$$

$$= \frac{x^2}{\lambda+1}\;[J_1(x)\,J_1(\lambda x) - J_0(x)\,J_0(\lambda x)] + \frac{2x}{(\lambda+1)^2(\lambda-1)}\;[\lambda\,J_0(x)\,J_1(\lambda x) - J_1(x)\,J_0(\lambda x)]$$

$$\int x^2\ [J_0(x)\, I_1(\lambda x) + J_1(x)\, I_0(\lambda x)]\ dx =$$

$$= \frac{x^2}{\lambda^2+1}\ [(\lambda-1)\, J_0(x)\, I_0(\lambda x) + (\lambda+1)\, J_1(x)\, I_1(\lambda x)] - \frac{2(\lambda-1)\, x}{(\lambda^2+1)^2}\ [\lambda\, J_0(x)\, I_1(\lambda x) + J_1(x)\, I_0(\lambda x)]$$

$$\int x^2\ [J_0(x)\, K_1(\lambda x) + J_1(x)\, K_0(\lambda x)]\ dx =$$

$$= -\frac{x^2}{\lambda^2+1}\ [(\lambda+1)\, J_0(x)\, K_0(\lambda x) + (\lambda-1)\, J_1(x)\, K_1(\lambda x)] - \frac{2(\lambda+1)\, x}{(\lambda^2+1)^2}\ [\lambda\, J_0(x)\, K_1(\lambda x) - J_1(x)\, K_0(\lambda x)]$$

$$\int x^2\ [I_0(x)\, I_1(\lambda x) + I_1(x)\, I_0(\lambda x)]\ dx =$$

$$= \frac{x^2}{\lambda+1}\ [I_0(x)\, I_0(\lambda x) + I_1(x)\, I_1(\lambda x)] - \frac{2x}{(\lambda+1)^2(\lambda-1)}\ [\lambda\, I_0(x)\, I_1(\lambda x) - I_1(x)\, I_0(\lambda x)]$$

$$\int x^2\ [I_0(x)\, K_1(\lambda x) + I_1(x)\, K_0(\lambda x)]\ dx =$$

$$= -\frac{x^2}{\lambda-1}\ [I_0(x)\, K_0(\lambda x) + I_1(x)\, K_1(\lambda x)] - \frac{2x}{(\lambda-1)^2(\lambda+1)}\ [\lambda\, I_0(x)\, K_1(\lambda x) + I_1(x)\, K_0(\lambda x)]$$

$$\int x^2\ [K_0(x)\, K_1(\lambda x) + K_1(x)\, K_0(\lambda x)]\ dx =$$

$$= -\frac{x^2}{\lambda+1}\ [K_0(x)\, K_0(\lambda x) + K_1(x)\, K_1(\lambda x)] - \frac{2x}{(\lambda+1)^2(\lambda-1)}\ [\lambda\, K_0(x)\, K_1(\lambda x) - K_1(x)\, K_0(\lambda x)]$$

$$\int x^2\ [J_0(x)\, J_1(\lambda x) - J_1(x)\, J_0(\lambda x)]\ dx =$$

$$= -\frac{x^2}{\lambda-1}\ [J_0(x)\, J_0(\lambda x) + J_1(x)\, J_1(\lambda x)] + \frac{2x}{(\lambda-1)^2(\lambda+1)}\ [\lambda\, J_0(x)\, J_1(\lambda x) - J_1(x)\, J_0(\lambda x)]$$

$$\int x^2\ [J_0(x)\, I_1(\lambda x) - J_1(x)\, I_0(\lambda x)]\ dx =$$

$$= \frac{x^2}{\lambda^2+1}\ [(\lambda+1)\, J_0(x)\, I_0(\lambda x) - (\lambda-1)\, J_1(x)\, I_1(\lambda x)] - \frac{2(\lambda+1)\, x}{(\lambda^2+1)^2}\ [\lambda\, J_0(x)\, I_1(\lambda x) + J_1(x)\, I_0(\lambda x)]$$

$$\int x^2\ [J_0(x)\, K_1(\lambda x) - J_1(x)\, K_0(\lambda x)]\ dx =$$

$$= -\frac{x^2}{\lambda^2+1}\ [(\lambda-1)\, J_0(x)\, K_0(\lambda x) - (\lambda+1)\, J_1(x)\, K_1(\lambda x)] - \frac{2(\lambda-1)\, x}{(\lambda^2+1)^2}\ [\lambda\, J_0(x)\, K_1(\lambda x) - J_1(x)\, K_0(\lambda x)]$$

$$\int x^2\ [I_0(x)\, I_1(\lambda x) - I_1(x)\, I_0(\lambda x)]\ dx =$$

$$= \frac{x^2}{\lambda-1}\ [I_0(x)\, I_0(\lambda x) - I_1(x)\, I_1(\lambda x)] - \frac{2x}{(\lambda-1)^2(\lambda+1)}\ [\lambda\, I_0(x)\, I_1(\lambda x) - I_1(x)\, I_0(\lambda x)]$$

$$\int x^2\ [I_0(x)\, K_1(\lambda x) - I_1(x)\, K_0(\lambda x)]\ dx =$$

$$= \frac{x^2}{\lambda+1}\ [I_1(x)\, K_1(\lambda x) - I_0(x)\, K_0(\lambda x)] - \frac{2x}{(\lambda+1)^2(\lambda-1)}\ [\lambda\, I_0(x)\, K_1(\lambda x) + I_1(x)\, K_0(\lambda x)]$$

$$\int x^2\ [K_0(x)\, K_1(\lambda x) - K_1(x)\, K_0(\lambda x)]\ dx =$$

$$= \frac{x^2}{\lambda-1}\ [K_1(x)\, K_1(\lambda x) - K_0(x)\, K_0(\lambda x)] - \frac{2x}{(\lambda-1)^2(\lambda+1)}\ [\lambda\, K_0(x)\, K_1(\lambda x) - K_1(x)\, K_0(\lambda x)]$$

aa

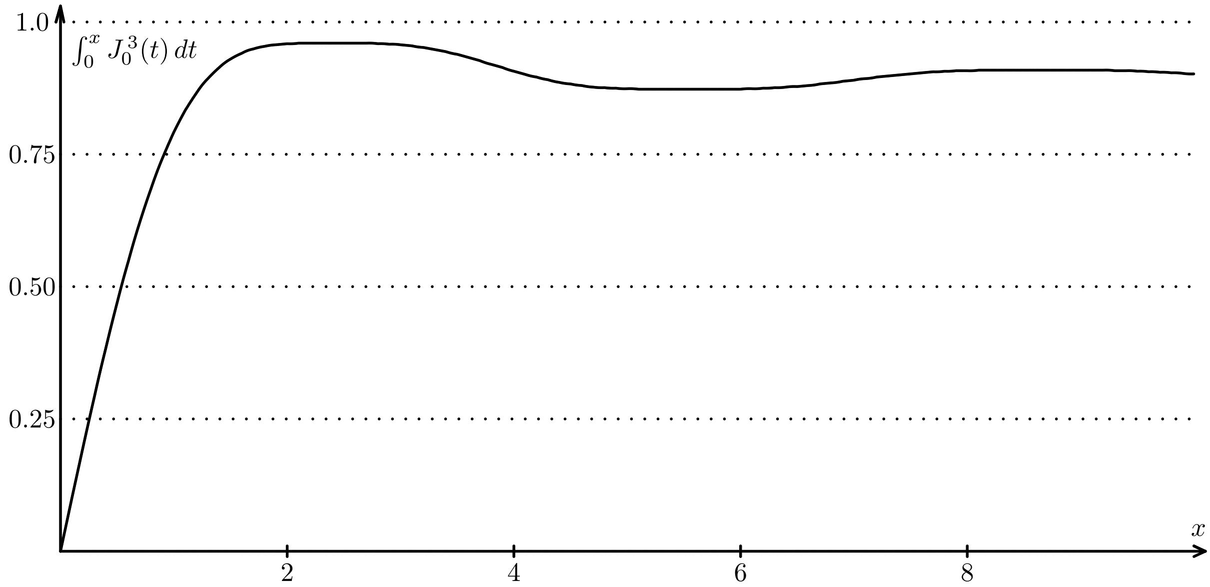
3. Products of three Bessel Functions

3.1. Integrals of the type $\int x^n\, Z_0^m(x)\, Z_1^{3-m}(x)\, dx$

These integrals are expressed by three basic integrals with $m = 0, 1, 3$ and the integral with $x^{-1}\, Z_1^3(x)$. One has obviously

$$\int J_0^2(x)\, J_1(x)\, dx \;=\; -\frac{1}{3}\, J_0^3(x) \quad , \qquad \int I_0^2(x)\, I_1(x)\, dx \;=\; \frac{1}{3}\, I_0^3(x) \; .$$

a) Basic integral $\int Z_0^3(x)\, dx$:



$$\int_0^\infty J_0^3(x)\, dx \;=\; \frac{\Gamma\left(\frac{1}{6}\right)}{3\,\Gamma\left(\frac{5}{6}\right)\cdot\Gamma^2\left(\frac{2}{3}\right)} \;=\; \frac{2\pi}{3\,\Gamma^2\left(\frac{5}{6}\right)\cdot\Gamma^2\left(\frac{2}{3}\right)} \;=\; 0.89644\;07887\;76762\;86423\;\ldots$$

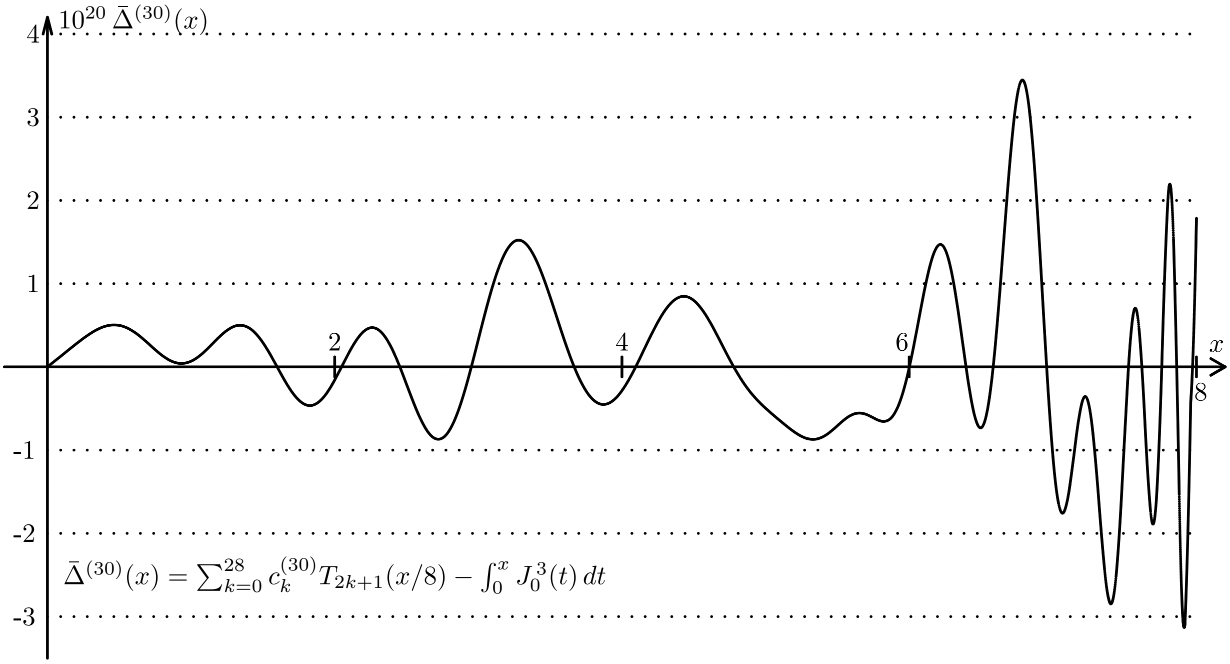
Formula 2.12.42.4 from [4] gives $2\sqrt{3}/3\pi = 0.36755\ldots$. This does not fit to the result of computations. The formula 2.12.42.18 offers $2\,\Gamma(1/6)/[3\,\Gamma(5/6)\cdot\Gamma^2(2/3)] = 2\cdot 0.896\ldots$.
With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x J_0^3(t)\, dt = \sum_{k=0}^{\infty} c_k^{(30)}\, T_{2k+1}\left(\frac{x}{8}\right) \; .$$

The first coefficients are

k	$c_k^{(30)}$	k	$c_k^{(30)}$
0	1.14145 15823 43066 65430	15	-0.00001 19647 92899 04163
1	-0.37698 69057 03625 95863	16	0.00000 19134 34477 76782
2	0.24193 82520 26895 89401	17	-0.00000 02652 90698 65709
3	-0.15672 12348 70401 19757	18	0.00000 00322 34936 59711
4	0.09593 25955 24494 24624	19	-0.00000 00034 64486 68661
5	-0.06183 39745 85568 16834	20	0.00000 00003 31986 30964
6	0.04061 90786 50027 57290	21	-0.00000 00000 28561 98591
7	-0.02860 28881 71908 58991	22	0.00000 00000 02219 71843
8	0.02158 63885 61883 43325	23	-0.00000 00000 00156 67592
9	-0.01457 50343 38917 46000	24	0.00000 00000 00010 09261
10	0.00781 65287 61390 63536	25	-0.00000 00000 00000 59593
11	-0.00326 40051 49421 93383	26	0.00000 00000 00000 03238
12	0.00107 89280 58820 94395	27	-0.00000 00000 00000 00162
13	-0.00028 89222 57879 12574	28	0.00000 00000 00000 00008
14	0.00006 40382 21736 47959	-	-

The given approximation differs from the true function as shown in the following figure:



Asymptotic formula:

$$\int_0^x J_0^3(t)\,dt \sim 0.89644\;07887\;76762\;86423\;\ldots +$$

$$+\sqrt{\frac{2}{\pi^3\,x}}\sum_{k=1}^{\infty}\frac{1}{x^k}\left[a_k^{(30)}\sin\left(3x+\frac{7-2k}{4}\pi\right)+b_k^{(30)}\sin\left(x+\frac{1-2k}{4}\pi\right)\right]$$

with the first values

k	$a_k^{(30)}$	$a_k^{(30)}$
1	1/6	0.16666 66666 66666 66667
2	7/48	0.14583 33333 33333 33333
3	379/2304	0.16449 65277 77777 77778
4	13141/55296	0.23764 82928 24074 07407
5	250513/589824	0.42472 50027 12673 61111
6	12913841/14155776	0.91226 65546 55852 14120
7	1565082415/679477248	2.30336 25034 63839 30724
8	36535718855/5435817984	6.72129 18023 63631 16606
9	23344744269635/1043677052928	22.36778 53260 66262 13087
10	2103860629922855/25048249270272	83.99232 24662 16381 92796
k	$b_k^{(30)}$	$b_k^{(30)}$
1	3/2	1.50000 00000 00000 00000
2	39/16	2.43750 00000 00000 00000
3	1635/256	6.38671 87500 00000 00000
4	46053/2048	22.48681 64062 50000 00000
5	6664257/65536	101.68849 18212 89062 5000
6	293433849/524288	559.68065 07110 595703 125
7	30538511055/8388608	3640.47420 68052 2918 70
8	1832502818925/67108864	27306.41989 29816 48445 1
9	996997642437465/4294967296	2 32131.60281 01055 41721
10	75773171001327165/34359738368	22 05289.52199 17166 1050

The first consecutive maxima and minima of

$$\Delta_n^{(30)}(x) = 0.896\ldots + \sqrt{\frac{2}{\pi^3\, x}} \sum_{k=1}^{n} \frac{1}{x^k} \left[a_k^{(30)} \sin\left(3x + \frac{7-2k}{4}\,\pi\right) + b_k^{(30)} \sin\left(x + \frac{1-2k}{4}\,\pi\right)\right] - \int_0^x J_0^3(t)\,dt \ :$$

i	x_i	$\Delta_1^{(30)}(x_i)$	x_i	$\Delta_2^{(30)}(x_i)$	x_i	$\Delta_3^{(30)}(x_i)$	x_i	$\Delta_4^{(30)}(x_i)$	x_i	$\Delta_5^{(30)}(x_i)$
1	3.953	-1.551E-02	2.356	3.014E-02	3.933	5.971E-03	2.356	-4.857E-02	3.928	-6.275E-03
2	7.084	4.314E-03	5.498	-2.889E-03	7.072	-6.255E-04	5.498	1.167E-03	7.069	2.440E-04
3	10.221	-1.832E-03	8.639	7.036E-04	10.213	1.380E-04	8.639	-1.302E-04	10.211	-2.818E-05
4	13.360	9.631E-04	11.781	-2.547E-04	13.354	-4.417E-05	11.781	2.690E-05	13.352	5.533E-06
5	16.500	-5.762E-04	14.923	1.153E-04	16.495	1.771E-05	14.923	-7.843E-06	16.494	-1.496E-06
6	19.641	3.758E-04	18.064	-6.022E-05	19.636	-8.262E-06	18.064	2.852E-06	19.635	5.014E-07
7	22.781	-2.607E-04	21.206	3.478E-05	22.778	4.297E-06	21.206	-1.211E-06	22.777	-1.962E-07
8	25.922	1.894E-04	24.347	-2.161E-05	25.919	-2.426E-06	24.347	5.757E-07	25.918	8.626E-08
9	29.064	-1.427E-04	27.489	1.421E-05	29.061	1.460E-06	27.489	-2.988E-07	29.060	-4.154E-08
10	32.205	1.106E-04	30.631	-9.769E-06	32.202	-9.244E-07	30.631	1.662E-07	32.202	2.152E-08

i	x_i	$\Delta_6^{(30)}(x_i)$	x_i	$\Delta_7^{(30)}(x_i)$	x_i	$\Delta_8^{(30)}(x_i)$	x_i	$\Delta_9^{(30)}(x_i)$	x_i	$\Delta_{10}^{(30)}(x_i)$
1	5.498	-1.017E-03	3.927	1.332E-02	5.498	1.579E-03	3.927	-4.858E-02	5.498	-3.900E-03
2	8.639	5.162E-05	7.069	-1.853E-04	8.639	-3.588E-05	7.069	2.333E-04	8.639	3.891E-05
3	11.781	-6.116E-06	10.210	1.115E-05	11.781	2.431E-06	10.210	-7.235E-06	11.781	-1.498E-06
4	14.923	1.154E-06	13.352	-1.345E-06	14.923	-2.973E-07	13.352	5.351E-07	14.923	1.186E-07
5	18.064	-2.934E-07	16.493	2.458E-07	18.064	5.293E-08	16.493	-6.616E-08	18.064	-1.479E-08
6	21.206	9.182E-08	19.635	-5.934E-08	21.206	-1.224E-08	19.635	1.152E-08	21.206	2.528E-09
7	24.347	-3.349E-08	22.777	1.750E-08	24.347	3.429E-09	22.777	-2.565E-09	24.347	-5.448E-10
8	27.489	1.374E-08	25.918	-6.004E-09	27.489	-1.114E-09	25.918	6.874E-10	27.489	1.403E-10
9	30.631	-6.192E-09	29.060	2.317E-09	30.631	4.071E-10	29.060	-2.129E-10	30.631	-4.162E-11
10	33.772	3.009E-09	32.201	-9.833E-10	33.772	-1.636E-10	32.201	7.404E-11	33.772	1.384E-11

In the case $x \geq 8$ one has $g_n^{(30)} \leq \Delta_n^{(30)}(x) \leq G_n^{(30)}$ with such values:

n	$g_n^{(30)}$	$G_n^{(30)}$	n	$g_n^{(30)}$	$G_n^{(30)}$
1	-1.832E-03	2.384E-03	6	5.162E-05	-6.116E-06
2	-2.547E-04	7.036E-04	7	-9.887E-05	1.115E-05
3	-3.729E-04	1.380E-04	8	-3.588E-05	2.431E-06
4	-1.302E-04	2.690E-05	9	1.129E-04	-7.235E-06
5	1.404E-04	-2.818E-05	10	3.891E-05	-1.498E-06

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{30}(x) = 0.896\ldots + \sum_{k=1}^{10} \frac{1}{x^{k+1/2}} \left[\tilde{a}_k^{(30)} \sin\left(3x + \frac{7-2k}{4}\,\pi\right) + \tilde{b}_k^{(30)} \sin\left(x + \frac{1-2k}{4}\,\pi\right)\right] \quad .$$

The values of the coefficients are

k	$\tilde{a}_k^{(30)}$	$\tilde{b}_k^{(30)}$
1	0.042328861175	0.380959694283
2	0.037041314549	0.619046345961
3	0.041506694202	1.619134748282
4	0.063280735888	5.687020985444
5	0.009657085591	24.391624641402
6	0.955932046063	129.687141434225
7	-12.477953789465	581.699380578201
8	57.157092072696	3774.819577903598
9	-530.170189213242	9530.689591084210
10	649.179657827779	65374.322605554143

With $8 \leq x \leq 30$ holds

$$-1.8 \cdot 10^{-9} \ \leq \ F_{30}(x) - \int_0^x J_0^3(t)\,dt \ \leq \ 1.1 \cdot 10^{-9} \quad .$$

Power series for the modified Bessel function:

$$\int_0^x I_0^3(t)\,dt \;=\; \sum_{k=0}^{\infty} d_k^{(30)}\,x^{2k+1} = x+\frac{1}{4}\,x^3+\frac{3}{64}\,x^5+\frac{31}{5376}\,x^7+\frac{71}{147456}\,x^9+\frac{47}{1638400}\,x^{11}+\frac{11723}{9201254400}\,x^{13}+\ldots$$

With $n \geq 1$ the following recurrence relation holds:

$$d_{n+1}^{(30)} = \frac{4\,\sigma_1^{(30)}(n,d) \;+\; 9\,\sigma_2^{(30)}(n,d)}{12(2n+3)(n+1)^2}$$

with

$$\sigma_1^{(30)}(n,d) = \sum_{k=1}^{n}(2k+1)k(2n-2k+3)(2n-5k+2)\cdot d_k^{(30)}\cdot d_{n-k+1}^{(30)}$$

and

$$\sigma_2^{(30)}(n,d) = \sum_{k=0}^{n}(2k+1)(2n-2k+1)\cdot d_k^{(30)}\cdot d_{n-k}^{(30)}\,.$$

Asymptotic formula for the modified Bessel function:

$$\int_0^x I_0^3(t)\,dt \sim \frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{\infty}\frac{c_k^{(30)}}{x^k}$$

with the first values

k	$c_k^{(30)}$	$c_k^{(30)}$
1	1/12	0.08333 33333 33333 3
2	7/96	0.07291 66666 66666 7
3	379/4608	0.08224 82638 88888 9
4	13141/110592	0.11882 41464 12037
5	250513/1179648	0.21236 25013 56337
6	12913841/28311552	0.45613 32773 27926
7	1565082415/1358954496	1.15168 12517 3192
8	36535718855/10871635968	3.36064 59011 8182
9	23344744269635/2087354105856	11.18389 26630 331
10	2103860629922855/50096498540544	41.99616 12331 082

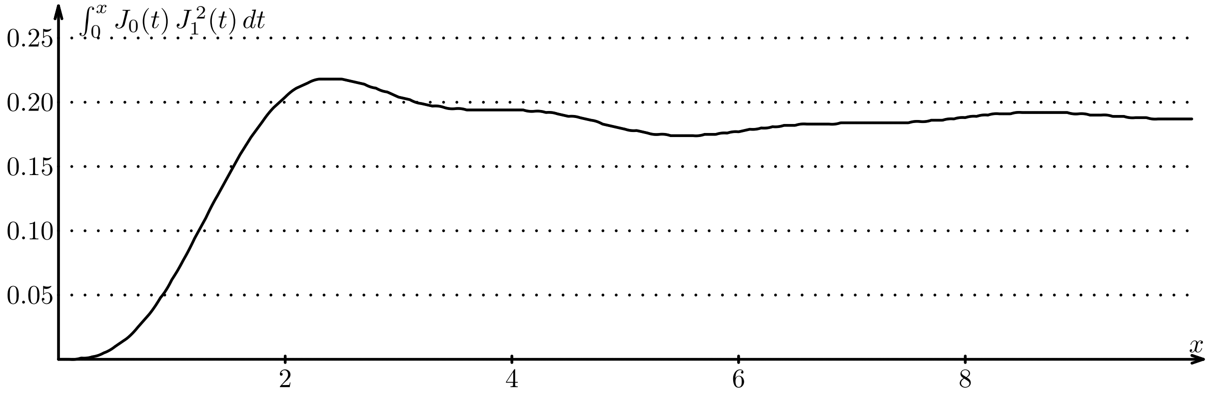
Let

$$\delta_n^{(30)}(x) = \left[\frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{n}\frac{c_k^{(30)}}{x^k}\right]\cdot\left[\int_0^x I_0^3(t)\,dt\right]^{-1} - 1$$

be the relative error, then one has the following values of $\delta_n^{(30)}(x)$:

n	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
1	-1.897E-01	-9.019E-02	-5.944E-02	-4.435E-02	-3.538E-02
2	-4.794E-02	-1.058E-02	-4.579E-03	-2.545E-03	-1.618E-03
3	-1.595E-02	-1.597E-03	-4.529E-04	-1.874E-04	-9.492E-05
4	-6.704E-03	-3.000E-04	-5.555E-05	-1.710E-05	-6.896E-06
5	-3.400E-03	-6.813E-05	-8.209E-06	-1.877E-06	-6.026E-07
6	-1.981E-03	-1.833E-05	-1.430E-06	-2.427E-07	-6.198E-08
7	-1.264E-03	-5.756E-06	-2.887E-07	-3.632E-08	-7.376E-09
8	-8.462E-04	-2.087E-06	-6.672E-08	-6.211E-09	-1.002E-09
9	-5.678E-04	-8.660E-07	-1.747E-08	-1.201E-09	-1.539E-10
10	-3.588E-04	-4.075E-07	-5.140E-09	-2.603E-10	-2.647E-11

b) Basic integral $\int Z_0(x)\, Z_1^2(x)\, dx$:



$$\int_0^\infty J_0(x)\, J_1^2(x)\, dx \quad = \quad 0.18578\ 75214\ 63065\ 82253\ \ldots$$

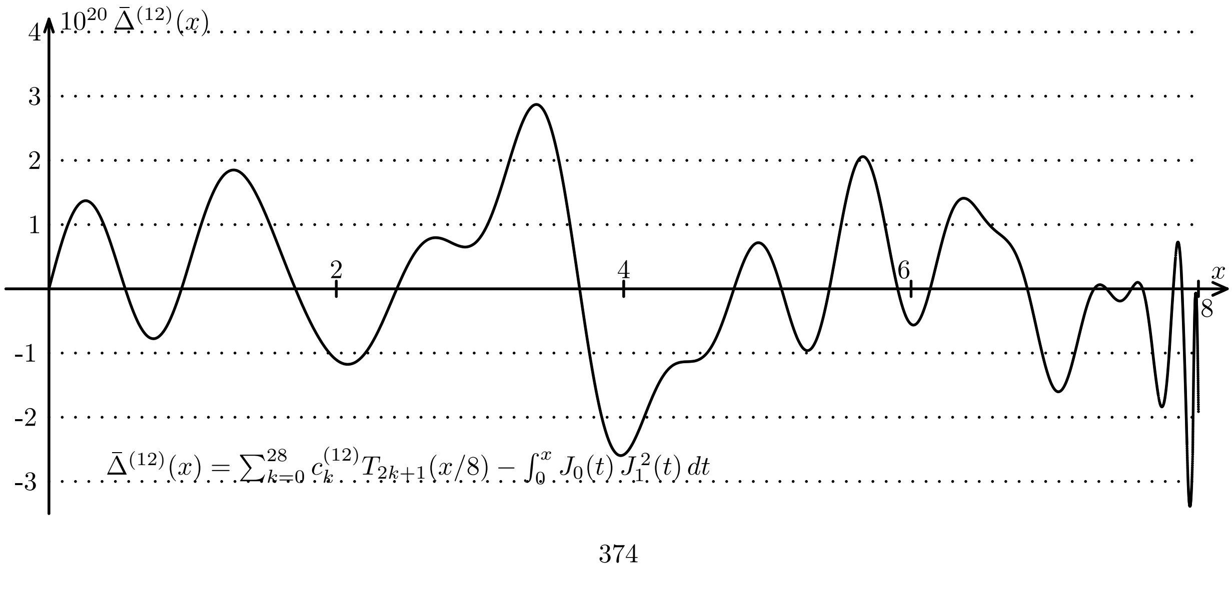
It differs from formula 2.12.42.4 in [4].
With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x J_0(t)\, J_1^2(t)\, dt = \sum_{k=0}^\infty c_k^{(12)}\, T_{2k+1}\left(\frac{x}{8}\right) \ .$$

The first coefficients are

k	$c_k^{(12)}$	k	$c_k^{(12)}$
0	0.23402 35970 33006 35445	15	0.00001 04741 31410 16206
1	-0.07444 03661 88704 59538	16	-0.00000 16973 09596 00078
2	0.04383 45295 19423 05343	17	0.00000 02378 37605 87389
3	-0.02088 20217 52543 47233	18	-0.00000 00291 52869 89193
4	0.00418 98619 72133 53415	19	0.00000 00031 56305 17549
5	0.00544 25004 57533 63387	20	-0.00000 00003 04352 15694
6	-0.00771 94350 97286 08113	21	0.00000 00000 26326 39770
7	0.00909 15741 08442 81722	22	-0.00000 00000 02055 66180
8	-0.01063 96487 30041 15062	23	0.00000 00000 00145 70197
9	0.00922 86127 36398 55548	24	-0.00000 00000 00009 42056
10	-0.00564 70826 05794 70935	25	0.00000 00000 00000 55809
11	0.00254 11445 11845 57857	26	-0.00000 00000 00000 03042
12	-0.00087 99927 23272 67696	27	0.00000 00000 00000 00153
13	0.00024 31559 72294 70777	28	-0.00000 00000 00000 00007
14	-0.00005 51197 58270 08008	-	-

The given approximation differs from the true function as shown in the following figure:



Asymptotic formula:

$$\int_0^x J_0(t)\,J_1^2(t)\,dt \sim 0.18578\ 75214\ 63065\ 82253\ \ldots\ + \sqrt{\frac{2}{\pi^3\,x}}\ \left\{ \frac{\left[a_0^{(12)}\ \sin\left(3x+\frac{\pi}{4}\right) + b_0^{(12)}\ \sin\left(x+\frac{7\pi}{4}\right)\right]}{x} + \right.$$

$$\left. + \sum_{k=2}^{\infty} \frac{1}{x^k}\ \left[a_k^{(12)}\ \sin\left(3x+\frac{7-2k}{4}\,\pi\right) + b_k^{(12)}\ \sin\left(x+\frac{1-2k}{4}\,\pi\right)\right]\right\}$$

with the first values

k	$a_k^{(12)}$	$a_k^{(12)}$
1	1/6	0.16666 66666 66666 66667
2	1/48	0.02083 33333 33333 33333
3	85/2304	0.03689 23611 11111 11111
4	3379/55296	0.06110 74942 12962 96296
5	69967/589824	0.11862 35215 92881 94444
6	3833063/14155776	0.27077 73137 97562 21065
7	487468417/679477248	0.71741 68354 19925 64260
8	11835711665/5435817984	2.17735 61402 97136 1890
9	7811427811325/1043677052928	7.48452 57825 78345 0049
10	722951995177505/25048249270272	28.86237 62633 79242 299
k	$b_k^{(12)}$	$b_k^{(12)}$
1	1/2	0.50000 00000 00000 00000
2	21/16	1.31250 00000 00000 00000
3	753/256	2.94140 62500 00000 00000
4	21375/2048	10.43701 17187 50000 0000
5	3061323/65536	46.71208 19091 79687 5000
6	134966187/524288	257.42757 22503 66210 938
7	14029169013/8388608	1672.40727 10275 65002 44
8	842027324535/67108864	12547.18489 25203 08494 6
9	458031686444595/4294967296	1 06643.81236 87624 46567
10	34812460139616855/34359738368	10 13175.93768 52090 6451

The first consecutive maxima and minima of

$$\Delta_n^{(12)}(x) = 0.185\ldots + \sqrt{\frac{2}{\pi^3\,x}}\ \sum_{k=1}^{n} \frac{1}{x^k}\ \left[a_k^{(12)}\ \sin(3x+\ldots) + b_k^{(12)}\ \sin(x+\ldots)\right] - \int_0^x J_0(t)\,J_1^2(t)\,dt \quad :$$

i	x_i	$\Delta_1^{(12)}(x_i)$	x_i	$\Delta_2^{(12)}(x_i)$	x_i	$\Delta_3^{(12)}(x_i)$	x_i	$\Delta_4^{(12)}(x_i)$	x_i	$\Delta_5^{(12)}(x_i)$
1	3.880	-8.324E-03	2.365	1.432E-02	3.923	2.771E-03	2.358	-2.242E-02	3.927	-2.887E-03
2	7.042	2.261E-03	5.503	-1.356E-03	7.066	-2.895E-04	5.499	5.375E-04	7.068	1.122E-04
3	10.192	-9.530E-04	8.643	3.293E-04	10.209	6.382E-05	8.640	-5.993E-05	10.210	-1.296E-05
4	13.338	4.995E-04	11.784	-1.191E-04	13.351	-2.042E-05	11.782	1.238E-05	13.352	2.544E-06
5	16.482	-2.984E-04	14.925	5.386E-05	16.492	8.184E-06	14.923	-3.609E-06	16.493	-6.876E-07
6	19.625	1.944E-04	18.066	-2.813E-05	19.634	-3.818E-06	18.065	1.313E-06	19.635	2.305E-07
7	22.768	-1.348E-04	21.207	1.624E-05	22.776	1.986E-06	21.206	-5.571E-07	22.776	-9.018E-08
8	25.911	9.790E-05	24.349	-1.009E-05	25.918	-1.121E-06	24.348	2.649E-07	25.918	3.965E-08
9	29.053	-7.371E-05	27.490	6.636E-06	29.059	6.745E-07	27.489	-1.375E-07	29.060	-1.910E-08
10	32.196	5.712E-05	30.632	-4.561E-06	32.201	-4.271E-07	30.631	7.645E-08	32.201	9.893E-09

i	x_i	$\Delta_6^{(12)}(x_i)$	x_i	$\Delta_7^{(12)}(x_i)$	x_i	$\Delta_8^{(12)}(x_i)$	x_i	$\Delta_9^{(12)}(x_i)$	x_i	$\Delta_{10}^{(12)}(x_i)$
1	2.356	8.374E-02	3.927	6.119E-03	2.356	-6.023E-01	3.927	-2.232E-02	2.356	7.281E+00
2	5.498	-4.673E-04	7.069	-8.515E-05	5.498	7.255E-04	7.069	1.072E-04	5.498	-1.792E-03
3	8.640	2.372E-05	10.210	5.122E-06	8.639	-1.649E-05	10.210	-3.324E-06	8.639	1.788E-05
4	11.781	-2.810E-06	13.352	-6.178E-07	11.781	1.117E-06	13.352	2.458E-07	11.781	-6.882E-07
5	14.923	5.304E-07	16.493	1.129E-07	14.923	-1.366E-07	16.493	-3.040E-08	14.923	5.448E-08
6	18.064	-1.348E-07	19.635	-2.726E-08	18.064	2.432E-08	19.635	5.293E-09	18.064	-6.794E-09
7	21.206	4.219E-08	22.777	8.042E-09	21.206	-5.621E-09	22.777	-1.179E-09	21.206	1.161E-09
8	24.347	-1.539E-08	25.918	-2.759E-09	24.347	1.575E-09	25.918	3.158E-10	24.347	-2.503E-10
9	27.489	6.316E-09	29.060	1.065E-09	27.489	-5.119E-10	29.060	-9.780E-11	27.489	6.446E-11
10	30.631	-2.845E-09	32.201	-4.518E-10	30.631	1.871E-10	32.201	3.402E-11	30.631	-1.912E-11

In the case $x \geq 8$ one has $g_n^{(12)} \leq \Delta_n^{(12)}(x) \leq G_n^{(12)}$ with such values:

n	$g_n^{(12)}$	$G_n^{(12)}$	n	$g_n^{(12)}$	$G_n^{(12)}$
1	-9.530E-04	1.314E-03	6	-2.810E-06	2.372E-05
2	-1.191E-04	3.293E-04	7	-4.543E-05	5.122E-06
3	-1.735E-04	6.382E-05	8	-1.649E-05	1.117E-06
4	-5.993E-05	1.238E-05	9	-3.324E-06	5.189E-05
5	6.457E-05	1.238E-05	10	1.788E-05	-6.882E-07

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{12}(x) = 0.18578\,75214\,63 + \sum_{k=1}^{10} \frac{1}{x^{k+1/2}} \left[\tilde{a}_k^{(12)} \sin\left(3x + \frac{7-2k+4\delta_{1k}}{4}\,\pi\right) + \tilde{b}_k^{(12)} \sin\left(x + \frac{1-2k}{4}\,\pi\right)\right] \quad .$$

Here δ_{kl} denotes the Kronecker symbol.
The values of the coefficients are

k	$\tilde{a}_k^{(12)}$	$\tilde{b}_k^{(12)}$
1	0.042329196006	0.126986297337
2	0.005292679875	0.333333960255
3	0.009245148401	0.745696621275
4	0.016862928651	2.639682546435
5	-0.014976122776	11.204590967086
6	0.401440149083	59.656792209975
7	-5.815950738680	267.223468914520
8	26.023951772804	1734.742283706764
9	-244.011181624276	4378.452764670006
10	296.167068374575	30038.714778003193

With $8 \leq x \leq 30$ holds

$$-8.1 \cdot 10^{-10} \;\leq\; F_{12}(x) - \int_0^x J_0(t)\, J_1^2(t)\, dx \;\leq\; 5.5 \cdot 10^{-10} \quad .$$

Power series for the modified Bessel functions:

$$\int_0^x I_0(t)\, I_1^2(t)\, dt \;=\; \sum_{k=1}^{\infty} d_k^{(12)}\, x^{2k+1} = \frac{x^3}{12} - \frac{x^5}{40} + \frac{5}{1344}\, x^7 - \frac{19}{55296}\, x^9 + \frac{707}{32440320}\, x^{11} - \frac{581}{575078400}\, x^{13} + \ldots$$

With $n \geq 1$ the coefficients $d_k^{(12)}$ are represented bei $d_k^{(03)}$ (see page 380):

$$d_k^{(12)} \;=\; \frac{4(k+1)(k+2)}{6k+3}\, d_{k+1}^{(03)} \quad .$$

Asymptotic formula for the modified Bessel function:

$$\int_0^x I_0(t)\, I_1^2(t)\, dt \sim \frac{\sqrt{2}\, e^{3x}}{\sqrt{\pi^3}\, x} \sum_{k=1}^{\infty} \frac{c_k^{(12)}}{x^k}$$

with the first values

k	$c_k^{(12)}$	$c_k^{(12)}$
1	1/12	0.08333 33333 33333 3
2	-1/96	-0.010416666666667
3	-85/4608	-0.018446180555556
4	-3379/110592	-0.030553747106481
5	-69967/1179648	-0.059311760796441
6	-3833063/28311552	-0.135388656898781
7	-487468417/1358954496	-0.358708417709963
8	-11835711665/10871635968	-1.088678070148568
9	-7811427811325/2087354105856	-3.742262891289173
10	-722951995177505/50096498540544	-14.431188131689621

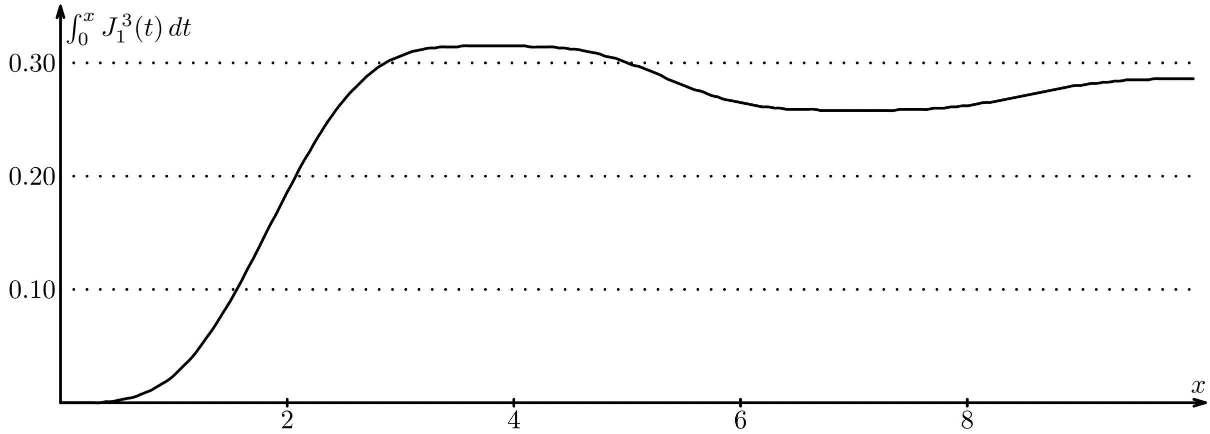
Let

$$\delta_n^{(12)}(x) = \left[\frac{\sqrt{2}\, e^{3x}}{\sqrt{\pi^3}\, x} \sum_{k=1}^{n} \frac{c_k^{(12)}}{x^k} \right] \cdot \left[\int_0^x I_0(t)\, I_1^2(t)\, dt \right]^{-1} - 1$$

be the relative error, then one has the following values of $\delta_n^{(12)}(x)$:

n	$x = 5$	$x = 10$	$x = 15$	$x = 20$	$x = 25$
1	4.085E-02	1.541E-02	9.532E-03	6.902E-03	5.409E-03
2	1.483E-02	2.715E-03	1.120E-03	6.084E-04	3.817E-04
3	5.617E-03	4.676E-04	1.265E-04	5.122E-05	2.561E-05
4	2.564E-03	9.526E-05	1.684E-05	5.070E-06	2.019E-06
5	1.379E-03	2.299E-05	2.643E-06	5.914E-07	1.875E-07
6	8.376E-04	6.492E-06	4.829E-07	8.020E-08	2.023E-08
7	5.508E-04	2.121E-06	1.014E-07	1.248E-08	2.503E-09
8	3.768E-04	7.943E-07	2.419E-08	2.204E-09	3.513E-10
9	2.571E-04	3.383E-07	6.500E-09	4.374E-10	5.537E-11
10	1.648E-04	1.624E-07	1.952E-09	9.679E-11	9.724E-12

c) Basic integral $\int Z_1^3(x)\, dx$:

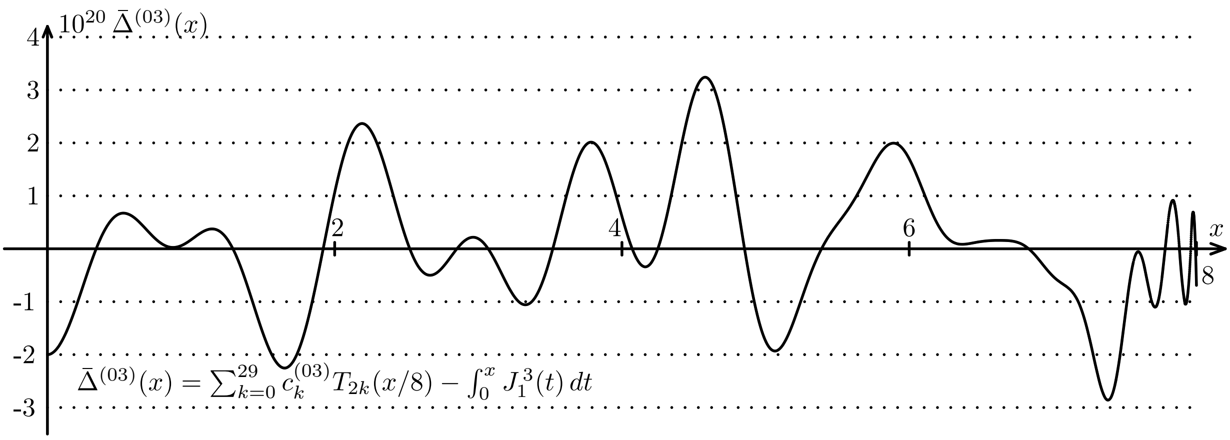


$$\int_0^\infty J_1^3(x)\, dx \quad = \quad \frac{\sqrt{3}}{2\pi} \quad = \quad 0.27566\ 44477\ 10896\ 02476\ \ldots$$

Formula 2.12.42.18 from [4] gives $2 \cdot 0.27566\ldots$.
With $-8 \le x \le 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x J_1^3(t)\, dt = \sum_{k=0}^{\infty} c_k^{(03)}\, T_{2k+1}\left(\frac{x}{8}\right) .$$

Using 30 items of the series, the given approximation differs from the true function as shown in the following figure:



The first coefficients of the series are

k	$c_k^{(03)}$	k	$c_k^{(03)}$
0	0.23821 24114 31419 64234	15	-0.00002 28518 02498 15054
1	0.05565 99710 46458 43436	16	0.00000 40330 04736 70912
2	-0.07298 97702 38741 38530	17	-0.00000 06104 34965 75114
3	0.06987 40694 66707 70829	18	0.00000 00802 87294 38422
4	-0.04171 62455 49522 66601	19	-0.00000 00092 76147 24624
5	0.01717 16060 47354 59698	20	0.00000 00009 50125 56893
6	-0.00484 82605 99791 99458	21	-0.00000 00000 86955 13371
7	-0.00029 03839 52557 94438	22	0.00000 00000 07159 24024
8	0.00492 11828 87493 16152	23	-0.00000 00000 00533 44393
9	-0.00739 18550 17718 86749	24	0.00000 00000 00036 16224
10	0.00611 65128 72946 06394	25	-0.00000 00000 00002 24086
11	-0.00338 52693 70801 23544	26	0.00000 00000 00000 12747
12	0.00137 35573 01229 79344	27	-0.00000 00000 00000 00668
13	-0.00043 18988 45664 76700	28	0.00000 00000 00000 00032
14	0.00010 92654 18291 14430	29	-0.00000 00000 00000 00001

Asymptotic formula:

$$\int_0^x J_1^3(t)\,dt \sim \frac{\sqrt{3}}{2\pi} + \sqrt{\frac{2}{\pi^3\,x}}\left\{\frac{a_1^{(03)}\sin(3x-\frac{\pi}{4}) + b_1^{(03)}\sin(x+\frac{5\pi}{4})}{x} + \right.$$

$$\left. + \sum_{k=2}^{\infty}\frac{1}{x^k}\left[a_k^{(03)}\,\sin\left(3x+\frac{5-2k}{4}\,\pi\right) + b_k^{(03)}\,\sin\left(x+\frac{7-2k}{4}\,\pi\right)\right]\right\}$$

with $\sqrt{3}/2\pi = 0.27566\ 44477\ 10896\ 02476\ \ldots$ (see [8], 13.46) and the first values

k	$a_k^{(03)}$	$a_k^{(03)}$
1	$1/6$	0.16666 66666 66666 66667
2	$5/48$	0.10416 66666 66666 66667
3	$173/2304$	0.07508 68055 55555 55556
4	$5735/55296$	0.10371 45543 98148 14815
5	$112415/589824$	0.19059 07524 95659 72222
6	$6113875/14155776$	0.43189 96712 01352 71991
7	$790059305/679477248$	1.16274 57833 58444 40225
8	$19738125085/5435817984$	3.63112 32537 76703 35181
9	$13496143234525/13496143234525$	12.93134 03956 34871 2675
10	$1298437733131525/25048249270272$	51.83746 45318 04589 8562
k	$b_k^{(03)}$	$b_k^{(03)}$
1	$3/2$	1.50000 00000 00000 00000
2	$27/16$	1.68750 00000 00000 00000
3	$891/256$	3.48046 87500 00000 00000
4	$25065/2048$	12.23876 95312 50000 0000
5	$3564945/65536$	54.39674 37744 14062 5000
6	$156773205/156773205$	299.02115 82183 83789 063
7	$16277745015/8388608$	1940.45841 87269 21081 54
8	$976536193185/67108864$	14551.52322 62760 40077 2
9	$531096920069625/4294967296$	1 23655.63774 23985 39558
10	$40362305845577625/34359738368$	11 74697.70617 25228 8503

The first consecutive maxima and minima of

$$\Delta_n^{(03)}(x) = \frac{\sqrt{3}}{2\pi} + \sqrt{\frac{2}{\pi^3\,x}}\,\left\{\frac{a_1^{(03)}\sin(3x-\frac{\pi}{4}) + b_1^{(03)}\sin(x+\frac{5\pi}{4})}{x} + \right.$$

$$\left. + \sum_{k=1}^{n} \frac{1}{x^k}\left[a_k^{(03)}\,\sin\left(3x + \frac{7-2k}{4}\,\pi\right) + b_k^{(03)}\,\sin\left(x + \frac{1-2k}{4}\,\pi\right)\right]\right\} - \int_0^x J_0^3(t)\,dt \quad :$$

i	x_i	$\Delta_1^{(03)}(x_i)$	x_i	$\Delta_2^{(03)}(x_i)$	x_i	$\Delta_3^{(03)}(x_i)$	x_i	$\Delta_4^{(03)}(x_i)$	x_i	$\Delta_5^{(03)}(x_i)$
1	2.023	2.977E-02	3.933	4.556E-03	2.338	-1.901E-02	3.927	-2.969E-03	2.354	4.614E-02
2	5.336	-4.810E-03	7.072	-7.718E-04	5.487	9.013E-04	7.069	1.899E-04	5.496	-5.376E-04
3	8.533	1.690E-03	10.213	2.352E-04	8.632	-1.477E-04	10.210	-3.029E-05	8.638	4.048E-05
4	11.702	-8.051E-04	13.354	-9.624E-05	11.775	4.028E-05	13.352	7.586E-06	11.780	-6.330E-06
5	14.860	4.533E-04	16.495	4.706E-05	14.918	-1.461E-05	16.493	-2.496E-06	14.922	1.484E-06
6	18.013	-2.838E-04	19.636	-2.593E-05	18.060	6.359E-06	19.635	9.863E-07	18.063	-4.510E-07
7	21.162	1.911E-04	22.778	1.557E-05	21.203	-3.145E-06	22.777	-4.448E-07	21.205	1.643E-07
8	24.309	-1.358E-04	25.919	-9.966E-06	24.345	1.709E-06	25.918	2.215E-07	24.347	-6.841E-08
9	27.455	1.005E-04	29.061	6.707E-06	27.486	-9.979E-07	29.060	-1.192E-07	27.488	3.156E-08
10	30.600	-7.684E-05	32.202	-4.698E-06	30.628	6.168E-07	32.201	6.828E-08	30.630	-1.579E-08

i	x_i	$\Delta_6^{(03)}(x_i)$	x_i	$\Delta_7^{(03)}(x_i)$	x_i	$\Delta_8^{(03)}(x_i)$	x_i	$\Delta_9^{(03)}(x_i)$	x_i	$\Delta_{10}^{(03)}(x_i)$
1	3.927	4.522E-03	2.356	-2.426E-01	3.927	-1.276E-02	2.356	2.291E+00	3.927	5.861E-02
2	7.069	-1.056E-04	5.497	6.337E-04	7.069	1.047E-04	5.498	-1.254E-03	7.069	-1.634E-04
3	10.210	8.813E-06	8.639	-2.157E-05	10.210	-4.529E-06	8.639	1.893E-05	10.210	3.623E-06
4	13.352	-1.356E-06	11.781	1.933E-06	13.352	4.279E-07	11.781	-9.676E-07	13.352	-2.095E-07
5	16.493	3.014E-07	14.922	-2.937E-07	16.493	-6.433E-08	14.923	9.523E-08	16.493	2.130E-08
6	19.635	-8.572E-08	18.064	6.248E-08	19.635	1.319E-08	18.064	-1.419E-08	19.635	-3.150E-09
7	22.777	2.911E-08	21.206	-1.680E-08	22.777	-3.380E-09	21.206	2.820E-09	22.777	6.096E-10
8	25.918	-1.130E-08	24.347	5.371E-09	25.918	1.025E-09	24.347	-6.931E-10	25.918	-1.444E-10
9	29.060	4.873E-09	27.489	-1.961E-09	29.060	-3.542E-10	27.489	2.005E-10	29.060	4.007E-11
10	32.201	-2.284E-09	30.630	7.955E-10	32.201	1.361E-10	30.631	-6.600E-11	32.201	-1.262E-11

In the case $x \geq 8$ one has $g_n^{(03)} \leq \Delta_n^{(03)}(x) \leq G_n^{(03)}$ with such values:

n	g_n	G_n	n	g_n	G_n
1	-8.051E-04	1.690E-03	6	-5.870E-05	8.813E-06
2	-4.559E-04	2.352E-04	7	-2.157E-05	1.933E-06
3	-1.477E-04	4.028E-05	8	-4.529E-06	5.333E-05
4	-3.029E-05	1.120E-04	9	-9.676E-07	1.893E-05
5	-6.330E-06	4.048E-05	10	-7.485E-05	3.623E-06

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{03}(x) = \frac{\sqrt{3}}{2\pi} + \sum_{k=1}^{10} \frac{1}{x^{k+1/2}}\left[\tilde{a}_k^{(03)}\,\sin\left(3x + \frac{5-2k+4\delta_{1k}}{4}\,\pi\right) + \tilde{b}_k^{(03)}\,\sin\left(x + \frac{7-2k}{4}\,\pi\right)\right] \quad .$$

Here δ_{kl} denotes the Kronecker symbol.
The values of the coefficients are

k	$\tilde{a}_k^{(12)}$	$\tilde{b}_k^{(12)}$
1	0.042329125987	0.380960729687
2	0.026455529094	0.428583324556
3	0.019031312665	0.882368927951
4	0.026093503230	3.106884244014
5	0.036754295203	13.006466163819
6	-0.015099663650	73.274117576600
7	-0.742020898226	297.966084182852
8	-21.408004618587	2526.040556452494
9	-6.362414618305	4245.043677286622
10	-1177.354351044844	55725.065225312970

With $8 \leq x \leq 30$ holds

$$-1.0\cdot 10^{-9} \;\leq\; F_{03}(x) \;\leq\; 4.5\cdot 10^{-10} \quad .$$

Power series for the modified Bessel function:

$$\int_0^x I_1^3(t)\,dt \;=\; \sum_{k=2}^{\infty} d_k^{(03)}\,x^{2k} = \frac{x^4}{32} + \frac{x^6}{128} + \frac{x^8}{1024} + \frac{19}{245760}\,x^{10} + \frac{101}{23592960}\,x^{12} + \frac{83}{471859200}\,x^{14} + \ldots$$

With $n \geq 3$ the following recurrence relation holds:

$$d_n^{(03)} = \frac{16\,\sigma_1^{(03)}(n,d) \;+\; 36\,\sigma_2^{(03)}(n,d)}{3n(n-1)(n-2)}$$

with

$$\sigma_1^{(03)}(n,d) = \sum_{k=3}^{n-1} k(n-k+2)(2nk-n+7k-5k^2)\cdot d_k^{(03)}\cdot d_{n-k+2}^{(03)}$$

and

$$\sigma_2^{(03)}(n,d) = \sum_{k=2}^{n-1} k(n-k+1)\cdot d_k^{(03)}\cdot d_{n-k+1}^{(03)}\,.$$

Asymptotic formula for the modified Bessel function:

$$\int_0^x I_1^3(t)\,dt \sim \frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{\infty}\frac{c_k^{(03)}}{x^k}$$

with the first values

k	$c_k^{(03)}$	$c_k^{(03)}$
1	1/12	0.08333 33333 33333 3
2	-5/96	-0.052083333333333
3	-173/4608	-0.037543402777778
4	-5735/110592	-0.051857277199074
5	-112415/1179648	-0.095295376247830
6	-6113875/28311552	-0.215949835600676
7	-790059305/1358954496	-0.581372891679222
8	-19738125085/10871635968	-1.815561626888352
9	-13496143234525/2087354105856	-6.465670197817436
10	-1298437733131525/50096498540544	-25.918732265902295

Let

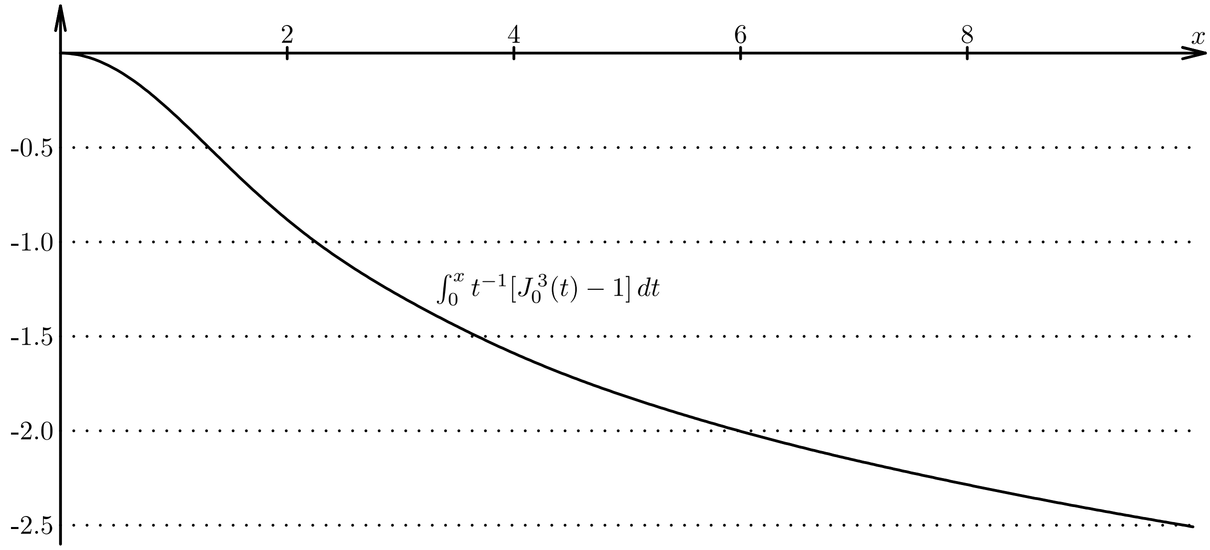
$$\delta_n^{(03)}(x) = \left[\frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{n}\frac{c_k^{(03)}}{x^k}\right]\cdot\left[\int_0^x I_1^3(t)\,dt\right]^{-1} - 1$$

be the relative error, then one has the following values of $\delta_n^{(03)}(x)$:

n	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
1	1.791E-01	7.271E-02	4.589E-02	3.355E-02	2.645E-02
2	3.170E-02	5.662E-03	2.315E-03	1.253E-03	7.841E-04
3	1.046E-02	8.293E-04	2.209E-04	8.875E-05	4.419E-05
4	4.587E-03	1.618E-04	2.801E-05	8.358E-06	3.311E-06
5	2.429E-03	3.912E-05	4.387E-06	9.712E-07	3.060E-07
6	1.451E-03	1.132E-05	8.179E-07	1.342E-07	3.362E-08
7	9.250E-04	3.840E-06	1.774E-07	2.154E-08	4.288E-09
8	5.962E-04	1.503E-06	4.399E-08	3.949E-09	6.244E-10
9	3.620E-04	6.710E-07	1.232E-08	8.161E-10	1.024E-10
10	1.742E-04	3.374E-07	3.863E-09	1.882E-10	1.874E-11

d) Basic integral $\int x^{-1} \cdot Z_0^3(x)\, dx$:

$$\int \frac{Z_0^3(x)\, dx}{x} \quad = \quad \ln|x| + \int_0^x \frac{Z_0^3(t) - 1}{t}\, dt$$



$$\int_0^x \frac{J_0^3(t) - 1}{t}\, dt = -\frac{1}{4}\, x^3 + \frac{3}{64}\, x^5 - \frac{31}{5376}\, x^7 + \frac{71}{147456}\, x^9 - \frac{47}{1638400}\, x^{11} + \frac{11723}{9201254400}\, x^{13} - \frac{2021}{46242201600}\, x^{15} +$$

$$+ \frac{1567}{1315333734400}\, x^{17} - \frac{5773279}{218624250662092800}\, x^{19} + \frac{3125957}{6443662124777472000}\, x^{21} - \frac{1114457}{148511069923442688000}\, x^{23} + \ldots$$

$$= -0.25x^3 + 0.046875x^5 - 0.00576\ 63690\ 47619x^7 + 0.00048\ 14995\ 65972x^9 -$$

$$-0.00002\ 86865\ 23438x^{11} + 0.00000\ 12740\ 65414x^{13} - 0.00000\ 00437\ 04667x^{15} + 0.00000\ 00011\ 91333x^{17} -$$

$$-0.00000\ 00000\ 26407x^{19} + 0.00000\ 00000\ 00485x^{21} - 0.00000\ 00000\ 00008x^{23} + \ldots$$

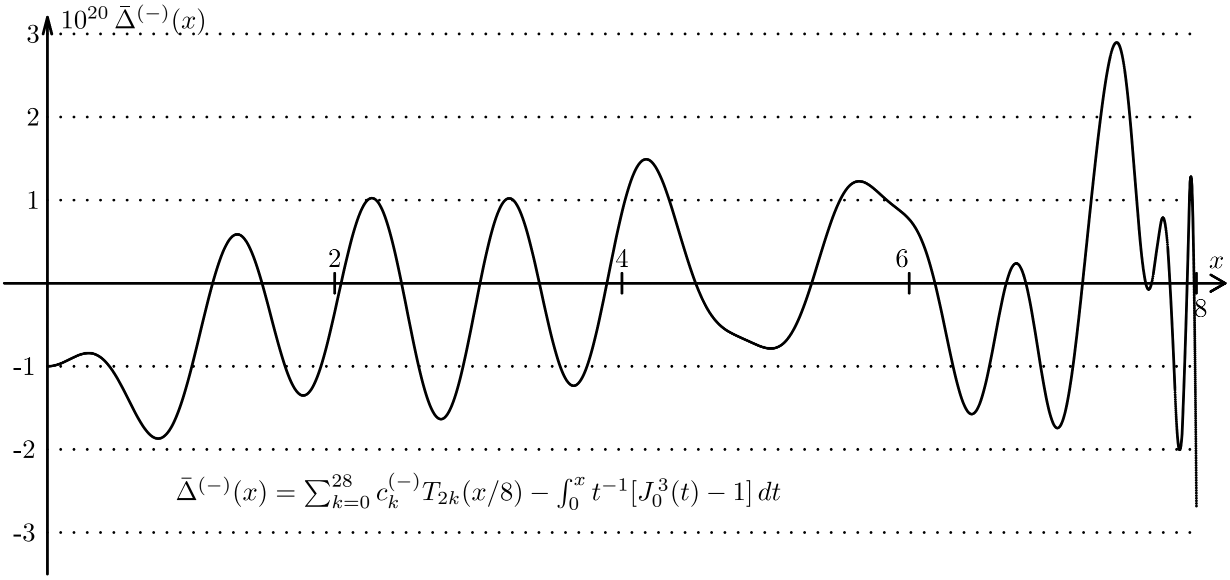
With $-8 \leq x \leq 8$ the following expansion in series of Chebyshev polynomials (based on [2], 9.7.) holds:

$$\int_0^x \frac{J_0^3(t) - 1}{t}\, dt = \sum_{k=0}^{\infty} c_k^{(-)}\, T_{2k}\left(\frac{x}{8}\right) .$$

The first coefficients are

k	$c_k^{(-)}$	k	$c_k^{(-)}$
0	-1.66487772839693011416	15	-0.00000370566539255354
1	-0.85731855220711666821	16	0.00000057633802528230
2	0.35797423128412846686	17	-0.00000007814654021909
3	-0.18824568501714899999	18	0.00000000932468761665
4	0.10690149363496148802	19	-0.00000000098725238622
5	-0.06393636091495798502	20	0.00000000009342113704
6	0.03911001491993894893	21	-0.00000000000795204305
7	-0.02409344095904841252	22	0.00000000000061237408
8	0.01437795892387628520	23	-0.00000000000004288320
9	-0.00768362174411199843	24	0.00000000000000274344
10	0.00345368891420790184	25	-0.00000000000000016101
11	-0.00127440919485714551	26	0.00000000000000000870
12	0.00038620719490337840	27	-0.00000000000000000043
13	-0.00009714124269423777	28	0.00000000000000000002
14	0.00002055168104878868	-	-

The given approximation differs from the true function as shown in the following figure:



Asymptotic formula:

$$\int_1^x \frac{J_0^3(t)\,dt}{t} \sim 0.11548\ 77825\ 09057\ 98226\ \ldots\ +$$

$$+\sqrt{\frac{2}{\pi^3\,x}}\sum_{k=1}^{\infty}\frac{1}{x^{k+1}}\left[a_k^{(01)}\ \sin\left(3x+\frac{7-2k}{4}\,\pi\right)+b_k^{(01)}\ \sin\left(x+\frac{1-2k}{4}\,\pi\right)\right]$$

with the first values

k	$a_k^{(01)}$	$a_k^{(01)}$
1	1/6	0.16666 66666 66666 66667
2	29/144	0.20138 88888 88888 88889
3	1921/6912	0.27792245370370370370
4	8527/18432	0.46261 93576 38888 88889
5	1621523/1769472	0.91638 80524 81192 12963
6	89993003/42467328	2.11911 14967 25200 13503
7	3821763071/679477248	5.62456 37101 89160 00555
8	275582100493/16307453952	16.89914 93892 39986 24700
9	177961856737289/3131031158784	56.83809 82852 95930 27586
k	$b_k^{(01)}$	$b_k^{(01)}$
1	3/2	1.50000 00000 00000 00000
2	63/16	3.93750 00000 00000 00000
3	3603/256	14.07421 87500 00000 00000
4	129981/2048	63.46728 51562 50000 00000
5	22909281/65536	349.56788 63525 39062 5000
6	1191489153/524288	2272.58520 69854 73632 813
7	143000089119/8388608	17046.93902 95743 94226 07
8	9724198215717/67108864	1 44901.84509 33247 80464 2
9	5912428624098201/4294967296	13 76594.56210 63469 44347

The first consecutive maxima and minima of

$$\Delta_n^{(-1)}(x) = 0.115\ldots + \sqrt{\frac{2}{\pi^3\,x}}\,\sum_{k=1}^{n} \frac{1}{x^{k+1}}\,\left[a_k^{(-1)}\,\sin\left(3x+\ldots\right) + b_k^{(-1)}\,\sin\left(x+\ldots\right)\right] - \int_1^x t^{-1}\cdot J_0^3(t)\,dt \quad :$$

i	x_i	$\Delta_1^{(-1)}(x_i)$	x_i	$\Delta_2^{(-1)}(x_i)$	x_i	$\Delta_3^{(-1)}(x_i)$	x_i	$\Delta_4^{(-1)}(x_i)$	x_i	$\Delta_5^{(-1)}(x_i)$
1	3.944	-5.284E-03	2.356	2.066E-02	3.929	3.477E-03	2.356	-5.340E-02	3.927	-5.296E-03
2	7.079	8.972E-04	5.498	-1.010E-03	7.070	-2.223E-04	5.498	6.258E-04	7.069	1.237E-04
3	10.217	-2.736E-04	8.639	1.668E-04	10.211	3.547E-05	8.639	-4.719E-05	10.210	-1.032E-05
4	13.357	1.120E-04	11.781	-4.561E-05	13.353	-8.883E-06	11.781	7.385E-06	13.352	1.589E-06
5	16.498	-5.477E-05	14.923	1.656E-05	16.494	2.923E-06	14.923	-1.732E-06	16.493	-3.531E-07
6	19.639	3.018E-05	18.064	-7.216E-06	19.635	-1.155E-06	18.064	5.264E-07	19.635	1.004E-07
7	22.780	-1.812E-05	21.206	3.571E-06	22.777	5.209E-07	21.206	-1.918E-07	22.777	-3.411E-08
8	25.921	1.160E-05	24.347	-1.941E-06	25.919	-2.594E-07	24.347	7.986E-08	25.918	1.324E-08
9	29.062	-7.806E-06	27.489	1.134E-06	29.060	1.396E-07	27.489	-3.685E-08	29.060	-5.709E-09
10	32.204	5.468E-06	30.631	-7.008E-07	32.202	-7.996E-08	30.631	1.844E-08	32.201	2.676E-09

i	x_i	$\Delta_6^{(-1)}(x_i)$	x_i	$\Delta_7^{(-1)}(x_i)$	x_i	$\Delta_8^{(-1)}(x_i)$	x_i	$\Delta_9^{(-1)}(x_i)$
1	2.356	2.837E-01	3.927	1.495E-02	2.356	-2.685E+00	3.927	-6.868E-02
2	5.498	-7.418E-04	7.069	-1.227E-04	5.498	1.470E-03	7.069	1.915E-04
3	8.639	2.526E-05	10.210	5.307E-06	8.639	-2.218E-05	10.210	-4.245E-06
4	11.781	-2.264E-06	13.352	-5.014E-07	11.781	1.134E-06	13.352	2.455E-07
5	14.923	3.441E-07	16.493	7.538E-08	14.923	-1.116E-07	16.493	-2.496E-08
6	18.064	-7.318E-08	19.635	-1.546E-08	18.064	1.663E-08	19.635	3.692E-09
7	21.206	1.968E-08	22.777	3.961E-09	21.206	-3.305E-09	22.777	-7.143E-10
8	24.347	-6.291E-09	25.918	-1.201E-09	24.347	8.122E-10	25.918	1.692E-10
9	27.489	2.297E-09	29.060	4.150E-10	27.489	-2.350E-10	29.060	-4.695E-11
10	30.631	-9.319E-10	32.201	-1.594E-10	30.631	7.734E-11	32.201	1.479E-11

In the case $x \geq 8$ one has $g_n^{(-1)} \leq \Delta_n^{(-1)}(x) \leq G_n^{(-1)}$ with the following values:

n	$g_n^{(-1)}$	$G_n^{(-1)}$	n	$g_n^{(-1)}$	$G_n^{(-1)}$
1	-2.736E-04	4.984E-04	6	-2.264E-06	2.526E-05
2	-4.561E-05	1.668E-04	7	-6.249E-05	5.307E-06
3	-1.301E-04	3.547E-05	8	-2.218E-05	1.134E-06
4	-4.719E-05	7.385E-06	9	-4.245E-06	8.772E-05
5	-1.032E-05	6.871E-05	-	-	-

The following sum gives on the interval $8 \leq x \leq 30$ a better aproximation than the asymptotic formula:

$$F_{-1}(x) = 0.115\ldots + \sum_{k=1}^{9} \frac{1}{x^{k+3/2}}\,\left[\tilde{a}_k^{(-1)}\,\sin\left(3x + \frac{7-2k}{4}\,\pi\right) + \tilde{b}_k^{(-1)}\,\sin\left(x + \frac{1-2k}{4}\,\pi\right)\right] \quad .$$

The values of the coefficients are

k	$\tilde{a}_k^{(-1)}$	$\tilde{b}_k^{(-1)}$
1	0.042328265593	0.380960857219
2	0.051124895140	0.999204230732
3	0.069839136706	3.573795779326
4	0.108172780150	15.423626931601
5	-0.035881146333	86.734624679653
6	-0.183965343668	370.667415902952
7	-39.511920554495	3145.668270195769
8	23.037919369082	5512.467112432406
9	-1996.507320270070	73698.065312438298

With $8 \leq x \leq 30$ holds

$$-2.0\cdot 10^{-9} \;\leq\; F_{-1}(x) - \int_1^x t^{-1}\cdot J_0^3(t)\,dt \;\leq\; 3.2\cdot 10^{-9} \quad .$$

Power series for the modified Bessel function:

$$\int_0^x \frac{I_0^3(t)-1}{t}\,dt = \sum_{k=1}^{\infty} d_k^{(-1)}\,x^{2k} = \frac{3}{8}\,x^2+\frac{15}{256}\,x^4+\frac{31}{4608}\,x^6+\frac{71}{131072}\,x^8+\frac{517}{16384000}\,x^{10}+\frac{11723}{8493465600}\,x^{12}+\ldots$$

From this:

$$\int_1^x I_0^3(t)\,dt = \ln|x| - 0.44089\ 58511\ 01198\ 85318 + \sum_{k=1}^{\infty} d_k^{(-1)}\,x^{2k}$$

With $n \geq 1$ the following recurrence relation holds:

$$d_{n+1}^{(-1)} = \frac{1}{24(n+1)^3}\left[16\sum_{k=1}^{n-1} k(n+1-k)(3n^2+5k^2-8kn+5n-6k+2)d_k^{(-1)}d_{n+1-k}^{(-1)} - \right.$$
$$\left. -36\sum_{k=1}^{n-1} k(n-k)d_k^{(-1)}d_{n-k}^{(-1)} - 36n\,d_n^{(-1)}\right]$$

Asymptotic formula for the modified Bessel function:

$$\int_1^x \frac{I_0^3(t)\,dt}{t} \sim \frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{\infty}\frac{c_k}{x^{k+1}}$$

with the first values

k	c_k	c_k
1	1/12	0.08333 33333 33333 3
2	29/288	0.10069 44444 44444 4
3	1921/13824	0.13896 12268 51851 9
4	8527/36864	0.2313 09678 81944 4
5	1621523/3538944	0.45819 40262 40596 1
6	89993003/84934656	1.05955 57483 62600 1
7	3821763071/1358954496	2.81228 18550 94580 0
8	275582100493/32614907904	8.44957 46946 19993 1
9	177961856737289/6262062317568	28.41904 91426 47965
10	5312592054074687/50096498540544	106.04717 31327 7115

Let

$$\delta_n(x) = \left[\frac{\sqrt{2}\,e^{3x}}{\sqrt{\pi^3}\,x}\sum_{k=1}^{n}\frac{c_k}{x^{k+1}} - \frac{\sqrt{2}\,e^3}{\sqrt{\pi^3}}\sum_{k=1}^{n}c_k\right]\cdot\left[\int_1^x t^{-1}I_0^3(t)\,dt\right]^{-1} - 1\,,$$

then one has the following values of $\delta_n(x)$:

n	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
1	-2.591E-01	-1.236E-01	-8.166E-02	-6.101E-02	-4.870E-02
2	-8.036E-02	-1.768E-02	-7.680E-03	-4.277E-03	-2.722E-03
3	-3.135E-02	-3.069E-03	-8.738E-04	-3.624E-04	-1.838E-04
4	-1.560E-02	-6.368E-04	-1.185E-04	-3.658E-05	-1.478E-05
5	-1.048E-02	-1.549E-04	-1.881E-05	-4.316E-06	-1.388E-06
6	-1.070E-02	-4.350E-05	-3.431E-06	-5.850E-07	-1.498E-07
7	-1.769E-02	-1.394E-05	-7.097E-07	-8.985E-08	-1.831E-08
8	-4.254E-02	-5.103E-06	-1.647E-07	-1.547E-08	-2.507E-09
9	-1.287E-01	-2.292E-06	-4.254E-08	-2.958E-09	-3.813E-10
10	-4.521E-01	-1.839E-06	-1.214E-08	-6.242E-10	-6.393E-11

e) Integrals of the type $\int x^n\, Z_0^3(x)\, dx$:

$$\int x\, J_0^3(x)\, dx = x\, J_0^2(x)\, J_1(x) + \frac{2x}{3}\, J_1^3(x) + \frac{4}{3} \int J_1^3(x)\, dx$$

$$\int x\, I_0^3(x)\, dx = x\, I_0^2(x)\, I_1(x) - \frac{2x}{3}\, I_1^3(x) - \frac{4}{3} \int I_1^3(x)\, dx$$

$$\int x^2\, J_0^3(x)\, dx =$$

$$= -\frac{x}{9}\, J_0^3(x) + x^2\, J_0^2(x)\, J_1(x) - \frac{2x}{3}\, J_0(x)\, J_1^2(x) + \frac{2x^2}{3}\, J_1^3(x) + \frac{1}{9} \int J_0^3(x)\, dx - \frac{2}{3} \int J_0(x)\, J_1^2(x)\, dx$$

$$\int x^2\, I_0^3(x)\, dx =$$

$$= \frac{x}{9}\, I_0^3(x) + x^2\, I_0^2(x)\, I_1(x) - \frac{2x}{3}\, I_0(x)\, I_1^2(x) - \frac{2x^2}{3}\, I_1^3(x) - \frac{1}{9} \int I_0^3(x)\, dx - \frac{2}{3} \int I_0(x)\, I_1^2(x)\, dx$$

$$\int x^3\, J_0^3(x)\, dx = \frac{2x^2}{3}\, J_0^3(x) + \frac{3x^3 - 4x}{3}\, J_0^2(x)\, J_1(x) + \frac{6x^3 - 8x}{9}\, J_1^3(x) - \frac{16}{9} \int J_1^3(x)\, dx$$

$$\int x^3\, I_0^3(x)\, dx = -\frac{2x^2}{3}\, I_0^3(x) + \frac{3x^3 + 4x}{3}\, I_0^2(x)\, I_1(x) - \frac{6x^3 + 8x}{9}\, I_1^3(x) - \frac{16}{9} \int I_1^3(x)\, dx$$

Let

$$\int x^n\, J_0^3(x)\, dx = \mathcal{P}_n(x)\, J_0^3(x) + \mathcal{Q}_n(x)\, J_0^2(x)\, J_1(x) + \mathcal{R}_n(x)\, J_0(x)\, J_1^2(x) + \mathcal{S}_n(x)\, J_1^3(x) +$$

$$+ \mathcal{U}_n \int J_0^3(x)\, dx + \mathcal{V}_n \int J_0(x)\, J_1(x)^2\, dx + \mathcal{W}_n \int J_1^3(x)\, dx + \mathcal{X}_n \int \frac{J_0^3(x)\, dx}{x}$$

and

$$\int x^n\, I_0^3(x)\, dx = \mathcal{P}_n^*(x)\, I_0^3(x) + \mathcal{Q}_n^*(x)\, I_0^2(x)\, I_1(x) + \mathcal{R}_n^*(x)\, I_0(x)\, I_1^2(x) + \mathcal{S}_n^*(x)\, I_1^3(x) +$$

$$+ \mathcal{U}_n^* \int I_0^3(x)\, dx + \mathcal{V}_n^* \int I_0(x)\, I_1(x)^2\, dx + \mathcal{W}_n^* \int I_1^3(x)\, dx + \mathcal{X}_n^* \int \frac{I_0^3(x)\, dx}{x} \,.$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_4(x) = \frac{39\, x^3 + 17\, x}{27}\,, \quad \mathcal{Q}_4(x) = \frac{3\, x^4 - 13\, x^2}{3}\,, \quad \mathcal{R}_4(x) = \frac{6\, x^3 + 28\, x}{9}\,, \quad \mathcal{S}_4(x) = \frac{6\, x^4 - 28\, x^2}{9}\,,$$

$$\mathcal{U}_4 = -\frac{17}{27}\,, \quad \mathcal{V}_4 = \frac{28}{9}\,, \quad \mathcal{W}_4 = 0\,.$$

$$\mathcal{P}_4^*(x) = -\frac{39\, x^3 - 17\, x}{27}\,, \quad \mathcal{Q}_4^*(x) = \frac{3\, x^4 + 13\, x^2}{3}\,, \quad \mathcal{R}_4^*(x) = \frac{6\, x^3 - 28\, x}{9}\,, \quad \mathcal{S}_4^*(x) = -\frac{6\, x^4 + 28\, x^2}{9}\,,$$

$$\mathcal{U}_4^* = -\frac{17}{27}\,, \quad \mathcal{V}_4^* = -\frac{28}{9}\,, \quad \mathcal{W}_0^* = 0$$

$$\mathcal{P}_5(x) = \frac{60\, x^4 - 160\, x^2}{27}\,, \quad \mathcal{Q}_5(x) = \frac{27\, x^5 - 240\, x^3 + 320\, x}{27}\,, \quad \mathcal{R}_5(x) = \frac{4x^4}{3}\,,$$

$$\mathcal{S}_5(x) = \frac{54\, x^5 - 552\, x^3 + 640\, x}{81}\,, \quad \mathcal{U}_5 = 0\,, \quad \mathcal{V}_5 = 0\,, \quad \mathcal{W}_5 = \frac{1280}{81}$$

$$\mathcal{P}_5^*(x) = -\frac{60\, x^4 + 160\, x^2}{27}\,, \quad \mathcal{Q}_5^*(x) = \frac{27\, x^5 + 240\, x^3 + 320\, x}{27}\,, \quad \mathcal{R}_5^*(x) = \frac{4\, x^4}{3}\,,$$

$$\mathcal{S}_5^*(x) = -\frac{54\, x^5 + 552\, x^3 + 640\, x}{81}\,, \quad \mathcal{U}_5^* = 0\,, \quad \mathcal{V}_5^* = 0\,, \quad \mathcal{W}_5^* = -\frac{1280}{81}$$

$$\mathcal{P}_6(x) = \frac{9\, x^5 - 69\, x^3 - 31\, x}{3}\,, \quad \mathcal{Q}_6(x) = x^6 - 15\, x^4 + 69\, x^2\,, \quad \mathcal{R}_6(x) = 2\, x^5 - 12\, x^3 - 50\, x\,,$$

$$\mathcal{S}_6(x) = \frac{2\, x^6 - 36\, x^4 + 150\, x^2}{3}\,, \quad \mathcal{U}_6 = \frac{31}{3}\,, \quad \mathcal{V}_6 = -50\,, \quad \mathcal{W}_6 = 0$$

$$\mathcal{P}_6^*(x) = -\frac{9\,x^5 + 69\,x^3 - 31\,x}{3}\,, \quad \mathcal{Q}_6^*(x) = x^6 + 15\,x^4 + 69\,x^2\,, \quad \mathcal{R}_6^*(x) = 2\,x^5 + 12\,x^3 - 50\,x\,,$$

$$\mathcal{S}_6^*(x) = -\frac{2\,x^6 + 36\,x^4 + 150\,x^2}{3}\,, \quad \mathcal{U}_6^* = -\frac{31}{3}\,, \quad \mathcal{V}_6^* = -50\,, \quad \mathcal{W}_6^* = 0$$

$$\mathcal{P}_7(x) = \frac{102\,x^6 - 1488\,x^4 + 3968\,x^2}{27}\,, \quad \mathcal{Q}_7(x) = \frac{27\,x^7 - 612\,x^5 + 5952\,x^3 - 7936\,x}{27}\,,$$

$$\mathcal{R}_7(x) = \frac{8\,x^6 - 112\,x^4}{3}\,, \quad \mathcal{S}_7(x) = \frac{54\,x^7 - 1512\,x^5 + 13920\,x^3 - 15872\,x}{81}\,,$$

$$\mathcal{U}_7 = 0\,, \quad \mathcal{V}_7 = 0\,, \quad \mathcal{W}_7 = -\frac{31744}{81}$$

$$\mathcal{P}_7^*(x) = -\frac{102\,x^6 + 1488\,x^4 + 3968\,x^2}{27}\,, \quad \mathcal{Q}_7^*(x) = \frac{27\,x^7 + 612\,x^5 + 5952\,x^3 + 7936\,x}{27}\,,$$

$$\mathcal{R}_7^*(x) = \frac{8\,x^6 + 112\,x^4}{3}\,, \quad \mathcal{S}_7^*(x) = -\frac{54\,x^7 + 1512\,x^5 + 13920\,x^3 + 15872\,x}{81}\,,$$

$$\mathcal{U}_7^* = 0\,, \quad \mathcal{V}_7^* = 0\,, \quad \mathcal{W}_7^* = -\frac{31744}{81}\,,$$

$$\mathcal{P}_8(x) = \frac{1107\,x^7 - 25947\,x^5 + 200427\,x^3 + 90373\,x}{243}\,, \quad \mathcal{Q}_8(x) = \frac{27\,x^8 - 861\,x^6 + 14415\,x^4 - 66809\,x^2}{27}\,,$$

$$\mathcal{R}_8(x) = \frac{270\,x^7 - 6516\,x^5 + 35346\,x^3 + 145400\,x}{81}\,, \quad \mathcal{S}_8(x) = \frac{54\,x^8 - 2172\,x^6 + 35346\,x^4 - 145400\,x^2}{81}$$

$$\mathcal{U}_8 = -\frac{90373}{243}\,, \quad \mathcal{V}_8 = \frac{145400}{81}\,, \quad \mathcal{W}_8 = 0$$

$$\mathcal{P}_8^*(x) = -\frac{1107\,x^7 + 25947\,x^5 + 200427\,x^3 - 90373\,x}{243}\,, \quad \mathcal{Q}_8^*(x) = \frac{27\,x^8 + 861\,x^6 + 14415\,x^4 + 66809\,x^2}{27}\,,$$

$$\mathcal{R}_8^*(x) = \frac{270\,x^7 + 6516\,x^5 + 35346\,x^3 - 145400\,x}{81}\,, \quad \mathcal{S}_8^*(x) = -\frac{54\,x^8 + 2172\,x^6 + 35346\,x^4 + 145400\,x^2}{81}$$

$$\mathcal{U}_8^* = -\frac{90373}{243}\,, \quad \mathcal{V}_8^* = -\frac{145400}{81}\,, \quad \mathcal{W}_8^* = 0$$

About recurrence relations for the previous and the following integrals see page 397.

$$\int \frac{J_0^3(x)\,dx}{x^2} = -\frac{J_0^3(x)}{x} + 3J_0^2(x)\,J_1(x) + 6\int J_0(x)\,J_1^2(x)\,dx - 3\int J_0^3(x)\,dx$$

$$\int \frac{I_0^3(x)\,dx}{x^2} = -\frac{I_0^3(x)}{x} - 3I_0^2(x)\,I_1(x) + 6\int I_0(x)\,I_1^2(x)\,dx + 3\int I_0^3(x)\,dx$$

$$\int \frac{J_0^3(x)\,dx}{x^3} = -\frac{x^2+1}{2x^2}\,J_0^3(x) + \frac{3}{4x}\,J_0^2(x)J_1(x) - \frac{3}{4}\,J_0(x)J_1^2(x) - \frac{3}{4}\int \frac{J_0^3(x)\,dx}{x} - \frac{3}{4}\int J_1^3(x)\,dx$$

$$\int \frac{I_0^3(x)\,dx}{x^3} = \frac{x^2-1}{2x^2}\,I_0^3(x) - \frac{3}{4x}\,I_0^2(x)I_1(x) - \frac{3}{4}\,I_0(x)i_1^2(x) + \frac{3}{4}\int \frac{I_0^3(x)\,dx}{x} + \frac{3}{4}\int I_1^3(x)\,dx$$

$$\int \frac{J_0^3(x)\,dx}{x^4} = \frac{x^2-1}{3x^3}\,J_0^3(x) - \frac{13\,x^2-3}{9x^2}\,J_0^2(x)\,J_1(x) - \frac{2}{9x}\,J_0(x)\,J_1^2(x) + \frac{2}{27}\,J_1^3(x) +$$

$$+\frac{13}{9}\int J_0^3(x)\,dx - \frac{28}{9}\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0^3(x)\,dx}{x^4} = -\frac{x^2+1}{3x^3}\,J_0^3(x) - \frac{13\,x^2+3}{9x^2}\,I_0^2(x)\,I_1(x) - \frac{2}{9x}\,I_0(x)\,I_1^2(x) - \frac{2}{27}\,I_1^3(x) +$$

$$+\frac{13}{9}\int I_0^3(x)\,dx + \frac{28}{9}\int I_0(x)\,I_1^2(x)\,dx$$

$$\mathcal{P}_{-5}(x) = \frac{23\,x^4 + 12\,x^2 - 32}{128\,x^4}\,, \quad \mathcal{Q}_{-5}(x) = -\frac{15\,x^2 - 12}{64\,x^3}\,, \quad \mathcal{R}_{-5}(x) = \frac{69\,x^2 - 24}{256\,x^2}\,, \quad \mathcal{S}_{-5}(x) = \frac{3}{128x}$$

$$\mathcal{U}_{-5} = 0\,, \quad \mathcal{V}_{-5} = 0\,, \quad \mathcal{W}_{-5} = \frac{69}{256}\,, \quad \mathcal{X}_{-5} = \frac{15}{64}$$

$$\mathcal{P}^*_{-5}(x) = \frac{23\,x^4 - 12\,x^2 - 32}{128\,x^4}\,, \quad \mathcal{Q}^*_{-5}(x) = -\frac{15\,x^2 + 12}{64\,x^3}\,, \quad \mathcal{R}^*_{-5}(x) = -\frac{69\,x^2 + 24}{256\,x^2}\,, \quad \mathcal{S}^*_{-5}(x) = -\frac{3}{128x}$$

$$\mathcal{U}^*_{-5} = 0\,, \quad \mathcal{V}^*_{-5} = 0\,, \quad \mathcal{W}^*_{-5} = \frac{69}{256}\,, \quad \mathcal{X}^*_{-5} = \frac{15}{64}$$

$$\mathcal{P}_{-6}(x) = -\frac{9\,x^4 - 5\,x^2 + 25}{125\,x^5}\,, \quad \mathcal{Q}_{-6}(x) = \frac{207\,x^4 - 45\,x^2 + 75}{625\,x^4}\,, \quad \mathcal{R}_{-6}(x) = \frac{36\,x^2 - 30}{625\,x^3}\,,$$

$$\mathcal{S}_{-6}(x) = -\frac{12\,x^2 - 6}{625\,x^2}\,, \quad \mathcal{U}_{-6} = -\frac{207}{625}\,, \quad \mathcal{V}_{-6} = \frac{18}{25}\,, \quad \mathcal{W}_{-6} = 0$$

$$\mathcal{P}^*_{-6}(x) = -\frac{9\,x^4 + 5\,x^2 + 25}{125\,x^5}\,, \quad \mathcal{Q}^*_{-6}(x) = -\frac{207\,x^4 + 45\,x^2 + 75}{625\,x^4}\,, \quad \mathcal{R}^*_{-6}(x) = -\frac{36\,x^2 + 30}{625\,x^3}\,,$$

$$\mathcal{S}^*_{-6}(x) = -\frac{12\,x^2 + 6}{625\,x^2}\,, \quad \mathcal{U}^*_{-6} = \frac{207}{625}\,, \quad \mathcal{V}^*_{-6} = \frac{18}{25}\,, \quad \mathcal{W}^*_{-6} = 0$$

$$\mathcal{P}_{-7}(x) = -\frac{145\,x^6 + 68\,x^4 - 96\,x^2 + 768}{4608\,x^6}\,, \quad \mathcal{Q}_{-7}(x) = \frac{93\,x^4 - 68\,x^2 + 192}{2304\,x^5}\,,$$

$$\mathcal{R}_{-7}(x) = -\frac{435\,x^4 - 168\,x^2 + 256}{9216\,x^4}\,, \quad \mathcal{S}_{-7}(x) = -\frac{63\,x^2 - 64}{13824\,x^3}$$

$$\mathcal{U}_{-7} = 0\,, \quad \mathcal{V}_{-7} = 0\,, \quad \mathcal{W}_{-7} = -\frac{145}{3072}\,, \quad \mathcal{X}_{-7} = -\frac{31}{768}$$

$$\mathcal{P}^*_{-7}(x) = \frac{145\,x^6 - 68\,x^4 - 96\,x^2 - 768}{4608\,x^6}\,, \quad \mathcal{Q}^*_{-7}(x) = -\frac{93\,x^4 + 68\,x^2 + 192}{2304\,x^5}\,,$$

$$\mathcal{R}^*_{-7}(x) = -\frac{435\,x^4 + 168\,x^2 + 256}{9216\,x^4}\,, \quad \mathcal{S}^*_{-7}(x) = -\frac{63\,x^2 + 64}{13824\,x^3}$$

$$\mathcal{U}^*_{-7} = 0\,, \quad \mathcal{V}^*_{-7} = 0\,, \quad \mathcal{W}^*_{-7} = \frac{145}{3072}\,, \quad \mathcal{X}^*_{-7} = \frac{31}{768}$$

$$\mathcal{P}_{-8}(x) = \frac{2883\,x^6 - 1435\,x^4 + 3675\,x^2 - 42875}{300125\,x^7}\,, \quad \mathcal{Q}_{-8}(x) = -\frac{66809\,x^6 - 14415\,x^4 + 21525\,x^2 - 91875}{1500625\,x^6}\,,$$

$$\mathcal{R}_{-8}(x) = -\frac{11782\,x^4 - 10860\,x^2 + 26250}{1500625\,x^5}\,, \quad \mathcal{S}_{-8}(x) = \frac{11782\,x^4 - 6516\,x^2 + 11250}{4501875\,x^4}\,,$$

$$\mathcal{U}_{-8} = \frac{66809}{1500625}\,, \quad \mathcal{V}_{-8} = -\frac{5816}{60025}\,, \quad \mathcal{W}_{-8} = 0$$

$$\mathcal{P}^*_{-8}(x) = -\frac{2883\,x^6 + 1435\,x^4 + 3675\,x^2 + 42875}{300125\,x^7}\,, \quad \mathcal{Q}^*_{-8}(x) = -\frac{66809\,x^6 + 14415\,x^4 + 21525\,x^2 + 91875}{1500625\,x^6}\,,$$

$$\mathcal{R}^*_{-8}(x) = -\frac{11782\,x^4 + 10860\,x^2 + 26250}{1500625\,x^5}\,, \quad \mathcal{S}^*_{-8}(x) = -\frac{11782\,x^4 + 6516\,x^2 + 11250}{4501875\,x^4}\,,$$

$$\mathcal{U}^*_{-8} = \frac{66809}{1500625}\,, \quad \mathcal{V}^*_{-8} = \frac{5816}{60025}\,, \quad \mathcal{W}^*_{-8} = 0$$

f) Integrals of the type $\int x^n\, Z_0^2(x)\, Z_1(x)\, dx$:

$$\int J_0^2(x)\, J_1(x)\, dx = -\frac{1}{3}\, J_0^3(x)$$

$$\int I_0^2(x)\, I_1(x)\, dx = \frac{1}{3}\, I_0^3(x)$$

$$\int x\, J_0^2(x)\, J_1(x)\, dx = -\frac{x}{3}\, J_0^3(x) + \frac{1}{3} \int J_0^3(x)\, dx$$

$$\int x\, I_0^2(x)\, I_1(x)\, dx = \frac{x}{3}\, I_0^3(x) - \frac{1}{3} \int I_0^3(x)\, dx$$

$$\int x^2\, J_0^2(x)\, J_1(x)\, dx = -\frac{x^2}{3}\, J_0^3(x) + \frac{2x}{3}\, J_0^2(x)\, J_1(x) + \frac{4x}{9}\, J_1^3(x) + \frac{8}{9} \int J_1^3(x)\, dx$$

$$\int x^2\, I_0^2(x)\, I_1(x)\, dx = \frac{x^2}{3}\, I_0^3(x) - \frac{2x}{3}\, I_0^2(x)\, I_1(x) + \frac{4x}{9}\, I_1^3(x) + \frac{8}{9} \int I_1^3(x)\, dx$$

$$\int x^3\, J_0^2(x)\, J_1(x)\, dx = -\frac{3x^3+x}{9}\, J_0^3(x) + x^2\, J_0^2(x)\, J_1(x) - \frac{2x}{3}\, J_0(x)\, J_1^2(x) + \frac{2x^2}{3}\, J_1^3(x) +$$

$$+\frac{1}{9} \int J_0^3(x)\, dx - \frac{2}{3} \int J_0(x)\, J_1^2(x)\, dx$$

$$\int x^3\, I_0^2(x)\, I_1(x)\, dx = \frac{3x^3-x}{9}\, I_0^3(x) - x^2\, I_0^2(x)\, I_1(x) + \frac{2x}{3}\, I_0(x)\, I_1^2(x) + \frac{2x^2}{3}\, I_1^3(x) +$$

$$+\frac{1}{9} \int I_0^3(x)\, dx + \frac{2}{3} \int I_0(x)\, I_1^2(x)\, dx$$

$$\int x^4\, J_0^2(x)\, J_1(x)\, dx = -\frac{3x^4-8x^2}{9}\, J_0^3(x) + \frac{12x^3-16x}{9}\, J_0^2(x)\, J_1(x) + \frac{24x^3-32x}{27}\, J_1^3(x) - \frac{64}{27} \int J_1^3(x)\, dx$$

$$\int x^4\, I_0^2(x)\, I_1(x)\, dx = \frac{3x^4+8x^2}{9}\, I_0^3(x) - \frac{12x^3+16x}{9}\, I_0^2(x)\, I_1(x) + \frac{24x^3+32x}{27}\, I_1^3(x) + \frac{64}{27} \int I_1^3(x)\, dx$$

Let

$$\int x^n\, J_0^2(x)\, J_1(x)\, dx = \mathcal{P}_n(x)\, J_0^3(x) + \mathcal{Q}_n(x)\, J_0^2(x)\, J_1(x) + \mathcal{R}_n(x)\, J_0(x)\, J_1^2(x) + \mathcal{S}_n(x)\, J_1^3(x) +$$

$$+\mathcal{U}_n \int J_0^3(x)\, dx + \mathcal{V}_n \int J_0(x)\, J_1(x)^2\, dx + \mathcal{W}_n \int J_1(x)^3\, dx + \mathcal{X}_n \int \frac{J_0^3(x)\, dx}{x}$$

and

$$\int x^n\, I_0^2(x)\, I_1(x)\, dx = \mathcal{P}_n^*(x)\, I_0^3(x) + \mathcal{Q}_n^*(x)\, I_0^2(x)\, I_1(x) + \mathcal{R}_n^*(x)\, I_0(x)\, I_1^2(x) + \mathcal{S}_n^*(x)\, I_1^3(x) +$$

$$+\mathcal{U}_n^* \int I_0^3(x)\, dx + \mathcal{V}_n^* \int I_0(x)\, I_1(x)^2\, dx + \mathcal{W}_n^* \int I_1(x)^3\, dx + \mathcal{X}_n^* \int \frac{I_0^3(x)\, dx}{x} .$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_5(x) = -\frac{27\,x^5 - 195\,x^3 - 85\,x}{81}\ , \quad \mathcal{Q}_5(x) = \frac{15\,x^4 - 65\,x^2}{9}\ , \quad \mathcal{R}_5(x) = \frac{30\,x^3 + 140\,x}{27}\ ,$$

$$\mathcal{S}_5(x) = \frac{30\,x^4 - 140\,x^2}{27}\ , \quad \mathcal{U}_5 = -\frac{85}{81}\ , \quad \mathcal{V}_5 = \frac{140}{27}\ , \quad \mathcal{W}_5 = 0$$

$$\mathcal{P}_5^*(x) = \frac{27\,x^5 + 195\,x^3 - 85\,x}{81}\ , \quad \mathcal{Q}_5^*(x) = -\frac{15\,x^4 + 65\,x^2}{9}\ , \quad \mathcal{R}_5^*(x) = -\frac{30\,x^3 - 140\,x}{27}\ ,$$

$$\mathcal{S}_5^*(x) = \frac{30\,x^4 + 140\,x^2}{27}\ , \quad \mathcal{U}_5^* = \frac{85}{81}\ , \quad \mathcal{V}_5^* = \frac{140}{27}\ , \quad \mathcal{W}_5^* = 0$$

$$\mathcal{P}_6(x) = -\frac{9\,x^6 - 120\,x^4 + 320\,x^2}{27} \; , \quad \mathcal{Q}_6(x) = \frac{54\,x^5 - 480\,x^3 + 640\,x}{27} \; , \quad \mathcal{R}_6(x) = \frac{8\,x^4}{3} \; ,$$

$$\mathcal{S}_6(x) = \frac{108\,x^5 - 1104\,x^3 + 1280\,x}{81} \; , \quad \mathcal{U}_6 = 0 \; , \quad \mathcal{V}_6 = 0 \; , \quad \mathcal{W}_6 = \frac{2560}{81}$$

$$\mathcal{P}_6^*(x) = \frac{9\,x^6 + 120\,x^4 + 320\,x^2}{27} \; , \quad \mathcal{Q}_6^*(x) = -\frac{54\,x^5 + 480\,x^3 + 640\,x}{27} \; , \quad \mathcal{R}_6^*(x) = -\frac{8x^4}{3} \; ,$$

$$\mathcal{S}_6^*(x) = \frac{108\,x^5 + 1104\,x^3 + 1280\,x}{81} \; , \quad \mathcal{U}_6^* = 0 \; , \quad \mathcal{V}_6^* = 0 \; , \quad \mathcal{W}_6^* = \frac{2560}{81}$$

$$\mathcal{P}_7(x) = -\frac{3\,x^7 - 63\,x^5 + 483\,x^3 + 217\,x}{9} \; , \quad \mathcal{Q}_7(x) = \frac{7\,x^6 - 105\,x^4 + 483\,x^2}{3} \; ,$$

$$\mathcal{R}_7(x) = \frac{14\,x^5 - 84\,x^3 - 350\,x}{3} \; , \quad \mathcal{S}_7(x) = \frac{14\,x^6 - 252\,x^4 + 1050\,x^2}{9} \; ,$$

$$\mathcal{U}_7 = \frac{217}{9} \; , \quad \mathcal{V}_7 = -\frac{350}{3} \; , \quad \mathcal{W}_7 = 0$$

$$\mathcal{P}_7^*(x) = \frac{3\,x^7 + 63\,x^5 + 483\,x^3 - 217\,x}{9} \; , \quad \mathcal{Q}_7^*(x) = -\frac{7\,x^6 + 105\,x^4 + 483\,x^2}{3} \; ,$$

$$\mathcal{R}_7^*(x) = -\frac{14\,x^5 + 84\,x^3 - 350\,x}{3} \; , \quad \mathcal{S}_7^*(x) = \frac{14\,x^6 + 252\,x^4 + 1050\,x^2}{9} \; ,$$

$$\mathcal{U}_7^* = \frac{217}{9} \; , \quad \mathcal{V}_7^* = \frac{350}{3} \; , \quad \mathcal{W}_7^* = 0$$

$$\mathcal{P}_8(x) = -\frac{27\,x^8 - 816\,x^6 + 11904\,x^4 - 31744\,x^2}{81} \; , \quad \mathcal{Q}_8(x) = \frac{216\,x^7 - 4896\,x^5 + 47616\,x^3 - 63488\,x}{81} \; ,$$

$$\mathcal{R}_8(x) = \frac{64\,x^6 - 896\,x^4}{9} \; , \quad \mathcal{S}_8(x) = \frac{432\,x^7 - 12096\,x^5 + 111360\,x^3 - 126976\,x}{243} \; ,$$

$$\mathcal{U}_8 = 0 \; , \quad \mathcal{V}_8 = 0 \; , \quad \mathcal{W}_8 = -\frac{253952}{243}$$

$$\mathcal{P}_8^*(x) = \frac{27\,x^8 + 816\,x^6 + 11904\,x^4 + 31744\,x^2}{81} \; , \quad \mathcal{Q}_8^*(x) = -\frac{216\,x^7 + 4896\,x^5 + 47616\,x^3 + 63488\,x}{81} \; ,$$

$$\mathcal{R}_8^*(x) = -\frac{64\,x^6 + 896\,x^4}{9} \; , \quad \mathcal{S}_8^*(x) = \frac{432\,x^7 + 12096\,x^5 + 111360\,x^3 + 126976\,x}{243} \; ,$$

$$\mathcal{U}_8^* = 0 \; , \quad \mathcal{V}_8^* = 0 \; , \quad \mathcal{W}_8^* = \frac{253952}{243}$$

About recurrence relations for the previous and the following integrals see page 397.

$$\int \frac{J_0^2(x)\,J_1(x)\,dx}{x} = -J_0^2(x)\,J_1(x) + \int J_0^3(x)\,dx - 2\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0^2(x)\,I_1(x)\,dx}{x} = -I_0^2(x)\,I_1(x) + \int I_0^3(x)\,dx + 2\int I_0(x)\,I_1^2(x)\,dx$$

$$\int \frac{J_0^2(x)J_1(x)\,dx}{x^2} = \frac{1}{3}\,J_0^3(x) - \frac{1}{2x}\,J_0^2(x)J_1(x) + \frac{1}{2}\,J_0(x)J_1^2(x) + \frac{1}{2}\int J_1^3(x)\,dx + \frac{1}{2}\int \frac{J_0^3(x)\,dx}{x}$$

$$\int \frac{I_0^2(x)I_1(x)\,dx}{x^2} = \frac{1}{3}\,I_0^3(x) - \frac{1}{2x}\,I_0^2(x)I_1(x) - \frac{1}{2}\,I_0(x)I_1^2(x) + \frac{1}{2}\int I_1^3(x)\,dx + \frac{1}{2}\int \frac{I_0^3(x)\,dx}{x}$$

$$\int \frac{J_0^2(x)\,J_1(x)\,dx}{x^3} = -\frac{1}{3x}\,J_0^3(x) + \frac{13x^2-3}{9x^2}\,J_0^2(x)\,J_1(x) + \frac{2}{9x}\,J_0(x)\,J_1^2(x) - \frac{2}{27}\,J_1^3(x) -$$
$$-\frac{13}{9}\int J_0^3(x)\,dx + \frac{28}{9}\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0^2(x)\,I_1(x)\,dx}{x^3} = -\frac{1}{3x}\,I_0^3(x) - \frac{13x^2+3}{9x^2}\,I_0^2(x)\,I_1(x) - \frac{2}{9x}\,I_0(x)\,I_1^2(x) - \frac{2}{27}\,I_1^3(x) +$$

$$+\frac{13}{9}\int I_0^3(x)\,dx+\frac{28}{9}\int I_0(x)\,I_1^2(x)\,dx$$

$$\mathcal{P}_{-4}(x)=-\frac{23\,x^2+12}{96\,x^2}\,,\quad \mathcal{Q}_{-4}(x)=\frac{5\,x^2-4}{16\,x^3}\,,\quad \mathcal{R}_{-4}(x)=-\frac{23\,x^2-8}{64\,x^2}\,,\quad \mathcal{S}_{-4}(x)=-\frac{1}{32x}\,,$$

$$\mathcal{U}_{-4}=0\,,\quad \mathcal{V}_{-4}=0\,,\quad \mathcal{W}_{-4}=-\frac{23}{64}\,,\quad \mathcal{X}_{-4}=-\frac{5}{16}$$

$$\mathcal{P}^*_{-4}(x)=\frac{23\,x^2-12}{96\,x^2}\,,\quad \mathcal{Q}^*_{-4}(x)=-\frac{5\,x^2+4}{16\,x^3}\,,\quad \mathcal{R}^*_{-4}(x)=-\frac{23\,x^2+8}{64\,x^2}\,,\quad \mathcal{S}^*_{-4}(x)=-\frac{1}{32x}\,,$$

$$\mathcal{U}^*_{-4}=0\,,\quad \mathcal{V}^*_{-4}=0\,,\quad \mathcal{W}^*_{-4}=\frac{23}{64}\,,\quad \mathcal{X}^*_{-4}=\frac{5}{16}$$

$$\mathcal{P}_{-5}(x)=\frac{9\,x^2-5}{75\,x^3}\,,\quad \mathcal{Q}_{-5}(x)=-\frac{69\,x^4-15\,x^2+25}{125\,x^4}\,,\quad \mathcal{R}_{-5}(x)=-\frac{12\,x^2-10}{125\,x^3}\,,$$

$$\mathcal{S}_{-5}(x)=\frac{4\,x^2-2}{125\,x^2}\,,\quad \mathcal{U}_{-5}=\frac{69}{125}\,,\quad \mathcal{V}_{-5}=-\frac{6}{5}\,,\quad \mathcal{W}_{-5}=0$$

$$\mathcal{P}^*_{-5}(x)=-\frac{9\,x^2+5}{75\,x^3}\,,\quad \mathcal{Q}^*_{-5}(x)=-\frac{69\,x^4+15\,x^2+25}{125\,x^4}\,,\quad \mathcal{R}^*_{-5}(x)=-\frac{12\,x^2+10}{125\,x^3}\,,$$

$$\mathcal{S}^*_{-5}(x)=-\frac{4\,x^2+2}{125\,x^2}\,,\quad \mathcal{U}^*_{-5}=\frac{69}{125}\,,\quad \mathcal{V}^*_{-5}=\frac{6}{5}\,,\quad \mathcal{W}^*_{-5}=0$$

$$\mathcal{P}_{-6}(x)=\frac{145\,x^4+68\,x^2-96}{2304\,x^4}\,,\quad \mathcal{Q}_{-6}(x)=-\frac{93\,x^4-68\,x^2+192}{1152\,x^5}\,,\quad \mathcal{R}_{-6}(x)=\frac{435\,x^4-168\,x^2+256}{4608\,x^4}\,,$$

$$\mathcal{S}_{-6}(x)=\frac{63\,x^2-64}{6912\,x^3}\,,\quad \mathcal{U}_{-6}=0\,,\quad \mathcal{V}_{-6}=0\,,\quad \mathcal{W}_{-6}=\frac{145}{1536}\,,\quad \mathcal{X}_{-6}=\frac{31}{384}$$

$$\mathcal{P}^*_{-6}(x)=\frac{145\,x^4-68\,x^2-96}{2304\,x^4}\,,\quad \mathcal{Q}^*_{-6}(x)=-\frac{93\,x^4+68\,x^2+192}{1152\,x^5}\,,\quad \mathcal{R}^*_{-6}(x)=-\frac{435\,x^4+168\,x^2+256}{4608\,x^4}\,,$$

$$\mathcal{S}^*_{-6}(x)=-\frac{63\,x^2+64}{6912\,x^3}\,,\quad \mathcal{U}^*_{-6}=0\,,\quad \mathcal{V}^*_{-6}=0\,,\quad \mathcal{W}^*_{-6}=\frac{145}{1536}\,,\quad \mathcal{X}^*_{-6}=\frac{31}{384}$$

$$\mathcal{P}_{-7}(x)=-\frac{2883\,x^4-1435\,x^2+3675}{128625\,x^5}\,,\quad \mathcal{Q}_{-7}(x)=\frac{66809\,x^6-14415\,x^4+21525\,x^2-91875}{643125\,x^6}\,,$$

$$\mathcal{R}_{-7}(x)=\frac{11782\,x^4-10860\,x^2+26250}{643125\,x^5}\,,\quad \mathcal{S}_{-7}(x)=-\frac{11782\,x^4-6516\,x^2+11250}{1929375\,x^4}\,,$$

$$\mathcal{U}_{-7}=-\frac{66809}{643125}\,,\quad \mathcal{V}_{-7}=\frac{5816}{25725}\,,\quad \mathcal{W}_{-7}=0$$

$$\mathcal{P}^*_{-7}(x)=-\frac{2883\,x^4+1435\,x^2+3675}{128625\,x^5}\,,\quad \mathcal{Q}^*_{-7}(x)=-\frac{66809\,x^6+14415\,x^4+21525\,x^2+91875}{643125\,x^6}\,,$$

$$\mathcal{R}^*_{-7}(x)=-\frac{11782\,x^4+10860\,x^2+26250}{643125\,x^5}\,,\quad \mathcal{S}^*_{-7}(x)=-\frac{11782\,x^4+6516\,x^2+11250}{1929375\,x^4}\,,$$

$$\mathcal{U}^*_{-7}=\frac{66809}{643125}\,,\quad \mathcal{V}^*_{-7}=\frac{5816}{25725}\,,\quad \mathcal{W}^*_{-7}=0$$

$$\mathcal{P}_{-8}(x)=-\frac{1331\,x^6+616\,x^4-768\,x^2+3072}{147456\,x^6}\,,\quad \mathcal{Q}_{-8}(x)=\frac{213\,x^6-154\,x^4+384\,x^2-2304}{18432\,x^7}\,,$$

$$\mathcal{R}_{-8}(x)=-\frac{3993\,x^6-1560\,x^4+2624\,x^2-9216}{294912\,x^6}\,,\quad \mathcal{S}_{-8}(x)=-\frac{585\,x^4-656\,x^2+1728}{442368\,x^5}\,,$$

$$\mathcal{U}_{-8}=0\,,\quad \mathcal{V}_{-8}=0\,,\quad \mathcal{W}_{-8}=-\frac{1331}{98304}\,,\quad \mathcal{X}_{-8}=-\frac{71}{6144}$$

$$\mathcal{P}^*_{-8}(x)=\frac{1331\,x^6-616\,x^4-768\,x^2-3072}{147456\,x^6}\,,\quad \mathcal{Q}^*_{-8}(x)=-\frac{213\,x^6+154\,x^4+384\,x^2+2304}{18432\,x^7}\,,$$

$$\mathcal{R}^*_{-8}(x)=-\frac{3993\,x^6+1560\,x^4+2624\,x^2+9216}{294912\,x^6}\,,\quad \mathcal{S}^*_{-8}(x)=-\frac{585\,x^4+656\,x^2+1728}{442368\,x^5}\,,$$

$$\mathcal{U}^*_{-8}=0\,,\quad \mathcal{V}^*_{-8}=0\,,\quad \mathcal{W}^*_{-8}=\frac{1331}{98304}\,,\quad \mathcal{X}^*_{-8}=\frac{71}{6144}$$

g) Integrals of the type $\int x^n\, Z_0(x)\, Z_1^2(x)\, dx$:

$$\int x\, J_0(x)\, J_1^2(x)\, dx = \frac{x}{3}\, J_1^3(x) + \frac{2}{3} \int J_1^3(x)\, dx$$

$$\int x\, I_0(x)\, I_1^2(x)\, dx = \frac{x}{3}\, I_1^3(x) + \frac{2}{3} \int I_1^3(x)\, dx$$

$$\int x^2\, J_0(x)\, J_1^2(x)\, dx = -\frac{2x}{9}\, J_0^3(x) - \frac{x}{3}\, J_0(x)\, J_1^2(x) + \frac{x^2}{3}\, J_0^3(x) + \frac{2}{9} \int J_0^3(x)\, dx - \frac{1}{3} \int J_0(x)\, J_1^2(x)\, dx$$

$$\int x^2\, I_0(x)\, I_1^2(x)\, dx = -\frac{2x}{9}\, I_0^3(x) + \frac{x}{3}\, I_0(x)\, I_1^2(x) + \frac{x^2}{3}\, I_0^3(x) + \frac{2}{9} \int I_0^3(x)\, dx + \frac{1}{3} \int I_0(x)\, I_1^2(x)\, dx$$

$$\int x^3\, J_0(x)\, J_1^2(x)\, dx = \frac{x^3}{3}\, J_1^3(x)$$

$$\int x^3\, I_0(x)\, I_1^2(x)\, dx = \frac{x^3}{3}\, I_1^3(x)$$

$$\int x^4\, J_0(x)\, J_1^2(x)\, dx = \frac{6\,x^3 + 4\,x}{27}\, J_0^3(x) - \frac{2x^2}{3}\, J_0^2(x)\, J_1(x) + \frac{3\,x^3 + 5\,x}{9}\, J_0(x)\, J_1^2(x) + \frac{3\,x^4 - 5\,x^2}{9}\, J_1^3(x) -$$
$$-\frac{4}{27} \int J_0^3(x)\, dx + \frac{5}{9} \int J_0(x)\, J_1^2(x)\, dx$$

$$\int x^4\, I_0(x)\, I_1^2(x)\, dx = \frac{6\,x^3 - 4\,x}{27}\, I_0^3(x) - \frac{2x^2}{3}\, I_0^2(x)\, I_1(x) - \frac{3\,x^3 - 5\,x}{9}\, I_0(x)\, I_1^2(x) + \frac{3\,x^4 + 5\,x^2}{9}\, I_1^3(x) +$$
$$+\frac{4}{27} \int I_0^3(x)\, dx + \frac{5}{9} \int I_0(x)\, I_1^2(x)\, dx$$

Let

$$\int x^n\, J_0(x)\, J_1^2(x)\, dx = \mathcal{P}_n(x)\, J_0^3(x) + \mathcal{Q}_n(x)\, J_0^2(x)\, J_1(x) + \mathcal{R}_n(x)\, J_0(x)\, J_1^2(x) + \mathcal{S}_n(x)\, J_1^3(x) +$$
$$+\mathcal{U}_n \int J_0^3(x)\, dx + \mathcal{V}_n \int J_0(x)\, J_1(x)^2\, dx + \mathcal{W}_n \int J_1(x)^3\, dx + \mathcal{X}_n \int \frac{J_0^3(x)\, dx}{x}$$

and

$$\int x^n\, I_0(x)\, I_1^2(x)\, dx = \mathcal{P}_n^*(x)\, I_0^3(x) + \mathcal{Q}_n^*(x)\, I_0^2(x)\, I_1(x) + \mathcal{R}_n^*(x)\, I_0(x)\, I_1^2(x) + \mathcal{S}_n^*(x)\, I_1^3(x) +$$
$$+\mathcal{U}_n^* \int I_0^3(x)\, dx + \mathcal{V}_n^* \int I_0(x)\, I_1(x)^2\, dx + \mathcal{W}_n^* \int I_1(x)^3\, dx + \mathcal{X}_n^* \int \frac{I_0^3(x)\, dx}{x}$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_5(x) = \frac{12\,x^4 - 32\,x^2}{27}\,, \quad \mathcal{Q}_5(x) = -\frac{48\,x^3 - 64\,x}{27}\,, \quad \mathcal{R}_5(x) = \frac{2\,x^4}{3}\,, \quad \mathcal{S}_5(x) = \frac{27\,x^5 - 132\,x^3 + 128\,x}{81}\,,$$
$$\mathcal{U}_5 = 0\,, \quad \mathcal{V}_5 = 0\,, \quad \mathcal{W}_5 = \frac{256}{81}$$

$$\mathcal{P}_5^*(x) = \frac{12\,x^4 + 32\,x^2}{27}\,, \quad \mathcal{Q}_5^*(x) = -\frac{48\,x^3 + 64\,x}{27}\,, \quad \mathcal{R}_5^*(x) = -\frac{2\,x^4}{3}\,, \quad \mathcal{S}_5^*(x) = \frac{27\,x^5 + 132\,x^3 + 128\,x}{81}\,,$$
$$\mathcal{U}_5^* = 0\,, \quad \mathcal{V}_5^* = 0\,, \quad \mathcal{W}_5^* = \frac{256}{81}$$

$$\mathcal{P}_6(x) = \frac{54\,x^5 - 444\,x^3 - 206\,x}{81}\,, \quad \mathcal{Q}_6(x) = -\frac{30\,x^4 - 148\,x^2}{9}\,, \quad \mathcal{R}_6(x) = \frac{27\,x^5 - 87\,x^3 - 325\,x}{27}\,,$$
$$\mathcal{S}_6(x) = \frac{9\,x^6 - 87\,x^4 + 325\,x^2}{27}\,, \quad \mathcal{U}_6 = \frac{206}{81}\,, \quad \mathcal{V}_6 = -\frac{325}{27}\,, \quad \mathcal{W}_6 = 0$$

$$\mathcal{P}_6^*(x) = \frac{54\,x^5 + 444\,x^3 - 206\,x}{81}\,, \quad \mathcal{Q}_6^*(x) = -\frac{30\,x^4 + 148\,x^2}{9}\,, \quad \mathcal{R}_6^*(x) = -\frac{27\,x^5 + 87\,x^3 - 325\,x}{27}\,,$$

$$\mathcal{S}_6^*(x) = \frac{9\,x^6 + 87\,x^4 + 325\,x^2}{27}\,, \quad \mathcal{U}_6^* = \frac{206}{81}\,, \quad \mathcal{V}_6^* = \frac{325}{27}\,, \quad \mathcal{W}_6^* = 0$$

$$\mathcal{P}_7(x) = \frac{24\,x^6 - 384\,x^4 + 1024\,x^2}{27}\,, \quad \mathcal{Q}_7(x) = -\frac{144\,x^5 - 1536\,x^3 + 2048\,x}{27}\,, \quad \mathcal{R}_7(x) = \frac{4\,x^6 - 32\,x^4}{3}\,,$$

$$\mathcal{S}_7(x) = \frac{27\,x^7 - 432\,x^5 + 3648\,x^3 - 4096\,x}{81}\,, \quad \mathcal{U}_7 = 0\,, \quad \mathcal{V}_7 = 0\,, \quad \mathcal{W}_7 = -\frac{8192}{81}$$

$$\mathcal{P}_7^*(x) = \frac{24\,x^6 + 384\,x^4 + 1024\,x^2}{27}\,, \quad \mathcal{Q}_7^*(x) = -\frac{144\,x^5 + 1536\,x^3 + 2048\,x}{27}\,, \quad \mathcal{R}_7^*(x) = -\frac{4\,x^6 + 32\,x^4}{3}\,,$$

$$\mathcal{S}_7^*(x) = \frac{27\,x^7 + 432\,x^5 + 3648\,x^3 + 4096\,x}{81}\,, \quad \mathcal{U}_7^* = 0\,, \quad \mathcal{V}_7^* = 0\,, \quad \mathcal{W}_7^* = \frac{8192}{81}$$

$$\mathcal{P}_8(x) = \frac{270\,x^7 - 7020\,x^5 + 54570\,x^3 + 24680\,x}{243}\,, \quad \mathcal{Q}_8(x) = -\frac{210\,x^6 - 3900\,x^4 + 18190\,x^2}{27}\,,$$

$$\mathcal{R}_8(x) = \frac{135\,x^7 - 1935\,x^5 + 9735\,x^3 + 39625\,x}{81}\,, \quad \mathcal{S}_8(x) = \frac{27\,x^8 - 645\,x^6 + 9735\,x^4 - 39625\,x^2}{81}\,,$$

$$\mathcal{U}_8 = -\frac{24680}{243}\,, \quad \mathcal{V}_8 = \frac{39625}{81}\,, \quad \mathcal{W}_8 = 0$$

$$\mathcal{P}_8^*(x) = \frac{270\,x^7 + 7020\,x^5 + 54570\,x^3 - 24680\,x}{243}\,, \quad \mathcal{Q}_8^*(x) = -\frac{210\,x^6 + 3900\,x^4 + 18190\,x^2}{27}\,,$$

$$\mathcal{R}_8^*(x) = -\frac{135\,x^7 + 1935\,x^5 + 9735\,x^3 - 39625\,x}{81}\,, \quad \mathcal{S}_8^*(x) = \frac{27\,x^8 + 645\,x^6 + 9735\,x^4 + 39625\,x^2}{81}\,,$$

$$\mathcal{U}_8^* = \frac{24680}{243}\,, \quad \mathcal{V}_n^* = \frac{39625}{81}\,, \quad \mathcal{W}_8^* = 0$$

About recurrence relations for the previous and the following integrals see page 397.

$$\int \frac{J_0(x)\,J_1^2(x)\,dx}{x} = -\frac{1}{2}\,J_0(x)\,J_1^2(x) - \frac{1}{3}J_0^3(x) - \frac{1}{2}\int J_1^3(x)\,dx$$

$$\int \frac{I_0(x)\,I_1^2(x)\,dx}{x} = -\frac{1}{2}\,I_0(x)\,I_1^2(x) + \frac{1}{3}I_0^3(x) + \frac{1}{2}\int I_1^3(x)\,dx$$

$$\int \frac{J_0(x)\,J_1^2(x)\,dx}{x^2} = -\frac{2}{3}\,J_0^2(x)\,J_1(x) - \frac{1}{3x}\,J_0(x)\,J_1^2(x) + \frac{1}{9}\,J_1^3(x) - \frac{5}{3}\int J_0(x)\,J_1^2(x)\,dx + \frac{2}{3}\int J_0^3(x)\,dx$$

$$\int \frac{I_0(x)\,I_1^2(x)\,dx}{x^2} = -\frac{2}{3}\,I_0^2(x)\,I_1(x) - \frac{1}{3x}\,I_0(x)\,I_1^2(x) - \frac{1}{9}\,I_1^3(x) + \frac{5}{3}\int I_0(x)\,I_1^2(x)\,dx + \frac{2}{3}\int I_0^3(x)\,dx$$

$$\int \frac{J_0(x)J_1^2(x)\,dx}{x^3} =$$

$$= \frac{11}{48}\,J_0^3(x) - \frac{1}{4x}\,J_0^2(x)J_1(x) + \frac{11\,x^2 - 8}{32\,x^2}\,J_0(x)J_1^2(x) + \frac{1}{16\,x}\,J_1^3(x) + \frac{11}{32}\int J_1^3(x)\,dx + \frac{1}{4}\int \frac{J_0^3(x)\,dx}{x}$$

$$\int \frac{I_0(x)I_1^2(x)\,dx}{x^3} =$$

$$= \frac{11}{48}\,I_0^3(x) - \frac{1}{4x}\,I_0^2(x)I_1(x) - \frac{11\,x^2 + 8}{32x^2}\,I_0(x)I_1^2(x) - \frac{1}{16\,x}\,I_1^3(x) + \frac{11}{32}\int I_3^3(x)\,dx + \frac{1}{4}\int \frac{I_0^3(x)\,dx}{x}$$

$$\int \frac{J_0(x)\,J_1^2(x)\,dx}{x^4} = -\frac{2}{15x}\,J_0^3(x) + \frac{148\,x^2 - 30}{225\,x^2}\,J_0^2(x)\,J_1(x) + \frac{29\,x^2 - 45}{225\,x^3}\,J_0(x)\,J_1^2(x) - \frac{29\,x^2 - 27}{675\,x^2}\,J_1^3(x) -$$

$$-\frac{148}{225}\int J_0^3(x)\,dx + \frac{13}{9}\int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_0(x)\,I_1^2(x)\,dx}{x^4} = -\frac{2}{15x}\,I_0^3(x) - \frac{148\,x^2 + 30}{225\,x^2}\,I_0^2(x)\,I_1(x) - \frac{29\,x^2 + 45}{225\,x^3}\,I_0(x)\,I_1^2(x) - \frac{29\,x^2 + 27}{675\,x^2}\,I_1^3(x) +$$

$$+\frac{148}{225}\int I_0^3(x)\,dx+\frac{13}{9}\int I_0(x)\,I_1^2(x)\,dx$$

$$\mathcal{P}_{-5}(x)=-\frac{19\,x^2+8}{192\,x^2}\,,\quad \mathcal{Q}_{-5}(x)=\frac{3\,x^2-2}{24\,x^3}\,,\quad \mathcal{R}_{-5}(x)=-\frac{57\,x^4-24\,x^2+64}{384\,x^4}\,,\quad \mathcal{S}_{-5}(x)=-\frac{9\,x^2-16}{576\,x^3}\,,$$

$$\mathcal{U}_{-5}=0\,,\quad \mathcal{V}_{-5}=0\,,\quad \mathcal{W}_{-5}=-\frac{19}{128}\,,\quad \mathcal{X}_{-5}=-\frac{1}{8}$$

$$\mathcal{P}^*_{-5}(x)=\frac{19\,x^2-8}{192\,x^2}\,,\quad \mathcal{Q}^*_{-5}(x)=-\frac{3\,x^2+2}{24\,x^3}\,,\quad \mathcal{R}^*_{-5}(x)=-\frac{57\,x^4+24\,x^2+64}{384\,x^4}\,,\quad \mathcal{S}^*_{-5}(x)=-\frac{9\,x^2+16}{576\,x^3}\,,$$

$$\mathcal{U}^*_{-5}=0\,,\quad \mathcal{V}^*_{-5}=0\,,\quad \mathcal{W}^*_{-5}=\frac{19}{128}\,,\quad \mathcal{X}^*_{-5}=\frac{1}{8}$$

$$\mathcal{P}_{-6}(x)=\frac{156\,x^2-70}{3675\,x^3}\,,\quad \mathcal{Q}_{-6}(x)=-\frac{3638\,x^4-780\,x^2+1050}{18375\,x^4}\,,\quad \mathcal{R}_{-6}(x)=-\frac{649\,x^4-645\,x^2+2625}{18375\,x^5}\,,$$

$$\mathcal{S}_{-6}(x)=\frac{649\,x^4-387\,x^2+1125}{55125\,x^4}\,,\quad \mathcal{U}_{-6}=\frac{3638}{18375}\,,\quad \mathcal{V}_{-6}=-\frac{317}{735}\,,\quad \mathcal{W}_{-6}=0$$

$$\mathcal{P}^*_{-6}(x)=-\frac{156\,x^2+70}{3675\,x^3}\,,\quad \mathcal{Q}^*_{-6}(x)=-\frac{3638\,x^4+780\,x^2+1050}{18375\,x^4}\,,\quad \mathcal{R}^*_{-6}(x)=-\frac{649\,x^4+645\,x^2+2625}{18375\,x^5}\,,$$

$$\mathcal{S}^*_{-6}(x)=-\frac{649\,x^4+387\,x^2+1125}{55125\,x^4}\,,\quad \mathcal{U}^*_{-6}=\frac{3638}{18375}\,,\quad \mathcal{V}^*_{-6}=\frac{317}{735}\,,\quad \mathcal{W}^*_{-6}=0$$

$$\mathcal{P}_{-7}(x)=\frac{751\,x^4+344\,x^2-384}{36864\,x^4}\,,\quad \mathcal{Q}_{-7}(x)=-\frac{60\,x^4-43\,x^2+96}{2304\,x^5}\,,$$

$$\mathcal{R}_{-7}(x)=\frac{2253\,x^6-888\,x^4+1600\,x^2-9216}{73728\,x^6}\,,\quad \mathcal{S}_{-7}(x)=\frac{333\,x^4-400\,x^2+1728}{110592\,x^5}\,,$$

$$\mathcal{U}_{-7}=0\,,\quad \mathcal{V}_{-7}=0\,,\quad \mathcal{W}_{-7}=\frac{751}{24576}\,,\quad \mathcal{X}_{-7}=\frac{5}{192}$$

$$\mathcal{P}^*_{-7}(x)=\frac{751\,x^4-344\,x^2-384}{36864\,x^4}\,,\quad \mathcal{Q}^*_{-7}(x)=-\frac{60\,x^4+43\,x^2+96}{2304\,x^5}\,,$$

$$\mathcal{R}^*_{-7}(x)=-\frac{2253\,x^6+888\,x^4+1600\,x^2+9216}{73728\,x^6}\,,\quad \mathcal{S}^*_{-7}(x)=-\frac{333\,x^4+400\,x^2+1728}{110592\,x^5}\,,$$

$$\mathcal{U}^*_{-7}=0\,,\quad \mathcal{V}^*_{-7}=0\,,\quad \mathcal{W}^*_{-7}=\frac{751}{24576}\,,\quad \mathcal{X}^*_{-7}=\frac{5}{192}$$

$$\mathcal{P}_{-8}(x)=-\frac{22758\,x^4-11060\,x^2+22050}{3472875\,x^5}\,,\quad \mathcal{Q}_{-8}(x)=\frac{528184\,x^6-113790\,x^4+165900\,x^2-551250}{17364375\,x^6}\,,$$

$$\mathcal{R}_{-8}(x)=\frac{93407\,x^6-87735\,x^4+249375\,x^2-1929375}{17364375\,x^7}\,,$$

$$\mathcal{S}_{-8}(x)=-\frac{93407\,x^6-52641\,x^4+106875\,x^2-643125}{52093125\,x^6}\,,$$

$$\mathcal{U}_{-8}=-\frac{528184}{17364375}\,,\quad \mathcal{V}_{-8}=\frac{45991}{694575}\,,\quad \mathcal{W}_{-8}=0$$

$$\mathcal{P}^*_{-8}(x)=-\frac{22758\,x^4+11060\,x^2+22050}{3472875\,x^5}\,,\quad \mathcal{Q}^*_{-8}(x)=-\frac{528184\,x^6+113790\,x^4+165900\,x^2+551250}{17364375\,x^6}\,,$$

$$\mathcal{R}^*_{-8}(x)=-\frac{93407\,x^6+87735\,x^4+249375\,x^2+1929375}{17364375\,x^7}\,,$$

$$\mathcal{S}^*_{-8}(x)=-\frac{93407\,x^6+52641\,x^4+106875\,x^2+643125}{52093125\,x^6}\,,$$

$$\mathcal{U}^*_{-8}=\frac{528184}{17364375}\,,\quad \mathcal{V}^*_{-8}=\frac{45991}{694575}\,,\quad \mathcal{W}^*_{-8}=0$$

h) Integrals of the type $\int x^n\, Z_1^3(x)\,dx$:

$$\int x\, J_1^3(x)\,dx = -\frac{2x}{3}\, J_0^3(x) - x\, J_0(x)\, J_1^2(x) + \frac{2}{3}\int J_0^3(x)\,dx - \int J_0(x)\, J_1^2(x)\,dx$$

$$\int x\, I_1^3(x)\,dx = -\frac{2x}{3}\, I_0^3(x) + x\, I_0(x)\, I_1^2(x) + \frac{2}{3}\int I_0^3(x)\,dx + \int I_0(x)\, I_1^2(x)\,dx$$

$$\int x^2\, J_1^3(x)\,dx = -\frac{2x^2}{3}\, J_0^3(x) + \frac{4x}{3}\, J_0^2\, J_1(x) - x^2\, J_0(x)\, J_1^2(x) + \frac{8x}{9}\, J_1^3(x) + \frac{16}{9}\int J_1^3(x)\,dx$$

$$\int x^2\, I_1^3(x)\,dx = -\frac{2x^2}{3}\, I_0^3(x) + \frac{4x}{3}\, I_0^2\, I_1(x) + x^2\, I_0(x)\, I_1^2(x) - \frac{8x}{9}\, I_1^3(x) - \frac{16}{9}\int I_1^3(x)\,dx$$

$$\int x^3\, J_1^3(x)\,dx = -\frac{6\,x^3+4\,x}{9}\, J_0^3(x) + 2x^2\, J_0^2\, J_1(x) - \frac{3\,x^3+5\,x}{3}\, J_0(x)\, J_1^2(x) + \frac{5x^2}{3}\, J_1^3(x) +$$

$$+\frac{4}{9}\int J_0^3(x)\,dx - \frac{5}{3}\int J_0(x)\, J_1^2(x)\,dx$$

$$\int x^3\, I_1^3(x)\,dx = \frac{-6\,x^3+4\,x}{9}\, I_0^3(x) + 2x^2\, I_0^2\, I_1(x) + \frac{3\,x^3-5\,x}{3}\, I_0(x)\, I_1^2(x) - \frac{5x^2}{3}\, I_1^3(x) -$$

$$-\frac{4}{9}\int I_0^3(x)\,dx - \frac{5}{3}\int I_0(x)\, I_1^2(x)\,dx$$

$$\int x^4\, J_1^3(x)\,dx =$$

$$= \frac{-6\,x^4+16\,x^2}{9}\, J_0^3(x) + \frac{24\,x^3-32\,x}{9}\, J_0^2\, J_1(x) - x^4\, J_0(x)\, J_1^2(x) + \frac{66\,x^3-64\,x}{27}\, J_1^3(x) - \frac{128}{27}\int J_1^3(x)\,dx$$

$$\int x^4\, I_1^3(x)\,dx =$$

$$= -\frac{6\,x^4+16\,x^2}{9}\, I_0^3(x) + \frac{24\,x^3+32\,x}{9}\, I_0^2\, I_1(x) + x^4\, I_0(x)\, I_1^2(x) - \frac{66\,x^3+64\,x}{27}\, I_1^3(x) - \frac{128}{27}\int I_1^3(x)\,dx$$

Let

$$\int x^n\, J_1^3(x)\,dx = \mathcal{P}_n(x)\, J_0^3(x) + \mathcal{Q}_n(x)\, J_0^2(x)\, J_1(x) + \mathcal{R}_n(x)\, J_0(x)\, J_1^2(x) + \mathcal{S}_n(x)\, J_1^3(x) +$$

$$+\mathcal{U}_n\int J_0^3(x)\,dx + \mathcal{V}_n\int J_0(x)\, J_1(x)^2\,dx + \mathcal{W}_n\int J_1^3(x)\,dx + \mathcal{X}_n\int \frac{I_0^3(x)\,dx}{x}$$

and

$$\int x^n\, I_1^3(x)\,dx = \mathcal{P}_n^*(x)\, I_0^3(x) + \mathcal{Q}_n^*(x)\, I_0^2(x)\, I_1(x) + \mathcal{R}_n^*(x)\, I_0(x)\, I_1^2(x) + \mathcal{S}_n^*(x)\, I_1^3(x) +$$

$$+\mathcal{U}_n^*\int I_0^3(x)\,dx + \mathcal{V}_n^*\int I_0(x)\, I_1(x)^2\,dx + \mathcal{W}_n^*\int I_1^3(x)\,dx + \mathcal{X}_n^*\int \frac{I_0^3(x)\,dx}{x}$$

If $\mathcal{X}_n = 0$ or $\mathcal{X}_n^* = 0$, then they are omitted from the following table.

$$\mathcal{P}_5(x) = -\frac{54\,x^5-444\,x^3-206\,x}{81}\,, \quad \mathcal{Q}_5(x) = \frac{30\,x^4-148\,x^2}{9}\,, \quad \mathcal{R}_5(x) = -\frac{27\,x^5-87\,x^3-325\,x}{27}\,,$$

$$\mathcal{S}_5(x) = \frac{87\,x^4-325\,x^2}{27}\,, \quad \mathcal{U}_5 = -\frac{206}{81}\,, \quad \mathcal{V}_5 = \frac{325}{27}\,, \quad \mathcal{W}_5 = 0$$

$$\mathcal{P}_5^*(x) = -\frac{54\,x^5+444\,x^3-206\,x}{81}\,, \quad \mathcal{Q}_5^*(x) = \frac{30\,x^4+148\,x^2}{9}\,, \quad \mathcal{R}_5^*(x) = \frac{27\,x^5+87\,x^3-325\,x}{27}\,,$$

$$\mathcal{S}_5^*(x) = -\frac{87\,x^4+325\,x^2}{27}\,, \quad \mathcal{U}_5^* = -\frac{206}{81}\,, \quad \mathcal{V}_5^* = -\frac{325}{27}\,, \quad \mathcal{W}_5^* = 0$$

$$\mathcal{P}_6(x) = -\frac{6\,x^6-96\,x^4+256\,x^2}{9}\,, \quad \mathcal{Q}_6(x) = \frac{36\,x^5-384\,x^3+512\,x}{9}\,, \quad \mathcal{R}_6(x) = -x^6+8\,x^4\,,$$

$$\mathcal{S}_6(x) = \frac{108\,x^5 - 912\,x^3 + 1024\,x}{27}\ , \quad \mathcal{U}_6 = 0\ , \quad \mathcal{V}_6 = 0\ , \quad \mathcal{W}_6 = \frac{2048}{27}$$

$$\mathcal{P}_6^*(x) = -\frac{6\,x^6 + 96\,x^4 + 256\,x^2}{9}\ , \quad \mathcal{Q}_6^*(x) = \frac{36\,x^5 + 384\,x^3 + 512\,x}{9}\ , \quad \mathcal{R}_6^*(x) = x^6 + 8\,x^4\ ,$$

$$\mathcal{S}_6^*(x) = -\frac{108\,x^5 + 912\,x^3 + 1024\,x}{27}\ , \quad \mathcal{U}_6^* = 0\ , \quad \mathcal{V}_6^* = 0\ , \quad \mathcal{W}_6^* = -\frac{2048}{27}$$

$$\mathcal{P}_7(x) = -\frac{54\,x^7 - 1404\,x^5 + 10914\,x^3 + 4936\,x}{81}\ , \quad \mathcal{Q}_7(x) = \frac{42\,x^6 - 780\,x^4 + 3638\,x^2}{9}\ ,$$

$$\mathcal{R}_7(x) = -\frac{27\,x^7 - 387\,x^5 + 1947\,x^3 + 7925\,x}{27}\ , \quad \mathcal{S}_7(x) = \frac{129\,x^6 - 1947\,x^4 + 7925\,x^2}{27}\ ,$$

$$\mathcal{U}_7 = \frac{4936}{81}\ , \quad \mathcal{V}_7 = -\frac{7925}{27}\ , \quad \mathcal{W}_7 = 0$$

$$\mathcal{P}_7^*(x) = -\frac{54\,x^7 + 1404\,x^5 + 10914\,x^3 - 4936\,x}{81}\ , \quad \mathcal{Q}_7^*(x) = \frac{42\,x^6 + 780\,x^4 + 3638\,x^2}{9}\ ,$$

$$\mathcal{R}_7^*(x) = \frac{27\,x^7 + 387\,x^5 + 1947\,x^3 - 7925\,x}{27}\ , \quad \mathcal{S}_7^*(x) = -\frac{129\,x^6 + 1947\,x^4 + 7925\,x^2}{27}\ ,$$

$$\mathcal{U}_7^* = -\frac{4936}{81}\ , \quad \mathcal{V}_7^* = -\frac{7925}{27}\ , \quad \mathcal{W}_7^* = 0$$

$$\mathcal{P}_8(x) = -\frac{54\,x^8 - 2064\,x^6 + 30720\,x^4 - 81920\,x^2}{81}\ , \quad \mathcal{Q}_8(x) = \frac{432\,x^7 - 12384\,x^5 + 122880\,x^3 - 163840\,x}{81}\ ,$$

$$\mathcal{R}_8(x) = -\frac{9\,x^8 - 200\,x^6 + 2368\,x^4}{9}\ , \quad \mathcal{S}_8(x) = \frac{1350\,x^7 - 31968\,x^5 + 288384\,x^3 - 327680\,x}{243}\ ,$$

$$\mathcal{U}_8 = 0\ , \quad \mathcal{V}_8 = 0\ , \quad \mathcal{W}_8 = -\frac{655360}{243}$$

$$\mathcal{P}_8^*(x) = -\frac{54\,x^8 + 2064\,x^6 + 30720\,x^4 + 81920\,x^2}{81}\ , \quad \mathcal{Q}_8^*(x) = \frac{432\,x^7 + 12384\,x^5 + 122880\,x^3 + 163840\,x}{81}\ ,$$

$$\mathcal{R}_8^*(x) = \frac{9\,x^8 + 200\,x^6 + 2368\,x^4}{9}\ , \quad \mathcal{S}_8^*(x) = -\frac{1350\,x^7 + 31968\,x^5 + 288384\,x^3 + 327680\,x}{243}\ ,$$

$$\mathcal{U}_8^* = 0\ , \quad \mathcal{V}_8^* = 0\ , \quad \mathcal{W}_8^* = -\frac{655360}{243}$$

About recurrence relations for the previous and the following integrals see page 397.

$$\int \frac{J_1^3(x)\,dx}{x} = -\frac{1}{3}\,J_1^3(x) + \int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_1^3(x)\,dx}{x} = -\frac{1}{3}\,I_1^3(x) + \int I_0(x)\,I_1^2(x)\,dx$$

$$\int \frac{J_1^3(x)\,dx}{x^2} = -\frac{1}{4}\,J_0^3(x) - \frac{3}{8}\,J_0(x)\,J_1^2(x) - \frac{1}{4x}\,J_1^3(x) - \frac{3}{8}\int J_1^3(x)\,dx$$

$$\int \frac{I_1^3(x)\,dx}{x^2} = \frac{1}{4}\,I_0^3(x) - \frac{3}{8}\,I_0(x)\,I_1^2(x) - \frac{1}{4x}\,I_1^3(x) + \frac{3}{8}\int I_1^3(x)\,dx$$

$$\int \frac{J_1^3(x)\,dx}{x^3} = -\frac{2}{5}\,J_0^2(x)\,J_1(x) - \frac{1}{5x}\,J_0(x)\,J_1^2(x) + \frac{x^2-3}{15x^2}\,J_1^3(x) + \frac{2}{5}\int J_0(x)^3\,dx - \int J_0(x)\,J_1^2(x)\,dx$$

$$\int \frac{I_1^3(x)\,dx}{x^3} = -\frac{2}{5}\,I_0^2(x)\,I_1(x) - \frac{1}{5x}\,I_0(x)\,I_1^2(x) - \frac{x^2+3}{15x^2}\,I_1^3(x) + \frac{2}{5}\int I_0(x)^3\,dx + \int I_0(x)\,I_1^2(x)\,dx$$

$$\int \frac{J_1^3(x)\,dx}{x^4} =$$

$$= \frac{11}{96}\,J_0^3(x) - \frac{1}{8x}\,J_0^2(x)J_1(x) + \frac{11\,x^2-8}{64\,x^2}\,J_0(x)J_1^2(x) + \frac{3\,x^2-16}{96\,x^3}\,J_0(x)J_1^3(x) + \frac{11}{64}\int J_1^3(x)\,dx + \frac{1}{8}\int \frac{J_0^3(x)\,dx}{x}$$

$$\int \frac{I_1^3(x)\,dx}{x^4} =$$

$$= \frac{11}{96}\,I_0^3(x) - \frac{1}{8x}\,I_0^2(x)I_1(x) - \frac{11\,x^2+8}{64\,x^2}\,I_0(x)I_1^2(x) - \frac{3\,x^2+16}{96\,x^3}\,I_0(x)I_1^3(x) + \frac{11}{64}\int I_1^3(x)\,dx + \frac{1}{8}\int \frac{I_0^3(x)\,dx}{x}$$

$$\mathcal{P}_{-5}(x) = -\frac{2}{35x}\,, \quad \mathcal{Q}_{-5}(x) = \frac{148\,x^2-30}{525\,x^2}\,, \quad \mathcal{R}_{-5}(x) = \frac{29\,x^2-45}{525\,x^3}\,, \quad \mathcal{S}_{-5}(x) = -\frac{29\,x^4-27\,x^2+225}{1575\,x^4}\,,$$

$$\mathcal{U}_{-5} = -\frac{148}{525}\,, \quad \mathcal{V}_{-5} = \frac{13}{21}\,, \quad \mathcal{W}_{-5} = 0$$

$$\mathcal{P}^*_{-5}(x) = -\frac{2}{35x}\,, \quad \mathcal{Q}^*_{-5}(x) = -\frac{148\,x^2+30}{525x^2}\,, \quad \mathcal{R}^*_{-5}(x) = -\frac{29\,x^2+45}{525x^3}\,, \quad \mathcal{S}^*_{-5}(x) = -\frac{29\,x^4+27\,x^2+225}{1575\,x^4}\,,$$

$$\mathcal{U}^*_{-5} = \frac{148}{525}\,, \quad \mathcal{V}^*_{-5} = \frac{13}{21}\,, \quad \mathcal{W}^*_{-5} = 0$$

$$\mathcal{P}_{-6}(x) = -\frac{19\,x^2+8}{512\,x^2}\,, \quad \mathcal{Q}_{-6}(x) = \frac{3\,x^2-2}{64\,x^3}\,, \quad \mathcal{R}_{-6}(x) = -\frac{57\,x^4-24\,x^2+64}{1024\,x^4}\,,$$

$$\mathcal{S}_{-6}(x) = -\frac{9\,x^4-16\,x^2+192}{1536\,x^5}\,, \quad \mathcal{U}_{-6} = 0\,, \quad \mathcal{V}_{-6} = 0\,, \quad \mathcal{W}_{-6} = -\frac{57}{1024}\,, \quad \mathcal{X}_{-6} = -\frac{3}{64}$$

$$\mathcal{P}^*_{-6}(x) = \frac{19\,x^2-8}{512\,x^2}\,, \quad \mathcal{Q}^*_{-6}(x) = -\frac{3\,x^2+2}{64\,x^3}\,, \quad \mathcal{R}^*_{-6}(x) = -\frac{57\,x^4+24\,x^2+64}{1024\,x^4}\,,$$

$$\mathcal{S}^*_{-6}(x) = -\frac{9\,x^4+16\,x^2+192}{1536\,x^5}\,, \quad \mathcal{U}^*_{-6} = 0\,, \quad \mathcal{V}^*_{-6} = 0\,, \quad \mathcal{W}^*_{-6} = \frac{57}{1024}\,, \quad \mathcal{X}^*_{-6} = \frac{3}{64}$$

$$\mathcal{P}_{-7}(x) = \frac{156\,x^2-70}{11025\,x^3}\,, \quad \mathcal{Q}_{-7}(x) = -\frac{3638\,x^4-780\,x^2+1050}{55125\,x^4}\,, \quad \mathcal{R}_{-7}(x) = -\frac{649\,x^4-645\,x^2+2625}{55125\,x^5}\,,$$

$$\mathcal{S}_{-7}(x) = \frac{649x^6-387x^4+1125x^2-18375}{165375\,x^6}\,, \quad \mathcal{U}_{-7} = \frac{3638}{55125}\,, \quad \mathcal{V}_{-7} = -\frac{317}{2205}\,, \quad \mathcal{W}_{-7} = 0$$

$$\mathcal{P}^*_{-7}(x) = -\frac{156\,x^2+70}{11025\,x^3}\,, \quad \mathcal{Q}^*_{-7}(x) = -\frac{3638\,x^4+780\,x^2+1050}{55125\,x^4}\,, \quad \mathcal{R}^*_{-7}(x) = -\frac{649\,x^4+645\,x^2+2625}{55125\,x^5}\,,$$

$$\mathcal{S}^*_{-7}(x) = -\frac{649\,x^6+387\,x^4+1125\,x^2+18375}{165375\,x^6}\,, \quad \mathcal{U}^*_{-7} = \frac{3638}{55125}\,, \quad \mathcal{V}^*_{-7} = \frac{317}{2205}\,, \quad \mathcal{W}^*_{-7} = 0$$

$$\mathcal{P}_{-8}(x) = \frac{751\,x^4+344\,x^2-384}{122880\,x^4}\,, \quad \mathcal{Q}_{-8}(x) = -\frac{60\,x^4-43\,x^2+96}{7680\,x^5}\,,$$

$$\mathcal{R}_{-8}(x) = \frac{2253\,x^6-888\,x^4+1600\,x^2-9216}{245760\,x^6}\,, \quad \mathcal{S}_{-8}(x) = \frac{333\,x^6-400\,x^4+1728\,x^2-36864}{368640\,x^7}\,,$$

$$\mathcal{U}_{-8} = 0\,, \quad \mathcal{V}_{-8} = 0\,, \quad \mathcal{W}_{-8} = \frac{751}{81920}\,, \quad \mathcal{X}_{-8} = \frac{1}{128}$$

$$\mathcal{P}^*_{-8}(x) = \frac{751\,x^4-344\,x^2-384}{122880\,x^4}\,, \quad \mathcal{Q}^*_{-8}(x) = -\frac{60\,x^4+43\,x^2+96}{7680\,x^5}\,,$$

$$\mathcal{R}^*_{-8}(x) = -\frac{2253\,x^6+888\,x^4+1600\,x^2+9216}{245760\,x^6}\,, \quad \mathcal{S}^*_{-8}(x) = -\frac{333\,x^6+400\,x^4+1728\,x^2+36864}{368640\,x^7}\,,$$

$$\mathcal{U}^*_{-8} = 0\,, \quad \mathcal{V}^*_{-8} = 0\,, \quad \mathcal{W}^*_{-8} = \frac{751}{81920}\,, \quad \mathcal{X}^*_{-8} = \frac{1}{128}$$

i) Recurrence Relations:

Let
$$\mathcal{J}_n^{(kl)} = \int x^n\, J_0^k(x)\, J_1^l(x)\, dx \qquad \text{and} \qquad \mathcal{I}_n^{(kl)} = \int x^n\, I_0^k(x)\, I_1^l(x)\, dx$$
with $k+l=3,\ k,l\geq 0$. Then the following formulas hold:

Ascending recurrence:

$$\mathcal{J}_{n+1}^{(30)} = x^{n+1}\left[J_0^2(x)\, J_1(x) + \frac{2}{3}\, J_1^3(x)\right] - n\,\mathcal{J}_n^{(21)} - \frac{2}{3}(n-2)\,\mathcal{J}_n^{(03)}$$
$$\mathcal{J}_{n+1}^{(21)} = \frac{n+1}{3}\,\mathcal{J}_n^{(30)} - \frac{x^{n+1}}{3}\, J_0^3(x)$$
$$\mathcal{J}_{n+1}^{(12)} = \frac{x^{n+1}}{3}\, J_1^3(x) - \frac{n-2}{3}\,\mathcal{J}_n^{(03)}$$
$$\mathcal{J}_{n+1}^{(03)} = -x^{n+1}\left[J_0(x)\, J_1^2(x) + \frac{2}{3}\, J_0^3(x)\right] + (n-1)\mathcal{J}_n^{(12)} + \frac{2}{3}(n+1)\mathcal{J}_n^{(30)}$$

$$\mathcal{I}_{n+1}^{(30)} = x^{n+1}\left[I_0^2(x)\, I_1(x) - \frac{2}{3}\, I_1^3(x)\right] - n\,\mathcal{I}_n^{(21)} + \frac{2}{3}(n-2)\,\mathcal{I}_n^{(03)}$$
$$\mathcal{I}_{n+1}^{(21)} = -\frac{n+1}{3}\,\mathcal{I}_n^{(30)} + \frac{x^{n+1}}{3}\, I_0^3(x)$$
$$\mathcal{I}_{n+1}^{(12)} = \frac{x^{n+1}}{3}\, I_1^3(x) - \frac{n-2}{3}\,\mathcal{I}_n^{(03)}$$
$$\mathcal{I}_{n+1}^{(03)} = x^{n+1}\left[I_0(x)\, I_1^2(x) - \frac{2}{3}\, I_0^3(x)\right] - (n-1)\mathcal{I}_n^{(12)} + \frac{2}{3}(n+1)\mathcal{I}_n^{(30)}$$

Descending recurrence with $n\leq -3$:

$$\mathcal{J}_n^{(30)} = \frac{x^{n+1} J_0^3(x) + 3\mathcal{J}_{n+1}^{(21)}}{n+1}\quad , \qquad \mathcal{J}_n^{(21)} = \frac{x^{n+1} J_0^2(x)\, J_1(x) + 2\mathcal{J}_{n+1}^{(12)} - \mathcal{J}_{n+1}^{(30)}}{n}$$
$$\mathcal{J}_n^{(12)} = -\frac{2\mathcal{J}_{n+1}^{(21)} - x^{n+1} J_0(x)\, J_1^2(x) - \mathcal{J}_{n+1}^{(03)}}{n-1}\quad , \qquad \mathcal{J}_n^{(03)} = \frac{x^{n+1} J_1^3(x) - 3\mathcal{J}_{n+1}^{(12)}}{n-2}$$

Holds
$$J_0^3(x) + 3\mathcal{J}_0^{(21)} = xJ_0^2(x)\, J_1(x) + 2\mathcal{J}_1^{(12)} - \mathcal{J}_1^{(30)} = 2\mathcal{J}_2^{(21)} - x^2 J_0(x)\, J_1^2(x) - \mathcal{J}_2^{(03)} =$$
$$= x^3 J_1^3(x) - 3\mathcal{J}_3^{(12)} = \text{const.}$$

$$\mathcal{I}_n^{(30)} = \frac{x^{n+1} I_0^3(x) - 3\mathcal{I}_{n+1}^{(21)}}{n+1}\quad , \qquad \mathcal{I}_n^{(21)} = \frac{x^{n+1} I_0^2(x)\, I_1(x) - 2\mathcal{I}_{n+1}^{(12)} - \mathcal{I}_{n+1}^{(30)}}{n}$$
$$\mathcal{I}_n^{(12)} = -\frac{2\mathcal{I}_{n+1}^{(21)} + \mathcal{I}_{n+1}^{(03)} - x^{n+1} I_0(x)\, I_1^2(x)}{n-1}\quad , \qquad \mathcal{I}_n^{(03)} = \frac{x^{n+1} I_1^3(x) - 3\mathcal{I}_{n+1}^{(12)}}{n-2}$$

Holds
$$I_0^3(x) - 3\mathcal{I}_0^{(21)} = xI_0^2(x)\, I_1(x) - 2\mathcal{I}_1^{(12)} - \mathcal{I}_1^{(30)} = x^2 I_0(x)\, I_1^2(x) - 2\mathcal{I}_2^{(21)} - \mathcal{I}_2^{(03)} =$$
$$= x^3 I_1^3(x) - 3\mathcal{I}_3^{(12)} = \text{const.}\ .$$

j) Different Functions and Different Arguments

Let $Z_\nu(x)$, $Z_\nu^*(x) \in \{J_\nu(x), Y_\nu(x), H_\nu^{(1)}(x), H_\nu^{(2)}(x)\}$ with $Z_\nu(x) \neq Z_\nu^*(x)$ and $Z_\nu(x)$ and $Z_{1-\nu}(x)$ as well as $Z_\nu^*(x)$ and $Z_{1-\nu}^*(x)$ of the same type, then holds

$$\int x^3\, Z_1^2(x) Z_0^*(x)\, dx =$$

$$= \frac{x^2}{3}\, \left\{-2\, Z_0(x)Z_1(x)[xZ_0^*(x) + 2Z_1^*(x)] + Z_1^2(x)[4Z_0^*(x) + xZ_1^*(x)] + 2x\, Z_0^2 x)Z_1^*(x)\right\}$$

and

$$\int x^3 Z_0(x)Z_1(x)Z_1^*(x)\, dx =$$

$$= \frac{x^2}{3}\, \left\{Z_0(x)Z_1(x)[xZ_0^*(x) + 2Z_1^*(x)] - Z_1^2(x)[2Z_0^*(x) - xZ_1^*(x)] - xZ_0^2(x)Z_1^*(x)\right\}\ .$$

$$\int x^3\, I_1^2(x)K_0(x)\, dx =$$

$$= \frac{1}{3}\left[-4x^2\, I_0(x)I_1(x)K_1(x) + 2\, I_0(x)I_1(x)K_0(x)x^3 - x^3 I_1^2(x)K_1(x) + 2\, x^3 I_0^2(x)K_1(x) - 4\, x^2 I_1^2(x)K_0(x)\right]$$

$$\int x^3\, K_1^2(x)I_0(x) =$$

$$= \frac{1}{3}\left[-2x^3\, I_0(x)K_0(x)K_1(x) - 4x^2\, I_1(x)K_0(x)K_1(x) + x^3 I_1(x)K_1^2(x) - 4\, x^2 I_0(x)K_1^2(x) - 2\, x^3 I_1(x)K_0^2(x)\right]$$

$$\int Z_1^2(x)Z_1^*(2x)\, dx = \frac{x}{2}\left\{\left[Z_0^2(x) - Z_1^2(x)\right]\ Z_1^*(2x) - 2Z_0(x)Z_1(x)Z_0^*(2x)\right\}$$

$$\int x\, Z_0^2(x)Z_0^*(2x)\, dx = \frac{x^2}{2}\left\{\left[Z_0^2(x) - Z_1^2(x)\right]\ Z_0^*(2x) + 2Z_0(x)Z_1(x)Z_1^*(2x)\right\}$$

$$\int x\, Z_1^2(x)Z_0^*(2x)\, dx =$$

$$= \frac{x}{2}\left\{x\ \left[Z_1^2(x) - Z_0^2(x)\right]\ Z_0^*(2x) + 2Z_0^2(x)Z_1^*(2x) - 2Z_0(x)Z_1(x)\left[Z_0^*(2x) + xZ_1^*(2x)\right]\right\}$$

$$\int x\, Z_1^2(x)Z_0^*\left(\sqrt{2}x\right)\, dx = \frac{\sqrt{2}x}{2} Z_0^2(x)Z_1^*\left(\sqrt{2}x\right) - xZ_0(x)Z_1(x)Z_0^*\left(\sqrt{2}x\right)$$

$$\int x\, Z_0(x)Z_1(x)Z_1^*(2x)\, dx = \frac{x}{2}\left\{x\ \left[Z_0^2(x) - Z_1^2(x)\right] Z_0^*(2x) + Z_0(x)Z_1^*(2x)\left[2x\, Z_1(x) - Z_0(x)\right]\right\}$$

$$\int x^2\, Z_0^2(x)Z_1^*(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{x\ \left[Z_0^2(x) - Z_1^2(x)\right]\ Z_1^*(2x) - 2Z_1^2(x)Z_0^*(2x) - 2Z_0(x)Z_1(x)\left[xZ_0^*(2x) - 2Z_1^*(2x)\right]\right\}$$

$$\int x^2\, Z_0^2(x)Z_1^*\left(\sqrt{2}x\right)\, dx = x^2\, Z_0(x)Z_1(x)Z_1^*\left(\sqrt{2}x\right) - \frac{\sqrt{2}x^2}{2} Z_1^2(x)Z_0^*\left(\sqrt{2}x\right)$$

$$\int x^2\, Z_1^2(x)Z_1^*(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{x\ \left[Z_1^2(x) - Z_0^2(x)\right]\ Z_1^*(2x) + 2Z_0(x)Z_1(x)\left[xZ_0^*(2x) + Z_1^*(2x)\right] - 4Z_1^2(x)Z_0^*(2x)\right\}$$

$$\int x^2\, Z_0(x)Z_1(x)Z_0^*(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{x\ \left[Z_1^2(x) - Z_0^2(x)\right] Z_1^*(2x) + 2Z_0(x)Z_1(2x)\left[x\, Z_0^*(2x) + Z_1^*(2x)\right] - Z_1^2(x)Z_0^*(2x)\right\}$$

$$\int x^3\, I_1^2(x)K_0(x)\, dx =$$

$$= \frac{x^2}{3}\ \left\{2\, I_0(x)I_1(x)[xK_0(x)-2K_1(x)] - I_1^2(x)[4K_0(x)+xK_1(x)] + 2xI_0^2(x)K_1(x)\right\}$$

$$\int x^3\, K_1^2(x)I_0(x)\, dx =$$

$$= -\frac{x^2}{3}\ \left\{2\, K_0(x)K_1(x)[xI_0(x)+2I_1(x)] + K_1^2(x)[4I_0(x)-xI_1(x)] + 2xK_0^2(x)I_1(x)\right\}$$

$$\int x^3 I_0(x)I_1(x)K_1(x)\, dx =$$

$$= \frac{x^2}{3}\ \left\{I_0(x)I_1(x)[xK_0(x)-2K_1(x)] + I_1^2(x)[xK_1(x)-2K_0(x)] + xI_0^2(x)K_1(x)\right\}$$

$$\int x^3 K_0(x)K_1(x)I_1(x)\, dx =$$

$$= -\frac{x^2}{3}\ \left\{K_0(x)K_1(x)[xI_0(x)+2I_1(x)] + K_1^2(x)[xI_1(x)+2I_0(x)] + xK_0^2(x)I_1(x)\right\}$$

$$\int I_1^2(x)K_1(2x)\, dx = -\frac{x}{2}\left[I_0^2(x)+I_1^2(x)\right]\ K_1(2x) + 2\, I_0(x)I_1(x)K_0(2x)$$

$$\int K_1^2(x)I_1(2x)\, dx = -\frac{x}{2}\left[K_0^2(x)+K_1^2(x)\right]\ I_1(2x) + 2\, K_0(x)K_1(x)I_0(2x)$$

$$\int x\, I_0^2(x)K_0(2x)\, dx = \frac{x^2}{2}\left\{\left[I_1^2(x)+I_0^2(x)\right]\ K_0(2x) + 2I_0(x)I_1(x)K_1(2x)\right\}$$

$$\int x\, K_0^2(x)I_0(2x)\, dx = \frac{x^2}{2}\left\{\left[K_1^2(x)+K_0^2(x)\right]\ I_0(2x) + 2K_0(x)K_1(x)I_1(2x)\right\}$$

$$\int x\, I_1^2(x)K_0(2x)\, dx =$$

$$= \frac{x}{2}\left\{x\ \left[I_0^2(x)+I_1^2(x)\right]\ K_0(2x) + 2I_0^2(x)K_1(2x) + 2I_0(x)I_1(x)\left[K_0(2x)+xK_1(2x)\right]\right\}$$

$$\int x\, K_1^2(x)I_0(2x)\, dx =$$

$$= \frac{x}{2}\left\{x\ \left[K_0^2(x)+K_1^2(x)\right]\ I_0(2x) - 2K_0^2(x)I_1(2x) - 2K_0(x)K_1(x)\left[I_0(2x)-xI_1(2x)\right]\right\}$$

$$\int x\, I_1^2(x)K_0\left(\sqrt{2}\,x\right)\ dx = \frac{\sqrt{2}\,x}{2}I_0^2(x)K_1\left(\sqrt{2}\,x\right) + xI_0(x)I_1(x)K_0\left(\sqrt{2}\,x\right)$$

$$\int x\, K_1^2(x)I_0\left(\sqrt{2}\,x\right)\ dx = -\frac{\sqrt{2}\,x}{2}K_0^2(x)I_1\left(\sqrt{2}\,x\right) - xK_0(x)K_1(x)I_0\left(\sqrt{2}\,x\right)$$

$$\int x\, I_0(x)I_1(x)K_1(2x)\, dx = \frac{x}{2}\left\{x\ \left[I_0^2(x)+I_1^2(x)\right]K_0(2x) + I_0(x)K_1(2x)\left[2x\, I_1(x)+I_0(x)\right]\right\}$$

$$\int x\, K_0(x)K_1(x)I_1(2x)\, dx = \frac{x}{2}\left\{x\ \left[K_0^2(x)+K_1^2(x)\right]I_0(2x) + K_0(x)I_1(2x)\left[2x\, K_1(x)-K_0(x)\right]\right\}$$

$$\int x^2\, I_0^2(x)K_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{x\ \left[I_0^2(x)+I_1^2(x)\right]\ K_1(2x) + 2I_1^2(x)K_0(2x) + 2I_0(x)I_1(x)\left[xK_0(2x)+2K_1(2x)\right]\right\}$$

$$\int x^2\, K_0^2(x)I_1(2x)\, dx =$$

$$= \frac{x^2}{6}\left\{x\ \left[K_0^2(x)+K_1^2(x)\right]\ I_1(2x) - 2K_1^2(x)I_0(2x) + 2K_0(x)K_1(x)\left[xI_0(2x)-2I_1(2x)\right]\right\}$$

$$\int x^2\, I_0^2(x)K_1\left(\sqrt{2}\,x\right)\ dx = x^2\, I_0(x)I_1(x)K_1\left(\sqrt{2}\,x\right) + \frac{\sqrt{2}\,x^2}{2}I_1^2(x)K_0\left(\sqrt{2}\,x\right)$$

$$\int x^2\, K_0^2(x) I_1\left(\sqrt{2}\, x\right)\, dx = -\, x^2\, K_0(x) K_1(x) I_1\left(\sqrt{2}\, x\right) - \frac{\sqrt{2}\, x^2}{2} K_1^2(x) I_0\left(\sqrt{2}\, x\right)$$

$$\int x^2\, I_1^2(x) K_1(2x)\, dx =$$

$$= \frac{x^2}{6} \left\{ x\, \left[I_0^2(x) + I_1^2(x) \right]\, K_1(2x) + 2 I_0(x) I_1(x) \left[x K_0(2x) - K_1(2x) \right] - 4 I_1^2(x) K_0(2x) \right\}$$

$$\int x^2\, K_1^2(x) I_1(2x)\, dx =$$

$$= \frac{x^2}{6} \left\{ x\, \left[K_1^2(x) + K_0^2(x) \right]\, I_1(2x) + 2 K_0(x) K_1(x) \left[x I_0(2x) + I_1(2x) \right] + 4 K_1^2(x) I_0(2x) \right\}$$

$$\int x^2\, I_0(x) I_1(x) K_0(2x)\, dx =$$

$$= \frac{x^2}{6} \left\{ x\, \left[I_0^2(x) + I_1^2(x) \right] K_1(2x) + 2 I_0(x) I_1(2x) \left[x\, K_0(2x) - K_1(2x) \right] - I_1^2(x) K_0(2x) \right\}$$

$$\int x^2\, K_0(x) K_1(x) I_0(2x)\, dx =$$

$$= \frac{x^2}{6} \left\{ x\, \left[K_0^2(x) + K_1^2(x) \right] I_1(2x) + 2 K_0(x) K_1(2x) \left[x\, I_0(2x) + I_1(2x) \right] + K_1^2(x) I_0(2x) \right\}$$

3.2. Integrals of the type $\int x^n\, Z_\kappa(\alpha x)\, Z_\mu(\beta x)\, Z_\nu(\gamma x)\, dx$

The general case is discussed in [12]. In the following some special solutions are given.

a) $x^n\, Z_\kappa(x)\, Z_\mu(x)\, Z_\nu(2x)$

With $\kappa, \mu, \nu \in \{0, 1\}$ the following integrals may be expressed by functions of the same kind:

$$\int x^{2n+1}\, Z_0^2(x)\, Z_0(2x)\, dx\;, \quad \int x^{2n+1}\, Z_0(x)\, Z_1(x)\, Z_1(2x)\, dx\;, \quad \int x^{2n+1}\, Z_1^2(x)\, Z_0(2x)\, dx\;, \quad n \geq 0,$$

and

$$\int x^{2n}\, Z_0(x)\, Z_1(x)\, Z_0(2x)\, dx\;, \quad \int x^{2n}\, Z_0^2(x)\, Z_1(2x)\, dx\;, \quad \int x^{2n}\, Z_1^2(x)\, Z_1(x)\, Z_1(2x)\, dx\;, \quad n \geq 1.$$

$$\int J_1^2(x) J_1(2x)\, dx = \frac{x}{2}\, \left[J_0^2(x) J_1(2x) - J_1^2(x) J_1(2x) - 2J_0(x) J_1(x) J_0(2x)\right]$$

$$\int I_1^2(x) I_1(2x)\, dx = -\frac{x}{2}\, \left[I_0^2(x) I_1(2x) + I_1^2(x) I_1(2x) - 2I_0(x) I_1(x) I_0(2x)\right]$$

$$\int K_1^2(x) K_1(2x)\, dx = -\frac{x}{2}\, \left[K_0^2(x) K_1(2x) + K_1^2(x) K_1(2x) - 2K_0(x) K_1(x) K_0(2x)\right]$$

$$\int x J_0^2(x) J_0(2x)\, dx = \frac{x^2}{2}\, \left[J_0^2(x) J_0(2x) - J_1^2(x) J_0(2x) + 2J_0(x) J_1(x) J_1(2x)\right]$$

$$\int x I_0^2(x) I_0(2x)\, dx = \frac{x^2}{2}\, \left[I_0^2(x) I_0(2x) + I_1^2(x) I_0(2x) - 2I_0(x) I_1(x) I_1(2x)\right]$$

$$\int x K_0^2(x) K_0(2x)\, dx = \frac{x^2}{2}\, \left[K_0^2(x) K_0(2x) + K_1^2(x) K_0(2x) - 2K_0(x) K_1(x) K_1(2x)\right]$$

$$\int x J_0(x) J_1(x) J_1(2x)\, dx = \frac{x}{2}\, \left[x\, J_0^2(x) J_0(2x) - J_0^2(x) J_1(2x) - x\, J_1^2(x) J_0(2x) + 2\, x J_0(x) J_1(x) J_1(2x)\right]$$

$$\int x I_0(x) I_1(x) I_1(2x)\, dx = \frac{x}{2}\, \left[-x\, I_0^2(x) I_0(2x) + I_0^2(x) I_1(2x) - x\, I_1^2(x) I_0(2x) + 2\, x I_0(x) I_1(x) I_1(2x)\right]$$

$$\int x K_0(x) K_1(x) K_1(2x)\, dx =$$

$$= \frac{x}{2}\, \left[-x\, K_0^2(x) K_0(2x) - K_0^2(x) K_1(2x) - x\, K_1^2(x) K_0(2x) + 2\, x K_0(x) K_1(x) K_1(2x)\right]$$

$$\int x\, J_1^2(x) J_0(2x)\, dx =$$

$$= \frac{x}{2}\, \left[-x\, J_0^2(x) J_0(2x) + 2J_0^2(x) J_1(2x) + x\, J_1^2(x) J_0(2x) - 2J_0(x) J_1(x) J_0(2x) - 2\, x J_0(x) J_1(x) J_1(2x)\right]$$

$$\int x\, I_1^2(x) I_0(2x)\, dx =$$

$$= \frac{x}{2}\, \left[x\, I_0^2(x) I_0(2x) - 2I_0^2(x) I_1(2x) + x\, I_1^2(x) I_0(2x) + 2I_0(x) I_1(x) I_0(2x) - 2\, x I_0(x) I_1(x) I_1(2x)\right]$$

$$\int x\, K_1^2(x) K_0(2x)\, dx =$$

$$= \frac{x}{2}\, \left[x\, K_0^2(x) K_0(2x) + 2K_0^2(x) K_1(2x) + x\, K_1^2(x) K_0(2x) - 2K_0(x) K_1(x) K_0(2x) - 2\, x K_0(x) K_1(x) K_1(2x)\right]$$

$$\int x^2 J_0(x)\, J_1(x)\, J_0(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[-x\, J_0^2(x) J_1(2x) - J_1^2(x) J_0(2x) + x\, J_1^2(x) J_1(2x) + 2\, x J_0(x) J_1(x) J_0(2x) + 2 J_0(x) J_1(x) J_1(2x)\right]$$

$$\int x^2 I_0(x)\, I_1(x)\, I_0(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[-x\, I_0^2(x) I_1(2x) - I_1^2(x) I_0(2x) - x\, I_1^2(x) I_1(2x) + 2\, x I_0(x) I_1(x) I_0(2x) + 2 I_0(x) I_1(x) I_1(2x)\right]$$

$$\int x^2 K_0(x)\, K_1(x)\, K_0(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[-x\, K_0^2(x) K_1(2x) + K_1^2(x) K_0(2x) - x\, K_1^2(x) K_1(2x) + 2\, x K_0(x) K_1(x) K_0(2x) - 2 K_0(x) K_1(x) K_1(2x)\right]$$

$$\int x^2\, J_0^2(x)\, J_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[x\, J_0^2(x) J_1(2x) - 2\, J_1^2(x) J_0(2x) - x\, J_1^2(x) J_1(2x) - 2\, x J_0(x) J_1(x) J_0(2x) + 4 J_0(x) J_1(x) J_1(2x)\right]$$

$$\int x^2\, I_0^2(x)\, I_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[x\, I_0^2(x) I_1(2x) - 2\, I_1^2(x) I_0(2x) + x\, I_1^2(x) I_1(2x) - 2\, x I_0(x) I_1(x) I_0(2x) + 4 I_0(x) I_1(x) I_1(2x)\right]$$

$$\int x^2\, K_0^2(x)\, K_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[x\, K_0^2(x) K_1(2x) + 2\, K_1^2(x) K_0(2x) + x\, K_1^2(x) K_1(2x) - 2\, x K_0(x) K_1(x) K_0(2x) - 4 K_0(x) K_1(x) K_1(2x)\right]$$

$$\int x^2\, J_1^2(x)\, J_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[-x J_0^2(x) J_1(2x) - 4\, J_1^2(x)\, J_0(2x) + x\, J_1^2(x)\, J_1(2x) + 2\, x J_0(x)\, J_1(x) J_0(2x) + 2 J_0(x)\, J_1(x) J_1(2x)\right]$$

$$\int x^2\, I_1^2(x)\, I_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,\left[x I_0^2(x) I_1(2x) + 4\, I_1^2(x)\, I_0(2x) + x\, I_1^2(x)\, I_1(2x) - 2\, x I_0(x)\, I_1(x) I_0(2x) - 2 I_0(x)\, I_1(x) I_1(2x)\right]$$

$$\int x^2\, K_1^2(x)\, K_1(2x)\, dx =$$

$$= \frac{x^2}{6}\,[x K_0^2(x) K_1(2x) - 4\, K_1^2(x)\, K_0(2x) + x\, K_1^2(x)\, K_1(2x) - 2\, x K_0(x)\, K_1(x) K_0(2x) + 2 K_0(x)\, K_1(x) K_1(2x)]$$

$$\int x^3 J_0^2(x)\, J_0(2x)\, dx = \frac{x^2}{10}\,[x^2\, J_0^2(x)\, J_0(2x) + 2x\, J_0(x)\, J_1(x)\, J_0(2x) + (2 - x^2)\, J_1^2(x)\, J_0(2x) + 2x\, J_0^2(x)\, J_1(2x) +$$

$$+ (2x^2 - 4)\, J_0(x)\, J_1(x)\, J_1(2x) + 2x\, J_1^2(x)\, J_1(2x)]$$

$$\int x^3 I_0^2(x)\, I_0(2x)\, dx = \frac{x^2}{10}\,[x^2\, I_0^2(x)\, I_0(2x) + 2x\, I_0(x)\, I_1(x)\, I_0(2x) + (x^2 + 2)\, I_1^2(x)\, I_0(2x) + 2x\, I_0^2(x)\, I_1(2x) -$$

$$-(2x^2 + 4)\, I_0(x)\, I_1(x)\, I_1(2x) - 2x\, I_1^2(x)\, I_1(2x)]$$

$$\int x^3 K_0^2(x)\, K_0(2x)\, dx = \frac{x^2}{10}\,[x^2\, K_0^2(x)\, K_0(2x) - 2x\, K_0(x)\, K_1(x)\, K_0(2x) + (x^2 + 2)\, K_1^2(x)\, K_0(2x) -$$

$$-2x\, K_0^2(x)\, K_1(2x) - (2x^2 + 4)\, K_0(x)\, K_1(x)\, K_1(2x) + 2x\, K_1^2(x)\, K_1(2x)]$$

$$\int x^3\, J_1^2(x)\, J_0(2x)\, dx = \frac{x^2}{30}\, [-3x^2\, J_0^2(x)\, J_0(2x) + 4x\, J_0(x)\, J_1(x)\, J_0(2x) + (3x^2+4)\, J_1^2(x)\, J_0(2x) +$$

$$+4x\, J_0^2(x)\, J_1(2x) - (6x^2+8)\, J_0(x)\, J_1(x)\, J_1(2x) + 14x\, J_1^2(x)\, J_1(2x)]$$

$$\int x^3\, I_1^2(x)\, I_0(2x)\, dx = \frac{x^2}{30}\, [3x^2\, I_0^2(x)\, I_0(2x) - 4x\, I_0(x)\, I_1(x)\, I_0(2x) + (3x^2-4)\, I_1^2(x)\, I_0(2x) +$$

$$-4x\, I_0^2(x)\, I_1(2x) - (6x^2-8)\, I_0(x)\, I_1(x)\, I_1(2x) + 14x\, I_1^2(x)\, I_1(2x)]$$

$$\int x^3\, K_1^2(x)\, K_0(2x)\, dx = \frac{x^2}{30}\, [3x^2\, K_0^2(x)\, K_0(2x) + 4x\, K_0(x)\, K_1(x)\, K_0(2x) + (3x^2-4)\, K_1^2(x)\, K_0(2x) +$$

$$+4x\, K_0^2(x)\, K_1(2x) - (6x^2-8)\, K_0(x)\, K_1(x)\, K_1(2x) - 14x\, K_1^2(x)\, K_1(2x)]$$

$$\int x^3 J_0(x)\, J_1(x)\, J_1(2x)\, dx = \frac{x^2}{30}\, [3x^2\, J_0^2(x)\, J_0(2x) - 4x\, J_0(x)\, J_1(x)\, J_0(2x) - (3x^2-4)\, J_1^2(x)\, J_0(2x) -$$

$$-4x\, J_0^2(x)\, J_1(2x) + (6x^2+8)\, J_0(x)\, J_1(x)\, J_1(2x) + x\, J_1^2(x)\, J_1(2x)]$$

$$\int x^3 I_0(x)\, I_1(x)\, I_1(2x)\, dx = \frac{x^2}{30}\, [-3x^2\, I_0^2(x)\, I_0(2x) + 4x\, I_0(x)\, I_1(x)\, I_0(2x) + (4-3x^2)\, I_1^2(x)\, I_0(2x) +$$

$$+4x\, I_0^2(x)\, I_1(2x) + (6x^2-8)\, I_0(x)\, I_1(x)\, I_1(2x) + x\, I_1^2(x)\, I_1(2x)]$$

$$\int x^3 K_0(x)\, K_1(x)\, K_1(2x)\, dx = \frac{x^2}{30}\, [-3x^2\, K_0^2(x)\, K_0(2x) - 4x\, K_0(x)\, K_1(x)\, K_0(2x) + (4-3x^2)\, K_1^2(x)\, K_0(2x) -$$

$$-4x\, K_0^2(x)\, K_1(2x) + (6x^2-8)\, K_0(x)\, K_1(x)\, K_1(2x) - x\, K_1^2(x)\, K_1(2x)]$$

$$\int x^4 J_0(x)\, J_1(x)\, J_0(2x)\, dx = \frac{x^2}{42}\, [-3x^2\, J_0^2(x)\, J_0(2x) + (6x^3+4x)\, J_0(x)\, J_1(x)\, J_0(2x) + 4\, J_1^2(x)\, J_0(2x) +$$

$$+(4x-3x^3)\, J_0^2(x)\, J_1(2x) + (6x^2-8)\, J_0(x)\, J_1(x)\, J_1(2x) + (3x^3+2x)\, J_1^2(x)\, J_1(2x)]$$

$$\int x^4 I_0(x)\, I_1(x)\, I_0(2x)\, dx = \frac{x^2}{42}\, [3x^2\, I_0^2(x)\, I_0(2x) + (6x^3-4x)\, I_0(x)\, I_1(x)\, I_0(2x) - 4\, I_1^2(x)\, I_0(2x) -$$

$$-(3x^3+4x)\, I_0^2(x)\, I_1(2x) + (6x^2+8)\, I_0(x)\, I_1(x)\, I_1(2x) + (2x-3x^3)\, I_1^2(x)\, I_1(2x)]$$

$$\int x^4 K_0(x)\, K_1(x)\, K_0(2x)\, dx = \frac{x^2}{42}\, [-3x^2\, K_0^2(x)\, K_0(2x) + (6x^3-4x)\, K_0(x)\, K_1(x)\, K_0(2x) + 4\, K_1^2(x)\, K_0(2x) -$$

$$-(3x^3+4x)\, K_0^2(x)\, K_1(2x) - (6x^2+8)\, K_0(x)\, K_1(x)\, K_1(2x) + (2x-3x^3)\, K_1^2(x)\, K_1(2x)]$$

$$\int x^4 J_0^2(x)\, J_1(2x)\, dx = \frac{x^2}{210}\, [-48x^2\, J_0^2(x)\, J_0(2x) - (30\, x^3 - 64\, x)\, J_0(x)\, J_1(x)\, J_0(2x) -$$

$$-(42\, x^2 - 64)\, J_1^2(x)\, J_0(2x) + (15\, x^3 + 64\, x)\, J_0^2(x)\, J_1(2x) + (54\, x^2 - 128)\, J_0(x)\, J_1(x)\, J_1(2x) -$$

$$-(15\, x^3 - 74\, x)\, J_1^2(x)\, J_1(2x)]$$

$$\int x^4 I_0^2(x)\, I_1(2x)\, dx = \frac{x^2}{210}\, [48x^2\, I_0^2(x)\, I_0(2x) - (30\, x^3 + 64\, x)\, I_0(x)\, I_1(x)\, I_0(2x) - (42\, x^2 + 64)\, I_1^2(x)\, I_0(2x) +$$

$$+(15\, x^3 - 64\, x)\, I_0^2(x)\, I_1(2x) + (54\, x^2 + 128)\, I_0(x)\, I_1(x)\, I_1(2x) + (15\, x^3 + 74\, x)\, I_1^2(x)\, I_1(2x)]$$

$$\int x^4 K_0^2(x)\, K_1(2x)\, dx = \frac{x^2}{210}\, [-48x^2\, K_0^2(x)\, K_0(2x) - (30\, x^3 + 64\, x)\, K_0(x)\, K_1(x)\, K_0(2x) +$$

$$+(42\, x^2 + 64)\, K_1^2(x)\, K_0(2x) + (15\, x^3 - 64\, x)\, K_0^2(x)\, K_1(2x) - (54\, x^2 + 128)\, K_0(x)\, K_1(x)\, K_1(2x) +$$

$$+(15\, x^3 + 74\, x)\, K_1^2(x)\, K_1(2x)]$$

$$\int x^4 J_1^2(x)\,J_1(2x)\,dx = \frac{x^2}{70}\,[\,-12x^2\,J_0^2(x)\,J_0(2x)+(10\,x^3+16\,x)\,J_0(x)\,J_1(x)\,J_0(2x)-(28\,x^2-16)\,J_1^2(x)\,J_0(2x)-$$

$$-(5\,x^3-16\,x)\,J_0^2(x)\,J_1(2x)-(4\,x^2+32)\,J_0(x)\,J_1(x)\,J_1(2x)+(5\,x^3+36\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^4 I_1^2(x)\,I_1(2x)\,dx = \frac{x^2}{70}\,[\,-12x^2\,I_0^2(x)\,I_0(2x)-(10\,x^3-16\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+(28\,x^2+16)\,I_1^2(x)\,I_0(2x)+$$

$$+(5\,x^3+16\,x)\,I_0^2(x)\,I_1(2x)+(4\,x^2-32)\,I_0(x)\,I_1(x)\,I_1(2x)+(5\,x^3-36\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^4 K_1^2(x)\,K_1(2x)\,dx = \frac{x^2}{70}\,[\,12x^2\,K_0^2(x)\,K_0(2x)-(10\,x^3-16\,x)\,K_0(x)\,K_1(x)\,K_0(2x)-$$

$$-(28\,x^2+16)\,K_1^2(x)\,K_0(2x)+(5\,x^3+16\,x)\,K_0^2(x)\,K_1(2x)-(4\,x^2-32)\,K_0(x)\,K_1(x)\,K_1(2x)+$$

$$+(5\,x^3-36\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^5 J_0^2(x)\,J_0(2x)\,dx = \frac{x^2}{630}\,[(35\,x^4+216\,x^2)\,J_0^2(x)\,J_0(2x)+(100\,x^3-288\,x)\,J_0(x)\,J_1(x)\,J_0(2x)-$$

$$-(35\,x^4-224\,x^2+288)\,J_1^2(x)\,J_0(2x)+(160\,x^3-288\,x)\,J_0^2(x)\,J_1(2x)+(70\,x^4-208\,x^2+576)\,J_0(x)\,J_1(x)\,J_1(2x)+$$

$$+(120\,x^3-368\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^5 I_0^2(x)\,I_0(2x)\,dx = \frac{x^2}{630}\,[(35\,x^4-216\,x^2)\,I_0^2(x)\,I_0(2x)+(100\,x^3+288\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+$$

$$+(35\,x^4+224\,x^2+288)\,I_1^2(x)\,I_0(2x)+(160x^3+288x)\,I_0^2(x)\,I_1(2x)-(70\,x^4+208\,x^2+576)\,I_0(x)\,I_1(x)\,I_1(2x)-$$

$$-(120\,x^3+368\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^5 K_0^2(x)\,K_0(2x)\,dx = \frac{x^2}{630}\,[(35\,x^4-216\,x^2)\,K_0^2(x)\,K_0(2x)-(100\,x^3+288\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+$$

$$+(35\,x^4+224\,x^2+288)\,K_1^2(x)\,K_0(2x)-(160\,x^3+288\,x)\,K_0^2(x)\,K_1(2x)-$$

$$-(70\,x^4+208\,x^2+576)\,K_0(x)\,K_1(x)\,K_1(2x)+(120\,x^3+368\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^5 J_1^2(x)\,J_0(2x)\,dx = \frac{x^2}{630}\,[(-35\,x^4+180\,x^2)\,J_0^2(x)\,J_0(2x)-(10\,x^3+240\,x)\,J_0(x)\,J_1(x)\,J_0(2x)02+$$

$$+(35\,x^4+280\,x^2-240)\,J_1^2(x)\,J_0(2x)+(110\,x^3-240\,x)\,J_0^2(x)\,J_1(2x)-(70\,x^4+80\,x^2-480)\,J_0(x)\,J_1(x)\,J_1(2x)+$$

$$+(240\,x^3-400\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^5 I_1^2(x)\,I_0(2x)\,dx = \frac{x^2}{630}\,[(35x^4+180x^2)\,I_0^2(x)\,I_0(2x)+(10\,x^3-240\,x)\,I_0(x)\,I_1(x)\,I_0(2x)+$$

$$+(35\,x^4-280\,x^2-240)\,I_1^2(x)\,I_0(2x)-(110\,x^3+240\,x)\,I_0^2(x)\,I_1(2x)-(70\,x^4-80\,x^2-480)\,I_0(x)\,I_1(x)\,I_1(2x)+$$

$$+(240\,x^3+400\,x)\,I_1^2(x)\,I_1(2x)]$$

$$\int x^5 K_1^2(x)\,K_0(2x)\,dx = \frac{x^2}{630}\,[(35\,x^4+180\,x^2)\,K_0^2(x)\,K_0(2x)-(10\,x^3-240\,x)\,K_0(x)\,K_1(x)\,K_0(2x)+$$

$$+(35\,x^4-280\,x^2-240)\,K_1^2(x)\,K_0(2x)+(110\,x^3+240\,x)\,K_0^2(x)\,K_1(2x)-$$

$$-(70\,x^4-80\,x^2-480)\,K_0(x)\,K_1(x)\,K_1(2x)-(240\,x^3+400\,x)\,K_1^2(x)\,K_1(2x)]$$

$$\int x^5 J_0(x)\,J_1(x)\,J_1(2x)\,dx = \frac{x^2}{630}\,[(35\,x^4-72\,x^2)\,J_0^2(x)\,J_0(2x)-(80\,x^3-96\,x)\,J_0(x)\,J_1(x)\,J_0(2x)-$$

$$-(35\,x^4+28\,x^2-96)\,J_1^2(x)\,J_0(2x)-(65\,x^3-96\,x)\,J_0^2(x)\,J_1(2x)+(70\,x^4+116\,x^2-192)\,J_0(x)\,J_1(x)\,J_1(2x)+$$

$$+(30\,x^3+76\,x)\,J_1^2(x)\,J_1(2x)]$$

$$\int x^5 I_0(x)\, I_1(x)\, I_1(2x)\, dx = \frac{x^2}{630}\, [-(35\, x^4 + 72\, x^2)\, I_0^2(x)\, I_0(2x) + (80\, x^3 + 96\, x)\, I_0(x)\, I_1(x)\, I_0(2x) -$$

$$-(35\, x^4 - 28\, x^2 - 96)\, I_1^2(x)\, I_0(2x) + (65\, x^3 + 96\, x)\, I_0^2(x)\, I_1(2x) + (70\, x^4 - 116\, x^2 - 192)\, I_0(x)\, I_1(x)\, I_1(2x) +$$
$$+(30\, x^3 - 76\, x)\, I_1^2(x)\, I_1(2x)]$$

$$\int x^5 K_0(x)\, K_1(x)\, K_1(2x)\, dx = \frac{x^2}{630}\, [-(35\, x^4 + 72\, x^2)\, K_0^2(x)\, K_0(2x) - (80\, x^3 + 96\, x)\, K_0(x)\, K_1(x)\, K_0(2x) -$$

$$-(35\, x^4 - 28\, x^2 - 96)\, K_1^2(x)\, K_0(2x) - (65\, x^3 + 96\, x)\, K_0^2(x)\, K_1(2x) +$$
$$+(70\, x^4 - 116\, x^2 - 192)\, K_0(x)\, K_1(x)\, K_1(2x) - (30\, x^3 - 76\, x)\, K_1^2(x)\, K_1(2x)]$$

Recurrence relations:

$$\int x^{2n+1}\, J_0^2(x)\, J_0(2x)\, dx = -\frac{n}{4n+1} \int x^{2n}\, J_1(2x)\, \left[3n\, J_0^2(x) + (n-1)\, J_1^2(x)\right]\, dx +$$
$$+\frac{x^{2n+1}}{2(4n+1)}[xJ_0^2(x)\, J_0(2x) - xJ_1^2(x)\, J_0(2x) + 3nJ_0^2(x)\, J_1(2x) + 2xJ_0(x)\, J_1(x)\, J_1(2x) + nJ_1^2(x)\, J_1(2x)]$$

$$\int x^{2n+1}\, J_0(x)\, J_1(x)\, J_1(2x)\, dx =$$
$$= \frac{1}{2(4n+1)} \int x^{2n}\, \left[-n(2n-1)\, J_0^2(x)J_1(2x) + 2n(4n+1)\, J_0(x)J_1(x)J_0(2x) + (n-1)(2n+1)J_1^2(x)J_1(2x)\right]\, dx +$$
$$+\frac{x^{2n+1}}{4(4n+1)}[(2n-1)J_0^2(x)\, J_1(2x) - 2(4n+1)\, J_0(x)J_1(x)J_0(2x) - (2n+1)J_1^2(x)\, J_1(2x) + 2xJ_0^2(x)\, J_0(2x) +$$
$$+4xJ_0(x)J_1(x)J_1(2x) - 2xJ_1^2(x)J_0(2x)]$$

$$\int x^{2n+1}\, J_1^2(x)\, J_0(2x)\, dx = -\frac{1}{4n+1} \int x^{2n}\, \left[n\,(n+1)J_0^2(x)J_1(2x) + (3n+1)(n-1)\, J_1^2(x)\, J_1(2x)\right]\, dx +$$
$$+\frac{x^{2n+1}}{2(4n+1)}[(n+1)J_0^2(x)J_1(2x) + (3n+1)J_1^2(x)J_1(2x) - xJ_0^2(x)\, J_0(2x) - 2xJ_0(x)J_1(x)J_1(2x) + xJ_1^2(x)J_0(2x)]$$

$$\int x^{2n+2}\, J_0(x)\, J_1(x)\, J_0(2x)\, dx =$$
$$= \frac{1}{4(4n+3)} \left\{-2(n+1)(2n+1) \int x^{2n+1}\, J_0^2(x)\, J_0(2x)\, dx - 4n(4n+3) \int x^{2n+1}\, J_0(x)\, J_1(x)\, J_1(2x)\, dx + \right.$$
$$+2n(2n+3) \int x^{2n+1}\, J_1^2(x)\, J_0(2x)\, dx + x^{2n+2}[(2n+1)J_0^2(x)J_0(2x) - 2xJ_0^2(x)J_1(2x) + 4xJ_0(x)J_1(x)J_0(2x) -$$
$$\left. -(2n+3)J_1^2(x)J_0(2x) + 2(4n+3)J_0(x)J_1(x)J_1(2x) + 2xJ_1^2(x)J_1(2x)]\right\}$$

$$\int x^{2n+2}\, J_0^2(x)\, J_1(2x)\, dx =$$
$$= \frac{1}{2(4n+3)} \left\{2(n+1)(3n+2) \int x^{2n+1} J_0^2(x)J_0(2x)\, dx + 2n^2 \int x^{2n+1} J_1^2(x)J_0(2x)\, dx + \right.$$
$$\left. +x^{2n+2}\left[-(3n+2)J_0^2(x)J_0(2x) + xJ_0^2(x)J_1(x) - 2xJ_0(x)J_1(x)J_0(2x) - nJ_1^2(x)J_0(2x) - xJ_1^2(x)J_1(2x)\right]\right\}$$

$$\int x^{2n+2}\, J_1^2(x)\, J_1(2x)\, dx =$$
$$= \frac{1}{2(4n+3)} \left\{2(n+1)^2 \int x^{2n+1}\, J_0^2(x)\, J_0(2x)\, dx + 6n(n+1) \int x^{2n+1}\, J_1^2(x)\, J_0(2x)\, dx + \right.$$
$$\left. +x^{2n+2}[-(n+1)J_0^2(x)J_0(2x) - xJ_0^2(x)J_1(2x) + 2xJ_0(x)J_1(x)J_0(2x) - 3(n+1)J_1^2(x)J_0(2x) + xJ_1^2(x)J_1(2x)]\right\}$$

$$\int x^{2n+1}\, I_0^2(x)\, I_0(2x)\, dx = \frac{1}{4n+1} \int x^{2n}[-3n^2\, I_0^2(x) I_1(2x) + n(n-1) I_1^2(x) I_1(2x)]\, dx +$$

$$+\frac{x^{2n+1}}{2(4n+1)}[x I_0^2(x)\, I_0(2x) + x I_1^2(x)\, I_0(2x) - 2x I_0(x)\, I_1(x)\, I_1(2x) + 3n I_0^2(x)\, I_1(2x) - n I_1^2(x)\, I_1(2x)]$$

$$\int x^{2n+1}\, I_0(x)\, I_1(x)\, I_1(2x)\, dx =$$

$$= \frac{1}{2(4n+1)} \int x^{2n}\, [n(2n-1)\, I_0^2(x) I_1(2x) - 2n(4n+1)\, I_0(x) I_1(x) I_0(2x) + (n-1)(2n+1) I_1^2(x) I_1(2x)]\, dx +$$

$$+\frac{x^{2n+1}}{4(4n+1)}[-(2n-1) I_0^2(x)\, I_1(2x) + 2(4n+1) I_0(x) I_1(x) I_0(2x) - (2n+1) I_1^2(x)\, I_1(2x) -$$

$$-2x I_0^2(x)\, I_0(2x) + 4x I_0(x) I_1(x) I_1(2x) - 2x I_1^2(x)\, I_0(2x)]$$

$$\int x^{2n+1}\, I_1^2(x)\, I_0(2x)\, dx = \frac{1}{4n+1} \int x^{2n}\, \left[n\,(n+1) I_0^2(x) I_1(2x) - (3n+1)(n-1)\, I_1^2(x)\, I_1(2x)\right]\, dx +$$

$$+\frac{x^{2n+1}}{2(4n+1)}[-(n+1) I_0^2(x) I_1(2x) + (3n+1) I_1^2(x) I_1(2x) + x I_0^2(x)\, I_0(2x) - 2x I_0(x) I_1(x) I_1(2x) + x I_1^2(x) I_0(2x)]$$

$$\int x^{2n+2}\, I_0(x)\, I_1(x)\, I_0(2x)\, dx =$$

$$= \frac{1}{2(4n+3)} \int x^{2n+1}\, [(n+1)(2n+1) I_0^2(x)\, I_0(2x) - 2n(4n+3)\, I_0(x)\, I_1(x)\, I_1(2x) + n(2n+3) I_1^2(x) I_0(2x)]\, dx +$$

$$+x^{2n+2}[-(2n+1)\, I_0^2(x)\, I_0(2x) + 2(4n+3)\, I_0(x)\, I_1(x)\, I_1(2x) - (2n+3)x\, I_1^2(x)\, I_0(2x) -$$

$$-2x\, I_0^2(x)\, I_1(2x) + 4x I_0(x)\, I_1(x)\, I_0(2x) - 2x\, I_1^2(x)\, I_1(2x)\}$$

$$\int x^{2n+2}\, I_0^2(x)\, I_1(2x)\, dx =$$

$$= \frac{1}{4n+3} \int x^{2n+1}\, [-(n+1)(3n+2) I_0^2(x) I_0(2x) + n^2 I_1^2(x) I_0(2x)]\, dx +$$

$$+x^{2n+2}[(3n+2)\, I_0^2(x)\, I_0(2x) - n\, I_1^2(x)\, I_0(2x) + x\; I_0^2(x)\, I_1(2x) - 2x\; I_0(x) I_1(x)\, I_0(2x) + x\, I_1^2(x)\, I_1(2x)]$$

$$\int x^{2n+2}\, I_1^2(x)\, I_1(2x)\, dx =$$

$$= \frac{1}{4n+3} \int x^{2n+1}\, [(n+1)^2\, I_0^2(x)\, I_0(2x) - 3n(n+1)\, I_1^2(x)\, I_0(2x)]\, dx +$$

$$+x^{2n+2}[-(n+1)\, I_0^2(x)\, I_0(2x) + 3(n+1)\, I_1^2(x)\, I_0(2x) + x\, I_0^2(x)\, I_1(2x) - 2x\, I_0(x)\, I_1(x)\, I_0(2x) + x\, I_1^2(x)\, I_1(2x)]$$

$$\int x^{2n+1}\,K_0^2(x)\,K_0(2x)\,dx=\frac{1}{4n+1}\int x^{2n}[3n^2\,K_0^2(x)K_1(2x)-n(n-1)K_1^2(x)K_1(2x)]\,dx+$$

$$+\frac{x^{2n+1}}{2(4n+1)}[xK_0^2(x)\,K_0(2x)+xK_1^2(x)\,K_0(2x)-2xK_0(x)\,K_1(x)\,K_1(2x)-3nK_0^2(x)\,K_1(2x)+nK_1^2(x)\,K_1(2x)]$$

$$\int x^{2n+1}\,K_0(x)\,K_1(x)\,K_1(2x)\,dx=$$

$$=\frac{1}{2(4n+1)}\int x^{2n}\,[-n(2n-1)\,K_0^2(x)K_1(2x)+2n(4n+1)\,K_0(x)K_1(x)K_0(2x)-(n-1)(2n+1)K_1^2(x)K_1(2x)]\,dx+$$

$$+\frac{x^{2n+1}}{4(4n+1)}[(2n-1)K_0^2(x)\,K_1(2x)-2(4n+1)K_0(x)K_1(x)K_0(2x)+(2n+1)K_1^2(x)\,K_1(2x)-$$

$$-2xK_0^2(x)\,K_0(2x)+4xK_0(x)K_1(x)K_1(2x)-2xK_1^2(x)\,K_0(2x)]$$

$$\int x^{2n+1}\,K_1^2(x)\,K_0(2x)\,dx=\frac{1}{4n+1}\int x^{2n}\,\left[-n\,(n+1)K_0^2(x)K_1(2x)+(3n+1)(n-1)\,K_1^2(x)\,K_1(2x)\right]\,dx+$$

$$+\frac{x^{2n+1}}{2(4n+1)}\,\Big[(n+1)K_0^2(x)K_1(2x)-(3n+1)K_1^2(x)K_1(2x)+xK_0^2(x)\,K_0(2x)-2xK_0(x)K_1(x)K_1(2x)+$$

$$+\,xK_1^2(x)K_0(2x)\Big]$$

$$\int x^{2n+2}\,K_0(x)\,K_1(x)\,K_0(2x)\,dx=\frac{1}{2(4n+3)}\cdot$$

$$\cdot\int x^{2n+1}\,[-(n+1)(2n+1)K_0^2(x)\,K_0(2x)+2n(4n+3)\,K_0(x)\,K_1(x)\,K_1(2x)-n(2n+3)\,K_1^2(x)K_0(2x)]\,dx+$$

$$+\frac{x^{2n+2}}{4(4n+3)}[(2n+1)\,K_0^2(x)\,K_0(2x)-2(4n+3)\,K_0(x)\,K_1(x)\,K_1(2x)+(2n+3)x\,K_1^2(x)\,K_0(2x)-$$

$$-2x\,K_0^2(x)\,K_1(2x)+4xK_0(x)\,K_1(x)\,K_0(2x)-2x\,K_1^2(x)\,K_1(2x)$$

$$\int x^{2n+2}\,K_0^2(x)\,K_1(2x)\,dx=$$

$$=\frac{1}{4n+3}\int x^{2n+1}\,[(n+1)(3n+2)K_0^2(x)K_0(2x)-n^2K_1^2(x)K_0(2x)]\,dx+$$

$$+\frac{x^{2n+2}}{2(4n+3)}\,\Big[-(3n+2)\,K_0^2(x)\,K_0(2x)+n\,K_1^2(x)\,K_0(2x)+x\;K_0^2(x)\,K_1(2x)-2x\;K_0(x)K_1(x)\,K_0(2x)+$$

$$+x\,K_1^2(x)\,K_1(2x)\Big]$$

$$\int x^{2n+2}\,K_1^2(x)\,K_1(2x)\,dx=$$

$$=\frac{1}{4n+3}\int x^{2n+1}\,[-(n+1)^2\,K_0^2(x)\,K_0(2x)+3n(n+1)\,K_1^2(x)\,K_0(2x)]\,dx+$$

$$+\frac{x^{2n+2}}{2(4n+3)}\,\Big[(n+1)\,K_0^2(x)\,K_0(2x)-3(n+1)\,K_1^2(x)\,K_0(2x)+x\,K_0^2(x)\,K_1(2x)-2x\,K_0(x)\,K_1(x)\,K_0(2x)\,+$$

$$+x\,K_1^2(x)\,K_1(2x)\Big]$$

b) $x^n\, Z_\kappa(\alpha x)\, Z_\mu(\beta x)\, Z_\nu((\alpha+\beta)x)$

Formulas were found for the following integrals only:

$$\int x^{2n+1}\, Z_0(\alpha x)\, Z_0(\beta x)\, Z_0((\alpha+\beta)x)\, dx\ , \quad \int x^{2n+1}\, Z_0(\alpha x)\, Z_1(\beta x)\, Z_1((\alpha+\beta)x)\, dx\ ,$$

$$\int x^{2n+1}\, Z_1(\alpha x)\, Z_1(\beta x)\, Z_0((\alpha+\beta)x)\, dx\ , \quad n \geq 0,$$

and

$$\int x^{2n}\, Z_0(\alpha x)\, Z_0(\beta x)\, Z_1((\alpha+\beta)x)\, dx\ , \quad \int x^{2n}\, Z_0(\alpha x)\, Z_1(\beta x)\, Z_0((\alpha+\beta)x)\, dx\ ,$$

$$\int x^{2n}\, Z_1(\alpha x)\, Z_1(\beta x)\, Z_1((\alpha+\beta)x)\, dx\ , \quad n \geq 1.$$

The integrals $\int x^n\, Z_\nu(\alpha x)\, Z_\nu(\beta x)\, Z_{1-\nu}((\alpha+\beta)x)\, dx$ and $\int x^n\, Z_{1-\nu}(\alpha x)\, Z_\nu(\beta x)\, Z_\nu((\alpha+\beta)x)\, dx$ may be expressed by each other. Nevertheless, they are both listed.

$$\int J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx = \frac{x}{2}\, [\, J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) -$$
$$- J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) - J_1(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) - J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)]$$

$$\int I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx = \frac{x}{2}\, [- I_0(\alpha x) I_0(\beta x) I_1((\alpha+\beta)x) +$$
$$+ I_0(\alpha x) I_1(\beta x) I_0((\alpha+\beta)x) + I_1(\alpha x) I_0(\beta x) I_0((\alpha+\beta)x) - I_1(\alpha x) I_1(\beta x) I_1((\alpha+\beta)x)]$$

$$\int K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx = \frac{x}{2}\, [- K_0(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x) +$$
$$+ K_0(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x) + K_1(\alpha x) K_0(\beta x) K_0((\alpha+\beta)x) - K_1(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x)]$$

$$\int x\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)\, dx = \frac{x^2}{2}\, [J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) +$$
$$+ J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x) + J_1(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) - J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x)]$$

$$\int x\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx = \frac{x^2}{2}\, [\, I_0(\alpha x) I_0(\beta x) I_0((\alpha+\beta)x) -$$
$$- I_0(\alpha x) I_1(\beta x) I_1((\alpha+\beta)x) - I_1(\alpha x) I_0(\beta x) I_1((\alpha+\beta)x) + I_1(\alpha x) I_1(\beta x) I_0((\alpha+\beta)x)]$$

$$\int x\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x)\, dx = \frac{x^2}{2}\, [\, K_0(\alpha x) K_0(\beta x) K_0((\alpha+\beta)x) -$$
$$- K_0(\alpha x) K_1(\beta x) K_1((\alpha+\beta)x) - K_1(\alpha x) K_0(\beta x) K_1((\alpha+\beta)x) + K_1(\alpha x) K_1(\beta x) K_0((\alpha+\beta)x)]$$

$$\int x\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx = \frac{x}{2\beta(\alpha+\beta)}\, [\beta(\alpha+\beta)x\, J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) -$$
$$- (\alpha+\beta)\, J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) - \beta\, J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) +$$
$$+ \beta(\alpha+\beta)x\, J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x) + \alpha\, J_1(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) +$$
$$+ \beta(\alpha+\beta)x\, J_1(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) - \beta(\alpha+\beta)x\, J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x)]$$

$$\int x\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx = \frac{x}{2\beta(\alpha+\beta)}\, [-\beta(\alpha+\beta)x\, I_0(\alpha x) I_0(\beta x) I_0((\alpha+\beta)x) +$$
$$+ (\alpha+\beta)\, I_0(\alpha x) I_0(\beta x) I_1((\alpha+\beta)x) + \beta\, I_0(\alpha x) I_1(\beta x) I_0((\alpha+\beta)x) +$$
$$+ \beta(\alpha+\beta)x\, I_0(\alpha x) I_1(\beta x) I_1((\alpha+\beta)x) - \alpha\, I_1(\alpha x) I_0(\beta x) I_0((\alpha+\beta)x) +$$

$$+\beta(\alpha+\beta)x\, I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x) - \beta(\alpha+\beta)x\, I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)]$$

$$\int x\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx = \frac{x}{2\beta(\alpha+\beta)}\ [-\beta(\alpha+\beta)x\, K_0(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)-$$

$$-(\alpha+\beta)K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) - \beta K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)+$$

$$+\beta(\alpha+\beta)xK_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x) + \alpha K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)+$$

$$+\beta(\alpha+\beta)xK_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) - \beta(\alpha+\beta)xK_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)]$$

$$\int x\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx = \frac{x}{2\alpha\beta}\ [-\alpha\beta x\, J_0(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x)+$$

$$+(\alpha+\beta)\, J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x) - \beta\, J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)-$$

$$-\alpha\beta x\, J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x) - \alpha\, J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x)-$$

$$-\alpha\beta x\, J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x) + \alpha\beta x\, J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)]$$

$$\int x\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx = \frac{x}{2\alpha\beta}\ [\alpha\beta x\, I_0(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)-$$

$$-(\alpha+\beta)\, I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x) + \beta\, I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)-$$

$$-\alpha\beta x\, I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x) + \alpha\, I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)-$$

$$-\alpha\beta x\, I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x) + \alpha\beta x\, I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)]$$

$$\int x\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x)\, dx = \frac{x}{2\alpha\beta}\ [\alpha\beta x\, K_0(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)+$$

$$+(\alpha+\beta)\, K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) - \beta\, K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)-$$

$$-\alpha\beta x\, K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x) - \alpha\, K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)-$$

$$-\alpha\beta x\, K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x) + \alpha\beta x\, K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)]$$

$$\int x^2\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x)\, dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\ [\alpha\beta\, x\ (\alpha+\beta)\ J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)-$$

$$-\alpha\beta\, x\ (\alpha+\beta)\ J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) + \alpha\ (\alpha+3\,\beta)\ J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)-$$

$$-\alpha\beta\ (\alpha+\beta)\ x\, J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x) + \beta\ (3\,\alpha+\beta)\ J_1(\alpha x)J_0(\beta x)\, J_1((\alpha+\beta)x)-$$

$$-\left(\alpha+\beta\right)^2\ J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x) - \alpha\beta\ (\alpha+\beta)\ x\, J_1(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)\Big]$$

$$\int x^2\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)\, dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\ [\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)\ + \alpha(\alpha+3\beta)\, I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x) + \beta(3\alpha+\beta)\, I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-$$

$$-(\alpha+\beta)^2\, I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)\big]$$

$$\int x^2\, K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x)\, dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\ [\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x) - \alpha(\alpha+3\beta)\, K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x) - \beta(3\alpha+\beta)\, K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)+$$

$$+(\alpha+\beta)^2 K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)\big]$$

$$\int x^2\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\, [-\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)+\alpha(2\alpha+3\beta)\, J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x)-\beta^2\, J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)-$$

$$-(\alpha+\beta)(2\alpha-\beta)\, J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)]$$

$$\int x^2\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\, [-\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)+\alpha(2\alpha+3\beta)\, I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)-\beta^2\, I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-$$

$$-(\alpha+\beta)(2\alpha-\beta)\, I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)]$$

$$\int x^2\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x)\, dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\, [-\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)-\alpha(2\alpha+3\beta)\, K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)+\beta^2\, K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)+$$

$$+(\alpha+\beta)(2\alpha-\beta)\, K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)]$$

$$\int x^2\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\, [-\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)+2\alpha^2\, J_0(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)J_0(\beta x)J_0((\alpha+\beta)x)+2\beta^2\, J_1(\alpha x)J_0(\beta x)J_1((\alpha+\beta)x)-$$

$$-2(\alpha+\beta)^2\, J_1(\alpha x)J_1(\beta x)J_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)J_1(\beta x)J_1((\alpha+\beta)x)\big]$$

$$\int x^2\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\, [\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)-2\alpha^2\, I_0(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)I_0(\beta x)I_0((\alpha+\beta)x)-2\beta^2\, I_1(\alpha x)I_0(\beta x)I_1((\alpha+\beta)x)+$$

$$+\, 2(\alpha+\beta)^2\, I_1(\alpha x)I_1(\beta x)I_0((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)I_1(\beta x)I_1((\alpha+\beta)x)\big]$$

$$\int x^2\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx = \frac{x^2}{6\alpha\beta(\alpha+\beta)}\, [\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)+2\alpha^2\, K_0(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)K_0(\beta x)K_0((\alpha+\beta)x)+2\beta^2\, K_1(\alpha x)K_0(\beta x)K_1((\alpha+\beta)x)-$$

$$-2(\alpha+\beta)^2\, K_1(\alpha x)K_1(\beta x)K_0((\alpha+\beta)x)+\, \alpha\beta(\alpha+\beta)x\, K_1(\alpha x)K_1(\beta x)K_1((\alpha+\beta)x)]$$

Let

$$\int x^n\, Z_\kappa(\alpha x)\, Z_\mu(\beta x)\, Z_\nu((\alpha+\beta)x)\, dx = {}^{\kappa\mu\nu}V_n \sum_{\kappa',\mu',\nu'=0}^{1} {}^{\kappa\mu\nu}P_{n,Z}^{\kappa'\mu'\nu'}(x)\, Z_{\kappa'}(\alpha x)\, Z_{\mu'}(\beta x)\, Z_{\nu'}((\alpha+\beta)x)\ ,$$

then holds

$$^{000}V_3 = \frac{x^2}{30\,\alpha^2\beta^2\,(\alpha+\beta)^2}$$

$$^{000}P_{3,J}^{000}(x) = 3\,\alpha^2\beta^2\,(\alpha+\beta)^2\ x^2$$

$$^{000}P_{3,J}^{001}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\left(\alpha^2+4\,\alpha\,\beta+\beta^2\right)\ x$$

$$^{000}P_{3,J}^{010}(x) = 2\,\alpha\,\beta\ (\alpha+\beta)\left(2\,\alpha^2+2\,\alpha\,\beta-\beta^2\right)\ x$$

$$^{000}P_{3,J}^{011}(x) = \alpha^2\,[3\,\beta^2\,(\alpha+\beta)^2\,x^2-4\,\alpha^2-10\,\alpha\,\beta-10\,\beta^2]$$

$$^{000}P_{3,J}^{100}(x) = -2\,\alpha\,\beta\ (\alpha+\beta)\left(\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\ x$$

$$^{000}P_{3,J}^{101}(x) = \beta^2\,[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-4\,\beta^2-10\,\alpha\,\beta-10\,\alpha^2]$$

$$^{000}P_{3,J}^{110}(x) = -\,(\alpha+\beta)^2\ [3\,\alpha^2\beta^2x^2-4\,\alpha^2+2\,\alpha\,\beta-4\,\beta^2]$$

$$^{000}P_{3,J}^{111}(x) = 4\,\alpha\,\beta\ (\alpha+\beta)\left(\alpha^2+\alpha\,\beta+\beta^2\right)x$$

$$^{000}P_{3,I}^{000}(x) = 3\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2$$

$$^{000}P_{3,I}^{001}(x) = 2\,\alpha\,\beta\ (\alpha+\beta)\left(\alpha^2+4\,\alpha\,\beta+\beta^2\right)\ x$$

$$^{000}P_{3,I}^{010}(x) = 2\,\alpha\,\beta\ (\alpha+\beta)\left(2\,\alpha^2+2\,\alpha\,\beta-\beta^2\right)\ x$$

$$^{000}P_{3,I}^{011}(x) = -\alpha^2\,[3\,\beta^2\,(\alpha+\beta)^2\,x^2+4\,\alpha^2+10\,\alpha\,\beta+10\,\beta^2]$$

$$^{000}P_{3,I}^{100}(x) = -2\,\alpha\,\beta\ (\alpha+\beta)\left(\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\ x$$

$$^{000}P_{3,I}^{101}(x) = -\beta^2\,[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+10\,\alpha^2+10\,\alpha\,\beta+4\,\beta^2]$$

$$^{000}P_{3,I}^{110}(x) = (\alpha+\beta)^2\ [3\,\alpha^2\beta^2x^2+4\,\alpha^2-2\,\alpha\,\beta+4\,\beta^2]$$

$$^{000}P_{3,I}^{111}(x) = -4\,\alpha\,\beta\ (\alpha+\beta)\left(\alpha^2+\alpha\,\beta+\beta^2\right)\ x$$

$$^{000}P_{3,K}^{000}(x) = 3\,\alpha^2\beta^2\ (\alpha+\beta)^2\ x^2$$

$$^{000}P_{3,K}^{001}(x) = -2\,\alpha\,\beta\ (\alpha+\beta)\left(\alpha^2+4\,\alpha\,\beta+\beta^2\right)\ x$$

$$^{000}P_{3,K}^{010}(x) = -2\,\alpha\,\beta\ (\alpha+\beta)\left(2\,\alpha^2+2\,\alpha\,\beta-\beta^2\right)\ x$$

$$^{000}P_{3,K}^{011}(x) = -\alpha^2\,[3\,\beta^2\,(\alpha+\beta)^2\,x^2+4\,\alpha^2+10\,\alpha\,\beta+10\,\beta^2]$$

$$^{000}P_{3,K}^{100}(x) = 2\,\alpha\,\beta\ (\alpha+\beta)\left(\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\ x$$

$$^{000}P_{3,K}^{101}(x) = -\beta^2\,[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+10\,\alpha^2+10\,\alpha\,\beta+4\,\beta^2]$$

$$^{000}P_{3,K}^{110}(x) = (\alpha+\beta)^2\ [3\,\alpha^2\beta^2x^2+4\,\alpha^2-2\,\alpha\,\beta+4\,\beta^2]$$

$$^{000}P_{3,K}^{111}(x) = 4\,\alpha\,\beta\ (\alpha+\beta)\left(\alpha^2+\alpha\,\beta+\beta^2\right)\ x$$

$$^{011}V_3 = \frac{x^2}{30\alpha^2\beta^2\,(\alpha+\beta)^2}$$

$$^{011}P_{3,J}^{000}(x) = 3\,\alpha^2\beta^2\,(\alpha+\beta)^2\ x^2$$

$$^{011}P_{3,J}^{001}(x) = -\alpha\,\beta\ (\alpha+\beta)\left(3\,\alpha^2+7\,\alpha\,\beta-2\,\beta^2\right)\ x$$

$$ {}^{011}P_{3,J}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(6\,\alpha^2+11\,\alpha\,\beta+2\,\beta^2\right)\,x $$
$$ {}^{011}P_{3,J}^{011}(x) = \alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2+6\,\alpha^2+20\,\alpha\,\beta+20\,\beta^2\right] $$
$$ {}^{011}P_{3,J}^{100}(x) = \alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+4\,\alpha\,\beta+4\,\beta^2\right)\,x $$
$$ {}^{011}P_{3,J}^{101}(x) = \beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-2\,\beta\,(5\,\alpha+2\,\beta)\right] $$
$$ {}^{011}P_{3,J}^{110}(x) = -\left(\alpha+\beta\right)^2\,\left[3\,\alpha^2\beta^2x^2+2\,(\alpha+\beta)\,(3\,\alpha-2\,\beta)\right] $$
$$ {}^{011}P_{3,J}^{111}(x) = -2\,\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\,x $$

$$ {}^{011}P_{3,I}^{000}(x) = -3\,\beta^2\alpha^2\,(\alpha+\beta)^2\,x^2 $$
$$ {}^{011}P_{3,I}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+7\,\alpha\,\beta-2\,\beta^2\right)\,x $$
$$ {}^{011}P_{3,I}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\left(6\,\alpha^2+11\,\alpha\,\beta+2\,\beta^2\right)\,x $$
$$ {}^{011}P_{3,I}^{011}(x) = \alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2-6\,\alpha^2-20\,\alpha\,\beta-20\,\beta^2\right] $$
$$ {}^{011}P_{3,I}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+4\,\alpha\,\beta+4\,\beta^2\right) $$
$$ {}^{011}P_{3,I}^{101}(x) = \beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+2\,\beta\,(5\,\alpha+2\,\beta)\right] $$
$$ {}^{011}P_{3,I}^{110}(x) = -\left(\alpha+\beta\right)^2\,\left[3\,\alpha^2\beta^2x^2-2\,(\alpha+\beta)\,(3\,\alpha-2\,\beta)\right] $$
$$ {}^{011}P_{3,I}^{111}(x) = -2\,\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\,x $$

$$ {}^{011}P_{3,K}^{000}(x) = -3\,\beta^2\alpha^2\,(\alpha+\beta)^2\,x^2 $$
$$ {}^{011}P_{3,K}^{001}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+7\,\alpha\,\beta-2\,\beta^2\right)\,x $$
$$ {}^{011}P_{3,K}^{010}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(6\,\alpha^2+11\,\alpha\,\beta+2\,\beta^2\right)\,x $$
$$ {}^{011}P_{3,K}^{011}(x) = \alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2-6\,\alpha^2-20\,\alpha\,\beta-20\,\beta^2\right] $$
$$ {}^{011}P_{3,K}^{100}(x) = \alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+4\,\alpha\,\beta+4\,\beta^2\right)\,x $$
$$ {}^{011}P_{3,K}^{101}(x) = \beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+2\,\beta\,(5\,\alpha+2\,\beta)\right] $$
$$ {}^{011}P_{3,K}^{110}(x) = -\left(\alpha+\beta\right)^2\,\left[3\,\alpha^2\beta^2x^2-2\,(\alpha+\beta)\,(3\,\alpha-2\,\beta)\right] $$
$$ {}^{011}P_{3,K}^{111}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2-2\,\alpha\,\beta-2\,\beta^2\right)\,x $$

$$ {}^{110}V_3 = \frac{x^2}{30\,\alpha^2\beta^2\,(\alpha+\beta)^2} $$

$$ {}^{110}P_{3,J}^{000}(x) = -3\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 $$
$$ {}^{110}P_{3,J}^{001}(x) = \alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+2\,\alpha\,\beta+3\,\beta^2\right)\,x $$
$$ {}^{110}P_{3,J}^{010}(x) = \alpha\,\beta\,(\alpha+\beta)\left(6\,\alpha^2+\alpha\,\beta-3\,\beta^2\right)\,x $$
$$ {}^{110}P_{3,J}^{011}(x) = -\alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2+2\,\alpha\,(3\,\alpha+5\,\beta)\right] $$
$$ {}^{110}P_{3,J}^{100}(x) = -\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2-\alpha\,\beta-6\,\beta^2\right)\,x $$
$$ {}^{110}P_{3,J}^{101}(x) = -\beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2+2\,\beta\,(5\,\alpha+3\,\beta)\right] $$
$$ {}^{110}P_{3,J}^{110}(x) = \left(\alpha+\beta\right)^2\,\left[3\,\alpha^2\beta^2x^2+6\,\alpha^2-8\,\alpha\,\beta+6\,\beta^2\right] $$
$$ {}^{110}P_{3,J}^{111}(x) = 2\,\alpha\,\beta\,(\alpha+\beta)\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)\,x $$

$$ {}^{110}P_{3,I}^{000}(x) = -3\,\beta^2\alpha^2\,(\alpha+\beta)^2\,x^2 $$

$$ {}^{110}P_{3,I}^{001}(x) = -\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+2\,\alpha\,\beta+3\,\beta^2\right)\,x $$
$$ {}^{110}P_{3,I}^{010}(x) = -\alpha\,\beta\;(\alpha+\beta)\left(6\,\alpha^2+\alpha\,\beta-3\,\beta^2\right)\,x $$
$$ {}^{110}P_{3,I}^{011}(x) = -\alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2-2\,\alpha\;(3\,\alpha+5\,\beta)\right] $$
$$ {}^{110}P_{3,I}^{100}(x) = \alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2-\alpha\,\beta-6\,\beta^2\right)\,x $$
$$ {}^{110}P_{3,I}^{101}(x) = -\beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-2\,\beta\;(5\,\alpha+3\,\beta)\right] $$
$$ {}^{110}P_{3,I}^{110}(x) = (\alpha+\beta)^2\;\left[3\,\alpha^2\beta^2x^2-6\,\alpha^2+8\,\alpha\,\beta-6\,\beta^2\right] $$
$$ {}^{110}P_{3,I}^{111}(x) = 2\,\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)\,x $$

$$ {}^{110}P_{3,K}^{000}(x) = 3\,\beta^2\alpha^2\,(\alpha+\beta)^2\;x^2 $$
$$ {}^{110}P_{3,K}^{001}(x) = \alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+2\,\alpha\,\beta+3\,\beta^2\right)\,x $$
$$ {}^{110}P_{3,K}^{010}(x) = \alpha\,\beta\;(\alpha+\beta)\left(6\,\alpha^2+\alpha\,\beta-3\,\beta^2\right)\,x $$
$$ {}^{110}P_{3,K}^{011}(x) = -\alpha^2\left[3\,\beta^2\,(\alpha+\beta)^2\,x^2-2\,\alpha\;(3\,\alpha+5\,\beta)\right] $$
$$ {}^{110}P_{3,K}^{100}(x) = -\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2-\alpha\,\beta-6\,\beta^2\right)\,x $$
$$ {}^{110}P_{3,K}^{101}(x) = -\beta^2\left[3\,\alpha^2\,(\alpha+\beta)^2\,x^2-2\,\beta\;(5\,\alpha+3\,\beta)\right] $$
$$ {}^{110}P_{3,K}^{110}(x) = (\alpha+\beta)^2\;\left[3\,\alpha^2\beta^2x^2-6\,\alpha^2+8\,\alpha\,\beta-6\,\beta^2\right] $$
$$ {}^{110}P_{3,K}^{111}(x) = -2\,\alpha\,\beta\;(\alpha+\beta)\left(3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2\right)\,x $$

$$ {}^{001}V_4 = \frac{x^2}{210\,\alpha^3\beta^3\,(\alpha+\beta)^3} $$
$$ {}^{001}P_{4,J}^{000}(x) = -6\,\alpha^2\beta^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\;x^2 $$
$$ {}^{001}P_{4,J}^{001}(x) = \alpha\beta\;(\alpha+\beta)\;x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+18\,\alpha^4+52\,\alpha^3\beta+116\,\alpha^2\beta^2+52\,\alpha\,\beta^3+18\,\beta^4\right] $$
$$ {}^{001}P_{4,J}^{010}(x) = -\alpha\beta\;(\alpha+\beta)\;x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-36\,\alpha^4-86\,\alpha^3\beta-58\,\alpha^2\beta^2+34\,\alpha\,\beta^3+18\,\beta^4\right] $$
$$ {}^{001}P_{4,J}^{011}(x) = \alpha^2\left[9\,\beta^2\,(\alpha+2\,\beta)\,(3\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2-4\,\alpha\;\left(9\,\alpha^3+35\,\beta\,\alpha^2+49\,\alpha\,\beta^2+35\,\beta^3\right)\right] $$
$$ {}^{001}P_{4,J}^{100}(x) = -\alpha\beta\;(\alpha+\beta)\;x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+18\,\alpha^4+34\,\alpha^3\beta-58\,\alpha^2\beta^2-86\,\alpha\,\beta^3-36\,\beta^4\right] $$
$$ {}^{001}P_{4,J}^{101}(x) = -\beta^2\left[9\,\alpha^2\,(2\,\alpha+\beta)\,(\alpha-3\,\beta)\,(\alpha+\beta)^2\,x^2+4\,\beta\;\left(35\,\alpha^3+49\,\beta\,\alpha^2+35\,\alpha\,\beta^2+9\,\beta^3\right)\right] $$
$$ {}^{001}P_{4,J}^{110}(x) = -\,(\alpha+\beta)^2\;\left[3\,\alpha^2\beta^2\left(9\,\alpha^2+10\,\alpha\,\beta+9\,\beta^2\right)x^2-4\;\left(9\,\alpha^2-10\,\alpha\,\beta+9\,\beta^2\right)(\alpha+\beta)^2\right] $$
$$ {}^{001}P_{4,J}^{111}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-36\,\alpha^4-86\,\alpha^3\beta-52\,\alpha^2\beta^2-86\,\alpha\,\beta^3-36\,\beta^4\right] $$

$$ {}^{001}P_{4,I}^{000}(x) = 6\,\alpha^2\beta^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\;x^2 $$
$$ {}^{001}P_{4,I}^{001}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-18\,\alpha^4-52\,\alpha^3\beta-116\,\alpha^2\beta^2-52\,\alpha\,\beta^3-18\,\beta^4\right] $$
$$ {}^{001}P_{4,I}^{010}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+36\,\alpha^4+86\,\alpha^3\beta+58\,\alpha^2\beta^2-34\,\alpha\,\beta^3-18\,\beta^4\right] $$
$$ {}^{001}P_{4,I}^{011}(x) = \alpha^2\left[9\,\beta^2\,(\alpha+2\,\beta)\,(3\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2+4\,\alpha\;\left(9\,\alpha^3+35\,\beta\,\alpha^2+49\,\alpha\,\beta^2+35\,\beta^3\right)\right] $$
$$ {}^{001}P_{4,I}^{100}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2-18\,\alpha^4-34\,\alpha^3\beta+58\,\alpha^2\beta^2+86\,\alpha\,\beta^3+36\,\beta^4\right] $$
$$ {}^{001}P_{4,I}^{101}(x) = -\beta^2[9\,\alpha^2\,(2\,\alpha+\beta)\,(\alpha-3\,\beta)\,(\alpha+\beta)^2\,x^2-4\,\beta\;\left(35\,\alpha^3+49\,\beta\,\alpha^2+35\,\alpha\,\beta^2+9\,\beta^3\right)] $$
$$ {}^{001}P_{4,I}^{110}(x) = -\,(\alpha+\beta)^2\;\left[3\,\alpha^2\beta^2\left(9\,\alpha^2+10\,\alpha\,\beta+9\,\beta^2\right)x^2+4\;\left(9\,\alpha^2-10\,\alpha\,\beta+9\,\beta^2\right)(\alpha+\beta)^2\right] $$
$$ {}^{001}P_{4,I}^{111}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\left[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2+36\,\alpha^4+86\,\alpha^3\beta+52\,\alpha^2\beta^2+86\,\alpha\,\beta^3+36\,\beta^4\right] $$

$$^{001}P^{000}_{4,K}(x) = -6\,\alpha^2\beta^2\,(\alpha+3\,\beta)\,(3\,\alpha+\beta)\,(\alpha+\beta)^2\;x^2$$

$$^{001}P^{001}_{4,K}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 18\,\alpha^4 - 52\,\alpha^3\beta - 116\,\alpha^2\beta^2 - 52\,\alpha\,\beta^3 - 18\,\beta^4]$$

$$^{001}P^{010}_{4,K}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 36\,\alpha^4 + 86\,\alpha^3\beta + 58\,\alpha^2\beta^2 - 34\,\alpha\,\beta^3 - 18\,\beta^4]$$

$$^{001}P^{011}_{4,K}(x) = -\alpha^2\,[9\,\beta^2\,(\alpha+2\,\beta)\,(3\,\alpha-\beta)\,(\alpha+\beta)^2\,x^2 + 4\,\alpha\;\left(9\,\alpha^3 + 35\,\beta\,\alpha^2 + 49\,\alpha\,\beta^2 + 35\,\beta^3\right)]$$

$$^{001}P^{100}_{4,K}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 18\,\alpha^4 - 34\,\alpha^3\beta + 58\,\alpha^2\beta^2 + 86\,\alpha\,\beta^3 + 36\,\beta^4]$$

$$^{001}P^{101}_{4,K}(x) = \beta^2\,[9\,\alpha^2\,(2\,\alpha+\beta)\,(\alpha-3\,\beta)\,(\alpha+\beta)^2\,x^2 - 4\,\beta\;\left(35\,\alpha^3 + 49\,\beta\,\alpha^2 + 35\,\alpha\,\beta^2 + 9\,\beta^3\right)]$$

$$^{001}P^{110}_{4,K}(x) = (\alpha+\beta)^2\;[3\,\alpha^2\beta^2\left(9\,\alpha^2 + 10\,\alpha\,\beta + 9\,\beta^2\right)x^2 + 4\;\left(9\,\alpha^2 - 10\,\alpha\,\beta + 9\,\beta^2\right)(\alpha+\beta)^2]$$

$$^{001}P^{111}_{4,K}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 36\,\alpha^4 + 86\,\alpha^3\beta + 52\,\alpha^2\beta^2 + 86\,\alpha\,\beta^3 + 36\,\beta^4]$$

$$^{010}V_4 = \frac{x^2}{210\,\alpha^3\beta^3\,(\alpha+\beta)^3}$$

$$^{010}P^{000}_{4,J}(x) = -6\,\alpha^2\beta^2\,(2\,\alpha+3\,\beta)\,(2\,\alpha-\beta)\,(\alpha+\beta)^2\;x^2$$

$$^{010}P^{001}_{4,J}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 24\,\alpha^4 - 60\,\alpha^3\beta - 52\,\alpha^2\beta^2 + 38\,\alpha\,\beta^3 + 18\,\beta^4]$$

$$^{010}P^{010}_{4,J}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 48\,\alpha^4 + 96\,\alpha^3\beta + 68\,\alpha^2\beta^2 + 20\,\alpha\,\beta^3 + 18\,\beta^4]$$

$$^{010}P^{011}_{4,J}(x) = \alpha^2\,[9\,\beta^2\,(\alpha+2\,\beta)\,(4\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2 - 4\,\alpha\;\left(12\,\alpha^3 + 42\,\beta\,\alpha^2 + 56\,\alpha\,\beta^2 + 35\,\beta^3\right)]$$

$$^{010}P^{100}_{4,J}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 24\,\alpha^4 - 36\,\alpha^3\beta - 16\,\alpha^2\beta^2 - 58\,\alpha\,\beta^3 - 36\,\beta^4]$$

$$^{010}P^{101}_{4,J}(x) = -\beta^2\,[3\,\alpha^2\left(8\,\alpha^2 + 8\,\alpha\,\beta + 9\,\beta^2\right)(\alpha+\beta)^2\,x^2 - 4\,\beta^2\left(28\,\alpha^2 + 28\,\alpha\,\beta + 9\,\beta^2\right)]$$

$$^{010}P^{110}_{4,J}(x) =$$

$$= -\,(\alpha+\beta)^2\;[9\,\alpha^2\beta^2\,(4\,\alpha+3\,\beta)\,(\alpha-\beta)\,x^2 - 4\;(\alpha+\beta)\left(12\,\alpha^3 - 6\,\beta\,\alpha^2 + 8\,\alpha\,\beta^2 - 9\,\beta^3\right)]$$

$$^{010}P^{111}_{4,J}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 48\,\alpha^4 + 96\,\alpha^3\beta - 10\,\alpha^2\beta^2 - 58\,\alpha\,\beta^3 - 36\,\beta^4]$$

$$^{010}P^{000}_{4,I}(x) = 6\,\alpha^2\beta^2\,(2\,\alpha+3\,\beta)\,(2\,\alpha-\beta)\,(\alpha+\beta)^2\;x^2$$

$$^{010}P^{001}_{4,I}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 24\,\alpha^4 + 60\,\alpha^3\beta + 52\,\alpha^2\beta^2 - 38\,\alpha\,\beta^3 - 18\,\beta^4]$$

$$^{010}P^{010}_{4,I}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 48\,\alpha^4 - 96\,\alpha^3\beta - 68\,\alpha^2\beta^2 - 20\,\alpha\,\beta^3 - 18\,\beta^4]$$

$$^{010}P^{011}_{4,I}(x) = \alpha^2\,[9\,\beta^2\,(\alpha+2\,\beta)\,(4\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2 + 4\,\alpha\;\left(12\,\alpha^3 + 42\,\beta\,\alpha^2 + 56\,\alpha\,\beta^2 + 35\,\beta^3\right)]$$

$$^{010}P^{100}_{4,I}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 24\,\alpha^4 + 36\,\alpha^3\beta + 16\,\alpha^2\beta^2 + 58\,\alpha\,\beta^3 + 36\,\beta^4]$$

$$^{010}P^{101}_{4,I}(x) = -\beta^2\,[3\,\alpha^2\left(8\,\alpha^2 + 8\,\alpha\,\beta + 9\,\beta^2\right)(\alpha+\beta)^2\,x^2 + 4\,\beta^2\left(28\,\alpha^2 + 28\,\alpha\,\beta + 9\,\beta^2\right)]$$

$$^{010}P^{110}_{4,I}(x) =$$

$$= -\,(\alpha+\beta)^2\;[9\,\alpha^2\beta^2\,(4\,\alpha+3\,\beta)\,(\alpha-\beta)\,x^2 + 4\;(\alpha+\beta)\left(12\,\alpha^3 - 6\,\beta\,\alpha^2 + 8\,\alpha\,\beta^2 - 9\,\beta^3\right)]$$

$$^{010}P^{111}_{4,I}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 48\,\alpha^4 - 96\,\alpha^3\beta + 10\,\alpha^2\beta^2 + 58\,\alpha\,\beta^3 + 36\,\beta^4]$$

$$^{010}P^{000}_{4,K}(x) = -6\,\alpha^2\beta^2\,(2\,\alpha+3\,\beta)\,(2\,\alpha-\beta)\,(\alpha+\beta)^2\;x^2$$

$$^{010}P^{001}_{4,K}(x) = -\alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 24\,\alpha^4 + 60\,\alpha^3\beta + 52\,\alpha^2\beta^2 - 38\,\alpha\,\beta^3 - 18\,\beta^4]$$

$$^{010}P^{010}_{4,K}(x) = \alpha\,\beta\;(\alpha+\beta)\;x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 48\,\alpha^4 - 96\,\alpha^3\beta - 68\,\alpha^2\beta^2 - 20\,\alpha\,\beta^3 - 18\,\beta^4]$$

$$^{010}P^{011}_{4,K}(x) =$$

$$= -\alpha^2\,[9\,\beta^2\,(\alpha+2\,\beta)\,(4\,\alpha+\beta)\,(\alpha+\beta)^2\,x^2 + 4\,\alpha\;\left(12\,\alpha^3 + 42\,\beta\,\alpha^2 + 56\,\alpha\,\beta^2 + 35\,\beta^3\right)]$$

$$ {}^{010}P^{100}_{4,K}(x) = \alpha\,\beta\ (\alpha+\beta)\ x\,[15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 24\,\alpha^4 + 36\,\alpha^3\beta + 16\,\alpha^2\beta^2 + 58\,\alpha\,\beta^3 + 36\,\beta^4] $$

$$ {}^{010}P^{101}_{4,K}(x) = \beta^2\,[3\,\alpha^2\left(8\,\alpha^2 + 8\,\alpha\,\beta + 9\,\beta^2\right)(\alpha+\beta)^2\,x^2 + 4\,\beta^2\left(28\,\alpha^2 + 28\,\alpha\,\beta + 9\,\beta^2\right)] $$

$$ {}^{010}P^{110}_{4,K}(x) = $$

$$ = (\alpha+\beta)^2\ [9\,\alpha^2\beta^2\,(4\,\alpha + 3\,\beta)\,(\alpha-\beta)\,x^2 + 4\ (\alpha+\beta)\left(12\,\alpha^3 - 6\,\beta\,\alpha^2 + 8\,\alpha\,\beta^2 - 9\,\beta^3\right)] $$

$$ {}^{010}P^{111}_{4,K}(x) = \alpha\,\beta\ (\alpha+\beta)\ x\,[-15\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 48\,\alpha^4 + 96\,\alpha^3\beta - 10\,\alpha^2\beta^2 - 58\,\alpha\,\beta^3 - 36\,\beta^4] $$

$$ {}^{111}V_4 = \frac{x^2}{70\,\alpha^3\beta^3\,(\alpha+\beta)^3} $$

$$ {}^{111}P^{000}_{4,J}(x) = -8\,\alpha^2\beta^2\left(\alpha^2 + \alpha\,\beta + \beta^2\right)(\alpha+\beta)^2\ x^2 $$

$$ {}^{111}P^{001}_{4,J}(x) = -\alpha\,x\beta\ (\alpha+\beta)\ [5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 8\,\alpha^4 - 20\,\alpha^3\beta - 8\,\alpha^2\beta^2 - 20\,\beta^3\alpha - 8\,\beta^4] $$

$$ {}^{111}P^{010}_{4,J}(x) = \alpha\beta\ (\alpha+\beta)\ x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 16\,\alpha^4 + 32\,\alpha^3\beta + 4\,\alpha^2\beta^2 - 12\,\beta^3\alpha - 8\,\beta^4] $$

$$ {}^{111}P^{011}_{4,J}(x) = 4\alpha^2\,[\beta^2\left(3\,\alpha^2 - 2\,\alpha\,\beta - 2\,\beta^2\right)(\alpha+\beta)^2\,x^2 - 2\,\alpha^2\left(2\,\alpha^2 + 7\,\alpha\,\beta + 7\,\beta^2\right)] $$

$$ {}^{111}P^{100}_{4,J}(x) = \alpha\,x\beta\ (\alpha+\beta)\ [5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 8\,\alpha^4 - 12\,\alpha^3\beta + 4\,\alpha^2\beta^2 + 32\,\beta^3\alpha + 16\,\beta^4] $$

$$ {}^{111}P^{101}_{4,J}(x) = -4\,\beta^2\,[\alpha^2\left(2\,\alpha^2 + 2\,\alpha\,\beta - 3\,\beta^2\right)(\alpha+\beta)^2\,x^2 + 2\,\beta^2\left(7\,\alpha^2 + 7\,\alpha\,\beta + 2\,\beta^2\right)] $$

$$ {}^{111}P^{110}_{4,J}(x) = -4\ (\alpha+\beta)^2\ [\alpha^2\beta^2\left(3\,\alpha^2 + 8\,\alpha\,\beta + 3\,\beta^2\right)x^2 - 2\ \left(2\,\alpha^2 - 3\,\alpha\,\beta + 2\,\beta^2\right)(\alpha+\beta)^2] $$

$$ {}^{111}P^{111}_{4,J}(x) = \alpha\,\beta\ (\alpha+\beta)\ x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 16\ \left(\alpha^2 + \alpha\,\beta + \beta^2\right)^2] $$

$$ {}^{111}P^{000}_{4,I}(x) = -8\,\alpha^2\beta^2\left(\alpha^2 + \alpha\,\beta + \beta^2\right)(\alpha+\beta)^2\ x^2 $$

$$ {}^{111}P^{001}_{4,I}(x) = \alpha\,\beta\ (\alpha+\beta)\ x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 8\,\alpha^4 + 20\,\alpha^3\beta + 8\,\alpha^2\beta^2 + 20\,\beta^3\alpha + 8\,\beta^4] $$

$$ {}^{111}P^{010}_{4,I}(x) = -\alpha\,\beta\ (\alpha+\beta)\ x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 16\,\alpha^4 - 32\,\alpha^3\beta - 4\,\alpha^2\beta^2 + 12\,\beta^3\alpha + 8\,\beta^4] $$

$$ {}^{111}P^{011}_{4,I}(x) = -4\,\alpha^2\,[\beta^2\left(3\,\alpha^2 - 2\,\alpha\,\beta - 2\,\beta^2\right)(\alpha+\beta)^2\,x^2 + 2\,\alpha^2\left(2\,\alpha^2 + 7\,\alpha\,\beta + 7\,\beta^2\right)] $$

$$ {}^{111}P^{100}_{4,I}(x) = -\alpha\,\beta\ (\alpha+\beta)\ x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 8\,\alpha^4 + 12\,\alpha^3\beta - 4\,\alpha^2\beta^2 - 32\,\beta^3\alpha - 16\,\beta^4] $$

$$ {}^{111}P^{101}_{4,I}(x) = 4\,\beta^2\,[\alpha^2\left(2\,\alpha^2 + 2\,\alpha\,\beta - 3\,\beta^2\right)(\alpha+\beta)^2\,x^2 - 2\,\beta^2\left(7\,\alpha^2 + 7\,\alpha\,\beta + 2\,\beta^2\right)] $$

$$ {}^{111}P^{110}_{4,I}(x) = 4\ (\alpha+\beta)^2\ [\alpha^2\beta^2\left(3\,\alpha^2 + 8\,\alpha\,\beta + 3\,\beta^2\right)x^2 + 2\ \left(2\,\alpha^2 - 3\,\alpha\,\beta + 2\,\beta^2\right)(\alpha+\beta)^2] $$

$$ {}^{111}P^{111}_{4,I}(x) = \alpha\,\beta\ (\alpha+\beta)\ x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 16\ \left(\alpha^2 + \alpha\,\beta + \beta^2\right)^2] $$

$$ {}^{111}P^{000}_{4,K}(x) = 8\,\alpha^2\beta^2\left(\alpha^2 + \alpha\,\beta + \beta^2\right)(\alpha+\beta)^2\ x^2 $$

$$ {}^{111}P^{001}_{4,K}(x) = \alpha\,\beta\ (\alpha+\beta)\ x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 8\,\alpha^4 + 20\,\alpha^3\beta + 8\,\alpha^2\beta^2 + 20\,\beta^3\alpha + 8\,\beta^4] $$

$$ {}^{111}P^{010}_{4,K}(x) = -\alpha\,\beta\ (\alpha+\beta)\ x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 16\,\alpha^4 - 32\,\alpha^3\beta - 4\,\alpha^2\beta^2 + 12\,\beta^3\alpha + 8\,\beta^4] $$

$$ {}^{111}P^{011}_{4,K}(x) = 4\,\alpha^2\,[\beta^2\left(3\,\alpha^2 - 2\,\alpha\,\beta - 2\,\beta^2\right)(\alpha+\beta)^2\,x^2 + 2\,\alpha^2\left(2\,\alpha^2 + 7\,\alpha\,\beta + 7\,\beta^2\right)] $$

$$ {}^{111}P^{100}_{4,K}(x) = -\alpha\,\beta\ (\alpha+\beta)\ x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 + 8\,\alpha^4 + 12\,\alpha^3\beta - 4\,\alpha^2\beta^2 - 32\,\beta^3\alpha - 16\,\beta^4] $$

$$ {}^{111}P^{101}_{4,K}(x) = -4\,\beta^2\,[\alpha^2\left(2\,\alpha^2 + 2\,\alpha\,\beta - 3\,\beta^2\right)(\alpha+\beta)^2\,x^2 - 2\,\beta^2\left(7\,\alpha^2 + 7\,\alpha\,\beta + 2\,\beta^2\right)] $$

$$ {}^{111}P^{110}_{4,K}(x) = -4\ (\alpha+\beta)^2\ [\alpha^2\beta^2\left(3\,\alpha^2 + 8\,\alpha\,\beta + 3\,\beta^2\right)x^2 + 2\ \left(2\,\alpha^2 - 3\,\alpha\,\beta + 2\,\beta^2\right)(\alpha+\beta)^2] $$

$$ {}^{111}P^{111}_{4,K}(x) = \alpha\,\beta\ (\alpha+\beta)\ x\,[5\,\alpha^2\beta^2\,(\alpha+\beta)^2\,x^2 - 16\ \left(\alpha^2 + \alpha\,\beta + \beta^2\right)^2] $$

Recurrence relations:

$$\int x^{2n+1} J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \Big\{ -4n^2\beta(2\alpha+\beta) \int x^{2n} J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x)\,dx -$$

$$-4n^2(\alpha^2-\beta^2) \int x^{2n} J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x)\,dx - 4n(n-1)\alpha^2 \int x^{2n} J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\,dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) + 2n\beta(2\alpha+\beta)\, J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) +$$

$$+2n(\alpha^2-\beta^2)\, J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) +$$

$$+ 2n\alpha^2\, J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)] \Big\}$$

$$\int x^{2n+1} J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \Big\{ 2n\beta(\alpha-2n\beta) \int x^{2n} J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x)\,dx +$$

$$+2n[(2n+1)\alpha+2n\beta](\alpha+\beta) \int x^{2n} J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x)\,dx +$$

$$+2(2n+1)(n-1)\alpha^2 \int x^{2n} J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\,dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) +$$

$$+\beta(2n\beta-\alpha)\, J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) - (\alpha+\beta)((2n+1)\alpha+2n\beta)\, J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, J_1(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) -$$

$$- \alpha\beta(\alpha+\beta)x\, J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) - (2n+1)\alpha^2\, J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)] \Big\}$$

$$\int x^{2n+1} J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \Big\{ -2n\beta[(2n+1)\beta+\alpha] \int x^{2n} J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x)\,dx -$$

$$-2n(2n+1)(\alpha^2-\beta^2) \int x^{2n} J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x)\,dx -$$

$$-2(n-1)\alpha[(2n+1)\alpha+(4n+1)\beta] \int x^{2n} J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)\,dx +$$

$$+x^{2n+1}[-\alpha\beta(\alpha+\beta)x\, J_0(\alpha x) J_0(\beta x) J_0((\alpha+\beta)x) + \beta[(2n+1)\beta+\alpha]\, J_0(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) +$$

$$+(2n+1)(\alpha^2-\beta^2)\, J_0(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_0(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, J_1(\alpha x) J_0(\beta x) J_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, J_1(\alpha x) J_1(\beta x) J_0((\alpha+\beta)x) +$$

$$+\alpha[(4n+1)\beta+(2n+1)\alpha]\, J_1(\alpha x) J_1(\beta x) J_1((\alpha+\beta)x)] \Big\}$$

$$\int x^{2n+2}\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ 2(n+1)\beta[(4n+3)\alpha+(2n+1)\beta] \int x^{2n+1}\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)\, dx - \right.$$

$$-2n(2n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx +$$

$$+2\alpha n[(2n+1)\alpha-\beta] \int x^{2n+1}\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\big[-\beta[(4n+3)\alpha+(2n+1)\beta]\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) +$$

$$+(2n+1)(\alpha^2-\beta^2)\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) -$$

$$\left. -\alpha[(2n+1)\alpha-\beta]\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)\, x\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\big] \right\}$$

$$\int x^{2n+2}\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ -2\beta^2(n+1)(2n+1) \int x^{2n+1}\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)\, dx - \right.$$

$$-2n(\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta] \int x^{2n+1}\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx +$$

$$+2n\alpha[2(n+1)\alpha+\beta] \int x^{2n+1}\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\big[(2n+1)\beta^2\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) + (\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta]\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) - \alpha[2(n+1)\alpha+\beta)]\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) +$$

$$\left. +\alpha\beta(\alpha+\beta)\, x\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\big] \right\}$$

$$\int x^{2n+2}\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ 4\beta^2(n+1)^2 \int x^{2n+1}\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x)\, dx - \right.$$

$$-4n(n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\, dx +$$

$$+4n(n+1)\alpha(\alpha+2\beta) \int x^{2n+1}\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\big[-2(n+1)\beta^2\, J_0(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_0(\beta x)\, J_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, J_0(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) + 2(n+1)(\alpha^2-\beta^2)\, J_0(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_0(\beta x)\, J_0((\alpha+\beta)x) - 2(n+1)\alpha(\alpha+2\beta)\, J_1(\alpha x)\, J_1(\beta x)\, J_0((\alpha+\beta)x) +$$

$$\left. +\alpha\beta(\alpha+\beta)x\, J_1(\alpha x)\, J_1(\beta x)\, J_1((\alpha+\beta)x)\big] \right\}$$

$$\int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ -4n^2\beta(2\alpha+\beta) \int x^{2n} I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)\, dx - \right.$$

$$-4n^2(\alpha^2-\beta^2) \int x^{2n}\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx + 4n(n-1)\alpha^2 \int x^{2n}\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) + 2n\beta(2\alpha+\beta)\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) +$$

$$+2n(\alpha^2-\beta^2)\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) -$$

$$\left. - 2n\alpha^2\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ -2n\beta(\alpha-2n\beta) \int x^{2n}\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)\, dx - \right.$$

$$-2n[(2n+1)\alpha+2n\beta](\alpha+\beta) \int x^{2n}\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx +$$

$$+2(2n+1)(n-1)\alpha^2 \int x^{2n}\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[-\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) -$$

$$-\beta(2n\beta-\alpha)\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) + (\alpha+\beta)((2n+1)\alpha+2n\beta)\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) -$$

$$\left. - \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) - (2n+1)\alpha^2\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)} \left\{ 2n\beta[(2n+1)\beta+\alpha] \int x^{2n}\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)\, dx + \right.$$

$$+2n(2n+1)(\alpha^2-\beta^2) \int x^{2n}\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx -$$

$$-2(n-1)\alpha[(2n+1)\alpha+(4n+1)\beta] \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx +$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) - \beta[(2n+1)\beta+\alpha]\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) -$$

$$-(2n+1)(\alpha^2-\beta^2)\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) +$$

$$\left. +\alpha[(4n+1)\beta+(2n+1)\alpha]\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)] \right\}$$

$$\int x^{2n+2}\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ -2(n+1)\beta[(4n+3)\alpha+(2n+1)\beta] \int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx - \right.$$

$$-2n(2n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx +$$

$$+2\alpha n[(2n+1)\alpha-\beta] \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\Big[\beta[(4n+3)\alpha+(2n+1)\beta]\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) +$$

$$+(2n+1)(\alpha^2-\beta^2)\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) -$$

$$\left. -\alpha[(2n+1)\alpha-\beta]\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)\, x\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\Big] \right\}$$

$$\int x^{2n+2}\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ 2\beta^2(n+1)(2n+1) \int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx - \right.$$

$$-2n(\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta] \int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx +$$

$$+2n\alpha[2(n+1)\alpha+\beta] \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\Big[-(2n+1)\beta^2\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) + (\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta]\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) - \alpha[2(n+1)\alpha+\beta)]\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) -$$

$$\left. -\alpha\beta(\alpha+\beta)\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\Big] \right\}$$

$$\int x^{2n+2}\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ 4\beta^2(n+1)^2 \int x^{2n+1}\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x)\, dx + \right.$$

$$+4n(n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\, dx -$$

$$-4n(n+1)\alpha(\alpha+2\beta) \int x^{2n+1}\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\Big[-2(n+1)\beta^2\, I_0(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_0(\beta x)\, I_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, I_0(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) - 2(n+1)(\alpha^2-\beta^2)\, I_0(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_0(\beta x)\, I_0((\alpha+\beta)x) + 2(n+1)\alpha(\alpha+2\beta)\, I_1(\alpha x)\, I_1(\beta x)\, I_0((\alpha+\beta)x) +$$

$$\left. +\alpha\beta(\alpha+\beta)x\, I_1(\alpha x)\, I_1(\beta x)\, I_1((\alpha+\beta)x)\Big] \right\}$$

$$\int x^{2n+1}\,K_0(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)}\,\Big\{4n^2\beta(2\alpha+\beta)\int x^{2n}K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)\,dx\,+$$

$$+4n^2(\alpha^2-\beta^2)\int x^{2n}\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx-4n(n-1)\alpha^2\int x^{2n}\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx+$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)-2n\beta(2\alpha+\beta)\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)-$$

$$-2n(\alpha^2-\beta^2)\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)+$$

$$+\,2n\alpha^2\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)]\Big\}$$

$$\int x^{2n+1}\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)}\,\Big\{2n\beta(\alpha-2n\beta)\int x^{2n}\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)\,dx+$$

$$+2n[(2n+1)\alpha+2n\beta](\alpha+\beta)\int x^{2n}\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx-$$

$$-2(2n+1)(n-1)\alpha^2\int x^{2n}\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx+$$

$$+x^{2n+1}[-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)+$$

$$+\beta(2n\beta-\alpha)\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)-(\alpha+\beta)((2n+1)\alpha+2n\beta)\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)+$$

$$+\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)-$$

$$-\,\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)+(2n+1)\alpha^2\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)]\Big\}$$

$$\int x^{2n+1}\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx =$$

$$= \frac{1}{2(4n+1)\alpha\beta(\alpha+\beta)}\,\Big\{-2n\beta[(2n+1)\beta+\alpha]\int x^{2n}\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)\,dx-$$

$$-2n(2n+1)(\alpha^2-\beta^2)\int x^{2n}\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)\,dx+$$

$$+2(n-1)\alpha[(2n+1)\alpha+(4n+1)\beta]\int x^{2n+1}\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)\,dx+$$

$$+x^{2n+1}[\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_0(\beta x)\,K_0((\alpha+\beta)x)+\beta[(2n+1)\beta+\alpha]\,K_0(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)+$$

$$+(2n+1)(\alpha^2-\beta^2)\,K_0(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)-\alpha\beta(\alpha+\beta)x\,K_0(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)-$$

$$-\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)\,K_0(\beta x)\,K_1((\alpha+\beta)x)+\alpha\beta(\alpha+\beta)x\,K_1(\alpha x)\,K_1(\beta x)\,K_0((\alpha+\beta)x)-$$

$$-\alpha[(4n+1)\beta+(2n+1)\alpha]\,K_1(\alpha x)\,K_1(\beta x)\,K_1((\alpha+\beta)x)]\Big\}$$

$$\int x^{2n+2}\, K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ 2(n+1)\beta[(4n+3)\alpha+(2n+1)\beta] \int x^{2n+1}\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x)\, dx + \right.$$

$$+2n(2n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx -$$

$$-2\alpha n[(2n+1)\alpha-\beta] \int x^{2n+1}\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\Big[-\beta[(4n+3)\alpha+(2n+1)\beta]\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) -$$

$$-(2n+1)(\alpha^2-\beta^2)\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, K_1(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x) +$$

$$\left. +\alpha[(2n+1)\alpha-\beta]\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)\, x\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\Big] \right\}$$

$$\int x^{2n+2}\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ -2\beta^2(n+1)(2n+1) \int x^{2n+1}\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x)\, dx + \right.$$

$$+2n(\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta] \int x^{2n+1}\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx -$$

$$-2n\alpha[2(n+1)\alpha+\beta] \int x^{2n+1}\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\Big[(2n+1)\beta^2\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x) - \alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) - (\alpha+\beta)[2(n+1)\alpha+(2n+1)\beta]\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x) +$$

$$+\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x) + \alpha[2(n+1)\alpha+\beta)]\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) -$$

$$\left. -\alpha\beta(\alpha+\beta)\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\Big] \right\}$$

$$\int x^{2n+2}\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx =$$

$$= \frac{1}{2(4n+3)\alpha\beta(\alpha+\beta)} \left\{ -4\beta^2(n+1)^2 \int x^{2n+1}\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x)\, dx - \right.$$

$$-4n(n+1)(\alpha^2-\beta^2) \int x^{2n+1}\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\, dx +$$

$$+4n(n+1)\alpha(\alpha+2\beta) \int x^{2n+1}\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x)\, dx +$$

$$+x^{2n+2}\Big[2(n+1)\beta^2\, K_0(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x) + \alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_0(\beta x)\, K_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, K_0(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) + 2(n+1)(\alpha^2-\beta^2)\, K_0(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x) -$$

$$-\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)\, K_0(\beta x)\, K_0((\alpha+\beta)x) - 2(n+1)\alpha(\alpha+2\beta)\, K_1(\alpha x)\, K_1(\beta x)\, K_0((\alpha+\beta)x) +$$

$$\left. +\alpha\beta(\alpha+\beta)x\, K_1(\alpha x)\, K_1(\beta x)\, K_1((\alpha+\beta)x)\Big] \right\}$$

c) $x^n\,Z_\kappa(\alpha x)\,Z_\mu(\beta x)\,Z_\nu(\sqrt{\alpha^2\pm\beta^2}\,x)$

Formulas with $\sqrt{\alpha^2+\beta^2}$ were found for the following integrals.
Replacing β by $\beta\,i$ one gets some modifications.

$$\int x\,J_1(\alpha x)\,J_1(\beta x)\,J_0(\sqrt{\alpha^2+\beta^2}\,x)\,dx=$$

$$=\frac{x}{2\alpha\beta}\left[\sqrt{\alpha^2+\beta^2}\,J_0(\alpha x)\,J_0(\beta x)\,J_1(\sqrt{\alpha^2+\beta^2}\,x)-\beta\,J_0(\alpha x)\,J_1(\beta x)\,J_0(\sqrt{\alpha^2+\beta^2}\,x)-\right.$$

$$\left.-\,\alpha\,J_1(\alpha x)\,J_0(\beta x)\,J_0(\sqrt{\alpha^2+\beta^2}\,x)\right]$$

$$\int x\,I_1(\alpha x)\,I_1(\beta x)\,I_0(\sqrt{\alpha^2+\beta^2}\,x)\,dx=$$

$$=\frac{x}{2\alpha\beta}\left[-\sqrt{\alpha^2+\beta^2}\,I_0(\alpha x)\,I_0(\beta x)\,I_1(\sqrt{\alpha^2+\beta^2}\,x)+\beta\,I_0(\alpha x)\,I_1(\beta x)\,I_0(\sqrt{\alpha^2+\beta^2}\,x)+\right.$$

$$\left.+\,\alpha\,I_1(\alpha x)\,I_0(\beta x)\,I_0(\sqrt{\alpha^2+\beta^2}\,x)\right]$$

$$\int x\,K_1(\alpha x)\,K_1(\beta x)\,K_0(\sqrt{\alpha^2+\beta^2}\,x)\,dx=$$

$$=-\frac{x}{2\alpha\beta}\left[\sqrt{\alpha^2+\beta^2}\,K_0(\alpha x)\,K_0(\beta x)\,K_1(\sqrt{\alpha^2+\beta^2}\,x)-\beta\,K_0(\alpha x)\,K_1(\beta x)\,K_0(\sqrt{\alpha^2+\beta^2}\,x)-\right.$$

$$\left.-\,\alpha\,K_1(\alpha x)\,K_0(\beta x)\,K_0(\sqrt{\alpha^2+\beta^2}\,x)\right]$$

$$\int x\,J_1(\alpha x)\,I_1(\beta x)\,J_0(\sqrt{\alpha^2-\beta^2}\,x)\,dx=$$

$$=\frac{x}{2\alpha\beta}\left[-\,\sqrt{\alpha^2-\beta^2}\,J_0(\alpha x)\,I_0(\beta x)\,J_1(\sqrt{\alpha^2-\beta^2}\,x)-\beta\,J_0(\alpha x)\,I_1(\beta x)\,J_0(\sqrt{\alpha^2-\beta^2}\,x)+\right.$$

$$\left.+\,\alpha\,J_1(\alpha x)\,I_0(\beta x)\,J_0(\sqrt{\alpha^2-\beta^2}\,x)\right]$$

$$\int x\,I_1(\alpha x)\,J_1(\beta x)\,I_0(\sqrt{\alpha^2-\beta^2}\,x)\,dx=$$

$$=\frac{x}{2\alpha\beta}\left[\sqrt{\alpha^2-\beta^2}\,I_0(\alpha x)\,J_0(\beta x)\,I_1(\sqrt{\alpha^2-\beta^2}\,x)+\beta\,I_0(\alpha x)\,J_1(\beta x)\,I_0(\sqrt{\alpha^2-\beta^2}\,x)-\right.$$

$$\left.-\,\alpha\,I_1(\alpha x)\,J_0(\beta x)\,I_0(\sqrt{\alpha^2-\beta^2}\,x)\right]$$

$$\int x^2\,J_0(\alpha x)\,J_0(\beta x)\,J_1(\sqrt{\alpha^2+\beta^2}\,x)\,dx=\frac{x^2}{2\alpha\beta}\left[\alpha\,J_0(\alpha\,x)J_1(\beta\,x)J_1(\sqrt{\alpha^2+\beta^2}x)+\right.$$

$$\left.+\beta\,J_1(\alpha\,x)J_0(\beta\,x)J_1(\sqrt{\alpha^2+\beta^2}x)-\sqrt{\alpha^2+\beta^2}J_1(\alpha\,x)J_1(\beta\,x)J_0(\sqrt{\alpha^2+\beta^2}x)\right]$$

$$\int x^2\,I_0(\alpha x)\,I_0(\beta x)\,I_1(\sqrt{\alpha^2+\beta^2}\,x)\,dx=\frac{x^2}{2\alpha\beta}\left[\alpha\,I_0(\alpha\,x)I_1(\beta\,x)I_1(\sqrt{\alpha^2+\beta^2}x)+\right.$$

$$\left.+\beta\,I_1(\alpha\,x)I_0(\beta\,x)I_1(\sqrt{\alpha^2+\beta^2}x)-\sqrt{\alpha^2+\beta^2}I_1(\alpha\,x)I_1(\beta\,x)I_0(\sqrt{\alpha^2+\beta^2}x)\right]$$

$$\int x^2\,K_0(\alpha x)\,K_0(\beta x)\,K_1(\sqrt{\alpha^2+\beta^2}\,x)\,dx=-\frac{x^2}{2\alpha\beta}\left[\alpha\,K_0(\alpha\,x)K_1(\beta\,x)K_1(\sqrt{\alpha^2+\beta^2}x)+\right.$$

$$\left.+\beta\,K_1(\alpha\,x)K_0(\beta\,x)K_1(\sqrt{\alpha^2+\beta^2}x)-\sqrt{\alpha^2+\beta^2}K_1(\alpha\,x)K_1(\beta\,x)K_0(\sqrt{\alpha^2+\beta^2}x)\right]$$

$$\int x^2\, J_0(\alpha x)\, I_0(\beta x)\, J_1(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x^2}{2\alpha\beta}\, \Big[\alpha\, J_0(\alpha\, x) I_1(\beta\, x) J_1(\sqrt{\alpha^2-\beta^2} x) +$$

$$+ \beta\, J_1(\alpha\, x) I_0(\beta\, x) J_1(\sqrt{\alpha^2-\beta^2} x) - \sqrt{\alpha^2-\beta^2} J_1(\alpha\, x) I_1(\beta\, x) J_0(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x^2\, I_0(\alpha x)\, J_0(\beta x)\, I_1(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x^2}{2\alpha\beta}\, \Big[\alpha\, I_0(\alpha\, x) J_1(\beta\, x) I_1(\sqrt{\alpha^2-\beta^2} x) +$$

$$+ \beta\, I_1(\alpha\, x) J_0(\beta\, x) I_1(\sqrt{\alpha^2-\beta^2} x) - \sqrt{\alpha^2-\beta^2} I_1(\alpha\, x) J_1(\beta\, x) I_0(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x\, J_0(\alpha x)\, J_1(\beta x)\, J_1(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2-\beta^2}}\, \Big[-\sqrt{\alpha^2-\beta^2}\, J_0(\alpha\, x) J_0(\beta\, x) J_1(\sqrt{\alpha^2-\beta^2} x) -$$

$$-\beta\, J_0(\alpha\, x) J_1(\beta\, x) J_0(\sqrt{\alpha^2-\beta^2} x) + \alpha\, J_1(\alpha\, x) J_0(\beta\, x) J_0(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x\, I_0(\alpha x)\, I_1(\beta x)\, I_1(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2-\beta^2}}\, \Big[\sqrt{\alpha^2-\beta^2}\, I_0(\alpha\, x) I_0(\beta\, x) I_1(\sqrt{\alpha^2-\beta^2} x) +$$

$$+\beta\, I_0(\alpha\, x) I_1(\beta\, x) I_0(\sqrt{\alpha^2-\beta^2} x) - \alpha\, I_1(\alpha\, x) I_0(\beta\, x) I_0(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x\, K_0(\alpha x)\, K_1(\beta x)\, K_1(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2-\beta^2}}\, \Big[-\sqrt{\alpha^2-\beta^2}\, K_0(\alpha\, x) K_0(\beta\, x) K_1(\sqrt{\alpha^2-\beta^2} x) -$$

$$-\beta\, K_0(\alpha\, x) K_1(\beta\, x) K_0(\sqrt{\alpha^2-\beta^2} x) + \alpha\, K_1(\alpha\, x) K_0(\beta\, x) K_0(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x\, J_0(\alpha x)\, I_1(\beta x)\, J_1(\sqrt{\alpha^2+\beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2+\beta^2}}\, \Big[\sqrt{\alpha^2+\beta^2}\, J_0(\alpha\, x) I_0(\beta\, x) J_1(\sqrt{\alpha^2+\beta^2} x) -$$

$$-\beta\, J_0(\alpha\, x) I_1(\beta\, x) J_0(\sqrt{\alpha^2+\beta^2} x) - \alpha\, J_1(\alpha\, x) I_0(\beta\, x) J_0(\sqrt{\alpha^2+\beta^2} x)\Big]$$

$$\int x\, I_0(\alpha x)\, J_1(\beta x)\, I_1(\sqrt{\alpha^2+\beta^2}\, x)\, dx = \frac{x}{2\beta\,\sqrt{\alpha^2+\beta^2}}\, \Big[-\sqrt{\alpha^2+\beta^2}\, I_0(\alpha\, x) J_0(\beta\, x) I_1(\sqrt{\alpha^2+\beta^2} x) +$$

$$+\beta\, I_0(\alpha\, x) J_1(\beta\, x) I_0(\sqrt{\alpha^2+\beta^2} x) + \alpha\, I_1(\alpha\, x) J_0(\beta\, x) I_0(\sqrt{\alpha^2+\beta^2} x)\Big]$$

$$\int x^2\, J_1(\alpha x)\, J_0(\beta x)\, J_0(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2-\beta^2}}\, \Big[\sqrt{\alpha^2-\beta^2}\, J_1(\alpha\, x) J_1(\beta\, x) J_0(\sqrt{\alpha^2-\beta^2} x) +$$

$$+\beta\, J_1(\alpha\, x) J_0(\beta\, x) J_1(\sqrt{\alpha^2-\beta^2} x) - \alpha\, J_0(\alpha\, x) J_1(\beta\, x) J_1(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x^2\, I_1(\alpha x)\, I_0(\beta x)\, I_0(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2-\beta^2}}\, \Big[\sqrt{\alpha^2-\beta^2}\, I_1(\alpha\, x) I_1(\beta\, x) I_0(\sqrt{\alpha^2-\beta^2} x) +$$

$$+\beta\, I_1(\alpha\, x) I_0(\beta\, x) I_1(\sqrt{\alpha^2-\beta^2} x) - \alpha\, I_0(\alpha\, x) I_1(\beta\, x) I_1(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x^2\, K_1(\alpha x)\, K_0(\beta x)\, K_0(\sqrt{\alpha^2-\beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2-\beta^2}}\, \Big[-\sqrt{\alpha^2-\beta^2}\, K_1(\alpha\, x) K_1(\beta\, x) K_0(\sqrt{\alpha^2-\beta^2} x) -$$

$$-\beta\, K_1(\alpha\, x) K_0(\beta\, x) K_1(\sqrt{\alpha^2-\beta^2} x) + \alpha\, K_0(\alpha\, x) K_1(\beta\, x) K_1(\sqrt{\alpha^2-\beta^2} x)\Big]$$

$$\int x^2\, J_1(\alpha x)\, I_0(\beta x)\, J_0(\sqrt{\alpha^2+\beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2+\beta^2}}\, \Big[\sqrt{\alpha^2+\beta^2}\, J_1(\alpha\, x) I_1(\beta\, x) J_0(\sqrt{\alpha^2+\beta^2} x)+$$

$$+\beta\, J_1(\alpha\, x) I_0(\beta\, x) J_1(\sqrt{\alpha^2+\beta^2} x) - \alpha\, J_0(\alpha\, x) I_1(\beta\, x) J_1(\sqrt{\alpha^2+\beta^2} x)\Big]$$

$$\int x^2\, I_1(\alpha x)\, J_0(\beta x)\, I_0(\sqrt{\alpha^2+\beta^2}\, x)\, dx = \frac{x^2}{2\beta\,\sqrt{\alpha^2+\beta^2}}\, \Big[\sqrt{\alpha^2+\beta^2}\, I_1(\alpha\, x) J_1(\beta\, x) I_0(\sqrt{\alpha^2+\beta^2} x)+$$

$$+\beta\, I_1(\alpha\, x) J_0(\beta\, x) I_1(\sqrt{\alpha^2+\beta^2} x) - \alpha\, I_0(\alpha\, x) J_1(\beta\, x) I_1(\sqrt{\alpha^2+\beta^2} x)\Big]$$

4. Products of four Bessel Functions

4.1. Integrals of the type $\int x^m Z_0^n(x) Z_1^{4-n}(x)\,dx$

4.1. a) Explicit Integrals

$$\int J_0^3(x)\,J_1(x)\,dx = -\frac{1}{4}\,J_0^4(x)$$

$$\int I_0^3(x)\,I_1(x)\,dx = \frac{1}{4}\,I_0^4(x)$$

$$\int K_0^3(x)\,K_1(x)\,dx = -\frac{1}{4}\,K_0^4(x)$$

$$\int J_0(x)J_1^3(x)\,dx = \frac{1}{4}\left[x^2J_0^4(x) - 2xJ_0^3(x)J_1(x) + 2x^2J_0^2(x)J_1^2(x) - 2xJ_0(x)J_1^3(x) + x^2J_1^4(x)\right]$$

$$\int I_0(x)I_1^3(x)\,dx = \frac{1}{4}\left[-x^2I_0^4(x) + 2xI_0^3(x)I_1(x) + 2x^2I_0^2(x)I_1^2(x) - 2xI_0(x)I_1^3(x) - x^2I_1^4(x)\right]$$

$$\int K_0(x)K_1^3(x)\,dx = \frac{1}{4}\left[x^2K_0^4(x) + 2xK_0^3(x)K_1(x) - 2x^2K_0^2(x)K_1^2(x) - 2xK_0(x)K_1^3(x) + x^2K_1^4(x)\right]$$

$$\int x^4\,J_0(x)\,J_1^3(x)\,dx = \frac{x^4}{4}\,J_1^4(x)$$ (See also p. 498.)

$$\int x^4\,I_0(x)\,I_1^3(x)\,dx = \frac{x^4}{4}\,I_1^4(x)$$

$$\int x^4\,K_0(x)\,K_1^3(x)\,dx = -\frac{x^4}{4}\,K_1^4(x)$$

$$\int \frac{J_0^2(x)J_1^2(x)}{x}\,dx = -\frac{x^2+1}{4}J_0^4(x) + \frac{x}{2}J_0^3(x)J_1(x) - \frac{x^2+1}{2}J_0^2(x)J_1^2(x) + \frac{x}{2}J_0(x)J_1^3(x) - \frac{x^2}{4}J_1^4(x)$$

$$\int \frac{I_0^2(x)I_1^2(x)}{x}\,dx = -\frac{x^2-1}{4}I_0^4(x) + \frac{x}{2}I_0^3(x)I_1(x) + \frac{x^2-1}{2}I_0^2(x)I_1^2(x) - \frac{x}{2}I_0(x)I_1^3(x) - \frac{x^2}{4}I_1^4(x)$$

$$\int \frac{K_0^2(x)K_1^2(x)}{x}\,dx = -\frac{x^2-1}{4}K_0^4(x) - \frac{x}{2}K_0^3(x)K_1(x) + \frac{x^2-1}{2}K_0^2(x)K_1^2(x) + \frac{x}{2}K_0(x)K_1^3(x) - \frac{x^2}{4}K_1^4(x)$$

$$\int \frac{J_1^4(x)}{x}\,dx = \frac{1}{4}\left[x^2J_0^4(x) - 2xJ_0^3(x)J_1(x) + 2x^2J_0^2(x)J_1^2(x) - 2xJ_0(x)J_1^3(x) + (x^2-1)J_1^4(x)\right]$$

$$\int \frac{I_1^4(x)}{x}\,dx = \frac{1}{4}\left[-x^2I_0^4(x) + 2xJ_I^3(x)I_1(x) + 2x^2I_0^2(x)I_1^2(x) - 2xI_0(x)I_1^3(x) - (x^2+1)I_1^4(x)\right]$$

$$\int \frac{K_1^4(x)}{x}\,dx =$$

$$= \frac{1}{4}\left[-x^2K_0^4(x) - 2xK_I^3(x)K_1(x) + 2x^2K_0^2(x)K_1^2(x) + 2xK_0(x)K_1^3(x) - (x^2+1)K_1^4(x)\right]$$

$$\int \frac{J_0(x)J_1^3(x)}{x^2}\,dx = -\frac{4x^2+3}{16}J_0^4(x) + \frac{x}{2}J_0^3(x)J_1(x) - \frac{4x^2+3}{8}J_0^2(x)J_1^2(x) + \frac{2x^2-1}{4x}J_0(x)J_1^3(x) - \frac{4x^2-1}{16}J_1^4(x)$$

$$\int \frac{I_0(x)I_1^3(x)}{x^2}\,dx = -\frac{4x^2-3}{16}I_0^4(x) + \frac{x}{2}I_0^3(x)I_1(x) + \frac{4x^2-3}{8}I_0^2(x)I_1^2(x) - \frac{2x^2+1}{4x}I_0(x)I_1^3(x) - \frac{4x^2+1}{16}I_1^4(x)$$

$$\int \frac{K_0(x)K_1^3(x)}{x^2}\,dx =$$

$$= \frac{4x^2-3}{16}K_0^4(x) + \frac{x}{2}K_0^3(x)K_1(x) - \frac{4x^2-3}{8}K_0^2(x)K_1^2(x) - \frac{2x^2+1}{4x}K_0(x)K_1^3(x) + \frac{4x^2+1}{16}K_1^4(x)$$

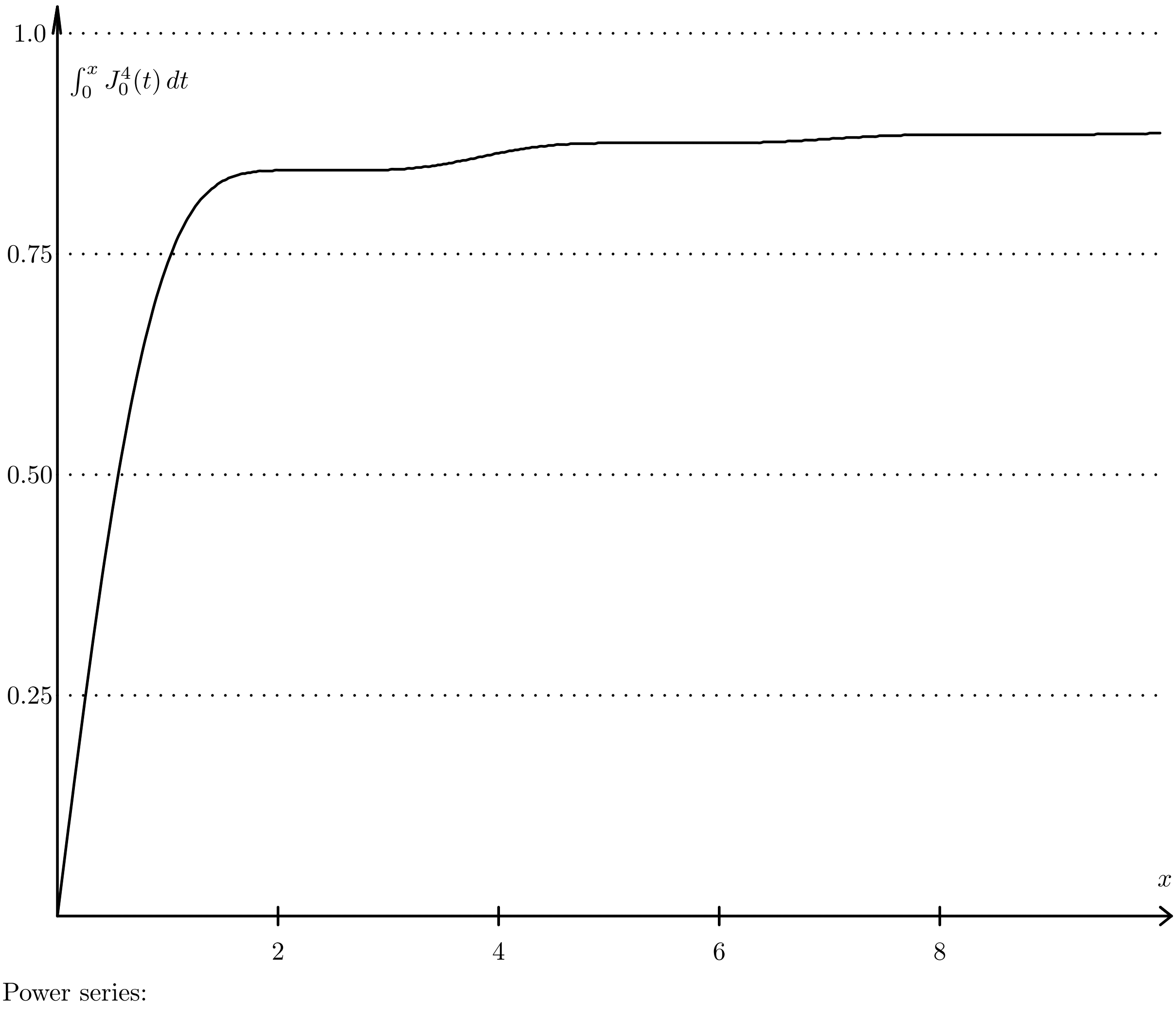
$$\int \frac{J_1^4(x)}{x^3}\,dx = -\frac{4x^2+3}{24}J_0^4(x) + \frac{x}{3}J_0^3(x)J_1(x) - \frac{4x^2+3}{12}J_0^2(x)J_1^2(x) + \frac{2x^2-1}{6x}J_0(x)J_1^3(x) - \frac{4x^4-x^2+4}{24x^2}J_1^4(x)$$

$$\int \frac{I_1^4(x)}{x^3}\,dx = -\frac{4x^2-3}{24}I_0^4(x) + \frac{x}{3}I_0^3(x)I_1(x) + \frac{4x^2-3}{12}I_0^2(x)I_1^2(x) - \frac{2x^2+1}{6x}I_0(x)I_1^3(x) - \frac{4x^4+x^2+4}{24x^2}I_1^4(x)$$

$$\int \frac{K_1^4(x)}{x^3}\,dx =$$

$$= -\frac{4x^2-3}{24}K_0^4(x) - \frac{x}{3}K_0^3(x)K_1(x) + \frac{4x^2-3}{12}K_0^2(x)K_1^2(x) + \frac{2x^2+1}{6x}K_0(x)K_1^3(x) - \frac{4x^4+x^2+4}{24x^2}K_1^4(x)$$

4.1. b) Basic Integral $Z_0^4(x)$



Power series:

$$\int_0^x J_0^4(t)\,dt = \sum_{k=0}^{\infty}(-1)^k a_k\,x^{2k+1} = x - \frac{1}{3}\,x^3 + \frac{7}{80}\,x^5 - \frac{1}{63}\,x^7 + \frac{679}{331776}\,x^9 - \frac{179}{921600}\,x^{11} + \frac{6049}{431308800}\,x^{13} - \dots$$

$$\int_0^x I_0^4(t)\,dt = \sum_{k=0}^{\infty} a_k\,x^{2k+1} = x + \frac{1}{3}\,x^3 + \frac{7}{80}\,x^5 + \frac{1}{63}\,x^7 + \frac{679}{331776}\,x^9 + \frac{179}{921600}\,x^{11} + \frac{6049}{431308800}\,x^{13} + \dots$$

k	a_k	a_k
0	1	1.00000 00000 00000 00000
1	$\frac{1}{3}$	0.33333 33333 33333 33333
2	$\frac{7}{80}$	0.08750 00000 00000 00000
3	$\frac{1}{63}$	0.01587 30158 73015 87302
4	$\frac{679}{331776}$	0.00204 65615 35493 82716
5	$\frac{179}{921600}$	0.00019 42274 30555 55556
6	$\frac{6049}{431308800}$	0.00001 40247 54421 88984
7	$\frac{9671}{12192768000}$	0.00000 07931 75101 83086
8	$\frac{16304551}{452803638067200}$	0.00000 00360 07994 70074
9	$\frac{7844077}{5856006714163200}$	0.00000 00013 39492 48744
10	$\frac{752932783}{18122799725936640000}$	0.00000 00000 41546 16253
11	$\frac{93524251}{8577508781850624000O}$	0.00000 00000 01090 34282
12	$\frac{36868956721}{1503674582974857216000000}$	0.00000 00000 00024 51924
13	$\frac{131084576323}{274450684884570939064320000}$	0.00000 00000 00000 47763
14	$\frac{134309549357}{16507700453797896482979840000}$	0.00000 00000 00000 00814
15	$\frac{242618760673}{1985193287331729792565248000000}$	0.00000 00000 00000 00012

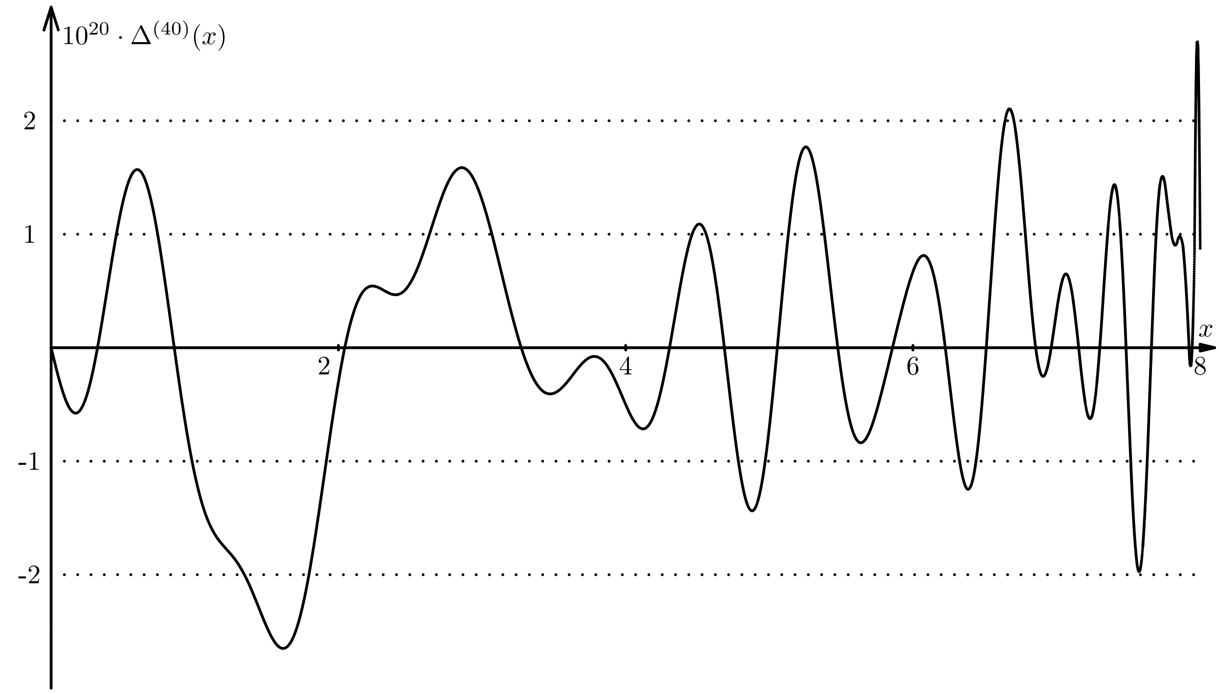
Approximation by Chebyshev polynomials: For $|x| \leq 8$ holds (based on [2], 9.7)

$$\int_0^x J_0^4(t)\,dt \;=\; \sum_{k=0}^{33} \alpha_k\, T_{2n+1}\left(\frac{x}{8}\right) + \Delta^{(40)}(x)$$

with the coefficients

k	α_k	k	α_k
0	1.11216 87232 77997 70374	17	-0.00004 87217 67090 45658
1	-0.34519 17415 46385 57003	18	0.00001 20357 15738 11243
2	0.19338 82074 76973 67514	19	-0.00000 25907 87311 63049
3	-0.12647 08183 95063 85502	20	0.00000 04907 99998 13736
4	0.09004 56892 27881 33025	21	-0.00000 00825 41454 11576
5	-0.06770 17817 94869 79248	22	0.00000 00124 18007 27445
6	0.04909 46023 85233 15906	23	-0.00000 00016 82526 91798
7	-0.03372 40353 97394 50397	24	0.00000 00002 06529 29192
8	0.02266 69978 75674 80247	25	-0.00000 00000 23088 90409
9	-0.01515 53930 28431 47066	26	0.00000 00000 02361 98127
10	0.01031 44234 66796 38411	27	-0.00000 00000 00222 04627
11	-0.00711 67527 37994 13547	28	0.00000 00000 00019 25615
12	0.00463 50732 52595 22669	29	-0.00000 00000 00001 54585
13	-0.00263 71375 49146 56248	30	0.00000 00000 00000 11524
14	0.00126 22032 52082 49660	31	-0.00000 00000 00000 00800
15	-0.00050 48484 79216 82394	32	0.00000 00000 00000 00052
16	0.00016 99635 21655 61120	33	-0.00000 00000 00000 00003

The derivation $\Delta^{(40)}(x)$:



Asymptotic formula:

$$\int_0^\infty J_0^4(x)\,dx \quad = \quad 0.90272\;85783\;23834\;82419\;\ldots$$

$$\int_0^x J_0^4(t)\,dt \sim 0.90272\;\ldots - \frac{1}{\pi^2}\left[\frac{3}{2x} + \frac{8\,\cos(2\,x)+\sin(4\,x)}{8x^2} - \frac{1-10\,\sin(2\,x)+\cos(4\,x)}{8\,x^3} + \ldots\right] =$$

$$= 0.90272\;\ldots - \frac{1}{\pi^2}\sum_{k=1}^\infty \frac{p_k + q_k\sin 2x + r_k \sin 4x + s_k\cos 2x + t_k\cos 4x}{n_k\cdot x^k} = 0.90272\;\ldots - \frac{1}{\pi^2}\sum_{k=1}^\infty \frac{\sigma_k(x)}{x^k}$$

k	p_k	q_k	r_k	s_k	t_k	n_k
1	3	0	0	0	0	2
2	0	0	1	8	0	8
3	-1	10	0	0	-1	8
4	0	0	-9	-138	0	64
5	168	-5770	0	0	245	1280
6	0	0	644	24095	0	2048
7	-27648	2053401	0	0	-35364	57344
8	0	0	-93636	-93636	0	65536
9	1042944	-134972229	0	0	1015836	262144
10	0	0	102333888	19644534099	0	8388608
11	-1979596800	394714074735	0	0	-1482416640	33554432
12	0	0	-48926574720	-17487139338315	0	268435456

Let

$$D_n(x) = 0.90272\;\ldots - \frac{1}{\pi^2}\sum_{k=1}^n \frac{\sigma_k(x)}{x^k} - \int_0^x J_0^4(t)\,dt$$

denote the derivation of the partial sums of the asymptotic series. The following table shows consecutive maxima and minima of this functions:

x_k	1.505	3.298	4.622	6.418	7.755	9.552	10.893
$D_1(x_k)$	-3.113E-02	8.502E-03	-4.513E-03	2.446E-03	-1.678E-03	1.126E-03	-8.628E-04
x_k	2.356	3.964	5.498	7.091	8.639	10.226	11.781
$D_2(x_k)$	-6.617E-03	1.749E-03	-6.893E-04	3.361E-04	-1.879E-04	1.152E-04	-7.558E-05

x_k	3.221	4.656	6.355	7.796	9.494	10.936	12.634
$D_3(x_k)$	-1.514E-03	3.813E-04	-1.230E-04	5.450E-05	-2.601E-05	1.463E-05	-8.457E-06
x_k	3.945	5.498	7.080	8.639	10.219	11.781	13.358
$D_4(x_k)$	-3.508E-04	7.283E-05	-2.295E-05	8.509E-06	-3.893E-06	1.881E-06	-1.046E-06
x_k	3.176	4.688	6.314	7.830	9.454	10.971	12.595
$D_5(x_k)$	6.526E-04	-7.890E-05	1.534E-05	-4.427E-06	1.511E-06	-6.268E-07	2.819E-07
x_k	3.937	5.498	7.076	8.639	10.216	11.781	13.356
$D_6(x_k)$	1.452E-04	-1.689E-05	3.288E-06	-8.541E-07	2.789E-07	-1.044E-07	4.469E-08
x_k	3.157	4.702	6.297	7.844	9.438	10.985	12.579
$D_7(x_k)$	-5.521E-04	3.208E-05	-3.752E-06	7.106E-07	-1.734E-07	5.326E-08	-1.868E-08
x_k	3.933	5.498	7.074	8.639	10.214	11.781	13.355
$D_8(x_k)$	-1.077E-04	6.921E-06	-8.432E-07	1.521E-07	-3.594E-08	1.033E-08	-3.490E-09
x_k	3.150	4.708	6.290	7.849	9.431	10.991	12.573
$D_9(x_k)$	7.860E-04	-2.183E-05	1.525E-06	-1.904E-07	3.320E-08	-7.650E-09	2.080E-09
x_k	3.931	5.498	7.072	8.639	10.213	11.781	13.354
$D_{10}(x_k)$	1.271E-04	-4.450E-06	3.409E-07	**-4.254E-08**	7.270E-09	-1.600E-09	4.100E-10

Holds $D_9(8) =$ -1.798E-7 and

$$\min \{D_n(x_k) \mid 8 \leq x\} = |D_{10}(8.639)| = 4.254E-8 \, .$$

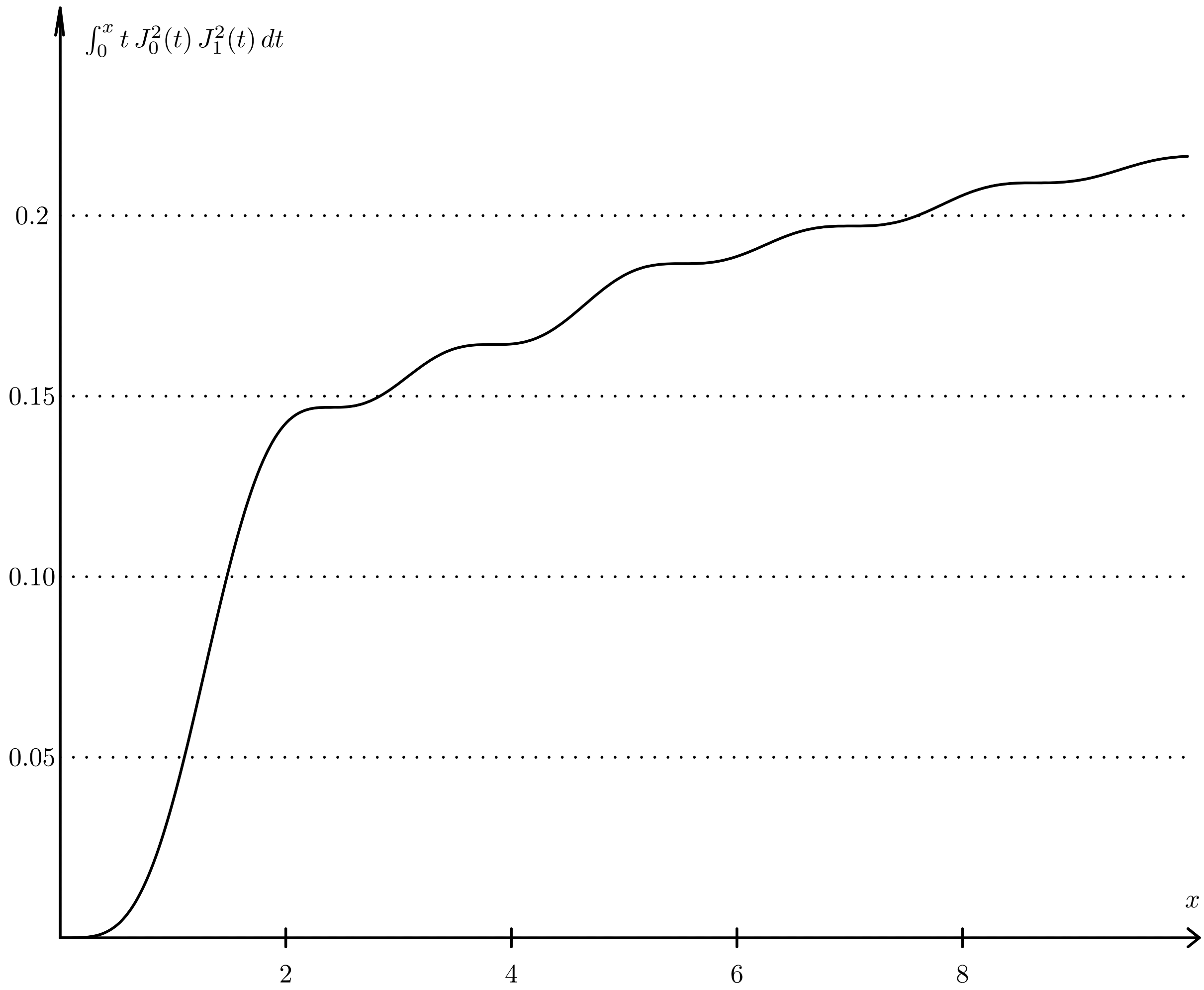
Therefore using the partial sum of the asymptotic series with $n = 10$ means to get the best uniform approximation of the integral with $x \geq 8$. It is the best way to continue the representation by the sum of Chebyshev polynomials given before.

$$\int_0^x I_0^4(t)\,dt \sim \frac{e^{4x}}{16\pi^2\,x^2}\left[1+\frac{1}{x}+\frac{9}{8x^2}+\frac{49}{32\,x^3}+\frac{161}{64\,x^4}+\frac{1263}{256\,x^5}+\frac{23409}{2048\,x^6}+\ldots\right] = \frac{e^{4x}}{16\pi^2\,x^2}\sum_{k=0}^{\infty}\frac{\mu_k}{x^k}$$

k	μ_k	μ_k	μ_k/μ_{k-1}
0	1	1.000 000 000 000	-
1	1	1.000 000 000 000	1.000
2	$\frac{9}{8}$	1.125 000 000 000	1.125
3	$\frac{49}{32}$	1.531 250 000 000	1.361
4	$\frac{161}{64}$	2.515 625 000 000	1.643
5	$\frac{1263}{256}$	4.933 593 750 000	1.961
6	$\frac{23409}{2048}$	11.430 175 781 25	2.317
7	$\frac{253959}{8192}$	31.000 854 492 19	2.712
8	$\frac{1598967}{16384}$	97.593 200 683 59	3.148
9	$\frac{2895345}{8192}$	353.435 668 945 3	3.622
10	$\frac{382238865}{262144}$	1 458.125 553 131	4.126
11	$\frac{7110791145}{1048576}$	6 781.378 884 315	4.651
12	$\frac{295087625775}{8388608}$	35 177.186 223 864	5.187

For a given $x >> 0$ the series can be used while $\mu_k/\mu_{k-1} \leq x$.

4.1. c) Basic Integral $x\, Z_0^2(x)\, Z_1^2(x)$



Power series:

$$\int_0^x t\, J_0^2(t)\, J_1^2(t)\, dt = \sum_{k=2}^{\infty} (-1)^k b_k\, x^{2k} = \frac{x^4}{16} - \frac{x^6}{32} + \frac{47}{6144}\, x^8 - \frac{43}{36864}\, x^{10} + \frac{17}{138240}\, x^{12} - \frac{211}{22118400}\, x^{14} + \dots$$

$$\int_0^x t\, I_0^2(t)\, I_1^2(t)\, dt = \sum_{k=2}^{\infty} b_k\, x^{2k} = \frac{x^4}{16} + \frac{x^6}{32} + \frac{47}{6144}\, x^8 + \frac{43}{36864}\, x^{10} + \frac{17}{138240}\, x^{12} + \frac{211}{22118400}\, x^{14} + \dots$$

k	b_k	b_k
2	$\frac{1}{4}$	0.25000 00000 00000 00000
3	$\frac{1}{32}$	0.03125 00000 00000 00000
4	$\frac{47}{6144}$	0.00764 97395 83333 33333
5	$\frac{43}{36864}$	0.00116 64496 52777 77778
6	$\frac{17}{138240}$	0.00012 29745 37037 03704
7	$\frac{211}{22118400}$	0.00000 95395 68865 74074
8	$\frac{540619}{951268147200}$	0.00000 05683 13993 89465

k	b_k	b_k
9	$\dfrac{1072333}{39953262182400}$	0.00000 00268 39685 70838
10	$\dfrac{19751801}{19177565847552000}$	0.00000 00010 29943 06770
11	$\dfrac{11307553}{345196185255936000}$	0.00000 00000 32756 88864
12	$\dfrac{88869497}{101257547675074560000}$	0.00000 00000 00877 65800
13	$\dfrac{402630853}{200489944396647628800000}$	0.00000 00000 00020 08235
14	$\dfrac{17384556227}{437870038562278421299200000}$	0.00000 00000 00000 39703
15	$\dfrac{16710855809}{24395616434184083472384000000}$	0.00000 00000 00000 00685
16	$\dfrac{58219427293829}{5595768915207408330561552384000000}$	0.00000 00000 00000 00010

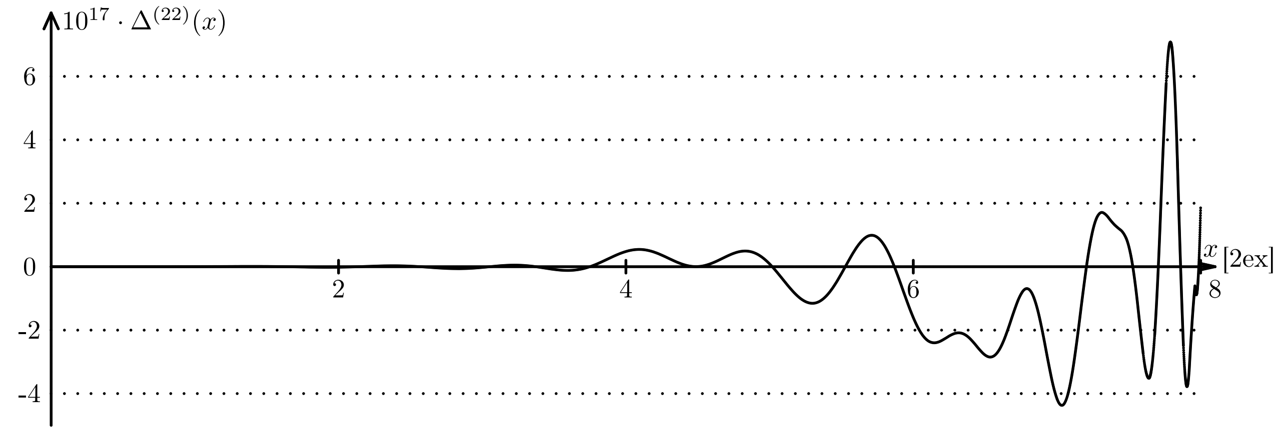
Approximation by Chebyshev polynomials: For $|x| \le 8$ holds (based on [2], 9.7)

$$\int_0^x t\, J_0^2(t)\, J_1^2(t)\, dt \;=\; x^4 \sum_{k=0}^{33} \beta_k\, T_{2n}\left(\frac{x}{8}\right) + \Delta^{(22)}(x)$$

with the coefficients

k	β_k	k	β_k
0	0.00654 92504 91232 31929	16	0.00000 13488 21251 71102
1	-0.01239 97369 78264 31272	17	-0.00000 03066 11870 00840
2	0.01094 53569 42769 10753	18	0.00000 00617 07706 64541
3	-0.00914 38524 55194 87209	19	-0.00000 00110 63931 41290
4	0.00727 27632 79880 71248	20	0.00000 00017 77850 51057
5	-0.00551 50237 23344 53306	21	-0.00000 00002 57472 65556
6	0.00398 42908 14902 62113	22	0.00000 00000 33780 93775
7	-0.00273 36991 93851 90098	23	-0.00000 00000 04034 52317
8	0.00177 13047 44123 60885	24	0.00000 00000 00440 55264
9	-0.00107 36173 62206 65294	25	-0.00000 00000 00044 16078
10	0.00059 89651 07157 49244	26	0.00000 00000 00004 07863
11	-0.00030 12533 67887 99958	27	-0.00000 00000 00000 34826
12	0.00013 40719 51669 75647	28	0.00000 00000 00000 02758
13	-0.00005 21809 55261 86499	29	-0.00000 00000 00000 00203
14	0.00001 76814 63326 68133	30	0.00000 00000 00000 00014
15	-0.00000 52208 90537 98925	31	-0.00000 00000 00000 00001

The derivation $\Delta^{(22)(x)}$:



Asymptotic formula:

$$\int_0^x t\,J_0^2(t)\,J_1^2(t)\,dt \sim 0.09947\ 25799\ 65044\ 03230\ldots+$$

$$+\frac{1}{\pi^2}\left[\frac{\ln x}{2}+\frac{\sin 4x}{8x}+\frac{\cos 4x-16\sin 2x-6}{8x^2}+\frac{\sin 4x+12\cos 2x}{32\,x^3}+\ldots\right]=$$

$$=0.09947\ \ldots+\frac{1}{\pi^2}\left[\frac{\ln x}{2}+\sum_{k=1}^{\infty}\frac{p_k+q_k\sin 2x+r_k\sin 4x+s_k\cos 2x+t_k\cos 4x}{n_k\cdot x^k}\right]=$$

$$=0.09947\ \ldots+\frac{1}{\pi^2}\left[\sigma_0(x)+\sum_{k=1}^{\infty}\frac{\sigma_k(x)}{x^k}\right]$$

k	p_k	q_k	r_k	s_k	t_k	n_k
1	0	0	1	0	0	8
2	-6	-16	0	0	1	32
3	0	0	1	12	0	32
4	18	180	0	0	-9	256
5	0	0	-51	-1356	0	1024
6	-576	-15483	0	0	357	4096
7	0	0	3015	182385	0	16384
8	99360	99360	0	0	-60300	131072
9	0	0	-703530	-88668135	0	524288
10	-137687040	-13633922835	0	0	75666960	16777216
11	0	0	1159159680	271539997785	0	67108864
12	67108864	12571439587875	0	0	-40044715200	536870912

Let

$$D_n(x)=0.09947\ \ldots+\frac{1}{\pi^2}\left[\sigma_0(x)+\sum_{k=1}^{n}\frac{\sigma_k(x)}{x^k}\right]-\int_0^x t\,J_0^2(t)\,J_1^2(t)\,dt$$

denote the derivation of the partial sums of the asymptotic series. The following table shows consecutive maxima and minima of this functions:

x_k	3.391	4.211	5.147	5.931	6.602	7.398	8.273
$D_1(x_k)$	-5.002E-04	6.375E-03	-2.955E-03	1.615E-03	-8.607E-04	2.793E-03	-1.744E-03
x_k	3.870	5.505	7.037	8.644	10.188	11.784	13.335
$D_2(x_k)$	4.475E-03	-8.864E-04	1.430E-03	-3.717E-04	6.919E-04	-2.023E-04	4.062E-04
x_k	3.248	4.676	6.384	7.800	9.520	10.934	12.658
$D_3(x_k)$	-1.021E-03	3.135E-04	-1.471E-04	7.402E-05	-4.504E-05	2.779E-05	-1.922E-05
x_k	3.901	5.502	7.053	8.642	10.199	11.783	13.343
$D_4(x_k)$	-2.742E-04	5.650E-05	-3.011E-05	1.011E-05	-7.214E-06	3.019E-06	-2.511E-06
x_k	3.201	4.711	6.340	7.840	9.477	10.975	12.616
$D_5(x_k)$	2.732E-04	-4.339E-05	1.169E-05	-3.970E-06	1.674E-06	-7.807E-07	4.107E-07
x_k	3.175	4.724	6.316	7.857	9.455	10.994	12.594
$D_6(x_k)$	-1.747E-04	1.386E-05	-2.198E-06	4.983E-07	-1.481E-07	5.176E-08	-2.095E-08
x_k	3.921	5.499	7.064	8.640	10.207	11.782	13.349
$D_7(x_k)$	-3.990E-05	3.253E-06	-5.342E-07	1.093E-07	-3.219E-08	9.988E-09	-3.994E-09
x_k	3.161	4.726	6.305	7.863	9.444	11.001	12.584
$D_8(x_k)$	2.032E-04	-7.880E-06	7.394E-07	-1.115E-07	2.348E-08	-6.142E-09	1.922E-09
x_k	3.924	5.499	7.066	8.640	10.208	11.782	13.350
$D_9(x_k)$	3.864E-05	-1.772E-06	1.785E-07	**-2.587E-08**	5.416E-09	-1.316E-09	4.030E-10
x_k	3.154	4.725	6.299	7.864	9.439	11.003	12.579
$D_{10}(x_k)$	-3.766E-04	7.014E-06	-3.886E-07	3.885E-08	-5.815E-09	1.138E-09	-2.759E-10

Holds

$$\min\ \{D_n(x_k)\,|\,8 \le x\ \} = |D_9(9.439)| = 2.587E-08\ .$$

Therefore using the partial sum of the asymptotic series with $n = 9$ means to get the best uniform approximation of the integral with $x \ge 8$. It is the best way to continue the representation by the sum of Chebyshev polynomials given before.

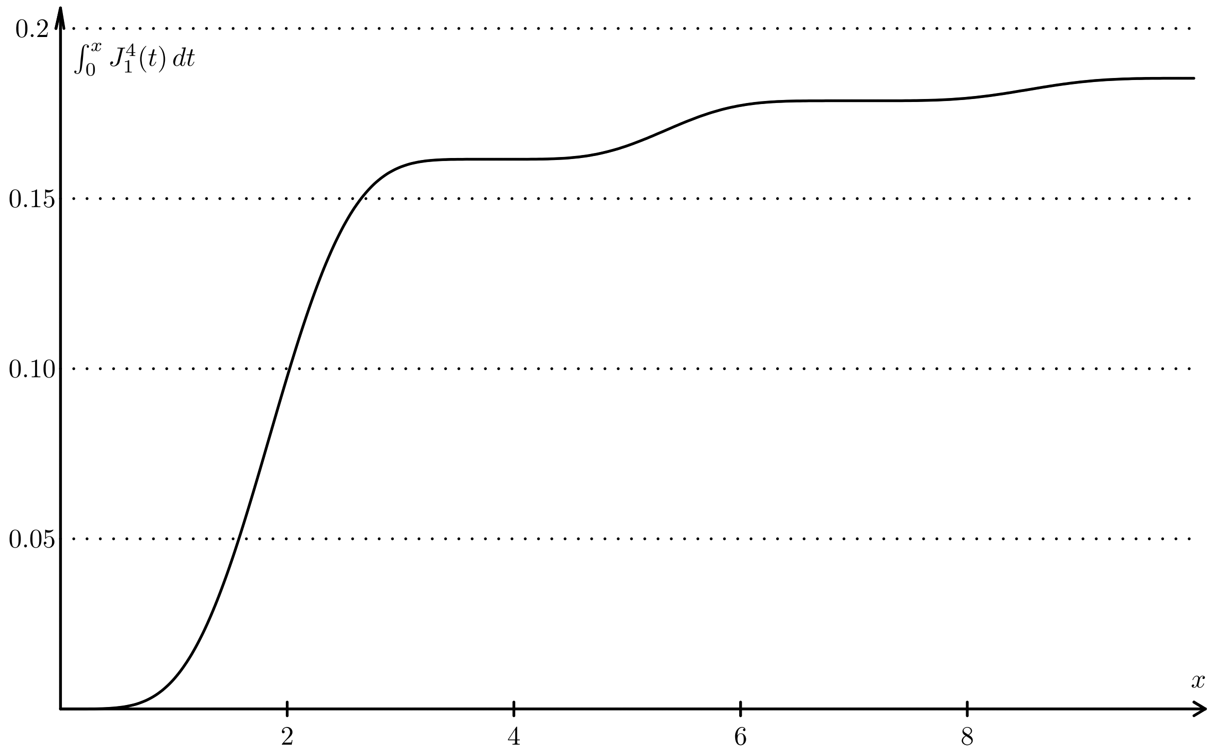
$$\int_0^x t\,I_0^2(t)\,I_1^2(t)\,dt \sim \frac{e^{4x}}{16\pi^2\,x}\left[1 - \frac{1}{4x} - \frac{1}{4x^2} - \frac{9}{32\,x^3} - \frac{51}{128\,x^4} - \frac{357}{512\,x^5} - \frac{3015}{2048\,x^6} - \dots\right] =$$

$$= \frac{e^{4x}}{16\pi^2\,x}\left(1 - \sum_{k=1}^{\infty}\frac{\mu_k}{x^k}\right)$$

k	μ_k	μ_k	μ_k/μ_{k-1}
1	$\frac{1}{4}$	0.250 000 000 000	-
2	$\frac{1}{4}$	0.250 000 000 000	1.000
3	$\frac{9}{32}$	0.281 250 000 000	1.125
4	$\frac{51}{128}$	0.398 437 500 000	1.417
5	$\frac{357}{512}$	0.697 265 625 000	1.750
6	$\frac{3015}{2048}$	1.472 167 968 750	2.111
7	$\frac{15075}{4096}$	3.680 419 921 875	2.500
8	$\frac{351765}{32768}$	10.735 015 869 14	2.917
9	$\frac{4729185}{131072}$	36.080 818 176 27	3.361
10	$\frac{9055935}{65536}$	138.182 601 928 7	3.830
11	$\frac{625698675}{1048576}$	596.712 756 156 9	4.318
12	$\frac{24131137275}{8388608}$	2 876.655 730 605	4.821

For a given $x >> 0$ the series can be used while $\mu_k/\mu_{k-1} \le x$.

4.1. d) Basic Integral $Z_1^4(x)$



Power series:

$$\int_0^x J_1^4(t)\,dt = \sum_{k=2}^{\infty}(-1)^k c_k\,x^{2k+1} = \frac{1}{80}\,x^5 - \frac{1}{224}\,x^7 + \frac{11}{13824}\,x^9 - \frac{37}{405504}\,x^{11} + \frac{11}{1474560}\,x^{13} - \frac{1223}{2654208000}\,x^{15} + \ldots$$

$$\int_0^x I_1^4(t)\,dt = \sum_{k=2}^{\infty} c_k\,x^{2k+1} = \frac{1}{80}\,x^5 + \frac{1}{224}\,x^7 + \frac{11}{13824}\,x^9 + \frac{37}{405504}\,x^{11} + \frac{11}{1474560}\,x^{13} + \frac{1223}{2654208000}\,x^{15} + \ldots$$

k	c_k	c_k
2	$\frac{1}{80}$	0.01250 00000 00000 00000
3	$\frac{1}{224}$	0.00446 42857 14285 71429
4	$\frac{11}{13824}$	0.00079 57175 92592 59259
5	$\frac{37}{405504}$	0.00009 12444 76010 10101
6	$\frac{11}{1474560}$	0.00000 74598 52430 55556
7	$\frac{1223}{2654208000}$	0.00000 04607 77753 66512
8	$\frac{45173}{2021444812800}$	0.00000 00223 46887 58949
9	$\frac{221467}{253037327155200}$	0.00000 00008 75234 50587
10	$\frac{2278819}{80545776559718400}$	0.00000 00000 28292 22210
11	$\frac{1524667}{1984878065221632000}$	0.00000 00000 00768 14139

k	c_k	c_k
12	$\dfrac{33739889}{189857901890764800000}$	0.00000 00000 00017 77113
13	$\dfrac{191964463}{5413228498709485977600000}$	0.00000 00000 00000 35462
14	$\dfrac{639303779}{1036590295371924425932800000}$	0.00000 00000 00000 00617
15	$\dfrac{1383459431}{14666940304673097457336320000}$	0.00000 00000 00000 00009

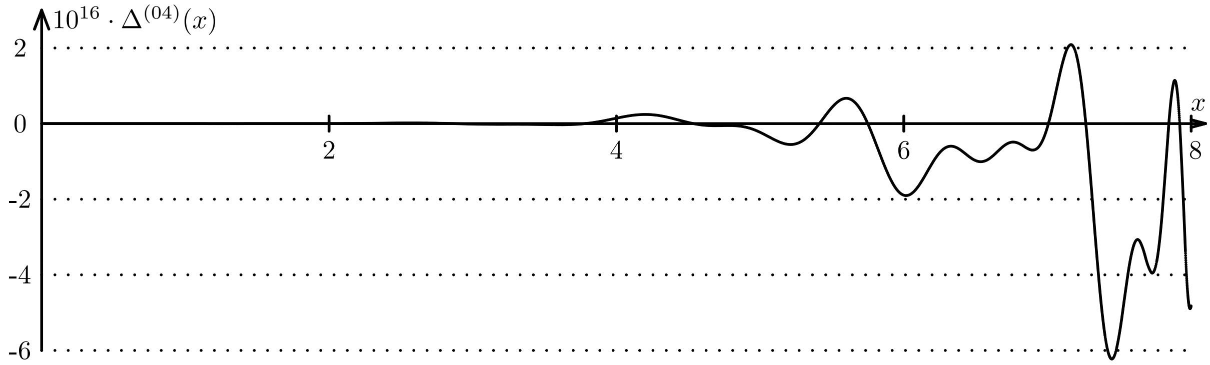
Approximation by Chebyshev polynomials: For $|x| \le 8$ holds (based on [2], 9.7)

$$\int_0^x J_1^4(t)\,dt \;=\; x^5 \sum_{k=0}^{33} \gamma_k\, T_{2n}\left(\frac{x}{8}\right) + \Delta^{(04)}(x)$$

with the coefficients

k	γ_k	k	γ_k
0	0.00153 17807 42787 63788	15	-0.00000 01214 49955 80695
1	-0.00288 55180 23979 94997	16	0.00000 00263 11808 30629
2	0.00246 63655 58097 67279	17	-0.00000 00050 87634 73551
3	-0.00194 01230 51814 45392	18	0.00000 00008 82115 64716
4	0.00141 41566 95709 63740	19	-0.00000 00001 37795 20459
5	-0.00095 77666 19168 38344	20	0.00000 00000 19482 53784
6	0.00060 29805 69397 05353	21	-0.00000 00000 02504 22692
7	-0.00035 21217 05640 59655	22	0.00000 00000 00293 84879
8	0.00018 96461 12672 32841	23	-0.00000 00000 00031 59998
9	-0.00009 33179 14008 29393	24	0.00000 00000 00003 12564
10	0.00004 14766 56943 42490	25	-0.00000 00000 00000 28533
11	-0.00001 64773 77688 70038	26	0.00000 00000 00000 02411
12	0.00000 58075 26406 73172	27	-0.00000 00000 00000 00189
13	-0.00000 18093 92101 58874	28	0.00000 00000 00000 00014
14	0.00000 04981 61958 19526	29	-0.00000 00000 00000 00001

The derivation $\Delta^{(04)(x)}$:



Asymptotic formula:

$$\int_0^\infty J_1^4(x)\,dx \quad = \quad 0.20025\ 27575\ 82806\ 70455\ \ldots$$

$$\int_0^x J_1^4(t)\,dt \sim 0.20025\ \ldots - \frac{1}{\pi^2}\left[\frac{3}{2x} + \frac{8\,\cos(2\,x) - \sin(4\,x)}{8x^2} - \frac{3 - 2\,\sin(2\,x) + \cos(4\,x)}{8\,x^3} + \ldots\right] =$$

$$= 0.20025\ \ldots - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{p_k + q_k \sin 2x + r_k \sin 4x + s_k \cos 2x + t_k \cos 4x}{n_k \cdot x^k} = 0.20025\ \ldots - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\sigma_k(x)}{x^k}$$

k	p_k	q_k	r_k	s_k	t_k	n_k
1	3	0	0	0	0	2
2	0	0	1	-8	0	8
3	3	-2	0	0	1	8
4	0	0	3	-6	0	64
5	-216	-30	0	0	-75	1280
6	0	0	-204	1173	0	2048
7	34560	65536	0	0	12348	57344
8	0	0	37116	-586503	0	65536
9	-1267200	-9916623	0	0	-9916623	262144
10	0	0	-52716096	1650714957	0	8388608
11	25778995200	376332256575	0	0	9405918720	369098752
12	0	0	369098752	-1633144646925	0	-1633144646925

Let

$$D_n(x) = 0.20025\ \ldots - \frac{1}{\pi^2}\sum_{k=1}^{n}\frac{\sigma_k(x)}{x^k} - \int_0^x J_0^4(t)\,dt$$

denote the derivation of the partial sums of the asymptotic series. The following table shows consecutive maxima and minima of this functions:

x_k	2.930	4.738	6.115	7.917	9.274	11.073	12.424
$D_1(x_k)$	-9.815E-03	5.061E-03	-2.533E-03	1.764E-03	-1.140E-03	8.883E-04	-6.457E-04
x_k	3.927	4.804	7.069	8.003	10.210	11.962	11.962
$D_2(x_k)$	-6.708E-06	5.006E-04	-5.760E-07	1.111E-04	-1.053E-07	3.014E-05	3.014E-05
x_k	4.118	4.967	5.948	7.057	7.182	8.102	9.122
$D_3(x_k)$	-7.413E-06	1.140E-05	-1.107E-05	-5.756E-07	-5.905E-07	2.135E-06	-1.971E-06
x_k	4.567	5.842	6.168	7.055	7.737	8.858	9.382
$D_4(x_k)$	-1.253E-05	-1.396E-06	-1.515E-06	-5.756E-07	-9.571E-07	-1.679E-07	-2.281E-07
x_k	4.932	6.114	8.037	9.272	11.165	12.418	14.300
$D_5(x_k)$	-2.582E-06	9.433E-07	-1.787E-07	8.754E-08	-2.737E-08	1.592E-08	-6.480E-09
x_k	3.932	5.630	7.072	8.735	10.212	11.855	13.353
$D_6(x_k)$	7.787E-06	-6.017E-07	1.910E-07	-3.239E-08	1.673E-08	-4.050E-09	2.700E-09
x_k	3.044	4.823	6.215	7.943	9.364	11.075	12.507
$D_7(x_k)$	-4.681E-05	1.729E-06	-2.866E-07	4.340E-08	-1.287E-08	3.440E-09	-1.340E-09
x_k	3.927	5.571	7.069	8.696	10.210	11.826	13.352
$D_8(x_k)$	-7.839E-06	4.320E-07	-6.420E-08	9.910E-09	-2.750E-09	7.500E-10	-2.800E-10
x_k	3.092	4.775	6.248	7.905	9.394	11.040	12.537
$D_9(x_k)$	7.406E-05	-1.525E-06	1.346E-07	-1.447E-08	**2.910E-09**	-5.800E-10	2.100E-10
x_k	3.927	5.542	7.069	8.677	10.210	11.812	13.352
$D_{10}(x_k)$	1.082E-05	-3.483E-07	2.989E-08	-3.330E-09	6.900E-10	-1.500E-10	2.000E-11

Note that $D_2(x)$, $D_3(x)$ and $D_4(x)$ do not have the regular behaviour of the other functions.
Holds

$$\max\{|D_{10}(x)| \mid x \geq 8\} = 4.00E-9\ .$$

Using the partial sum of the asymptotic series with $n = 10$ means to get the best uniform approximation of the integral with $x \geq 8$. It is the best way to continue the representation by the sum of Chebyshev polynomials given before.

$$\int_0^x I_1^4(t)\,dt \sim \frac{e^{4x}}{16\pi^2\,x^2}\left[1 - \frac{1}{x} - \frac{3}{8x^2} - \frac{15}{32\,x^3} - \frac{51}{64\,x^4} - \frac{441}{256\,x^5} - \frac{9279}{2048\,x^6} - \ldots\right] = \frac{e^{4x}}{16\pi^2\,x^2}\left(1 - \sum_{k=1}^{\infty}\frac{\mu_k}{x^k}\right)$$

k	μ_k	μ_k	μ_k/μ_{k-1}
1	1	1.000 000 000 000	-
2	$\frac{3}{8}$	0.375 000 000 000	0.375
3	$\frac{15}{32}$	0.468 750 000 000	1.250
4	$\frac{51}{64}$	0.796 875 000 000	1.700
5	$\frac{441}{256}$	1.722 656 250 000	2.162
6	$\frac{9279}{2048}$	4.530 761 718 750	2.630
7	$\frac{115137}{8192}$	14.054 809 570 31	3.102
8	$\frac{823689}{16384}$	50.273 986 816 41	3.577
9	$\frac{1670085}{8192}$	203.867 797 851 6	4.055
10	$\frac{242464455}{262144}$	924.928 493 499 8	4.537
11	$\frac{4871010735}{1048576}$	4 645.357 832 909	5.022
12	$\frac{214767448785}{8388608}$	25 602.274 988 294	5.511

For a given $x >> 0$ the series can be used while $\mu_k/\mu_{k-1} \leq x$.

4.1. e) Integrals of $x^m\, Z_0^4(x)$

With the basic integrals

$$\mathcal{I}_0^{(40)}(x) = \int J_0^4(x)\,dx\,, \quad \mathcal{I}_1^{(22)}(x) = \int x\,J_0^2(x)\,J_1{}^2(x)dx\,, \quad \mathcal{I}_0^{(04)}(x) = \int J_1^4(x)\,dx$$

and

$$\mathcal{I}_0^{*(40)}(x) = \int I_0^4(x)\,dx\,, \quad \mathcal{I}_1^{*(22)}(x) = \int x\,I_0^2(x)\,I_1{}^2(x)dx\,, \quad \mathcal{I}_0^{*(04)}(x) = \int I_1^4(x)\,dx$$

holds

$$\int x\; J_0^4(x)\,dx = x\,J_0^3(x)\,J_1(x) + 3\mathcal{I}_1^{(22)}(x)$$

$$\int x\; I_0^4(x)\,dx = x\,I_0^3(x)\,I_1(x) - 3\mathcal{I}_1^{*(22)}(x)$$

$$\int x^2\; J_0^4(x)\,dx = \frac{12x^3 - x}{32}\,J_0^4(x) - \frac{x^2}{8}\,J_0^3(x)\,J_1(x) + \frac{3x^3}{4}\,J_0^2(x)\,J_1^2(x) - \frac{3x^2}{8}\,J_0(x)\,J_1^3(x) +$$
$$+ \frac{12x^3 - 3x}{32}\,J_1^4(x) + \frac{1}{32}\,\mathcal{I}_0^{(40)}(x) - \frac{9}{32}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^2\; I_0^4(x)\,dx = \frac{12x^3 + x}{32}\,I_0^4(x) - \frac{x^2}{8}\,I_0^3(x)\,I_1(x) - \frac{3x^3}{4}\,I_0^2(x)\,I_1^2(x) + \frac{3x^2}{8}\,I_0(x)\,I_1^3(x) +$$
$$+ \frac{12x^3 + 3x}{32}\,I_1^4(x) - \frac{1}{32}\,\mathcal{I}_0^{*(40)}(x) + \frac{9}{32}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^3\; J_0^4(x)\,dx = \frac{3x^4 + 2x^2}{16}\,J_0^4(x) + \frac{x^3 - x}{4}\,J_0^3(x)\,J_1(x) + \frac{3x^4}{8}\,J_0^2(x)\,J_1^2(x) + \frac{3x^4}{16}\,J_1^4(x) - \frac{3}{4}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^3\; I_0^4(x)\,dx = \frac{3\,x^4 - 2\,x^2}{16}\,I_0^4(x) + \frac{x^3 + x}{4}\,I_0^3(x)\,I_1(x) - \frac{3x^4}{8}\,I_0^2(x)\,I_1^2(x) + \frac{3x^4}{16}\,I_1^4(x) - \frac{3}{4}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^4\ J_0^4(x)\,dx = \frac{32\,x^5-12\,x^3+9\,x}{256}\,J_0^4(x)+\frac{24\,x^4+9\,x^2}{64}\,J_0^3(x)\,J_1(x)+\frac{8\,x^5-21\,x^3}{32}\,J_0^2(x)\,J_1^2(x)+$$
$$+\frac{8\,x^4+23\,x^2}{64}\,J_0(x)\,J_1^3(x)+\frac{32\,x^5-92\,x^3+23\,x}{256}\,J_1^4(x)-\frac{9}{256}\,\mathcal{I}_0^{(40)}(x)+\frac{69}{256}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^4\ I_0^4(x)\,dx = \frac{32\,x^5+12\,x^3+9\,x}{256}\,I_0^4(x)+\frac{24\,x^4-9\,x^2}{64}\,I_0^3(x)\,I_1(x)-\frac{8\,x^5+21\,x^3}{32}\,I_0^2(x)\,I_1^2(x)-$$
$$-\frac{8\,x^4-23\,x^2}{64}\,I_0(x)\,I_1^3(x)+\frac{32\,x^5+92\,x^3+23\,x}{256}\,I_1^4(x)-\frac{9}{256}\,\mathcal{I}_0^{*(40)}(x)+\frac{69}{256}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^5\ J_0^4(x)\,dx = \frac{6\,x^6+7\,x^4-14\,x^2}{64}\,J_0^4(x)+\frac{7\,x^5-7\,x^3+7\,x}{16}\,J_0^3(x)\,J_1(x)+\frac{6\,x^6-21\,x^4}{32}\,J_0^2(x)\,J_1^2(x)+$$
$$+\frac{3x^5}{16}\,J_0(x)\,J_1^3(x)+\frac{6\,x^6-27\,x^4}{64}\,J_1^4(x)+\frac{21}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^5\ I_0^4(x)\,dx = \frac{6\,x^6-7\,x^4-14\,x^2}{64}\,I_0^4(x)+\frac{7\,x^5+7\,x^3+7\,x}{16}\,I_0^3(x)\,I_1(x)-\frac{6\,x^6+21\,x^4}{32}\,I_0^2(x)\,I_1^2(x)-$$
$$-\frac{3x^5}{16}\,I_0(x)\,I_1^3(x)+\frac{6\,x^6+27\,x^4}{64}\,I_1^4(x)-\frac{21}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^6\ J_0^4(x)\,dx = \frac{768\,x^7+2496\,x^5+1668\,x^3-1251\,x}{10240}\,J_0^4(x)+\frac{1216\,x^6-3120\,x^4-1251\,x^2}{2560}\,J_0^3(x)\,J_1(x)+$$
$$+\frac{192\,x^7-896\,x^5+2757\,x^3}{1280}\,J_0^2(x)\,J_1^2(x)+\frac{576\,x^6-1328\,x^4-3089\,x^2}{2560}\,J_0(x)\,J_1^3(x)+$$
$$+\frac{768\,x^7-5312\,x^5+12356\,x^3-3089\,x}{10240}\,J_1^4(x)+\frac{1251}{10240}\,\mathcal{I}_0^{(40)}(x)-\frac{9267}{10240}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^6\ I_0^4(x)\,dx = \frac{768\,x^7-2496\,x^5+1668\,x^3+1251\,x}{10240}\,I_0^4(x)+\frac{1216\,x^6+3120\,x^4-1251\,x^2}{2560}\,I_0^3(x)\,I_1(x)-$$
$$-\frac{192\,x^7+896\,x^5+2757\,x^3}{1280}\,I_0^2(x)\,I_1^2(x)-\frac{576\,x^6+1328\,x^4-3089\,x^2}{2560}\,I_0(x)\,I_1^3(x)+$$
$$+\frac{768\,x^7+5312\,x^5+12356\,x^3+3089\,x}{10240}\,I_1^4(x)-\frac{1251}{10240}\,\mathcal{I}_0^{*(40)}(x)+\frac{9267}{10240}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^4(x)\,dx}{x^2} = -\frac{6x^2+1}{x}\,J_0^4(x)+4\,J_0^3(x)\,J_1(x)-12x\,J_0^2(x)\,J_1^2(x)-6x\,J_1^4(x)+2\mathcal{I}_0^{(40)}(x)-18\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^4(x)\,dx}{x^2} = \frac{6x^2-1}{x}\,I_0^4(x)-4\,I_0^3(x)\,I_1(x)-12x\,I_0^2(x)\,I_1^2(x)+6x\,I_1^4(x)-2\mathcal{I}_0^{*(40)}(x)+18\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^4(x)\,dx}{x^4} = \frac{40\,x^4+4\,x^2-3}{9x^3}\,J_0^4(x) - \frac{24\,x^2-4}{9x^2}\,J_0^3(x)\,J_1(x) + \frac{80\,x^2-4}{9x}\,J_0^2(x)\,J_1^2(x) +$$
$$+\frac{8}{27}\,J_0(x)\,J_1^3(x) + \frac{40x}{9}\,J_1^4(x) - \frac{16}{9}\,\mathcal{I}_0^{(40)}(x) + \frac{368}{27}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^4(x)\,dx}{x^4} = \frac{40\,x^4-4\,x^2-3}{9x^3}\,I_0^4(x) - \frac{24\,x^2+4}{9x^2}\,I_0^3(x)\,I_1(x) - \frac{80\,x^2+4}{9x}\,I_0^2(x)\,I_1^2(x) -$$
$$-\frac{8}{27}\,I_0(x)\,I_1^3(x) + \frac{40x}{9}\,I_1^4(x) - \frac{16}{9}\,\mathcal{I}_0^{*(40)}(x) + \frac{368}{27}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^4(x)\,dx}{x^6} = -\frac{42752\,x^6+3800\,x^4-1500\,x^2+5625}{28125\,x^5}\,J_0^4(x) + \frac{4992\,x^4-760\,x^2+900}{5625\,x^4}\,J_0^3(x)\,J_1(x) -$$
$$-\frac{85504\,x^4-4880\,x^2+2700}{28125\,x^3}\,J_0^2(x)\,J_1^2(x) - \frac{10624\,x^2-3240}{84375\,x^2}\,J_0(x)\,J_1^3(x) - \frac{42752\,x^2+216}{28125\,x}\,J_1^4(x) +$$
$$+\frac{17792}{28125}\,\mathcal{I}_0^{(40)}(x) - \frac{395392}{84375}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^4(x)\,dx}{x^6} = \frac{42752\,x^6-3800\,x^4-1500\,x^2-5625}{28125\,x^5}\,I_0^4(x) - \frac{4992\,x^4+760\,x^2+900}{5625\,x^4}\,I_0^3(x)\,I_1(x) -$$
$$-\frac{85504\,x^4+4880\,x^2+2700}{28125\,x^3}\,I_0^2(x)\,I_1^2(x) - \frac{10624\,x^2+3240}{84375\,x^2}\,I_0(x)\,I_1^3(x) + \frac{42752\,x^2-216}{28125\,x}\,I_1^4(x) -$$
$$-\frac{17792}{28125}\,\mathcal{I}_0^{*(40)}(x) + \frac{395392}{84375}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. f) Integrals of $x^m\,Z_0^3(x)\,Z_1(x)$

Explicit and basic integrals are omitted.
With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 438 holds

$$\int x\,J_0^3(x)\,J_1(x)\,dx = -\frac{x}{4}\,J_0^4(x) + \frac{1}{4}\,\mathcal{I}_0^{(40)}(x)$$

$$\int x\,I_0^3(x)\,I_1(x)\,dx = \frac{x}{4}\,I_0^4(x) - \frac{1}{4}\,\mathcal{I}_0^{*(40)}(x)$$

$$\int x^2\,J_0^3(x)\,J_1(x)\,dx = -\frac{x^2}{4}\,J_0^4(x) + \frac{x}{2}\,J_0^3(x)\,J_1(x) + \frac{3}{2}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^2\,I_0^3(x)\,I_1(x)\,dx = \frac{x^2}{4}\,I_0^4(x) - \frac{x}{2}\,I_0^3(x)\,I_1(x) + \frac{3}{2}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^3\,J_0^3(x)\,J_1(x)\,dx = \frac{4\,x^3-3\,x}{128}\,J_0^4(x) - \frac{3x^2}{32}\,J_0^3(x)\,J_1(x) + \frac{9x^3}{16}\,J_0^2(x)\,J_1^2(x) - \frac{9x^2}{32}\,J_0(x)\,J_1^3(x) +$$
$$+\frac{36\,x^3-9\,x}{128}\,J_1^4(x) + \frac{3}{128}\,\mathcal{I}_0^{(40)}(x) - \frac{27}{128}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^3\,I_0^3(x)\,I_1(x)\,dx = -\frac{4\,x^3+3\,x}{128}\,I_0^4(x) + \frac{3x^2}{32}\,I_0^3(x)\,I_1(x) + \frac{9x^3}{16}\,I_0^2(x)\,I_1^2(x) - \frac{9x^2}{32}\,I_0(x)\,I_1^3(x) -$$
$$-\frac{36\,x^3+9\,x}{128}\,I_1^4(x) + \frac{3}{128}\,\mathcal{I}_0^{*(40)}(x) - \frac{27}{128}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^4\ J_0^3(x)\,J_1(x)\,dx = -\frac{x^4-2\,x^2}{16}\ J_0^4(x)+\frac{x^3-x}{4}\ J_0^3(x)\,J_1(x)+\frac{3x^4}{8}\ J_0^2(x)\,J_1^2(x)+\frac{3x^4}{16}\ J_1^4(x)-\frac{3}{4}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^4\ I_0^3(x)\,I_1(x)\,dx = \frac{x^4+2\,x^2}{16}\ I_0^4(x)-\frac{x^3+x}{4}\ I_0^3(x)\,I_1(x)+\frac{3x^4}{8}\ I_0^2(x)\,I_1^2(x)-\frac{3x^4}{16}\ I_1^4(x)+\frac{3}{4}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^5\ J_0^3(x)\,J_1(x)\,dx = -\frac{96\,x^5+60\,x^3-45\,x}{1024}\ J_0^4(x)+\frac{120\,x^4+45\,x^2}{256}\ J_0^3(x)\,J_1(x)+$$
$$+\frac{40\,x^5-105\,x^3}{128}\ J_0^2(x)\,J_1^2(x)+\frac{40\,x^4+115\,x^2}{256}\ J_0(x)\,J_1^3(x)+\frac{160\,x^5-460\,x^3+115\,x}{1024}\ J_1^4(x)-$$
$$-\frac{45}{1024}\,\mathcal{I}_0^{(40)}(x)+\frac{345}{1024}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^5\ I_0^3(x)\,I_1(x)\,dx = \frac{96\,x^5-60\,x^3-45\,x}{1024}\ I_0^4(x)-\frac{120\,x^4-45\,x^2}{256}\ I_0^3(x)\,I_1(x)+$$
$$+\frac{40\,x^5+105\,x^3}{128}\ I_0^2(x)\,I_1^2(x)+\frac{40\,x^4-115\,x^2}{256}\ I_0(x)\,I_1^3(x)-\frac{160\,x^5+460\,x^3+115\,x}{1024}\ I_1^4(x)+$$
$$+\frac{45}{1024}\,\mathcal{I}_0^{*(40)}(x)-\frac{345}{1024}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^6\ J_0^3(x)\,J_1(x)\,dx = -\frac{14\,x^6-21\,x^4+42\,x^2}{128}\ J_0^4(x)+\frac{21\,x^5-21\,x^3+21\,x}{32}\ J_0^3(x)\,J_1(x)+$$
$$+\frac{18\,x^6-63\,x^4}{64}\ J_0^2(x)\,J_1^2(x)+\frac{9x^5}{32}\ J_0(x)\,J_1^3(x)+\frac{18\,x^6-81\,x^4}{128}\ J_1^4(x)+\frac{63}{32}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^6\ I_0^3(x)\,I_1(x)\,dx = \frac{14\,x^6+21\,x^4+42\,x^2}{128}\ I_0^4(x)-\frac{21\,x^5+21\,x^3+21\,x}{32}\ I_0^3(x)\,I_1(x)+$$
$$+\frac{18\,x^6+63\,x^4}{64}\ I_0^2(x)\,I_1^2(x)+\frac{9x^5}{32}\ I_0(x)\,I_1^3(x)-\frac{18\,x^6+81\,x^4}{128}\ I_1^4(x)+\frac{63}{32}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int \frac{J_0^3(x)\,J_1(x)\,dx}{x} = \frac{3x}{2}\ J_0^4(x)-J_0^3(x)\,I_1(x)+3x\,J_0^2(x)\,J_1^2(x)+\frac{3x}{2}\ J_1^4(x)-\frac{1}{2}\,\mathcal{I}_0^{(40)}(x)+\frac{9}{2}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^3(x)\,I_1(x)\,dx}{x} = \frac{3x}{2}\ I_0^4(x)-I_0^3(x)\,I_1(x)-3x\,I_0^2(x)\,I_1^2(x)+\frac{3x}{2}\ I_1^4(x)-\frac{1}{2}\,\mathcal{I}_0^{*(40)}(x)+\frac{9}{2}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^3(x)\,J_1(x)\,dx}{x^3} = -\frac{10\,x^2+1}{3x}\ J_0^4(x)+\frac{6\,x^2-1}{3x^2}\ J_0^3(x)\,J_1(x)-\frac{20\,x^2-1}{3x}\ J_0^2(x)\,J_1^2(x)-\frac{2}{9}\ J_0(x)\,J_1^3(x)-$$
$$-\frac{10x}{3}\ J_1^4(x)+\frac{4}{3}\,\mathcal{I}_0^{(40)}(x)-\frac{92}{9}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^3(x)\,I_1(x)\,dx}{x^3} = \frac{10\,x^2-1}{3x}\ I_0^4(x)-\frac{6\,x^2+1}{3x^2}\ I_0^3(x)\,I_1(x)-\frac{20\,x^2+1}{3x}\ I_0^2(x)\,I_1^2(x)-\frac{2}{9}\ I_0(x)\,I_1^3(x)+$$
$$+\frac{10x}{3}\ I_1^4(x)-\frac{4}{3}\,\mathcal{I}_0^{*(40)}(x)+\frac{92}{9}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^3(x)\,J_1(x)\,dx}{x^5} = \frac{10688\,x^4 + 950\,x^2 - 375}{5625\,x^3}\,J_0^4(x) - \frac{1248\,x^4 - 190\,x^2 + 225}{1125\,x^4}\,J_0^3(x)\,J_1(x) +$$

$$+\frac{21376\,x^4 - 1220\,x^2 + 675}{5625\,x^3}\,J_0^2(x)\,J_1^2(x) + \frac{2656\,x^2 - 810}{16875\,x^2}\,J_0(x)\,J_1^3(x) + \frac{10688\,x^2 + 54}{5625\,x}\,J_1^4(x) -$$

$$-\frac{4448}{5625}\,\mathcal{I}_0^{(40)}(x) + \frac{98848}{16875}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^3(x)\,I_1(x)\,dx}{x^5} = \frac{10688\,x^4 - 950\,x^2 - 375}{5625\,x^3}\,I_0^4(x) - \frac{1248\,x^4 + 190\,x^2 + 225}{1125\,x^4}\,I_0^3(x)\,I_1(x) -$$

$$-\frac{21376\,x^4 + 1220\,x^2 + 675}{5625\,x^3}\,I_0^2(x)\,I_1^2(x) - \frac{2656\,x^2 + 810}{16875\,x^2}\,I_0(x)\,I_1^3(x) + \frac{10688\,x^2 - 54}{5625\,x}\,I_1^4(x) -$$

$$-\frac{4448}{5625}\,\mathcal{I}_0^{*(40)}(x) + \frac{98848}{16875}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. g) Integrals of $x^m\, Z_0^2(x)\, Z_1^2(x)$

Explicit and basic integrals are omitted.
With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 438 holds

$$\int J_0^2(x)\,J_1^2(x)\,dx = -\frac{x}{2}\,J_0^4(x) - x\,J_0^2(x)\,J_1^2(x) - \frac{x}{2}\,J_1^4(x) + \frac{1}{2}\,\mathcal{I}_0^{(40)}(x) - \frac{3}{2}\,\mathcal{I}_0^{(04)}(x)$$

$$\int I_0^2(x)\,I_1^2(x)\,dx = \frac{x}{2}\,I_0^4(x) - x\,I_0^2(x)\,I_1^2(x) + \frac{x}{2}\,I_1^4(x) - \frac{1}{2}\,\mathcal{I}_0^{*(40)}(x) + \frac{3}{2}\,\mathcal{I}_0^{*(04)}(x)$$

x^1 : Basic integral.

$$\int x^2\; J_0^2(x)\,J_1^2(x)\,dx = \frac{4\,x^3 - 3\,x}{32}\,J_0^4(x) - \frac{3x^2}{8}\,J_0^3(x)\,J_1(x) + \frac{x^3}{4}\,J_0^2(x)\,J_1^2(x) - \frac{x^2}{8}\,J_0(x)\,J_1^3(x) +$$

$$+\frac{4\,x^3 - x}{32}\,J_1^4(x) + \frac{3}{32}\,\mathcal{I}_0^{(40)}(x) - \frac{3}{32}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^2\; I_0^2(x)\,I_1^2(x)\,dx = -\frac{4\,x^3 + 3\,x}{32}\,I_0^4(x) + \frac{3x^2}{8}\,I_0^3(x)\,I_1(x) + \frac{x^3}{4}\,I_0^2(x)\,I_1^2(x) - \frac{x^2}{8}\,I_0(x)\,I_1^3(x) -$$

$$-\frac{4\,x^3 + x}{32}\,I_1^4(x) + \frac{3}{32}\,\mathcal{I}_0^{*(40)}(x) - \frac{3}{32}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^3\; J_0^2(x)\,J_1^2(x)\,dx = \frac{x^4 - 2\,x^2}{16}\,J_0^4(x) - \frac{x^3 - x}{4}\,J_0^3(x)\,J_1(x) + \frac{x^4}{8}\,J_0^2(x)\,J_1^2(x) + \frac{x^4}{16}\,J_1^4(x) + \frac{3}{4}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^3\; I_0^2(x)\,I_1^2(x)\,dx = -\frac{x^4 + 2\,x^2}{16}\,I_0^4(x) + \frac{x^3 + x}{4}\,I_0^3(x)\,I_1(x) + \frac{x^4}{8}\,I_0^2(x)\,I_1^2(x) - \frac{x^4}{16}\,I_1^4(x) - \frac{3}{4}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^4\; J_0^2(x)\,J_1^2(x)\,dx = \frac{32\,x^5 + 12\,x^3 - 9\,x}{768}\,J_0^4(x) - \frac{40\,x^4 + 9\,x^2}{192}\,J_0^3(x)\,J_1(x) + \frac{8\,x^5 + 33\,x^3}{96}\,J_0^2(x)\,J_1^2(x) +$$

$$+\frac{8\,x^4 - 31\,x^2}{192}\,J_0(x)\,J_1^3(x) + \frac{32\,x^5 + 124\,x^3 - 31\,x}{768}\,J_1^4(x) + \frac{3}{256}\,\mathcal{I}_0^{(40)}(x) - \frac{31}{256}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^4\, I_0^2(x)\, I_1^2(x)\, dx = -\frac{32\,x^5 - 12\,x^3 + 9\,x}{768}\, I_0^4(x) + \frac{40\,x^4 - 9\,x^2}{192}\, I_0^3(x)\, I_1(x) + \frac{8\,x^5 - 33\,x^3}{96}\, I_0^2(x)\, I_1^2(x) +$$

$$+\frac{8\,x^4 + 31\,x^2}{192}\, I_0(x)\, I_1^3(x) - \frac{32\,x^5 - 124\,x^3 - 31\,x}{768}\, I_1^4(x) - \frac{3}{256}\, \mathcal{I}_0^{*(40)}(x) + \frac{31}{256}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int x^5\, J_0^2(x)\, J_1^2(x)\, dx = \frac{2\,x^6 - 3\,x^4 + 6\,x^2}{64}\, J_0^4(x) - \frac{3\,x^5 - 3\,x^3 + 3\,x}{16}\, J_0^3(x)\, J_1(x) + \frac{2\,x^6 + 9\,x^4}{32}\, J_0^2(x)\, J_1^2(x) +$$

$$+\frac{x^5}{16}\, J_0(x)\, J_1^3(x) + \frac{2\,x^6 + 7\,x^4}{64}\, J_1^4(x) - \frac{9}{16}\, \mathcal{I}_1^{(22)}(x)$$

$$\int x^5\, I_0^2(x)\, I_1^2(x)\, dx = -\frac{2\,x^6 + 3\,x^4 + 6\,x^2}{64}\, I_0^4(x) + \frac{3\,x^5 + 3\,x^3 + 3\,x}{16}\, I_0^3(x)\, I_1(x) + \frac{2\,x^6 - 9\,x^4}{32}\, I_0^2(x)\, I_1^2(x) +$$

$$+\frac{x^5}{16}\, I_0(x)\, I_1^3(x) - \frac{2\,x^6 - 7\,x^4}{64}\, I_1^4(x) - \frac{9}{16}\, \mathcal{I}_1^{*(22)}(x)$$

$$\int x^6\, J_0^2(x)\, J_1^2(x)\, dx = \frac{256\,x^7 - 768\,x^5 - 444\,x^3 + 333\,x}{10240}\, J_0^4(x) - \frac{448\,x^6 - 960\,x^4 - 333\,x^2}{2560}\, J_0^3(x)\, J_1(x) +$$

$$+\frac{64\,x^7 + 368\,x^5 - 831\,x^3}{1280}\, J_0^2(x)\, J_1^2(x) + \frac{192\,x^6 + 224\,x^4 + 887\,x^2}{2560}\, J_0(x)\, J_1^3(x) +$$

$$+\frac{256\,x^7 + 896\,x^5 - 3548\,x^3 + 887\,x}{10240}\, J_1^4(x) - \frac{333}{10240}\, \mathcal{I}_0^{(40)}(x) + \frac{2661}{10240}\, \mathcal{I}_0^{(04)}(x)$$

$$\int x^6\, I_0^2(x)\, I_1^2(x)\, dx = -\frac{256\,x^7 + 768\,x^5 - 444\,x^3 - 333\,x}{10240}\, I_0^4(x) + \frac{448\,x^6 + 960\,x^4 - 333\,x^2}{2560}\, I_0^3(x)\, I_1(x) +$$

$$+\frac{64\,x^7 - 368\,x^5 - 831\,x^3}{1280}\, I_0^2(x)\, I_1^2(x) + \frac{192\,x^6 - 224\,x^4 + 887\,x^2}{2560}\, I_0(x)\, I_1^3(x) -$$

$$-\frac{256\,x^7 - 896\,x^5 - 3548\,x^3 - 887\,x}{10240}\, I_1^4(x) - \frac{333}{10240}\, \mathcal{I}_0^{*(40)}(x) + \frac{2661}{10240}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^2(x)\, J_1^2(x)\, dx}{x^2} = \frac{4x}{3}\, J_0^4(x) - \frac{2}{3}\, J_0^3(x)\, J_1(x) + \frac{8\,x^2 - 1}{3x}\, J_0^2(x)\, J_1^2(x) + \frac{2}{9}\, J_0(x)\, J_1^3(x) + \frac{4x}{3}\, J_1^4(x) -$$

$$-\frac{2}{3}\, \mathcal{I}_0^{(40)}(x) + \frac{38}{9}\, \mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^2(x)\, I_1^2(x)\, dx}{x^2} = \frac{4x}{3}\, I_0^4(x) - \frac{2}{3}\, I_0^3(x)\, I_1(x) - \frac{8\,x^2 + 1}{3x}\, I_0^2(x)\, I_1^2(x) - \frac{2}{9}\, I_0(x)\, I_1^3(x) + \frac{4x}{3}\, I_1^4(x) -$$

$$-\frac{2}{3}\, \mathcal{I}_0^{*(40)}(x) + \frac{38}{9}\, \mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^2(x)\, J_1^2(x)\, dx}{x^4} = -\frac{632\,x^2 + 50}{375\,x}\, J_0^4(x) + \frac{72\,x^2 - 10}{75\,x^2}\, J_0^3(x)\, J_1(x) - \frac{1264\,x^4 - 80\,x^2 + 75}{375\,x^3}\, J_0^2(x)\, J_1^2(x) -$$

$$-\frac{184\,x^2 - 90}{1125x^2}\, J_0(x)\, J_1^3(x) - \frac{632\,x^2 + 6}{375\,x}\, J_1^4(x) + \frac{272}{375}\, \mathcal{I}_0^{(40)}(x) - \frac{5872}{1125}\, \mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^2(x)\,I_1^2(x)\,dx}{x^4} = \frac{632\,x^2-50}{375\,x}\,I_0^4(x) - \frac{72\,x^2+10}{75x^2}\,I_0^3(x)\,I_1(x) - \frac{1264\,x^4+80\,x^2+75}{375x^3}\,I_0^2(x)\,I_1^2(x) -$$
$$- \frac{184\,x^2+90}{1125\,x^2}\,I_0(x)\,I_1^3(x) + \frac{632\,x^2-6}{375\,x}\,I_1^4(x) - \frac{272}{375}\,\mathcal{I}_0^{*(40)}(x) + \frac{5872}{1125}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0^2(x)\,J_1^2(x)\,dx}{x^6} = \frac{1485184\,x^4+124600\,x^2-36750}{1929375\,x^3}\,J_0^4(x) - \frac{171264\,x^4-24920\,x^2+22050}{385875\,x^4}\,J_0^3(x)\,J_1(x) +$$
$$+ \frac{2970368\,x^6-178960\,x^4+113400\,x^2-275625}{1929375\,x^5}\,J_0^2(x)\,J_1^2(x) + \frac{401408\,x^4-163080\,x^2+236250}{5788125\,x^4}\,J_0(x)\,J_1^3(x) +$$
$$+ \frac{1485184\,x^4+10872\,x^2-11250}{1929375\,x^3}\,J_1^4(x) - \frac{628864}{1929375}\,\mathcal{I}_0^{(40)}(x) + \frac{13768064}{5788125}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0^2(x)\,I_1^2(x)\,dx}{x^6} = \frac{1485184\,x^4-124600\,x^2-36750}{1929375\,x^3}\,I_0^4(x) - \frac{171264\,x^4+24920\,x^2+22050}{385875\,x^4}\,I_0^3(x)\,I_1(x) -$$
$$- \frac{2970368\,x^6+178960\,x^4+113400\,x^2+275625}{1929375\,x^5}\,I_0^2(x)\,I_1^2(x) - \frac{401408\,x^4+163080\,x^2+236250}{5788125\,x^4}\,I_0(x)\,I_1^3(x) +$$
$$+ \frac{1485184\,x^4-10872\,x^2-11250}{1929375\,x^3}\,I_1^4(x) - \frac{628864}{1929375}\,\mathcal{I}_0^{*(40)}(x) + \frac{13768064}{5788125}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. h) Integrals of $x^m\,Z_0(x)\,Z_1^3(x)$

Explicit and basic integrals are omitted.
With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 438 holds

$$\int x\,J_0(x)\,J_1^3(x)\,dx = \frac{x}{4}\,J_1^4(x) + \frac{3}{4}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x\,J_0(x)\,I_1^3(x)\,dx = \frac{x}{4}\,I_1^4(x) + \frac{3}{4}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^2\,J_0(x)\,J_1^3(x)\,dx = -\frac{x^2}{4}\,J_0^4(x) + \frac{x}{2}\,J_0^3(x)\,J_1(x) - \frac{x^2}{2}\,J_0^2(x)\,J_1^2(x) + \frac{3}{2}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^2\,I_0(x)\,I_1^3(x)\,dx = -\frac{x^2}{4}\,I_0^4(x) + \frac{x}{2}\,I_0^3(x)\,I_1(x) + \frac{x^2}{2}\,I_0^2(x)\,I_1^2(x) - \frac{3}{2}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^3\,J_0(x)\,J_1^3(x)\,dx = \frac{12\,x^3-9\,x}{128}\,J_0^4(x) - \frac{9x^2}{32}\,J_0^3(x)\,J_1(x) + \frac{3x^3}{16}\,J_0^2(x)\,J_1^2(x) - \frac{11x^2}{32}\,J_0(x)\,J_1^3(x) +$$
$$+ \frac{44\,x^3-11\,x}{128}\,J_1^4(x) + \frac{9}{128}\,\mathcal{I}_0^{(40)}(x) - \frac{33}{128}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^3\,I_0(x)\,I_1^3(x)\,dx = \frac{12\,x^3+9\,x}{128}\,I_0^4(x) - \frac{9x^2}{32}\,I_0^3(x)\,I_1(x) - \frac{3x^3}{16}\,I_0^2(x)\,I_1^2(x) + \frac{11x^2}{32}\,I_0(x)\,I_1^3(x) +$$
$$+ \frac{44\,x^3+11\,x}{128}\,I_1^4(x) - \frac{9}{128}\,\mathcal{I}_0^{*(40)}(x) + \frac{33}{128}\,\mathcal{I}_0^{*(04)}(x)$$

About $x^4\,Z_(x)\,Z_1^3(x)$ see page 426.

$$\int x^5\, J_0(x)\, J_1^3(x)\, dx = -\frac{32\,x^5+36\,x^3-27\,x}{1024}\, J_0^4(x) + \frac{40\,x^4+27\,x^2}{256}\, J_0^3(x)\, J_1(x) - \frac{8\,x^5+39\,x^3}{128}\, J_0^2(x)\, J_1^2(x) +$$

$$+\frac{56\,x^4+53\,x^2}{256}\, J_0(x)\, J_1^3(x) + \frac{224\,x^5-212\,x^3+53\,x}{1024}\, J_1^4(x) - \frac{27}{1024}\,\mathcal{I}_0^{(40)}(x) + \frac{159}{1024}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^5\, I_0(x)\, I_1^3(x)\, dx = -\frac{32\,x^5-36\,x^3-27\,x}{1024}\, I_0^4(x) + \frac{40\,x^4-27\,x^2}{256}\, I_0^3(x)\, I_1(x) + \frac{8\,x^5-39\,x^3}{128}\, I_0^2(x)\, I_1^2(x) -$$

$$-\frac{56\,x^4-53\,x^2}{256}\, I_0(x)\, I_1^3(x) + \frac{224\,x^5+212\,x^3+53\,x}{1024}\, I_1^4(x) - \frac{27}{1024}\,\mathcal{I}_0^{*(40)}(x) + \frac{159}{1024}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^6\, J_0(x)\, J_1^3(x)\, dx = -\frac{6\,x^6-9\,x^4+18\,x^2}{128}\, J_0^4(x) + \frac{9\,x^5-9\,x^3+9\,x}{32}\, J_0^3(x)\, J_1(x) -$$

$$-\frac{6\,x^6+27\,x^4}{64}\, J_0^2(x)\, J_1^2(x) + \frac{13x^5}{32}\, J_0(x)\, J_1^3(x) + \frac{26\,x^6-53\,x^4}{128}\, J_1^4(x) + \frac{27}{32}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^6\, I_0(x)\, I_1^3(x)\, dx = -\frac{6\,x^6+9\,x^4+18\,x^2}{128}\, I_0^4(x) + \frac{9\,x^5+9\,x^3+9\,x}{32}\, I_0^3(x)\, I_1(x) +$$

$$+\frac{6\,x^6-27\,x^4}{64}\, I_0^2(x)\, I_1^2(x) - \frac{13x^5}{32}\, I_0(x)\, I_1^3(x) + \frac{26\,x^6+53\,x^4}{128}\, I_1^4(x) - \frac{27}{32}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int \frac{J_0(x)\, J_1^3(x)\, dx}{x} = -\frac{x}{2}\, J_0^4(x) - x\, J_0^2(x)\, J_1^2(x) - \frac{1}{3}\, J_0(x)\, J_1^3(x) - \frac{x}{2}\, J_1^4(x) + \frac{1}{2}\,\mathcal{I}_0^{(40)}(x) - \frac{11}{6}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0(x)\, I_1^3(x)\, dx}{x} = \frac{x}{2}\, I_0^4(x) - x\, I_0^2(x)\, I_1^2(x) - \frac{1}{3}\, I_0(x)\, I_1^3(x) + \frac{x}{2}\, I_1^4(x) - \frac{1}{2}\,\mathcal{I}_0^{*(40)}(x) + \frac{11}{6}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0(x)\, J_1^3(x)\, dx}{x^3} = \frac{22x}{25}\, J_0^4(x) - \frac{2}{5}\, J_0^3(x)\, J_1(x) + \frac{44\,x^2-5}{25x}\, J_0^2(x)\, J_1^2(x) + \frac{14\,x^2-15}{75x^2}\, J_0(x)\, J_1^3(x) +$$

$$+\frac{22\,x^2+1}{25x}\, J_1^4(x) - \frac{12}{25}\,\mathcal{I}_0^{(40)}(x) + \frac{212}{75}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0(x)\, I_1^3(x)\, dx}{x^3} = \frac{22x}{25}\, I_0^4(x) - \frac{2}{5}\, I_0^3(x)\, I_1(x) - \frac{44\,x^2+5}{25x}\, I_0^2(x)\, I_1^2(x) - \frac{14\,x^2+15}{75x^2}\, I_0(x)\, I_1^3(x) +$$

$$+\frac{22\,x^2-1}{25x}\, I_1^4(x) - \frac{12}{25}\,\mathcal{I}_0^{*(40)}(x) + \frac{212}{75}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_0(x)\, J_1^3(x)\, dx}{x^5} = -\frac{4864\,x^2+350}{6125\,x}\, J_0^4(x) + \frac{544\,x^2-70}{1225\,x^2}\, J_0^3(x)\, J_1(x) - \frac{9728\,x^4-660\,x^2+525}{6125\,x^3}\, J_0^2(x)\, J_1^2(x) -$$

$$-\frac{1568\,x^4-930\,x^2+2625}{18375\,x^4}\, J_0(x)\, J_1^3(x) - \frac{4864\,x^4+62\,x^2-125}{6125\,x^3}\, J_1^4(x) + \frac{2144}{6125}\,\mathcal{I}_0^{(40)}(x) - \frac{45344}{18375}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_0(x)\, I_1^3(x)\, dx}{x^5} = \frac{4864\,x^2-350}{6125\,x}\, I_0^4(x) - \frac{544\,x^2+70}{1225\,x^2}\, I_0^3(x)\, I_1(x) - \frac{9728\,x^4+660\,x^2+525}{6125\,x^3}\, I_0^2(x)\, I_1^2(x) -$$

$$-\frac{1568\,x^4+930\,x^2+2625}{18375\,x^4}\, I_0(x)\, I_1^3(x) + \frac{4864\,x^4-62\,x^2-125}{6125\,x^3}\, I_1^4(x) - \frac{2144}{6125}\,\mathcal{I}_0^{*(40)}(x) + \frac{45344}{18375}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. i) Integrals of $x^m\, Z_1^4(x)$

Explicit and basic integrals are omitted.
With the basic integrals $\mathcal{I}_0^{(40)}(x)$, $\mathcal{I}_1^{(22)}(x)$, $\mathcal{I}_0^{(04)}(x)$ and $\mathcal{I}_0^{*(40)}(x)$, $\mathcal{I}_1^{*(22)}(x)$, $\mathcal{I}_0^{*(04)}(x)$ as defined on page 438 holds

$$\int x\, J_1^4(x)\,dx = -\frac{x^2}{2}\, J_0^4(x) + x\, J_0^3(x)\, J_1(x) - x^2\, J_0^2(x)\, J_1^2(x) - \frac{x^2}{2}\, J_1^4(x) + 3\mathcal{I}_1^{(22)}(x)$$

$$\int x\, I_1^4(x)\,dx = -\frac{x^2}{2}\, I_0^4(x) + x\, I_0^3(x)\, I_1(x) + x^2\, I_0^2(x)\, I_1^2(x) - \frac{x^2}{2}\, I_1^4(x) - 3\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^2\, J_1^4(x)\,dx = \frac{12\,x^3 - 9\,x}{32}\, J_0^4(x) - \frac{9x^2}{8}\, J_0^3(x)\, J_1(x) + \frac{3x^3}{4}\, J_0^2(x)\, J_1^2(x) - \frac{11x^2}{8}\, J_0(x)\, J_1^3(x) +$$
$$+ \frac{12\,x^3 - 11\,x}{32}\, J_1^4(x) + \frac{9}{32}\,\mathcal{I}_0^{(40)}(x) - \frac{33}{32}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^2\, I_1^4(x)\,dx = \frac{12\,x^3 + 9\,x}{32}\, I_0^4(x) - \frac{9x^2}{8}\, I_0^3(x)\, I_1(x) - \frac{3x^3}{4}\, I_0^2(x)\, I_1^2(x) + \frac{11x^2}{8}\, I_0(x)\, I_1^3(x) +$$
$$+ \frac{12\,x^3 + 11\,x}{32}\, I_1^4(x) - \frac{9}{32}\,\mathcal{I}_0^{*(40)}(x) + \frac{33}{32}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^3\, J_1^4(x)\,dx =$$
$$= \frac{3\,x^4 - 6\,x^2}{16}\, J_0^4(x) - \frac{3\,x^3 - 3\,x}{4}\, J_0^3(x)\, J_1(x) + \frac{3x^4}{8}\, J_0^2(x)\, J_1^2(x) - x^3\, J_0(x)\, J_1^3(x) + \frac{3x^4}{16}\, J_1^4(x) + \frac{9}{4}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^3\, I_1^4(x)\,dx =$$
$$= \frac{3\,x^4 + 6\,x^2}{16}\, I_0^4(x) - \frac{3\,x^3 + 3\,x}{4}\, I_0^3(x)\, I_1(x) - \frac{3x^4}{8}\, I_0^2(x)\, I_1^2(x) + x^3\, I_0(x)\, I_1^3(x) + \frac{3x^4}{16}\, I_1^4(x) + \frac{9}{4}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^4\, J_1^4(x)\,dx = \frac{32\,x^5 + 36\,x^3 - 27\,x}{256}\, J_0^4(x) - \frac{40\,x^4 + 27\,x^2}{64}\, J_0^3(x)\, J_1(x) + \frac{8\,x^5 + 39\,x^3}{32}\, J_0^2(x)\, J_1^2(x) -$$
$$- \frac{56\,x^4 + 53\,x^2}{64}\, J_0(x)\, J_1^3(x) + \frac{32\,x^5 + 212\,x^3 - 53\,x}{256}\, J_1^4(x) + \frac{27}{256}\,\mathcal{I}_0^{(40)}(x) - \frac{159}{256}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^4\, I_1^4(x)\,dx = \frac{32\,x^5 - 36\,x^3 - 27\,x}{256}\, I_0^4(x) - \frac{40\,x^4 - 27\,x^2}{64}\, I_0^3(x)\, I_1(x) - \frac{8\,x^5 - 39\,x^3}{32}\, I_0^2(x)\, I_1^2(x) +$$
$$+ \frac{56\,x^4 - 53\,x^2}{64}\, I_0(x)\, I_1^3(x) + \frac{32\,x^5 - 212\,x^3 - 53\,x}{256}\, I_1^4(x) + \frac{27}{256}\,\mathcal{I}_0^{*(40)}(x) - \frac{159}{256}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int x^5\, J_1^4(x)\,dx = \frac{6\,x^6 - 9\,x^4 + 18\,x^2}{64}\, J_0^4(x) - \frac{9\,x^5 - 9\,x^3 + 9\,x}{16}\, J_0^3(x)\, J_1(x) + \frac{6\,x^6 + 27\,x^4}{32}\, J_0^2(x)\, J_1^2(x) -$$
$$- \frac{13x^5}{16}\, J_0(x)\, J_1^3(x) + \frac{6\,x^6 + 53\,x^4}{64}\, J_1^4(x) - \frac{27}{16}\,\mathcal{I}_1^{(22)}(x)$$

$$\int x^5\, I_1^4(x)\,dx = \frac{6\,x^6+9\,x^4+18\,x^2}{64}\,I_0^4(x) - \frac{9\,x^5+9\,x^3+9\,x}{16}\,I_0^3(x)\,I_1(x) - \frac{6\,x^6-27\,x^4}{32}\,I_0^2(x)\,I_1^2(x) +$$
$$+\frac{13x^5}{16}\,I_0(x)\,I_1^3(x) + \frac{6\,x^6-53\,x^4}{64}\,I_1^4(x) + \frac{27}{16}\,\mathcal{I}_1^{*(22)}(x)$$

$$\int x^6\, J_1^4(x)\,dx = \frac{768\,x^7-3264\,x^5-2412\,x^3+1809\,x}{10240}\,J_0^4(x) - \frac{1344\,x^6-4080\,x^4-1809\,x^2}{2560}\,J_0^3(x)\,J_1(x) +$$
$$+\frac{192\,x^7+864\,x^5-3663\,x^3}{1280}\,J_0^2(x)\,J_1^2(x) - \frac{1984\,x^6-2352\,x^4-4251\,x^2}{2560}\,J_0(x)\,J_1^3(x) +$$
$$+\frac{768\,x^7+9408\,x^5-17004\,x^3+4251\,x}{10240}\,J_1^4(x) - \frac{1809}{10240}\,\mathcal{I}_0^{(40)}(x) + \frac{12753}{10240}\,\mathcal{I}_0^{(04)}(x)$$

$$\int x^6\, I_1^4(x)\,dx = \frac{768\,x^7+3264\,x^5-2412\,x^3-1809\,x}{10240}\,I_0^4(x) - \frac{1344\,x^6+4080\,x^4-1809\,x^2}{2560}\,I_0^3(x)\,I_1(x) -$$
$$-\frac{192\,x^7-864\,x^5-3663\,x^3}{1280}\,I_0^2(x)\,I_1^2(x) + \frac{1984\,x^6+2352\,x^4-4251\,x^2}{2560}\,I_0(x)\,I_1^3(x) +$$
$$+\frac{768\,x^7-9408\,x^5-17004\,x^3-4251\,x}{10240}\,I_1^4(x) + \frac{1809}{10240}\,\mathcal{I}_0^{*(40)}(x) - \frac{12753}{10240}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_1^4(x)\,dx}{x^2} =$$
$$= -\frac{2x}{5}\,J_0^4(x) - \frac{4x}{5}\,J_0^2(x)\,J_1^2(x) - \frac{4}{15}\,J_0(x)\,J_1^3(x) - \frac{2\,x^2+1}{5x}\,J_1^4(x) + \frac{2}{5}\,\mathcal{I}_0^{(40)}(x) - \frac{22}{15}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_1^4(x)\,dx}{x^2} =$$
$$= \frac{2x}{5}\,I_0^4(x) - \frac{4x}{5}\,I_0^2(x)\,I_1^2(x) - \frac{4}{15}\,I_0(x)\,I_1^3(x) + \frac{2\,x^2-1}{5x}\,I_1^4(x) - \frac{2}{5}\,\mathcal{I}_0^{*(40)}(x) + \frac{22}{15}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_1^4(x)\,dx}{x^4} = \frac{88x}{175}\,J_0^4(x) - \frac{8}{35}\,J_0^3(x)\,J_1(x) + \frac{176\,x^2-20}{175x}\,J_0^2(x)\,J_1^2(x) + \frac{56\,x^2-60}{525\,x^2}\,J_0(x)\,J_1^3(x) +$$
$$+\frac{88\,x^4+4\,x^2-25}{175\,x^3}\,J_1^4(x) - \frac{48}{175}\,\mathcal{I}_0^{(40)}(x) + \frac{848}{525}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_1^4(x)\,dx}{x^4} = \frac{88x}{175}\,I_0^4(x) - \frac{8}{35}\,I_0^3(x)\,I_1(x) - \frac{176\,x^2+20}{175x}\,I_0^2(x)\,I_1^2(x) - \frac{56\,x^2+60}{525\,x^2}\,I_0(x)\,I_1^3(x) +$$
$$+\frac{88\,x^4-4\,x^2-25}{175\,x^3}\,I_1^4(x) - \frac{48}{175}\,\mathcal{I}_0^{*(40)}(x) + \frac{848}{525}\,\mathcal{I}_0^{*(04)}(x)$$

$$\int \frac{J_1^4(x)\,dx}{x^6} = -\frac{19456\,x^2+1400}{55125\,x}\,J_0^4(x) + \frac{2176\,x^2-280}{11025\,x^2}\,J_0^3(x)\,J_1(x) -$$
$$-\frac{38912\,x^4-2640\,x^2+2100}{55125\,x^3}\,J_0^2(x)\,J_1^2(x) - \frac{6272\,x^4-3720\,x^2+10500}{165375\,x^4}\,J_0(x)\,J_1^3(x) -$$
$$-\frac{19456\,x^6+248\,x^4-500\,x^2+6125}{55125\,x^5}\,J_1^4(x) + \frac{8576}{55125}\,\mathcal{I}_0^{(40)}(x) - \frac{181376}{165375}\,\mathcal{I}_0^{(04)}(x)$$

$$\int \frac{I_1^4(x)\,dx}{x^6} = \frac{19456\,x^2 - 1400}{55125\,x}\,I_0^4(x) - \frac{2176\,x^2 + 280}{11025\,x^2}\,I_0^3(x)\,I_1(x) -$$

$$-\frac{38912\,x^4 + 2640\,x^2 + 2100}{55125\,x^3}\,I_0^2(x)\,I_1^2(x) - \frac{6272\,x^4 + 3720\,x^2 + 10500}{165375\,x^4}\,I_0(x)\,I_1^3(x) +$$

$$+\frac{19456\,x^6 - 248\,x^4 - 500\,x^2 - 6125}{55125\,x^5}\,I_1^4(x) - \frac{8576}{55125}\,\mathcal{I}_0^{*(40)}(x) + \frac{181376}{165375}\,\mathcal{I}_0^{*(04)}(x)$$

4.1. j) Recurrence relations

With the integrals

$$\mathcal{I}_n^{(pq)}(x) \quad = \quad \int x^n\,J_0^p(x)\,J_1^q(x)\,dx\,, \qquad p+q=4$$

and

$$\mathcal{I}_n^{*(pq)}(x) \quad = \quad \int x^n\,I_0^p(x)\,I_1^q(x)\,dx\,, \qquad p+q=4$$

holds

Ascending recurrence relations:

$$\mathcal{I}_{n+1}^{(40)}(x) = \frac{x^{n+1}}{8n}\left[3x\,J_0^4(x) + (5n-6)\,J_0^3(x)\,J_1(x) + 6x\,J_0^2(x)\,J_1^2(x) + 3(n-2)\,J_0(x)\,J_1^3(x) + 3x\,J_1^4(x)\right] +$$

$$-\frac{5n-6}{8}\,\mathcal{I}_n^{(31)}(x) - \frac{3(n-2)^2}{8n}\,\mathcal{I}_n^{(13)}(x)$$

$$\mathcal{I}_{n+1}^{(31)}(x) = -\frac{x^{n+1}}{4}\,J_0^4(x) + \frac{n+1}{4}\,\mathcal{I}_n^{(40)}(x)$$

$$\mathcal{I}_{n+1}^{(22)}(x) = \frac{x^{n+1}}{8n}\left[x\,J_0^4(x) - (n+2)\,J_0^3(x)\,J_1(x) + 2x\,J_0^2(x)\,J_1^2(x) + (n-2)\,J_0(x)\,J_1^3(x) + x\,J_1^4(x)\right] +$$

$$+\frac{n+2}{8}\,\mathcal{I}_n^{(31)}(x) - \frac{(n-2)^2}{8n}\mathcal{I}_n^{(13)}(x)$$

$$\mathcal{I}_{n+1}^{(13)}(x) = -\frac{x^{n+1}}{4}\left[J_0^4(x) + 2\,J_0^2(x)\,J_1^2(x)\right] + \frac{n+1}{4}\,\mathcal{I}_n^{(40)}(x) + \frac{n-1}{2}\,\mathcal{I}_n^{(22)}(x)$$

$$\mathcal{I}_{n+1}^{(04)}(x) = \frac{x^{n+1}}{8n}\left[3x\,J_0^4(x) - 3(n+2)\,J_0^3(x)\,J_1(x) + 6x\,J_0^2(x)\,J_1^2(x) - (5n+6)\,J_0(x)\,J_1^3(x) + 3x\,J_1^4(x)\right] +$$

$$+\frac{3(n+2)}{8}\,\mathcal{I}_n^{(31)}(x) + \frac{(5n+6)(n-2)}{8n}\,\mathcal{I}_n^{(13)}(x)$$

$$\mathcal{I}_{n+1}^{*(40)}(x) = \frac{x^{n+1}}{8n}\left[3x\,I_0^4(x) + (5n-6)\,I_0^3(x)\,I_1(x) - 6x\,I_0^2(x)\,I_1^2(x) - 3(n-2)\,I_0(x)\,I_1^3(x) + 3x\,I_1^4(x)\right] -$$

$$-\frac{5n-6}{8}\,\mathcal{I}_n^{*(31)}(x) + \frac{3(n-2)^2}{8n}\,\mathcal{I}_n^{*(13)}(x)$$

$$\mathcal{I}_{n+1}^{*(31)}(x) = \frac{x^{n+1}}{4}\,I_0^4(x) - \frac{n+1}{4}\,\mathcal{I}_n^{*(40)}(x)$$

$$\mathcal{I}_{n+1}^{*(22)}(x) = \frac{x^{n+1}}{8n}\left[-x\,I_0^4(x) + (n+2)\,I_0^3(x)\,I_1(x) + 2x\,I_0^2(x)\,I_1^2(x) + (n-2)\,I_0(x)\,I_1^3(x) - x\,I_1^4(x)\right] -$$

$$-\frac{n+2}{8}\,\mathcal{I}_n^{*(31)}(x)-\frac{(n-2)^2}{8n}\mathcal{I}_n^{*(13)}(x)$$

$$\mathcal{I}_{n+1}^{*(13)}(x)=\frac{x^{n+1}}{4}\,\left[-I_0^4(x)+2\,I_0^2(x)\,I_1^2(x)\right]+\frac{n+1}{4}\,\mathcal{I}_n^{*(40)}(x)-\frac{n-1}{2}\,\mathcal{I}_n^{*(22)}(x)$$

$$\mathcal{I}_{n+1}^{*(04)}(x)=\frac{x^{n+1}}{8n}\,\left[3x\,I_0^4(x)-3(n+2)\,I_0^3(x)\,I_1(x)-6x\,I_0^2(x)\,I_1^2(x)+(5n+6)\,I_0(x)\,I_1^3(x)+3x\,I_1^4(x)\right]+$$

$$+\frac{3(n+2)}{8}\,\mathcal{I}_n^{*(31)}(x)-\frac{(5n+6)(n-2)}{8n}\,\mathcal{I}_n^{*(13)}(x)$$

Descending recurrence relations:

$$\mathcal{I}_{-n-1}^{(40)}(x)=-\frac{J_0^4(x)}{n\,x^n}-\frac{4}{n}\,\mathcal{I}_{-n}^{(31)}(x)$$

$$\mathcal{I}_{-n-1}^{(31)}(x)=-\frac{J_0^3(x)\,J_1(x)}{(n+1)\,x^n}+\frac{1}{n+1}\,\left[\mathcal{I}_{-n}^{(40)}(x)-3\,\mathcal{I}_{-n}^{(22)}(x)\right]$$

$$\mathcal{I}_{-n-1}^{(22)}(x)=-\frac{J_0^2(x)\,J_1^2(x)}{(n+2)\,x^n}+\frac{2}{n+2}\,\left[\mathcal{I}_{-n}^{(31)}(x)-\mathcal{I}_{-n}^{(13)}(x)\right]$$

$$\mathcal{I}_{-n-1}^{(13)}(x)=-\frac{J_0(x)\,J_1^3(x)}{(n+3)\,x^n}+\frac{1}{n+3}\,\left[3\,\mathcal{I}_{-n}^{(22)}(x)-\mathcal{I}_{-n}^{(04)}(x)\right]$$

$$\mathcal{I}_{-n-1}^{(04)}(x)=-\frac{J_1^4(x)}{(n+4)\,x^n}+\frac{4}{n+4}\,\mathcal{I}_{-n}^{(13)}(x)$$

$$\mathcal{I}_{-n-1}^{*(40)}(x)=-\frac{I_0^4(x)}{n\,x^n}+\frac{4}{n}\,\mathcal{I}_{-n}^{*(31)}(x)$$

$$\mathcal{I}_{-n-1}^{*(31)}(x)=-\frac{I_0^3(x)\,I_1(x)}{(n+1)\,x^n}+\frac{1}{n}\,\mathcal{I}_{-n}^{*(40)}(x)+\frac{3}{n+1}\,\mathcal{I}_{-n}^{*(22)}(x)$$

$$\mathcal{I}_{-n-1}^{*(22)}(x)=-\frac{I_0^2(x)\,I_1(x)}{(n+2)\,x^n}+\frac{2}{n+2}\,\left[\mathcal{I}_{-n}^{*(31)}(x)+\mathcal{I}_{-n}^{*(13)}(x)\right]$$

$$\mathcal{I}_{-n-1}^{*(13)}(x)=-\frac{I_0(x)\,I_1^3(x)}{(n+3)\,x^n}+\frac{1}{n+3}\,\left[3\,\mathcal{I}_{-n}^{*(22)}(x)+\mathcal{I}_{-n}^{*(04)}(x)\right]$$

$$\mathcal{I}_{-n-1}^{*(04)}(x)=-\frac{I_1^4(x)}{(n+4)\,x^n}+\frac{4}{n+4}\,\mathcal{I}_{-n}^{*(13)}(x)$$

4.2. Different Functions and Different Arguments

a) $x^m Z_0^p(x) Z_1^{2-p}(x)\,[Z_0^*(x)]^q\,[Z_1^*(x)]^{2-q}$, $p,q \in \{0,1,2\}$

Holds (see [1], 9.6.14)

$$x^2\,[I_0^2(x)K_1^2(x) + 2I_0(x)I_1(x)K_0(x)K_1(x) + I_1^2(x)K_0^2(x)\,] \;=\; 1 \;.$$

Therefore each multiple of this expression may be added to the antiderivatives, concerning to this functions.

$$\int I_1^2(x)\,K_0(x)\,K_1(x)\,dx = \frac{x}{4}\,[\,x\,I_0^2(x)K_0^2(x) - 2\,I_0(x)I_1(x)K_0^2(x) +$$
$$+\,2x\,I_0(x)I_1(x)K_0(x)K_1(x) - 2\,I_1^2(x)K_0(x)K_1(x) + x\;I_1^2(x)K_1^2(x)]$$

$$\int I_0(x)I_1(x)K_1^2(x)\,dx = -\frac{x}{4}\,[\,x\,I_0^2(x)K_0^2(x) + 2I_0^2(x)K_0(x)K_1(x) +$$
$$+2x\,I_0(x)I_1(x)K_0(x)K_1(x) + 2I_0(x)I_1(x)K_1^2(x) + x\,I_1^2(x)K_1^2(x)]$$

$$\int x^3\,J_0^2(x)I_0^2(x)\,dx = \frac{x^2}{8}[\,x^2\,J_0^2(x)I_0^2(x) + 2x\,J_0^2(x)I_0(x)I_1(x) - (x^2+1)\,J_0^2(x)I_1^2(x) +$$
$$+2x\,J_0(x)J_1(x)I_0^2(x) - 2J_0(x)J_1(x)I_0(x)I_1(x) + (x^2-1)\,J_1^2(x)I_0^2(x) - x^2\,J_1^2(x)I_1^2(x)\,]$$

$$\int x^3\,J_0^2(x)K_0^2(x)\,dx = \frac{x^2}{8}[\,x^2\,J_0^2(x)K_0^2(x) - 2x\,J_0^2(x)K_0(x)K_1(x) - (x^2+1)\,J_0^2(x)K_1^2(x) +$$
$$+2x\,J_0(x)J_1(x)K_0^2(x) + 2J_0(x)J_1(x)K_0(x)K_1(x) + (x^2-1)\,J_1^2(x)K_0^2(x) - x^2\,J_1^2(x)K_1^2(x)\,]$$

$$\int x^3\,J_1^2(x)I_1^2(x)\,dx = \frac{x^2}{8}[-x^2\,J_0^2(x)I_0^2(x) + 2x\,J_0^2(x)I_0(x)I_1(x) + (x^2-1)\,J_0^2(x)I_1^2(x) +$$
$$+2x\,J_0(x)J_1(x)I_0^2(x) - 2J_0(x)J_1(x)I_0(x)I_1(x) - 4x\,J_0(x)J_1(x)I_1^2(x) - (x^2+1)\,J_1^2(x)I_0^2(x) +$$
$$+\,4x\,J_1^2(x)I_0(x)I_1(x) + x^2\,J_1^2(x)I_1^2(x)\,]$$

$$\int x^3\,J_1^2(x)K_1^2(x)\,dx = \frac{x^2}{8}[-x^2\,J_0^2(x)K_0^2(x) - 2x\,J_0^2(x)K_0(x)I_1(x) + (x^2-1)\,J_0^2(x)K_1^2(x) +$$
$$+2x\,J_0(x)J_1(x)K_1^2(x) + 2J_0(x)J_1(x)K_0(x)K_1(x) - 4x\,J_0(x)J_1(x)K_1^2(x) - (x^2+1)\,J_1^2(x)K_0^2(x) -$$
$$-\,4x\,J_1^2(x)K_0(x)K_1(x) + x^2\,J_1^2(x)K_1^2(x)\,]$$

$$\int x^4\,I_1^2(x)\,K_0(x)\,K_1(x)\,dx = \frac{x^3}{12}\,[x^2\,I_0^2(x)K_0(x)K_1(x) + 2x\,I_0^2(x)K_1^2(x) + x^2\,I_0(x)I_1(x)K_0^2(x) +$$
$$+(x^2-4)\,I_0(x)I_1(x)K_1^2(x) - 2x\,I_1^2(x)K_0^2(x) + (x^2-4)\,I_1^2(x)K_0(x)K_1(x) - 3x\,I_1^2(x)K_1^2(x)\,]$$

$$\int x^4\,I_0(x)\,I_1(x)\,K_1^2(x)\,dx = \frac{x^3}{12}\,[\,x^2\,I_0^2(x)K_0(x)K_1(x) + 2x\,I_0^2(x)K_1^2(x) + x^2\,I_0(x)I_(x)K_0^2(x) +$$
$$+(x^2-4)\,I_0(x)I_(x)K_1^2(x) - 2x\,I_1^2(x)K_0^2(x) + (x^2-4)\,I_1^2(x)K_0(x)K_1(x) + 3x\,I_1^2(x)K_1^2(x)\,]$$

$$\int x^5\,J_0(x)J_1(x)I_0(x)I_1(x)\,dx = \frac{x^2}{8}\,[\,x^2\,J_0^2(x)I_0^2(x) - x(x^2+2)\,J_0^2(x)I_0(x)I_1(x) + (x^2+1)\,J_0^2(x)I_1^2(x) +$$
$$+x(x^2-2)\,J_0(x)J_1(x)I_0^2(x) + 2J_0(x)J_1(x)I_0(x)I_1(x) + x^3\,J_0(x)J_1(x)I_1^2(x) - (x^2-1)\,J_1^2(x)I_0^2(x) +$$
$$+x^3\,J_1^2(x)I_0(x)I_1(x)\,]$$

$$\int x^5\,J_0(x)J_1(x)K_0(x)K_1(x)\,dx = \frac{x^2}{8}\,[-x^2\,J_0^2(x)K_0^2(x) - x(x^2+2)\,J_0^2(x)K_0(x)K_1(x) - (x^2+1)\,J_0^2(x)K_1^2(x) -$$

$$-x(x^2-2)\,J_0(x)J_1(x)K_0^2(x)+2J_0(x)J_1(x)K_0(x)K_1(x)-x^3\,J_0(x)J_1(x)K_1^2(x)+(x^2-1)\,J_1^2(x)K_0^2(x)+$$
$$+x^3\,J_1^2(x)K_0(x)K_1(x)\,]$$

$$\int\frac{I_1^2(x)\,K_1^2(x)\,dx}{x}=\frac{1}{4}\,\{-x^2\,I_0^2(x)\,K_0^2(x)-x\,I_0^2(x)\,K_0(x)K_1(x)+x^2\,I_0^2(x)K_1^2(x)+$$
$$+x\,I_0(x)I_1(x)[K_0^2(x)-K_1^2(x)]+x^2\,I_1^2(x)K_0^2(x)+x\,I_1^2(x)K_0(x)K_1(x)-(x^2+1)I_1^2(x)K_1^2(x)\}$$
$$\int\frac{I_0(x)\,I_1(x)\,K_0(x)\,K_1(x)\,dx}{x}=\frac{1}{4}\,\{(x^2-1)\,I_0^2(x)K_0^2(x)+x\,I_0^2(x)K_0(x)K_1(x)-$$
$$-x^2\,I_0^2(x)K_1^2(x)-x\,I_0(x)I_1(x)K_0^2(x)-2\,I_0(x)I_1(x)K_0(x)K_1(x)+x\,I_0(x)I_1(x)K_1^2(x)-$$
$$-x^2\,I_1^2(x)K_0^2(x)-x\,I_1^2(x)K_0(x)K_1(x)+x^2\,I_1^2(x)K_1^2(x)\}$$

$$\int\frac{I_1^2(x)\,K_1^2(x)\,dx}{x^3}=\frac{1}{24\,x^2}\,\Big\{-(4x^4-3x^2)\,J_0^2(x)K_0^2(x)-4x^3\,J_0^2(x)K_0(x)K_1(x)-x^2\,J_0^2(x)K_1^2(x)+$$
$$+4x^3\,J_0(x)J_1(x)K_0^2(x)-(8x^4-4x^2)\,J_0(x),J_1(x)K_0(x)K_1(x)-(4x^3+2x)\,J_0(x)J_1(x)K_1^2(x)-$$
$$-x^2\,J_1^2(x)K_0^2(x)+(4x^2+2x)\,J_1^2(x)K_0(x)K_1(x)-(4x^4+x^2+4)\,J_1^2(x)K_1^2(x)\Big\}$$

Recurrence relations:

$$\int x^{4n+3}\,J_0^2(x)\,I_0^2(x)\,dx=\frac{x^{4n}}{8(2n+1)}\,\Big\{[x^4-2n^2(12n^2+5n+1)]\,J_0^2(x)\,I_0^2(x)+$$
$$+[(6n+2)x^3+n(12n^2+5n+1)x]\,J_0^2(x)\,I_0(x)\,I_1(x)-[x^4+(8n^2+5n+1)x^2]\,J_0^2(x)\,I_1^2(x)+$$
$$+[(6n+2)x^3-n(12n^2+5n+1)x]\,J_0(x)\,J_1(x)\,I_0^2(x)-2(n+1)(4n+1)\,x^2\,J_0(x)\,J_1(x)\,I_0(x)\,I_1(x)-$$
$$-[2nx^3-n(20n^2+15n+3)x]\,J_0(x)\,J_1(x)\,I_1^2(x)+[x^4-(8n^2+5n+1)x^2]\,J_1^2(x)\,I_0^2(x)+$$
$$+[2nx^3+n(20n^2+15n+3)x]\,J_1^2(x)\,I_0(x)\,I_1(x)-[x^4+(2n-1)(20n^2+15n+3)n]\,J_1^2(x)\,I_1^2(x)\,\Big\}+$$
$$+\frac{(12n^2+5n+1)n^3}{2n+1}\int x^{4n-1}\,J_0^2(x)\,I_0^2(x)\,dx+\frac{(n-1)(2n-1)(20n^2+15n+3)n}{2(2n+1)}\int x^{4n-1}\,J_1^2(x)\,I_1^2(x)\,dx$$

$$\int x^{4n+3}\,J_0^2(x)\,K_0^2(x)\,dx=\frac{x^{4n}}{2n+1}\,\Big\{[x^4-2n^2(12n^2+5n+1)]\,J_0^2(x)\,K_0^2(x)-$$
$$-[(6n+2)x^3+n(12n^2+5n+1)x]\,J_0^2(x)\,K_0(x)\,K_1(x)-[x^4+(8n^2+5n+1)x^2]\,J_0^2(x)\,K_1^2(x)+$$
$$+[(6n+2)x^3-n(12n^2+5n+1)x]\,J_0(x)\,J_1(x)\,K_0^2(x)+2(n+1)(4n+1)\,x^2\,J_0(x)\,J_1(x)\,K_0(x)\,K_1(x)-$$
$$-[2nx^3-n(20n^2+15n+3)x]\,J_0(x)\,J_1(x)\,K_1^2(x)+[x^4-(8n^2+5n+1)x^2]\,J_1^2(x)\,K_0^2(x)+$$
$$-[2nx^3+n(20n^2+15n+3)x]\,J_1^2(x)\,K_0(x)\,K_1(x)-[x^4+(2n-1)(20n^2+15n+3)n]\,J_1^2(x)\,K_1^2(x)\,\Big\}+$$
$$+\frac{(12n^2+5n+1)n^3}{2n+1}\int x^{4n-1}\,J_0^2(x)\,K_0^2(x)\,dx+\frac{(n-1)(2n-1)(20n^2+15n+3)n}{2(2n+1)}\int x^{4n-1}\,J_1^2(x)\,K_1^2(x)\,dx$$

$$\int x^{4n+3}\,J_1^2(x)\,I_1^2(x)\,dx=\frac{x^{4n}}{2n+1}\,\Big\{-[x^4+2n^2(20n^2+15n+1)]\,J_0^2(x)\,I_0^2(x)+$$
$$+[(2n+2)x^3+n(20n^2+15n+1)x]\,J_0^2(x)\,I_0(x)\,I_1(x)+[x^4-(8n^2+7n+1)x^2]\,J_0^2(x)\,I_1^2(x)+$$
$$+[(2n+2)x^3-n(20n^2+15n+1)x]\,J_0(x)\,J_1(x)\,I_0^2(x)+(8n^2+2n-2)\,x^2\,J_0(x)\,J_1(x)\,I_0(x)\,I_1(x)-$$
$$-[(6n+4)x^3-n(4n+3)(3n+1)x]\,J_0(x)\,J_1(x)\,I_1^2(x)-[x^4+(8n^2+7n+1)x^2]\,J_1^2(x)\,I_0^2(x)+$$

$$+[(6n+4)x^3+n(4n+3)(3n+1)x]\,J_1^2(x)\,I_0(x)\,I_1(x)+[x^4-n(4n+3)(3n+1)(2n-1)]\,J_1^2(x)\,I_1^2(x)\,\Big\}+$$

$$+\frac{(20n^2+15n+1)n^3}{2n+1}\int x^{4n-1}\,J_0^2(x)\,I_0^2(x)\,dx+\frac{(4n+3)(3n+1)(2n-1)n(n-1)}{2(2n+1)}\int x^{4n-1}\,J_1^2(x)\,I_1^2(x)\,dx$$

$$\int x^{4n+3}\,J_1^2(x)\,K_1^2(x)\,dx=\frac{x^{4n}}{2n+1}\Big\{-[x^4+2n^2(20n^2+15n+1)]\,J_0^2(x)\,K_0^2(x)-$$

$$-[(2n+2)x^3+n(20n^2+15n+1)x]\,J_0^2(x)\,K_0(x)\,K_1(x)+[x^4-(8n^2+7n+1)x^2]\,J_0^2(x)\,K_1^2(x)+$$

$$+[(2n+2)x^3-n(20n^2+15n+1)x]\,J_0(x)\,J_1(x)\,K_0^2(x)-(8n^2+2n-2)\,x^2\,J_0(x)\,J_1(x)\,K_0(x)\,K_1(x)-$$

$$-[(6n+4)x^3-n(4n+3)(3n+1)x]\,J_0(x)\,J_1(x)\,K_1^2(x)-[x^4+(8n^2+7n+1)x^2]\,J_1^2(x)\,K_0^2(x)-$$

$$-[(6n+4)x^3+n(4n+3)(3n+1)x]\,J_1^2(x)\,K_0(x)\,K_1(x)+[x^4-n(4n+3)(3n+1)(2n-1)]\,J_1^2(x)\,K_1^2(x)\,\Big\}+$$

$$+\frac{(20n^2+15n+1)n^3}{2n+1}\int x^{4n-1}\,J_0^2(x)\,K_0^2(x)\,dx+\frac{(4n+3)(3n+1)(2n-1)n(n-1)}{2(2n+1)}\int x^{4n-1}\,J_1^2(x)\,K_1^2(x)\,dx$$

$$\int x^{4n+1}\,J_0(x)\,J_1(x)I_0(x)I_1(x)\,dx\;=\;\frac{x^{4n-3}}{8(4n-3)}\Big\{(4n-3)x^3\,J_0^2(x)I_0^2(x)-$$

$$-[(4n-3)x^4+4(2n-1)^2(n-1)x^2]\,J_0^2(x)I_0(x)I_1(x)+(4n-3)(2n-1)x^3\,J_0^2(x)I_1^2(x)+$$

$$+[(4n-3)x^4-4(2n-1)^2(n-1)x^2]\,J_0(x)J_1(x)I_0^2(x)+8(2n-1)^3(n-1)x\,J_0(x)J_1(x)I_0(x)I_1(x)+$$

$$+[(4n-3)x^4-2(2n-1)^3x^2]\,J_0(x)J_1(x)I_1^2(x)-(4n-3)(2n-1)x^3\,J_1^2(x)I_0^2(x)+$$

$$+[(4n-3)x^4+2(2n-1)^3x^2]\,J_1^2(x)I_0(x)I_1(x)\Big\}-\frac{4(2n-1)^3(n-1)^2}{4n-3}\int x^{4n-3}\,J_0(x)\,J_1(x)I_0(x)I_1(x)\,dx+$$

$$+\frac{8n^3-20n^2+13n-2}{2(4n-3)}\int x^{4n-1}\,J_0^2(x)I_0^2(x)\,dx-\frac{(2n-1)^3}{2(4n-3)}\int x^{4n-1}\,J_1^2(x)I_1^2(x)\,dx$$

$$\int x^{4n+1}\,J_0(x)\,J_1(x)K_0(x)K_1(x)\,dx\;=\;\frac{x^{4n-3}}{8(4n-3)}\Big\{-(4n-3)x^3\,J_0^2(x)K_0^2(x)-$$

$$-[(4-3)x^4+4(2n-1)^2(n-1)x^2]\,J_0^2(x)K_0(x)K_1(x)-(4n-3)(2n-1)x^3\,J_0^2(x)K_1^2(x)-$$

$$-[(4-3)x^4-4(2n-1)^2(n-1)x^2]\,J_0(x)J_1(x)K_0^2(x)+8(2n-1)^3(n-1)x\,J_0(x)J_1(x)K_0(x)K_1(x)-$$

$$-[(4-3)x^4-2(2n-1)^3x^2]\,J_0(x)J_1(x)K_1^2(x)+(4n-3)(2n-1)x^3\,J_1^2(x)K_0^2(x)+$$

$$+[(4n-3)x^4+2(2n-1)^3x^2]\,J_1^2(x)K_0(x)K_1(x)\Big\}-\frac{4(2n-1)^3(n-1)^2}{4n-3}\int x^{4n-3}\,J_0(x)\,J_1(x)K_0(x)K_1(x)\,dx-$$

$$-\frac{8n^3-20n^2+13n-2}{2(4n-3)}\int x^{4n-1}\,J_0^2(x)K_0^2(x)\,dx+\frac{(2n-1)^3}{2(4n-3)}\int x^{4n-1}\,J_1^2(x)K_1^2(x)\,dx$$

b) $x^m Z_0^p(x) Z_1^{2-p}(x)\,[Z_0^*(\alpha x)]^q\,[Z_1^*(\alpha x)]^{2-q}$, $p,q\in\{0,1,2\}$

$$\int x^3\,J_0^2(x)J_1^2(\sqrt{3}x)\,dx=\frac{x^2}{8}\,\Big[x^2\,J_0^2(x)J_0^2(\sqrt{3}x)-2\sqrt{3}\,x\,J_0^2(x)J_0(\sqrt{3}x)J_1(\sqrt{3}x)+$$

$$+(x^2+3)\,J_0^2(x)J_1^2(\sqrt{3}x)+2x\,J_0(x)J_1(x)J_0^2(\sqrt{3}x)-2\sqrt{3}\,J_0(x)J_1(x)J_0(\sqrt{3}x)J_1(\sqrt{3}x)+$$

$$+(x^2+1)\,J_1^2(x)J_0^2(\sqrt{3}x)+x^2\,J_1^2(x)J_1^2(\sqrt{3}x)\Big]$$

$$\int x^3\,I_0^2(x)I_1^2(\sqrt{3}x)\,dx=\frac{x^2}{8}\,\Big[-x^2\,I_0^2(x)I_0^2(\sqrt{3}x)+2\sqrt{3}\,x\,I_0^2(x)I_0(\sqrt{3}x)I_1(\sqrt{3}x)+$$

$$+(x^2-3)\,I_0^2(x)I_1^2(\sqrt{3}x)-2x\,I_0(x)I_1(x)I_0^2(\sqrt{3}x)+2\sqrt{3}\,I_0(x)I_1(x)I_0(\sqrt{3}x)I_1(\sqrt{3}x)+$$

$$+(x^2-1)\,I_1^2(x)I_0^2(\sqrt{3}x)-x^2\,I_1^2(x)I_1^2(\sqrt{3}x)\Big]$$

$$\int x^3\,K_0^2(x)K_1^2(\sqrt{3}x)\,dx=\frac{x^2}{8}\,\Big[-x^2\,K_0^2(x)K_0^2(\sqrt{3}x)-2\sqrt{3}\,x\,K_0^2(x)K_0(\sqrt{3}x)K_1(\sqrt{3}x)+$$

$$+(x^2-3)\,K_0^2(x)K_1^2(\sqrt{3}x)+2x\,K_0(x)K_1(x)K_0^2(\sqrt{3}x)+2\sqrt{3}\,K_0(x)K_1(x)K_0(\sqrt{3}x)K_1(\sqrt{3}x)+$$

$$+(x^2-1)\,K_1^2(x)K_0^2(\sqrt{3}x)-x^2\,K_1^2(x)K_1^2(\sqrt{3}x)\Big]$$

$$\int x^3\,I_0^2(x)K_1^2(\sqrt{3}x)\,dx=\frac{x^2}{8}\,[-x^2\,I_0^2(x)K_0^2(\sqrt{3}x)-2\sqrt{3}x\,I_0^2(x)K_0(\sqrt{3}x)K_1(\sqrt{3}x)+$$

$$+(x^2-3)\,I_0^2(x)K_1^2(\sqrt{3}x)-2x\,I_0(x)I_1(x)K_0^2(\sqrt{3}x)-2\sqrt{3}\,I_0(x)I_1(x)K_0(\sqrt{3}x)K_1(\sqrt{3}x)+$$

$$+(x^2-1)\,I_1^2(x)K_0^2(\sqrt{3}x)-x^2\,I_1^2(x)K_1^2(\sqrt{3}x)]$$

$$\int x^3\,K_0^2(x)I_1^2(\sqrt{3}x)\,dx=\frac{x^2}{8}\,[-x^2\,K_0^2(x)I_0^2(\sqrt{3}x)+2\sqrt{3}x\,K_0^2(x)I_0(\sqrt{3}x)K_1(\sqrt{3}x)+$$

$$+(x^2-3)\,K_0^2(x)I_1^2(\sqrt{3}x)+2x\,K_0(x)K_1(x)I_0^2(\sqrt{3}x)-2\sqrt{3}\,K_0(x)K_1(x)I_0(\sqrt{3}x)I_1(\sqrt{3}x)+$$

$$+(x^2-1)\,K_1^2(x)I_0^2(\sqrt{3}x)-x^2\,K_1^2(x)I_1^2(\sqrt{3}x)]$$

$$\int x^4\,J_0^2(x)I_0\left(\sqrt{3}\,x\right)I_1\left(\sqrt{3}\,x\right)\,dx=\frac{x^2}{384}\,\Big[20\sqrt{3}\,x^2\,J_0^2(x)I_0^2\left(\sqrt{3}\,x\right)-60x\,J_0^2(x)I_0\left(\sqrt{3}\,x\right)I_1\left(\sqrt{3}\,x\right)+$$

$$+3\sqrt{3}\,(12x^2+5)\,J_0^2(x)I_1^2\left(\sqrt{3}\,x\right)-20\sqrt{3}\,x\,J_0(x)J_1(x)I_0^2\left(\sqrt{3}\,x\right)+$$

$$+6(8x^2+5)\,J_0(x)J_1(x)I_0\left(\sqrt{3}\,x\right)I_1\left(\sqrt{3}\,x\right)-12\sqrt{3}\,x\,J_0(x)J_1(x)I_1^2\left(\sqrt{3}\,x\right)-$$

$$-\sqrt{3}\,(4x^2-5)\,J_1^2(x)I_0^2\left(\sqrt{3}\,x\right)-12x\,J_1^2(x)I_0\left(\sqrt{3}\,x\right)I_1\left(\sqrt{3}\,x\right)+12\sqrt{3}\,x^2\,J_1^2(x)I_1^2\left(\sqrt{3}\,x\right)\Big]$$

$$\int x^4\,J_0^2(x)K_0\left(\sqrt{3}\,x\right)K_1\left(\sqrt{3}\,x\right)\,dx=\frac{x^2}{384}\,\Big[-20\sqrt{3}\,x^2\,J_0^2(x)K_0^2\left(\sqrt{3}\,x\right)-60x\,J_0^2(x)K_0\left(\sqrt{3}\,x\right)K_1\left(\sqrt{3}\,x\right)-$$

$$-3\sqrt{3}\,(12x^2+5)\,J_0^2(x)K_1^2\left(\sqrt{3}\,x\right)+20\sqrt{3}\,x\,J_0(x)J_1(x)K_0^2\left(\sqrt{3}\,x\right)+$$

$$+6(8x^2+5)\,J_0(x)J_1(x)K_0\left(\sqrt{3}\,x\right)K_1\left(\sqrt{3}\,x\right)+12\sqrt{3}\,x\,J_0(x)J_1(x)K_1^2\left(\sqrt{3}\,x\right)+$$

$$+\sqrt{3}\,(4x^2-5)\,J_1^2(x)K_0^2\left(\sqrt{3}\,x\right)-12x\,J_1^2(x)K_0\left(\sqrt{3}\,x\right)K_1\left(\sqrt{3}\,x\right)-12\sqrt{3}\,x^2\,J_1^2(x)K_1^2\left(\sqrt{3}\,x\right)\Big]$$

$$\int x^4\,I_0^2(x)J_0\left(\sqrt{3}\,x\right)J_1\left(\sqrt{3}\,x\right)\,dx=\frac{x^2}{384}\,\Big[-20\sqrt{3}\,x^2\,I_0^2(x)J_0^2\left(\sqrt{3}\,x\right)+60x\,I_0^2(x)J_0\left(\sqrt{3}\,x\right)J_1\left(\sqrt{3}\,x\right)-$$

$$+3\sqrt{3}\,(12x^2-5)\,I_0^2(x)J_1^2\left(\sqrt{3}\,x\right)+20\sqrt{3}\,x\,I_0(x)I_1(x)J_0^2\left(\sqrt{3}\,x\right)+$$

$$+6(8x^2-5)\,I_0(x)I_1(x)J_0\left(\sqrt{3}\,x\right)J_1\left(\sqrt{3}\,x\right)-12\sqrt{3}\,x\,I_0(x)I_1(x)J_1^2\left(\sqrt{3}\,x\right)-$$

$$-\sqrt{3}\,(4x^2+5)\,I_1^2(x)J_0^2\left(\sqrt{3}\,x\right)-12x\,I_1^2(x)J_0\left(\sqrt{3}\,x\right)J_1\left(\sqrt{3}\,x\right)-12\sqrt{3}\,x^2\,I_1^2(x)J_1^2\left(\sqrt{3}\,x\right)\Big]$$

$$\int x^4\,K_0^2(x)J_0\left(\sqrt{3}\,x\right)J_1\left(\sqrt{3}\,x\right)\,dx=\frac{x^2}{384}\,\Big[-20\sqrt{3}\,x^2\,K_0^2(x)J_0^2\left(\sqrt{3}\,x\right)+60x\,K_0^2(x)J_0\left(\sqrt{3}\,x\right)J_1\left(\sqrt{3}\,x\right)-$$

$$+3\sqrt{3}\,(12x^2-5)\,K_0^2(x)J_1^2\left(\sqrt{3}\,x\right)-20\sqrt{3}\,x\,K_0(x)K_1(x)J_0^2\left(\sqrt{3}\,x\right)-$$

$$-6(8x^2-5)\,K_0(x)K_1(x)J_0\left(\sqrt{3}\,x\right)J_1\left(\sqrt{3}\,x\right)+12\sqrt{3}\,x\,K_0(x)K_1(x)J_1^2\left(\sqrt{3}\,x\right)-$$

$$-\sqrt{3}\,(4x^2+5)\,K_1^2(x)J_0^2\left(\sqrt{3}\,x\right)-12x\,K_1^2(x)J_0\left(\sqrt{3}\,x\right)J_1\left(\sqrt{3}\,x\right)-12\sqrt{3}\,x^2\,K_1^2(x)J_1^2\left(\sqrt{3}\,x\right)\Big]$$

The following integrals with real values of γ may be expressed by sums of the type

$$\int x^n\, U_\nu^p(x) U_\nu^{2-p}(x)\, W_\mu^q(\gamma x) W_\mu^{2-q}(\gamma x)\, dx =$$

$$= \sum_{m=0}^{n+1} \sum_{i,j=0}^{2} \sum_{\varrho,\sigma,\tau,\omega=0}^{1} \alpha_{mij}^{(\mu\nu;\varrho\,\sigma\tau\omega)}\, x^m\, U_\varrho^i(x) U_\sigma^{2-i}(x)\, W_\tau^j(\gamma x) W_\omega^{2-j}(\gamma x)\,, \quad p,q \in \{0,1,2\}$$

with $\alpha_{mij}^{(\mu\nu;\varrho\,\sigma\tau\omega)}$ depending from $U_\nu(x)$ and $W_\mu(\gamma x)$.

$$\int x^5\, f(x)\, dx \;:\; f(x) = J_0^2(x)\, J_1^2(\zeta x)\,,\; I_0^2(x)\, I_1^2(\zeta x)\,,\; I_0^2(x)\, K_1^2(\zeta x)\,,\; K_0^2(x)\, I_1^2(\zeta x)\,,\; K_0^2(x)\, K_1^2(\zeta x)$$

$$\text{with} \quad \zeta \;=\; \frac{\sqrt{2\sqrt{34}+1}}{3} \;=\; 1.18611\,89649$$

$$\int x^5\, f(x)\, dx \;:\; f(x) = J_0^2(x)\, I_1^2(\zeta^* x)\,,\; J_0^2(x)\, K_1^2(\zeta^* x)\,,\; I_0^2(x)\, J_1^2(\zeta^* x)\,,\; K_0^2(x)\, J_1^2(\zeta^* x)$$

$$\text{with} \quad \zeta^* \;=\; \frac{\sqrt{2\sqrt{34}-1}}{3} \;=\; 1.08841\,90262$$

$$\int x^5\, f(x)\, dx \;:\; f(x) = J_1^2(x)\, J_0^2(\eta x)\,,\; I_1^2(x)\, I_0^2(\eta x)\,,\; I_1^2(x)\, K_0^2(\eta x)\,,\; K_1^2(x)\, I_0^2 \eta x)\,,\; K_1^2(x)\, K_0^2(\eta x)$$

$$\text{with} \quad \eta \;=\; \frac{\sqrt{30\sqrt{34}-15}}{15} \;=\; 0.84308\,57524$$

$$\int x^5\, f(x)\, dx \;:\; f(x) = J_1^2(x)\, I_0^2(\eta^* x)\,,\; J_1^2(x)\, K_0^2(\eta^* x)\,,\; I_1^2(x)\, J_0^2(\eta^* x)\,,\; K_1^2(x)\, I_0^2(\eta^* x)$$

$$\text{with} \quad \eta^* \;=\; \frac{\sqrt{30\sqrt{34}+15}}{15} \;=\; 0.91876\,37995$$

$$\int x^6\, f(x)\, dx \;:\; f(x) = J_1^2(x)\, J_0(\lambda x)\, J_1(\lambda x)\,,\; I_1^2(x)\, I_0(\lambda x)\, I_1(\lambda x)\,,\; K_1^2(x) I_0(\lambda x) I_1(\lambda x)\,,\; K_1^2(x)\, I_0(\lambda x)\, I_1(\lambda x)\,,$$

$$K_1^2(x) K_0(\lambda x) I_1(\lambda x)\,, \quad \text{with} \quad \lambda \;=\; \frac{\sqrt{10\sqrt{19}-5}}{5} \;=\; 1.24240\,07314$$

$$\int x^6\, f(x)\, dx \;:\; f(x) = J_1^2(x)\, I_0(\lambda^* x)\, I_1(\lambda^* x)\,,\; J_1^2(x)\, K_0(\lambda^* x)\, K_1(\lambda^* x)\,,\; I_1^2(x) J_0(\lambda^* x) J_1(\lambda^* x)\,,$$

$$K_1^2(x)\, J_0(\lambda^* x)\, J_1(\lambda^* x)\,, \quad \text{with} \quad \lambda^* \;=\; \frac{\sqrt{10\sqrt{19}+5}}{5} \;=\; 1.39411\,60559$$

$$\int x^6\, f(x)\, dx \;:\; f(x) = J_0(x)\, J_1(x)\, J_1^2(\nu x)\,,\; I_0(x)\, I_1(x)\, I_1^2(\nu x)\,,\; I_0(x)\, I_1(x)\, K_1^2(\nu x)\,,\; K_0(x)\, K_1(x)\, I_1^2(\nu x)\,,$$

$$K_0(x)\, K_1(x)\, K_1^2(\nu x)\,, \quad \text{with} \quad \nu \;=\; \frac{\sqrt{30\sqrt{19}+15}}{15} \;=\; 0.80489\,32802$$

$$\int x^6\, f(x)\, dx \;:\; f(x) = J_0(x)\, J_1(x)\, I_1^2(\nu^* x)\,,\; J_0(x)\, J_1(x)\, K_1^2(\nu^* x)\,,\; I_0(x)\, I_1(x)\, J_1^2(\nu^* x)\,,\; K_0(x)\, K_1(x)\, J_1^2(\nu^* x)$$

$$\text{with} \quad \nu^* \;=\; \frac{\sqrt{30\sqrt{19}-15}}{15} \;=\; 0.71730\,03967$$

$$\int x^7\, f(x)\, dx \;:\; f(x) = J_0^2(x)\, J_1^2(\alpha x)\,,\; I_0^2(x)\, I_1^2(\alpha x)\,,\; I_0^2(x)\, K_1^2(\alpha x)\,,\; K_0^2(x)\, I_1^2(\alpha x)\,,\; K_0^2(x)\, K_1^2(\alpha x)$$

$$\text{with} \quad \alpha \;=\; \frac{\sqrt{30\beta^3+135\beta^2+14670\beta}}{75\beta} \;=\; 1.07488\,70261\,,\; \beta = \sqrt[3]{4617513+3750\,\sqrt{1516182}} = 209.80\ldots$$

$$\int x^7\, f(x)\, dx \;:\; f(x) = J_1^2(x)\, J_0^2(\delta x)\,,\; I_1^2(x)\, I_0^2(\delta x)\,,\; I_1^2(x)\, K_0^2(\delta x)\,,\; K_1^2(x)\, I_0^2(\delta x)\,,\; K_1^2(x)\, K_0^2(\delta x)$$

$$\text{with} \quad \delta \;=\; \frac{1}{\alpha} \;=\; \frac{\sqrt{210\varepsilon^3-105\varepsilon^2-9870\varepsilon}}{1055\varepsilon} \;=\; 0.93033\,03284\,,\; \varepsilon = \sqrt[3]{51715+42\,\sqrt{1516182}} = 46.94\ldots$$

$$\int x^8\, f(x)\, dx \;:\; f(x) = J_1^2(x)\, J_0(\phi x)\, J_1(\phi x)\,,\; I_1^2(x)\, I_0(\phi x)\, I_1(\phi x)\,,\; I_1^2(x)\, K_0(\phi x)\, K_1(\phi x)\,,$$

$$K_1^2(x)\, I_0(\phi x)\, I_1(\phi x)\,,\; K_1^2(x)\, K_0(\phi x)\, K_1(\phi x)$$

$$\text{with} \quad \phi \;=\; \frac{\sqrt{70\psi^3-35\psi^2-1820\psi}}{35\,\psi} \;=\; 1.27774\,26063\,,\; \psi = \sqrt[3]{13418+210\,\sqrt{4083}} = 29.939\ldots$$

See previous formula: $\int x^8 f(x)\,dx \;:\; f(x) = J_0(x)\,J_1(x)\,J_1^2(\phi^* x)\,,\; I_0(x)\,I_1(x)\,I_1^2(\phi^* x)\,,$

$$K_0(x)\,K_1(x)\,I_1^2(\phi^* x)\,,\; I_0(x)\,I_1(x)\,K_1^2(\phi^* x)\,,\; K_0(x)\,K_1(x)\,K_1^2(\phi^* x)$$

$$\text{with}\quad \phi^* \;=\; \frac{1}{\phi} \;=\; \frac{\sqrt{14\psi^3+27\psi^2+644\psi}}{35\,\psi} \;=\; 0.78263\,02380$$

c) $x^m Z_0^p(x) Z_1^{3-p}(x)\, Z_\nu^* \alpha x)$, $p \in \{0,1,2,3\}$:

$$\int \frac{J_1^3(x)\,J_1(3x)\,dx}{x} = \frac{1}{8}\,\{ [-6x^2\,J_0^3(x) - 9x\,J_0^2(x)\,J_1(x) + 18x^2\,J_0(x)\,J_1^2(x) - 3x\,J_1^3(x)]\,J_0(3x) +$$
$$+ [\,7x\,J_0^3(x) - 18x^2\,J_0^2(x)\,J_1(x) - 3x\,J_0(x)\,J_1^2(x)\,J_0(x) + (6x^2-2)\,J_1^3(x)]\,J_1(3x)\,\}$$

$$\int \frac{I_1^3(x)\,I_1(3x)\,dx}{x} = \frac{1}{8}\,\{ [6x^2\,I_0^3(x) + 9x\,I_0^2(x)\,I_1(x) + 18x^2\,I_0(x)\,I_1^2(x) - 3x\,I_1^3(x)]\,I_0(3x) -$$
$$- [\,7x\,I_0^3(x) + 18x^2\,I_0^2(x)\,I_1(x) + 3x\,I_0(x)\,I_1^2(x)\,I_0(x) + (6x^2+2)\,I_1^3(x)]\,I_1(3x)\,\}$$

$$\int \frac{I_1^3(x)\,K_1(3x)\,dx}{x} = \frac{1}{8}\,\{ [-6x^2\,I_0^3(x) - 9x\,I_0^2(x)\,I_1(x) - 18x^2\,I_0(x)\,I_1^2(x) + 3x\,I_1^3(x)]\,K_0(3x) -$$
$$- [\,7x\,I_0^3(x) + 18x^2\,I_0^2(x)\,I_1(x) + 3x\,I_0(x)\,I_1^2(x)\,I_0(x) + (6x^2+2)\,I_1^3(x)]\,I_1(3x)\,\}$$

$$\int \frac{K_1^3(x)\,I_1(3x)\,dx}{x} = \frac{1}{8}\,\{ [-6x^2\,K_0^3(x) + 9x\,K_0^2(x)\,K_1(x) - 18x^2\,K_0(x)\,K_1^2(x) - 3x\,K_1^3(x)]\,I_0(3x) +$$
$$+ [\,7x\,K_0^3(x) - 18x^2\,K_0^2(x)\,K_1(x) + 3x\,K_0(x)\,K_1^2(x)\,K_0(x) - (6x^2+2)\,K_1^3(x)]\,I_1(3x)\,\}$$

$$\int J_1^3(x)\,J_0(3x)\,dx = \frac{1}{4}\,\{ [-3x^2\,J_0^3(x) - 3x\,J_0^2(x)J_1(x) + 9x^2\,J_0(x)J_1^2(x) - 2x\,J_1^3(x)]\,J_0(3x) +$$
$$+ [3x\,J_0^3(x) - 9x^2\,J_0^2(x)J_1(x) + 3x^2\,J_1^3(x)]\,J_1(3x)\,\}$$

$$\int I_1^3(x)\,I_0(3x)\,dx = \frac{1}{4}\,\{ [\,3x^2\,I_0^3(x) + 3x\,I_0^2(x)I_1(x) + 9x^2\,I_0(x)I_1^2(x) - 2x\,I_1^3(x)]\,I_0(3x) -$$
$$- [3x\,I_0^3(x) + 9x^2\,I_0^2(x)I_1(x) + 3x^2\,I_1^3(x)]\,I_1(3x)\,\}$$

$$\int K_1^3(x)\,K_0(3x)\,dx = \frac{1}{4}\,\{ [-3x^2\,K_0^3(x) + 3x\,K_0^2(x)K_1(x) - 9x^2\,K_0(x)K_1^2(x) - 2x\,K_1^3(x)]\,K_0(3x) -$$
$$- [3x\,K_0^3(x) - 9x^2\,K_0^2(x)K_1(x) - 3x^2\,K_1^3(x)]\,K_1(3x)\,\}$$

$$\int I_1^3(x)\,K_0(3x)\,dx = \frac{1}{4}\,\{ [\,3x^2\,I_0^3(x) + 3x\,I_0^2(x)I_1(x) + 9x^2\,I_0(x)I_1^2(x) - 2x\,I_1^3(x)]\,K_0(3x) +$$
$$+ [3x\,I_0^3(x) + 9x^2\,I_0^2(x)I_1(x) + 3x^2\,I_1^3(x)]\,K_1(3x)\,\}$$

$$\int K_1^3(x)\,I_0(3x)\,dx = \frac{1}{4}\,\{ [-3x^2\,K_0^3(x) + 3x\,K_0^2(x)K_1(x) - 9x^2\,K_0(x)K_1^2(x) - 2x\,K_1^3(x)]\,I_0(3x) +$$
$$+ [3x\,K_0^3(x) - 9x^2\,K_0^2(x)K_1(x) - 3x^2\,K_1^3(x)]\,I_1(3x)\,\}$$

$$\int J_0(x)\,J_1^2(x)\,J_1(3x)\,dx = \frac{x}{12}\,\{ [-3x\,J_0^3(x) - 9\,J_0^2(x)J_1(x) + 9x\,J_0(x)J_1^2(x)]\,J_0(3x) +$$
$$+ [\,5\,J_0^3(x) - 9x\,J_0^2(x)J_1(x) - 6\,J_0(x)J_1^2(x) + 3x\,J_1^3(x)]\,J_1(3x)\,\}$$

$$\int I_0(x)\,I_1^2(x)\,I_1(3x)\,dx = \frac{x}{12}\,\{ [\,3x\,I_0^3(x) + 9\,I_0^2(x)I_1(x) + 9x\,I_0(x)I_1^2(x)]\,I_0(3x) -$$
$$- [\,5\,I_0^3(x) + 9x\,I_0^2(x)I_1(x) + 6\,I_0(x)I_1^2(x) + 3x\,I_1^3(x)]\,I_1(3x)\,\}$$

$$\int K_0(x)\,K_1^2(x)\,K_1(3x)\,dx = \frac{x}{12}\,\{ [-3x\,K_0^3(x) + 9\,K_0^2(x)K_1(x) - 9x\,K_0(x)K_1^2(x)]\,K_0(3x) -$$

$$-[\,5\,K_0^3(x) - 9x\,K_0^2(x)K_1(x) + 6\,K_0(x)K_1^2(x) - 3x\,K_1^3(x)\,]\,K_1(3x)\,\Big\}$$

$$\int I_0(x)\,I_1^2(x)\,K_1(3x)\,dx = -\frac{x}{12}\,\Big\{\,[\,3x\,I_0^3(x) + 9\,I_0^2(x)I_1(x) + 9x\,I_0(x)I_1^2(x)\,]\,K_0(3x) +$$

$$+[\,5\,I_0^3(x) + 9x\,I_0^2(x)I_1(x) + 6\,I_0(x)I_1^2(x) + 3x\,I_1^3(x)\,]\,K_1(3x)\,\Big\}$$

$$\int K_0(x)\,K_1^2(x)\,I_1(3x)\,dx = \frac{x}{12}\,\Big\{\,[\,3x\,K_0^3(x) - 9\,K_0^2(x)K_1(x) + 9x\,K_0(x)K_1^2(x)\,]\,I_0(3x) -$$

$$-[\,5\,K_0^3(x) - 9x\,K_0^2(x)K_1(x) + 6\,K_0(x)K_1^2(x) - 3x\,K_1^3(x)\,]\,I_1(3x)\,\Big\}$$

$$\int x\,J_0(x)\,J_1^2(x)\,J_0(\sqrt{3}\,x)\,dx = \frac{J_0^2(x)\,x}{6}\left[\sqrt{3}\,J_0(x)\,J_1(\sqrt{3}\,x) - 3\,J_1(x)\,J_0(\sqrt{3}\,x)\right]$$

$$\int x\,I_0(x)\,I_1^2(x)\,I_0(\sqrt{3}\,x)\,dx = \frac{I_0^2(x)\,x}{6}\left[3\,I_1(x)\,I_0(\sqrt{3}\,x) - \sqrt{3}\,I_0(x)\,I_1(\sqrt{3}\,x)\right]$$

$$\int x\,I_0(x)\,I_1^2(x)\,K_0(\sqrt{3}\,x)\,dx = \frac{I_0^2(x)\,x}{6}\left[3\,I_1(x)\,K_0(\sqrt{3}\,x) + \sqrt{3}\,I_0(x)\,K_1(\sqrt{3}\,x)\right]$$

$$\int x\,K_0(x)\,K_1^2(x)\,K_0(\sqrt{3}\,x)\,dx = \frac{K_0^2(x)\,x}{6}\left[\sqrt{3}\,K_0(x)\,K_1(\sqrt{3}\,x) - 3\,K_1(x)\,K_0(\sqrt{3}\,x)\right]$$

$$\int x\,K_0(x)\,K_1^2(x)\,I_0(\sqrt{3}\,x)\,dx = -\,\frac{K_0^2(x)\,x}{6}\left[\sqrt{3}\,K_0(x)\,I_1(\sqrt{3}\,x) + 3\,K_1(x)\,I_0(\sqrt{3}\,x)\right]$$

$$\int x^3\,J_0^2(x)\,J_1(x)\,J_1(\sqrt{3}\,x)\,dx = \frac{J_1^2(x)\,x^3}{6}\left[3\,J_0(x)\,J_1(\sqrt{3}\,x) - \sqrt{3}\,J_1(x)\,J_0(\sqrt{3}\,x)\right]$$

$$\int x^3\,I_0^2(x)\,I_1(x)\,I_1(\sqrt{3}\,x)\,dx = \frac{I_1^2(x)\,x^3}{6}\left[3\,I_0(x)\,I_1(\sqrt{3}\,x) - \sqrt{3}\,I_1(x)\,I_0(\sqrt{3}\,x)\right]$$

$$\int x^3\,I_0^2(x)\,I_1(x)\,K_1(\sqrt{3}\,x)\,dx = \frac{I_1^2(x)\,x^3}{6}\left[3\,I_0(x)\,K_1(\sqrt{3}\,x) + \sqrt{3}\,I_1(x)\,K_0(\sqrt{3}\,x)\right]$$

$$\int x^3\,K_0^2(x)\,K_1(x)\,K_1(\sqrt{3}\,x)\,dx = \frac{K_1^2(x)\,x^3}{6}\left[\sqrt{3}\,K_1(x)\,K_0(\sqrt{3}\,x) - 3\,K_0(x)\,K_1(\sqrt{3}\,x)\right]$$

$$\int x^3\,K_0^2(x)\,K_1(x)\,I_1(\sqrt{3}\,x)\,dx = -\,\frac{K_1^2(x)\,x^3}{6}\left[\sqrt{3}\,K_1(x)\,I_0(\sqrt{3}\,x) + 3\,K_0(x)\,I_1(\sqrt{3}\,x)\right]$$

d) $x^m Z_0^p(x) Z_1^{2-p}(x)\,Z_\mu^*\alpha x)\,Z_\nu^{**}\alpha x)$, $p \in \{0,1,2\}$:

$$\int x^3\,J_0^2(x)\,I_0(x)\,K_0(x)\,dx = \frac{x^2}{8}\,\bigg\{\,\big[x^2\,J_0^2(x)I_0(x) + x\,J_0^2(x)I_1(x) + 2x\,J_0(x)J_1(x)I_0(x) -$$

$$-J_0(x)J_1(x)I_1(x) + (x^2-1)\,J_1^2(x)I_0(x)\big]\;K_0(x) - \big[x\,J_0^2(x)I_0(x) - (x^2+1)\,J_0^2(x)I_1(x) -$$

$$-J_0(x)J_1(x)I_0(x) - x^2\,J_1^2(x)I_1(x)\big]\;K_1(x)\bigg\}$$

$$\int x^3\,I_0^2(x)\,J_0(x)\,Y_0(x)\,dx = \frac{x^2}{8}\,\bigg\{\,\big[x^2\,I_0^2(x)J_0(x) + x\,I_0^2(x)J_1(x) + 2x\,I_0(x)I_1(x)J_0(x) -$$

$$-I_0(x)I_1(x)J_1(x) - (x^2+1)\,I_1^2(x)J_0(x)\big]\;Y_0(x) + \big[x\,I_0^2(x)J_0(x) + (x^2-1)\,I_0^2(x)J_1(x) -$$

$$-I_0(x)I_1(x)J_0(x) - x^2\,I_1^2(x)J_1(x)\big]\;Y_1(x)\bigg\}$$

$$\int x^3\,K_0^2(x)\,J_0(x)\,Y_0(x)\,dx = \frac{x^2}{8}\,\bigg\{\,\big[x^2\,K_0^2(x)J_0(x) + x\,K_0^2(x)J_1(x) - 2x\,K_0(x)K_1(x)J_0(x) +$$

$$+K_0(x)K_1(x)J_1(x) - (x^2+1)\,K_1^2(x)J_0(x)\big]\;Y_0(x) + \big[x\,K_0^2(x)J_0(x) + (x^2-1)\,K_0^2(x)J_1(x) +$$

$$+K_0(x)K_1(x)J_0(x) - x^2\,K_1^2(x)J_1(x)\big]\;Y_1(x)\Big\}$$

$$\int x^3\,J_1^2(x)\,I_1(x)\,K_1(x)\,dx = \frac{x^2}{8}\,\Big\{\big[x^2\,J_0^2(x)I_0(x) - x\,J_0^2(x)I_1(x) - 2x\,J_0(x)J_1(x)I_0(x) +$$

$$+J_0(x)J_1(x)I_1(x) + (x^2+1)\,J_1^2(x)I_0(x) - 2x\;J_1^2(x)I_1(x)\big]\;K_0(x) + \big[x\,J_0^2(x)I_0(x) +$$

$$+(x^2-1)\,J_0^2(x)I_1(x) - J_0(x)J_1(x)I_0(x) - 4x\,J_0(x)J_1(x)I_1(x) + 2x\,J_1^2(x)I_0(x) + x^2\,J_1^2(x)I_1(x)\big]\;K_1(x)\Big\}$$

$$\int x^3\,I_1^2(x)\,J_1(x)\,Y_1(x)\,dx = \frac{x^2}{8}\,\Big\{\big[-x^2\,I_0^2(x)J_0(x) + x\,I_0^2(x)J_1(x) + 2x\,I_0(x)I_1(x)J_0(x) -$$

$$-I_0(x)I_1(x)J_1(x) + (x^2-1)\,I_1^2(x)J_0(x) - 2x\;I_1^2(x)J_1(x)\big]\;Y_0(x) + \big[x\,I_0^2(x)J_0(x) -$$

$$-(x^2+1)\,I_0^2(x)J_1(x) - I_0(x)I_1(x)J_0(x) + 4x\,I_0(x)I_1(x)J_1(x) - 2x\,I_1^2(x)J_0(x) + x^2\,I_1^2(x)J_1(x)\big]\;Y_1(x)\Big\}$$

$$\int x^3\,K_1^2(x)\,J_1(x)\,Y_1(x)\,dx = \frac{x^2}{8}\,\Big\{\big[-x^2\,K_0^2(x)J_0(x) + x\,K_0^2(x)J_1(x) - 2x\,K_0(x)K_1(x)J_0(x) +$$

$$+K_0(x)K_1(x)J_1(x) + (x^2-1)\,K_1^2(x)J_0(x) - 2x\;K_1^2(x)J_1(x)\big]\;Y_0(x) + \big[x\,K_0^2(x)J_0(x) -$$

$$-(x^2+1)\,K_0^2(x)J_1(x) + K_0(x)K_1(x)J_0(x) - 4x\,K_0(x)K_1(x)J_1(x) - 2x\,K_1^2(x)J_0(x) + x^2\,K_1^2(x)J_1(x)\big]\;Y_1(x)\Big\}$$

$$\int x^5\,I_1(x)\,K_0(x)\,J_0(x)\,J_1(x)\,dx = \frac{x^2}{48}\,\Big\{\big[6x^2\,J_0^2(x)I_0(x) - (7x^3+6x)\,J_0^2(x)I_1(x) + 6x(x^2-2)\,J_0(x)J_1(x)I_0(x) +$$

$$+(16x^2+6)\,J_0(x)J_1(x)I_1(x) - (6x^2-6)\,J_1^2(x)I_0(x) + (11x^3-16x)\,J_1^2(x)I_1(x)\big]\;K_0(x) - \big[(x^3-6x)\,J_0^2(x)I_0(x) +$$

$$+(6x^2+6)\,J_0^2(x)I_1(x) - (16x^2-6)\,J_0(x)J_1(x)I_0(x) + 6x^3\,J_0(x)J_1(x)I_1(x) - (5x^3-16x)\,J_1^2(x)I_0(x)\big]\;K_1(x)\Big\}$$

$$\int x^5\,K_1(x)\,I_0(x)\,J_0(x)\,J_1(x)\,dx = \frac{x^2}{48}\,\Big\{\big[-6x^2\,J_0^2(x)K_0(x) - (7x^3+6x)\,J_0^2(x)K_1(x) - 6x(x^2-2)\,J_0(x)J_1(x)K_0(x) +$$

$$+(16x^2+6)\,J_0(x)J_1(x)K_1(x) + (6x^2-6)\,J_1^2(x)K_0(x) + (11x^3-16x)\,J_1^2(x)K_1(x)\big]\;I_0(x) - \big[(x^3-6x)\,J_0^2(x)K_0(x) -$$

$$-(6x^2+6)\,J_0^2(x)K_1(x) - (16x^2-6)\,J_0(x)J_1(x)K_0(x) - 6x^3\,J_0(x)J_1(x)K_1(x) - (5x^3-16x)\,J_1^2(x)K_0(x)\big]\;I_1(x)\Big\}$$

$$\int x^5\,J_1(x)\,Y_0(x)\,I_0(x)\,I_1(x)\,dx = \frac{x^2}{48}\,\Big\{\big[6x^2\,I_0^2(x)J_0(x) + (7x^3-6x)\,I_0^2(x)J_1(x) - 6x(x^2+2)\,I_0(x)I_1(x)J_0(x) -$$

$$-(16x^2-6)\,I_0(x)I_1(x)J_1(x) + (6x^2+6)\,I_1^2(x)J_0(x) + (11x^3+16x)\,I_1^2(x)J_1(x)\big]\;Y_0(x) - \big[(x^3+6x)\,I_0^2(x)J_0(x) +$$

$$+(6x^2-6)\,I_0^2(x)J_1(x) - (16x^2+6)\,I_0(x)I_1(x)J_0(x) - 6x^3\,I_0(x)I_1(x)J_1(x) + (5x^3+16x)\,I_1^2(x)J_0(x)\big]\;Y_1(x)\Big\}$$

$$\int x^5\,J_1(x)\,Y_0(x)\,K_0(x)\,K_1(x)\,dx = \frac{x^2}{48}\,\Big\{\big[-6x^2\,K_0^2(x)J_0(x) - (7x^3-6x)\,K_0^2(x)J_1(x) - 6x(x^2+2)\,K_0(x)K_1(x)J_0(x) -$$

$$-(16x^2+6)\,K_0(x)K_1(x)J_1(x) - (6x^2+6)\,K_1^2(x)J_0(x) - (11x^3+16x)\,K_1^2(x)J_1(x)\big]\;Y_0(x) + \big[(x^3+6x)\,K_0^2(x)J_0(x) +$$

$$+(6x^2-6)\,K_0^2(x)J_1(x) + (16x^2+6)\,K_0(x)K_1(x)J_0(x) + 6x^3\,K_0(x)K_1(x)J_1(x) + (5x^3+16x)\,K_1^2(x)J_0(x)\big]\;Y_1(x)\Big\}$$

5. Quotients

In the following formulas $J_\nu(x)$ may be substituted by $Y_\nu(x)$ or $H_\nu^{(p)}(x)$, $p = 1, 2$.
Integrals are omitted, when $f(x)$ turns out to be of a very special kind or when the antiderivative is expressed by Whittaker or other hypergeometric functions.

5.1. Denominator $p\,(x)\,Z_0(x) + q(x)\,Z_1(x)$

a) Typ $f(x)\,Z_\nu(x)\,/\,[p\,(x)\,Z_0(x) + q(x)\,Z_1(x)]$:

$$\int \frac{J_1(x)\,dx}{J_0(x)} = -\ln|J_0(x)|\,, \quad \int \frac{J_0(x)\,dx}{J_1(x)} = \ln|x\,J_1(x)|$$

$$\int \frac{I_1(x)\,dx}{I_0(x)} = \ln\,I_0(x)\,, \quad \int \frac{I_0(x)\,dx}{I_1(x)} = \ln\,[x\,I_1(x)]$$

$$\int \frac{K_1(x)\,dx}{K_0(x)} = -\ln\,K_0(x)\,, \quad \int \frac{K_0(x)\,dx}{K_1(x)} = -\ln\,[x\,K_1(x)]\,, \quad x > 0$$

$$\int \frac{(x^2+a^2-2a)\,J_1(x)\,dx}{x[x\,J_0(x) - a\,J_1(x)]} = -\,\ln|x^a\,J_0(x) - ax^{a-1}\,J_1(x)|$$

$$\int \frac{(x^2-a^2+2a)\,I_1(x)\,dx}{x[x\,I_0(x) - a\,I_1(x)]} = \ln|x^a\,I_0(x) - ax^{a-1}\,I_1(x)|$$

$$\int \frac{(x^2-a^2+2a)\,K_1(x)\,dx}{x[x\,K_0(x) + a\,K_1(x)]} = -\,\ln|x^a\,K_0(x) + ax^{a-1}\,K_1(x)|$$

$$\int \frac{\cos x\,J_1(x)\,dx}{x[\sin x\,J_0(x) - \cos x\,J_1(x)]} = \ln|\sin x\,J_0(x) - \cos x\,J_1(x)|$$

$$\int \frac{(2x\,\sin x + \cos x)\,I_1(x)\,dx}{x[\sin x\,I_0(x) - \cos x\,I_1(x)]} = \ln|\sin x\,I_0(x) - \cos x\,I_1(x)|$$

$$\int \frac{(2x\,\sin x + \cos x)\,K_1(x)\,dx}{x[\sin x\,K_0(x) + \cos x\,K_1(x)]} = -\,\ln|\sin x\,K_0(x) + \cos x\,K_1(x)|$$

$$\int \frac{\sin x\,J_1(x)\,dx}{x[\cos x\,J_0(x) + \sin x\,J_1(x)]} = -\ln|\cos x\,J_0(x) + \sin x\,J_1(x)|$$

$$\int \frac{(2x\,\cos x - \sin x)\,I_1(x)\,dx}{x[\cos x\,I_0(x) + \sin x\,I_1(x)]} = \ln|\cos\,I_0(x) + \sin x\,I_1(x)|$$

$$\int \frac{(2x\,\cos x - \sin x)\,K_1(x)\,dx}{x[\cos x\,K_0(x) - \sin x\,K_1(x)]} = -\ln|\cos\,K_0(x) - \sin x\,K_1(x)|$$

5.2. Denominator $[p\,(x)\,Z_0(x) + q(x)\,Z_1(x)]^2$

a) Typ $f(x)\,Z_\mu(x)/[p\,(x)\,Z_0(x) + q(x)\,Z_1(x)]^2$:

$$\int \frac{[(a^2+b^2)x + ab]\exp(-\frac{ax}{b})\,J_0(x)\,dx}{x^2\,[a\,J_0(x) + b\,J_1(x)]^2} = -\,\frac{b\exp(-\frac{ax}{b})}{x\,[a\,J_0(x) + b\,J_1(x)]}$$

$$\int \frac{[(a^2-b^2)x - ab]\exp(\frac{ax}{b})\,I_0(x)\,dx}{x^2\,[a\,I_0(x) + b\,I_1(x)]^2} = \frac{b\exp(\frac{ax}{b})}{x\,[a\,I_0(x) + b\,I_1(x)]}$$

$$\int \frac{[(a^2-b^2)x + ab]\exp(-\frac{ax}{b})\,K_0(x)\,dx}{x^2\,[a\,K_0(x) + b\,K_1(x)]^2} = -\,\frac{b\exp(-\frac{ax}{b})}{x\,[a\,I_0(x) + b\,I_1(x)]}$$

$$\int \frac{[(a^2+b^2)x + ab]\exp(\frac{bx}{a})\,J_1(x)\,dx}{x\,[a\,J_0(x) + b\,J_1(x)]^2} = \frac{a\exp(\frac{bx}{a})}{a\,J_0(x) + b\,J_1(x)}$$

$$\int \frac{[(a^2-b^2)x-ab]\exp(\frac{bx}{a})\, I_1(x)\, dx}{x\,[a\, I_0(x)+b\, I_I(x)]^2} = -\,\frac{a\exp(\frac{bx}{a})}{a\, I_0(x)+b\, I_1(x)}$$

$$\int \frac{[(a^2-b^2)x+ab]\exp(-\frac{bx}{a})\, K_1(x)\, dx}{x\,[a\, K_0(x)+b\, K_I(x)]^2} = \frac{a\exp(-\frac{bx}{a})}{a\, K_0(x)+b\, K_1(x)}$$

$$\int \frac{[a^2x^2+b^2+2ab]\exp(-\frac{ax^2}{2b})\, J_0(x)\, dx}{x[ax\, J_0(x)+b\, J_1(x)]^2} = -\,\frac{b\exp(\frac{-ax^2}{2b})}{x\,[ax\, J_0(x)+b\, J_1(x)]}$$

$$\int \frac{[a^2x^2-b^2-2ab]\exp(\frac{ax^2}{2b})\, I_0(x)\, dx}{x[ax\, I_0(x)+b\, I_1(x)]^2} = \frac{b\exp(\frac{ax^2}{2b})}{x\,[ax\, I_0(x)+b\, I_1(x)]}$$

$$\int \frac{[a^2x^2-b^2+2ab]\exp(-\frac{ax^2}{2b})\, K_0(x)\, dx}{x[ax\, K_0(x)+b\, K_1(x)]^2} = -\,\frac{b\exp(-\frac{ax^2}{2b})}{x\,[ax\, K_0(x)+b\, K_1(x)]}$$

$$\int \frac{x^{b/a}\,[a^2x^2+b^2+2ab]\, J_1(x)\, dx}{[ax\, J_0(x)+b\, J_1(x)]^2} = \frac{a\, x^{1+b/a}}{ax\, J_0(x)+b\, J_1(x)}$$

$$\int \frac{x^{b/a}\,[a^2x^2-b^2-2ab]\, I_1(x)\, dx}{[ax\, I_0(x)+b\, I_1(x)]^2} = -\,\frac{a\, x^{1+b/a}}{ax\, I_0(x)+b\, I_1(x)}$$

$$\int \frac{x^{-b/a}\,[a^2x^2-b^2+2ab]\, K_1(x)\, dx}{[ax\, K_0(x)+b\, K_1(x)]^2} = \frac{a\, x^{1-b/a}}{ax\, K_0(x)+b\, K_1(x)}$$

$$\int \frac{x^{-a/b}\,[a^2+b^2x^2]\, J_0(x)\, dx}{x\,[a\, J_0(x)+bx\, J_1(x)]^2} = -\,\frac{b\, x^{-a/b}}{a\, J_0(x)+bx\, J_1(x)}$$

$$\int \frac{x^{a/b}\,[a^2-b^2x^2]\, I_0(x)\, dx}{x\,[a\, I_0(x)+bx\, I_1(x)]^2} = \frac{b\, x^{a/b}}{a\, I_0(x)+bx\, I_1(x)}$$

$$\int \frac{x^{-a/b}\,[a^2-b^2x^2]\, K_0(x)\, dx}{x\,[a\, K_0(x)+bx\, K_1(x)]^2} = -\,\frac{b\, x^{-a/b}}{a\, K_0(x)+bx\, K_1(x)}$$

$$\int \frac{[a^2+b^2x^2]\exp(\frac{bx^2}{2a})\, J_1(x)\, dx}{[a\, J_0(x)+bx\, J_1(x)]^2} = \frac{a\exp(\frac{bx^2}{2a})}{a\, J_0(x)+bx\, J_1(x)}$$

$$\int \frac{[b^2x^2-a^2]\exp(\frac{bx^2}{2a})\, I_1(x)\, dx}{[a\, I_0(x)+bx\, I_1(x)]^2} = \frac{a\exp(\frac{bx^2}{2a})}{a\, I_0(x)+bx\, I_1(x)}$$

$$\int \frac{[b^2x^2-a^2]\exp(-\frac{bx^2}{2a})\, K_1(x)\, dx}{[a\, K_0(x)+bx\, K_1(x)]^2} = -\,\frac{a\exp(-\frac{bx^2}{2a})}{a\, K_0(x)+bx\, K_1(x)}$$

$$\int \frac{[(a^2+c^2)x^3+(2ab+ac+2cd)x^2+(b^2+2ad+d^2)x+bd]\,(cx+d)^{(da-cb)/c^2}\exp(-\frac{ax}{c})\, J_0(x)\, dx}{x^2\,[(ax+b)\, J_0(x)+(cx+d)\, J_1(x)]^2} =$$
$$= -\,\frac{(cx+d)^{1+(da-cb)/c^2}\exp(-\frac{ax}{c})}{x\,[(ax+b)\, J_0(x)+(cx+d)\, J_1(x)]}$$

$$\int \frac{[(a^2-c^2)x^3+(2ab-ac-2cd)x^2+(b^2-2ad-d^2)x-bd]\,(cx+d)^{(cb-da)/c^2}\exp(\frac{ax}{c})\, I_0(x)\, dx}{x^2\,[(ax+b)\, I_0(x)+(cx+d)\, I_1(x)]^2} =$$
$$= \frac{(cx+d)^{1+(cb-da)/c^2}\exp(\frac{ax}{c})}{x\,[(ax+b)\, I_0(x)+(cx+d)\, I_1(x)]}$$

$$\int \frac{[(a^2-c^2)x^3+(2ab+ac-2cd)x^2+(b^2+2ad-d^2)x+bd]\,(cx+d)^{(da-cb)/c^2}\exp(-\frac{ax}{c})\, K_0(x)\, dx}{x^2\,[(ax+b)\, K_0(x)+(cx+d)\, K_1(x)]^2} =$$
$$= -\,\frac{(cx+d)^{1+(da-cb)/c^2}\exp(-\frac{ax}{c})}{x\,[(ax+b)\, K_0(x)+(cx+d)\, K_1(x)]}$$

$$\int \frac{[(a^2+c^2)x^3+(2ab+ac+2cd)x^2+(b^2+2ad+d^2)x+bd]\,(ax+b)^{(da-cb)/a^2}\exp(\frac{cx}{a})\,J_1(x)\,dx}{x\,[(ax+b)\,J_0(x)+(cx+d)\,J_1(x)]^2} =$$

$$= \frac{(ax+b)^{1+(da-cb)/a^2}\exp(\frac{cx}{a})}{(ax+b)\,J_0(x)+(cx+d)\,J_1(x)}$$

$$\int \frac{[(a^2-c^2)x^3+(2ab-ac-2cd)x^2+(b^2-2ad-d^2)x-bd]\,(ax+b)^{(da-cb)/a^2}\exp(\frac{cx}{a})\,I_1(x)\,dx}{x\,[(ax+b)\,I_0(x)+(cx+d)\,I_1(x)]^2} =$$

$$= -\frac{(ax+b)^{1+(da-cb)/c^2}\exp(\frac{cx}{a})}{(ax+b)\,I_0(x)+(cx+d)\,I_1(x)}$$

$$\int \frac{[(a^2-c^2)x^3+(2ab+ac-2cd)x^2+(b^2+2ad-d^2)x+bd]\,(ax+b)^{(cb-da)/c^2}\exp(-\frac{cx}{a})\,K_1(x)\,dx}{x\,[(ax+b)\,K_0(x)+(cx+d)\,K_1(x)]^2} =$$

$$= \frac{(ax+b)^{1+(cb-da)/a^2}\exp(-\frac{cx}{a})}{(ax+b)\,K_0(x)+(cx+d)\,K_1(x)}$$

$$\int \frac{[2a^2\,x^{3/2}+2b^2\sqrt{x}+3ab]\exp(-\frac{2a}{3b}\,x^{3/2})\,J_0(x)\,dx}{x^{3/2}\,[a\sqrt{x}\,J_0(x)+b\,J_1(x)]^2} = -\frac{2b\exp(-\frac{2a}{3b}\,x^{3/2})}{x[a\sqrt{x}\,J_0(x)+b\,J_1(x)]}$$

$$\int \frac{[2a^2\,x^{3/2}-2b^2\sqrt{x}-3ab]\exp(\frac{2ax}{3b}\,x^{3/2})\,I_0(x)\,dx}{x^{3/2}\,[a\sqrt{x}\,I_0(x)+b\,I_1(x)]^2} = \frac{2b\exp(\frac{2a}{3b}\,x^{3/2})}{x[a\sqrt{x}\,I_0(x)+b\,I_1(x)]}$$

$$\int \frac{[2a^2\,x^{3/2}-2b^2\sqrt{x}+3ab]\exp(-\frac{2a}{3b}\,x^{3/2})\,K_0(x)\,dx}{x^{3/2}\,[a\sqrt{x}\,K_0(x)+b\,K_1(x)]^2} = -\frac{2b\exp(-\frac{2a}{3b}\,x^{3/2})}{x[a\sqrt{x}\,K_0(x)+b\,K_1(x)]}$$

$$\int \frac{[2a^2\,x^{3/2}+2b^2\sqrt{x}+3ab]\exp(\frac{2b}{a}\,\sqrt{x})\,J_1(x)\,dx}{\sqrt{x}\,[a\sqrt{x}\,J_0(x)+b\,J_1(x)]^2} = \frac{2a\,\sqrt{x}\exp(\frac{2b}{a}\,\sqrt{x})}{a\sqrt{x}\,J_0(x)+b\,J_1(x)}$$

$$\int \frac{[2a^2\,x^{3/2}-2b^2\sqrt{x}-3ab]\exp(\frac{2b}{a}\,\sqrt{x})\,I_1(x)\,dx}{\sqrt{x}\,[a\sqrt{x}\,I_0(x)+b\,I_1(x)]^2} = -\frac{2a\,\sqrt{x}\exp(\frac{2b}{a}\,\sqrt{x})}{a\sqrt{x}\,I_0(x)+b\,I_1(x)}$$

$$\int \frac{[2a^2\,x^{3/2}-2b^2\sqrt{x}+3ab]\exp(-\frac{2b}{a}\,\sqrt{x})\,K_1(x)\,dx}{\sqrt{x}\,[a\sqrt{x}\,K_0(x)+b\,K_1(x)]^2} = \frac{2a\,\sqrt{x}\exp(-\frac{2b}{a}\,\sqrt{x})}{a\sqrt{x}\,K_0(x)+b\,K_1(x)}$$

$$\int \frac{[2b^2\,x^{3/2}+2a^2\sqrt{x}+ab]\exp(-\frac{2a}{b}\,\sqrt{x})\,J_0(x)\,dx}{x^{3/2}\,[a\,J_0(x)+b\sqrt{x}\,J_1(x)]^2} = -\frac{2b\exp(-\frac{2a}{b}\,\sqrt{x})}{\sqrt{x}\,[a\,J_0(x)+b\sqrt{x}\,J_1(x)]}$$

$$\int \frac{[2b^2\,x^{3/2}-2a^2\sqrt{x}+ab]\exp(\frac{2a}{b}\,\sqrt{x})\,I_0(x)\,dx}{x^{3/2}\,[a\,I_0(x)+b\sqrt{x}\,I_1(x)]^2} = -\frac{2b\exp(\frac{2a}{b}\,\sqrt{x})}{\sqrt{x}\,[a\,I_0(x)+b\sqrt{x}\,I_1(x)]}$$

$$\int \frac{[2b^2\,x^{3/2}-2a^2\sqrt{x}-ab]\exp(-\frac{2a}{b}\,\sqrt{x})\,K_0(x)\,dx}{x^{3/2}\,[a\,K_0(x)+b\sqrt{x}\,K_1(x)]^2} = \frac{2b\exp(-\frac{2a}{b}\,\sqrt{x})}{\sqrt{x}\,[a\,K_0(x)+b\sqrt{x}\,K_1(x)]}$$

$$\int \frac{[2b^2\,x^{3/2}+2a^2\sqrt{x}+ab]\exp(\frac{2b}{3a}\,x^{3/2})\,J_1(x)\,dx}{\sqrt{x}\,[a\,J_0(x)+b\sqrt{x}\,J_1(x)]^2} = \frac{2a\exp(\frac{2b}{3a}\,x^{3/2})}{a\,J_0(x)+b\sqrt{x}\,J_1(x)}$$

$$\int \frac{[2b^2\,x^{3/2}-2a^2\sqrt{x}+ab]\exp(\frac{2b}{3a}\,x^{3/2})\,I_1(x)\,dx}{\sqrt{x}\,[a\,I_0(x)+b\sqrt{x}\,I_1(x)]^2} = \frac{2a\exp(\frac{2b}{3a}\,x^{3/2})}{a\,I_0(x)+b\sqrt{x}\,I_1(x)}$$

$$\int \frac{[2b^2\,x^{3/2}-2a^2\sqrt{x}-ab]\exp(-\frac{2b}{3a}\,x^{3/2})\,K_1(x)\,dx}{\sqrt{x}\,[a\,K_0(x)+b\sqrt{x}\,K_1(x)]^2} = -\frac{2a\exp(-\frac{2b}{3a}\,x^{3/2})}{a\,K_0(x)+b\sqrt{x}\,K_1(x)}$$

$$\int \frac{[2a^2\,x^3+5ab\,\sqrt{x}+2b^2]\exp(-\frac{2a}{5b}\,x^{5/2})\,J_0(x)\,dx}{x\,[ax^{3/2}\,J_0(x)+b\,J_1(x)]^2}=-\,\frac{2b\exp(-\frac{2a}{5b}\,x^{5/2})}{x\,[ax^{3/2}\,J_0(x)+b\,J_1(x)]}$$

$$\int \frac{[2a^2\,x^3-5ab\,\sqrt{x}-2b^2]\exp(\frac{2a}{5b}\,x^{5/2})\,I_0(x)\,dx}{x\,[ax^{3/2}\,I_0(x)+b\,I_1(x)]^2}=\frac{2b\exp(\frac{2a}{5b}\,x^{5/2})}{x\,[ax^{3/2}\,I_0(x)+b\,I_1(x)]}$$

$$\int \frac{[2a^2\,x^3+5ab\,\sqrt{x}-2b^2]\exp(-\frac{2a}{5b}\,x^{5/2})\,K_0(x)\,dx}{x\,[ax^{3/2}\,K_0(x)+b\,K_1(x)]^2}=-\,\frac{2b\exp(-\frac{2a}{5b}\,x^{5/2})}{x\,[ax^{3/2}\,K_0(x)+b\,K_1(x)]}$$

$$\int \frac{[2a^2\,x^3+5ab\,\sqrt{x}+2b^2]\exp(-2b/a\sqrt{x}\,)\,J_1(x)\,dx}{[ax^{3/2}\,J_0(x)+b\,J_1(x)]^2}=\frac{2a\,x^{3/2}\exp(-2b/a\sqrt{x}\,)}{ax^{3/2}\,J_0(x)+b\,J_1(x)}$$

$$\int \frac{[2a^2\,x^3-5ab\,\sqrt{x}-2b^2]\exp(-2b/a\sqrt{x}\,)\,I_1(x)\,dx}{[ax^{3/2}\,I_0(x)+b\,I_1(x)]^2}=-\frac{2a\,x^{3/2}\exp(-2b/a\sqrt{x}\,)}{ax^{3/2}\,I_0(x)+b\,I_1(x)}$$

$$\int \frac{[2a^2\,x^3+5ab\,\sqrt{x}-2b^2]\exp(2b/a\sqrt{x}\,)\,K_1(x)\,dx}{[ax^{3/2}\,K_0(x)+b\,K_1(x)]^2}=\frac{2a\,x^{3/2}\exp(2b/a\sqrt{x}\,)}{ax^{3/2}\,K_0(x)+b\,K_1(x)}$$

$$\int \frac{[2b^2\,x^3-ab\,\sqrt{x}+2a^2]\exp(2a/b\sqrt{x}\,)\,J_0(x)\,dx}{x\,[a\,J_0(x)+bx^{3/2}\,J_1(x)]^2}=-\,\frac{2b\,\sqrt{x}\exp(2a/b\sqrt{x}\,)}{a\,J_0(x)+bx^{3/2}\,J_1(x)}$$

$$\int \frac{[2b^2\,x^3-ab\,\sqrt{x}-2a^2]\exp(-2a/b\sqrt{x}\,)\,I_0(x)\,dx}{x\,[a\,I_0(x)+bx^{3/2}\,I_1(x)]^2}=-\,\frac{2b\,\sqrt{x}\exp(-2a/b\sqrt{x}\,)}{a\,I_0(x)+bx^{3/2}\,I_1(x)}$$

$$\int \frac{[2b^2\,x^3+ab\,\sqrt{x}-2a^2]\exp(2a/b\sqrt{x}\,)\,K_0(x)\,dx}{x\,[a\,K_0(x)+bx^{3/2}\,K_1(x)]^2}=\frac{2b\,\sqrt{x}\exp(2a/b\sqrt{x}\,)}{a\,K_0(x)+bx^{3/2}\,K_1(x)}$$

$$\int \frac{[2b^2\,x^3-ab\,\sqrt{x}+2a^2]\exp(2bx^{5/2}/5a)\,J_1(x)\,dx}{[a\,J_0(x)+bx^{3/2}\,J_1(x)]^2}=\frac{2a\exp(2bx^{5/2}/5a)}{a\,J_0(x)+bx^{3/2}\,J_1(x)}$$

$$\int \frac{[2b^2\,x^3-ab\,\sqrt{x}-2a^2]\exp(2bx^{5/2}/5a)\,I_1(x)\,dx}{[a\,I_0(x)+bx^{3/2}\,I_1(x)]^2}=\frac{2a\exp(2bx^{5/2}/5a)}{a\,I_0(x)+bx^{3/2}\,I_1(x)}$$

$$\int \frac{[2b^2\,x^3+ab\,\sqrt{x}-2a^2]\exp(-2bx^{5/2}/5a)\,K_1(x)\,dx}{[a\,K_0(x)+bx^{3/2}\,K_1(x)]^2}=-\,\frac{2a\exp(-2bx^{5/2}/5a)}{a\,K_0(x)+bx^{3/2}\,K_1(x)}$$

b) Typ $f(x)\,Z_0^n(x)\,Z_1^{2-n}(x)/[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]^2\,,\ n=0,1,2:$

$$\int \frac{[(a^2+b^2)x+ab]\cdot\exp\left(-\frac{2ax}{b}\right)\cdot J_0^2(x)\,dx}{x^2\,[a\,J_0(x)\,+b\,J_1(x)]^2}=-\mathrm{Ei}\left(-\frac{2ax}{b}\right)-\frac{b\,J_0(x)}{x\,[a\,J_0(x)\,+b\,J_1(x)]}\cdot\exp\left(-\frac{2ax}{b}\right)$$

with the exponential integral $\mathrm{Ei}\,(x)$ (see page 499).

$$\int \frac{[(a^2-b^2)x-ab]\cdot\exp\left(\frac{2ax}{b}\right)\cdot I_0^2(x)\,dx}{x^2\,[a\,I_0(x)\,+b\,I_1(x)]^2}=-\mathrm{Ei}\left(\frac{2ax}{b}\right)+\frac{b\,I_0(x)}{x\,[a\,I_0(x)\,+b\,I_1(x)]}\cdot\exp\left(\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\cdot\exp\left(-\frac{2ax}{b}\right)\cdot K_0^2(x)\,dx}{x^2\,[a\,K_0(x)\,+b\,K_1(x)]^2}=-\mathrm{Ei}\left(-\frac{2ax}{b}\right)-\frac{b\,K_0(x)}{x\,[a\,K_0(x)\,+b\,K_1(x)]}\cdot\exp\left(-\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2+b^2)x+ab]\cdot\exp\left(-\frac{a^2-b^2}{ab}\,x\right)\cdot J_0(x)\cdot J_1(x)\,dx}{x\,[a\,J_0(x)\,+b\,J_1(x)]^2}=-\,\frac{ab\,[b\,J_0(x)+a\,J_1(x)]}{(a^2-b^2)\,[aJ_0(x)\,+\,bJ_1(x)]}\cdot\exp\left(-\frac{a^2-b^2}{ab}\,x\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\cdot\exp\left(\frac{a^2+b^2}{ab}\,x\right)\cdot I_0(x)\cdot I_1(x)\,dx}{x\,[a\,I_0(x)\,+b\,I_1(x)]^2}=-\,\frac{ab\,[b\,J_0(x)-a\,J_1(x)]}{(a^2+b^2)\,[a\,I_0(x)\,+b\,I_1(x)]}\cdot\exp\left(\frac{a^2+b^2}{ab}\,x\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\cdot \exp\,(-\frac{a^2+b^2}{ab}\,x)\cdot K_0(x)\cdot K_1(x)\,dx}{x\,[a\,K_0(x)\;+b\,K_1(x)]^2} = \frac{ab\,[b\,K_0(x)-a\,K_1(x)]}{(a^2+b^2)\,[a\,K_0(x)\;+b\,K_1(x)]}\cdot \exp\left(-\frac{a^2+b^2}{ab}\,x\right)$$

$$\int \frac{[(a^2+b^2)x+ab]\cdot \exp\,(\frac{2bx}{a})\cdot J_1^2(x)\,dx}{[a\,J_0(x)\;+\;b\,J_1(x)]^2} = \frac{a\,[a(a-2bx)\,J_0(x)+b(a+2bx)\,J_1(x)]}{4b^2\,[a\,J_0(x)\;+b\,J_1(x)]}\cdot\exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\cdot \exp\,(\frac{2bx}{a})\cdot I_1^2(x)\,dx}{[a\,I_0(x)\;+b\,I_1(x)]^2} = -\,\frac{a\,[a(a-2bx)\,I_0(x)+b(a+2bx)\,I_1(x)]}{4b^2\,[a\,I_0(x)\;+b\,I_1(x)]}\cdot\exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\cdot \exp\,(-\frac{2bx}{a})\cdot K_1^2(x)\,dx}{[a\,K_0(x)\;+b\,K_1(x)]^2} = -\,\frac{a\,[a(a+2bx)\,K_0(x)+b(a-2bx)\,K_1(x)]}{4b^2\,[a\,K_0(x)\;+b\,K_1(x)]}\cdot\exp\left(-\frac{2bx}{a}\right)$$

$$\int \frac{[a^2x^2+b^2+2ab]\cdot \exp\,(-\frac{ax^2}{b})\cdot J_0^2(x)\,dx}{x\,[ax\,J_0(x)\;+b\,J_1(x)]^2} = -\frac{1}{2}\,\mathrm{Ei}\,\left(-\frac{ax^2}{b}\right)-\frac{b\,J_0(x)}{x\,[ax\,J_0(x)\;+b\,J_1(x)]}\cdot\exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2-b^2-2ab]\cdot \exp\,(\frac{ax^2}{b})\cdot I_0^2(x)\,dx}{x\,[ax\,I_0(x)\;+b\,I_1(x)]^2} = -\frac{1}{2}\,\mathrm{Ei}\,\left(\frac{ax^2}{b}\right)+\frac{b\,I_0(x)}{x\,[ax\,I_0(x)\;+b\,I_1(x)]}\cdot\exp\left(\frac{ax^2}{b}\right)$$

$$\int \frac{[(a^2x^2-b^2+2ab]\cdot \exp\,(-\frac{ax^2}{b})\cdot K_0^2(x)\,dx}{x\,[ax\,K_0(x)\;+b\,K_1(x)]^2} = -\frac{1}{2}\,\mathrm{Ei}\,\left(-\frac{ax^2}{b}\right)-\frac{b\,K_0(x)}{x\,[ax\,K_0(x)\;+b\,K_1(x)]}\cdot\exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{b(a+b)\,(a^2x^2+2a+b)\,x^{1+2b/a}\;\;J_1^2(x)\,dx}{[ax\,J_0(x)+b\,J_1(x)]^2} = -\,\frac{a\,x^{2+2b/a}\,[ax\,J_0(x)-(2a+b)\,J_1(x)]}{2\,[ax\,J_0(x)+b\,J_1(x)]}$$

$$\int \frac{(b(a+b)\,(a^2x^2-2a-b)\,x^{1+2b/a}\;\;I_1^2(x)\,dx}{[ax\,I_0(x)+b\,I_1(x)]^2} = \frac{a\,x^{2+2b/a}\,[ax\,I_0(x)-(2a+b)\,I_1(x)]}{2\,[ax\,I_0(x)+b\,I_1(x)]}$$

$$\int \frac{b(a-b)\,(a^2x^2+2a-b)\,x^{1-2b/a}\;\;K_1^2(x)\,dx}{[ax\,K_0(x)+b\,K_1(x)]^2} = \frac{a\,x^{2-2b/a}\,[ax\,K_0(x)+(2a-b)\,K_1(x)]}{2\,[ax\,K_0(x)+b\,K_1(x)]}$$

$$\int \frac{(a^2+b^2x^2)\,x^{-1-2a/b}\;\;J_0^2(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^2} = -\,\frac{b\,x^{-2a/b}\,[a\,J_0(x)-bx\,J_1(x)]}{2a\,[a\,J_0(x)+bx\,J_1(x)]}$$

$$\int \frac{(a^2-b^2x^2)\,x^{-1+2a/b}\;\;I_0^2(x)\,dx}{[a\,I_0(x)+bx\,I_1(x)]^2} = \frac{b\,x^{2a/b}\,[a\,I_0(x)-bx\,I_1(x)]}{2a\,[a\,I_0(x)+bx\,I_1(x)]}$$

$$\int \frac{(a^2-b^2x^2)\,x^{-1-2a/b}\;\;K_0^2(x)\,dx}{[a\,K_0(x)+bx\,K_1(x)]^2} = -\,\frac{b\,x^{-2a/b}\,[a\,K_0(x)-bx\,K_1(x)]}{2a\,[a\,K_0(x)+bx\,K_1(x)]}$$

$$\int \frac{x(a^2+b^2x^2)\cdot \exp\,(\frac{bx^2}{a})\cdot J_1^2(x)\,dx}{[a\,J_0(x)\;+bx\,J_1(x)]^2} = -\,\frac{a\,[a\,J_0(x)-bx\,J_1(x)]}{2b\,[a\,J_0(x)\;+bx\,J_1(x)]}\cdot\exp\left(\frac{bx^2}{a}\right)$$

$$\int \frac{x(a^2-b^2x^2)\cdot \exp\,(\frac{bx^2}{a})\cdot I_1^2(x)\,dx}{[a\,I_0(x)\;+bx\,I_1(x)]^2} = \frac{a\,[a\,I_0(x)-bx\,I_1(x)]}{2b\,[a\,I_0(x)\;+bx\,I_1(x)]}\cdot\exp\left(\frac{bx^2}{a}\right)$$

$$\int \frac{x(a^2-b^2x^2)\cdot \exp\,(-\frac{bx^2}{a})\cdot K_1^2(x)\,dx}{[a\,K_0(x)\;+bx\,K_1(x)]^2} = -\,\frac{a\,[a\,K_0(x)-bx\,K_1(x)]}{2b\,[a\,K_0(x)\;+bx\,K_1(x)]}\cdot\exp\left(-\frac{bx^2}{a}\right)$$

5.3. Denominator $[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]^3$

$$\int \frac{[(a^2+b^2)x+ab]\exp(-\frac{2ax}{b})\,J_0(x)\,dx}{x^3\,[a\,J_0(x)+b\,J_1(x)]^3} = -\,\frac{b}{2x^2\,[a\,J_0(x)+b\,J_1(x)]^2}\,\exp\left(-\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\exp(\frac{2ax}{b})\,I_0(x)\,dx}{x^3\,[a\,I_0(x)+b\,I_1(x)]^3} = \frac{b}{2x^2\,[a\,I_0(x)+b\,I_1(x)]^2}\,\exp\left(\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\exp(-\frac{2ax}{b})\,K_0(x)\,dx}{x^3\,[a\,K_0(x)+b\,K_1(x)]^3} = -\,\frac{b}{2x^2\,[a\,K_0(x)+b\,K_1(x)]^2}\exp\left(-\frac{2ax}{b}\right)$$

$$\int \frac{[(a^2+b^2)x+ab]\exp(\frac{2bx}{a})\,J_1(x)\,dx}{x\,[a\,J_0(x)+b\,J_1(x)]^3} = \frac{a}{2\,[a\,J_0(x)+b\,J_1(x)]^2}\exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\exp(\frac{2bx}{a})\,I_1(x)\,dx}{x\,[a\,I_0(x)+b\,I_1(x)]^3} = -\,\frac{a}{2\,[a\,I_0(x)+b\,I_1(x)]^2}\exp\left(\frac{2bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\exp(-\frac{2bx}{a})\,K_1(x)\,dx}{x\,[a\,K_0(x)+b\,K_1(x)]^3} = \frac{a}{2\,[a\,K_0(x)+b\,K_1(x)]^2}\exp\left(-\frac{2bx}{a}\right)$$

$$\int \frac{[a^2x^2+b^2+2ab]\exp(-\frac{ax^2}{b})\,J_0(x)\,dx}{x^2\,[ax\,J_0(x)+b\,J_1(x)]^3} = -\,\frac{b}{2x^2\,[ax\,J_0(x)+b\,J_1(x)]^2}\exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2-b^2-2ab]\exp(\frac{ax^2}{b})\,I_0(x)\,dx}{x^2\,[ax\,I_0(x)+b\,I_1(x)]^3} = \frac{b}{2x^2\,[ax\,I_0(x)+b\,I_1(x)]^2}\exp\left(\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2-b^2+2ab]\exp(-\frac{ax^2}{b})\,K_0(x)\,dx}{x^2\,[ax\,K_0(x)+b\,K_1(x)]^3} = -\,\frac{b}{2x^2\,[ax\,J_0(x)+b\,J_1(x)]^2}\exp\left(-\frac{ax^2}{b}\right)$$

$$\int \frac{[a^2x^2+b^2+2ab]\,x^{1+2b/a}\,J_1(x)\,dx}{[ax\,J_0(x)+b\,J_1(x)]^3} = \frac{ax^{2+2b/a}}{2\,[ax\,J_0(x)+b\,J_1(x)]^2}$$

$$\int \frac{[a^2x^2-b^2-2ab]\,x^{1+2b/a}\,I_1(x)\,dx}{[ax\,I_0(x)+b\,I_1(x)]^3} = -\,\frac{ax^{2+2b/a}}{2\,[ax\,I_0(x)+b\,I_1(x)]^2}$$

$$\int \frac{[a^2x^2-b^2+2ab]\,x^{1-2b/a}\,K_1(x)\,dx}{[ax\,K_0(x)+b\,K_1(x)]^3} = \frac{ax^{2-2b/a}}{2\,[ax\,K_0(x)+b\,K_1(x)]^2}$$

$$\int \frac{[a^2+b^2x^2]\,x^{-1-2a/b}\,J_0(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^3} = -\,\frac{bx^{-2a/b}}{2\,[a\,J_0(x)+bx\,J_1(x)]^2}$$

$$\int \frac{[a^2-b^2x^2]\,x^{-1+2a/b}\,I_0(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^3} = \frac{bx^{2a/b}}{2\,[a\,I_0(x)+bx\,I_1(x)]^2}$$

$$\int \frac{[a^2-b^2x^2]\,x^{-1-2a/b}\,K_0(x)\,dx}{[a\,K_0(x)+bx\,K_1(x)]^3} = -\,\frac{bx^{-2a/b}}{2\,[a\,K_0(x)+bx\,K_1(x)]^2}$$

$$\int \frac{[a^2+b^2x^2]\exp(\frac{b}{a}\,x^2)\,J_1(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^3} = \frac{a}{2\,[a\,J_0(x)+bx\,J_1(x)]^2}\exp\left(\frac{b\,x^2}{a}\right)$$

$$\int \frac{[a^2-b^2x^2]\exp(\frac{b}{a}\,x^2)\,I_1(x)\,dx}{[a\,I_0(x)+bx\,I_1(x)]^3} = -\,\frac{a}{2\,[a\,I_0(x)+bx\,I_1(x)]^2}\exp\left(\frac{b\,x^2}{a}\right)$$

$$\int \frac{[a^2-b^2x^2]\exp(-\frac{b}{a}\,x^2)\,K_1(x)\,dx}{[a\,K_0(x)+bx\,K_1(x)]^3} = \frac{a}{2\,[a\,K_0(x)+bx\,K_1(x)]^2}\exp\left(-\frac{b\,x^2}{a}\right)$$

5.4. Denominator $[p\,(x)\,Z_0(x)+q(x)\,Z_1(x)]^4$

$$\int \frac{[(a^2+b^2)x+ab]\exp(-\frac{3ax}{b})\,J_0(x)\,dx}{x^4\,[a\,J_0(x)+b\,J_1(x)]^4} = -\,\frac{b}{3x^3\,[a\,J_0(x)+b\,J_1(x)]^3}\exp\left(-\frac{3ax}{b}\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\exp(\frac{3ax}{b})\,I_0(x)\,dx}{x^4\,[a\,I_0(x)+b\,I_1(x)]^4} = \frac{b}{3x^3\,[a\,I_0(x)+b\,I_1(x)]^3}\exp\left(\frac{3ax}{b}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\exp(-\frac{3ax}{b})\,K_0(x)\,dx}{x^4\,[a\,K_0(x)+b\,K_1(x)]^4} = -\frac{b}{3x^3\,[a\,J_0(x)+b\,J_1(x)]^3}\exp\left(-\frac{3ax}{b}\right)$$

$$\int \frac{[(a^2+b^2)x+ab]\exp(\frac{3bx}{a})\,J_1(x)\,dx}{x\,[a\,J_0(x)+b\,J_1(x)]^4} = \frac{a}{3\,[a\,J_0(x)+b\,J_1(x)]^3}\exp\left(\frac{3bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x-ab]\exp(\frac{3bx}{a})\,I_1(x)\,dx}{x\,[a\,I_0(x)+b\,I_1(x)]^4} = -\frac{a}{3\,[a\,I_0(x)+b\,I_1(x)]^3}\exp\left(\frac{3bx}{a}\right)$$

$$\int \frac{[(a^2-b^2)x+ab]\exp(-\frac{3bx}{a})\,K_1(x)\,dx}{x\,[a\,K_0(x)+b\,K_1(x)]^4} = \frac{a}{3\,[a\,K_0(x)+b\,K_1(x)]^3}\exp\left(-\frac{3bx}{a}\right)$$

$$\int \frac{[a^2x^2+b^2+2ab]\exp(-\frac{3ax^2}{2b})\,J_0(x)\,dx}{x^3\,[ax\,J_0(x)+b\,J_1(x)]^4} = -\frac{b}{3x^3\,[ax\,J_0(x)+b\,J_1(x)]^3}\exp\left(-\frac{3ax^2}{2b}\right)$$

$$\int \frac{[a^2x^2-b^2-2ab]\exp(\frac{3ax^2}{2b})\,I_0(x)\,dx}{x^3\,[ax\,I_0(x)+b\,I_1(x)]^4} = \frac{b}{3x^3\,[ax\,I_0(x)+b\,I_1(x)]^3}\exp\left(\frac{3ax^2}{2b}\right)$$

$$\int \frac{[a^2x^2-b^2+2ab]\exp(-\frac{3ax^2}{2b})\,K_0(x)\,dx}{x^3\,[ax\,K_0(x)+b\,K_1(x)]^4} = -\frac{b}{3x^3\,[ax\,K_0(x)+b\,K_1(x)]^3}\exp\left(-\frac{3ax^2}{2b}\right)$$

$$\int \frac{[a^2x^2+b^2+2ab]\,x^{2+3b/a}\;J_1(x)\,dx}{[ax\,J_0(x)+b\,J_1(x)]^4} = \frac{a\,x^{3+3b/a}}{3\,[ax\,J_0(x)+b\,J_1(x)]^3}$$

$$\int \frac{[a^2x^2-b^2-2ab]\,x^{2+3b/a}\;I_1(x)\,dx}{[ax\,I_0(x)+b\,I_1(x)]^4} = -\frac{a\,x^{3+3b/a}}{3\,[ax\,I_0(x)+b\,I_1(x)]^3}$$

$$\int \frac{[a^2x^2-b^2+2ab]\,x^{2-3b/a}\;K_1(x)\,dx}{[ax\,K_0(x)+b\,K_1(x)]^4} = \frac{a\,x^{3-3b/a}}{3\,[ax\,K_0(x)+b\,K_1(x)]^3}$$

$$\int \frac{(a^2+b^2x^2)\,x^{-1-3a/b}\;J_0(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^4} = -\frac{b\,x^{-3a/b}}{3\,[a\,J_0(x)+bx\,J_1(x)]^3}$$

$$\int \frac{(a^2-b^2x^2)\,x^{-1+3a/b}\;I_0(x)\,dx}{[a\,I_0(x)+bx\,I_1(x)]^4} = \frac{b\,x^{3a/b}}{3\,[a\,I_0(x)+bx\,I_1(x)]^3}$$

$$\int \frac{(a^2-b^2x^2)\,x^{-1-3a/b}\;K_0(x)\,dx}{[a\,K_0(x)+bx\,K_1(x)]^4} = -\frac{b\,x^{-3a/b}}{3\,[a\,K_0(x)+bx\,K_1(x)]^3}$$

$$\int \frac{[a^2+b^2x^2]\exp(\frac{3bx^2}{2a})\,J_1(x)\,dx}{[a\,J_0(x)+bx\,J_1(x)]^4} = \frac{a}{3\,[a\,J_0(x)+bx\,J_1(x)]^3}\exp\left(\frac{3bx^2}{2a}\right)$$

$$\int \frac{[a^2-b^2x^2]\exp(\frac{3bx^2}{2a})\,I_1(x)\,dx}{[a\,I_0(x)+bx\,I_1(x)]^4} = -\frac{a}{3\,[a\,I_0(x)+bx\,I_1(x)]^3}\exp\left(\frac{3bx^2}{2a}\right)$$

$$\int \frac{[a^2-b^2x^2]\exp(-\frac{3bx^2}{2a})\,K_1(x)\,dx}{[a\,K_0(x)+bx\,K_1(x)]^4} = \frac{a}{3\,[a\,K_0(x)+bx\,K_1(x)]^3}\exp\left(-\frac{3bx^2}{2a}\right)$$

5.5. Denominator $p(x)\,Z_0^2(x)+q(x)\,Z_0(x)\,Z_1(x)+r(x)\,Z_1^2(x)$

a) Typ $f(x)\,Z_0^n(x)\,Z_1^{2-n}(x)/[p(x)\,Z_0^2(x)+q(x)\,Z_1^2(x)],\ n=0,1,2:$

$$\int\frac{J_0^2(x)\,dx}{x\,[J_0^2(x)\,+J_1^2(x)]}=\frac{1}{2}\,\ln\{x^2\,[J_0^2(x)\,+J_1^2(x)]\}$$

$$\int\frac{I_0^2(x)\,dx}{x\,[I_0^2(x)\,-I_1^2(x)]}=\frac{1}{2}\,\ln\{x^2\,[I_0^2(x)\,-I_1^2(x)]\}$$

$$\int\frac{K_0^2(x)\,dx}{x\,[K_0^2(x)\,-K_1^2(x)]}=\frac{1}{2}\,\ln\{x^2\,[K_1^2(x)\,-K_0^2(x)]\}$$

$$\int\frac{(x^2-a)\,J_0(x)\,J_1(x)\,dx}{a\,J_0^2(x)\,+x^2\,J_1^2(x)}=\frac{1}{2}\,\ln[a\,J_0^2(x)\,+x^2\,J_1^2(x)]$$

$$\int\frac{(x^2+a)\,I_0(x)\,I_1(x)\,dx}{a\,I_0^2(x)\,+x^2\,I_1^2(x)}=\frac{1}{2}\,\ln[a\,I_0^2(x)\,+x^2\,I_1^2(x)]$$

$$\int\frac{(x^2+a)\,K_0(x)\,K_1(x)\,dx}{a\,K_0^2(x)\,+x^2\,K_1^2(x)}=-\frac{1}{2}\,\ln[a\,K_0^2(x)\,+x^2\,K_1^2(x)]$$

$$\int\frac{J_1^2(x)\,dx}{x\,[J_0^2(x)\,+J_1^2(x)]}=-\frac{1}{2}\,\ln[J_0^2(x)\,+J_1^2(x)]$$

$$\int\frac{I_1^2(x)\,dx}{x\,[I_0^2(x)\,-I_1^2(x)]}=\frac{1}{2}\,\ln[I_0^2(x)\,-I_1^2(x)]$$

$$\int\frac{K_1^2(x)\,dx}{x\,[K_0^2(x)\,-K_1^2(x)]}=\frac{1}{2}\,\ln[K_0^2(x)\,-K_1^2(x)]$$

b) Typ $f(x)\,Z_0^n(x)\,Z_1^{2-n}(x)/[p(x)\,Z_0^2(x)+q(x)\,Z_0(x)\,Z_1(x)+r(x)\,Z_1^2(x)],\ n=0,1,2:$

$$\int\frac{x\,J_0^2(x)\,dx}{x^2\,J_0^2(x)+2x\,J_0(x)\,J_1(x)+(x^2-2)\,J_1^2(x)}=\frac{1}{6}\,\ln\left|x^4\,J_0^2(x)+2x^3\,J_0(x)\,J_1(x)+(x^4-2x^2)\,J_1^2(x)\right|$$

$$\int\frac{x\,I_0^2(x)\,dx}{x^2\,I_0^2(x)+2x\,I_0(x)\,I_1(x)-(x^2+2)\,I_1^2(x)}=\frac{1}{6}\,\ln\left|x^4\,I_0^2(x)+2x^3\,I_0(x)\,I_1(x)-(x^4+2x^2)\,I_1^2(x)\right|$$

$$\int\frac{x\,K_0^2(x)\,dx}{x^2\,K_0^2(x)-2x\,K_0(x)\,K_1(x)-(x^2+2)\,K_1^2(x)}=\frac{1}{6}\,\ln\left|x^4\,K_0^2(x)-2x^3\,K_0(x)\,K_1(x)-(x^4+2x^2)\,K_1^2(x)\right|$$

$$\int\frac{J_0(x)J_1(x)\,dx}{x[xJ_0^2(x)x-J_0(x)J_1(x)+xJ_1^2(x)]}=\ln\left|xJ_0^2(x)x-J_0(x)J_1(x)+xJ_1^2(x)\right|$$

$$\int\frac{I_0(x)I_1(x)\,dx}{x[xI_0^2(x)x-I_0(x)I_1(x)-xI_1^2(x)]}=\ln\left|xI_0^2(x)x-I_0(x)I_1(x)-xI_1^2(x)\right|$$

$$\int\frac{K_0(x)K_1(x)\,dx}{x[xK_0^2(x)x+K_0(x)K_1(x)+xK_1^2(x)]}=-\ln\left|xK_0^2(x)x+K_0(x)K_1(x)-xK_1^2(x)\right|$$

$$\int\frac{(1+2\ln x)x\,J_0(x)\,J_1(x)\,dx}{xJ_0^2(x)-2J_0(x)J_1(x)-2x\ln xJ_1^2(x)}=-\frac{1}{2}\ln\left|x\left[xJ_0^2(x)-2J_0(x)J_1(x)-2x\ln xJ_1^2(x)\right]\right|$$

$$\int\frac{(1+2\ln x)x\,I_0(x)\,I_1(x)\,dx}{xI_0^2(x)-2I_0(x)I_1(x)+2x\ln xI_1^2(x)}=\frac{1}{2}\ln\left|x\left[xI_0^2(x)-2I_0(x)I_1(x)+2x\ln xI_1^2(x)\right]\right|$$

$$\int\frac{(1+2\ln x)x\,K_0(x)\,K_1(x)\,dx}{xK_0^2(x)+2K_0(x)K_1(x)-2x\ln xK_1^2(x)}=-\frac{1}{2}\ln\left|x\left[xK_0^2(x)+2K_0(x)K_1(x)+2x\ln xK_1^2(x)\right]\right|$$

$$\int \frac{(8x^2+3)\,J_0(x)J_1(x)\,dx}{x[xJ_0^2(x)-3J_0(x)J_1(x)-3xJ_1^2(x)]} = -\ln\left|x^2\left(xJ_0^2(x)-3J_0(x)J_1(x)-3xJ_1^2(x)\right)\right|$$

$$\int \frac{(8x^2-3)\,I_0(x)I_1(x)\,dx}{x[xI_0^2(x)-3I_0(x)I_1(x)+3xI_1^2(x)]} = \ln\left|x^2\left(xI_0^2(x)-3I_0(x)I_1(x)+3xI_1^2(x)\right)\right|$$

$$\int \frac{(8x^2-3)\,K_0(x)K_1(x)\,dx}{x[xK_0^2(x)+3K_0(x)K_1(x)+3xK_1^2(x)]} = -\ln\left|x^2\left(xK_0^2(x)+3K_0(x)K_1(x)+3xK_1^2(x)\right)\right|$$

$$\int \frac{x^2\,J_0(x)\,J_1(x)\,dx}{x^2\,J_0^2(x)-4x\,J_0(x)\,J_1(x)-(2x^2-4)\,J_1^2(x)} = -\frac{1}{6}\,\ln\left|x^4\,J_0^2(x)-4x^3\,J_0(x)\,J_1(x)-(2x^4-4x^2)\,J_1^2(x)\right|$$

$$\int \frac{x^2\,I_0(x)\,I_1(x)\,dx}{x^2\,I_0^2(x)+4x\,I_0(x)\,I_1(x)+(2x^2+4)\,I_1^2(x)} = -\frac{1}{6}\,\ln\left|x^4\,I_0^2(x)+4x^3\,I_0(x)\,I_1(x)+(2x^4+4x^2)\,I_1^2(x)\right|$$

$$\int \frac{x^2\,K_0(x)\,K_1(x)\,dx}{x^2\,K_0^2(x)+4x\,K_0(x)\,K_1(x)+(2x^2+4)\,K_1^2(x)} = -\frac{1}{6}\,\ln\left|x^4\,K_0^2(x)+4x^3\,K_0(x)\,K_1(x)+(2x^4+4x^2)\,K_1^2(x)\right|$$

$$\int \frac{J_1^2(x)\,dx}{xJ_0^2(x)-2J_0(x)J_1(x)+xJ_1^2(x)} = \frac{1}{2}\ln\left[x^2\,J_0^2(x)-2x\,J_0(x)J_1(x)+x^2\,J_1^2(x)\right]$$

$$\int \frac{I_1^2(x)\,dx}{xI_0^2(x)-2I_0(x)I_1(x)-xI_1^2(x)} = -\frac{1}{2}\ln\left[x^2\,I_1^2(x)+2x\,I_0(x)I_1(x)-x^2\,I_0^2(x)\right]$$

$$\int \frac{K_1^2(x)\,dx}{xK_0^2(x)+2K_0(x)K_1(x)-xK_1^2(x)} = -\frac{1}{2}\ln\left[x^2\,K_1^2(x)-2x\,K_0(x)K_1(x)-x^2\,K_0^2(x)\right]$$

$$\int \frac{x\,J_1^2(x)\,dx}{x^2\,J_0^2(x)-4x\,J_0(x)\,J_1(x)+(x^2+4)\,J_1^2(x)} = \frac{1}{6}\,\ln\left[x^4\,J_0^2(x)-4x^3\,J_0(x)\,J_1(x)+(x^4+4x^2)\,J_1^2(x)\right]$$

$$\int \frac{x\,I_1^2(x)\,dx}{x^2\,I_0^2(x)-4x\,I_0(x)\,I_1(x)-(x^2-4)\,I_1^2(x)} = -\frac{1}{6}\,\ln\left[4x^3\,I_0(x)\,I_1(x)+(x^4-4x^2)\,I_1^2(x)-x^4\,I_0^2(x)\right]$$

$$\int \frac{x\,K_1^2(x)\,dx}{x^2\,K_0^2(x)+4x\,K_0(x)\,K_1(x)-(x^2-4)\,K_1^2(x)} = -\frac{1}{6}\,\ln\left[x^4\,K_0^2(x)+4x^3\,K_0(x)\,K_1(x)-(x^4-4x^2)\,K_1^2(x)\right]$$

5.6. Denominator $\sqrt{a(x)\,Z_0(x)+b(x)\,Z_1(x)+p\,(x)\,Z_0^2(x)+q(x)\,Z_0(x)\,Z_1(x)+r(x)\,Z_1^2(x)}$

Generally: From

$$\int \varphi(x)\,Z_0^m(x)\,Z_1^n(x)\,dx \quad = \quad a(x)\,Z_0(x)+b(x)\,Z_1(x)+p\,(x)\,Z_0^2(x)+q(x)\,Z_0(x)\,Z_1(x)+r(x)\,Z_1^2(x)$$

follows

$$\int \frac{\varphi(x)\,Z_0^m(x)\,Z_1^n(x)\,dx}{\sqrt{a(x)\,Z_0(x)+b(x)\,Z_1(x)+p\,(x)\,Z_0^2(x)+q(x)\,Z_0(x)\,Z_1(x)+r(x)\,Z_1^2(x)}} =$$

$$= 2\sqrt{a(x)\,Z_0(x)+b(x)\,Z_1(x)+p\,(x)\,Z_0^2(x)+q(x)\,Z_0(x)\,Z_1(x)+r(x)\,Z_1^2(x)}\,.$$

Therefore the formulas from 1. and 2. give a lot of integrals of this kind.
Some special cases:

$$\int \frac{J_1(x)\,dx}{\sqrt{J_0(x)}} = -2\sqrt{J_0(x)}\,, \qquad \int \frac{\sqrt{x}\,J_0(x)\,dx}{\sqrt{J_1(x)}} = 2\sqrt{x\,J_1(x)}$$

$$\int \frac{I_1(x)\,dx}{\sqrt{I_0(x)}} = 2\sqrt{I_0(x)}\,, \qquad \int \frac{\sqrt{x}\,I_0(x)\,dx}{\sqrt{I_1(x)}} = 2\sqrt{x\,I_1(x)}$$

$$\int \frac{K_1(x)\,dx}{\sqrt{K_0(x)}} = -2\,\sqrt{K_0(x)}\,, \qquad \int \frac{\sqrt{x}\,K_0(x)\,dx}{\sqrt{K_1(x)}} = -2\,\sqrt{x\,K_1(x)}$$

$$\int \frac{x^2\,J_1(x)\,dx}{\sqrt{2x\,J_1(x) - x^2\,J_0(x)}} = 2\,\sqrt{2x\,J_1(x) - x^2\,J_0(x)} \quad \text{(at least for } |x| \le 5.1356)$$

$$\int \frac{x^2\,I_1(x)\,dx}{\sqrt{x^2\,I_0(x) - 2x\,I_1(x)}} = 2\,\sqrt{x^2\,J_0(x) - 2x\,J_1(x)}$$

$$\int \frac{x^2\,K_1(x)\,dx}{\sqrt{x^2\,K_0(x) + 2x\,K_1(x)}} = -2\,\sqrt{x^2\,K_0(x) + 2x\,K_1(x)}$$

$$\int \frac{x\,\ln x\,J_0(x)\,dx}{\sqrt{J_0(x) + x\,\ln x\,J_1(x)}} = 2\,\sqrt{J_0(x) + x\,\ln x\,J_1(x)} \quad \text{(at least for } 0 < x \le 3.6265)$$

$$\int \frac{x\,\ln x\,I_0(x)\,dx}{\sqrt{I_0(x) - x\,\ln x\,I_1(x)}} = -2\,\sqrt{I_0(x) - x\,\ln x\,I_1(x)} \quad \text{(at least for } 0 < x \le 2.0230)$$

$$\int \frac{x\,\ln x\,K_0(x)\,dx}{\sqrt{K_0(x) + x\,\ln x\,K_1(x)}} = -2\,\sqrt{K_0(x) + x\,\ln x\,K_1(x)}$$

$$\int \frac{\sin x\,J_0(x)\,dx}{\sqrt{x\,\sin x\,J_0(x) - x\,\cos x\,J_1(x)}} = 2\,\sqrt{x\,\sin x\,J_0(x) - x\,\cos x\,J_1(x)}$$

$$\int \frac{\cos x\,J_0(x)\,dx}{\sqrt{x\,\cos x\,J_0(x) + x\,\sin x\,J_1(x)}} = 2\,\sqrt{x\,\cos x\,J_0(x) + x\,\sin x\,J_1(x)}$$

$$\int \frac{J_1^2(x)\,dx}{x\,\sqrt{J_0^2(x) + J_1^2(x)}} = -\,\sqrt{J_0^2(x) + J_1^2(x)}\,, \qquad \int \frac{I_1^2(x)\,dx}{x\,\sqrt{I_0^2(x) - I_1^2(x)}} = \sqrt{I_0^2(x) - I_1^2(x)}$$

$$\int \frac{K_1^2(x)\,dx}{x\,\sqrt{K_0^2(x) - K_1^2(x)}} = \sqrt{K_0^2(x) - K_1^2(x)}$$

$$\int \frac{J_0^2(x)\,dx}{\sqrt{J_0^2(x) + J_1^2(x)}} = x\,\sqrt{J_0^2(x) + J_1^2(x)}\,, \qquad \int \frac{I_0^2(x)\,dx}{\sqrt{I_0^2(x) - I_1^2(x)}} = x\,\sqrt{I_0^2(x) - I_1^2(x)}$$

$$\int \frac{K_0^2(x)\,dx}{x\,\sqrt{K_0^2(x) - K_1^2(x)}} = x\,\sqrt{K_0^2(x) - K_1^2(x)}$$

$$\int \frac{J_0(x)\,J_1(x)\,dx}{x\,\sqrt{x\,J_0^2(x) - J_0(x)\,J_1(x) + x\,J_1^2(x)}} = 2\,\sqrt{x\,J_0^2(x) - J_0(x)\,J_1(x) + x\,J_1^2(x)}$$

$$\int \frac{I_0(x)\,I_1(x)\,dx}{x\,\sqrt{x\,I_0^2(x) - I_0(x)\,I_1(x) - x\,I_1^2(x)}} = 2\,\sqrt{x\,I_0^2(x) - I_0(x)\,I_1(x) - x\,I_1^2(x)}$$

$$\int \frac{K_0(x)\,K_1(x)\,dx}{x\,\sqrt{x\,K_0^2(x) + K_0(x)\,K_1(x) - x\,K_1^2(x)}} = -2\,\sqrt{x\,K_0^2(x) + K_0(x)\,K_1(x) - x\,K_1^2(x)}$$

$$\int \frac{x\,J_1^2(x)\,dx}{\sqrt{x^2\,J_0^2(x) - 2x\,J_0(x)\,J_1(x) + x\,J_1^2(x)}} = \sqrt{x^2\,J_0^2(x) - 2x\,J_0(x)\,J_1(x) + x^2\,J_1^2(x)}$$

$$\int \frac{x\, I_1^2(x)\, dx}{\sqrt{x^2\, I_1^2(x) + 2x\, I_0(x)\, I_1(x) - x\, I_0^2(x)}} = \sqrt{x^2\, I_1^2(x) + 2x\, I_0(x)\, I_1(x) - x^2\, I_0^2(x)}$$

$$\int \frac{x\, K_1^2(x)\, dx}{\sqrt{x^2\, K_0^2(x) + 2x\, K_0(x)\, K_1(x) - x^2\, K_1^2(x)}} = -\sqrt{x^2\, K_0^2(x) + 2x\, K_0(x)\, K_1(x) - x^2\, K_1^2(x)}$$

$$\int \frac{x^2\, J_1^2(x)\, dx}{\sqrt{x^2\, J_0^2(x) - 4x\, J_0(x)\, J_1(x) + (x^2+4)\, J_1^2(x)}} = \frac{x}{3}\sqrt{x^2\, J_0^2(x) - 4x\, J_0(x)\, J_1(x) + (x^2+4)\, J_1^2(x)}$$

$$\int \frac{x^2\, I_1^2(x)\, dx}{\sqrt{(x^2-4)\, I_1^2(x) + 4x\, I_0(x)\, I_1(x) - x^2\, I_0^2(x)}} = \frac{x}{3}\sqrt{(x^2-4)\, I_1^2(x) + 4x\, I_0(x)\, I_1(x) - x^2\, I_0^2(x)}$$

$$\int \frac{x^2\, K_1^2(x)\, dx}{\sqrt{x^2\, K_0^2(x) - 4x\, K_0(x)\, K_1(x) - (x^2-4)\, K_1^2(x)}} = -\frac{x}{3}\sqrt{x^2\, K_0^2(x) - 4x\, K_0(x)\, K_1(x) - (x^2-4)\, K_1^2(x)}$$

$$\int \frac{x^2\, J_0^2(x)\, dx}{\sqrt{x^2\, J_0^2(x) + 2x\, J_0(x)\, J_1(x) + (x^2-2)\, J_1^2(x)}} = \frac{x}{3}\sqrt{x^2\, J_0^2(x) + 2x\, J_0(x)\, J_1(x) + (x^2-2)\, J_1^2(x)}$$

$$\int \frac{x^2\, I_0^2(x)\, dx}{\sqrt{x^2\, I_0^2(x) + 2x\, I_0(x)\, I_1(x) - (x^2+2)\, I_1^2(x)}} = \frac{x}{3}\sqrt{x^2\, I_0^2(x) + 2x\, I_0(x)\, I_1(x) - (x^2+2)\, I_1^2(x)}$$

$$\int \frac{x^2\, K_0^2(x)\, dx}{\sqrt{(x^2+2)\, K_0^2(x) + 2x\, K_0(x)\, K_1(x) - x^2\, K_1^2(x)}} = -\frac{x}{3}\sqrt{(x^2+2)\, K_1^2(x) + 2x\, K_0(x)\, K_1(x) - x^2\, K_0^2(x)}$$

$$\int \frac{x^3\, J_0(x)\, J_1(x)\, dx}{\sqrt{(2x^2-4)\, J_1^2(x) + 4x\, J_0(x)\, J_1(x) - x^2\, J_0^2(x)}} = \frac{x}{3}\sqrt{(2x^2-4)\, J_1^2(x) + 4x\, J_0(x)\, J_1(x) - x^2\, J_0^2(x)}$$

at least for $|x| \leq 2.9878$

$$\int \frac{x^3\, I_0(x)\, I_1(x)\, dx}{\sqrt{x^2\, I_0^2(x) - 4x\, I_0(x)\, I_1(x) + (2x^2+4)\, I_1^2(x)}} = \frac{x}{3}\sqrt{x^2\, I_0^2(x) - 4x\, I_0(x)\, I_1(x) + (2x^2+4)\, I_1^2(x)}$$

$$\int \frac{x^3\, K_0(x)\, K_1(x)\, dx}{\sqrt{x^2\, K_0^2(x) + 4x\, K_0(x)\, K_1(x) + (2x^2+4)\, K_1^2(x)}} = -\frac{x}{3}\sqrt{x^2\, K_0^2(x) + 4x\, K_0(x)\, K_1(x) + (2x^2+4)\, K_1^2(x)}$$

6. Some Gaussian Quadrature Formulas for special Defined Integrals

About the origin of the formulas see [15], 7.1.

$$\int_0^a \varrho(x)\, f(x)\, dx = \sum_{k=1}^{n} A_k^{(n)}\, f(x_k^{(n)}) + R_n(f)$$

Assuming that the continous derivative $d^{2n}f/dx^{2n}$ exists one has with some $\xi_n \in [0,a]$ depending from $f(x)$ the error

$$R_n(f) = r_n \cdot f^{(2n)}(\xi_n)\ .$$

The given values of r_n are rounded to up.
Let z_0 denote the first positive zero of $J_0(x)$ and z_1 the first positive zero of $J_1(x)$.
$z_0 = 2.40482\ 55576\ 95772\ 76862\,,\ z_1 = 3.83170\ 59702\ 07512\ 31561$
Holds $\varrho(x) = J_0(x) > 0$ for $0 < x < z_0$ and $\varrho(x) = J_1(x) > 0$ for $0 < x < z_1$.
In any case one has

$$\sum_{k=1}^{n} A_k^{(n)} \ = \ A_1^{(1)} \ = \ \int_0^a \varrho(x)\, dx\ .$$

6.1. $\varrho(x) = J_0(x),\ a = z_0$:

n	k	x_k	$A_k^{(n)}$
1	1	0.84911 86375 95859 14440	1.47030 00433 84178 98213
2	1	0.39882 92282 29946 50719	0.90108 88277 42624 27265
	2	1.56194 85061 59024 25690	0.56921 12156 41554 70947
3	1	0.22634 80927 64316 44350	0.54778 48193 39959 52854
	2	1.00516 39594 78741 19902	0.70092 58605 26759 67124
	3	1.89505 35380 34339 27541	0.22158 93635 17459 78235
4	1	0.14473 82282 55696 21841	0.35943 22951 47741 12189
	2	0.68411 14453 73497 09001	0.60043 63758 36197 57058
	3	1.41258 23653 31484 08156	0.41175 50421 69162 76320
	4	2.06769 20930 02856 90479	0.09867 63302 31077 52645
5	1	0.10021 38764 64392 85496	0.25188 30914 84562 78217
	2	0.49030 87468 48216 27800	0.47696 66584 93069 36682
	3	1.06603 72772 05082 94676	0.45505 16639 82458 95258
	4	1.67579 09335 27784 59726	0.23687 69005 56173 29393
	5	2.16668 25320 32995 02545	0.04952 17288 67914 58662
6	1	0.07339 09066 03133 18519	0.18567 01891 22829 58772
	2	0.36659 94270 55653 86888	0.37603 00607 89693 99979
	3	0.82318 89118 45400 56078	0.42592 25681 22988 96567
	4	1.35083 69247 90597 31458	0.31454 44042 01812 98980
	5	1.85067 67803 21263 94621	0.14082 51792 75388 66431
	6	2.22814 70218 71287 87049	0.02730 76418 71464 77484
7	1	0.05602 45260 34280 52744	0.14228 77583 04715 82762
	2	0.28359 36021 81776 57522	0.29980 85241 57316 96860
	3	0.65055 43979 28046 67146	0.37494 72554 54351 71927
	4	1.09825 71874 06190 64557	0.33591 91612 03562 33066
	5	1.56062 92196 48622 52565	0.21370 34786 04027 46305
	6	1.97116 65464 72516 80454	0.08742 58793 99324 47881
	7	2.26873 98995 48321 17386	0.01620 79862 60880 19412
8	1	0.04415 16822 70185 67264	0.11241 45281 58959 86690
	2	0.22550 46441 44908 65760	0.24287 21983 42402 78068
	3	0.52500 76324 59107 67743	0.32298 95029 80519 02245
	4	0.90401 79209 12731 39428	0.32551 60301 08802 50497
	5	1.31711 94427 36933 38632	0.25319 54398 14890 36268
	6	1.71683 41508 13325 77892	0.14651 48161 74850 95088
	7	2.05706 85469 40771 30526	0.05660 02199 54023 77307
	8	2.29687 90390 82795 50272	0.01019 73078 49729 72049

n	r_n
1	2.360E-01
2	6.834E-03
3	8.069E-05
4	5.145E-07
5	2.050E-09
6	5.578E-12
7	1.103E-14
8	1.656E-17
9	1.950E-20
10	1.851E-23

n	k	x_k	$A_k^{(n)}$
9	1	0.03568 24385 35339 63793	0.09100 54880 68636 79312
	2	0.18339 95455 20560 72825	0.19994 03942 89885 35994
	3	0.43153 27709 86789 28064	0.27682 48640 75388 09817
	4	0.75378 95449 42819 65529	0.30111 60165 22402 67682
	5	1.11817 72467 97612 79458	0.26594 41351 42704 29766
	6	1.49018 46734 69815 92962	0.18838 00567 60179 29507
	7	1.83510 82326 65444 36893	0.10231 86185 89166 42240
	8	2.12018 77212 80538 58048	0.03804 60533 61272 04146
	9	2.31715 56799 89430 39540	0.00672 44165 74543 99748
10	1	0.02943 20801 70948 03192	0.07515 49753 94236 45992
	2	0.15196 93001 27994 54755	0.16706 34344 69764 88751
	3	0.36038 97622 40754 97575	0.23778 12425 00351 80276
	4	0.63628 72652 14219 79340	0.27240 08422 34623 59420
	5	0.95662 39779 19819 83159	0.26197 96107 07124 04222
	6	1.29588 08007 77129 56440	0.21151 59005 73181 40869
	7	1.62769 77336 04089 43949	0.14039 29837 65469 47691
	8	1.92626 72552 05830 23170	0.07297 54521 24144 36484
	9	2.16779 36949 02068 73559	0.02642 69230 91999 99596
	10	2.33223 52003 55296 43725	0.00460 86785 23282 94911

6.2. $\varrho(x) = J_1(x),\ a = z_1$:

n	k	x_k	$A_k^{(n)}$
1	1	1.87876 52796 87720 77862	1.40275 93957 02552 97210
2	1	1.06490 35971 77076 73955	0.71352 73057 09344 42983
	2	2.72131 53320 91136 68635	0.68923 20899 93208 54226
3	1	0.67234 43602 14129 71914	0.32906 60341 43142 50045
	2	1.89442 60157 28575 44303	0.76668 74947 14353 05601
	3	3.13276 49538 77173 30814	0.30700 58668 45057 41563
4	1	0.45894 55203 26475 31235	0.16288 77917 03040 35012
	2	1.36622 66845 01665 12141	0.55523 92326 02955 25023
	3	2.43260 54162 85590 13131	0.53670 44151 79575 17804
	4	3.35654 13570 69680 31486	0.14792 79562 16982 19371
5	1	0.33180 20337 36165 55678	0.08770 25071 91829 47300
	2	1.02142 19523 37440 45833	0.36410 09808 94805 82001
	3	1.90084 44312 86510 28562	0.53080 78354 68009 88152
	4	2.78606 68373 94159 50371	0.34208 41472 38969 07093
	5	3.48948 15673 50310 05946	0.07806 39249 08938 72664
6	1	0.25050 63052 28711 97552	0.05081 72006 85742 58212
	2	0.78800 95790 83595 25210	0.23625 18296 43115 36822
	3	1.50976 00320 21313 92188	0.43333 57977 70745 12177
	4	2.29683 53747 01485 20854	0.42085 76435 00112 46757
	5	3.02585 79200 18626 93204	0.21694 12820 42452 78091
	6	3.57417 42088 49931 43235	0.04455 56420 60384 65150
7	1	0.19558 40010 41249 50480	0.03128 48720 61092 70863
	2	0.62427 48418 07360 43163	0.15598 42462 27833 87320
	3	1.22041 72865 27573 05119	0.32922 48592 89470 36004
	4	1.90430 89567 09341 85832	0.40649 88779 46399 56710
	5	2.59089 22472 42858 90095	0.31201 30064 45854 17579
	6	3.19412 16451 13656 70328	0.14063 79161 58331 59436
	7	3.63118 89959 66345 37392	0.02711 56175 73570 69297

n	r_n
1	4.810E-01
2	3.317E-02
3	9.615E-04
4	1.525E-05
5	1.520E-07
6	1.040E-09
7	5.177E-12
8	1.960E-14
9	5.833E-17
10	1.400E-19

n	k	x_k	$A_k^{(n)}$
8	1	0.15682 33177 08037 94485	0.02024 22670 07056 04918
	2	0.50569 47441 49987 41001	0.10565 63995 87912 13878
	3	1.00301 79183 33490 80019	0.24426 85534 74077 52574
	4	1.59383 56846 69951 78381	0.35160 77800 21817 38025
	5	2.21766 91829 95139 45587	0.34288 84302 64882 18094
	6	2.81224 79584 90927 04488	0.22683 43760 90276 89708
	7	3.31591 27752 75352 15183	0.09387 42676 40371 70507
	8	3.67129 80952 12594 29179	0.01738 73216 16159 09505
9	1	0.12848 64993 58978 72698	0.01364 69714 56644 03898
	2	0.41739 43521 28271 20996	0.07350 42895 51720 43702
	3	0.83678 02396 08545 23386	0.18078 17286 84913 48106
	4	1.34780 40848 24007 81937	0.28859 28656 90951 59125
	5	1.90647 50319 97646 23886	0.32952 79063 95977 39406
	6	2.46660 60964 63479 03024	0.27546 79821 25950 98159
	7	2.98165 22148 56115 11947	0.16506 19568 17501 99304
	8	3.40651 53800 04878 64863	0.06453 73457 74434 07420
	9	3.70053 85543 16270 69345	0.01163 83492 04458 98090
10	1	0.10716 10473 47871 95337	0.00952 20553 84942 54239
	2	0.35003 94839 50296 02288	0.05244 74001 75843 56752
	3	0.70744 95101 06619 77663	0.13474 30147 12181 03556
	4	1.15132 70571 55910 48530	0.23129 32570 81504 53353
	5	1.64907 06313 31381 24496	0.29468 70039 45311 30494
	6	2.16577 21273 78569 27320	0.28831 22736 46923 12496
	7	2.66569 90324 63164 53996	0.21677 46332 60589 06045
	8	3.11347 56523 36544 98403	0.12130 15002 95402 68207
	9	3.47554 69151 98466 54666	0.04560 49082 00260 04102
	10	3.72249 03211 68481 97545	0.00807 33489 99595 07965

6.3. $\varrho(x) = 1/I_0(x),\ a = +\infty$:

n	k	x_k	$A_k^{(n)}$
1	1	1.47310 83641 91783 60064	2.08323 32771 13127 95218E+00
2	1	0.85994 11440 15675 54142	1.68527 96108 11361 25049E+00
	2	4.06978 80826 72085 70164	3.97953 66630 17667 01697E-01
3	1	0.59755 34675 56868 22131	1.31493 75299 05595 50844E+00
	2	2.78391 18750 36898 84897	7.34241 33470 80618 57238E-01
	3	7.01875 89825 32937 25977	3.40544 12499 47058 65003E-02
4	1	0.45331 39149 12805 13818	1.05306 26097 75956 07647E+00
	2	2.13558 95289 43486 97404	9.02512 08172 41694 57004E-01
	3	5.12185 75626 39841 77113	1.25633 78345 36953 72818E-01
	4	10.16728 24266 09517 18024	2.02480 21593 07045 89394E-03
5	1	0.36298 25783 15813 27329	8.68535 68333 69535 98993E-01
	2	1.73485 64987 24408 50688	9.61293 58956 45974 27811E-01
	3	4.08840 02959 78222 62846	2.39874 58102 66442 74731E-01
	4	7.73113 12361 47051 67671	1.34317 96791 85615 68688E-02
	5	13.44227 21087 98979 85098	9.76263 93076 49377 79330E-05
6	1	0.30151 68788 09093 54645	7.34498 70505 50255 17754E-01
	2	1.46024 60077 14868 15583	9.61035 32456 10017 93250E-01
	3	3.41763 04991 38670 51478	3.48456 38665 83890 94377E-01
	4	6.33346 62336 46446 50357	3.81567 86992 36035 92301E-02
	5	10.52504 11270 02343 63533	1.08197 49482 90461 70158E-03
	6	16.80563 48913 20209 83671	4.09889 80607 25870 74037E-06
7	1	0.25719 28130 60929 86946	6.33881 44217 97228 59427E-01
	2	1.25963 30870 75378 31156	9.31488 31470 62746 57780E-01

n	r_n
1	1.6585E+00
2	7.0079E-01
3	2.4628E-01
4	7.9264E-02
5	2.4229E-02
6	7.1584E-03
7	2.0646E-03
8	5.8481E-04
9	1.6336E-04
10	4.5125E-05
11	1.2352E-05
12	3.3556E-06
13	9.0574E-07
14	2.4315E-07
15	6.4962E-08
16	1.7284E-08
17	4.5819E-09
18	1.2107E-09
19	3.1896E-10
20	8.3812E-11

n	k	x_k	$A_k^{(n)}$
7	3	2.94166 10663 04726 06097	4.38933 48845 04731 08992E-01
	4	5.39356 68489 20765 95279	7.44472 41592 36002 46747E-02
	5	8.78880 43993 86755 55756	4.41076 58145 08736 67563E-03
	6	13.45335 60054 46439 20504	7.18685 05049 87430 42218E-05
	7	20.23479 55816 33220 25763	1.55864 73869 03289 68045E-07
8	1	0.22382 55043 26641 84575	5.56099 10653 91139 21916E-01
	2	1.10647 82752 34951 19485	8.88787 82968 39983 60656E-01
	3	2.58436 85632 76880 28626	5.08553 53144 02145 98416E-01
	4	4.70885 05701 98697 06913	1.18035 55282 55366 71375E-01
	5	7.59021 39321 60880 65158	1.13448 73435 36217 60953E-02
	6	11.40223 93494 86021 53061	4.08233 55336 79379 06574E-04
	7	16.48452 76170 86259 22249	4.14413 61949 17370 74910E-06
	8	23.71516 09184 23344 83410	5.49933 93684 47005 37355E-09
9	1	0.19786 00369 14523 66909	4.94440 51627 31082 07742E-01
	2	0.98569 93319 61811 69138	8.41423 21472 70868 10612E-01
	3	2.30538 20387 32673 00735	5.58940 80157 05057 61094E-01
	4	4.18429 63508 70013 91124	1.64603 39803 40941 82698E-01
	5	6.69919 82101 70259 59420	2.24319 35514 11512 14634E-02
	6	9.95733 80255 39896 41979	1.36119 16622 48907 36712E-03
	7	14.13944 41319 09732 96650	3.20048 45227 02075 27588E-05
	8	19.59733 34118 88961 32538	2.14303 79889 28013 64811E-07
	9	27.23669 59380 24852 24124	1.82943 04765 20376 86128E-10
10	1	0.17711 56423 97524 12307	4.44513 03136 31333 02666E-01
	2	0.88803 66578 43806 85033	7.93753 88903 44936 66633E-01
	3	2.08104 76942 91994 03659	5.93207 63696 14256 10194E-01
	4	3.76801 32238 68663 82652	2.10783 71608 19690 23756E-01
	5	6.00541 80730 87554 18790	3.75118 50528 63945 89448E-02
	6	8.86595 94139 19861 31376	3.32466 28587 70682 41334E-03
	7	12.46050 83159 44928 10647	1.36276 17323 31997 09691E-04
	8	16.97656 44317 08342 69118	2.20394 54774 80609 50108E-06
	9	22.77674 86309 55387 95072	1.01601 82754 63036 60968E-08
	10	30.79217 32438 36959 09844	5.80277 26254 72929 16728E-12
11	1	0.16018 40732 43669 50607	4.03348 44352 97055 46462E-01
	2	0.80746 92703 73028 95900	7.47929 01247 46292 47512E-01
	3	1.89649 63966 21548 18534	6.14630 77895 77580 75406E-01
	4	3.42881 44687 59253 39133	2.54320 09845 64691 23391E-01
	5	5.44741 77984 68983 37362	5.59529 71022 15700 46515E-02
	6	8.00500 68225 11687 45009	6.63265 63792 66595 03786E-03
	7	11.17583 66584 35510 17542	4.07355 27613 87280 27609E-04
	8	15.07513 18072 77806 86275	1.18238 79718 54825 67379E-05
	9	19.89629 04302 75847 93868	1.36688 49315 11474 12701E-07
	10	26.01170 94256 55737 06102	4.48614 99351 02539 13965E-10
	11	34.37619 66481 08291 10274	1.76938 77995 62063 33483E-13
12	1	0.14611 73545 20290 89504	3.68881 34335 38035 09694E-01
	2	0.73990 39940 13802 08123	7.04918 45677 12125 72799E-01
	3	1.74187 28908 06294 91991	6.26131 66137 89107 07303E-01
	4	3.14665 86677 15665 14494	2.93894 53087 41582 15708E-01
	5	4.98760 99785 49526 97749	7.69031 03552 60341 52164E-02
	6	7.30492 71373 76661 33365	1.14963 08791 99677 70919E-02
	7	10.15146 37752 01265 03270	9.64077 07084 00038 77862E-04
	8	13.60447 11545 58435 45345	4.28746 10047 35948 07643E-05
	9	17.78297 91419 14740 13150	9.12916 46956 93519 31248E-07
	10	22.88562 61019 32681 15961	7.77441 67541 65373 79328E-09
	11	29.29381 00001 95018 12274	1.86638 48860 63727 20488E-11
	12	37.98461 93168 33240 91666	5.21863 36019 75917 95168E-15

n	k	x_k	$A_k^{(n)}$
13	1	0.13425 50380 27303 80239	3.39636 28567 38182 92222E-01
	2	0.68245 75410 59493 24648	6.65067 19837 87014 32669E-01
	3	1.61036 75991 01197 74839	6.30129 05378 09777 07538E-01
	4	2.90800 73616 73815 16400	3.28879 30949 02618 98334E-01
	5	4.60145 92726 82080 21664	9.94737 85275 95154 64618E-02
	6	6.72262 07972 62321 50547	1.79883 15868 92125 98412E-02
	7	9.31087 32636 68920 29233	1.93559 90768 39172 04272E-03
	8	12.42117 12422 80738 76056	1.19682 58612 04504 72000E-04
	9	16.13346 94166 56511 54028	3.98261 98002 43364 48335E-06
	10	20.57012 86738 67955 49273	6.39499 41605 12798 60607E-08
	11	25.93454 22225 66569 97872	4.11056 04793 29286 78987E-10
	12	32.61649 94844 82393 51855	7.38146 58307 45769 33764E-13
	13	41.61417 83189 55310 02056	1.49592 58432 91979 64958E-16
14	1	0.12412 35024 01519 46313	3.14534 43473 64052 60643E-01
	2	0.63303 88747 84683 84412	6.28395 20173 10277 47209E-01
	3	1.49711 41186 70135 92765	6.28548 76328 74977 53472E-01
	4	2.70335 20682 59560 06270	3.59112 85929 28693 63603E-01
	5	4.27215 66516 85373 48654	1.22849 78211 97800 62292E-01
	6	6.22961 87808 71788 90693	2.60589 26471 15769 76338E-02
	7	8.60617 08327 35875 13140	3.44261 53472 13955 47290E-03
	8	11.44263 49232 60657 93149	2.77318 69537 61425 99840E-04
	9	14.79600 15096 71288 55480	1.30382 06272 12641 70938E-05
	10	18.74857 11575 46466 42858	3.33080 72437 60163 26403E-07
	11	23.42568 57947 36286 48412	4.12435 91606 79732 35779E-09
	12	29.03512 17071 69171 43548	2.04163 56095 95167 13485E-11
	13	35.97456 36671 91710 09365	2.79458 63893 34225 15035E-14
	14	45.26225 50884 12593 82359	4.18333 95129 83616 74022E-18
15	1	0.11537 46292 89824 71598	2.92770 51083 72634 36404E-01
	2	0.59009 41372 49014 10508	5.94760 41949 85423 03488E-01
	3	1.39853 62799 33491 40523	6.22889 82310 37096 63073E-01
	4	2.52580 20387 25507 82808	3.84726 56084 42637 95263E-01
	5	3.98774 53045 51999 92710	1.46341 17660 36315 13573E-01
	6	5.80620 59931 10407 73272	3.55662 48808 80022 05087E-02
	7	8.00542 42055 55383 50348	5.58307 04285 05918 20542E-03
	8	10.61675 03998 87471 61509	5.59437 60082 41506 59624E-04
	9	13.68259 38942 96465 16443	3.47318 37173 65909 56708E-05
	10	17.26171 04664 34657 71430	1.27183 53474 21104 23709E-06
	11	21.43847 62799 51204 43481	2.54663 96890 68140 36422E-08
	12	26.34094 95508 86292 57198	2.47707 47804 50993 90259E-10
	13	32.18099 48801 10863 99426	9.60483 52163 62055 56134E-13
	14	39.36377 76829 74052 74079	1.01844 51939 01947 12815E-15
	15	48.92671 27604 95452 67276	1.14476 93206 92297 17514E-19
16	1	0.10774 70470 55089 54043	2.73732 31937 35321 13530E-01
	2	0.55244 44841 70336 47880	5.63947 40529 60060 11346E-01
	3	1.31194 35191 97163 02109	6.14302 90677 66635 01004E-01
	4	2.37023 25973 22764 16613	4.06021 35622 16613 90480E-01
	5	3.73945 92865 92399 51558	1.69398 45633 33126 21248E-01
	6	5.43822 02958 03763 87301	4.63089 15401 62282 24508E-02
	7	7.48632 90267 85785 98210	8.42410 21482 52498 44033E-03
	8	9.90851 34476 51226 19183	1.01455 43983 50648 10313E-03
	9	12.73738 38375 34519 02402	7.92668 10958 90051 12688E-05
	10	16.01673 88143 87477 76408	3.87971 07559 37448 52069E-06
	11	19.80688 30418 99917 08446	1.12826 53908 14368 75182E-07
	12	24.19406 42765 41461 36918	1.80144 83838 24885 53738E-09
	13	29.30885 02858 51883 65788	1.39809 21391 25850 55575E-11
	14	35.36695 34779 47364 01617	4.30850 75453 78611 92682E-14

n	k	x_k	$A_k^{(n)}$
16	15	42.78066 57679 76313 44929	3.58891 14317 46527 18272E-17
	16	52.60578 23642 15958 19733	3.07310 30831 11171 10925E-21
17	1	0.10104 08427 65278 80867	2.5694680 25111 22663 963E-01
	2	0.51917 94551 03901 06808	5.35715 09287 15816 73329E-01
	3	1.23527 04734 55752 29031	6.03663 19926 45469 89284E-01
	4	2.23274 84351 08036 70251	4.23384 35147 40159 12063E-01
	5	3.52071 01874 84599 13000	1.91606 70074 44652 49698E-01
	6	5.11518 54901 32972 08123	5.80547 65979 34460 95715E-02
	7	7.03273 56785 62990 27197	1.19996 76592 56192 05135E-02
	8	9.29330 27669 83982 04817	1.69179 79213 02206 77390E-03
	9	11.92261 92779 23887 09118	1.60497 98410 95881 14665E-04
	10	14.95442 33178 97557 82410	9.98982 71654 99244 11979E-06
	11	18.43372 62060 28286 51806	3.92608 35995 54776 36064E-07
	12	22.42221 72198 73902 63807	9.21476 62747 58801 57667E-09
	13	27.00785 74958 27573 15199	1.19036 50920 43721 57724E-10
	14	32.32355 83826 53527 81407	7.47046 04794 47637 32317E-13
	15	38.58867 91419 03110 00771	1.85291 14902 29186 97014E-15
	16	46.22233 05290 95322 49684	1.22746 47264 99308 22634E-18
	17	56.29798 11374 99377 55190	8.10959 19251 22896 93879E-23
18	1	0.09510 05977 37383 93914	2.42043 04628 91498 14273E-01
	2	0.48958 48192 45577 27056	5.09821 98896 20008 42937E-01
	3	1.16690 39255 32362 03136	5.91631 94391 78447 09143E-01
	4	2.11033 28546 37268 65674	4.37234 84319 43126 02464E-01
	5	3.32644 16867 73014 94428	2.12669 66232 96168 27748E-01
	6	4.82916 27015 13234 60608	7.05631 56950 12644 41276E-02
	7	6.63261 00518 18435 97316	1.63124 02132 07746 90606E-02
	8	8.75317 46080 91423 43666	2.63699 09534 15223 94195E-03
	9	11.21159 31036 52924 75538	2.95526 70618 19665 07468E-04
	10	14.03453 35801 07361 32159	2.25412 13444 47682 33236E-05
	11	17.25671 69148 29159 47195	1.13731 73130 46325 23101E-06
	12	20.92423 12784 52780 23310	3.64404 47217 29801 81564E-08
	13	25.10002 33193 19776 77583	6.99753 90759 26522 37767E-10
	14	29.87364 16776 66012 49096	7.40530 73640 84068 55709E-12
	15	35.38020 48416 82775 54427	3.80197 91969 76759 50383E-14
	16	41.84254 76803 86953 15049	7.67425 98735 18849 25877E-17
	17	49.68632 87540 25906 22949	4.08720 69889 91214 20030E-20
	18	60.00205 25865 34322 56257	2.10735 76031 44958 60757E-24
19	1	0.08980 37262 52980 46836	2.28726 38988 43270 17122E-01
	2	0.46309 25257 06752 34221	4.86038 82920 60577 20360E-01
	3	1.10556 45736 45439 31186	5.78706 00780 89615 31609E-01
	4	2.00061 14081 57986 19158	4.47990 92511 15363 39919E-01
	5	3.15270 34524 79786 27152	2.32390 27837 51755 64731E-01
	6	4.57401 27707 91662 80347	8.36007 50149 08103 42593E-02
	7	6.27677 08241 71084 40074	2.13378 80453 36852 96966E-02
	8	8.27467 64135 28942 25733	3.88956 98717 74668 63004E-03
	9	10.58473 56030 41355 74140	5.03894 69140 34802 94285E-04
	10	13.22847 24997 11077 50020	4.57759 52159 74264 46806E-05
	11	16.23337 20711 37637 68943	2.85400 25899 93395 67050E-06
	12	19.63502 36864 97387 20883	1.18422 93629 38134 32777E-07
	13	23.48048 68720 92954 24666	3.13350 74176 43894 27768E-09
	14	27.83386 47937 40746 97548	4.98102 20023 32822 08837E-11
	15	32.78619 11686 12358 97316	4.36542 61836 94301 03780E-13
	16	38.47467 71586 79388 97007	1.85234 79774 00335 47443E-15
	17	45.12548 48934 17643 83697	3.07270 83285 12987 93677E-18
	18	53.17057 92043 10194 86299	1.32848 33482 74570 87930E-21
	19	63.71692 16151 42070 83818	5.40051 73307 24298 93073E-26

n	k	x_k	$A_k^{(n)}$
20	1	0.08505 22847 77512 81376	2.16759 96527 18443 98610E-01
	2	0.43924 52269 39711 11523	4.64154 29303 29039 14247E-01
	3	1.05022 42843 55284 07713	5.65256 76320 68209 72287E-01
	4	1.90168 83890 12679 75957	4.56050 00140 29186 08882E-01
	5	2.99636 12298 33040 87319	2.50651 22555 73577 59250E-01
	6	4.34490 55369 45574 63631	9.69516 72932 58398 78809E-02
	7	5.95807 34412 45303 81992	2.70302 29691 55609 59717E-02
	8	7.84748 62694 81668 17656	5.48054 01945 31370 35646E-03
	9	10.02730 00739 25051 08539	8.06538 08252 87162 54611E-04
	10	12.51518 95450 95246 16594	8.53232 69232 13004 22810E-05
	11	15.33337 09624 76190 01874	6.38303 42760 44748 55966E-06
	12	18.51004 90149 71846 94667	3.29789 54014 28967 64092E-07
	13	22.08159 23381 77859 43412	1.13919 56395 21810 61208E-08
	14	26.09594 82220 78327 75313	2.51706 72432 94389 60205E-10
	15	30.61829 47309 51067 21661	3.34608 58768 86504 04970E-12
	16	35.74106 56547 44686 86057	2.45184 46606 68983 97812E-14
	17	41.60346 71155 48917 89408	8.67646 74904 26718 45567E-17
	18	48.43485 82065 83453 12955	1.19318 88319 74143 10492E-19
	19	56.67329 27047 88901 11925	4.22445 44125 46876 54175E-23
	20	67.44166 00351 73204 04679	1.36659 32364 04848 41167E-27

Beside of this, one can change the integrand

$$\int_0^\infty \frac{f(x)\,dx}{I_0(x)} = \int_0^\infty e^{-x} \cdot \frac{e^x\,f(x)\,dx}{I_0(x)} = \int_0^\infty e^{-x}\,g(x)\,dx$$

and use the Laguerre quadrature formula, see [1], table 25.9, or [15], 7, §5.

6.4. $\varrho(x) = x/I_1(x),\ a = +\infty$:

n	k	x_k	$A_k^{(n)}$
1	1	1.90092 64439 51218 29515	5.57879 62425 91488 49304E+00
2	1	1.08367 71021 84824 81582	4.38035 40854 48893 50147E+00
	2	4.88800 55213 19570 74678	1.19844 21571 42594 99157E+00
3	1	0.73660 65808 05804 67494	3.31678 24277 45580 98803E+00
	2	3.35589 05608 45704 02617	2.14021 10104 85699 93997E+00
	3	8.04089 54162 99366 93147	1.21802 80436 02075 65038E-01
4	1	0.54909 32780 41874 86549	2.59515 37074 12723 20927E+00
	2	2.56313 51769 13390 74216	2.54021 93701 92317 61602E+00
	3	5.91578 46614 23914 72780	4.34948 04168 02392 60897E-01
	4	11.31727 11430 38162 35263	8.47512 33062 08406 85077E-03
5	1	0.43359 20498 07089 83947	2.10252 41354 09818 13464E+00
	2	2.06782 53942 91622 97285	2.61928 24340 74398 39526E+00
	3	4.73420 95953 16940 08263	8.01870 44452 08635 46708E-01
	4	8.67181 77243 86136 33142	5.46503 33754 93851 51809E-02
	5	14.68204 65958 47513 14824	4.68894 83146 99012 42603E-04
6	1	0.35611 94175 44391 11194	1.75326 53892 34415 30169E+00
	2	1.72748 65700 21750 22226	2.54526 86035 48041 97242E+00
	3	3.95706 62737 25370 33152	1.12439 85520 44271 29559E+00
	4	7.13138 70779 06145 23651	1.50773 59910 04288 45818E-01
	5	11.57193 72777 10390 02737	5.06787 90470 33512 88008E-03
	6	18.11285 55361 04817 96732	2.22196 17297 56464 00009E-05

n	r_n
1	6.8095E+00
2	3.8096E+00
3	1.6542E+00
4	6.3135E-01
5	2.2267E-01
6	7.4431E-02
7	2.3931E-02
8	7.4696E-03
9	2.2780E-03
10	6.8175E-04
11	2.0090E-04
12	5.8435E-05
13	1.6810E-05
14	4.7895E-06
15	1.3534E-06
16	3.7962E-07
17	1.0580E-07
18	2.9315E-08
19	8.0805E-09
20	2.2170E-09

n	k	x_k	$A_k^{(n)}$
7	1	0.30093 24278 45085 69781	1.49603 48108 06419 35584E+00
	2	1.47926 98422 77346 85389	2.40782 92656 80863 94835E+00
	3	3.40061 51880 28689 91373	1.36909 05160 48545 63599E+00
	4	6.08468 76479 93126 23068	2.85295 30414 01232 80422E-01
	5	9.69971 75173 08222 39446	2.01646 28700 88238 63651E-02
	6	14.58168 35730 06913 20767	3.80776 22820 87085 04770E-04
	7	21.59502 11725 61537 30124	9.40986 44517 75668 73460E-07
8	1	0.25982 15861 14023 98536	1.30021 35098 62688 20443E+00
	2	1.29051 90748 78159 73330	2.25057 56832 15284 20760E+00
	3	2.98037 27293 86215 57365	1.53681 13693 36939 61834E+00
	4	5.31620 45587 91872 27136	4.38486 04493 67816 04557E-01
	5	8.39660 57162 72409 08595	5.05664 64289 01325 17506E-02
	6	12.40141 85046 91767 57707	2.11864 34154 79350 76345E-03
	7	17.67778 64801 12627 86854	2.44909 57810 40497 63436E-05
	8	25.11842 24741 07837 35898	3.65774 91850 62963 41949E-08
9	1	0.22812 01602 57049 20606	1.14694 17007 99114 40241E+00
	2	1.14241 96473 86939 33810	2.09366 22634 33855 12774E+00
	3	2.65097 38759 30901 33649	1.64073 07764 39416 42654E+00
	4	4.72390 40129 33954 56073	5.92964 33556 10300 04748E-01
	5	7.42171 33002 38933 49359	9.73942 70895 97410 17863E-02
	6	10.85524 37337 00686 54184	6.91575 78871 62547 98093E-03
	7	15.21000 04273 18988 64228	1.85739 35083 48938 84304E-04
	8	20.84387 65276 89476 872477	1.39689 54042 38833 58547E-06
	9	28.67577 37786 13452 1088	1.32869 67491 15548 36842E-09
10	1	0.20299 42239 14476 91883	1.02415 15421 17538 27138E+00
	2	1.02332 30738 84366 08828	1.94577 49721 31247 82190E+00
	3	2.38553 59433 31778 63210	1.69598 72371 19121 00743E+00
	4	4.25160 94130 88215 00852	7.36985 94275 35495 85922E-01
	5	6.65873 80820 41007 43107	1.58582 77367 86244 07644E-01
	6	9.68118 80022 37699 17271	1.65229 60983 16207 59208E-02
	7	13.43335 58576 02544 49372	7.76607 98530 67080 50893E-04
	8	18.10633 28868 20582 98137	1.41333 97005 47605 36203E-05
	9	24.06799 53584 77546 08196	7.23802 52562 57270 77864E-08
	10	32.26164 33201 57378 73335	4.56805 76171 07899 78364E-11
11	1	0.18263 04864 83627 43693	9.23836 71904 96782 96111E-01
	2	0.92562 14773 15925 72372	1.81009 24747 48703 29797E+00
	3	2.16698 43743 39014 29715	1.71586 45810 04931 69373E+00
	4	3.86531 61757 69599 77555	8.64108 30210 25000 45531E-01
	5	6.04250 58882 47067 52765	2.30316 13353 11877 93966E-01
	6	8.75100 43097 40259 49705	3.22242 67121 32028 62378E-02
	7	12.06783 89264 67311 14377	2.27817 71998 91711 60567E-03
	8	16.11077 40770 76069 90348	7.46250 68185 20262 84485E-05
	9	21.07616 22790 32198 60777	9.59297 19080 83397 25954E-07
	10	27.34112 60037 36447 47148	3.46639 90429 33775 13680E-09
	11	35.87186 37499 91945 70164	1.50031 40025 62464 00223E-12

n	k	x_k	$A_k^{(n)}$
12	1	0.16581 84149 34893 01092	8.40510 38167 82832 52359E-01
	2	0.84413 47422 35141 81925	1.68720 50776 99610 77218E+00
	3	1.98389 61552 90900 73402	1.71086 19387 32777 21212E+00
	4	3.54303 28183 14316 42320	9.71766 09925 24013 06742E-01
	5	5.53293 41939 58026 41126	3.08308 99309 13817 62522E-01
	6	7.99189 16408 98558 95920	5.45830 35600 03962 23116E-02
	7	10.97496 21273 05790 74634	5.28811 93131 77252 78013E-03
	8	14.56138 77167 06038 56652	2.66221 31159 45905 75766E-04
	9	18.87218 79867 03588 66613	6.31651 53158 52671 15436E-06
	10	24.10857 34001 31836 77730	5.92414 36455 20294 88465E-08
	11	30.65628 82855 56295 92605	1.55423 00674 78141 52520E-10
	12	39.50316 10944 75031 03203	4.74076 59403 26782 32870E-14
13	1	0.15172 09656 69748 69709	7.70303 31030 72038 54347E-01
	2	0.77521 53803 05182 66542	1.57654 12394 11007 49679E+00
	3	1.82831 31718 33829 89098	1.68885 99868 38878 92549E+00
	4	3.26981 79612 92254 08461	1.05985 84497 87257 01342E+00
	5	5.10372 86667 52835 33742	3.88613 26228 31052 69661E-01
	6	7.35854 23038 91676 48744	8.34531 39157 80976 27141E-02
	7	10.07533 97499 17937 81306	1.04084 48080 48278 50891E-02
	8	13.31073 39386 49576 95839	7.30759 48611 98793 59380E-04
	9	17.14617 48568 17478 94329	2.71627 95103 27817 55480E-05
	10	21.70569 75445 12511 14210	4.81059 46541 10076 48553E-07
	11	27.19501 04402 59828 67631	3.37846 47243 29153 02875E-09
	12	34.00795 86310 00891 19152	6.58865 71402 45545 26188E-12
	13	43.15291 15906 52203 69163	1.44916 55598 03483 72780E-15
14	1	0.13974 19755 15420 63750	7.10417 73203 00250 26258E-01
	2	0.71622 27069 09992 90498	1.47706 28835 74657 53765E+00
	3	1.69450 76360 83725 69623	1.65560 61896 80837 69121E+00
	4	3.03512 21451 77954 91652	1.12967 40234 59076 78102E+00
	5	4.73679 46544 79036 77389	4.67998 93985 64714 67475E-01
	6	6.82090 90679 44619 33219	1.18138 75179 45749 23045E-01
	7	9.31911 06710 38222 55851	1.81433 00668 41328 20563E-02
	8	12.27365 16427 43577 93376	1.66429 17029 82511 29648E-03
	9	15.74272 92333 83146 84370	8.76225 57357 06569 17583E-05
	10	19.80986 11934 05386 16587	2.47358 81951 38672 03441E-06
	11	24.60186 08902 80635 40837	3.34986 54669 46832 99629E-08
	12	30.32863 29325 07734 09742	1.79976 82401 02485 67824E-10
	13	37.39168 41037 68481 47067	2.66113 70945 01156 12124E-13
	14	46.81897 67468 02428 08507	4.30413 69217 08399 81613E-17
15	1	0.12944 59909 70655 32725	6.58785 55175 23673 08526E-01
	2	0.66519 95212 79471 04904	1.38760 00701 54105 52914E+00
	3	1.57824 50580 49626 11705	1.61521 85949 73862 45224E+00
	4	2.83126 23871 80336 16915	1.18316 93106 86030 55014E+00
	5	4.41920 90467 35600 21727	5.44052 48358 72216 97667E-01
	6	6.35810 74803 73339 91951	1.57598 44175 88482 49067E-01
	7	8.67292 39016 60601 58466	2.88335 32951 94168 26260E-02
	8	11.39626 24104 49368 71879	3.29882 56018 99399 52010E-03
	9	14.57155 44751 98277 99396	2.29902 41964 43298 25548E-04
	10	18.25850 85329 66054 13589	9.32208 88494 01103 54188E-06
	11	22.54254 51754 39149 14403	2.04448 00405 39708 48002E-07
	12	27.55305 21583 72071 92548	2.15968 42465 14730 39172E-09
	13	33.50387 85120 49154 65273	9.03755 35229 23521 42124E-12
	14	40.80381 73704 02561 91452	1.03038 62292 53824 48713E-14
	15	50.49958 81164 05870 05943	1.24647 79469 41853 01156E-18

n	k	x_k	$A_k^{(n)}$
16	1	0.12050 79415 15258 89867	6.13848 04122 67591 65599E-01
	2	0.62066 56625 08150 77856	1.30700 66327 48993 85514E+00
	3	1.47632 38358 54147 93907	1.57061 13553 92316 34492E+00
	4	2.65249 82112 12349 72398	1.22252 29063 71576 03995E+00
	5	4.14146 29822 34015 93011	6.15121 11603 67569 26458E-01
	6	5.95508 48775 57607 19988	2.00631 57257 83869 04032E-01
	7	8.11341 38310 59652 69639	4.26322 65581 65935 43411E-02
	8	10.64227 96454 82015 78616	5.87676 23800 55871 57043E-03
	9	13.57518 77705 22631 88563	5.16622 58868 01224 18631E-04
	10	16.95672 18653 14170 51604	2.80568 46478 00316 35615E-05
	11	20.84798 54051 42478 38956	8.95170 01188 38881 61406E-07
	12	25.33615 31511 91699 98001	1.55396 32265 37841 92793E-08
	13	30.55301 67599 05799 89276	1.30197 04890 24018 33974E-10
	14	36.71615 76081 55651 53775	4.30911 62333 89480 53329E-13
	15	44.24132 96468 36568 427279	3.84387 90016 31205 31873E-16
	16	54.19326 47281 59815 3518	3.52996 64618 09935 22344E-20
17	1	0.11268 05070 90514 70619	5.74409 96441 98996 75074E-01
	2	0.58148 19363 93944 26154	1.23422 68271 19063 62250E+00
	3	1.38627 66071 56451 94955	1.52382 52215 31215 56905E+00
	4	2.49444 70534 65469 53345	1.24988 06158 65411 25413E+00
	5	3.89638 59657 73092 85475	6.80190 27375 18185 26985E-01
	6	5.60067 19451 74715 19680	2.46021 26903 46547 82579E-01
	7	7.62361 48702 21752 35997	5.95128 47777 72501 94606E-02
	8	9.98611 61583 57966 28229	9.62513 92649 78887 43343E-03
	9	12.71471 05607 54129 23432	1.02967 18302 75250 78174E-03
	10	15.84380 77015 18140 73615	7.12546 25546 42055 09829E-05
	11	19.41906 30334 09479 88615	3.07762 45701 70328 78023E-06
	12	23.50286 25406 35435 20291	7.86412 74316 62426 28305E-08
	13	28.18400 91347 28364 24808	1.09763 67108 48653 56666E-09
	14	33.59655 57114 81808 48625	7.39864 74422 56827 94817E-12
	15	39.96163 49095 14659 56351	1.96253 84550 89017 07373E-14
	16	47.70167 54936 75406 78520	1.38730 03373 26444 66270E-17
	18	57.89875 25927 00286 66261	9.79916 76691 81293 94448E-22
18	1	0.10577 23536 59493 09390	5.39540 63267 02459 22130E-01
	2	0.54675 79497 40323 18705	1.16831 81466 36623 83675E+00
	3	1.30617 02677 90030 10718	1.47627 38431 30882 00897E+00
	4	2.35370 08028 79135 76391	1.26722 33034 69746 39486E+00
	5	3.67845 81536 79567 36184	7.38744 90313 42053 99802E-01
	6	5.28638 06253 87291 89679	2.92630 25470 34988 83934E-01
	7	7.19085 18137 74202 01819	7.92969 32327 12803 82910E-02
	8	9.40908 02647 12176 57935	1.47346 75191 04239 33202E-02
	9	11.96253 27053 50536 23815	1.86589 59087 75452 49873E-03
	10	14.87846 77956 83950 27243	1.58535 43630 82247 10802E-04
	11	18.19214 63590 68617 62553	8.80590 35267 79218 52365E-06
	12	21.95020 90775 77593 30426	3.07617 08266 33089 36088E-07
	13	26.21622 66811 17311 24214	6.38935 42160 12024 69779E-09
	14	31.08052 46354 78533 14894	7.26684 73421 54168 19213E-11
	15	36.67929 65171 17607 29626	3.98947 32416 94697 91346E-13
	16	43.23706 94882 65510 09815	8.58002 76715 64891 35614E-16
	17	51.18269 30224 00870 00314	4.86073 30557 42781 86337E-19
	18	61.61497 95088 95795 88267	2.67189 59670 22201 32818E-23

n	k	x_k	$A_k^{(n)}$
19	1	0.09963 32533 09493 45685	5.08505 31923 32606 74502E-01
	2	0.51578 81892 66425 87120	1.10845 31321 96293 19093E+00
	3	1.23446 83634 59535 15413	1.42892 33549 17506 45767E+00
	4	2.22756 64979 69750 44139	1.27630 95162 38110 06240E+00
	5	3.48335 48491 41814 10363	7.90640 21268 46017 28694E-01
	6	5.00563 00404 34267 10177	3.39456 26866 41214 78857E-01
	7	6.80543 07006 75789 30304	1.01691 60254 35115 79217E-01
	8	8.89712 75723 74730 73187	2.13459 50545 78248 47117E-02
	9	11.29839 70298 93961 75370	3.13066 49873 33977 84773E-03
	10	14.03129 37604 27562 37085	3.17382 58468 06526 28588E-04
	11	17.12376 87958 80788 55100	2.18205 89559 21471 46506E-05
	12	20.61186 74257 00503 97141	9.88589 67827 28419 52912E-07
	13	24.54313 76273 11855 70592	2.83281 18845 89148 67659E-08
	14	28.98225 22704 94350 28390	4.84368 13905 42571 50180E-10
	15	34.02097 05217 71464 05579	4.54117 18561 56457 77477E-12
	16	39.79752 35004 47495 87281	2.05250 41221 31651 98671E-14
	17	46.53969 55198 05683 08519	3.61578 82663 26144 90206E-17
	18	54.68252 88116 71966 33213	1.65817 53857 39938 48249E-20
	19	65.34102 06297 85367 89560	7.16817 65977 17821 58388E-25
20	1	0.09414 36852 23372 53328	4.80716 77573 20573 33278E-01
	2	0.48800 67553 91856 09831	1.05391 17367 91682 62825E+00
	3	1.16993 41367 03273 46122	1.38242 17877 52637 32306E+00
	4	2.11388 63334 40152 74886	1.27866 13175 80795 65095E+00
	5	3.30763 56328 78697 87240	8.35992 39752 30861 62944E-01
	6	4.75323 01269 55180 96814	3.85657 02806 02831 69791E-01
	7	6.45979 09780 34895 70767	1.26327 59006 61872 06006E-01
	8	8.43945 95619 37962 63992	2.95429 16782 08162 35627E-02
	9	10.70700 97915 73924 31270	4.93059 52245 73287 87978E-03
	10	13.28060 08225 13213 67186	5.83091 65704 17439 70424E-04
	11	16.18286 04631 82173 70235	4.81791 65454 37409 60346E-05
	12	19.44238 11536 23540 10367	2.72184 87650 28383 79702E-06
	13	23.09592 95955 11327 78215	1.01946 70294 54528 80014E-07
	14	27.19189 42547 91221 98975	2.42535 50147 35696 36657E-09
	15	31.79598 05566 63296 64187	3.45145 29769 17765 30117E-11
	16	37.00130 58866 53656 67670	2.69457 21676 91554 67848E-13
	17	42.94804 95698 63208 84145	1.01222 19420 23565 59028E-15
	18	49.86713 13106 61625 83708	1.47397 91282 77088 21820E-18
	19	58.19957 98835 44491 29848	5.52125 59306 49427 86483E-22
	20	69.07607 09213 85364 92701	1.89494 96597 75905 75955E-26

6.5. $\varrho(x) = K_0(x),\ a = +\infty$:

$$A_1^{(1)} = \int_0^\infty K_0(x)\,dx = \int_0^\infty \varrho(x)\,dx = \frac{\pi}{2}$$

Generally (see [4], 2.16.2.2):

$$\int_0^\infty x^{2n+1} K_0(x)\,dx = \left(\frac{(2n)!}{n!}\right)^2 , \quad \int_0^\infty x^{2n} K_0(x)\,dx = \frac{\pi}{2^{2n+1}} \left(\frac{(2n)!}{n!}\right)^2$$

n	k	x_k	$A_k^{(n)}$
1	1	0.63661 97723 67581 34308	1.57079 63267 94896 61923E+00
2	1	0.36721 86882 16808 67238	1.39995 12372 93972 84811E+00
	2	2.84416 56970 81442 72853	1.70845 08950 09237 71119E-01
3	1	0.26096 12883 46442 72853	1.22944 21664 78323 37814E+00
	2	1.88024 25951 92307 23720	3.31389 46564 20531 53151E-01
	3	5.62692 59843 49789 79049	9.96469 46745 20087 94482E-03

n	k	x_k	$A_k^{(n)}$
4	1	0.20346 78616 08216 51366	1.09483 09833 31706 91533E+00
	2	1.42025 85051 16463 38592	4.37739 19226 58012 04252E-01
	3	4.01398 76795 63093 10778	3.77769 61179 76693 12798E-02
	4	8.67787 18602 95781 85143	4.49190 01762 15683 70873E-04
5	1	0.16724 81117 75635 19008	9.88881 05969 949954 3270E-01
	2	1.14562 94187 88403 73001	5.03750 00939 28931 22366E-01
	3	3.16287 39844 90265 02804	7.51092 39652 71542 59028E-02
	4	6.49392 78146 52040 42058	3.03857 89524 58977 43499E-03
	5	11.88748 74318 93429 03860	1.74390 97329 55025 71627E-05
6	1	0.14226 07913 50015 95177	9.03772 753858 39543 9448E-01
	2	0.96187 99865 97519 53802	5.43482 69975 45733 51846E-01
	3	2.62201 32377 19443 00296	1.14485 47202 58102 84508E-01
	4	5.26789 04874 99131 39894	8.85829 44081 57636 25731E-03
	5	9.19710 80556 60770 25097	1.96493 93094 09694 52219E-04
	6	15.20345 83591 31339 31927	6.12817 01893 77195 58116E-07
7	1	0.12394 43636 01194 07490	8.33919 89568 75937 18633E-01
	2	0.82988 83221 59694 48432	5.66346 20956 66190 73155E-01
	3	2.24414 54746 20568 87466	1.51866 97126 90954 23456E-01
	4	4.45570 42365 48512 14226	1.78361 62975 61136 50738E-02
	5	7.62115 63463 58316 49173	8.16158 13396 80016 96570E-04
	6	12.05770 47887 46960 73658	1.09091 25532 15799 63666E-05
	7	18.59639 89087 07247 56285	2.00364 76879 22110 73556E-08
8	1	0.10992 12217 79063 47820	7.75487 94104 82396 66325E-01
	2	0.73030 99184 96626 27276	5.78280 43327 62226 41733E-01
	3	1.96392 13601 62659 60774	1.85527 69447 78725 48967E-01
	4	3.87067 52344 22834 48264	2.92897 69299 20942 43596E-02
	5	6.54552 79256 07758 08568	2.14705 89526 54715 50129E-03
	6	10.15637 02498 02973 16178	6.28886 01672 33874 87484E-05
	7	15.03608 50747 98943 61488	5.40518 99097 91903 55114E-07
	8	22.04801 25531 10893 98810	6.20034 30440 66445 76386E-10
9	1	0.09882 78556 23114 63690	7.25811 60449 17844 26489E-01
	2	0.65242 44716 71066 36745	5.83041 21905 83546 68687E-01
	3	1.74726 54508 81345 84395	2.14974 71416 28028 82723E-01
	4	3.42646 28324 00874 32952	4.23961 25905 71691 85370E-02
	5	5.75292 01007 91584 35353	4.35508 12282 78554 50495E-03
	6	8.82893 56824 62934 01474	2.13330 88358 08940 90902E-04
	7	12.83142 52914 87200 07925	4.22653 88832 24398 80839E-06
	8	18.10638 59719 76917 82996	2.45071 26720 89884 24315E-08
	9	25.54606 35566 72559 75616	1.83683 28901 32509 76716E-11
10	1	0.08982 51394 65073 51273	6.82994 89619 94860 36260E-01
	2	0.58979 23377 43165 66686	5.83042 63419 81469 41508E-01
	3	1.57448 14702 83904 92456	2.40305 65626 39476 58196E-01
	4	3.07648 51680 49643 48192	5.64166 06837 67699 52586E-02
	5	5.14011 28674 71344 69787	7.48668 81075 14947 72824E-03
	6	7.83334 36939 12404 28440	5.31342 50451 01792 72256E-04
	7	11.26446 66379 46739 00956	1.82468 37982 20892 53018E-05
	8	15.61778 56548 08941 93161	2.54810 60245 23016 36845E-07
	9	21.25071 58718 07071 55246	1.03450 40580 33080 17366E-09
	10	29.08192 75401 85443 74479	5.25141 74849 18088 84324E-13
11	1	0.08236 77885 11932 84267	6.45656 12245 15324 32620E-01
	2	0.53830 42174 60810 24499	5.79866 59479 16994 90525E-01
	3	1.43332 57406 93156 32340	2.61869 96123 31840 38039E-01
	4	2.79303 70825 51615 23874	7.07683 11295 16245 38708E-02
	5	4.65010 81703 82186 44747	1.14955 85174 84005 90127E-02
	6	7.05248 30651 19581 69108	1.08295 47372 78113 37447E-03
	7	10.07243 67672 01059 87103	5.54001 06526 25586 03069E-05
	8	13.82312 70627 43583 20473	1.38290 62069 38244 65083E-06
	9	18.49514 52561 69282 34372	1.40572 85846 63155 07890E-08
	10	24.45613 73102 10957 46901	4.11664 16272 01518 94937E-11
	11	32.64927 52185 13629 24635	1.45747 80918 09705 44138E-14

n	r_n
1	4.6709E-01
2	1.2218E-01
3	3.1239E-02
4	7.9193E-03
5	1.9991E-03
6	5.0338E-04
7	1.2656E-04
8	3.1783E-05
9	7.9758E-06
10	2.0003E-06
11	5.0146E-07
12	1.2567E-07
13	3.1483E-08
14	7.8855E-09
15	1.9747E-09
16	4.9444E-10
17	1.2378E-10
18	3.0985E-11
19	7.7552E-12
20	1.9409E-12

n	k	x_k	$A_k^{(n)}$
12	1	0.07608 57882 50235 63758	6.12765 46804 50956 12505E-01
	2	0.49521 11095 73286 20413	5.74569 06792 87910 51939E-01
	3	1.31576 02596 15865 36831	2.80099 99156 55699 22646E-01
	4	2.55847 37878 47117 54569	8.50248 36738 95814 23795E-02
	5	4.24831 19117 23435 69224	1.62781 15740 64017 67714E-02
	6	6.42060 11559 58385 67849	1.92028 95070 27545 46607E-03
	7	9.12651 33155 54329 64769	1.33382 38197 61008 91425E-04
	8	12.44182 68474 11704 59340	5.07933 68891 68808 01790E-06
	9	16.48378 26734 16144 05241	9.48282 82781 20804 58381E-08
	10	21.44849 55480 31779 15317	7.20106 97808 86132 95653E-10
	11	27.71296 43975 44468 97338	1.55874 40769 12198 79104E-12
	12	36.24331 06958 33002 44128	3.94450 18083 93934 48913E-16
13	1	0.07071 90377 02890 09607	5.83539 67509 87336 90331E-01
	2	0.45860 28154 46093 47768	5.67866 58120 70556 01925E-01
	3	1.21627 76917 77038 65095	2.95430 91088 04601 42422E-01
	4	2.36096 77216 74187 01875	9.88917 14822 79633 78337E-02
	5	3.91229 73216 52154 83253	2.17030 79294 34218 88261E-02
	6	5.89718 96444 74715 05026	3.07679 16187 92118 60211E-03
	7	8.35347 14273 44963 05193	2.72769 30595 49304 57578E-04
	8	11.33497 56978 23805 88480	1.43803 46693 84918 11724E-05
	9	14.92039 85019 19729 11584	4.18209 18105 32645 01122E-07
	10	19.23051 24236 51812 29334	5.97619 29749 14475 02083E-09
	11	24.46640 19811 69534 70532	3.46371 56342 79385 79301E-11
	12	31.01373 96474 88543 70097	5.65646 84480 67588 05362E-14
	13	39.86030 43545 88647 31036	1.04462 88975 82268 15012E-17
14	1	0.06607 92542 87197 05588	5.57372 16924 85212 42542E-01
	2	0.42710 98238 23474 02059	5.60252 11422 07917 17142E-01
	3	1.13097 22321 82868 98669	3.08265 61805 57273 89040E-01
	4	2.19226 73382 20556 28418	1.12176 92995 23107 03927E-01
	5	3.62679 43065 21450 26432	2.76325 24758 20712 50276E-02
	6	5.45563 27696 16088 09515	4.56709 46057 15565 19263E-03
	7	7.70768 37932 07981 76937	4.94626 45737 19349 95842E-04
	8	10.42286 91489 38411 11683	3.38322 27050 15478 35954E-05
	9	13.65723 57500 28335 63387	1.38549 60545 13256 27717E-06
	10	17.49197 68176 73898 52420	3.14212 15796 72591 01046E-08
	11	22.05099 76317 16098 97004	3.50350 47684 73894 10118E-10
	12	27.53993 42961 55962 96847	1.57802 10894 89395 47540E-12
	13	34.35258 73659 97028 34683	1.97838 98315 53400 48757E-15
	14	43.49729 27023 45869 92421	2.71466 96074 78691 64974E-19
15	1	0.06202 67216 56829 93108	5.33785 61428 41967 65974E-01
	2	0.39972 41701 83036 76645	5.52068 65251 72263 02545E-01
	3	1.05699 38427 49686 81902	3.18962 03512 84810 42727E-01
	4	2.04642 73190 92256 48638	1.24764 35585 43809 99258E-01
	5	3.38100 07046 29827 80444	3.39344 92783 74848 32671E-02
	6	5.07758 78900 45886 99813	6.38953 27442 89064 45733E-03
	7	7.15885 71243 61857 47109	8.18421 70743 05650 14719E-04
	8	9.65540 04705 05974 30396	6.93599 04527 53483 27028E-05
	9	12.60888 84012 86229 6717	3.73845 99172 19009 07461E-06
	10	16.07725 14867 41133 4419	1.21209 96048 29684 18562E-07
	11	20.14387 34572 83771 8962	2.18138 00928 92865 22647E-09
	12	24.93549 37021 98155 1603	1.92894 31807 61151 09915E-11
	13	30.66197 25884 29149 5671	6.85674 74694 94063 76532E-14
	14	37.72478 78416 60054 6250	6.69959 94575 86869 67315E-17
	15	47.15187 79364 92118 7526	6.93798 54584 51528 91456E-21

n	k	x_k	$A_k^{(n)}$
16	1	0.05845 55372 21741 90827	5.12398 97828 24980 30109E-01
	2	0.37568 72334 39906 17826	5.43556 84006 22710 13939E-01
	3	0.99221 24183 79768 10384	3.27831 12737 88954 85770E-01
	4	1.91904 84039 05059 30728	1.36592 31077 52451 14384E-01
	5	3.16703 64343 22832 98882	4.04898 35728 80860 18119E-02
	6	4.74993 69417 07809 51661	8.52972 72471 65033 94304E-03
	7	6.68591 40666 06970 28299	1.26055 16557 91431 57365E-03
	8	8.99902 60905 19025 10603	1.27922 41797 43917 11098E-04
	9	11.72128 57786 32225 71077	8.64950 87035 86202 74297E-06
	10	14.89584 81102 63064 06202	3.73844 17015 79761 26789E-07
	11	18.58228 30606 79187 73852	9.75102 73875 07110 13097E-09
	12	22.86593 52398 18237 01423	1.41338 41210 46256 43540E-10
	13	27.87614 74736 54444 62334	1.00511 24047 25424 91424E-12
	14	33.82674 03290 29660 10594	2.85758 94619 61185 55447E-15
	15	41.12648 71481 47071 90168	2.20480 08096 82244 34789E-18
	16	50.82208 97106 51380 47657	1.74708 68736 05635 14885E-22
17	1	0.05528 39384 49480 43527	4.92904 20239 68144 51947E-01
	2	0.35441 69531 51994 00550	5.34886 44937 14509 85124E-01
	3	0.93500 29925 12892 27695	3.35139 75898 05364 52714E-01
	4	1.80680 15382 88129 63998	1.47637 17598 72659 95231E-01
	5	2.97900 54135 99924 55812	4.71951 19510 23620 13130E-02
	6	4.46301 53159 54606 25724	1.09642 65870 73264 52139E-02
	7	6.27364 82825 70851 45937	1.83352 35007 93961 40736E-03
	8	8.43022 70295 47868 57390	2.17055 73250 97601 00932E-04
	9	10.95798 80822 02237 30529	1.77664 66189 03449 20752E-05
	10	13.89015 09529 02044 46076	9.73982 86192 73511 58079E-07
	11	17.27115 63082 99002 69734	3.42581 14862 95769 36234E-08
	12	21.16202 94973 54077 50970	7.28723 68994 78349 51778E-10
	13	25.64989 93602 26927 64232	8.61665 71186 32423 24470E-12
	14	30.86652 88506 31900 23348	4.98794 87726 61842 82428E-14
	15	37.02948 02520 25281 72003	1.14754 27132 10513 46062E-16
	16	44.55449 35356 52848 71046	7.07310 43279 39294 67009E-20
	17	54.50628 71469 34181 86587	4.34140 20303 26578 61963E-24
18	1	0.05244 77763 36315 37947	4.75049 36170 94391 61993E-01
	2	0.33545 91787 22788 81970	5.26177 53367 05108 95638E-01
	3	0.88410 37948 63112 25659	3.41115 43099 00213 32056E-01
	4	1.70711 87707 79246 39255	1.57901 27582 60306 92807E-01
	5	2.81240 09277 34346 96024	5.39631 33558 46400 76760E-02
	6	4.20952 63680 82544 87336	1.36639 71115 66936 62612E-02
	7	5.91076 62789 01552 80013	2.54568 99908 64119 80587E-03
	8	7.93191 38639 06449 05652	3.44396 96856 34760 90837E-04
	9	10.29328 43810 66327 07869	3.32046 54241 33551 19655E-05
	10	13.02111 78590 21461 68536	2.22509 30770 15061 06076E-06
	11	16.14967 63560 28832 02517	1.00260 22221 41657 63342E-07
	12	19.72453 49873 80131 41917	2.90627 77029 87864 79828E-09
	13	23.80803 27082 31004 10326	5.10153 26100 41244 34015E-11
	14	28.48894 32279 16209 51783	4.97598 82314 24418 65710E-13
	15	33.90130 03289 27646 71933	2.36979 28322 63522 81209E-15
	16	40.26622 33830 91028 17282	4.45762 36204 90814 48037E-18
	17	48.00613 09669 30128 60047	2.21764 59845 04620 31068E-21
	18	58.20308 77149 13742 62961	1.06596 93966 42652 02303E-25

n	k	x_k	$A_k^{(n)}$
19	1	0.04989 60014 98579 50851	4.58626 30375 50839 49035E-01
	2	0.31845 42914 36804 78656	5.17514 87310 07827 34763E-01
	3	0.83851 97847 67946 61174	3.45951 46017 96783 14191E-01
	4	1.61798 60924 58835 32227	1.67404 09358 26898 80834E-01
	5	2.66371 48644 77377 72736	6.07220 58053 34455 38196E-02
	6	3.98384 68712 38533 49398	1.65965 62041 39052 58023E-02
	7	5.58867 83699 80381 17492	3.40138 60171 79818 31050E-03
	8	7.49131 01634 79651 55560	5.17257 31800 34503 88836E-04
	9	9.70838 18295 76167 63129	5.74912 67890 69611 04744E-05
	10	12.26105 76044 62880 39432	4.57738 63782 87995 88400E-06
	11	15.17644 40030 92128 32890	2.54328 65804 64940 17864E-07
	12	18.48971 92629 74483 79034	9.53023 97439 52719 10220E-09
	13	22.24747 58766 46394 39881	2.30180 32231 88183 26307E-10
	14	26.51325 47719 88677 73519	3.36881 41855 94665 80540E-12
	15	31.37736 20555 77664 55285	2.73725 96800 61577 73980E-14
	16	36.97597 81257 41968 67297	1.08264 44220 28783 83146E-16
	17	43.53362 06443 13237 46821	1.68042 80088 56465 04065E-19
	18	51.47913 14546 35200 07968	6.81030 60663 84850 21280E-23
	19	61.91131 45372 42250 19831	2.58904 73853 78064 52002E-27
20	1	0.04758 74732 90308 41469	4.43461 43282 32543 14523E-01
	2	0.30311 37763 21477 43670	5.08957 98448 17851 86090E-01
	3	0.79745 54744 95703 80756	3.49811 93139 02086 11244E-01
	4	1.53780 08284 60233 09296	1.76175 99976 93896 03887E-01
	5	2.53017 29993 00353 07096	6.74139 64002 52878 27112E-02
	6	3.78156 65505 02668 26180	1.97286 89602 80813 52524E-02
	7	5.30071 95587 05193 67773	4.40131 96964 46547 68197E-03
	8	7.09863 83106 18815 53745	7.42281 00233 39835 23845E-04
	9	9.18915 24364 80462 27781	9.34744 37364 03996 74263E-05
	10	11.58962 94603 79221 02031	8.64665 17797 29898 17523E-06
	11	14.32196 73814 70182 54303	5.75281 65369 07948 60809E-07
	12	17.41403 23735 34440 80445	2.67944 20557 37631 54519E-08
	13	20.90182 03492 32541 92613	8.43570 66518 07113 22174E-10
	14	24.83284 88398 12755 09887	1.71402 54797 10832 98297E-11
	15	29.27177 10224 04163 83103	2.11070 38243 72213 03558E-13
	16	34.31033 27763 68327 90130	1.44114 07746 29346 55507E-15
	17	40.08675 58623 39557 21251	4.77382 61494 70700 94276E-18
	18	46.82881 76112 93230 86917	6.16490 01076 61841 52309E-21
	19	54.97155 44057 84266 09992	2.05233 04396 84907 43568E-24
	20	65.62995 66351 52843 35501	6.22629 43329 15960 54863E-29

6.6. $\varrho(x) = x\,K_1(x),\ a = +\infty$:

$$A_1^{(1)} = \int_0^\infty \varrho(x)\,dx = \int_0^\infty x\,K_1(x) = \frac{\pi}{2}$$

Generally (see [4], 2.16.2.2):

$$\int_0^\infty x^{2n}\,K_1(x)\,dx = 2^{2n-1}\cdot(n-1)!\cdot n!\ ,\qquad \int_0^\infty x^{2n+1}\,K_1(x)\,dx = \frac{(2n+1)\pi}{2^{2n+1}}\ \left[\frac{(2n)!}{n!}\right]^2$$

n	k	x_k	$A_k^{(n)}$
1	1	1.27323 95447 35162 68615	1.57079 63267 94896 61923E+00
2	1	0.74305 22262 16061 16025	1.30479 67070 28191 82079E+00
	2	3.87394 50191 21039 03172	2.65999 61976 67047 98445E-01
3	1	0.52168 27597 89904 29612	1.05211 62538 90912 89179E+00
	2	2.62336 93985 57700 47071	4.97265 30640 38416 65952E-01
	3	6.84660 75156 66516 27043	2.14147 66500 14206 14914E-02

n	k	x_k	$A_k^{(n)}$
4	1	0.40018 47015 51560 42913	8.66703 30990 81875 34070E-01
	2	2.00016 10660 16531 33603	6.23771 27033 86611 14827E-01
	3	4.97139 27241 52934 35253	7.90909 45769 43586 07813E-02
	4	10.01253 28397 40784 03250	1.23080 07786 12109 55317E-03
5	1	0.32363 88725 84816 14340	7.31210 05749 23358 11893E-01
	2	1.61901 47633 08521 23874	6.79454 75968 29900 38549E-01
	3	3.95372 67918 64134 43445	1.51924 69608 73695 57916E-01
	4	7.59182 52397 09581 26657	8.14878 44437 36022 05872E-03
	5	13.30067 01389 66033 34763	5.80290 88465 18881 44975E-05
6	1	0.27112 84786 90390 66807	6.29634 13630 07178 28785E-01
	2	1.36020 30891 71974 96350	6.94612 51236 84209 14109E-01
	3	3.29530 90638 73467 56630	2.22739 30970 16442 53385E-01
	4	6.20600 31288 85294 54650	2.31665 59810 76526 18513E-02
	5	10.39518 27264 58860 68312	6.41411 42253 10588 82024E-04
	6	16.67436 73526 09663 16988	2.39719 08173 02218 68215E-06
7	1	0.23294 56143 39065 25361	5.51338 66161 82775 50869E-01
	2	1.17252 37346 82439 74500	6.87730 53496 64918 07521E-01
	3	2.82950 16234 79021 84032	2.83746 93334 82650 69224E-01
	4	5.27559 56452 40216 48665	4.53260 74341 84259 94833E-02
	5	8.66831 18077 02509 82237	2.61211 60540 39081 20071E-03
	6	13.33141 41396 58552 85176	4.19164 37517 51036 23839E-05
	7	20.11191 17058 09985 17087	9.00284 63000 57183 63611E-08
8	1	0.20397 48894 58634 01613	4.89459 96806 89061 69109E-01
	2	1.03005 58781 94077 80391	6.69317 14253 39300 00694E-01
	3	2.48087 41941 98543 11664	3.32884 93611 79231 90843E-01
	4	4.59873 20779 36216 28452	7.21733 91594 07290 40879E-02
	5	7.47743 80842 20706 12413	6.72079 53107 80816 21148E-03
	6	11.28795 29370 03213 67733	2.37702 57941 82696 12571E-04
	7	16.36928 41908 74956 06831	2.38744 48113 29320 08705E-06
	8	23.59924 87349 22313 69065	3.14505 39393 53541 85625E-09
9	1	0.18126 84389 15648 42222	4.39490 90261 50307 24272E-01
	2	0.91817 70677 48712 40208	6.45265 31303 34105 30372E-01
	3	2.20945 98929 04439 22473	3.70725 38928 93324 15815E-01
	4	4.08083 89388 20775 63153	1.01197 03565 03194 52191E-01
	5	6.59293 20841 30858 44078	1.33071 38959 52830 57294E-02
	6	9.84949 79013 77007 68308	7.92021 27073 25527 08429E-04
	7	14.03061 35222 63121 60650	1.84036 09183 83180 71518E-05
	8	19.48783 07396 73432 91775	1.22263 58532 54546 15555E-07
	9	27.12669 68326 95996 38478	1.03773 48088 12983 79039E-10
10	1	0.16300 98883 23973 70132	3.98389 89032 21983 53801E-01
	2	0.82798 20439 45748 17241	6.18875 52460 31986 95808E-01
	3	1.99185 15533 24522 10870	3.98802 59489 90536 72842E-01
	4	3.67032 42163 31792 43804	1.30412 27231 14442 48428E-01
	5	5.90474 52216 14969 65279	2.23019 08350 44417 94572E-02
	6	8.76365 04023 64749 01570	1.93460 48616 16689 16750E-03
	7	12.35717 19874 28777 90262	7.82706 92759 76688 87697E-05
	8	16.87253 70611 85799 01459	1.25500 03930 68355 16144E-06
	9	22.67222 71483 26936 63048	5.75051 88795 52425 00154E-09
	10	30.68727 03407 28512 62001	3.26906 49310 48024 92910E-12
11	1	0.14802 01716 80606 32420	3.64046 35894 40608 82703E-01
	2	0.75372 63409 40471 54604	5.91989 17155 46087 21398E-01
	3	1.81333 88473 41009 40787	4.18837 95521 26883 24411E-01
	4	3.33620 87514 37616 90319	1.58458 87672 58725 51691E-01
	5	5.35162 30699 91243 30186	3.33611 74774 07040 38105E-02
	6	7.90751 48604 25548 08528	3.86215 82662 34901 76295E-03

n	r_n
1	1.0830E+00
2	4.3245E-01
3	1.4736E-01
4	4.6482E-02
5	1.4003E-02
6	4.0914E-03
7	1.1696E-03
8	3.2886E-04
9	9.1294E-05
10	2.5086E-05
11	6.8351E-06
12	1.8494E-06
13	4.9739E-07
14	1.3310E-07
15	3.5455E-08
16	9.4083E-09
17	2.4881E-09
18	6.5594E-10
19	1.7246E-10
20	4.5231E-11

n	k	x_k	$A_k^{(n)}$
11	7	11.07728 07190 57598 27482	2.33830 13116 58405 63057E-04
	8	14.97586 22093 04222 32590	6.72371 28369 20002 36654E-06
	9	19.79651 58038 45290 89097	7.72210 40272 61430 00757E-08
	10	25.91155 83902 80647 88316	2.52218 70378 08700 08567E-10
	11	34.27574 44303 51516 46881	9.90964 94176 11235 52752E-14
12	1	0.13550 12344 41430 00080	3.34957 31889 66947 70301E-01
	2	0.69153 38292 34865 86629	5.65618 04390 03721 32997E-01
	3	1.66417 52135 56618 84628	4.32423 09072 22671 67082E-01
	4	3.05859 88926 74440 06148	1.84508 24143 63883 59350E-01
	5	4.89612 33588 65534 60084	4.60091 56172 58716 49233E-02
	6	7.21168 17931 81328 50864	6.70224 85062 15391 46485E-03
	7	10.05711 89973 50724 78003	5.53348 96996 37408 73390E-04
	8	13.50938 94431 54412 15057	2.43588 11738 90700 39962E-05
	9	17.68737 80151 92493 26736	5.15005 01839 56806 02636E-07
	10	22.78964 23506 60967 81067	4.36321 43003 09761 09629E-09
	11	29.19753 12820 88090 80022	1.04333 81373 04782 72641E-11
	12	37.88809 74213 00986 84791	2.90787 25479 42870 65400E-15
13	1	0.12489 40380 26847 65722	3.10027 15650 50424 08689E-01
	2	0.63869 23840 04785 71735	5.40297 34143 64286 59800E-01
	3	1.53763 03096 05961 24199	4.40916 47367 90044 56037E-01
	4	2.82406 32183 02466 40932	2.08126 38807 69944 13714E-01
	5	4.51381 87567 64957 49853	5.97433 68195 47438 89832E-02
	6	6.63315 76074 30019 71031	1.05040 34753 78076 14595E-02
	7	9.22027 66066 77671 10778	1.11132 20796 66346 81891E-03
	8	12.32981 47186 19108 58290	6.79616 71489 35846 94291E-05
	9	16.04157 81168 77968 30500	2.24432 82705 88558 44820E-06
	10	20.47784 53674 25377 25884	3.58389 29808 58105 30519E-08
	11	25.84196 05719 68285 76716	2.29404 74871 36690 38192E-10
	12	32.52368 13014 34338 92813	4.10596 42529 77972 12402E-13
	13	41.52116 01334 52084 37399	8.29813 56177 04438 26063E-17
14	1	0.11579 55603 31983 04984	2.88440 39235 01225 10767E-01
	2	0.59324 66282 97401 46671	5.16286 64756 69644 78472E-01
	3	1.42889 89231 93548 70167	4.45432 40195 49992 82462E-01
	4	2.62317 18452 37517 29188	2.29149 61222 63532 38616E-01
	5	4.18798 26476 32830 87944	7.40975 50162 62148 91993E-02
	6	6.14355 35581 33541 93011	1.52468 69128 90872 81005E-02
	7	8.51893 93960 80480 72054	1.97787 58444 72546 75239E-03
	8	11.35462 13992 56973 67525	1.57446 84357 94236 43889E-04
	9	14.70743 77571 44318 16612	7.34195 33794 09368 19389E-06
	10	18.65960 49563 73929 82988	1.86453 62121 30241 07286E-07
	11	23.33641 50320 24180 86531	2.29851 99108 06291 99207E-09
	12	28.94561 27927 64575 47741	1.13389 08232 48324 21252E-11
	13	35.88486 14167 37054 25003	1.54774 75211 58744 84689E-14
	14	45.17238 57659 17141 79357	2.31134 17495 84001 80082E-18
15	1	0.10790 80763 13319 86078	2.69578 67925 15948 89153E-01
	2	0.55375 08311 16300 19019	4.93686 11875 52279 52492E-01
	3	1.33445 44184 57049 01395	4.46865 59287 99928 17860E-01
	4	2.44909 24701 61111 97205	2.47588 12384 97186 84211E-01
	5	3.90672 23408 36861 87497	8.86721 57040 84484 15680E-02
	6	5.72321 67368 53608 99129	2.08570 56195 23791 01094E-02
	7	7.92123 69408 77090 19395	3.21067 25918 39851 71094E-03
	8	10.53175 96644 71593 40058	3.17651 88901 71256 81331E-04
	9	13.59703 77456 90283 45064	1.95486 92927 90280 82554E-05
	10	17.17574 11357 16614 59126	7.11334 46866 52253 17138E-07

n	k	x_k	$A_k^{(n)}$
15	11	21.35219 51239 24769 61825	1.41761 07681 13139 83996E-08
	12	26.25442 60912 99548 94310	1.37386 56419 04055 94984E-10
	13	32.09427 75730 22983 09895	5.31171 14877 92596 79397E-13
	14	39.27689 97792 36098 30276	5.61878 53715 41179 40313E-16
	15	48.83969 34266 66835 35369	6.30246 52343 78156 01057E-20
16	1	0.10100 69801 05273 59084	2.52965 31747 50120 03541E-01
	2	0.51911 28563 13616 45195	4.72504 56770 59483 09143E-01
	3	1.25164 77556 92375 23089	4.45926 04015 86807 72613E-01
	4	2.29674 44467 23562 57049	2.63557 05429 28285 57855E-01
	5	3.66132 25341 45394 96524	1.03143 90144 97480 47955E-01
	6	5.35803 41125 11840 44355	2.72260 52367 18929 64866E-02
	7	7.40491 35300 43818 07708	4.85028 63812 00297 24203E-03
	8	9.82627 39469 90379 22599	5.76269 92763 35286 73788E-04
	9	12.65456 41598 23772 30061	4.46047 57095 86824 35821E-05
	10	15.93349 52578 28808 35910	2.16852 65254 73854 30843E-06
	11	19.72332 00185 04293 24443	6.27472 93697 28905 30326E-08
	12	24.11025 38064 86109 20930	9.97995 22669 19378 58473E-10
	13	29.22484 31996 69297 24509	7.72178 25324 04031 85035E-12
	14	35.28278 59879 44415 68516	2.37373 77247 49370 48845E-14
	15	42.69636 30267 78655 70615	1.97317 15012 12108 91797E-17
	16	52.52135 85176 54245 21449	1.68644 65424 92387 65262E-21
17	1	0.09491 97045 93268 51804	2.38227 31954 53734 62000E-01
	2	0.48849 22190 94355 15597	4.52699 76990 75181 41286E-01
	3	1.17844 92319 31176 71675	4.43173 51817 75763 91474E-01
	4	2.16226 61158 30859 16544	2.77229 43013 85586 73080E-01
	5	3.44523 52783 29924 46007	1.17263 39745 56891 67890E-01
	6	5.03756 89141 08675 95228	3.42265 33881 01378 74061E-02
	7	6.95385 84282 31349 42031	6.91875 74685 68513 86127E-03
	8	9.21358 17827 38345 17639	9.61482 81783 69843 03284E-04
	9	11.84230 38347 85703 78658	9.03127 71154 38431 31552E-05
	10	14.87367 34873 00956 61152	5.58128 13804 00104 41523E-06
	11	18.35264 91837 27458 47804	2.18184 09655 26329 12865E-07
	12	22.34088 72004 05025 17573	5.10004 69620 35051 69459E-09
	13	26.92632 71707 66066 14923	6.56711 09940 76257 99666E-11
	14	32.24186 58310 66076 95843	4.11070 82545 06010 64392E-13
	15	38.50685 19796 39485 83392	1.01740 57897 23259 24412E-15
	16	46.14038 80977 89321 01477	6.72751 98285 59372 04753E-19
	17	56.21593 39615 00002 01086	4.43737 27348 60009 73400E-23
18	1	0.08951 15218 42739 22779	2.25068 93478 88053 37470E-01
	2	0.46123 15267 99495 75196	4.34202 47093 03279 96267E-01
	3	1.11327 67435 77675 96595	4.39047 85180 99021 66172E-01
	4	2.04266 69141 88483 95432	2.88805 29937 17423 57225E-01
	5	3.25343 68700 42084 74614	1.30846 97943 74951 13544E-01
	6	4.75391 43806 36348 38078	4.17249 70997 85409 07326E-02
	7	6.55606 90517 67818 64914	9.42048 91602 32293 53826E-03
	8	8.67577 08617 67460 81282	1.49976 75495 48363 50580E-03
	9	11.13358 11153 36928 45850	1.66320 07664 82024 93047E-04
	10	13.95607 69549 29680 65909	1.25904 44866 65004 50922E-05
	11	17.17792 54318 47819 54803	6.31685 15320 37460 64630E-07
	12	20.84518 11030 46007 83024	2.01525 54330 87005 34970E-08
	13	25.02076 87404 92430 80115	3.85675 19644 86190 08601E-10
	14	29.79422 22022 68575 70274	4.07042 35731 11522 51285E-12
	15	35.30064 95909 46173 66889	2.08515 68817 30318 13752E-14
	16	41.76287 80826 71110 48058	4.20100 72251 56674 06056E-17
	17	49.60655 98927 49700 47300	2.23378 11408 94102 02791E-20
	18	59.92219 23303 80667 46412	1.15002 33187 36859 80268E-24

n	k	x_k	$A_k^{(n)}$
19	1	0.08467 57428 69625 57876	2.13252 82238 07867 69190E-01
	2	0.43680 91938 30642 21849	4.16930 69414 63964 78581E-01
	3	1.05487 84111 29863 03043	4.33894 02610 59827 97589E-01
	4	1.93559 30541 13338 17607	2.98492 16973 55174 92989E-01
	5	3.08200 34604 02562 76916	1.43766 37827 27286 47009E-01
	6	4.50095 78337 25544 46554	4.95906 01844 13681 30777E-02
	7	6.20238 99654 77136 28362	1.23445 77895 98192 05649E-02
	8	8.19941 43995 90419 71677	2.21416 14007 61623 16356E-03
	9	10.50885 20568 11083 87346	2.83677 63255 92573 28611E-04
	10	13.15213 43585 73558 33108	2.55656 64570 08346 12384E-05
	11	16.15669 15373 80291 29224	1.58451 30442 22846 17363E-06
	12	19.55807 87518 25318 91564	6.54491 87702 56977 82068E-08
	13	23.40333 32632 07228 67122	1.72564 97974 54819 80938E-09
	14	27.75654 32247 57812 39062	2.73528 32706 06104 44431E-11
	15	32.70873 18851 75129 56490	2.39166 15209 97605 16759E-13
	16	38.39710 28303 07996 10750	1.01287 11679 61481 66686E-15
	17	45.04781 23994 93262 67823	1.67740 17451 02110 18241E-18
	18	53.09282 04525 71012 98094	7.24176 44043 83098 25513E-22
	19	63.63908 25042 41825 40426	2.93995 27407 10624 93386E-26
20	1	0.08032 68069 17182 13656	2.02586 43295 11212 61066E-01
	2	0.41480 60999 25382 56967	4.00798 20975 66672 50003E-01
	3	1.00225 04257 70922 81099	4.27982 41483 21280 52829E-01
	4	1.83916 54776 63101 91822	3.06493 12348 75283 57200E-01
	5	2.92782 24329 31169 12013	1.55938 29630 23620 50101E-01
	6	4.27389 11871 87051 25530	5.77012 83427 30490 54186E-02
	7	5.88569 81279 20380 85274	1.56678 19298 26053 84309E-02
	8	7.77421 18616 44572 97326	3.12309 66770 96927 52929E-03
	9	9.95339 13227 06180 68174	4.54264 21112 06737 17825E-04
	10	12.44081 65505 12031 55850	4.76544 36627 85608 63537E-05
	11	15.25864 82168 16141 31867	3.54283 16268 64862 69683E-06
	12	18.43505 62259 30492 38744	1.82173 74302 05496 64422E-07
	13	22.00638 65626 66761 14240	6.26935 52047 63201 41252E-09
	14	26.02057 12541 48349 89849	1.38108 71868 54252 24043E-10
	15	30.54277 77556 74500 59140	1.83149 73504 49301 14188E-12
	16	35.66543 22529 92912 33008	1.33932 59165 55104 55188E-14
	17	41.52773 52174 54048 14487	4.73148 42654 30849 31867E-17
	18	48.35904 12988 27233 22478	6.49721 74393 97126 49071E-20
	19	56.59740 03163 95923 59642	2.29733 79208 30993 90137E-23
	20	67.36569 66550 22990 34316	7.42272 66505 52206 28973E-28

6.7. $\varrho(x) = x^{-1/2}\, J_0(x),\ a = z_0$:

About z_0 see page 469.

n	k	x_k	$A_k^{(n)}$
1	1	0.53300 32202 42271 24134	2.36605 28163 78191 13115
2	1	0.21420 43489 09592 65574	1.74275 63856 32515 76369
	2	1.42437 48715 81250 50056	0.62329 64307 45675 36746
3	1	0.11331 99138 35413 17635	1.31105 54657 20433 27765
	2	0.86563 34174 25391 30585	0.84823 89258 92190 64389
	3	1.82958 01885 68362 01502	0.20675 84247 65567 20962
4	1	0.06974 41200 73133 50102	1.04093 64996 65801 84902
	2	0.56800 47180 44512 93370	0.82746 89016 40055 96983
	3	1.32394 75455 50030 49946	0.41346 52697 85388 28893
	4	2.03252 12706 23853 99685	0.08418 21452 86945 02337

n	k	x_k	$A_k^{(n)}$
5	1	0.04715 89450 25845 36129	0.86065 29427 93541 17127
	2	0.39721 32513 85306 35330	0.74956 40873 71111 34882
	3	0.97755 97028 18634 11835	0.50175 53049 46998 20027
	4	1.61852 36488 18102 42390	0.21414 95521 35801 25935
	5	2.14589 58724 24774 25260	0.03993 09291 30739 15144
6	1	0.03398 72780 65417 43442	0.73277 82433 19726 57861
	2	0.29189 95669 94265 16295	0.66891 42322 63755 08754
	3	0.74265 34123 61218 27013	0.51816 73511 49412 30530
	4	1.28607 34570 43235 25348	0.30600 61860 59737 26311
	5	1.81225 61375 63041 10336	0.11899 52839 93991 24447
	6	2.21493 03013 90218 99196	0.02119 15195 91568 65213
7	1	0.02564 71332 89465 94061	0.63765 60772 21229 42181
	2	0.22296 12772 52611 71453	0.59796 17757 14779 94053
	3	0.57964 92886 41091 84376	0.50253 70297 21431 12890
	4	1.03366 36667 06804 95097	0.35344 84918 41095 35754
	5	1.51300 97659 82497 99909	0.19171 60111 99272 33079
	6	1.94440 29691 16280 87956	0.07049 68038 89184 09972
	7	2.25984 75299 16014 65912	0.01223 66267 91198 85187
8	1	0.02003 68306 70609 10903	0.56423 95331 61849 48775
	2	0.17559 26637 57557 12875	0.53801 57941 39862 66305
	3	0.46326 96991 12343 91074	0.47495 12742 87121 50488
	4	0.84302 05815 69277 87344	0.37062 57454 31309 54824
	5	1.26619 53003 21934 03633	0.24237 10388 34860 01517
	6	1.68126 09930 95811 51358	0.12419 50690 95070 31846
	7	2.03778 96537 62525 61748	0.04411 26893 39386 34209
	8	2.29062 32109 53947 98248	0.00754 16720 88731 25151
9	1	0.01608 37515 92576 60303	0.50590 53536 43440 01865
	2	0.14173 60173 11551 60864	0.48772 39685 53204 03386
	3	0.37785 97521 40256 76577	0.44434 29447 57093 61070
	4	0.69766 04052 20519 66895	0.37036 84890 46321 32384
	5	1.06736 62646 29156 73844	0.27202 34341 32465 12099
	6	1.44999 06083 11280 63014	0.16870 39044 05801 51962
	7	1.80804 05562 11950 15132	0.08319 06579 49199 00272
	8	2.10589 14539 25235 21582	0.02889 96397 91291 61431
	9	2.31259 37744 46845 90727	0.00489 44240 99374 88645
10	1	0.01319 43790 06575 19374	0.45845 99663 39955 80607
	2	0.11674 00159 86152 49608	0.44535 56200 97232 64779
	3	0.31359 86521 31574 96609	0.41444 06310 70260 30672
	4	0.58526 85943 13472 47066	0.36094 56243 59298 53108
	5	0.90773 58902 80707 33918	0.28620 40555 56274 84593
	6	1.25394 98526 63370 29077	0.20050 69980 54101 44670
	7	1.59572 35904 52709 05445	0.11967 27110 80593 46044
	8	1.90529 51273 49255 13265	0.05747 85217 00159 27101
	9	2.15692 44580 64621 08855	0.01967 65894 99312 05944
	10	2.32880 91678 81470 12027	0.00331 20986 21002 75597

n	r_n
1	3.3618E-01
2	9.3291E-03
3	1.0824E-04
4	6.8366E-07
5	2.7076E-09
6	7.3415E-12
7	1.4475E-14
8	2.1678E-17
9	2.5493E-20
10	2.4162E-23

6.8. $\varrho(x) = x^{-1/2}\, J_1(x),\ a = z_1$: About z_1 see page 469.

n	k	x_k	$A_k^{(n)}$
1	1	1.62333 70297 72799 39168	1.15005 63206 53692 45663
2	1	0.85548 11815 35298 19713	0.64064 81525 62703 26637
	2	2.58901 73258 92346 73880	0.50940 81680 90989 19027
3	1	0.51725 99744 42046 17896	0.33630 11951 36750 72989
	2	1.74064 75394 86289 89244	0.60442 80291 46581 86241
	3	3.06160 89050 20550 79914	0.20932 70963 70359 86433

n	k	x_k	$A_k^{(n)}$
4	1	0.34353 10780 86336 09607	0.19015 02790 25383 87835
	2	1.22483 06896 79854 40023	0.47830 23671 99480 87362
	3	2.32924 34566 07168 20709	0.38627 19292 63470 62465
	4	3.31521 99064 51764 11396	0.09533 17451 65357 08002
5	1	0.24379 86743 15095 54567	0.11608 57238 38106 18938
	2	0.89943 98608 01193 00991	0.34523 89859 76630 95573
	3	1.79142 28661 87139 62176	0.40950 90459 83430 27053
	4	2.71593 46530 01834 24707	0.23087 91569 25582 70416
	5	3.46374 39913 25978 28293	0.04834 34079 29942 33684
6	1	0.18164 5011 97923 261867	0.07551 20019 43543 79213
	2	0.68465 61168 15453 05535	0.24652 92889 08755 92623
	3	1.40527 83906 06974 66406	0.36052 97323 61107 80388
	4	2.21450 48484 23529 70213	0.30094 96211 85898 46865
	5	2.97698 58982 53126 89610	0.13973 21500 96356 49395
	6	3.55718 75611 02302 61569	0.02680 35261 58029 97180
7	1	0.14042 74927 48541 34646	0.05167 98108 61533 94865
	2	0.53684 39465 91875 56558	0.17850 09247 93955 01159
	3	1.12480 20726 81764 02734	0.29582 58772 76257 60679
	4	1.81946 39889 99092 29871	0.30948 91155 06034 78859
	5	2.52877 58488 58524 00880	0.21112 56927 19209 49463
	6	3.15906 51351 40798 47350	0.08747 48025 72456 35889
	7	3.61943 86737 38481 84337	0.01596 00969 24245 24749
8	1	0.11174 27619 01052 87885	0.03684 46175 78582 22633
	2	0.43137 24471 02076 69556	0.13195 63451 73936 29281
	3	0.91711 33648 34932 00454	0.23680 64585 95797 64844
	4	1.51143 35095 70517 98976	0.28551 68023 30947 90048
	5	2.14968 38809 49919 28417	0.24480 36530 39780 23276
	6	2.76481 46645 66847 76682	0.14721 96638 22807 21962
	7	3.29007 35822 61214 92883	0.05684 60770 24187 11913
	8	3.66285 29289 89593 51309	0.01006 27030 87653 81707
9	1	0.09099 90494 62333 07323	0.02715 75333 14324 96017
	2	0.35375 12259 91805 74317	0.09967 60720 21663 57695
	3	0.76015 95213 71319 89327	0.18864 63076 71184 34547
	4	1.27014 48646 59255 63432	0.24993 36217 19169 40767
	5	1.83721 26245 15676 18759	0.24867 99946 96993 76711
	6	2.41209 44351 25542 54518	0.18732 79839 92138 67902
	7	2.94489 24140 66618 47167	0.10371 91684 65173 97360
	8	3.38699 99640 25704 49915	0.03826 97820 22759 94426
	9	3.69427 46275 60637 56869	0.00664 58567 50283 80239
10	1	0.07552 29676 82866 15777	0.02057 62904 73694 12292
	2	0.29510 14424 10791 61732	0.07683 68220 18724 23826
	3	0.63922 48569 29456 39244	0.15089 70725 75745 97658
	4	1.07928 93212 36406 47434	0.21339 36849 14620 00276
	5	1.58118 89158 96933 16465	0.23516 11881 65637 25122
	6	2.10799 98085 23450 92370	0.20582 10988 25625 40795
	7	2.62171 76115 44622 38207	0.14192 26453 68502 20293
	8	3.08455 54007 99910 63628	0.07429 36633 38319 21842
	9	3.46048 70096 79160 88696	0.02659 35792 64679 58296
	10	3.71772 06400 54232 32254	0.00456 02757 0814 445264

n	r_n
1	4.2639E-01
2	3.0564E-02
3	9.0610E-04
4	1.4574E-05
5	1.4677E-07
6	1.0113E-09
7	5.0649E-12
8	1.9266E-14
9	5.7533E-17
10	1.3847E-19

6.9. $\varrho(x) = 1/[\sqrt{x}\, I_0(x)],\ a = +\infty$:

n	k	x_k	$A_k^{(n)}$
1	1	0.83785 88793 38414 23349	2.74235 45178 63014 77106E+00
2	1	0.43908 47518 81085 06054	2.36596 65158 23181 85050E+00
	2	3.34454 40760 70051 67391	3.76388 00203 98329 20568E-01
3	1	0.28907 08638 64741 64556	2.02388 12887 22597 08171E+00
	2	2.23386 71660 16809 31944	6.91559 54322 77021 53509E-01
	3	6.23507 36415 99816 75810	2.69136 85912 71553 58426E-02
4	1	0.21260 25320 89340 49995	1.77537 20719 94970 77159E+00
	2	1.68808 53754 80834 50298	8.66640 86569 18770 53851E-01
	3	4.48976 18955 23031 02177	9.89453 83403 58888 16226E-02
	4	9.34861 19743 37825 73377	1.39619 67725 78064 00457E-03
5	1	0.16693 47314 29841 09282	1.59177 43661 74531 53623E+00
	2	1.35613 77013 63248 32739	9.51001 25146 52416 42111E-01
	3	3.55477 14692 87897 13414	1.90266 24500 83182 89846E-01
	4	7.04487 68033 13852 21148	9.25226 48194 04714 26144E-03
	5	12.60002 48956 91170 90484	6.03903 95518 58862 00983E-05
6	1	0.13683 39623 49863 40202	1.45112 46353 90900 26893E+00
	2	1.13151 55204 55347 43488	9.84049 71803 20406 04292E-01
	3	2.95449 42126 41471 59996	2.80105 38296 93904 16177E-01
	4	5.73823 58186 26913 70244	2.64027 90668 65190 77618E-02
	5	9.80073 83098 93011 39258	6.69673 25583 59611 29601E-04
	6	15.94623 46324 14805 47669	2.31754 61956 12775 10627E-06
7	1	0.11561 08188 17466 71116	1.33976 04964 95570 21574E+00
	2	0.96917 13443 94269 03349	9.88976 73347 52161 61542E-01
	3	2.53181 73822 24275 95132	3.58871 54104 47113 89844E-01
	4	4.86651 37348 65497 88689	5.19660 47352 69590 18358E-02
	5	8.14853 01346 92692 15020	2.73891 25832 55959 71272E-03
	6	12.70065 29781 25231 35328	4.07052 81476 63911 67849E-05
	7	19.36225 78992 66042 28334	8.16300 88503 27240 40159E-08
8	1	0.09989 71066 24431 90436	1.24914 39379 29196 13471E+00
	2	0.84638 59774 62496 26728	9.78739 41825 02968 15719E-01
	3	2.21635 77506 80886 53907	4.23805 98980 90219 89546E-01
	4	4.23526 40932 35319 82072	8.33427 59055 71607 87029E-02
	5	7.01475 29039 87931 56757	7.08839 99757 48432 11331E-03
	6	10.72746 72005 76616 50799	2.31834 10245 35982 72269E-04
	7	15.70968 96918 86696 69776	2.17604 61803 49568 79803E-06
	8	22.83218 21993 31520 54836	2.69440 13724 33021 12419E-09
9	1	0.08782 32793 79286 13261	1.17374 88283 79164 23365E+00
	2	0.75034 79465 19983 49185	9.60550 71956 80579 37821E-01
	3	1.97115 62503 00629 21310	4.75358 88702 87038 36988E-01
	4	3.75394 52658 51748 09782	1.17768 88845 89269 23802E-01
	5	6.17582 39505 57788 73729	1.41336 46720 46360 51577E-02
	6	9.34396 50502 00730 18638	7.76596 46516 29124 51518E-04
	7	13.43728 00883 92547 16738	1.68458 15126 55644 95207E-05
	8	18.80468 13669 15026 51820	1.05342 91068 15358 15100E-07
	9	26.34518 40061 34450 46127	8.44980 83204 16378 15706E-11
10	1	0.07827 28621 47227 06144	1.10986 08851 53118 08527E+00
	2	0.67324 88423 05449 72969	9.38456 12109 54537 07997E-01
	3	1.77473 02056 03525 91166	5.15205 45661 36600 63647E-01
	4	3.37342 34254 06738 69368	1.52979 21657 40233 09037E-01
	5	5.52504 84092 52099 61174	2.38708 96437 07063 64724E-02
	6	8.30282 91949 09476 51212	1.90885 92869 49810 60245E-03
	7	11.81652 94064 14378 96116	7.19921 13303 88936 35335E-05
	8	16.25204 36390 51999 45506	1.08587 62201 59336 62440E-06

n	r_n
1	1.3707E+00
2	4.4698E-01
3	1.3234E-01
4	3.7502E-02
5	1.0359E-02
6	2.8137E-03
7	7.5515E-04
8	2.0088E-04
9	5.3063E-05
10	1.3941E-05
11	3.6457E-06
12	9.4984E-07
13	2.4669E-07
14	6.3894E-08
15	1.6510E-08
16	4.2573E-09
17	1.0958E-09
18	2.8160E-10
19	7.2259E-11
20	1.8518E-11

n	k	x_k	$A_k^{(n)}$
10	9	21.96940 04563 19921 14727	4.71067 30005 26619 33312E-09
	10	29.89353 31514 85036 63092	2.54210 88128 25038 37860E-12
11	1	0.07053 99999 83512 77505	1.05489 60139 04956 75547E+00
	2	0.61004 75534 37368 96557	9.14740 49206 38368 27555E-01
	3	1.61366 04432 30809 64814	5.45305 91207 97341 56739E-01
	4	3.06433 16679 02746 54194	1.87352 84218 49187 43199E-01
	5	5.00326 41608 44470 75040	3.59996 46145 73304 69162E-02
	6	7.48400 82426 71890 76110	3.83743 62816 46179 93210E-03
	7	10.58014 23446 12643 38759	2.16268 10254 57849 23171E-04
	8	14.40584 93853 50826 97230	5.84338 10364 96677 92958E-06
	9	19.15312 28799 25036 50046	6.35211 72098 51401 69006E-08
	10	25.19199 62818 21852 84255	1.97360 79465 81476 65422E-10
	11	33.47149 64875 68342 85725	7.38864 80580 84848 03874E-14
12	1	0.06415 75831 08829 81046	1.00700 12145 26011 46455E+00
	2	0.55734 11301 72896 13890	8.90695 15787 50121 66738E-01
	3	1.47909 69640 72841 89091	5.67507 53359 31718 07158E-01
	4	2.80789 01978 02687 43892	2.19843 44708 79800 15379E-01
	5	4.57442 97790 07897 32747	5.00613 12140 78376 98079E-02
	6	6.81989 23031 29557 38317	6.70920 04357 93044 01431E-03
	7	9.59675 86969 07902 26598	5.14949 31323 41885 02276E-04
	8	12.98154 32243 99638 44877	2.12740 78256 85527 30972E-05
	9	17.09237 44708 04447 31753	4.25376 62310 90118 42704E-07
	10	22.12663 16996 78442 78810	3.42831 71520 88602 38486E-09
	11	28.46352 50811 89378 55926	7.82911 27554 90490 93212E-12
	12	37.07469 21730 64671 39603	2.08578 88370 83335 59934E-15
13	1	0.05880 47054 14231 03423	9.64812 41184 01688 57969E-01
	2	0.51274 88754 81092 50853	8.67041 91530 55864 79668E-01
	3	1.36494 88940 31487 96026	5.83405 36400 31409 49579E-01
	4	2.59146 79911 13109 57092	2.49849 01461 10972 69668E-01
	5	4.21510 22370 53205 49277	6.55463 18130 54892 97696E-02
	6	6.26871 18319 33401 80136	1.05968 42746 27013 29943E-02
	7	8.79153 69769 17795 37213	1.04108 21490 65877 45301E-03
	8	11.83798 71019 39707 45845	5.96786 63706 10684 45816E-05
	9	15.48737 78649 63059 79930	1.86197 19053 59604 21866E-06
	10	19.86127 42704 86860 29954	2.82683 78516 26232 76148E-08
	11	25.16191 93027 80047 43936	1.72849 79323 19655 95034E-10
	12	31.77705 47760 24655 41931	2.96440 59481 79005 13432E-13
	13	40.69968 79527 13730 51273	5.74251 47116 95505 11921E-17
14	1	0.05425 39087 89343 80428	9.27302 32050 71984 48226E-01
	2	0.47455 55653 86126 31239	8.44171 73955 66015 00198E-01
	3	1.26687 73047 29570 65213	5.94318 26725 99696 09663E-01
	4	2.40622 82617 01611 72580	2.77084 81281 83997 42416E-01
	5	3.90927 18587 72617 82577	8.19622 96455 93635 94174E-02
	6	5.80295 19313 75440 43701	1.55039 05477 13022 74288E-02
	7	8.11774 06191 83192 41822	1.86583 46478 52110 63710E-03
	8	10.89405 05637 62806 95985	1.39071 06911 02622 87206E-04
	9	14.18838 96576 78219 19997	6.12066 20303 37265 58685E-06
	10	18.08250 04485 55887 72616	1.47662 34487 57863 18662E-07
	11	22.70100 71613 18542 14315	1.73821 06642 33104 22780E-09
	12	28.25062 04661 50380 28480	8.21985 04190 27438 77820E-12
	13	35.12709 29202 76088 52703	1.07815 39099 65071 30516E-14
	14	44.34373 99085 13806 79964	1.54696 55835 41448 55935E-18
15	1	0.05033 96484 93611 74191	8.93681 78383 82150 56678E-01
	2	0.44149 45355 61550 91405	8.22281 70637 83245 44484E-01
	3	1.18170 20796 10598 02459	6.01311 63917 08678 88325E-01
	4	2.24578 90804 79951 11086	3.01480 04927 29971 32311E-01

n	k	x_k	$A_k^{(n)}$
15	5	3.64558 31676 72122 11371	9.88708 46609 89987 76166E-02
	6	5.40360 58261 12771 27861	2.13784 99978 39289 81354E-02
	7	7.54427 68199 93117 59054	3.05070 82342 63692 68113E-03
	8	10.09864 18609 01642 87251	2.82325 69623 70241 30216E-04
	9	13.10873 06379 44931 52545	1.63820 27250 29325 54221E-05
	10	16.63288 02687 35463 28172	5.65795 87616 19577 44269E-07
	11	20.75496 92952 13253 62080	1.07603 73743 55941 76261E-08
	12	25.60239 81402 62209 93387	9.99445 62503 41686 39523E-11
	13	31.38603 36839 78348 01305	3.71516 82257 74718 66134E-13
	14	38.50920 64588 56724 75626	3.78563 01512 29551 08172E-16
	15	48.00461 62232 48903 36675	4.08839 40705 74581 94985E-20
16	1	0.04693 86743 41535 21491	8.63333 93652 96860 23657E-01
	2	0.41261 04422 51348 62355	8.01454 89811 42288 32454E-01
	3	1.10703 60576 88197 49213	6.05235 14993 46732 21990E-01
	4	2.10541 86353 37042 96066	3.23100 65816 13393 48300E-01
	5	3.41573 28743 67796 22249	1.15902 89042 17644 31974E-01
	6	5.05705 13064 45714 27186	2.81294 69538 66977 32084E-02
	7	7.04947 12174 38421 48005	4.64264 20041 90333 89315E-03
	8	9.41750 19007 75805 78592	5.15505 13080 93159 82192E-04
	9	12.19344 70594 88688 09092	3.75865 69632 09029 95107E-05
	10	15.42077 62906 06039 04006	1.73290 74721 09030 0 1555E-06
	11	19.15942 41673 19499 16280	4.78165 80026 82556 31829E-08
	12	23.49518 68836 05650 10253	7.28534 02997 05295 65603E-10
	13	28.55802 17356 44737 79249	5.41916 27823 42083 16469E-12
	14	34.56269 93611 68951 86069	1.60578 10384 84084 17686E-14
	15	41.91976 05104 62209 03146	1.28854 03316 49386 50080E-17
	16	51.68047 45718 26612 53605	1.06232 90180 53104 63667E-21
17	1	0.04395 74004 89201 64086	8.35769 13513 78360 36824E-01
	2	0.38716 94754 13196 59084	7.81707 73253 73530 78511E-01
	3	1.04105 04705 04732 80649	6.06761 76981 13855 74756E-01
	4	1.98152 98833 93422 57144	3.42094 46207 33031 83893E-01
	5	3.21349 52694 85265 25350	1.32761 00935 34728 31954E-01
	6	4.75323 25881 02115 45827	3.56414 52730 77139 35724E-02
	7	6.61765 95150 48738 56779	6.67191 97063 13170 76493E-03
	8	8.82657 30163 18250 12085	8.65839 00053 37679 32435E-04
	9	11.40545 42042 47815 62808	7.65445 33260 50147 12270E-05
	10	14.38777 39764 56914 89655	4.48222 78594 72742 48475E-06
	11	17.81825 46194 73023 83363	1.66967 89400 40399 32923E-07
	12	21.75825 24098 91806 88766	3.73651 32570 44060 71476E-09
	13	26.29531 30179 28326 98882	4.62388 32901 34734 85460E-11
	14	31.56177 56863 63035 62311	2.78997 14651 94168 35259E-13
	15	37.77610 50400 71368 22398	6.67083 97312 04041 51487E-16
	16	45.35573 38233 24114 25250	4.26616 28738 58295 81539E-19
	17	55.36977 45488 48370 79726	2.71882 71906 74269 83163E-23
18	1	0.04132 35301 42217 08330	8.10593 39385 99763 44109E-01
	2	0.36459 89120 89034 47003	7.63018 32660 10878 18263E-01
	3	0.98231 95370 35169 81982	6.06422 72347 35026 51230E-01
	4	1.87135 14036 37302 61519	3.58653 61237 98163 37533E-01
	5	3.03410 29418 39121 13751	1.49214 67664 34060 13499E-01
	6	4.48454 39748 74990 36008	4.37872 88144 87847 12315E-02
	7	6.23718 93285 12511 27368	9.15226 07274 94193 22586E-03
	8	8.30834 91499 25592 68182	1.35976 04126 45591 25442E-03
	9	10.71855 26681 39997 90984	1.41811 14222 28062 29620E-04
	10	13.49424 56979 56810 61892	1.01638 17638 33618 25828E-05
	11	16.66991 52836 64362 79466	4.85563 88223 18507 62632E-07
	12	20.29139 12852 79392 85579	1.48212 15537 09749 02316E-08

n	k	x_k	$A_k^{(n)}$
18	13	24.42131 63464 77122 16347	2.72463 49254 58483 28605E-10
	14	29.14885 26265 58222 05397	2.77120 42399 02475 02821E-12
	15	34.60857 83125 97233 94196	1.37158 09801 69220 96680E-14
	16	41.02247 40769 64778 59611	2.67475 90724 55622 59598E-17
	17	48.81458 52214 01242 62878	1.37781 94559 20592 42938E-20
	18	59.07121 36825 39115 29354	6.86401 77307 79894 74427E-25
19	1	0.03898 03562 71368 49879	7.87485 82291 97113 70439E-01
	2	0.34444 54015 02802 59155	7.45343 63063 67357 38345E-01
	3	0.92971 45539 55720 24664	6.04636 82755 63333 33421E-01
	4	1.77270 63076 22712 87726	3.72989 66165 03886 79478E-01
	5	2.87383 88681 82139 20906	1.65092 17512 29175 28267E-01
	6	4.24511 43323 96603 68162	5.24373 93134 17180 79066E-02
	7	5.89918 47404 38950 54779	1.20823 54747 74180 45553E-02
	8	7.84972 47349 83705 68710	2.02126 46268 22927 45332E-03
	9	10.11356 94123 17741 28824	2.43364 28570 95409 21754E-04
	10	12.71205 48366 26147 31251	2.07499 09253 03005 75297E-05
	11	15.67247 17818 70432 54543	1.22370 13734 02318 56206E-06
	12	19.03020 17052 26851 66549	4.83296 32165 36942 51508E-08
	13	22.83206 98675 24472 93596	1.22338 44988 21186 79971E-09
	14	27.14189 86085 20695 08615	1.86805 09474 20085 50971E-11
	15	32.05035 89412 38899 03434	1.57786 27045 83482 30147E-13
	16	37.69414 86000 80923 18418	6.46926 48546 02091 20207E-16
	17	44.29861 07498 73835 89956	1.03880 35518 77212 56168E-18
	18	52.29415 48959 33321 67231	4.35119 16277 56276 37884E-22
	19	62.78367 97289 57088 22119	1.71162 55984 98571 67037E-26
20	1	0.03688 27938 52145 53112	7.66182 20311 94767 63926E-01
	2	0.32634 55451 77488 62264	7.28629 81353 82507 01840E-01
	3	0.88232 99220 86783 42277	6.01734 29038 03125 54957E-01
	4	1.68385 97599 63504 90604	3.85317 56176 07040 09180E-01
	5	2.72975 94655 77380 14365	1.80271 46168 14126 92854E-01
	6	4.03033 19203 27460 63148	6.14663 53896 45247 83905E-02
	7	5.59674 90726 24659 93081	1.54481 86987 11828 41271E-02
	8	7.44065 76116 08519 80262	2.87071 31800 46175 19975E-03
	9	9.57607 41626 12234 57987	3.92150 81663 56887 42618E-04
	10	12.02051 23110 44478 63242	3.88933 84441 16347 85715E-05
	11	14.79602 38227 49529 13227	2.74946 63525 44250 00617E-06
	12	17.93064 13419 43736 08058	1.35094 92260 23527 91125E-07
	13	21.46054 59511 56938 26951	4.46104 14747 18418 59411E-09
	14	25.43346 98752 90004 04458	9.46271 67559 91476 57187E-11
	15	29.91432 92532 30155 52760	1.21186 12837 48134 70940E-12
	16	34.99521 50428 17277 13384	8.57877 04849 96334 47430E-15
	17	40.81484 30202 01584 59672	2.93928 09045 43461 50774E-17
	18	47.60178 44078 38070 04327	3.91950 99655 78150 77367E-20
	19	55.79258 99568 25017 79931	1.34641 33692 82563 35606E-23
	20	66.50621 42680 65338 79445	4.22039 16526 42243 12037E-28

6.10. $\varrho(x) = \sqrt{x}/I_1(x)$ $a = +\infty$:

n	k	x_k	$A_k^{(n)}$
1	1	1.10110 24421 80541 00065	6.37652 23909 80087 39091E+00
2	1	0.55711 42396 03342 91668	5.39323 411766274209435E+00
	2	4.08482 12694 65392 36526	9.83288 273317345296557E-01
3	1	0.35735 62271 99135 49799	4.53784 503413913176070E+00
	2	2.72752 92391 67193 43458	1.75296 036747845603157E+00
	3	7.21358 20857 99896 43390	8.57169 893624995986445E-02

n	k	x_k	$A_k^{(n)}$
4	1	0.25785 99877 51895 61457	3.93392 415862198024968E+00
	2	2.04438 20294 00355 60941	2.13380 932007827524482E+00
	3	5.23543 34949 84875 81908	3.03470 623610724142307E-01
	4	10.46972 99642 28818 83172	5.31828 866910775410541E-03
5	1	0.19953 41155 09389 05764	3.49635 312774791191622E+00
	2	1.62640 91404 16455 87215	2.28341 974121038893780E+00
	3	4.15195 06386 55132 16847	5.62400 607742396834248E-01
	4	7.95238 66611 39501 36262	3.40808 667897986646255E-02
	5	13.81888 34066 74105 27877	2.68047 489591038011914E-04
6	1	0.16166 32543 00866 77731	3.16585 751341765873097E+00
	2	1.34416 23691 53849 66968	2.31422 363030943400120E+00
	3	3.44664 51686 57620 52750	7.99418 954674395608503E-01
	4	6.50156 78671 05504 68509	9.41243 063652407598978E-02
	5	10.82299 17715 10652 82760	2.88624 598656275724581E-03
	6	17.23741 69002 16510 37099	1.17402 267955330985051E-05
7	1	0.13529 09787 71329 96152	2.90700 947470594082471E+00
	2	1.14130 59926 48108 89074	2.28677 774486158595877E+00
	3	2.94563 79020 94530 21324	9.91795 477424458139796E-01
	4	5.52289 62800 62942 55312	1.79252 909403789158225E-01
	5	9.03317 87061 15823 06466	1.14854 934907636031384E-02
	6	13.80975 37204 99792 84454	2.00826 667683998612803E-04
	7	20.70969 15592 79148 31956	4.64425 865707663428746E-07
8	1	0.11596 82330 19652 00486	2.69821 224137409219821E+00
	2	0.98894 12896 85349 53176	2.23207 345030869682923E+00
	3	2.56974 94208 49498 78224	1.13789 046384869993400E+00
	4	4.80836 00566 24235 16677	2.78307 176908881807626E-01
	5	7.79426 57829 36289 00532	2.89095 394997635012587E-02
	6	11.70581 66048 99430 62513	1.11741 865795581770222E-03
	7	16.88742 02443 92513 62128	1.20833 822421872819590E-05
	8	24.22494 06606 90657 05189	1.69997 551156004450172E-08
9	1	0.10125 42981 91384 41041	2.52573 206917976750601E+00
	2	0.87063 13330 23182 87504	2.16588 251194900252651E+00
	3	2.27678 48983 48680 24873	1.24397 111284187272337E+00
	4	4.26011 04830 70586 96418	3.81145 299926825842591E-01
	5	6.87139 33372 33629 16937	5.60425 846430508601499E-02
	6	10.22024 52292 07280 54262	3.65650 556087530854193E-03
	7	14.49080 31047 47536 66773	9.16565 671048565650052E-05
	8	20.03835 52088 99821 29126	6.49726 376402130827486E-07
	9	27.77544 32172 11957 69353	5.85211 365049264935367E-10
10	1	0.08970 64355 75237 44316	2.38046 159728726541120E+00
	2	0.77632 73728 15689 79311	2.09623 896530199839349E+00
	3	2.04189 10572 11614 06015	1.31804 657264391436749E+00
	4	3.82457 76191 75084 58274	4.80581 699613219357299E-01
	5	6.15166 92080 79120 07450	9.20273 764421817176136E-02
	6	9.09610 56348 72333 34541	8.77564 560576484287848E-03
	7	12.77149 17997 43084 15546	3.83925 584012486871029E-04
	8	17.36755 82292 36178 62714	6.57655 075430878109996E-06
	9	23.24976 47903 41637 58305	3.19318 113936299781082E-08
	10	31.35546 63070 17235 91950	1.91651 116644630569271E-11
11	1	0.08042 07131 91695 41949	2.25613 253297607163766E+00
	2	0.69954 47236 64323 80099	2.02720 064259184462717E+00
	3	1.84936 08340 13756 62646	1.36749 807904441088510E+00
	4	3.46949 35723 14472 20367	5.72317 522192146599889E-01
	5	5.57208 26734 29737 70500	1.34988 018540903148897E-01

n	r_n
1	5.1749E+00
2	2.2879E+00
3	8.4804E-01
4	2.8737E-01
5	9.2119E-02
6	2.8429E-02
7	8.5330E-03
8	2.5077E-03
9	7.2472E-04
10	2.0664E-04
11	5.8260E-05
12	1.6273E-05
13	4.5089E-06
14	1.2407E-06
15	3.3934E-07
16	9.2320E-08
17	2.4999E-08
18	6.7406E-09
19	1.8107E-09
20	4.8473E-10

n	k	x_k	$A_k^{(n)}$
11	6	8.20801 53539 81111 92532	1.72203 483160740173843E-02
	7	11.45380 50065 10763 30736	1.13004 682412722979535E-03
	8	15.42638 88070 89290 38690	3.47755 319640679868806E-05
	9	20.32084 83456 13753 87694	4.23504 227380009337597E-07
	10	26.51206 06078 26708 71707	1.45771 565354287646226E-09
	11	34.96062 83081 58185 15223	6.02143 477030633218492E-13
12	1	0.07280 37286 81968 57024	2.14828 981908039720657E+00
	2	0.63591 70858 72837 89040	1.96075 116704999311031E+00
	3	1.68873 12703 84698 17281	1.39839 763907136870805E+00
	4	3.17407 89714 90305 86143	6.54241 118066105487409E-01
	5	5.09402 13630 93974 73940	1.82695 190847742572496E-01
	6	7.48501 03077 21508 36527	2.93848 725790320761251E-02
	7	10.40171 95983 61437 39654	2.63537 889598178798178E-03
	8	13.92284 47232 88162 30983	1.24388 199690853648086E-04
	9	18.16861 82768 45927 89243	2.79219 082749744434256E-06
	10	23.33907 52191 17061 66672	2.49363 715410093796203E-08
	11	29.81785 85749 63671 79499	6.25582 834341862426206E-11
	12	38.58749 87579 10499 46920	1.82664 262094919072330E-14
13	1	0.06645 07454 06401 14813	2.05367 849374243519044E+00
	2	0.58240 13073 17667 19181	1.89777 939916917372157E+00
	3	1.55274 14598 13174 74162	1.41548 889949855596413E+00
	4	2.92426 82330 45217 82823	7.25697 685066545817664E-01
	5	4.69223 77851 70377 90681	2.33003 015744332918833E-01
	6	6.88304 68406 69071 85151	4.53029 685359281092942E-02
	7	9.53743 91644 68041 53742	5.21698 902142726076533E-03
	8	12.71164 77066 62263 51936	3.42702 851753335026628E-04
	9	16.48658 65241 25650 54494	1.20332 469990291734238E-05
	10	20.98556 29311 26900 67924	2.02738 924072707375438E-07
	11	26.41319 29108 61618 25998	1.36146 434213356178322E-09
	12	33.16134 13217 08022 95580	2.54709 149062953417079E-12
	13	42.23333 69151 93914 10371	5.37686 912907849918753E-16
14	´1	0.06107 67556 07820 00433	1.96986 311941741441912E+00
	2	0.53681 46055 75160 80216	1.83859 430099743045942E+00
	3	1.43618 44627 80695 77398	1.42238 404277416378343E+00
	4	2.71016 22020 05554 98444	7.86914 647316816688612E-01
	5	4.34940 88208 93980 00368	2.84073 279259311310001E-01
	6	6.37298 94605 26794 22656	6.47149 947121310027648E-02
	7	8.81220 51852 71869 09339	9.15391 914030655810852E-03
	8	11.70903 39127 80651 25866	7.84094 895079975943602E-04
	9	15.12128 57825 86701 69869	3.89344 759612549676305E-05
	10	19.13198 00478 48797 92742	1.04440 603232553329040E-06
	11	23.86725 75145 92530 04894	1.35156 656889294427823E-08
	12	29.53600 00082 01895 32526	6.96748 080739975308992E-11
	13	36.53783 68716 51409 62333	9.91005 799918111935461E-14
	14	45.89591 50349 46191 42531	1.54184 483538531378199E-17
15	1	0.05647 55622 39084 02778	1.89498 342138684298727E+00
	2	0.49755 26963 79607 48830	1.78320 066621431921338E+00
	3	1.33522 42581 82392 21901	1.42179 874813386780959E+00
	4	2.52457 17254 81198 54521	8.38601 902863703493683E-01
	5	4.05319 53890 49401 34151	3.34453 517949455552433E-01
	6	5.93463 42012 50238 87468	8.71618 746253783229563E-02
	7	8.19346 93079 23915 84882	1.46532 047544314873877E-02
	8	10.86209 91340 30533 87539	1.56248 406294492952267E-03
	9	13.98366 47946 43829 72323	1.02541 561164311883004E-04
	10	17.61751 01043 17320 41558	3.94596 379650048754315E-06

n	k	x_k	$A_k^{(n)}$
15	11	21.84859 62184 61314 19847	8.26237 033551160931118E-08
	12	26.80568 53565 98958 76098	8.37104 743687229832439E-10
	13	32.70166 22465 30158 84713	3.37097 730471788516532E-12
	14	39.94353 10683 80242 70694	3.70577 258653731135142E-15
	15	49.57339 53322 64776 70683	4.32087 812124102247466E-19
16	1	0.05249 44855 86916 71880	1.82759 259030826363408E+00
	2	0.46341 14767 04995 23034	1.73144 890855240352003E+00
	3	1.24697 02840 91749 74294	1.41576 262767920929240E+00
	4	2.36213 85443 75412 87202	8.81693 714542536106186E-01
	5	3.79454 03313 56801 21260	3.83071 021300234504762E-01
	6	5.55344 98030 88859 67496	1.12075 279424638278238E-01
	7	7.65847 06491 49111 27009	2.18344 234164984154807E-02
	8	10.13527 04617 80100 16640	2.80009 793949878315547E-03
	9	13.01713 14513 23633 28407	2.31445 346204296716262E-04
	10	16.34835 85193 21744 68642	1.19138 370948535716367E-05
	11	20.18971 88454 15288 87544	3.62552 458432029072214E-07
	12	24.62796 21674 42025 34889	6.03226 772318362974483E-09
	13	29.79429 40386 12983 55229	4.86241 648939677243898E-11
	14	35.90538 11478 18679 53779	1.55252 355201797592851E-13
	15	43.37526 56974 93202 19078	1.33809 011565342365167E-16
	16	53.26424 21050 02040 66949	1.18647 951489020740132E-20
17	1	0.04901 81759 94754 29725	1.76654 739131876879398E+00
	2	0.43347 05323 19831 25020	1.68311 709450027416124E+00
	3	1.16920 21628 09934 45268	1.40578 924972047271701E+00
	4	2.21878 08586 89773 86831	9.17191 637290092395025E-01
	5	3.56662 87791 11569 12946	4.29186 344073688322291E-01
	6	5.21867 49517 11217 61833	1.38850 189361722746115E-01
	7	7.19071 12869 13664 21885	3.07296 018923243953537E-02
	8	9.50350 21755 45125 70942	4.61564 796040458295011E-03
	9	12.18340 77586 81649 49839	4.63584 168458111875655E-04
	10	15.26461 79441 41408 62571	3.03698 563826866465906E-05
	11	18.79252 99327 87674 25363	1.24983 848886156143910E-06
	12	22.82921 96806 12304 36529	3.05858 276045552018472E-08
	13	27.46309 41967 48355 22126	4.10506 037484829426888E-10
	14	32.82765 34686 86767 65537	2.66912 515492824494944E-12
	15	39.14315 80923 95606 27294	6.84497 736885258924687E-15
	16	46.83039 32578 69262 22576	4.68354 129426992425834E-18
	17	56.96715 75498 32131 99607	3.19927 004984363000883E-22
18	1	0.04595 79384 07398 09915	1.71093 169025460040540E+00
	2	0.40701 48963 92958 58808	1.63795 584924322665508E+00
	3	1.10018 58063 74526 02527	1.39300 713626367945790E+00
	4	2.09133 07052 11463 96731	9.46074 700383566220305E-01
	5	3.36422 51142 86071 36228	4.72332 761633673288710E-01
	6	4.92214 64353 74291 73653	1.66896 551925399841828E-01
	7	6.77788 52427 28592 04275	4.12932 896477377596662E-02
	8	8.94852 43442 20715 08229	7.11421 290715016583569E-03
	9	11.45539 20267 02232 30139	8.44633 514386556466790E-04
	10	14.32557 00335 46529 71688	6.78553 304659718959788E-05
	11	17.59411 39544 21867 39826	3.58753 386612212522263E-06
	12	21.30742 66932 14032 00983	1.19922 280356250192343E-07
	13	25.52879 15877 34835 39594	2.39366 390999424568976E-09
	14	30.34816 27015 48443 51186	2.62510 544089330309225E-11
	15	35.90120 89340 99651 50287	1.39334 621802858103962E-13
	16	42.41162 25323 47176 65759	2.90261 984780515106580E-16
	17	50.30667 02224 92425 97280	1.59422 226066065645042E-19
	18	60.68103 39501 36842 02153	8.48671 571920735818166E-24

n	k	x_k	$A_k^{(n)}$
19	1	0.04324 45046 61213 83922	1.66000 209853671503324E+00
	2	0.38348 12126 85224 03696	1.59571 262174068252872E+00
	3	1.03854 71664 91653 04793	1.37825 832864668249805E+00
	4	1.97728 93832 63958 81793	9.69252 313172287370685E-01
	5	3.18323 47600 59475 28690	5.12255 903180287909877E-01
	6	4.65754 58279 40744 42715	1.95672 051620070009868E-01
	7	6.41059 50385 37399 11157	5.34178 832377399801404E-02
	8	8.45662 79291 29908 25271	1.03800 170861886852380E-02
	9	10.81320 96191 10665 50430	1.42538 087483794486694E-03
	10	13.50225 01821 74652 22745	1.36473 656991310132541E-04
	11	16.55153 34215 03435 53542	8.92195 415860053689826E-06
	12	19.99691 54731 91950 41979	3.86461 906718051290345E-07
	13	23.88572 46618 92014 41161	1.06346 935906297586382E-08
	14	28.28236 56188 41746 53988	1.75250 137537424875605E-10
	15	33.27824 68380 58750 24599	1.58810 800565768692416E-12
	16	39.01109 90586 40590 73466	6.95343 148602807335064E-15
	17	45.70790 43830 00489 03304	1.18850 345604299334947E-17
	18	53.80217 70579 72177 73522	5.29169 757304889423800E-21
	19	64.40491 73812 25443 75920	2.21821 150428783252674E-25
20	1	0.04082 30253 38156 11533	1.61314 859882099069590E+00
	2	0.36241 98913 69484 41895	1.55614 435918096669021E+00
	3	0.98318 34870 79496 23058	1.36217 169238524474321E+00
	4	1.87465 80821 61782 29018	9.87543 268908263541835E-01
	5	3.02040 57391 88331 09831	5.48860 188319514959432E-01
	6	4.41989 77956 70103 07708	2.24699 964130963043233E-01
	7	6.08152 16090 55625 46003	6.69504 879630584595820E-02
	8	8.01728 55804 79574 77279	1.44723 626181635203882E-02
	9	10.24187 26950 43467 26014	2.25860 543705652159561E-03
	10	12.77332 26304 44417 98172	2.51978 311825296293492E-04
	11	15.63412 43889 01261 54565	1.97782 611502342820577E-05
	12	18.85271 57831 99048 90014	1.06738 641817972591112E-06
	13	22.46568 91261 81524 80853	3.83651 351958084207565E-08
	14	26.52122 87391 84444 25577	8.79156 730318820236839E-10
	15	31.08478 80124 39808 41333	1.20878 413889599144451E-11
	16	36.24915 13694 23395 00124	9.14040 286848493069803E-14
	17	42.15401 89152 15979 17369	3.33204 535570785317385E-16
	18	49.02953 72584 18913 88463	4.71468 852721916298103E-19
	19	57.31525 68850 98845 69556	1.71682 286270280683535E-22
	20	68.13797 91326 12018 17299	5.72033 430916686829512E-27

6. Miscellaneous:

$$\int x^n \cdot J_0(x) \cdot J_1^{n-1}(x)\,dx = \frac{x^n}{n}\,J_1^n(x)\,, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots ...$$

$$\int \frac{J_0(x)\,dx}{x^n\,J_1^{n+1}(x)} = -\,\frac{1}{n\,x^n\,J_1^n(x}\,, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots ...$$

$$\int x^n \cdot I_0(x) \cdot I_1^{n-1}(x)\,dx = \frac{x^n}{n}\,I_1^n(x)\,, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots ...$$

$$\int \frac{I_0(x)\,dx}{x^n\,I_1^{n+1}(x)} = -\,\frac{1}{n\,x^n\,I_1^n(x)}\,, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots ...$$

$$\int x^n \cdot K_0(x) \cdot I_1^{n-1}(x)\,dx = -\frac{x^n}{n}\,K_1^n(x)\,, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots ...$$

$$\int \frac{K_0(x)\,dx}{x^n\,K_1^{n+1}(x)} = \frac{1}{n\,x^n\,K_1^n(x)}\,, \quad n = \pm 1,\ \pm 2,\ \pm 3,\ \ldots ...$$

$$\int \frac{\Phi(x)}{x}\,dx = \Lambda_1(x) - x\,J_0(x) - \Phi(x)$$

$$\int \frac{x^5\,J_1(px^2+i)\,dx}{px^2+i} = \frac{1}{2p^3}\,\left[J_1(px^2+i) - (px^2-i)J_0(px^2+i)\right]$$

7. Used special functions and defined functions:

Used functions:

$J_\nu(x)$	Bessel function of the first kind	
$I_\nu(x)$	Modified Bessel function (of the first kind)	
$Y_\nu(x)$	Bessel function of the second kind, Neumann's function, Weber's function	
$K_\nu(x)$	Modified Bessel function (of the second [third] kind), MacDonald Function	
$H_\nu^{(p)}(x)$, $p=1,2$	Bessel function of the third kind, Hankel function	
$\Gamma(x)$	Gamma function $\Gamma(x)=\int_0^\infty t^{x-1}\,e^{-t}\,dt$	[1] 6.1, [2] 1.1, [4] II.1.,[5] V, [7]
$\Gamma(x,z)$	Upper incomplete Gamma function $\Gamma(x,z)=\int_z^\infty t^{x-1}\,e^{-t}\,dt$	[1] 6.5, [6] 8.35
$\mathrm{Ei}\,(x)$	Exponential integral $E_1(x)=\int_x^\infty \frac{e^{-t}\,dt}{t}=\int_1^\infty \frac{e^{-xt}\,dt}{t}$	[1] 5.1, [4] II.5.
$\mathbf{H}_\nu(x)$	Struve functions	[1] 12.1, [2] 10.1, [5] XIII 2., [7] 8.55
$\mathbf{L}_\nu(x)$	Modified Struve functions	[1] 12.2, [2] 10.1, [7] 8.55
$Ji_0(x)$	$Ji_0(x)=\int_x^\infty \frac{J_0(t)\,dt}{t}$	[1] 11.1.19, [9]
$s_{\mu,\nu}(x)$	Lommel functions	[2] 10.1, [7] 8.57
$P_n^m(x)$	(Associated) Legendre functions of the first kind	[1] 8, [5] XII 3.1.
$P_n(x)$	Legendre polynom	$w(x)\equiv 1$
$T_n(x)$	Chebyshev polynom, first kind	$w(x)=1/\sqrt{1-x^2}$
$U_n(x)$	Chebyshev polynom, second kind	$w(x)=\sqrt{1-x^2}$
$C_n^{(\alpha)}(x)$	Ultraspherical (Gegenbauer) Polynomials	$w(x)=(1-x^2)^{\alpha-1/2}$
$C_n^{(\alpha,\beta)}(x)$	Jacobi Polynomials	$w(x)=(1-x)^\alpha(1+x)^\beta$
$L_n(x)$	Laguerre polynom	$w(x)=e^{-x}$
$L_n^{(\alpha)}(x)$	Generalized Laguerre polynom	$w(x)=x^\alpha\,e^{-x}$
$H_n(x)$	Hermite polynom	$w(x)=\exp(-x^2)$
$\mathbf{E}(x),\ \mathbf{K}(x)$	Complete elliptic integrals	[1], 17.3., [5] IX 30., [7] 8.11

Defined functions:

Function	Page
$\Phi(x) = \frac{\pi x}{2}[J_1(x) \cdot \mathbf{H}_0(x) - J_0(x) \cdot \mathbf{H}_1(x)]$	9
$\Phi_Y(x) = \frac{\pi x}{2}[Y_1(x) \cdot \mathbf{H}_0(x) - Y_0(x) \cdot \mathbf{H}_1(x)]$	9
$\Phi_H^{(1)} = \frac{\pi x}{2}[H_1^{(1)}(x) \cdot \mathbf{H}_0(x) - H_0^{(1)}(x) \cdot \mathbf{H}_1(x)]$	9
$\Phi_H^{(2)} = \frac{\pi x}{2}[H_1^{(2)}(x) \cdot \mathbf{H}_0(x) - H_0^{(2)}(x) \cdot \mathbf{H}_1(x)]$	9
$\Psi(x) = \frac{\pi x}{2}[I_0(x) \cdot \mathbf{L}_1(x) - I_1(x) \cdot \mathbf{L}_0(x)]$	9
$\Psi_K(x) = \frac{\pi x}{2}[K_0(x) \cdot \mathbf{L}_1(x) + K_1(x) \cdot \mathbf{L}_0(x)]$	9
$\Theta(x) = \int_0^x J_0^2(t)\,dt$	271
$\Omega(x) = \int_0^x I_0^2(t)\,dt$	271
$\Lambda_0(x) = \int_0^x J_0(t)\,dt$	119
$\Lambda_0^* = \int_0^x I_0(t)\,dt(x)$	121
$\Lambda_1(x) = \int_0^x t^{-1} \cdot \Lambda_0(t)\,dt$	121
$\Lambda_1^*(x) = \int_0^x t^{-1} \cdot \Lambda_0^*(t)\,dt$	123
$\Theta_0(x;\gamma) = \int_0^x J_0(t)\,J_0(\gamma t)\,dt$	303
$\Omega_0(x;\gamma) = \int_0^x I_0(t)\,I_0(\gamma t)\,dt$	303
$\Theta_1(x;\gamma) = \int_0^x J_1(t)\,J_1(\gamma t)\,dt$	305
$\Omega_1(x;\gamma) = \int_0^x I_1(t)\,I_1(\gamma t)\,dt$	305
$\mathfrak{H}_p(x,a),\ p = 0,1$	76
$\mathfrak{H}_p^*(x,a),\ p = 0,1$	82